Generalizing mutational equations for uniqueness of some nonlocal 1^{st}–order geometric evolutions

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Abstract. The mutational equations of Aubin extend ordinary differential equations to metric spaces (with compact balls). In first–order geometric evolutions, however, the topological boundary need not be continuous in the sense of Painlevé–Kuratowski.
So this paper suggests a generalization of Aubin's mutational equations that extends classical notions of dynamical systems and functional analysis beyond the traditional border of vector spaces: Distribution–like solutions are introduced in a set just supplied with a countable family of (possibly non-symmetric) distance functions. Moreover their existence is proved by means of Euler approximations and a form of "weak" sequential compactness (although no continuous linear forms are available beyond topological vector spaces).
This general framework is applied to a first-order geometric example, i.e. compact subsets of $\mathbb{R}^N$ evolving according to the nonlocal properties of both the current set and its proximal normal cones. Here neither regularity assumptions about the boundaries nor the inclusion principle are required. In particular, we specify sufficient conditions for the uniqueness of these solutions.

Keywords. Mutational equation, timed ostensible metric (non–symmetric distance with time orientation), reachable set of differential inclusion, proximal normal cone, interior and exterior ball condition on sets

Contents

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1 Introduction

Whenever different types of evolutions meet, they usually do not have an obvious vector space structure in common providing a basis for differential calculus. In particular, "shapes and images are basically sets, not even smooth" as Aubin stated [1]. So he regards this obstacle as a starting point for extending ordinary differential equations to metric spaces – the so–called *mutational equations* [1, 2, 3].
Aubin's initial idea is to replace the directional vector $v \in \mathbb{R}^N$ by the corresponding "elementary deformation" $(h, x) \longmapsto x + h\,v$. Generally speaking, such a so-called *transition* ϑ specifies the state $\vartheta(h, x)$ that the initial point x reaches at time h. Choosing the continuity assumptions about $(h, x) \longmapsto \vartheta(h, x)$ appropriately, Aubin was free to dispense with any (affine-) linear structure and followed the same track as for ordinary differential equations - but now in a metric space. (A brief summary of his approach is given here in § 2.)

The main contribution of this paper is to generalize his theory of mutational equations as an analytical framework for dynamical systems beyond the traditional border of vector spaces – and even beyond metric spaces. In a word, we weaken the initial assumptions and implement more "degrees of freedom" (in regard to parameters, for example) so that Euler approximations and a suitable form of sequential compactness still ensure existence and estimates of (generalized) solutions. The key motivation is mostly provided by evolving compact subsets of the Euclidean space – always avoiding both regularity conditions on their topological boundaries and restricting to the popular inclusion principle (as e.g. in [8, 9]).

In comparison to more popular concepts like viscosity solutions, an essential advantage of mutational equations (in every facet so far) is that *the main existence results hold for systems.* So they provide a common basis for solving completely different types of evolutions simultaneously – after each component has been verified separately to fit in this framework. In particular, the exact relationship between (generalized) "mutational solutions" and more popular concepts (like weak or mild solutions of PDEs) is a characteristic feature of each component and belongs to such a verification.
Meanwhile some non-geometric examples have been investigated as potential applications of generalized mutational equations – such as mild solutions of semilinear evolution equations in reflexive Banach spaces [23] and distributional solutions of nonlinear transport equations for Radon measures on $\mathbb{R}^N$ with compact support [19].
Time–dependent compact sets in $\mathbb{R}^N$, however, may be regarded as the most challenging example, because they are lacking any linear structure. Indeed, we cannot simply exploit the well-known conclusions of functional analysis for generalizing existence results, but we have to "re-interpret" them beyond vector spaces. "Distributions" and "weakly compact" represent such analytical terms generalized in this paper.

From now on, we focus on evolving compact subsets of $\mathbb{R}^N$ as key motivation for generalizing mutational equations. In his original concept [1, 2, 3], Aubin uses reachable sets of differential inclusions as transitions on the metric space $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$ of nonempty compact subsets of $\mathbb{R}^N$

supplied with the Pompeiu–Hausdorff distance $d\!\!l$.
However this approach (also called *morphological transitions*) can hardly be applied to geometric evolutions depending on the topological boundary explicitly. Indeed, roughly speaking, "holes" of sets might disappear while evolving along differential inclusions and thus, strictly speaking, the topological boundary need not be "continuous" (in the sense of Painlevé–Kuratowski) with respect to time.

This obstacle has been the motivation in [20] for extending mutational equations to a nonempty set E and an *ostensible metric*, i.e. a distance function $q : E \times E \longrightarrow [0, \infty[$ satisfying just the triangle inequality and $q(x, x) = 0$ for each $x \in E$.
Two new aspects were introduced for handling the lacking continuity properties of shape evolutions: Firstly, we dispensed with the symmetry of the distance function – introducing so-called ostensible metrics. Secondly, distribution-like solutions were defined although continuous linear forms are not available beyond the traditional border of vector spaces. Indeed, as general key idea of distributions, we regard: "Select an important property and then try to preserve it (at least) for all 'test elements' specified before." In the classical sense, this feature is partial integration. For an ostensible metric space (E, q), however, we preferred the "structural estimate" of the distance $h \mapsto q(\vartheta(h, x), \ \tau(h, y))$ between two transitions ϑ, τ starting in any points $x, y \in E$, i.e.

$$q(\vartheta(h, x), \ \tau(h, y)) \ \le \ (q(x, y) \ + \ h \cdot Q(\vartheta, \tau)) \ \cdot \ e^{\alpha(\tau)\, h}$$

The first point x was to be restricted to a given "test set" $D \subset E$.
These notions led to so–called *forward transitions* on an ostensible metric space and *right–hand forward solutions* of generalized mutational equations. The main points of this concept are summarized in § 3 and, Aubin's original version of mutational equations proves to be the special case with $D = E$.
As geometric example in [20, 23], the evolution of compact sets was prescribed as a function of the current set and all its normal cones (at the boundary). So its non-local properties "up to first order" could be taken into consideration – without regularity restriction of the boundary and dispensing with the popular inclusion principle. The "test elements" were provided by all compact N–dimensional submanifolds of $\mathbb{R}^N$ with $C^{1,1}$ boundary and, reachable sets of differential inclusions (again) induced the forward transitions. This example required us to investigate the regularity of reachable sets more intensively (see e.g. Appendix A below). In particular, sufficient conditions on the differential inclusions were specified for preserving the initial $C^{1,1}$ regularity of boundaries shortly.

However, this example also demonstrates a weakness of the "forward" concept. In regard to uniqueness, the results of [20] require the assumption that, roughly speaking, compact sets with $C^{1,1}$ boundary do not loose their regularity too quickly (see Proposition 3.18 in [20] as counterpart of Proposition 5.6 here). However this condition is not obvious to verify for the geometric example of [20, 23].

This lack of uniqueness has motivated us to generalize mutational equations once more in this paper. If we cannot dispense with the condition how long "test elements" preserve this feature, then we "expand" the set of test elements. For the previous geometric example in particular, it might be helpful to take *all* compact subsets into consideration (and not just the compact N–dimensional submanifolds of $\mathbb{R}^N$ with $C^{1,1}$ boundary). Then the missing continuity of topological boundaries forms an obstacle to right–hand forward solutions, though.
For overcoming this obstacle, we return to the basic notions when extending mutational equations: Right–hand forward solutions (in [20, 23]) can be interpreted as a form of *distributional solution*. Indeed, they do not rely on partial integration, but preserve a fixed structural estimate while comparing with the evolutions of all test elements shortly.
So an essential new idea of this paper is similar to Petrov–Galerkin methods in numerics: *The test elements need not belong to the same set as the values of solutions.* Just some continuity properties have to be preserved when comparing their evolutions (along transitions) with each other.
Furthermore, the "mode" of comparing evolving states is changed in several respects:

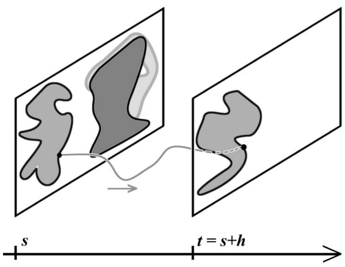
1. More than one distance function can be taken into consideration simultaneously. Now a (at most) countable family of ostensible metrics is given. This modification has already proved to be useful for adapting the weak topology of a Banach space – as e.g. for semilinear evolution equations in a reflexive Banach space [23].
   For the evolution of compact sets here in § 6, we use this additional freedom to consider only boundary points with exterior balls of radius $\geq \varepsilon$ (for given $\varepsilon > 0$). Assuming this family to be (at most) countable has essentially the advantage that Cantor's diagonal construction can be applied easily for obtaining statements "for all $\varepsilon \in \mathcal{J}$" – without the proofs becoming significantly more complicated. Moreover, as additional degrees of freedom, all parameters of transitions may now depend on the index $\varepsilon \in \mathcal{J}$ of the current ostensible metric.

2. Whenever compact sets of $\mathbb{R}^N$ are expanding, connected components of their topological boundaries might disappear. So, in other words, information about the boundaries is "lost irreversibly" while expanding. This phenomenon holds for reachable sets of differential inclusions since the flow along a differential inclusion can always be interpreted as superposition of a flow along an ODE and an expansion. It has already motivated us to dispense with the symmetry of distance functions (and introducing *ostensible metrics*) in [20, 23].
   The distinction between "earlier" and "later" state can be implemented in a very strict way and, the whole theory of mutational equations is based only on comparing "later" states with "earlier" ones (but not necessarily vice versa). For distinguishing between "earlier" and "later", however, we require an additional real component indicating time. So the tuple $\widetilde{E} \stackrel{\text{Def.}}{=} \mathbb{R} \times E$ will here play the role of the basic set $E \neq \emptyset$ and, $\widetilde{\mathcal{D}} \stackrel{\text{Def.}}{=} \mathbb{R} \times \mathcal{D}$ consists of all "test elements" $z \in \mathcal{D} \neq \emptyset$ supplied with a supplementary time component. (As mentioned before, $\mathcal{D} \subset E$ is *not* supposed here any longer.) From now on, every notation with tilde refers to such tuples with separate (real) time component and, set $\pi_1 : \widetilde{\mathcal{D}} \cup \widetilde{E} \longrightarrow \mathbb{R},\ (t, x) \longmapsto t.$

Let us combine this notion with point (1.) before: Some (at most) countable index set $\mathcal{J} \neq \emptyset$ is given and for each $\varepsilon \in \mathcal{J}$, a generalized distance function $\widetilde{q}_\varepsilon : (\widetilde{\mathcal{D}} \cup \widetilde{E}) \times (\widetilde{\mathcal{D}} \cup \widetilde{E}) \longrightarrow [0, \infty[$ ought to play the role of an ostensible metric. Paying regard to the distinction between "earlier" and "later" states, however, we now demand (only) the so-called *timed triangle inequality*, i.e.

$$\widetilde{q}_\varepsilon((r,x),\ (t,z)) \ \leq\ \widetilde{q}_\varepsilon((r,x),\ (s,y)) \ +\ \widetilde{q}_\varepsilon((s,y),\ (t,z))$$

for all $(r,x), (s,y), (t,z) \in \widetilde{\mathcal{D}} \cup \widetilde{E}$ with $r \leq s \leq t$.
This "analytical modesty" (in regard to the triangle inequality) proves to be very helpful in the geometric example of § 6. Indeed, the difference of time components will decide about the minimal radius of an exterior ball so that the corresponding boundary point is taken into account (for the distance) and, it will indicate how long an evolution has already lasted.
The latter is implemented by the evolution of the time component: The time component of all elements in $\widetilde{E}$ have to increase at constant speed 1 whereas the time components of "test elements" in $\widetilde{\mathcal{D}}$ are allowed to grow more slowly (or even stay constant as sketched on the right).



3. Two parameters of a transition ϑ play an essential role. The first one concerns the continuity of ϑ with respect to the initial point: $q(\vartheta(h,z),\ \vartheta(h,y)) \leq q(z,y) \cdot e^{\alpha \cdot h}$.
The second one specifies how long a given "test element" stays in the "test set", i.e. $\vartheta(h,z) \in D$ for $z \in D$ and all $h \in [0, \mathcal{T}]$.
For so-called forward transitions [20], α was chosen independently from both initial points $z \in D, y \in E$ and, the time parameter $\mathcal{T}$ depended only on the "test element" $z \in D$. Now we permit further dependences: Both parameters may depend on the "test element" and the index $\varepsilon \in \mathcal{J}$ of distance. To be more precise, for each index $\varepsilon \in \mathcal{J}$ and "test element" $\widetilde{z} \in \widetilde{\mathcal{D}}$, a transition $\widetilde{\vartheta}$ now requires parameters $\alpha_\varepsilon(\widetilde{\vartheta}, \widetilde{z}) \in [0, \infty[$ and $\mathbb{T}_\varepsilon = \mathbb{T}_\varepsilon(\widetilde{\vartheta}, \widetilde{z}) \in]0,1]$ such that

$$\begin{cases} \quad \widetilde{\vartheta}(s,\widetilde{z}),\ \ \widetilde{\vartheta}(s+h,\widetilde{z}) & \in \quad \widetilde{\mathcal{D}} \\ \widetilde{q}_\varepsilon(\vartheta(s+h,\widetilde{z}),\ \widetilde{\vartheta}(h,\widetilde{y})) & \leq \quad \widetilde{q}_\varepsilon(\widetilde{\vartheta}(s,\widetilde{z}),\ \widetilde{y}) \ \cdot\ e^{\alpha_\varepsilon \cdot h} \end{cases}$$

for all $0 \leq s \leq s+h \leq \mathbb{T}_\varepsilon$ and $\widetilde{y} \in \widetilde{E}$ with $s + \pi_1\, \widetilde{z} \leq \pi_1 \widetilde{y}$.
So $\alpha_\varepsilon(\widetilde{\vartheta}, \widetilde{z})$ is now to "cover" the estimate for all "test elements" $\widetilde{\vartheta}(s,\widetilde{z}) \in \widetilde{\mathcal{D}}$ with $0 \leq s \leq \mathbb{T}_\varepsilon$. This additional freedom does not complicate any proof and, to be honest, it is less relevant for the example of § 6 since we will find a uniform bound of α_ε in Proposition 6.11.
The key point for our geometric example here is rather that the time parameter may depend on the index $\varepsilon \in \mathcal{J}$. Indeed, the results about reachable sets in Appendix A imply that $\mathbb{T}_\varepsilon$ will be chosen only as function of $\varepsilon \in \mathcal{J}$ – and not of the "test element" (see Proposition 6.11 below). Such a uniform lower bound of $\mathbb{T}_\varepsilon$ (for each fixed $\varepsilon \in \mathcal{J}$) opens the door to exploiting the estimates about uniqueness (see Propositions 5.6,6.14). That is a substantial progress of this paper in comparison with earlier results (as [20, 23]).

Together with the Petrov-Galerkin possibility $\widetilde{\mathcal{D}} \not\subset \widetilde{E}$, these three points summarize the new aspects from the *forward* transitions (presented in [20, 23]) to so-called *timed sleek transitions* (introduced here in § 4).
In fact, we obtain existence and stability results in essentially the same way as in [20, 23] because the proofs only require us to focus on how indices and parameters depend on each other. Furthermore, the earlier concept of "forward solutions" [20] and Aubin's original version of mutational equations are just special cases again. So in particular, the examples of [20, 23] and [1] fulfill the assumptions about "timed right–hand sleek solutions" presented here in § 4.

For completing the list of new results about mutational equations in this paper, we focus on suitable compactness in $(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$. Indeed, all proofs of existence here are based on Euler approximations in combination with some sequential compactness. In regard to mutational equations, we are free to benefit from potential smoothening effects of transitions (on elements such as compact subsets) and thus introduced the term "transitionally compact" in [20, 23]. This concept is easy to extend to the tuple $(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ and its timed sleek transitions (see Definition 4.12).
Considering the geometric example of § 6, however, it is a very restrictive hypothesis since the boundary points of different sets with exterior balls of fixed radius will be compared (as we explain in a moment). In fact, the appropriate choice of topology (in regard to compactness) is an old analytical problem even in Banach spaces. Weak compactness has proved to be a useful alternative. Here we want to extend this notion to sets without linear structure. In particular, no continuous linear forms are available in $\widetilde{E}$ or $\widetilde{\mathcal{D}}$. As an alternative starting point, we seize a well-known representation of the norm in a Banach space $(X, \|\cdot\|_X)$

$$\|z\|_X \;=\; \sup\;\{\, y^*(z) \mid y^* : X \longrightarrow \mathbb{R} \text{ linear, continuous, } \;\|y^*\|_{X^*} \le 1 \,\}.$$

More generally, each $\widetilde{q}_\varepsilon : (\widetilde{\mathcal{D}} \cup \widetilde{E}) \times (\widetilde{\mathcal{D}} \cup \widetilde{E}) \longrightarrow [0, \infty[$ is now assumed to be supremum with respect to an additional parameter $\kappa \in \mathcal{I}$: $\quad \widetilde{q}_\varepsilon \;=\; \sup_{\kappa \in \mathcal{I}} \; \widetilde{q}_{\varepsilon,\kappa}$.
Here $\widetilde{q}_{\varepsilon,\kappa} : (\widetilde{\mathcal{D}} \cup \widetilde{E}) \times (\widetilde{\mathcal{D}} \cup \widetilde{E}) \longrightarrow [0, \infty[\;\; (\varepsilon \in \mathcal{J}, \kappa \in \mathcal{I})$ is a countable family of functions that need not satisfy the timed triangle inequality separately – in contrast to each $\widetilde{q}_\varepsilon$ $(\varepsilon \in \mathcal{J})$.
We assume instead that each $\kappa \in \mathcal{I}$ has counterparts $\kappa', \kappa'' \in \mathcal{I}$ fulfilling

$$\widetilde{q}_{\varepsilon,\kappa}(\widetilde{y}_1, \widetilde{y}_3) \;\le\; \widetilde{q}_{\varepsilon,\kappa'}(\widetilde{y}_1, \widetilde{y}_2) \;+\; \widetilde{q}_{\varepsilon,\kappa''}(\widetilde{y}_2, \widetilde{y}_3)$$

for all $\widetilde{y}_1, \widetilde{y}_2, \widetilde{y}_3 \in \widetilde{\mathcal{D}} \cup \widetilde{E}$ with $\pi_1 \widetilde{y}_1 \;\le\; \pi_1 \widetilde{y}_2 \;\le\; \pi_1 \widetilde{y}_3$.
In regard to sequential compactness (and the wanted "convergent subsequences"), the key point now presented in § 5 is: Right–convergence with respect to each $\widetilde{q}_\varepsilon$ can be reduced to right–convergence with respect to each $\widetilde{q}_{\varepsilon,\kappa}$, i.e. not necessarily uniformly in $\kappa \in \mathcal{I}$. This modification comes into play only when proving that the limit of some Euler approximations is a solution (see Proposition 5.1) and, it again results from answering a question being typical for generalizing mutational equations, i.e. which "sufficiently large" index of a subsequence may depend on which parameter or index of distance. In fact, the weaker demand of convergence is easily verified in § 6 thanks to the geometric results about proximal normals and exterior balls in Appendix B.

In § 6, the new concept is applied to first–order geometric evolutions, i.e. the evolution of nonempty compact subsets of $\mathbb{R}^N$ may depend on nonlocal properties of both the current subset and its proximal normal cones at the boundary. As mentioned briefly before, we dispense with $C^{1,1}$ regularity of the "test elements" (in contrast to [20, 23]) and, we distinguish between the basic set $\widetilde{E}$ and the set $\widetilde{\mathcal{D}}$ of test elements by additional components instead:

$$\begin{aligned}\widetilde{E} &:= \mathbb{R} \times \{1\} \times \mathcal{K}(\mathbb{R}^N),\\ \widetilde{\mathcal{D}} &:= \mathbb{R} \times \{0\} \times \mathcal{K}(\mathbb{R}^N).\end{aligned}$$

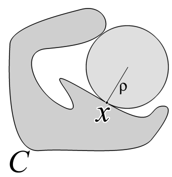
The second (auxiliary) component is just to specify how the first component (indicating time) evolves along a transition $\widetilde{\vartheta}$. Indeed, for a set–valued map $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ and a compact set $K \subset \mathbb{R}^N$, let $\vartheta_F(h, K) \subset \mathbb{R}^N$ denote the reachable set of K and the differential inclusion $\dot{x}(\cdot) \in F(x(\cdot))$ (a.e.) at time $h \geq 0$. Then we distinguish between

$$\widetilde{\vartheta}_F(h, \widetilde{K}) := (t+h,\ 1,\ \vartheta_F(h,K)) \qquad \text{for } \widetilde{K} = (t,1,K) \in \widetilde{E}$$

and

$$\widetilde{\vartheta}_F(h, \widetilde{K}) := (t,\ 0,\ \vartheta_F(h,K)) \qquad \text{for } \widetilde{K} = (t,0,K) \in \widetilde{\mathcal{D}}.$$

The real time component comes into play only for comparing proximal normal cones: For a closed subset $C \subset \mathbb{R}^N$, $x \in \partial C$ and any $\rho > 0$, let $N^P_{C,\rho}(x) \subset \mathbb{R}^N$ consist of all proximal normal vectors $\eta \in N^P_C(x) \setminus \{0\}$ with the proximal radius $\geq \rho$ (thus it might be empty) and set ${}^\flat N^P_{C,\rho}(x) := N^P_{C,\rho}(x) \cap \mathbb{B}_1(0)$ (see Definition 6.2).



With $d\!l$ denoting the Pompeiu–Hausdorff distance between compact subsets of $\mathbb{R}^N$, we define the following distance function for any $\varepsilon \in [0,1]$ and $(s,\mu,C),\ (t,\nu,D) \in \widetilde{\mathcal{D}} \cup \widetilde{E}$

$$\begin{aligned}&\widetilde{q}_{\mathcal{K},\varepsilon}((s,\mu,C),\ (t,\nu,D)) := \\ &\quad d\!l(C,D) + \\ &\quad \limsup_{\kappa \downarrow 0} \int_\varepsilon^\infty \psi(\rho+\kappa+200\,\Lambda\,|t-s|) \cdot \operatorname{dist}\Big(\text{Graph } {}^\flat N^P_{D,\,(\rho+\kappa+200\,\Lambda\,|t-s|)},\ \text{Graph } {}^\flat N^P_{C,\,\rho}\Big)\, d\rho\end{aligned}$$

with a fixed nonincreasing weight function $\psi \in C_0^\infty([0,2[)$, $\psi \geq 0$, and a parameter $\Lambda > 0$ (related with the differential inclusions inducing transitions).
This distance function $\widetilde{q}_{\mathcal{K},\varepsilon} : (\widetilde{\mathcal{D}} \cup \widetilde{E}) \times (\widetilde{\mathcal{D}} \cup \widetilde{E}) \longrightarrow [0,\infty[$ is motivated by the features of reachable sets of differential inclusions: Roughly speaking, when considering an arbitrary compact set $K \subset \mathbb{R}^N$ while evolving along a differential inclusion $\dot{x}(\cdot) \in F(x(\cdot))$ (a.e.), its exterior balls do not change very much for short times if the Hamiltonian function of F is C^2. To be more precise, Appendix A provides a connection between the exterior balls of $\vartheta_F(t,K)$ and K (and vice versa) for small times $t > 0$:

Lemma 1.1 *Assume for the set–valued map $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ that its values are nonempty, compact, convex and that its Hamiltonian is $C^2(\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}))$ with $\|\mathcal{H}_F\|_{C^2(\mathbb{R}^N \times\, \partial\mathbb{B}_1)} < \Lambda$. Then for every radius $r_0 \in\,]0,2]$, there exists some time $\tau = \tau(r_0, \Lambda) > 0$ such that for any $K \in \mathcal{K}(\mathbb{R}^N)$, $r \in [r_0, 2]$ and $t \in [0, \tau[$,*

1. *each $x_1 \in \partial\vartheta_F(t,K)$ and $\nu_1 \in N^P_{\vartheta_F(t,K)}(x_1)$ with proximal radius r are linked to some $x_0 \in \partial K$ and $\nu_0 \in N^P_K(x_0)$ with proximal radius $\geq r - 81\,\Lambda\, t$ by a solution to $\dot{x}(\cdot) \in F(x(\cdot))$ a.e. and its adjoint arc, respectively.*

2. *each* $x_0 \in \partial K$ *and* $\nu_0 \in N^P_K(x_0)$ *with proximal radius* r *are linked to some* $x_1 \in \partial \vartheta_F(t, K)$ *and* $\nu_1 \in N^P_{\vartheta_F(t,K)}(x_1)$ *with proximal radius* $\geq\ r - 81\,\Lambda\, t$ *by a solution to* $\dot{x}(\cdot) \in F(x(\cdot))$ *a.e. and its adjoint arc, respectively.*

So the difference of more than $200\,\Lambda\, t$ (in respect to proximal radii) proves to have two advantages. Firstly, a form of equi–Lipschitz continuity with respect to time

$$\widetilde{q}_{\mathcal{K},\varepsilon}\left(\widetilde{\vartheta}_F(s,\,\widetilde{K}),\ \widetilde{\vartheta}_F(t,\,\widetilde{K})\right) \ \leq\ \Lambda\,(1 + \|\psi\|_{L^1}\ (e^{\Lambda} + 1))\ \cdot\ |t - s|$$

holds for every initial element $\widetilde{K} \in \widetilde{E}$ and any times $0\ \leq\ s\ <\ t\ \leq\ 1$ (Lemma 6.8). Secondly, we can compare the evolution of arbitrary elements $\widetilde{K}_1 = (t_1, 0, K_1) \in \widetilde{\mathcal{D}}$, $\widetilde{K}_2 = (t_2, 1, K_2) \in \widetilde{E}$ with $t_1 \leq t_2$ while evolving along two set–valued maps F, G that satisfy the conditions of Lemma 1.1. The different features of their time components, in particular, lead to the estimate

$$\widetilde{q}_{\mathcal{K},\varepsilon}\left(\widetilde{\vartheta}_F(h, \widetilde{K}_1),\ \widetilde{\vartheta}_G(h, \widetilde{K}_2)\right)\ \leq\ e^{C(\Lambda,\varepsilon)\cdot h}\cdot\left(\widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{K}_1,\ \widetilde{K}_2)\ +\ \mathrm{C}\ h\ \|\mathcal{H}_F - \mathcal{H}_G\|_{C^1(\mathbb{R}^N\times\,\partial\mathbb{B}_1)}\right)$$

(Lemma 6.9). Thus the required continuity properties of timed sleek transitions are fulfilled — without any regularity assumptions about the compact subsets.

In regard to $\widetilde{q}_\varepsilon$, the additional limit superior with respect to $\kappa \downarrow 0$ has a geometric motivation. Appendix B investigates the proximal normal subsets $N^P_{K_n,\,\rho}(\cdot)$ for a converging sequence $(K_n)_{n\,\in\,\mathbb{N}}$ of compact subsets. Indeed, Proposition B.1 states for $K = \mathrm{Lim}_{n\,\to\,\infty}\, K_n\ \in\ \mathcal{K}(\mathbb{R}^N)$

$$\begin{array}{llcll} & \mathrm{Limsup}_{n\,\to\,\infty}\ \mathrm{Graph}\ {}^\flat N^P_{K_n,\rho} & \subset & \mathrm{Graph}\ {}^\flat N^P_{K,\rho} & \text{for any}\ \ \rho > 0,\\ \text{but} & \mathrm{Graph}\ {}^\flat N^P_{K,\,\rho} & \subset & \mathrm{Liminf}_{n\,\to\,\infty}\ \mathrm{Graph}\ {}^\flat N^P_{K_n,\,r} & \text{for any}\ \ 0 < r < \rho \end{array}$$

and in general, we cannot dispense with the restriction $r < \rho$. Thus, it does not seem advisable to compare proximal subsets of identical proximal radii with each other when verifying a form of sequential compactness. Here the weaker demand on right-convergence introduced in § 5 before proves to be useful (Proposition 6.12). In other words, the comparison of proximal normal subsets is rather "epigraphical" (than pointwise) with respect to the proximal radius.

So the theory of mutational equations and their "timed right-hand sleek solutions" implies the following results about existence and uniqueness. In particular, the proposition about uniqueness is the main new geometric feature of this paper in comparison with [20, 23].

Proposition 1.2 (Existence due to continuity)
Let $\mathrm{LIP}^{(C^2)}_\Lambda(\mathbb{R}^N, \mathbb{R}^N)$ *denote the set of all set–valued maps satisfying the hypotheses of Lemma 1.1. Using the abbreviations* $\widetilde{E} \stackrel{\text{Def.}}{=} \mathbb{R}\times\{1\}\times\mathcal{K}(\mathbb{R}^N)$, $\widetilde{\mathcal{D}} \stackrel{\text{Def.}}{=} \mathbb{R}\times\{0\}\times\mathcal{K}(\mathbb{R}^N)$, *regard the maps* $\widetilde{\vartheta}_F$ *of all set–valued map* $F \in \mathrm{LIP}^{(C^2)}_\Lambda(\mathbb{R}^N, \mathbb{R}^N)$ *as timed sleek transitions on* $(\widetilde{E},\ \widetilde{\mathcal{D}},\ (\widetilde{q}_{\mathcal{K},\varepsilon})_{\varepsilon\in\,]0,1]\cap\mathbb{Q}})$. *For* $\widetilde{f} : \widetilde{E}\times[0,T] \longrightarrow \mathrm{LIP}^{(C^2)}_\Lambda(\mathbb{R}^N, \mathbb{R}^N)$, *suppose*

$$\|\mathcal{H}_{\widetilde{f}(\widetilde{K},t)} - \mathcal{H}_{\widetilde{f}(\widetilde{K}_m,t_m)}\|_{C^1(\mathbb{R}^N\times\,\partial\mathbb{B}_1)} \stackrel{m\to\infty}{\longrightarrow} 0$$

whenever $0 \leq t_m - t \to 0$ *and* $\widetilde{q}_{\mathcal{K},0}(\widetilde{K}, \widetilde{K}_m) \to 0$ $(\widetilde{K}, \widetilde{K}_m \in \widetilde{E},\ \ \pi_1\,\widetilde{K} \leq \pi_1\,\widetilde{K}_m)$.

Then for every initial element $\widetilde{K}_0 = (0, 1, K_0) \in \widetilde{E}$, *there exists a timed right–hand sleek solution* $\widetilde{K}\ :\ [0,T[\ \longrightarrow\ \widetilde{E}$ *of the generalized mutational equation* $\overset{\circ}{\widetilde{K}}(\cdot)\ \ni\ \widetilde{f}(\widetilde{K}(\cdot),\,\cdot)$ *with* $\widetilde{K}(0) = \widetilde{K}_0$, *i.e. satisfying* *a)* $K(0) = K_0$ *and* $\widetilde{K}(\cdot)$ *is Lipschitz continuous in forward time direction w.r.t.* $\widetilde{q}_{\mathcal{K},0}$, *i.e.* $\widetilde{q}_{\mathcal{K},0}(\widetilde{K}(s),\ \widetilde{K}(t)) \ \le\ const(\Lambda, T) \cdot (t-s)$ *for all* $0 \le s < t < T$.

b) $\limsup_{h \downarrow 0} \frac{1}{h} \cdot \Big(\widetilde{q}_{\mathcal{K},\varepsilon}\Big(\widetilde{\vartheta}_{\widetilde{f}(\widetilde{K}(t),\,t)}(h,\ \widetilde{Z}),\ \ \widetilde{K}(t+h) \Big) \ -\ \widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{Z},\ \widetilde{K}(t)) \cdot e^{10\,\Lambda\, e^{2\,\Lambda} \cdot h} \Big) \ \le\ 0$
for every $\varepsilon \in\,]0,1] \cap \mathbb{Q}$, *time* $t \in [0, T[$ *and test set* $\widetilde{Z} \in\ ]-\infty, t] \times \{0\} \times \mathcal{K}(\mathbb{R}^N)$.
In particular, $\limsup_{h \downarrow 0} \ \frac{1}{h} \cdot d\!\!l \Big(\vartheta_{\widetilde{f}(\widetilde{K}(t),\,t)}(h,\ K(t)),\ \ K(t+h) \Big) \ = \ 0$ *for all* t.

Proposition 1.3 (Uniqueness due to Lipschitz continuity)
For $\widetilde{f} : (\widetilde{\mathcal{D}} \cup \widetilde{E}) \times [0, T] \longrightarrow \mathrm{LIP}^{(C^2)}_{\Lambda}(\mathbb{R}^N, \mathbb{R}^N)$, *suppose that there exist a modulus* $\widehat{\omega}(\cdot)$ *of continuity and a constant* $L \ge 0$ *satisfying*

$$\|\mathcal{H}_{\widetilde{f}(\widetilde{Z},s)} - \mathcal{H}_{\widetilde{f}(\widetilde{K},t)}\|_{C^1(\mathbb{R}^N \times \partial \mathbb{B}_1)} \ \le\ L \cdot \widetilde{q}_{\mathcal{K},0}(\widetilde{Z}, \widetilde{K}) \ +\ \widehat{\omega}(t-s)$$

for all $0 \le s \le t \le T$ *and* $\widetilde{Z} \in \widetilde{\mathcal{D}}$, $\widetilde{K} \in \widetilde{E}$ $(\pi_1\, \widetilde{Z} \le \pi_1\, \widetilde{K})$.

Then for every initial element $\widetilde{K}_0 \in \widetilde{E}$, *the timed right–hand sleek solution* $\widetilde{K} : [0, T[\longrightarrow \widetilde{E}$ *of the generalized mutational equation* $\overset{\circ}{\widetilde{K}}(\cdot) \ni \widetilde{f}(\widetilde{K}(\cdot),\ \cdot)$ *with* $\widetilde{K}(0) = \widetilde{K}_0$ *is unique.*

Finally, let us give a brief overview of this paper and its structure being already reflected by this introduction. § 2 summarizes Aubin's original proposal how to extend ordinary differential equations to a metric space. In § 3, the key definitions presented in [20] are summarized. Seizing the notion of distributions in vector spaces, they lead to so–called *forward* transitions and *right–hand forward* solutions. In particular, the summary is to serve as a motivation for pointing out the new features of this paper.
Then in § 4, we introduce timed sleek transitions and follow basically the same steps up to existence and uniqueness results about so–called *timed right–hand sleek solutions.* § 5 provides a first proposal how to extend "weakly compact" from Banach spaces to the framework of mutational equations. In a word, an additional assumption about the structure of distance functions enables us to weaken the convergence demands on "converging subsequences" in regard to sequential compactness – but still preserving existence results.
The subsequent paragraph 6 contains the example of first–order geometric evolutions using the countable family of distance functions $(\widetilde{q}_{\mathcal{K},\varepsilon})_{\varepsilon \in\,]0,1] \cap \mathbb{Q}}$. In particular, we verify that reachable sets of maps in $\mathrm{LIP}^{(C^2)}_{\Lambda}(\mathbb{R}^N, \mathbb{R}^N)$ induce timed sleek transitions and investigate some required properties of sequential compactness. Appendix A provides the key tools of reachable sets (of differential inclusions) quoted here in Lemma 1.1. In the end, Appendix B relates the proximal normal subsets $N^P_{K_n,\,\rho}(\cdot)$ of a convergent sequence $(K_n)_{n \in \mathbb{N}}$ in $\mathcal{K}(\mathbb{R}^N)$ with its limit $K = \mathrm{Lim}_{n \to \infty} K_n$.

2 A brief introduction to mutational equations of Aubin

An approach to evolution problems in metric spaces is the *mutational analysis* of Jean–Pierre Aubin presented in [1, 2]. It proves to be the more general background of "shape derivatives" introduced by Jean Céa, Jean–Paul Zolésio et al. (see e.g. [12]) and has similarities to "quasi-differential equations" of Panasyuk [25, 26, 27].

Roughly speaking, the starting point consists in extending the terms "direction" and "velocity" from vector spaces to metric spaces. Then the basic idea of first–order approximation leads to a definition of derivative for curves in a metric space and step by step, we can follow the same track as for ordinary differential equations.

Let us now describe the mutational approach in more detail: In a vector space like $\mathbb{R}^N$, each vector v defines a continuous function

$$[0,\infty[\, \times \mathbb{R}^N \longrightarrow \mathbb{R}^N, \qquad (h,x) \longmapsto x + h\, v$$

mapping the time h and the initial point x to its final point — similar to the topological notion of a homotopy. This concept does not really require addition or scalar multiplication and thus can be applied to every metric space (M,d) instead:

Definition 2.1 ([1]) *Let (M,d) be a metric space.*
A map $\vartheta : [0,1] \times M \longrightarrow M$ is called transition *on (M,d) if it satisfies*

1. $\vartheta(0,x) = x \qquad \forall\ x \in M,$
2. $\limsup_{h \downarrow 0} \frac{1}{h} \cdot d(\vartheta(h,\ \vartheta(t,x)),\ \vartheta(t+h,\ x)) = 0 \qquad \forall\ x \in M,\ t < 1,$
3. $\alpha(\vartheta) := \sup_{x \neq y} \limsup_{h \downarrow 0} \left(\frac{d(\vartheta(h,\,x),\ \vartheta(h,\,y)) - d(x,y)}{h \quad d(x,y)} \right)^+ < \infty,$
4. $\beta(\vartheta) := \sup_{x \in M} \limsup_{h \downarrow 0} \frac{1}{h} \cdot d(x,\ \vartheta(h,\,x)) < \infty$

with the abbreviation $(r)^+ := \max(0,r)$ for $r \in \mathbb{R}$.

Condition (1.) guarantees that the second argument x represents the initial point at time $t=0$. Moreover condition (2.) can be regarded as a weakened form of the semigroup property. Finally the parameters $\alpha(\vartheta),\ \beta(\vartheta)$ imply the continuity of ϑ with respect to both arguments. In particular, condition (4.) together with Gronwall's Lemma ensures the uniform Lipschitz continuity of ϑ with respect to time: $d(\vartheta(s,x),\ \vartheta(t,x)) \le \beta(\vartheta) \cdot |t-s|$ for all $s,t \in [0,1],\ x \in M$.

Obviously the function $[0,1] \times \mathbb{R}^N \longrightarrow \mathbb{R}^N,\ (h,x) \longmapsto x + h\, v$ mentioned before fulfills the conditions on a transition on $(\mathbb{R}^N,\ |\cdot|)$. Let us give some further examples:

1.) Leaving vector spaces like $\mathbb{R}^N$, we consider the set $\mathcal{K}(\mathbb{R}^N)$ of all nonempty compact subsets of $\mathbb{R}^N$ supplied with the so–called *Pompeiu–Hausdorff distance*

$$d\!\!l(K_1,K_2) := \max \Big\{ \sup_{x \in K_1} \operatorname{dist}(x,K_2),\quad \sup_{y \in K_2} \operatorname{dist}(y,K_1) \Big\}$$

It has the advantage that any closed ball of $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$ is compact (see e.g. [1], [29]). Supposing $f : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ again to be bounded and Lipschitz continuous, transitions are defined as *reachable sets* of the vector field f,

$$\begin{array}{rcl}\vartheta_f: \ [0,1]\times\mathcal{K}(\mathbb{R}^N) & \longrightarrow & \mathcal{K}(\mathbb{R}^N) \\ (t,\ K_0) & \longmapsto & \{\, x(t) \mid \exists\ x(\cdot) \in C^1([0,t],\mathbb{R}^N):\ \frac{d}{dt}\,x(\cdot) = f(x(\cdot)), \\ & & \quad x(0) \in K_0\}.\end{array}$$

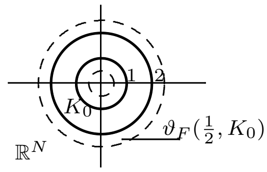
The Theorem of Cauchy–Lipschitz ensures that ϑ_f is a transition on $(\mathcal{K}(\mathbb{R}^N),\ d\!\!l)$ and, $\alpha(\vartheta_f) \le \mathrm{Lip}\ f,\ \beta(\vartheta_f) \le \|f\|_{L^\infty}$ (see [1], Proposition 3.5.2). In this regard, we find a close relation to the *velocity method* of Céa et al. in shape optimization.

2.) Now more than one velocity is admitted at every point of $\mathbb{R}^N$, i.e. strictly speaking, we consider the differential inclusion $\frac{d}{dt}\ x(\cdot) \in F(x(\cdot))$ (a.e.) with a set–valued map $F: \mathbb{R}^N \leadsto \mathbb{R}^N$ instead of the ODE $\frac{d}{dt}\ x(\cdot) = f(x(\cdot))$. For every bounded Lipschitz continuous map $F: \mathbb{R}^N \leadsto \mathbb{R}^N$ with convex values in $\mathcal{K}(\mathbb{R}^N)$,

$$\begin{array}{rcl}\vartheta_F: \ [0,1]\times\mathcal{K}(\mathbb{R}^N) & \longrightarrow & \mathcal{K}(\mathbb{R}^N) \\ (t,\ K_0) & \longmapsto & \{\, x(t) \mid \exists\ x(\cdot) \in W^{1,1}([0,t],\mathbb{R}^N): \\ & & \quad \frac{d}{dt}\,x(\cdot) \in F(x(\cdot))\ \text{a.e.}, \quad x(0) \in K_0\}\end{array}$$

is a transition on $(\mathcal{K}(\mathbb{R}^N),\ d\!\!l)$ — as a consequence of well-known Filippov's Theorem about differential inclusions (see [1], Proposition 3.7.3).

In contrast to example (1.), the reachable set $\vartheta_F(t, K_0)$ of a set–valued map F might change its topological features. $F(\cdot) := \mathbb{B}_1 \stackrel{\text{Def.}}{=} \{\, v \in \mathbb{R}^N \mid |v| \le 1\}$, for example, leads to the expansion with constant speed 1 in all directions and makes the "hole" of the annulus $K_0 := \{\, x \mid 1 \le |x| \le 2\,\} \subset \mathbb{R}^N$ disappear at time 1.



This phenomenon cannot occur in the examples of ordinary differential equations (with Lipschitz right–hand side) since their evolutions are reversible in time.

A transition $\vartheta: [0,1]\times M \longrightarrow M$ provides a first–order approximation of a curve $x(\cdot): [0,T[\longrightarrow M$ at time $t \in [0,T[$ if $\limsup_{h\downarrow 0}\ \frac{1}{h}\cdot d(\vartheta(h,\ x(t)),\ \ x(t+h)) = 0.$

Naturally ϑ need not be unique in general and thus,

Definition 2.2 *Let (M,d) be a metric space and $x(\cdot): [0,T[\longrightarrow M$ a curve. The so–called* mutation *of $x(\cdot)$ at time $t \in [0,T[$, abbreviated as $\overset{\circ}{x}(t)$, consists of all transitions ϑ satisfying* $\limsup_{h\downarrow 0}\ \frac{1}{h}\cdot d(\vartheta(h,\ x(t)),\ \ x(t+h)) = 0.$

Remark 2.3 As an immediate consequence of Definition 2.1 (2), every transition ϑ belongs to its own mutation in the sense of $\vartheta \in \overset{\circ}{x}(t)$ for $x(\cdot) := \vartheta(\cdot, x_0)$ with any $x_0 \in M,\ t \in [0,1[$.

A *mutational equation* is based on a given function f of time $t \in [0,T[$ and state $x \in M$ whose values are transitions on (M,d), i.e. $f: M\times[0,T[\longrightarrow \Theta(M,d),\ (x,t) \longmapsto f(x,t)$, and, we look for a Lipschitz curve $x(\cdot): [0,T[\longrightarrow (M,d)$ such that $f(x(t),t)$ belongs to its mutation $\overset{\circ}{x}(t)$ for almost every time $t \in [0,T[$ (see [1], Definition 1.3.1).

The Theorem of Cauchy–Lipschitz and its proof suggest Euler method for constructing solutions of mutational equations. In this context we need an upper estimate of the distance between two points while evolving along two (different) transitions.
First of all, a distance between two transitions $\vartheta, \tau : [0,1] \times M \longrightarrow M$ has to be defined and, it is based on comparing the evolution of one and the same initial point:

Definition 2.4 ([1], Definition 1.1.2) *Let (M, d) be a metric space. For any two transitions ϑ, τ on (M, d), define* $D(\vartheta, \tau) := \sup_{x \in M} \limsup_{h \downarrow 0} \frac{1}{h} \cdot d(\vartheta(h, x), \ \tau(h, x))$.

As an immediate consequence of triangle inequality, $D(\vartheta, \tau) \le \beta(\vartheta) + \beta(\tau) < \infty$.
Considering the preceding example of $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$ and reachable sets ϑ_F, ϑ_G of bounded Lipschitz maps $F, G : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$, Filippov's Theorem implies $D(\vartheta_F, \vartheta_G) \le \sup_{x \in \mathbb{R}^N} d\!\!l(F(x), G(x))$ (see [1], Proposition 3.7.3). In general, these definitions lead to the substantial estimate:

Lemma 2.5 ([1], Lemma 1.1.3) *For any transitions ϑ, τ on a metric space (M, d) and initial points $x, y \in M$, the distance at each time $h \in [0,1]$ satisfies*

$$d(\vartheta(h, x), \ \tau(h, y)) \ \le \ d(x, y) \cdot e^{\alpha(\vartheta)\, h} \ + \ h \ D(\vartheta, \tau) \cdot \frac{e^{\alpha(\vartheta)\, h} - 1}{\alpha(\vartheta)\ h}. \qquad (*)$$

The proof of this inequality provides an excellent insight into the basic technique for drawing global conclusions from local properties: Due to the definition of transitions, the distance $\psi : [0,1] \longrightarrow [0, \infty[, \ h \longmapsto d(\vartheta(h, x), \tau(h, y))$ is a Lipschitz continuous function of time and satisfies

$$\begin{aligned}
&\lim_{h \downarrow 0} \frac{\psi(t+h) - \psi(t)}{h} \\
&= \lim_{h \downarrow 0} \frac{1}{h} \cdot \Big(d\Big(\vartheta(t+h, x), \ \tau(t+h, y)\Big) - d\Big(\vartheta(t, x), \ \tau(t, y)\Big)\Big) \\
&\le \limsup_{h \downarrow 0} \frac{1}{h} \cdot \Big(d\Big(\vartheta(t+h, x), \ \vartheta(h, \vartheta(t, x))\Big) + \\
&\qquad d\Big(\vartheta(h, \vartheta(t,x)), \ \vartheta(h, \tau(t,y))\Big) - d\Big(\vartheta(t, x), \ \tau(t, y)\Big) + \\
&\qquad d\Big(\vartheta(h, \tau(t,y)), \ \tau(h, \tau(t,y))\Big) + \\
&\qquad d\Big(\tau(h, \tau(t,y)), \ \tau(t+h, \ y)\Big) \Big) \\
&\le 0 + \alpha(\vartheta) \cdot \psi(t) + D(\vartheta, \tau) + 0
\end{aligned}$$

for almost every $t \in [0,1[$ (i.e. every t at which the limit on the left–hand side exists). So the estimate results from well–known Gronwall's Lemma about Lipschitz continuous functions. In fact, Gronwall's Lemma proves to be the key analytical tool for all these conclusions of mutational analysis and, its integral version holds even for continuous functions (see [1], Lemma 8.3.1).

Considering now mutational equations, Lemma 2.5 is laying the foundations for proving the convergence of Euler method. It leads to the following mutational counterpart of the Theorem of Cauchy–Lipschitz (quoted from Theorem 1.4.2 in [1, Aubin 99]) – ensuring existence, uniqueness as well as continuity with respect to the right–hand side.

Theorem 2.6 ([1]) *Assume that the closed bounded balls of the metric space* (M,d) *are compact. Let* f *be a function from* M *to a set* $\Theta(M,d)$ *of transitions on* (M,d) *satisfying*

1. $\exists\ \lambda > 0 : \quad D(f(x),\ f(y)) \ \le\ \lambda \cdot d(x,y) \qquad \forall\ x,y \in M$
2. $A := \sup\limits_{x \in M} \ \alpha(f(x)) \ <\ \infty.$

Suppose for $y : [0,T[\longrightarrow M$ *that its mutation* $\overset{\circ}{y}(t)$ *is nonempty for each* t.

Then for every initial value $x_0 \in M$, *there exists a unique solution* $x(\cdot) : [0,T[\longrightarrow M$ *of the mutational equation* $\overset{\circ}{x}(t) \ni f(x(t))$, *i.e.* $x(\cdot)$ *is Lipschitz continuous and for almost every* $t \in [0,T[$,

$$\limsup_{h \downarrow 0} \ \tfrac{1}{h} \cdot d(x(t+h),\ f(x(t))\,(h, x(t))) \ = \ 0,$$

satisfying, in addition, $x(0) = x_0$ *and the inequality (for every* $t \in [0,T[$)

$$d(x(t),\ y(t)) \ \le\ d(x_0,\ y(0)) \cdot e^{(A+\lambda)\,t} \ + \ \int_0^t e^{(A+\lambda)\,(t-s)} \cdot \inf_{\vartheta \in \overset{\circ}{y}(s)} D(f(y(s)),\ \vartheta) \ \ ds. \qquad \square$$

Further results about mutational analysis and its application to compact subsets of the Euclidean space can be found in [6, 13, 14, 15, 18, 24].

3 Right–hand forward solutions of mutational equations: Previous definitions.

Generalizing the mutational equations of Aubin in metric spaces [1, 2, 3], the so–called *right–hand forward solutions* were defined in [20] and sufficient conditions ensure their existence (see also [23]). In this section, we summarize the main points – in preparation for the new steps of generalization in § 4. This modification is to weaken the restriction of "uniform" continuity on transitions and leads to so–called *timed sleek transitions* in Definition 4.1.

As a first step, we specify the mathematical environment of our considerations.
Similarly to metric spaces, a nonempty set E is to be supplied with a distance function. Motivated by the geometric examples depending on the boundaries [20], however, we again dispense with the symmetry of distance functions. Just for linguistic reasons, we prefer the following definition to the equivalent term "quasi-pseudo-metric" used in topology [28].

Definition 3.1 *Let* E *be a nonempty set.*
$q : E \times E \longrightarrow [0,\infty[$ *is called* ostensible metric *on* E *if it satisfies:*

$$q(z,\ z) \ = \ 0 \qquad (\textit{reflexive})$$
$$q(x,\ z) \ \le \ q(x,\ y) \ + \ q(y,\ z) \qquad (\textit{triangle inequality})$$

for all $x,y,z \in E$. (E,q) *is called* ostensible metric space.

General assumptions for § 3.

1. Let E denote a nonempty set and $D \subset E$ a fixed subset of "test elements".
2. $q : E \times E \longrightarrow [0,\infty[$ is an ostensible metric on E.

Now we specify tools for describing deformations in the tuple (E, D, q). $\vartheta : [0,1] \times E \longrightarrow E$ is to specify which point $\vartheta(t,x) \in E$ is reached from the initial point $x \in E$ after time t. Of course, ϑ has to fulfill some regularity conditions so that it may form the basis for a calculus of differentiation. Following [20], we define

Definition 3.2 *A map* $\vartheta : [0,1] \times E \longrightarrow E$ *is a so–called* forward transition *on* (E, D, q) *if it fulfills*

1. $\vartheta(0, \cdot) \;=\; \mathrm{Id}_E,$
2. $\limsup_{h \downarrow 0} \;\; \frac{1}{h} \cdot q(\vartheta(h,\; \vartheta(t,x)),\;\; \vartheta(t+h,\; x)) \;=\; 0 \qquad \forall\; x \in E,\; t \in [0,1[,$
   $\limsup_{h \downarrow 0} \;\; \frac{1}{h} \cdot q(\vartheta(t+h,\; x),\;\; \vartheta(h,\; \vartheta(t,x))) \;=\; 0 \qquad \forall\; x \in E,\; t \in [0,1[,$
3. $\exists\; \alpha^{\mapsto}(\vartheta) < \infty : \sup_{z \in D,\, y \in E} \; \limsup_{h \downarrow 0} \left(\frac{q\left(\vartheta(h,z),\, \vartheta(h,y)\right) \,-\, q(z,y)}{h \quad q(z,y)} \right)^{+} \le \alpha^{\mapsto}(\vartheta)$
4. $\exists\; \beta(\vartheta) < \infty : \;\; q(\vartheta(s,y),\;\; \vartheta(t,y)) \;\; \le \;\; \beta(\vartheta) \cdot (t-s) \quad \forall\; s < t \le 1,\;\; y \in E,$
5. $\forall\; z \in D \quad \exists\; \mathcal{T}_\Theta = \mathcal{T}_\Theta(\vartheta, z) \in \,]0,1] : \quad \vartheta(t,z) \in D \quad \forall\; t \in [0, \mathcal{T}_\Theta],$
6. $\limsup_{h \downarrow 0} \;\; q(\vartheta(t-h,\; z),\; y) \;\; \ge \;\; q(\vartheta(t,z),\; y) \qquad\qquad \forall\; z \in D,\; y \in E,\; t \le \mathcal{T}_\Theta$

Remark 3.3 The term "forward" and the symbol $\mapsto$ (representing the time axis) indicate that states at time $t{+}h$ are usually compared with elements at time t for $h \downarrow 0$.

Condition (2.) can be regarded as a weakened form of the semigroup property. It consists of two demands as q need not be symmetric. Condition (3.) specifies the continuity property of ϑ with respect to the initial point. In particular, the first argument of q is restricted to elements z of the "test set" D and, $\alpha^{\mapsto}(\vartheta)$ may be chosen larger than necessary. Thus, it is easier to define $\alpha^{\mapsto}(\cdot) < \infty$ uniformly in some applications like the first–order geometric example of [20, 23]. In condition (4.), all $\vartheta(\cdot, y) : [0,1] \longrightarrow E \;\; (y \in E)$ are supposed to be equi–Lipschitz–continuous (in forward time direction).
Condition (5.) guarantees that every "test element" $z \in D$ stays in the "test set" D for short times at least. This assumption is required because estimates using the parameter $\alpha^{\mapsto}(\cdot)$ can be ensured only within this period. Further conditions on $\mathcal{T}_\Theta(\vartheta, \cdot) > 0$ are avoidable for proving existence of solutions, but they are used for uniqueness (in [20], Proposition 3.18).
Condition (6.) forms the basis for applying Gronwall's Lemma that has been extended to semicontinuous functions in [22] (see subsequent Lemma 4.5 here). Indeed, every function $y : [0,1] \longrightarrow E$ with $q(y(t{-}h),\, y(t)) \longrightarrow 0$ (for $h \downarrow 0$ and each t) satisfies

$$q\Big(\vartheta(t,z),\; y(t)\Big) \;\;\le\;\; \limsup_{h \downarrow 0} \;\; q\Big(\vartheta(t-h,z),\;\; y(t-h)\Big).$$

for all elements $z \in D$ and times $t \in \;]\,0,\; \mathcal{T}_\Theta(\vartheta, z)]$.

Remark 3.4 Transitions on a metric space (M, d) according to Definition 2.1 (introduced by Aubin in [1, 2]) prove to be a special case of forward transitions on (M, M, d).

Definition 3.5
$\Theta^{\mapsto}(E,D,q)$ denotes a set of forward transitions on (E,D,q) assuming

$$Q^{\mapsto}(\vartheta,\tau) \;:=\; \sup_{z\in D,\, y\in E}\; \limsup_{h\downarrow 0} \left(\frac{q(\vartheta(h,z),\ \tau(h,y)) \;-\; q(z,y)\cdot e^{\alpha^{\mapsto}(\tau)\, h}}{h}\right)^{+}$$

to be finite for all $\vartheta,\tau\in\Theta^{\mapsto}(E,D,q)$.

These definitions enable us to compare any element $y\in E$ with each "test element" $z\in D$ while evolving along two forward transitions. The key idea of right–hand forward solutions is to preserve the structural estimate of the next proposition while extending mutational equations to ostensible metrics and "distributional" features (in regard to a test set D).
All statements in this paragraph have already been proved in [20] in detail. Many of these steps will be presented in a more general framework in subsequent §§ 4, 5 and thus, we dispense with technical details here.

Proposition 3.6 *Let $\vartheta,\tau\in\Theta^{\mapsto}(E,D,q)$ be forward transitions, $z\in D$, $y\in E$ and $0\le t_1\le t_2\le 1$, $h\ge 0$ (with $t_1+h<\mathcal{T}_\Theta(\vartheta,z)$).*
Then the following estimate holds

$$q(\vartheta(t_1{+}h,z),\ \tau(t_2{+}h,y)) \;\le\; \Big(q(\vartheta(t_1,z),\ \tau(t_2,y)) \;+\; h\cdot Q^{\mapsto}(\vartheta,\tau)\Big)\cdot e^{\alpha^{\mapsto}(\tau)\ h}.$$

The next step is to define the term "right–hand forward primitive" for a curve $\vartheta(\cdot): [0,T]\longrightarrow\Theta^{\mapsto}(E,D,q)$ of forward transitions. Roughly speaking, a curve $x(\cdot):[0,T[\longrightarrow E$ represents a primitive of $\vartheta(\cdot)$ if at each time $t\in[0,T[$, the forward transition $\vartheta(t)$ can be interpreted as a first–order approximation of $x(t+\cdot)$. Combining this notion with the key estimate of Proposition 3.6, a vague meaning of "first–oder approximation" is provided: Comparing $x(t+\cdot)$ with $\vartheta(t)(\cdot,z)$ (for any test element $z\in D$), the same estimate ought to hold as if the factor $Q^{\mapsto}(\cdot,\cdot)$ was 0. It motivates the following definition with the expression "right–hand" indicating that $x(\cdot)$ appears in the second argument of the ostensible metric q in condition (1.).

Definition 3.7 *The curve $x(\cdot):[0,T[\longrightarrow(E,q)$ is called* right–hand forward primitive of *a map $\vartheta(\cdot):[0,T[\longrightarrow\Theta^{\mapsto}(E,D,q)$, abbreviated to $\overset{\circ}{x}(\cdot)\ni\vartheta(\cdot)$, if*

1. $\forall\ t\in[0,T[\qquad \exists\ \widehat{\alpha}^{\mapsto}(t)\ge\alpha^{\mapsto}(\vartheta(t)):$
$$\limsup_{h\downarrow 0}\ \tfrac{1}{h}\Big(q(\vartheta(t)\,(h,z),\ x(t{+}h)) \;-\; q(z,\ x(t))\cdot e^{\widehat{\alpha}^{\mapsto}(t)\cdot h}\Big)\;\le\;0 \quad \textit{for every } z\in D,$$
2. *$x(\cdot)$ is uniformly continuous in time direction with respect to q,*
i.e. there is $\omega(x,\cdot):]0,T[\longrightarrow[0,\infty[$ such that $\limsup_{h\downarrow 0}\ \omega(x,h)\;=\;0,$
$q(x(s),\ x(t))\;\le\;\omega(x,\ t-s)$ *for* $0\le s<t<T.$

Remark 3.8 Forward transitions induce their own primitives. To be more precise, every constant function $\vartheta(\cdot):[0,1[\longrightarrow\Theta^{\mapsto}(E,D,q)$ with $\vartheta(\cdot)=\vartheta_0$ has the right–hand forward primitives $[0,1[\longrightarrow E,\quad t\longmapsto\vartheta_0(t,x)$ with any $x\in E$ — as an immediate consequence of Proposition 3.6 and $Q^{\mapsto}(\vartheta_0, \vartheta_0) = 0$. This property is easy to extend to piecewise constant functions $[0, T[\ \longrightarrow \Theta^{\mapsto}(E, D, q)$ and so it forms the basis for Euler approximations.

Definition 3.9 *For* $f : E \times [0, T[\ \longrightarrow \Theta^{\mapsto}(E, D, q)$ *given,* $x : [0, T[\ \longrightarrow E$ *is a* right–hand forward solution *of the generalized mutational equation* $\overset{\circ}{x}(\cdot) \ni f(x(\cdot), \cdot)$ *if* $x(\cdot)$ *is right–hand forward primitive of* $f(x(\cdot), \cdot) : [0, T[\ \longrightarrow \Theta^{\mapsto}(E, D, q)$.

In [20] (and [23]), these definitions form a basis for extending evolution equations to ostensible metric spaces. A special kind of compactness (so–called *transitional compactness*) proves to be sufficient for the existence of these solutions if the right–hand side $f(\cdot, \cdot)$ is continuous. So a common environment for completely different types of evolutions is provided as the examples of [20, 23] show.

4 Weaker conditions on continuity and test elements: Timed right–hand sleek solutions.

Similarly to semigroups in Banach spaces however, the assumptions about (uniform) continuity might form severe obstacles in applications. With regard to forward transitions ϑ, a bound of the parameter $\alpha^{\mapsto}(\vartheta)$ is often difficult to verify. Thus, we want to weaken the "uniform" character of continuity assumptions. In particular, the choice of $\alpha^{\mapsto}$, $\mathcal{T}_\Theta$ ought to be more flexible without losing Euler method as track to the final aim of existence. As second key aspect, we dispense with the assumption $D \subset E$ (similarly to the notion of Petrov–Galerkin methods).

The third new aspect is motivated by the geometric example of § 6: A countable family $(q_\varepsilon)_{\varepsilon \in \mathcal{J}}$ of distance functions is now to play the role of the single ostensible metric q so far. Supposing the index set $\mathcal{J} \neq \emptyset$ as countable has the analytical advantage that we can apply Cantor's diagonal construction when selecting subsequences appropriately. The parameters of transitions are now free to depend on $\varepsilon \in \mathcal{J}$, of course.

Last but not least, we introduce a separate time component, i.e. the Cartesian products $\widetilde{E} := \mathbb{R} \times E$, $\widetilde{\mathcal{D}} := \mathbb{R} \times \mathcal{D}$ replace the nonempty sets E, D of § 3, respectively. Time as supplementary information about each element provides two advantages. First, we can distinguish explicitly between "earlier" and "later" states and, the whole theory of generalized mutational equations is respecting such a strict distinction (as we will see in this paragraph). The second difference is even more useful in regard to the geometric example in § 6 here. Indeed, the additional time component enables us to detect how long "deformations" along transitions have been lasting. (Roughly speaking, we obtain an estimate how long we have been "losing" geometric information about the boundaries, see Appendix A.)

In the following, all notations with tilde refer to tuples with separate (real) time components.

General assumptions for § 4.

1. Let E and $\mathcal{D}$ denote nonempty sets (not necessarily $\mathcal{D} \subset E$), $\widetilde{E} \stackrel{\text{Def.}}{=} \mathbb{R} \times E$, $\widetilde{\mathcal{D}} \stackrel{\text{Def.}}{=} \mathbb{R} \times \mathcal{D}$, $\pi_1 : (\widetilde{\mathcal{D}} \cup \widetilde{E}) \longrightarrow \mathbb{R},\ (t, x) \longmapsto t.$
2. $\mathcal{J} \neq \emptyset$ abbreviates a (at most) countable index set.
3. $\widetilde{q}_\varepsilon : (\widetilde{\mathcal{D}} \cup \widetilde{E}) \times (\widetilde{\mathcal{D}} \cup \widetilde{E}) \longrightarrow [0, \infty[$ satisfies the *timed* triangle inequality (for each index $\varepsilon \in \mathcal{J}$), i.e. for all $(r, x), (s, y), (t, z) \in \widetilde{\mathcal{D}} \cup \widetilde{E}$ with $r \le s \le t$: $\widetilde{q}_\varepsilon((r,x),\ (t,z)) \ \le\ \widetilde{q}_\varepsilon((r,x),\ (s,y)) \ +\ \widetilde{q}_\varepsilon((s,y),\ (t,z)).$
4. $i_{\widetilde{\mathcal{D}}} : \widetilde{\mathcal{D}} \longrightarrow \widetilde{E}$ fulfills $\widetilde{q}_\varepsilon(\widetilde{z},\ i_{\widetilde{\mathcal{D}}}\,\widetilde{z}) = 0,\ \ \pi_1\,\widetilde{z} = \pi_1\, i_{\widetilde{\mathcal{D}}}\,\widetilde{z}$ for every $\widetilde{z} \in \widetilde{\mathcal{D}}, \varepsilon \in \mathcal{J}$.

Definition 4.1 *A map* $\widetilde{\vartheta} : [0,1] \times (\widetilde{\mathcal{D}} \cup \widetilde{E}) \longrightarrow (\widetilde{\mathcal{D}} \cup \widetilde{E})$ *is called* timed sleek transition *on* $(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ *if it fulfills for each* $\varepsilon \in \mathcal{J}$

1. $\widetilde{\vartheta}(0, \cdot) \ =\ \mathrm{Id}_{\widetilde{\mathcal{D}} \cup \widetilde{E}}$,

2. $\limsup\limits_{h \downarrow 0}\ \frac{1}{h} \cdot \widetilde{q}_\varepsilon(\widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t, \widetilde{x})),\ \ \widetilde{\vartheta}(t+h,\ \widetilde{x})) \ =\ 0 \qquad \forall\ \widetilde{x} \in \widetilde{\mathcal{D}} \cup \widetilde{E},\ t \in [0,1[,$

   $\limsup\limits_{h \downarrow 0}\ \frac{1}{h} \cdot \widetilde{q}_\varepsilon(\widetilde{\vartheta}(t+h,\ \widetilde{x}),\ \ \widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t, \widetilde{x}))) \ =\ 0 \qquad \forall\ \widetilde{x} \in \widetilde{\mathcal{D}} \cup \widetilde{E},\ t \in [0,1[,$

3′. $\forall\ \widetilde{z} \in \widetilde{\mathcal{D}} \quad \exists\ \ \alpha_\varepsilon(\widetilde{\vartheta}, \widetilde{z}) \in [0, \infty[, \quad \mathbb{T}_\varepsilon = \mathbb{T}_\varepsilon(\widetilde{\vartheta}, \widetilde{z}) \in\,]0, 1]:$

   $\limsup\limits_{h \downarrow 0} \left(\frac{\widetilde{q}_\varepsilon\left(\widetilde{\vartheta}(t+h, \widetilde{z}),\ \widetilde{\vartheta}(h, \widetilde{y})\right) \ -\ \widetilde{q}_\varepsilon(\widetilde{\vartheta}(t, \widetilde{z}),\ \widetilde{y})}{h} \right)^+ \le \alpha_\varepsilon(\widetilde{\vartheta}, \widetilde{z}) \cdot \widetilde{q}_\varepsilon(\widetilde{\vartheta}(t, \widetilde{z}),\ \widetilde{y}) \quad \forall\ \ 0 \le t < \mathbb{T}_\varepsilon,\ \ \widetilde{y} \in \widetilde{E}\ \ (t + \pi_1\,\widetilde{z} \le \pi_1\,\widetilde{y}),$

4. $\exists\ \beta_\varepsilon(\widetilde{\vartheta}) \in [0, \infty[: \quad \widetilde{q}_\varepsilon(\widetilde{\vartheta}(s, \widetilde{y}),\ \ \widetilde{\vartheta}(t, \widetilde{y})) \ \le\ \beta_\varepsilon(\widetilde{\vartheta}) \cdot (t - s) \qquad \forall\ \ s < t \le 1,\ \ \widetilde{y} \in \widetilde{E},$

5. $\forall\ \widetilde{z} \in \widetilde{\mathcal{D}}: \qquad \widetilde{\vartheta}(t, \widetilde{z}) \in \widetilde{\mathcal{D}} \qquad \forall\ \ t \in [0,\ \mathbb{T}_\varepsilon(\widetilde{\vartheta}, \widetilde{z})],$

6. $\limsup\limits_{h \downarrow 0}\ \ \widetilde{q}_\varepsilon(\widetilde{\vartheta}(t-h,\ \widetilde{z}),\ \widetilde{y}) \ \ \ge\ \ \widetilde{q}_\varepsilon(\widetilde{\vartheta}(t, \widetilde{z}),\ \widetilde{y}) \qquad \forall\ \ \widetilde{z} \in \widetilde{\mathcal{D}},\ \widetilde{y} \in \widetilde{E},\ t \le \mathbb{T}_\varepsilon\ \ (t + \pi_1\,\widetilde{z} \le \pi_1\,\widetilde{y}),$

7′. $\widetilde{\vartheta}(h,\ (t,y)) \ \in\ \{t+h\} \times E \ \subset\ \widetilde{E} \qquad \forall\ \ (t,y) \in \widetilde{E}, \ \ h \in [0,1],$

   $\pi_1\, \widetilde{\vartheta}(h,\ (t,z)) \ \le \ \ t+h$ *nondecreasing w.r.t.* $h \qquad \forall\ \ (t,z) \in \widetilde{\mathcal{D}}, \ \ h \in [0,1],$

8′. $\limsup\limits_{h \downarrow 0}\ \frac{1}{h} \cdot \widetilde{q}_\varepsilon(\widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t, i_{\widetilde{\mathcal{D}}}\,\widetilde{z})),\ \ \widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t, \widetilde{z}))) \ =\ 0 \qquad \forall\ \ \widetilde{z} \in \widetilde{\mathcal{D}},\ t < \mathbb{T}_\varepsilon(\widetilde{\vartheta}, \widetilde{z}).$

So in comparison with Definition 3.2 of a *forward transition*, some features are changed: Roughly speaking, key new properties of *sleek* transitions $\widetilde{\vartheta}$ are that $\alpha_\varepsilon(\widetilde{\vartheta}, \widetilde{z})$ may depend on the test element $\widetilde{z} \in \widetilde{\mathcal{D}}$ (together with $\varepsilon \in \mathcal{J}$) and, $\mathbb{T}_\varepsilon(\widetilde{\vartheta}, \widetilde{z})$ can depend on $\varepsilon \in \mathcal{J}$ additionally. To be more precise, we introduce additional "degrees of freedom" in comparison to § 3:
Firstly, in condition (3'.), the parameter $\alpha_\varepsilon(\widetilde{\vartheta}, \widetilde{z})$ (with any $\widetilde{z} \in \widetilde{\mathcal{D}}, \varepsilon \in \mathcal{J}$ fixed) is chosen "uniformly" for comparing the evolution of any $\widetilde{y} \in \widetilde{E}$ with the elements $\widetilde{\vartheta}(t, \widetilde{z}) \in \widetilde{\mathcal{D}}$ $(0 \le t < \mathbb{T}_\varepsilon(\vartheta, \widetilde{z}))$ — whereas condition (3.) of Definition 3.2 takes all $y \in E$ and every "test element" $z \in D$ into consideration for $\alpha^{\mapsto}(\vartheta) < \infty$.
Secondly, we take into account that the "test set" $\widetilde{\mathcal{D}}$ need not be a subset of $\widetilde{E}$. In § 3, the distance function q was supposed to be an ostensible metric and thus, reflexive in particular. To be more precise, $q(z,z) = 0$ for all $z \in D \subset E$ formed the basis for

1.) the triangle inequality of $Q^{\mapsto}$ (see [22], Remarks 11, 18 (iv)) and
2.) estimating the distance between a forward transition $\vartheta(\cdot, z)$ and a right–hand forward solution (see [20], Lemma 3.17).

Although we might dispense with such a triangle inequality of transitions, the second point will be relevant for proving estimates between solutions such as Proposition 5.6 later. So we need a further relation between every test element $\widetilde{z} \in \widetilde{\mathcal{D}}$ and its counterpart $i_{\widetilde{\mathcal{D}}}\, \widetilde{z} \in \widetilde{E}$ — in addition to the general assumption $\widetilde{q}_\varepsilon(\widetilde{z},\, i_{\widetilde{\mathcal{D}}}\, \widetilde{z}) = 0$. Condition (8'.) bridges this gap for each timed sleek transition and, (only) here $i_{\widetilde{\mathcal{D}}}\, \widetilde{z} \in \widetilde{E}$ occurs in the first argument of $\widetilde{q}_\varepsilon$ whereas $\widetilde{z} \in \widetilde{\mathcal{D}}$ appears in the second one.
Finally, condition (7'.) is restricting the time component of $\widetilde{\vartheta}(\cdot, \widetilde{z})$ (for every test element $\widetilde{z} \in \widetilde{\mathcal{D}}$) just qualitatively. This additional "degree of freedom" will prove to be an important advantage for the geometric example in § 6.

The common aim of these different approaches is to preserve the structural estimate stated in Proposition 3.6. So first the counterpart of $Q^{\mapsto}(\vartheta, \tau)$ is introduced and then we obtain the corresponding estimate in exactly the same way as in [20].

Definition 4.2
$\widetilde{\Theta}(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ denotes a set of timed sleek transitions on $(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ assuming

$$\widetilde{Q}_\varepsilon(\widetilde{\vartheta}, \widetilde{\tau}; \widetilde{z}) \;:=\; \sup_{\substack{t \,\le\, \mathbb{T}_\varepsilon(\widetilde{\vartheta}, \widetilde{z}),\ \widetilde{y} \in \widetilde{E} \\ t + \pi_1 \widetilde{z} \,\le\, \pi_1 \widetilde{y}}} \ \limsup_{h \downarrow 0} \left(\frac{\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t+h, \widetilde{z}),\ \widetilde{\tau}(h, \widetilde{y})\big) - \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t, \widetilde{z}),\ \widetilde{y}\big) \cdot e^{\alpha_\varepsilon(\widetilde{\tau}, \widetilde{z}) \cdot h}}{h} \right)^+$$

to be finite for all $\widetilde{\vartheta}, \widetilde{\tau} \in \widetilde{\Theta}(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}}),\ \widetilde{z} \in \widetilde{\mathcal{D}},\ \varepsilon \in \mathcal{J}$.

Remark 4.3 The triangle inequality for $\widetilde{Q}_\varepsilon(\cdot, \cdot; \widetilde{z})$ cannot be expected to hold in general. Indeed for any timed sleek transitions $\widetilde{\vartheta}_1, \widetilde{\vartheta}_2, \widetilde{\vartheta}_3 \in \widetilde{\Theta}(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ and $\widetilde{z} \in \widetilde{\mathcal{D}},\ \widetilde{y} \in \widetilde{E},\ t \in [0, \mathbb{T}_\varepsilon(\widetilde{\vartheta}_1, \widetilde{z})]$ with $t + \pi_1 \widetilde{z} \le \pi_1 \widetilde{y}$, the timed triangle inequality of $\widetilde{q}_\varepsilon$ leads to

$$\begin{aligned} & \tfrac{1}{h} \cdot \Big(\widetilde{q}_\varepsilon(\widetilde{\vartheta}_1(t+h, \widetilde{z}), \quad \widetilde{\vartheta}_3(h, \widetilde{y})) \quad - \ \widetilde{q}_\varepsilon(\widetilde{\vartheta}_1(t, \widetilde{z}),\ \widetilde{y}) \ \cdot e^{\alpha_\varepsilon(\widetilde{\vartheta}_3, \widetilde{z})\, h} \Big) \\ \le \quad & \tfrac{1}{h} \cdot \ \widetilde{q}_\varepsilon(\widetilde{\vartheta}_1(t+h, \widetilde{z}), \quad \widetilde{\vartheta}_2(h,\ i_{\widetilde{\mathcal{D}}}\, \widetilde{\vartheta}_1(t, \widetilde{z}))) \\ + \ & \tfrac{1}{h} \cdot \ \widetilde{q}_\varepsilon(\widetilde{\vartheta}_2(h,\ i_{\widetilde{\mathcal{D}}}\, \widetilde{\vartheta}_1(t, \widetilde{z})) \quad \widetilde{\vartheta}_2(h,\ \widetilde{\vartheta}_1(t, \widetilde{z}))) \\ + \ & \tfrac{1}{h} \cdot \Big(\widetilde{q}_\varepsilon(\widetilde{\vartheta}_2(h,\ \widetilde{\vartheta}_1(t, \widetilde{z})), \quad \widetilde{\vartheta}_3(h, \widetilde{y})) \quad - \ \widetilde{q}_\varepsilon(\widetilde{\vartheta}_1(t, \widetilde{z}),\ \widetilde{y}) \ \cdot e^{\alpha_\varepsilon(\widetilde{\vartheta}_3, \widetilde{z})\, h} \Big). \end{aligned}$$

Supposing now $\alpha_\varepsilon(\widetilde{\vartheta}_3, \widetilde{z}) \ge \alpha_\varepsilon(\widetilde{\vartheta}_3,\ \widetilde{\vartheta}_1(t, \widetilde{z}))$ in addition, we conclude from condition (8'.) on timed sleek transitions (Definition 4.1) and $\widetilde{q}_\varepsilon(\widetilde{\vartheta}_1(t, \widetilde{z}),\ i_{\widetilde{\mathcal{D}}}\, \widetilde{\vartheta}_1(t, \widetilde{z})) = 0$

$$\begin{aligned} & \limsup_{h \downarrow 0} \ \tfrac{1}{h} \cdot \Big(\widetilde{q}_\varepsilon(\widetilde{\vartheta}_1(t+h, \widetilde{z}),\ \widetilde{\vartheta}_3(h, \widetilde{y})) \ - \ \widetilde{q}_\varepsilon(\widetilde{\vartheta}_1(t, \widetilde{z}),\ \widetilde{y}) \ \cdot e^{\alpha_\varepsilon(\widetilde{\vartheta}_3, \widetilde{z})\, h} \Big) \\ & \le \ \ \widetilde{Q}_\varepsilon(\widetilde{\vartheta}_1, \widetilde{\vartheta}_2; \widetilde{z}) \ + \ 0 \ + \ \widetilde{Q}_\varepsilon(\widetilde{\vartheta}_2, \widetilde{\vartheta}_3;\ \widetilde{\vartheta}_1(t, \widetilde{z})). \end{aligned}$$

Thus, $\widetilde{Q}_\varepsilon$ satisfies a form of triangle inequality – but with the appropriate "test elements" (as third arguments).

Proposition 4.4 *Let $\widetilde{\vartheta}, \widetilde{\tau} : [0,1] \times \widetilde{E} \longrightarrow \widetilde{E}$ be timed sleek transitions on $(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$. Furthermore suppose $\varepsilon \in \mathcal{J},\ \ \widetilde{z} \in \widetilde{\mathcal{D}},\ \ \widetilde{y} \in \widetilde{E}$ and $0 \le t_1 \le t_2 \le 1,\ h \ge 0$ with $\pi_1 \widetilde{z} \le \pi_1 \widetilde{y}$, $t_1 + h < \mathbb{T}_\varepsilon(\widetilde{\vartheta}, \widetilde{z})$. Then,*

$$\widetilde{q}_\varepsilon(\widetilde{\vartheta}(t_1{+}h, \widetilde{z}),\ \widetilde{\tau}(t_2{+}h, \widetilde{y})) \ \le \ \Big(\widetilde{q}_\varepsilon(\widetilde{\vartheta}(t_1, \widetilde{z}),\ \widetilde{\tau}(t_2, \widetilde{y})) \ + \ h \cdot \widetilde{Q}_\varepsilon(\widetilde{\vartheta}, \widetilde{\tau}; \widetilde{z}) \Big) \cdot e^{\alpha_\varepsilon(\widetilde{\tau}, \widetilde{z})\ h}.$$

Proof is based on the subsequent version of Gronwall's Lemma for semicontinuous functions. The auxiliary function $\varphi_\varepsilon : h \longmapsto \widetilde{q}_\varepsilon(\widetilde{\vartheta}(t_1 + h, \widetilde{z}),\ \widetilde{\tau}(t_2 + h, \widetilde{y}))$ satisfies $\varphi_\varepsilon(h) \le \limsup_{k \downarrow 0} \ \varphi_\varepsilon(h - k)$ due to property (6.) of Definition 4.1.

Moreover it fulfills for any $h \in [0,1[$ with $t_1 + h < \mathbb{T}_\varepsilon(\widetilde{\vartheta}, \widetilde{z})$

$$\limsup_{k \downarrow 0} \frac{\varphi_\varepsilon(h+k) - \varphi_\varepsilon(h)}{k} \le \alpha_\varepsilon(\widetilde{\tau}, \widetilde{z}) \cdot \varphi_\varepsilon(h) + \widetilde{Q}_\varepsilon(\widetilde{\vartheta}, \widetilde{\tau}; \widetilde{z}).$$

Indeed, for all $k > 0$ sufficiently small, the timed triangle inequality leads to

$$\begin{aligned} \varphi_\varepsilon(h+k) \le \;& \widetilde{q}_\varepsilon(\widetilde{\vartheta}(t_1+h+k,\, \widetilde{z}), \quad \widetilde{\tau}(k,\; \widetilde{\tau}(t_2+h,\, \widetilde{y}))) \\ &+ \widetilde{q}_\varepsilon(\widetilde{\tau}(k, \widetilde{\tau}(t_2+h, \widetilde{y})), \quad \widetilde{\tau}(t_2+h+k, \quad \widetilde{y})\,) \\ \le \;& \widetilde{Q}_\varepsilon(\widetilde{\vartheta}, \widetilde{\tau}; \widetilde{z}) \cdot k + \varphi_\varepsilon(h)\, e^{\alpha_\varepsilon(\widetilde{\tau}, \widetilde{z})\, k} + o(k) . \end{aligned}$$

□

Lemma 4.5 (Lemma of Gronwall for semicontinuous functions [22, 23]**)**
Let $\psi : [a,b] \longrightarrow \mathbb{R}$, $f, g \in C^0([a,b[, \mathbb{R})$ *satisfy* $f(\cdot) \ge 0$ *and*

$$\begin{aligned} \psi(t) &\le \limsup_{h \downarrow 0} \psi(t-h), && \forall\ t \in]a,\, b], \\ \psi(t) &\ge \limsup_{h \downarrow 0} \psi(t+h), && \forall\ t \in [a,\, b[, \\ \limsup_{h \downarrow 0} \frac{\psi(t+h) - \psi(t)}{h} &\le f(t) \cdot \limsup_{h \downarrow 0} \psi(t-h) + g(t) && \forall\ t \in]a,\, b[. \end{aligned}$$

Then, for every $t \in [a,b]$, *the function* $\psi(\cdot)$ *fulfills the upper estimate*

$$\psi(t) \le \psi(a) \cdot e^{\mu(t)} + \int_a^t e^{\mu(t) - \mu(s)}\, g(s)\, ds \qquad \text{with } \mu(t) := \int_a^t f(s)\, ds.$$

Corollary 4.6 $\widetilde{Q}_\varepsilon(\widetilde{\vartheta}, \widetilde{\vartheta};\, \widetilde{z}) = 0$ *for all* $\widetilde{\vartheta} \in \widetilde{\Theta}(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$, $\widetilde{z} \in \widetilde{\mathcal{D}}$, $\varepsilon \in \mathcal{J}$.

Proof. Similarly to Proposition 4.4, the definition of $\alpha_\varepsilon(\widetilde{\tau}, \widetilde{z})$ ensures for the auxiliary function $\varphi_\varepsilon : h \longmapsto \widetilde{q}_\varepsilon(\widetilde{\vartheta}(t+h, \widetilde{z}),\ \widetilde{\vartheta}(h, \widetilde{y}))$ the estimate $\limsup_{k \downarrow 0} \frac{\varphi_\varepsilon(h+k) - \varphi_\varepsilon(h)}{k} \le \alpha_\varepsilon(\widetilde{\tau}, \widetilde{z}) \cdot \varphi_\varepsilon(h)$ with any $h \in [0,1[$ satisfying $t + h < \mathbb{T}_\varepsilon(\widetilde{\vartheta}, \widetilde{z})$. Due to Gronwall's Lemma 4.5, it implies

$$\widetilde{q}_\varepsilon(\widetilde{\vartheta}(t+h, \widetilde{z}),\ \widetilde{\vartheta}(h, \widetilde{y})) = \varphi_\varepsilon(h) \le \varphi_\varepsilon(0) \cdot e^{\alpha_\varepsilon(\widetilde{\tau}, \widetilde{z}) \cdot h}$$

□

The structural estimate of Proposition 4.4 provides the key tool for extending the forward solutions of mutational equations mentioned in § 3 to this generalized environment. Again we seize the basic idea of comparing the evolutions of "test elements" along transitions and the wanted curve so that the structural estimate is preserved. To be more precise, we now focus on comparing the evolutions with the initial points $\widetilde{\vartheta}_0(s, \widetilde{z}) \in \widetilde{\mathcal{D}}$, $0 \le s < \mathbb{T}_\varepsilon(\widetilde{\vartheta}_0, \widetilde{z})$, for any $\widetilde{z} \in \widetilde{\mathcal{D}}$ fixed (and the current transition $\widetilde{\vartheta}_0$). So the counterpart of Definition 3.7 is

Definition 4.7 *The curve* $\widetilde{x}(\cdot) : [0,T[\longrightarrow (\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ *is called* timed right–hand sleek primitive of *a map* $\widetilde{\vartheta}(\cdot) : [0,T[\longrightarrow \widetilde{\Theta}(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$, *abbreviated to* $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{\vartheta}(\cdot)$, *if for each* ε,

1. $\forall\ t \in [0,T[\quad \forall\ \widetilde{z} \in \widetilde{\mathcal{D}}$ *with* $\pi_1 \widetilde{z} \le \pi_1 \widetilde{x}(t)$: $\exists\ \widehat{\alpha}_\varepsilon(t, \widetilde{z}) \ge \alpha_\varepsilon(\widetilde{\vartheta}(t), \widetilde{z})$

$$\limsup_{h \downarrow 0} \frac{1}{h} \Big(\widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t)\,(s+h, \widetilde{z}),\ \widetilde{x}(t+h)\Big) - \widetilde{q}_\varepsilon\Big(\widetilde{\vartheta}(t)\,(s, \widetilde{z}),\ \widetilde{x}(t)\Big) \cdot e^{\widehat{\alpha}_\varepsilon(t, \widetilde{z}) \cdot h} \Big) \le 0$$

for every $s \in [0,\ \mathbb{T}_\varepsilon(\widetilde{\vartheta}(t), \widetilde{z})[$ *with* $s + \pi_1 \widetilde{z} \le \pi_1 \widetilde{x}(t)$,

2. $\widetilde{x}(\cdot)$ *is uniformly continuous in time direction with respect to* $\widetilde{q}_\varepsilon$,
i.e. there is $\omega_\varepsilon(\widetilde{x},\cdot) :]0,T[\longrightarrow [0,\infty[$ *such that* $\limsup_{h\downarrow 0} \ \omega_\varepsilon(\widetilde{x},h) = 0$,
$\widetilde{q}_\varepsilon(\widetilde{x}(s),\ \widetilde{x}(t)) \ \le\ \omega_\varepsilon(\widetilde{x},\ t-s)$ *for* $0 \le s < t < T$.

3. $\pi_1\,\widetilde{x}(t) \ = \ t \ + \ \pi_1\,\widetilde{x}(0)$ *for all* $t \in [0,T[$.

Remark 4.8 Timed sleek transitions induce their own sleek primitives — as a direct consequence of Definition 4.1, Proposition 4.4 and Corollary 4.6 (exactly as in Remark 3.8 about *forward* transitions). Correspondingly, each piecewise constant $\widetilde{\vartheta} : [0,T[\longrightarrow \widetilde{\Theta}(\widetilde{E},\widetilde{\mathcal{D}},(\widetilde{q}_\varepsilon))$ has a timed right–hand sleek primitive that is also defined piecewise.

Definition 4.9 *For* $\widetilde{f} : \widetilde{E} \times [0,T[\longrightarrow \widetilde{\Theta}(\widetilde{E},\widetilde{\mathcal{D}},(\widetilde{q}_\varepsilon))$ *given, a map* $\widetilde{x} : [0,T[\longrightarrow \widetilde{E}$ *is a* timed right–hand sleek solution *of the generalized mutational equation* $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot),\cdot)$ *if* $\widetilde{x}(\cdot)$ *is timed right–hand sleek primitive of* $\widetilde{f}(\widetilde{x}(\cdot),\cdot) : [0,T[\longrightarrow \widetilde{\Theta}(\widetilde{E},\widetilde{\mathcal{D}},(\widetilde{q}_\varepsilon))$.

Considering right–hand *forward* solutions in [20] (and [23]), the main steps can now be applied to the new *sleek* versions. Two features have to be taken into account appropriately: firstly, $\widetilde{\mathcal{D}} \not\subset \widetilde{E}$ in general (but $i_{\widetilde{\mathcal{D}}} : \widetilde{\mathcal{D}} \longrightarrow \widetilde{E}$ "links" counterparts) and secondly, the dependence of parameters $\alpha_\varepsilon(\cdot,\widetilde{z})$, $\mathbb{T}_\varepsilon(\cdot,\widetilde{z})$ and $\widetilde{Q}_\varepsilon(\cdot,\cdot;\widetilde{z})$ on the test element $\widetilde{z}$ and $\varepsilon \in \mathcal{J}$.

Proposition 4.10 *Suppose* $\widetilde{\psi} \in \widetilde{\Theta}(\widetilde{E},\widetilde{\mathcal{D}},(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$, $t_1 \in [0,1[$, $t_2 \in [0,T[$, $\widetilde{z} \in \widetilde{\mathcal{D}}$.
Let $\widetilde{x}(\cdot) : [0,T[\longrightarrow \widetilde{E}$ *be a timed sleek primitive of* $\widetilde{\vartheta}(\cdot) : [0,T[\longrightarrow \widetilde{\Theta}(\widetilde{E},\widetilde{\mathcal{D}},(\widetilde{q}_\varepsilon))$ *such that for each* $\varepsilon \in \mathcal{J}$, $t \in [0,T[$, *their parameters fulfill*

$$\begin{cases} \sup_{0\le s\le \min\{t,\mathbb{T}_\varepsilon(\widetilde{\psi},\widetilde{z})\}} & \widehat{\alpha}_\varepsilon(t,\widetilde{\psi}(s,\widetilde{z})) \ \le\ M_\varepsilon(t), \\ & \widetilde{Q}_\varepsilon(\widetilde{\psi},\ \widetilde{\vartheta}(t);\ \widetilde{z}) \ \le\ c_\varepsilon(t) \end{cases}$$

with upper semicontinuous $M_\varepsilon,\ c_\varepsilon : [0,T[\longrightarrow [0,\infty[$. *Set* $\mu_\varepsilon(h) := \int_{t_2}^{t_2+h} M_\varepsilon(s)\,ds$.
Then, for every $\varepsilon \in \mathcal{J}$ *and* $h \in]0,T[$ *with* $t_1 + h < \mathbb{T}_\varepsilon(\widetilde{\psi},\widetilde{z})$, $t_1 + \pi_1\,\widetilde{z} \le \pi_1\,\widetilde{x}(t_2)$,

$$\begin{aligned} &\widetilde{q}_\varepsilon(\widetilde{\psi}(t_1{+}h,\ \widetilde{z}),\ \ \widetilde{x}(t_2{+}h)) \\ \le\ \ &\widetilde{q}_\varepsilon(\widetilde{\psi}(t_1,\widetilde{z}),\ \widetilde{x}(t_2)) \ \cdot\ e^{\mu_\varepsilon(h)} \ +\ \int_0^h e^{\mu_\varepsilon(h)-\mu_\varepsilon(s)}\ c_\varepsilon(t_2{+}s)\ ds. \end{aligned}$$

Proof. We follow the same track as in the proof of Proposition 4.4 and consider the function $\varphi_\varepsilon : h \longmapsto \widetilde{q}_\varepsilon(\widetilde{\psi}(t_1{+}h,\widetilde{z}),\ \widetilde{x}(t_2{+}h))$. Firstly, $\varphi_\varepsilon(h) \le \limsup_{k\downarrow 0} \varphi_\varepsilon(h-k)$ results from condition (6.) on sleek transitions (Definition 4.1) and the continuity of $\widetilde{x}(\cdot)$.

Furthermore we prove for any $h \in [0,T[$ with $t_1 + h < \mathbb{T}_\varepsilon(\widetilde{\psi},\widetilde{z})$,

$$\limsup_{k\downarrow 0} \frac{\varphi_\varepsilon(h{+}k) - \varphi_\varepsilon(h)}{k} \ \le\ M_\varepsilon(t_2{+}h)\cdot\varphi_\varepsilon(h) + c_\varepsilon(t_2{+}h).$$

In particular, this inequality implies $\varphi_\varepsilon(h) \ge \limsup_{k\downarrow 0} \varphi_\varepsilon(h+k)$ since its right–hand side is finite. Thus, the claim results from Gronwall's Lemma 4.5 – after approximating $M_\varepsilon(\cdot)$, $c_\varepsilon(\cdot)$ by continuous functions from above.

For all small $k > 0$, the timed triangle inequality and Proposition 4.4 lead to

$$\begin{array}{rcll}
\varphi_\varepsilon(h+k) & \le & \widetilde{q}_\varepsilon(\widetilde{\psi}(t_1+h+k,\ \widetilde{z}), & \widetilde{\vartheta}(t_2+h)\,(k,\ i_{\widetilde{\mathcal{D}}}\,\widetilde{\psi}(t_1+h,\ \widetilde{z}))) \\
& + & \widetilde{q}_\varepsilon(\widetilde{\vartheta}(t_2+h)\,(k,\ i_{\widetilde{\mathcal{D}}}\,\widetilde{\psi}(t_1+h,\widetilde{z})), & \widetilde{\vartheta}(t_2+h)\,(k,\quad \widetilde{\psi}(t_1+h,\ \widetilde{z}))) \\
& + & \widetilde{q}_\varepsilon(\widetilde{\vartheta}(t_2+h)\,(k,\quad \widetilde{\psi}(t_1+h,\widetilde{z})), & \widetilde{x}(t_2+h+k)) \\
& \le & \widetilde{Q}_\varepsilon(\widetilde{\psi},\ \widetilde{\vartheta}(t_2+h);\ \widetilde{z}) \qquad e^{M_\varepsilon(t_2+h)\cdot k}\cdot k \ + \ o(k) & \\
& + & \varphi_\varepsilon(h)\,e^{\widehat{\alpha}_\varepsilon(t_2+h,\,\widetilde{\psi}(t_1+h,\,\widetilde{z}))\cdot k} \qquad\qquad + \ o(k) & \\
& \le & \varphi_\varepsilon(h)\,e^{M_\varepsilon(t_2+h)\cdot k} \quad + \ c_\varepsilon(t_2+h) \quad \cdot k \ + \ o(k) &
\end{array}$$

since $t_1+h+k < \mathbb{T}_\varepsilon(\widetilde{\psi},\widetilde{z})$ implies $\widetilde{\psi}(t_1+h,\widetilde{z}),\ \widetilde{\psi}(t_1+h+k,\widetilde{z}) \in \widetilde{\mathcal{D}}$. □

With the objective of using Euler method for the existence of sleek solutions, we first have to specify an adequate type of convergence preserving the solution property. Assumptions (5.ii), (5.iii) of the next proposition might be subsumed under the term "two–sided graphically convergent". Obviously, it is weaker than pointwise convergence (with respect to time) and consists of two conditions with the limit function appearing in both arguments of $\widetilde{q}_\varepsilon$. Admitting vanishing "time perturbations" $\delta_j, \delta'_j \ge 0$ exemplifies the basic idea that the first argument of $\widetilde{q}_\varepsilon$ usually refers to the earlier element whereas the second argument mostly represents the later point.

Proposition 4.11 (Convergence Theorem)
Suppose the following properties of

$$\begin{array}{rcll}
\widetilde{f}_m,\ \widetilde{f}: & \widetilde{E}\times[0,T[& \longrightarrow & \widetilde{\Theta}(\widetilde{E},\widetilde{\mathcal{D}},(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}}) \qquad (m\in\mathbb{N}) \\
\widetilde{x}_m,\ \widetilde{x}: & [0,T[& \longrightarrow & \widetilde{E}\ :
\end{array}$$

1. $M_\varepsilon(\widetilde{z}) := \sup_{m,t,\widetilde{y}} \{\, \alpha_\varepsilon(\widetilde{f}_m(\widetilde{y},t),\ \widetilde{f}(\widetilde{x}(t),t)(h,\widetilde{z})) \mid 0 \le h < \mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t),\widetilde{z}) \} < \infty,$
2. $\sup_{\substack{\widetilde{v}=\widetilde{f}(\widetilde{x}(t),t)\,(\tau,\widetilde{z}) \\ \tau<\mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t),\widetilde{z})}} \widetilde{Q}_\varepsilon\left(\widetilde{f}_m(\widetilde{y},t),\ \widetilde{f}_m(\widetilde{y}_m,t_m);\ \widetilde{v}\right) \overset{m\to\infty}{\longrightarrow} 0$
   whenever $t_m - t \downarrow 0,\quad \widetilde{q}_\delta(\widetilde{y},\widetilde{y}_m) \longrightarrow 0\ \ (\pi_1\,\widetilde{y} \le \pi_1\,\widetilde{y}_m)$ *for all* $\delta\in\mathcal{J}$,
3. $\overset{\circ}{\widetilde{x}}_m(\cdot) \ni \widetilde{f}_m(\widetilde{x}_m(\cdot),\cdot)$ *in* $[0,T[$ *(in the sense of Definition 4.9),*
4. $\widehat{\omega}_\varepsilon(h) := \sup_m \omega_\varepsilon(\widetilde{x}_m,h) \overset{h\downarrow 0}{\longrightarrow} 0$ *(uniform moduli of continuity w.r.t.* $\widetilde{q}_\varepsilon$*),*
5. $\forall\ t_1,t_2\in[0,T[,\quad t_3\in\,]0,T[\quad \exists\ (m_j)_{j\in\mathbb{N}}$ *with* $m_j \nearrow \infty$ *and*
   (i) $\limsup_{j\to\infty} \widetilde{Q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t_1),t_1),\ \widetilde{f}_{m_j}(\widetilde{x}(t_1),t_1);\ \widetilde{z}\Big) = 0$
   (ii) $\exists\ (\delta'_j)_{j\in\mathbb{N}}:\ \delta'_j \searrow 0,\quad \widetilde{q}_{\varepsilon'}(\widetilde{x}(t_2),\ \widetilde{x}_{m_j}(t_2+\delta'_j)) \longrightarrow 0 \qquad \forall\,\varepsilon'\in\mathcal{J},$
   $\pi_1\,\widetilde{x}_{m_j}(t_2+\delta'_j) \searrow \pi_1\,\widetilde{x}(t_2),$
   (iii) $\exists\ (\delta_j)_{j\in\mathbb{N}}:\ \delta_j \searrow 0,\quad \widetilde{q}_\varepsilon(\widetilde{x}_{m_j}(t_3-\delta_j),\ \widetilde{x}(t_3)) \longrightarrow 0$
   $\pi_1\,\widetilde{x}_{m_j}(t_3-\delta_j) \nearrow \pi_1\,\widetilde{x}(t_3).$

for each $\varepsilon\in\mathcal{J}$ *and* $\widetilde{z}\in\widetilde{\mathcal{D}}$.
Then, $\widetilde{x}(\cdot)$ *is a timed right–hand sleek solution of* $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot),\cdot)$ *in* $[0,T[$.

Proof. The uniform continuity of $\widetilde{x}(\cdot)$ results from assumption (4.):
Each $\widetilde{x}_m(\cdot)$ satisfies $\widetilde{q}_\varepsilon(\widetilde{x}_m(t_1),\ \widetilde{x}_m(t_2)) \le \widehat{\omega}_\varepsilon(t_2 - t_1)$ for $t_1 < t_2 < T$.
Let $\varepsilon \in \mathcal{J}$, $0 \le t_1 < t_2 < T$ be arbitrary and choose $(\delta_j')_{j\in\mathbb{N}}$, $(\delta_j)_{j\in\mathbb{N}}$, for t_1, t_2 (according to condition (5.ii), (5.iii)). For all $j \in \mathbb{N}$ large enough, we obtain $t_1 + \delta_j' < t_2 - \delta_j$ and so,

$$\begin{aligned} \widetilde{q}_\varepsilon(\widetilde{x}(t_1),\ \widetilde{x}(t_2)) \ &\le\ \widetilde{q}_\varepsilon(\widetilde{x}(t_1),\ \widetilde{x}_{m_j}(t_1+\delta_j')) \ +\ \widetilde{q}_\varepsilon(\widetilde{x}_{m_j}(t_1+\delta_j'),\ \widetilde{x}_{m_j}(t_2-\delta_j)) \\ &\quad + \widetilde{q}_\varepsilon(\widetilde{x}_{m_j}(t_2-\delta_j),\ \widetilde{x}(t_2)) \\ &\le\ o(1) \qquad\qquad +\ \widehat{\omega}_\varepsilon(t_2 - t_1) \qquad \text{for } j \longrightarrow \infty. \end{aligned}$$

Now let $\varepsilon \in \mathcal{J}$, $\widetilde{z} \in \widetilde{D}$ and $t \in [0,T[$, $0 \le s < s+h < \mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t),\ \widetilde{z})$ be chosen arbitrarily with $s + \pi_1\,\widetilde{z} \le \pi_1\,\widetilde{x}(t)$. Condition (6.) of Definition 4.1 guarantees for each $k \in\,]0,h[$ sufficiently small

$$\widetilde{q}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t)\,(s+h,\widetilde{z}),\ \ \widetilde{x}(t+h)) \ \le\ \widetilde{q}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t)\,(s+h-k,\widetilde{z}),\ \ \widetilde{x}(t+h)) \ +\ h^2.$$

According to hypothesis (5.), there always exist sequences $(m_j)_{j\in\mathbb{N}}$, $(\delta_j)_{j\in\mathbb{N}}$, $(\delta_j')_{j\in\mathbb{N}}$ satisfying $m_j \nearrow \infty$, $\delta_j \downarrow 0$, $\delta_j' \downarrow 0$, $\delta_j + \delta_j' < k$ and

$$\begin{cases} \widetilde{Q}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t), & \widetilde{f}_{m_j}(\widetilde{x}(t),t);\ \widetilde{z}) & \le & h^2, \\ \widetilde{q}_\varepsilon(\widetilde{x}_{m_j}(t+h-\delta_j), & \widetilde{x}(t+h)) & \longrightarrow & 0 \\ \widetilde{q}_{\varepsilon'}(\widetilde{x}(t), & \widetilde{x}_{m_j}(t+\delta_j')) & \longrightarrow & 0 & \forall\, \varepsilon' \in \mathcal{J}, \\ & \pi_1\,\widetilde{x}_{m_j}(t+h-\delta_j) & \nearrow & \pi_1\,\widetilde{x}(t+h), \\ & \pi_1\,\widetilde{x}_{m_j}(t+\delta_j') & \searrow & \pi_1\,\widetilde{x}(t). \end{cases}$$

Thus, Proposition 4.10 and Remark 4.3 imply for all large $j \in \mathbb{N}$ (depending on $\varepsilon, \widetilde{z}, t, h, k$),

$$\begin{aligned} &\widetilde{q}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t)\,(s+h,\ \widetilde{z}),\ \ \widetilde{x}(t+h)) \\ \le\ & \widetilde{q}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t)\,(s+h-k,\ \widetilde{z}),\ \ \widetilde{x}_{m_j}(t+\delta_j' + h-k)) \\ &+\ \widetilde{q}_\varepsilon(\widetilde{x}_{m_j}(t+\delta_j' + h-k),\ \ \widetilde{x}_{m_j}(t+h - \delta_j)) \\ &+\ \widetilde{q}_\varepsilon(\widetilde{x}_{m_j}(t+h - \delta_j),\ \ \widetilde{x}(t+h)) \ +\ h^2 \\ \le\ & \widetilde{q}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t)\,(s,\widetilde{z}),\ \ \widetilde{x}_{m_j}(t+\delta_j'))\ \cdot\ e^{M_\varepsilon(\widetilde{z})\cdot(h-k)} \ + \\ &+\ \int_0^{h-k} e^{M_\varepsilon(\widetilde{z})\cdot(h-k-\sigma)}\ \widetilde{Q}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t),\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j},\cdot)|_{t+\delta_j'+\sigma}\,;\ \widetilde{z})\ d\sigma \\ &+\ \widehat{\omega}_\varepsilon(k - \delta_j - \delta_j') \\ &+\ \widetilde{q}_\varepsilon(\widetilde{x}_{m_j}(t+h - \delta_j),\ \ \widetilde{x}(t+h)) \ +\ h^2 \\ \le\ & \Big(\ \widetilde{q}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t)\,(s,\widetilde{z}),\ \ \widetilde{x}(t))\ +\ \widetilde{q}_\varepsilon(\widetilde{x}(t),\ \ \widetilde{x}_{m_j}(t+\delta_j'))\Big)\ \cdot\ e^{M_\varepsilon(\widetilde{z})\cdot(h-k)}\ + \\ &+\ \int_0^{h} e^{M_\varepsilon(\widetilde{z})\cdot(h-\sigma)}\ \ \widetilde{Q}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t),\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j},\cdot)|_{t+\delta_j'+\sigma}\,;\ \widetilde{z})\ d\sigma \\ &+\ \widehat{\omega}_\varepsilon(k) \ +\ 2\,h^2 \\ \le\ & \widetilde{q}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t)\,(s,\widetilde{z}),\ \widetilde{x}(t))\ \cdot\ e^{M_\varepsilon(\widetilde{z})\,h} + \ \widehat{\omega}_\varepsilon(k)\ \ +\ 3\,h^2 \\ &+\ \int_0^{h} e^{M_\varepsilon(\widetilde{z})\cdot(h-\sigma)}\Big(h^2 + \sup_{\substack{\widetilde{v}\,=\,\widetilde{f}(\widetilde{x}(t),t)\,(\tau,\widetilde{z}) \\ \tau\,<\,\mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t),\widetilde{z})}} \widetilde{Q}_\varepsilon(\widetilde{f}_{m_j}(\widetilde{x}(t),t),\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j},\cdot)|_{t+\delta_j'+\sigma}\,;\widetilde{v})\Big)d\sigma \end{aligned}$$

Now $j \longrightarrow \infty$ and then $k \longrightarrow 0$ provide the estimate

$$\begin{aligned}
&\widetilde{q}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t)\,(s+h,\ \widetilde{z}),\ \ \widetilde{x}(t+h)) \\
&\le\ \ \widetilde{q}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t)\,(s,\widetilde{z}),\ \widetilde{x}(t))\cdot e^{M_\varepsilon(\widetilde{z})\,h}\ +\ 0\ +\ \text{const}\cdot h^2 \\
&\quad +\ h\ e^{M_\varepsilon(\widetilde{z})\,h}\cdot \limsup_{j\longrightarrow\infty}\ \sup_{0\le\sigma\le h}\ \sup_{\substack{\widetilde{v}=\widetilde{f}(\widetilde{x}(t),t)\,(\tau,\widetilde{z})\\ \tau<\mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t),\widetilde{z})}}\ \widetilde{Q}_\varepsilon(\widetilde{f}_{m_j}(\widetilde{x}(t),t),\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j},\cdot)|_{t+\delta'_j+\sigma}\,;\,\widetilde{v}).
\end{aligned}$$

Finally, convergence assumptions (2.),(5.ii) and equi–continuity of $(\widetilde{x}_m)$ ensure indirectly

$$\limsup_{h\downarrow 0}\ \limsup_{j\longrightarrow\infty}\ \sup_{0\le\sigma\le h}\ \sup_{\substack{\widetilde{v}=\widetilde{f}(\widetilde{x}(t),t)\,(\tau,\widetilde{z})\\ \tau<\mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t),\widetilde{z})}}\ \widetilde{Q}_\varepsilon(\widetilde{f}_{m_j}(\widetilde{x}(t),t),\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j},\cdot)|_{t+\delta'_j+\sigma}\,;\,\widetilde{v})\ =\ 0$$

and thus,

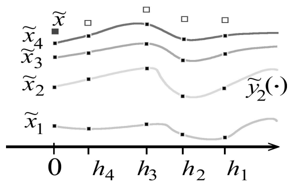
$$\limsup_{h\downarrow 0}\ \tfrac{1}{h}\Big(\widetilde{q}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t)\,(s+h,\widetilde{z}),\ \widetilde{x}(t+h))\ -\ \widetilde{q}_\varepsilon(\widetilde{f}(\widetilde{x}(t),t)\,(s,\widetilde{z}),\ \widetilde{x}(t))\cdot e^{M_\varepsilon(\widetilde{z})\,h}\Big)\ \le\ 0. \qquad\square$$

Similarly to ordinary differential equations, the convergence of approximations to a wanted solution usually results from assumptions about completeness or compactness. Here we prefer a suitable form of compactness since more than one distance function is involved. Still aiming to apply Convergence Theorem 4.11 to Euler approximations, we introduce the following term:

Definition 4.12 *Let $\widetilde{\Theta}$ denote a nonempty set of maps $[0,1]\times\widetilde{E}\longrightarrow\widetilde{E}$.*
$(\widetilde{E},\ (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}},\ \widetilde{\Theta})$ *is called* timed transitionally compact *if it fulfills:*

Let $(\widetilde{x}_n)_{n\in\mathbb{N}}$, $(h_j)_{j\in\mathbb{N}}$ be any sequences in $\widetilde{E}$, $]0,1[$, respectively and $\widetilde{v}\in\widetilde{E}$ with $\sup_n |\pi_1\,\widetilde{x}_n|<\infty$, $\sup_n \widetilde{q}_\varepsilon(\widetilde{v},\widetilde{x}_n)<\infty$ for each $\varepsilon\in\mathcal{J}$, $h_j\longrightarrow 0$. Moreover suppose $\widetilde{\vartheta}_n:[0,1]\longrightarrow\widetilde{\Theta}$ to be piecewise constant (for each $n\in\mathbb{N}$) such that all curves $\widetilde{\vartheta}_n(t)(\cdot,\widetilde{x}):[0,1]\longrightarrow\widetilde{E}$ have a common modulus of continuity $(n\in\mathbb{N},\ t\in[0,1],\ \widetilde{x}\in\widetilde{E})$.
Each $\widetilde{\vartheta}_n$ induces a function $\widetilde{y}_n(\cdot):[0,1]\longrightarrow\widetilde{E}$ with $\widetilde{y}_n(0)=\widetilde{x}_n$ in the same (piecewise) way as timed sleek transitions induce their own primitives according to Remark 4.8 (i.e. using $\widetilde{\vartheta}_n(t_m)\,(\cdot,\ \widetilde{y}_n(t_m))$ in each interval $]t_m,t_{m+1}]$ in which $\widetilde{\vartheta}_n(\cdot)$ is constant).
Then there exist a sequence $n_k\nearrow\infty$ and $\widetilde{x}\in\widetilde{E}$ satisfying for each $\varepsilon\in\mathcal{J}$,

$$\begin{aligned}
\lim_{k\to\infty}\quad & \pi_1\,\widetilde{x}_{n_k} && = \pi_1\,\widetilde{x},\\
\limsup_{k\longrightarrow\infty}\quad & \widetilde{q}_\varepsilon(\widetilde{x}_{n_k},\ \widetilde{x}) && = 0,\\
\limsup_{j\longrightarrow\infty}\ \sup_{k\ge j}\quad & \widetilde{q}_\varepsilon(\widetilde{x},\ \ \widetilde{y}_{n_k}(h_j)) && = 0.
\end{aligned}$$



A nonempty subset $\widetilde{F}\subset\widetilde{E}$ is called timed transitionally compact in $(\widetilde{E},\ (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}},\ \widetilde{\Theta})$ *if the same property holds for any sequence $(\widetilde{x}_n)_{n\in\mathbb{N}}$ in $\widetilde{F}$ (but $\widetilde{x}\in\widetilde{F}$ is* not *required).*

Proposition 4.13 (Existence of timed right–hand sleek solutions)

Assume that the tuple $(\widetilde{E},\ (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}},\ \widetilde{\Theta}(\widetilde{E},\widetilde{\mathcal{D}},(\widetilde{q}_\varepsilon)))$ is timed transitionally compact.
Furthermore let $\widetilde{f}:\widetilde{E}\times[0,T]\longrightarrow\widetilde{\Theta}(\widetilde{E},\widetilde{\mathcal{D}},(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ fulfill for every $\varepsilon\in\mathcal{J}$, $\widetilde{z}\in\widetilde{\mathcal{D}}$

1. $M_\varepsilon(\widetilde{z})\ :=\ \sup_{t_1,t_2,\widetilde{y}_1,\widetilde{y}_2}\ \{\alpha_\varepsilon(\widetilde{f}(\widetilde{y}_1,t_1),\ \widetilde{f}(\widetilde{y}_2,t_2)(h,\widetilde{z}))\mid 0\le h<\mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{y}_2,t_2),\widetilde{z})\}\ <\ \infty,$
2. $c_\varepsilon\ :=\ \sup_{t,\widetilde{y}}\ \beta_\varepsilon(\widetilde{f}(\widetilde{y},t))\ <\ \infty,$
3. $\widetilde{Q}_\varepsilon\Big(\widetilde{f}(\widetilde{y},t),\ \widetilde{f}(\widetilde{y}_m,t_m);\ \widetilde{z}\Big)\ \longrightarrow\ 0$
   whenever $0\le t_m-t\longrightarrow 0$ and $\widetilde{q}_\delta(\widetilde{y},\widetilde{y}_m)\longrightarrow 0$ $(\pi_1\,\widetilde{y}\le\pi_1\,\widetilde{y}_m)$ for each $\delta\in\mathcal{J}$.

Then for every $\widetilde{x}_0 \in \widetilde{E}$, *there is a timed right–hand sleek solution* $\widetilde{x} : [0, T[\longrightarrow \widetilde{E}$ *of the generalized mutational equation* $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot), \cdot)$ *with* $\widetilde{x}(0) = \widetilde{x}_0$.

Remark 4.14 The basic notion of its proof is easy to sketch. Indeed adapting the existence proof of *forward* solutions (in [20]) to *sleek* solutions here, we again start with Euler approximations $\widetilde{x}_n(\cdot) : [0, T[\longrightarrow \widetilde{E}$ $(n \in \mathbb{N})$,

$$\begin{array}{lllll} h_n & := \frac{T}{2^n}, & t_n^j & := j\, h_n & \text{for } j = 0 \ldots 2^n, \\ \widetilde{x}_n(0) & := \widetilde{x}_0, & \widetilde{x}_0(\cdot) & := \widetilde{x}_0, & \\ \widetilde{x}_n(t) & := \widetilde{f}(\widetilde{x}_n(t_n^j),\, t_n^j)\,(t - t_n^j,\ \widetilde{x}_n(t_n^j)) & & & \text{for } t \in]t_n^j,\, t_n^{j+1}],\ j \le 2^n, \end{array}$$

and then use Cantor diagonal construction (as $\mathcal{J}$ is assumed to be countable) in combination with timed transitional compactness. This leads to a function $\widetilde{x}(\cdot) : [0, T[\longrightarrow \widetilde{E}$ with the property: For each $j \in \mathbb{N}$, there exist $K_j \in \mathbb{N}$ and $N_j \in \mathbb{N}$ (depending on K_j) such that $N_j > K_j > N_{j-1}$ and

$$\begin{cases} \widetilde{q}_\varepsilon(\widetilde{x}_{N_j}(s - 2\, h_{K_j}),\ \ \widetilde{x}(s)\,) & \le \frac{1}{j} \\ \widetilde{q}_\varepsilon(\widetilde{x}(t),\ \ \widetilde{x}_{N_j}(t + 2\, h_{K_j})\,) & \le \frac{1}{j} \end{cases}$$

for every $s, t \in [0, T[$ and $\varepsilon \in \{\varepsilon_1, \varepsilon_2 \ldots \varepsilon_j\} \subset \mathcal{J}$. Due to Convergence Theorem 4.11 for $\widetilde{x}_{N_j}(\cdot + 2\, h_{N_j} + 2\, h_{K_j})$, the limit function $\widetilde{x}(\cdot)$ is a timed right–hand sleek solution. Here we dispense with further details because Proposition 5.4 below states a similar existence result under weaker compactness assumptions and will be proved in Remark 5.5.

Remark 4.15 Due to the fixed initial point $\widetilde{x}_0$, the compactness hypothesis can be weakened slightly. We only need that all Euler approximations $\widetilde{x}_n(t)$ $(0 < t < T,\ n \in \mathbb{N})$ are contained in a set $\widetilde{F} \subset \widetilde{E}$ that is transitionally compact in $(\widetilde{E},\ (\widetilde{q}_\varepsilon),\ \widetilde{\Theta}(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon)))$. This modification is useful if each transition $\widetilde{\vartheta} \in \widetilde{\Theta}(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon))$ has all values in $\widetilde{F}$ after any positive time, i.e. $\widetilde{\vartheta}(t, \widetilde{x}) \in \widetilde{F}$ for all $0 < t \le 1$, $\widetilde{x} \in \widetilde{E}$. In particular, it does not require additional assumptions about the initial value $\widetilde{x}_0 \in \widetilde{E}$.

5 Weaker conditions on compactness: Extending "weak" compactness beyond vector spaces.

Considering the geometric example of § 6, however, timed transitional compactness might be a very restrictive hypothesis. In fact, the appropriate choice of topology (in regard to compactness) is an old analytical problem even in Banach spaces. Weak compactness has proved to be useful alternative. Here we want to extend this notion to sets without linear structure. In particular, no continuous linear forms are available in $\widetilde{E}$ or $\widetilde{\mathcal{D}}$. As an alternative starting point, we seize a well-known representation of the norm on a Banach space $(X, \|\cdot\|_X)$

$$\|z\|_X = \sup\ \{\, y^*(z) \mid y^* : X \longrightarrow \mathbb{R} \text{ linear, continuous, } \|y^*\|_{X^*} \le 1\,\}.$$

More generally, each distance function $\widetilde{q}_\varepsilon : (\widetilde{\mathcal{D}} \cup \widetilde{E}) \times (\widetilde{\mathcal{D}} \cup \widetilde{E}) \longrightarrow [0,\infty[$ is now assumed to be supremum with respect to an additional parameter $\kappa \in \mathcal{I}$: $\widetilde{q}_\varepsilon = \sup_{\kappa \in \mathcal{I}} \widetilde{q}_{\varepsilon,\kappa}$.
Here $\widetilde{q}_{\varepsilon,\kappa} : (\widetilde{\mathcal{D}} \cup \widetilde{E}) \times (\widetilde{\mathcal{D}} \cup \widetilde{E}) \longrightarrow [0,\infty[$ $(\varepsilon \in \mathcal{J}, \kappa \in \mathcal{I})$ is a countable family of functions that need not satisfy the timed triangle inequality separately – in contrast to each $\widetilde{q}_\varepsilon$ $(\varepsilon \in \mathcal{J})$.
We assume instead that each $\kappa \in \mathcal{I}$ has counterparts $\kappa', \kappa'' \in \mathcal{I}$ fulfilling

$$\widetilde{q}_{\varepsilon,\kappa}(\widetilde{y}_1, \widetilde{y}_3) \;\le\; \widetilde{q}_{\varepsilon,\kappa'}(\widetilde{y}_1, \widetilde{y}_2) + \widetilde{q}_{\varepsilon,\kappa''}(\widetilde{y}_2, \widetilde{y}_3)$$

for all $\widetilde{y}_1, \widetilde{y}_2, \widetilde{y}_3 \in \widetilde{\mathcal{D}} \cup \widetilde{E}$ with $\pi_1 \widetilde{y}_1 \le \pi_1 \widetilde{y}_2 \le \pi_1 \widetilde{y}_3$.
For extracting subsequences of Euler approximations, the key point now is: Supposing right–convergence with respect to each $\widetilde{q}_\varepsilon$ can be replaced by the hypothesis of right–convergence with respect to each $\widetilde{q}_{\varepsilon,\kappa}$ (and the latter might be easier to verify as § 6 exemplifies). In particular, assumption (5.iii) of the preceding Convergence Theorem 4.11 is modified.

Proposition 5.1 (Convergence Theorem II)
Assume $\widetilde{q}_\varepsilon = \sup_{\kappa \in \mathcal{I}} \widetilde{q}_{\varepsilon,\kappa}$ *with (at most) countably many* $\widetilde{q}_{\varepsilon,\kappa} : (\widetilde{\mathcal{D}} \cup \widetilde{E})^2 \longrightarrow [0,\infty[$ $(\varepsilon \in \mathcal{J}, \kappa \in \mathcal{I})$ *such that each* $\kappa \in \mathcal{I}$ *has counterparts* $\kappa', \kappa'' \in \mathcal{I}$ *fulfilling*

$$\widetilde{q}_{\varepsilon,\kappa}(\widetilde{y}_1, \widetilde{y}_3) \;\le\; \widetilde{q}_{\varepsilon,\kappa'}(\widetilde{y}_1, \widetilde{y}_2) + \widetilde{q}_{\varepsilon,\kappa''}(\widetilde{y}_2, \widetilde{y}_3)$$

for all $\widetilde{y}_1, \widetilde{y}_2, \widetilde{y}_3 \in \widetilde{\mathcal{D}} \cup \widetilde{E}$ *with* $\pi_1 \widetilde{y}_1 \le \pi_1 \widetilde{y}_2 \le \pi_1 \widetilde{y}_3$.
In addition to hypotheses (1.)–(4.) of Proposition 4.11, suppose for all $\varepsilon \in \mathcal{J}$ *and*

$$\begin{array}{llcll} \widetilde{f}_m, \;\widetilde{f} : & \widetilde{E} \times [0,T[& \longrightarrow & \widetilde{\Theta}(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}}) & (m \in \mathbb{N}) \\ \widetilde{x}_m, \;\widetilde{x} : & [0,T[& \longrightarrow & \widetilde{E} \; : & \end{array}$$

5'. $\forall\; t_1, t_2 \in [0,T[,\;\; t_3 \in]0,T[\;\;\; \exists\; (m_j)_{j \in \mathbb{N}}$ *with* $m_j \nearrow \infty$ *and*

(i) $\limsup_{j \longrightarrow \infty} \; \widetilde{Q}_\varepsilon\Big(\widetilde{f}(\widetilde{x}(t_1), t_1),\; \widetilde{f}_{m_j}(\widetilde{x}(t_1), t_1);\; \widetilde{z}\Big) \;\le\; 0 \qquad \forall\; \widetilde{z} \in \widetilde{\mathcal{D}},$

(ii) $\exists\; (\delta_j')_{j \in \mathbb{N}} : \; \delta_j' \searrow 0, \quad \widetilde{q}_{\varepsilon'}(\widetilde{x}(t_2),\; \widetilde{x}_{m_j}(t_2 + \delta_j')) \longrightarrow 0 \quad \forall\; \varepsilon' \in \mathcal{J}$
$\qquad \pi_1\, \widetilde{x}_{m_j}(t_2 + \delta_j') \;\searrow\; \pi_1\, \widetilde{x}(t_2).$

(iii') $\exists\; (\delta_j)_{j \in \mathbb{N}} : \; \delta_j \searrow 0, \quad \widetilde{q}_{\varepsilon,\kappa}(\widetilde{x}_{m_j}(t_3 - \delta_j),\; \widetilde{x}(t_3)) \longrightarrow 0 \quad \forall\; \kappa \in \mathcal{I}$
$\qquad \pi_1\, \widetilde{x}_{m_j}(t_3 - \delta_j) \;\nearrow\; \pi_1\, \widetilde{x}(t_3),$

Then, $\widetilde{x}(\cdot)$ *is a timed right–hand sleek solution of* $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot), \cdot)$ *in* $[0,T[$.

Proof differs from the proof of Proposition 4.11 only in the additional supremum with respect to $\kappa \in \mathcal{I}$. Indeed, following the same track, the sufficiently large index $j \in \mathbb{N}$ (of the approximating sequences) now depends on $\kappa \in \mathcal{I}$ and its counterparts $\kappa', \kappa'' \in \mathcal{I}$ in addition. □

Remark 5.2 Assumption (5.iii') of Proposition 5.1 can be interpreted as a form of weak convergence and, similar forms of convergence suggest themselves in hypothesis (5.ii) or the continuity assumption (2.), for example.
There is an analytical obstacle, however: The "structural estimate" of Proposition 4.10 applied to $\widetilde{q}_\varepsilon(\widetilde{f}(\widetilde{x}(t), t)\,(s + \cdot,\, \widetilde{z}),\; \widetilde{x}_{m_j}(t + \delta_j' + \cdot))$ plays a key role in the proof of Proposition 4.11 and, a similar inequality is not immediately available for $\widetilde{q}_{\varepsilon,\kappa}$.
The geometric example in § 6 has the advantage that we do not need this "weak" convergence in both arguments. Extending mutational equations in this regard is work in progress.

Now timed transitional compactness is now adapted for this modified condition on right–convergence and, we obtain the corresponding result about existence:

Definition 5.3 *Let $\widetilde{\Theta}$ denote a nonempty set of maps $[0,1] \times \widetilde{E} \longrightarrow \widetilde{E}$. Suppose $\widetilde{q}_\varepsilon = \sup_{\kappa\in\mathcal{I}} \widetilde{q}_{\varepsilon,\kappa}$ with (at most) countably many $\widetilde{q}_{\varepsilon,\kappa} : (\widetilde{\mathcal{D}} \cup \widetilde{E}) \times (\widetilde{\mathcal{D}} \cup \widetilde{E}) \longrightarrow [0,\infty[$ $(\varepsilon \in \mathcal{J}, \kappa \in \mathcal{I})$. $(\widetilde{E},\ (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}},\ (\widetilde{q}_{\varepsilon,\kappa})_{\substack{\varepsilon\in\mathcal{J}\\ \kappa\in\mathcal{I}}},\ \widetilde{\Theta})$ is called* right-weakly transitionally compact *if it fulfills:*
Let $(\widetilde{x}_n)_{n\in\mathbb{N}}$, $(h_j)_{j\in\mathbb{N}}$ and $\widetilde{\vartheta}_n : [0,1] \longrightarrow \widetilde{\Theta}$, $\widetilde{y}_n(\cdot) : [0,1] \longrightarrow \widetilde{E}$ (for each $n \in \mathbb{N}$) satisfy the assumptions of Definition 4.12. Then there exist a sequence $n_k \nearrow \infty$ and $\widetilde{x} \in \widetilde{E}$ satisfying for each $\varepsilon \in \mathcal{J}$, $\kappa \in \mathcal{I}$

$$\begin{aligned} \lim_{k\to\infty} \ \pi_1\, \widetilde{x}_{n_k} &= \pi_1\, \widetilde{x}, \\ \limsup_{k\longrightarrow\infty} \ \widetilde{q}_{\varepsilon,\kappa}(\widetilde{x}_{n_k},\ \widetilde{x}) &= 0, \\ \limsup_{j\longrightarrow\infty}\ \sup_{k\geq j}\ \widetilde{q}_\varepsilon(\widetilde{x},\ \ \widetilde{y}_{n_k}(h_j)) &= 0. \end{aligned}$$

Proposition 5.4 (Existence of timed right–hand sleek solutions II)
Assume $\widetilde{q}_\varepsilon = \sup_{\kappa\in\mathcal{I}} \widetilde{q}_{\varepsilon,\kappa}$ with (at most) countably many $\widetilde{q}_{\varepsilon,\kappa} : (\widetilde{\mathcal{D}} \cup \widetilde{E})^2 \longrightarrow [0,\infty[$ $(\varepsilon \in \mathcal{J}, \kappa \in \mathcal{I})$ such that each $\kappa \in \mathcal{I}$ has counterparts $\kappa', \kappa'' \in \mathcal{I}$ fulfilling

$$\widetilde{q}_{\varepsilon,\kappa}(\widetilde{y}_1,\widetilde{y}_3) \ \leq\ \widetilde{q}_{\varepsilon,\kappa'}(\widetilde{y}_1,\widetilde{y}_2) \ +\ \widetilde{q}_{\varepsilon,\kappa''}(\widetilde{y}_2,\widetilde{y}_3)$$

for all $\widetilde{y}_1,\widetilde{y}_2,\widetilde{y}_3 \in \widetilde{\mathcal{D}} \cup \widetilde{E}$ with $\pi_1\,\widetilde{y}_1 \leq \pi_1\,\widetilde{y}_2 \leq \pi_1\,\widetilde{y}_3$.
Moreover, let $\Big(\widetilde{E},\ (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}},\ (\widetilde{q}_{\varepsilon,\kappa})_{\substack{\varepsilon\in\mathcal{J}\\ \kappa\in\mathcal{I}}},\ \widetilde{\Theta}(\widetilde{E},\widetilde{\mathcal{D}},(\widetilde{q}_\varepsilon))\Big)$ be right-weakly transitionally compact and $\widetilde{f} : \widetilde{E} \times [0,T] \longrightarrow \widetilde{\Theta}(\widetilde{E},\widetilde{\mathcal{D}},(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ fulfill for every $\varepsilon \in \mathcal{J}$, $\widetilde{z} \in \widetilde{\mathcal{D}}$

1. *$M_\varepsilon(\widetilde{z}) := \sup_{t_1,t_2,\widetilde{y}_1,\widetilde{y}_2} \{\alpha_\varepsilon(\widetilde{f}(\widetilde{y}_1,t_1),\ \widetilde{f}(\widetilde{y}_2,t_2)(h,\widetilde{z})) \mid 0 \leq h < \mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{y}_2,t_2),\widetilde{z})\} < \infty$,*
2. *$c_\varepsilon := \sup_{t,\widetilde{y}} \ \beta_\varepsilon(\widetilde{f}(\widetilde{y},t)) < \infty$,*
3. *$\widetilde{Q}_\varepsilon\Big(\widetilde{f}(\widetilde{y},t),\ \widetilde{f}(\widetilde{y}_m,t_m);\ \widetilde{z}\Big) \stackrel{m\to\infty}{\longrightarrow} 0$ whenever $0 \leq t_m - t \longrightarrow 0$ and $\widetilde{q}_\delta(\widetilde{y},\widetilde{y}_m) \longrightarrow 0$ $(\pi_1\,\widetilde{y} \leq \pi_1\,\widetilde{y}_m)$ for each $\delta \in \mathcal{J}$.*

Then for every $\widetilde{x}_0 \in \widetilde{E}$, there is a timed right–hand sleek solution $\widetilde{x} : [0,T[\longrightarrow \widetilde{E}$ of the generalized mutational equation $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot),\cdot)$ with $\widetilde{x}(0) = \widetilde{x}_0$.

Remark 5.5 The proof is based on the same idea as for Proposition 4.13. Indeed, starting with the Euler approximations $\widetilde{x}_n(\cdot) : [0,T[\longrightarrow \widetilde{E}$ $(n \in \mathbb{N})$ of Remark 4.14, we conclude from the compactness hypothesis in combination with Cantor diagonal construction ($\mathcal{J} = \{\varepsilon_{j_1}, \varepsilon_{j_2} \ldots\}$, $\mathcal{I} = \{\kappa_{i_1}, \kappa_{i_2} \ldots\}$ are assumed to be countable at the most):
With Q_K denoting the finite set $]0,T[\cap \mathbb{N} \cdot h_K$ of time steps for each $K \in \mathbb{N}$, there are sequences $m_k, n_k \nearrow \infty$ of indices and a function $\widetilde{x} : \bigcup_{K\in\mathbb{N}} Q_K \longrightarrow \widetilde{E}$ such that $m_k \leq n_k$,

$$\begin{cases} \sup_{l\geq k} \ \widetilde{q}_{\varepsilon,\kappa}(\widetilde{x}_{n_l}(t),\ \ \widetilde{x}(t)) \ \leq\ \frac{1}{k} \\ \sup_{l\geq k} \ \widetilde{q}_\varepsilon(\widetilde{x}(s),\ \ \widetilde{x}_{n_l}(s + \frac{h_{m_k}}{2})) \ \leq\ \frac{1}{k} \end{cases}$$

for every $K \in \mathbb{N}$ and all $\varepsilon \in \{\varepsilon_{j_1} \ldots \varepsilon_{j_K}\} \subset \mathcal{J}$, $\kappa \in \{\kappa_{i_1} \ldots \kappa_{i_K}\} \subset \mathcal{I}$, $s,t \in Q_K$, $k \geq K$.
In particular, $\widetilde{q}_{\varepsilon,\kappa}(\widetilde{x}(s),\ \widetilde{x}(t)) \leq c_\varepsilon \cdot (t-s)$ for any $s,t \in \bigcup_K Q_K$ with $s < t$ and all $\varepsilon \in \mathcal{J}$, $\kappa \in \mathcal{I}$. The supremum with respect to $\kappa \in \mathcal{I}$ implies $\widetilde{q}_\varepsilon(\widetilde{x}(s),\ \widetilde{x}(t)) \leq c_\varepsilon \cdot (t-s)$.

Moreover, the sequence $(\widetilde{x}_{n_k}(\cdot))_{k\in\mathbb{N}}$ fulfills for all $\varepsilon \in \mathcal{J}$, $\kappa \in \mathcal{I}$, $K \in \mathbb{N}$, $t \in Q_K$ and sufficiently large $k, l \in \mathbb{N}$ (depending merely on ε, κ, K)

$$\widetilde{q}_{\varepsilon,\kappa}(\widetilde{x}_{n_k}(t),\ \widetilde{x}_{n_l}(t + \tfrac{h_{m_k}}{2})) \ \le\ \tfrac{1}{k} + \tfrac{1}{l}\,.$$

For extending $\widetilde{x}(\cdot)$ to $t \in]0,T[\setminus \bigcup_K Q_K$, we apply the hypothesis of right-weak compactness to $((\widetilde{x}_{n_k}(t))_{k\in\mathbb{N}}$ and obtain a subsequence $n_{l_j} \nearrow \infty$ of indices (depending on t) and an element $\widetilde{x}(t) \in \widetilde{E}$ satisfying for every $\varepsilon \in \mathcal{J}$, $\kappa \in \mathcal{I}$

$$\left\{\begin{array}{rlcl} & \widetilde{q}_{\varepsilon,\kappa}(\widetilde{x}_{n_{l_j}}(t),\quad \widetilde{x}(t)) & \longrightarrow & 0, \\ \sup\limits_{i\ge j} & \widetilde{q}_{\varepsilon}(\widetilde{x}(t),\quad \widetilde{x}_{n_{l_i}}(t + \frac{h_{m_j}}{2})) & \longrightarrow & 0 \end{array}\right. \qquad \text{for } j \longrightarrow \infty.$$

It implies $\quad \widetilde{q}_\varepsilon(\widetilde{x}(s),\ \widetilde{x}(t)) \le c_\varepsilon \cdot (t-s) \quad$ for all $s \in [0,t[\ \cap \bigcup_K Q_K$ and
$\widetilde{q}_\varepsilon(\widetilde{x}(t),\ \widetilde{x}(s')) \le c_\varepsilon \cdot (s'-t) \quad$ for all $s' \in]t,T[\ \cap \bigcup_K Q_K$.

The following convergence is even uniform in t (but not necessarily in ε, κ)

$$\left\{\begin{array}{lll} \limsup\limits_{K\longrightarrow\infty}\ \limsup\limits_{k\longrightarrow\infty} & \widetilde{q}_{\varepsilon,\kappa}(\widetilde{x}_{n_k}(t-2\,h_K),\quad \widetilde{x}(t)\,) & = 0, \\ \limsup\limits_{K\longrightarrow\infty}\ \limsup\limits_{k\longrightarrow\infty} & \widetilde{q}_{\varepsilon}(\widetilde{x}(t),\quad \widetilde{x}_{n_k}(t+2\,h_K)\,) & = 0. \end{array}\right.$$

Indeed, for $\varepsilon \in \mathcal{J}, \kappa \in \mathcal{I}$ and $K \in \mathbb{N}$ fixed arbitrarily, there are $s = s(t,K,\varepsilon,\kappa) \in Q_K$ and $K' = K'(\varepsilon,\kappa,K) \in \mathbb{N}$ with $t - 2\,h_K < s \le t - h_K$, $K' \ge K$ and

$$\widetilde{q}_{\varepsilon,\kappa''}(\widetilde{x}_{n_k}(s),\ \widetilde{x}_{n_l}(s + \tfrac{h_{m_k}}{2})) \le \tfrac{1}{k} + \tfrac{1}{l} \qquad \text{for all } k,l \ge K'.$$

So for any $k, l_j \ge K'$, we conclude from $h_{m_{l_j}} < \frac{1}{2}\,h_{l_j} \le \frac{1}{2}\,h_K$ and the triangle hypothesis (with a suitable $\widehat{\kappa} = \widehat{\kappa}(\varepsilon,\kappa) \in \mathcal{I}$)

$$\begin{array}{rcll} \widetilde{q}_{\varepsilon,\kappa}(\widetilde{x}_{n_k}(t-2\,h_K),\ \widetilde{x}(t)) & \le & & \widetilde{q}_\varepsilon(\widetilde{x}_{n_k}(t-2\,h_K),\quad \widetilde{x}_{n_k}(s)) \\ & & + & \widetilde{q}_{\varepsilon,\kappa''}(\widetilde{x}_{n_k}(s),\quad \widetilde{x}_{n_{l_j}}(s + \frac{h_{m_k}}{2}))) \\ & & + & \widetilde{q}_\varepsilon(\widetilde{x}_{n_{l_j}}(s + \frac{h_{m_k}}{2}),\quad \widetilde{x}_{n_{l_j}}(t)) \\ & & + & \widetilde{q}_{\varepsilon,\widehat{\kappa}}(\widetilde{x}_{n_{l_j}}(t),\quad \widetilde{x}(t)) \\ & \le & & c_\varepsilon \cdot h_K + \frac{1}{k} + \frac{1}{l_j} + c_\varepsilon \cdot 2\,h_K + \widetilde{q}_{\varepsilon,\widehat{\kappa}}(\widetilde{x}_{n_{l_j}}(t), \widetilde{x}(t)) \end{array}$$

and $j \longrightarrow \infty$ leads to the estimate $\widetilde{q}_{\varepsilon,\kappa}(\widetilde{x}_{n_k}(t-2\,h_K),\ \widetilde{x}(t)) \le 3\,c_\varepsilon\,h_K + \frac{2}{K}$.
The proof of $\limsup\limits_{K\longrightarrow\infty}\ \limsup\limits_{k\longrightarrow\infty}\ \widetilde{q}_\varepsilon(\widetilde{x}(t),\ \widetilde{x}_{n_k}(t+2\,h_K)\,) = 0$ is based on the continuity of $\widetilde{x}(\cdot)$ analogously (with $s' = s'(t,K,\varepsilon) \in Q_K$ satisfying $t + h_K \le s' < t + 2\,h_K$).

Similarly to Remark 4.14, we summarize the construction of $\widetilde{x}(\cdot)$ in the following notation: For each $j \in \mathbb{N}$, there exist $K_j \in \mathbb{N}$ (depending on j, K_{j-1}, N_{j-1}) and $N_j \in \mathbb{N}$ (depending on j, K_j) such that $\quad N_j > K_j > N_{j-1} \quad$ and

$$\left\{\begin{array}{lcl} \widetilde{q}_{\varepsilon,\kappa}(\widetilde{x}_{N_j}(s - 2\,h_{K_j}),\quad \widetilde{x}(s)\,) & \le & \frac{1}{j} \\ \widetilde{q}_\varepsilon(\widetilde{x}(t),\quad \widetilde{x}_{N_j}(t + 2\,h_{K_j})\,) & \le & \frac{1}{j} \end{array}\right.$$

for every $s,t \in [0,T[$ and $\varepsilon \in \{\varepsilon_{i_1} \ldots \varepsilon_{i_j}\} \subset \mathcal{J}$, $\kappa \in \{\kappa_{i_1} \ldots \kappa_{i_j}\} \subset \mathcal{I}$.
So Convergence Theorem II (Proposition 5.1) ensures that $\widetilde{x}(\cdot)$ is timed right–hand sleek solution of the generalized mutational equation $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot), \cdot)$ with $\widetilde{x}(0) = \widetilde{x}_0$. □

For concluding the existence from timed transitional compactness, we do not need any assumptions about the time parameter $\mathbb{T}_\varepsilon(\cdot, \widetilde{z}) > 0$ of timed sleek transitions.

The situation changes, however, for estimating the distance between solutions. Indeed, the definition of timed right–hand sleek solutions is based on comparisons with earlier elements (merely) of the form $\widetilde{\vartheta}(s, \widetilde{z}) \in \widetilde{\mathcal{D}}$ for $\widetilde{z} \in \widetilde{\mathcal{D}}$, $0 \leq s < \mathbb{T}_\varepsilon(\widetilde{\vartheta}, \widetilde{z})$. So two sleek solutions $\widetilde{x}(\cdot)$, $\widetilde{y}(\cdot)$ of the same initial value problem can hardly be compared with each other directly and, we need an auxiliary function instead — like, for example,

$$\varphi_\varepsilon(t) \;:=\; \inf_{\substack{\widetilde{z} \in \widetilde{\mathcal{D}} \\ \pi_1 \widetilde{z} < \pi_1 \widetilde{x}(t)}} \Big(\widetilde{q}_\varepsilon(\widetilde{z},\, \widetilde{x}(t)) \;+\; \widetilde{q}_\varepsilon(\widetilde{z},\, \widetilde{y}(t)) \Big).$$

Proposition 5.6 *Assume for the function* $\widetilde{f} : (\widetilde{\mathcal{D}} \cup \widetilde{E}) \times [0,T] \longrightarrow \widetilde{\Theta}(\widetilde{E}, \widetilde{\mathcal{D}}, (\widetilde{q}_\varepsilon))$, *the curves* $\widetilde{x}, \widetilde{y} : [0,T[\longrightarrow \widetilde{E}$ *and some* $\varepsilon \in \mathcal{J}$

1. $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot), \cdot)$, $\quad \overset{\circ}{\widetilde{y}}(\cdot) \ni \widetilde{f}(\widetilde{y}(\cdot), \cdot)$ *in* $[0,T[$ *(in the sense of Definition 4.7)*
   $\pi_1 \widetilde{x}(0) = \pi_1 \widetilde{y}(0) = 0$,
2. $M_\varepsilon \geq \sup\limits_{\widetilde{v} \in \widetilde{\mathcal{D}} \cup \widetilde{E},\, t<T,\, \widetilde{z} \in \widetilde{\mathcal{D}}} \{\alpha_\varepsilon(\widetilde{f}(\widetilde{v}, t),\, \widetilde{z}),\; \widehat{\alpha}_\varepsilon(t, \widetilde{x}(\cdot), \widetilde{z}),\; \widehat{\alpha}_\varepsilon(t, \widetilde{y}(\cdot), \widetilde{z})\}$,
3. $\exists\; \widehat{\omega}_\varepsilon(\cdot), L_\varepsilon, R_\varepsilon :\; \widetilde{Q}_\varepsilon(\widetilde{f}(\widetilde{z}, s),\, \widetilde{f}(\widetilde{v}, t);\, \widetilde{z}) \;\leq\; R_\varepsilon + L_\varepsilon \cdot \widetilde{q}_\varepsilon(\widetilde{z}, \widetilde{v}) + \widehat{\omega}_\varepsilon(t-s)$
   for all $0 \leq s \leq t \leq T$ *and* $\widetilde{v} \in \widetilde{E}$, $\widetilde{z} \in \widetilde{\mathcal{D}}$ *with* $\pi_1 \widetilde{z} \leq \pi_1 \widetilde{v}$,
   $\widehat{\omega}_\varepsilon(\cdot) \geq 0$ *nondecreasing,* $\quad \limsup\limits_{s \downarrow 0} \; \widehat{\omega}_\varepsilon(s) = 0$.
4. $\forall\; t \in [0,T[:$ *the infimum* $\varphi_\varepsilon(t) := \inf\limits_{\widetilde{z} \in \widetilde{\mathcal{D}},\, \pi_1 \widetilde{z} < t} \big(\widetilde{q}_\varepsilon(\widetilde{z},\, \widetilde{x}(t)) + \widetilde{q}_\varepsilon(\widetilde{z},\, \widetilde{y}(t))\big) < \infty$
   can be approximated by a minimizing sequence $(\widetilde{z}_j)_{j \in \mathbb{N}}$ *in* $\widetilde{\mathcal{D}}$ *with*
   $\pi_1 \widetilde{z}_j \;\leq\; \pi_1 \widetilde{z}_{j+1} \;<\; t, \qquad \dfrac{\sup_{k>j}\; \widetilde{q}_\varepsilon(\widetilde{z}_j, \widetilde{z}_k)}{\mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{z}_j, t),\, \widetilde{z}_j)} \;\longrightarrow\; 0 \quad (j \longrightarrow \infty).$

Then, $\quad \varphi_\varepsilon(t) \;\leq\; \varphi_\varepsilon(0) \cdot e^{(L_\varepsilon + M_\varepsilon)\cdot t} \;+\; 2\, R_\varepsilon\, t \cdot e^{(L_\varepsilon + M_\varepsilon)\cdot t}$ $\quad$ *for every time* $t \in [0,T[$.

Proof is based on a further subdifferential version of Gronwall's Lemma quoted in Lemma 5.7. $\varphi_\varepsilon(\cdot)$ satisfies $\varphi_\varepsilon(t) \;\leq\; \liminf\limits_{h \downarrow 0}\; \varphi_\varepsilon(t-h)$ for every $t \in]0,T[$ due to the timed triangle inequality and the continuity of $\widetilde{x}(\cdot)$, $\widetilde{y}(\cdot)$ (in time direction).

For showing $\qquad \liminf\limits_{h \downarrow 0} \frac{\varphi_\varepsilon(t+h) - \varphi_\varepsilon(t)}{h} \;\leq\; (L_\varepsilon + M_\varepsilon)\; \varphi_\varepsilon(t) \;+\; 2\, R_\varepsilon,$

let $(\widetilde{z}_j)_{j \in \mathbb{N}}$ denote a minimizing sequence in $\widetilde{\mathcal{D}}$ such that

$$\begin{cases} \pi_1\, \widetilde{z}_j \;\leq\; \pi_1\, \widetilde{z}_k \;<\; t, & \\ \widetilde{q}_\varepsilon(\widetilde{z}_j, \widetilde{z}_k) \qquad \leq\; \frac{1}{2\,j} \cdot \mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{z}_j, t),\, \widetilde{z}_j) & \text{for all } j < k, \\ \widetilde{q}_\varepsilon(\widetilde{z}_j,\, \widetilde{x}(t)) \;+\; \widetilde{q}_\varepsilon(\widetilde{z}_j,\, \widetilde{y}(t)) \;\longrightarrow\; \varphi_\varepsilon(t) & (j \longrightarrow \infty). \end{cases}$$

Now for every time $h \in]0,\, \mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{z}_j, t),\, \widetilde{z}_j)[$, Proposition 4.10 implies

$$\begin{aligned} &\widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{z}_j, t)\,(h, \widetilde{z}_j),\;\; \widetilde{x}(t+h)\Big) \\ &\leq\; \widetilde{q}_\varepsilon\Big(\widetilde{z}_j,\; \widetilde{x}(t)\Big) \cdot e^{M_\varepsilon\, h} \;+ \int_0^h e^{M_\varepsilon \cdot (h-s)}\; \Big(R_\varepsilon + L_\varepsilon \cdot \widetilde{q}_\varepsilon\Big(\widetilde{z}_j,\; \widetilde{x}(t+s)\Big) + \widehat{\omega}_\varepsilon(s)\Big)\, ds. \end{aligned}$$

Setting the abbreviation $h_j := \frac{1}{2}\, \mathbb{T}_\varepsilon(\widetilde{f}(\widetilde{z}_j, t),\, \widetilde{z}_j) > 0$, the approximating properties of $(\widetilde{z}_j)_{j \in \mathbb{N}}$ and the timed triangle inequality guarantee for any index $k > j$ and time $h \in]0, h_j]$

$$\widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{z}_j,t)\,(h,\widetilde{z}_j),\ \ \widetilde{x}(t{+}h)\Big)$$
$$\le\ \widetilde{q}_\varepsilon\Big(\widetilde{z}_k,\ \widetilde{x}(t)\Big)\cdot e^{M_\varepsilon\,h}\ +\ \tfrac{e^{M_\varepsilon\,h}-1}{M_\varepsilon}\ \Big(L_\varepsilon\cdot\widetilde{q}_\varepsilon(\widetilde{z}_k,\ \widetilde{x}(t))\ +L_\varepsilon\cdot\tfrac{1}{j}\,h_j\ +\ R_\varepsilon\Big)$$
$$+\ \ \tfrac{1}{j}\,h_j\ \ \cdot e^{M_\varepsilon\,h}\ +\int_0^h e^{M_\varepsilon\cdot(h-s)}\ \Big(L_\varepsilon\cdot\omega_\varepsilon(\widetilde{x},s)\ \ \ +\widehat{\omega}_\varepsilon(s)\Big)\ ds.$$

The corresponding estimate for $\widetilde{q}_\varepsilon\Big(\widetilde{f}(\widetilde{z}_j,t)\,(h,\widetilde{z}_j),\ \widetilde{y}(t{+}h)\Big)$ and $k\longrightarrow\infty,\ h:=h_j,\ j\longrightarrow\infty$ lead to $\liminf_{h\downarrow 0}\ \frac{\varphi_\varepsilon(t+h)\,-\,\varphi_\varepsilon(t)}{h}\ \le\ (L_\varepsilon{+}M_\varepsilon)\ \varphi_\varepsilon(t)\ +\ 2\,R_\varepsilon.$ □

Lemma 5.7 (Lemma of Gronwall for semicontinuous functions II [22, 23]**)**
Let $\psi:[a,b]\longrightarrow\mathbb{R},\ f,g\in C^0([a,b[,\mathbb{R})$ *satisfy* $f(\cdot)\ge 0$ *and*

$$\begin{aligned}\psi(t)\ &\le\ \liminf_{h\downarrow 0}\ \psi(t-h), && \forall\ t\in\,]a,\,b],\\ \psi(t)\ &\ge\ \liminf_{h\downarrow 0}\ \psi(t+h), && \forall\ t\in[a,\,b[,\\ \liminf_{h\downarrow 0}\ \tfrac{\psi(t+h)-\psi(t)}{h}\ &\le\ f(t)\ \cdot\ \liminf_{h\downarrow 0}\ \psi(t-h)\ +\ g(t) && \forall\ t\in\,]a,\,b[.\end{aligned}$$

Then, for every $t\in[a,b]$, *the function* $\psi(\cdot)$ *fulfills the upper estimate*

$$\psi(t)\ \le\ \psi(a)\ \cdot\ e^{\mu(t)}\ +\ \int_a^t e^{\mu(t)-\mu(s)}\ g(s)\ ds\qquad\text{with } \mu(t):=\int_a^t f(s)\ ds.$$

6 Example of first–order geometric evolutions

Now the concept of timed right–hand *sleek* solutions is applied to the evolution of compact subsets of $\mathbb{R}^N$. As key feature of *first–order* geometric evolutions, they may depend on nonlocal properties of the current compact set and its normal cones at the boundary.

Remark 6.1 (An earlier geometric example considering normal cones)
In [20, 23], such a geometric example is given for right–hand *forward* solutions. Indeed, the set $\mathcal{K}(\mathbb{R}^N)$ of all nonempty compact subsets of $\mathbb{R}^N$ is supplied with the ostensible metric

$$q_{\mathcal{K},N}(K_1,K_2)\ :=\ d\!l(K_1,K_2)\ +\ \text{dist}(\text{Graph}\ {}^\flat N_{K_2},\ \text{Graph}\ {}^\flat N_{K_1})$$

with $d\!l$ denoting the Pompeiu–Hausdorff distance on $\mathcal{K}(\mathbb{R}^N)$,
$N_K(x)$ the limiting normal cone of $K\in\mathcal{K}(\mathbb{R}^N)$ at $x\in\partial K$ (Def. 6.2),
${}^\flat N_K(x)\ :=\ N_K(x)\cap\mathbb{B}\ =\ \{v\in N_K(x):|v|\le 1\}$.

$\mathcal{K}_{C^{1,1}}(\mathbb{R}^N)$ consisting of all nonempty compact subsets with $C^{1,1}$ boundary is used for "test elements". Then for any parameter $\lambda>0$ fixed, the set–valued maps $F:\mathbb{R}^N\leadsto\mathbb{R}^N$ satisfying

1. $F:\mathbb{R}^N\leadsto\mathbb{R}^N$ has nonempty compact convex values,
2. $\mathcal{H}_F(x,p):=\sup_{v\in F(x)}\ p\cdot v$ belongs to $C^{1,1}(\mathbb{R}^N\times(\mathbb{R}^N\setminus\{0\}))$,
3. $\|\mathcal{H}_F\|_{C^{1,1}(\mathbb{R}^N\times\,\partial\mathbb{B}_1)}\ \stackrel{\text{Def.}}{=}\ \|\mathcal{H}_F\|_{C^1(\mathbb{R}^N\times\,\partial\mathbb{B}_1)}\ +\ \text{Lip}\ D\mathcal{H}_F|_{\mathbb{R}^N\times\,\partial\mathbb{B}_1}\ <\ \lambda$

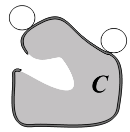
induce forward transitions on $(\mathcal{K}(\mathbb{R}^N),\ \mathcal{K}_{C^{1,1}}(\mathbb{R}^N),\ q_{\mathcal{K},N})$ by means of their reachable sets $\vartheta_F(t,K)\ :=\ \{\,x(t)\mid x(\cdot)\in W^{1,1}([0,t],\ \mathbb{R}^N),\ x(0)\in K,\ \dot{x}(\cdot)\in F(x(\cdot))\text{ a.e.}\}$.

Under stronger assumptions about the Hamiltonian $\mathcal{H}_F$, the required properties of transitional compactness are also verified in [20], Proposition 4.25 and, so we obtain the existence of right–hand forward solutions in the sense of Definition 3.9 (see [23], § 4.4.4 alternatively).
The estimates between forward solutions do not provide uniqueness though. Indeed, the smooth sets of $\mathcal{K}_{C^{1,1}}(\mathbb{R}^N)$ stay smooth for short times while evolving along such a differential inclusion, but there is no obvious lower bound of this period satisfying the approximating hypothesis such as condition (4.) of Proposition 5.6.

In this section, we introduce another distance function for describing evolutions of compact subsets of $\mathbb{R}^N$ in Definition 6.3. In contrast to the earlier example of Remark 6.1, the substantial idea is now to

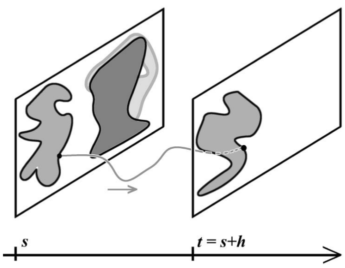
1. use *all* nonempty compact subsets of $\mathbb{R}^N$ as "test elements" (instead of $\mathcal{K}_{C^{1,1}}(\mathbb{R}^N)$), but

2. take only the proximal normals with an exterior ball of radius $\geq \varepsilon$ into consideration simultaneously. Choosing the parameter ε here as rational positive number induces a countable family of (generalized) distance functions specified in Definition 6.3.



   The geometric advantage is that Proposition A.1 provides an upper estimate how fast these exterior balls can shrink (at most) and thus, the corresponding time parameter $\mathbb{T}_\varepsilon(\cdot,\cdot)$ may depend on ε, but not on the "test set".

3. "record" the period $h > 0$ how long the compact set $K(s+h) \subset \mathbb{R}^N$ and the "test set" $\vartheta_F(h, K(s))$ have been evolving while being compared. This period determines the radii of exterior balls that are related with each other for calculating the "distance" between these two sets (see Definition 6.3).
   The additional time component is to provide information about period h : The compact set $K(s{+}h)$ is supplied with a linearly increasing time component whereas all "test sets" preserve their initial time components. Then the wanted period results from their difference.



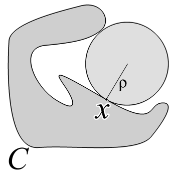
   For implementing this notion in the framework of timed sleek transitions, we introduce an additional component being either 0 (for "test sets") or 1 (otherwise) and indicating the growth of the time component while evolving (see Definition 6.7).

So now we consider

$$\begin{array}{rcl} E & := & \{1\} \times \mathcal{K}(\mathbb{R}^N), \\ \mathcal{D} & := & \{0\} \times \mathcal{K}(\mathbb{R}^N) \end{array} \qquad \text{and thus,} \qquad \begin{array}{rcl} \widetilde{E} & := & \mathbb{R} \times \{1\} \times \mathcal{K}(\mathbb{R}^N), \\ \widetilde{\mathcal{D}} & := & \mathbb{R} \times \{0\} \times \mathcal{K}(\mathbb{R}^N). \end{array}$$

In comparison with the earlier geometric example mentioned in Remark 6.1 and using *forward* transitions [20], the main advantage of the *sleek* concept here is the uniqueness stated in Proposition 6.14.
From now on, fix the parameter $\Lambda > 0$ arbitrarily. It is used for both the distance function $\widetilde{q}_{\mathcal{K},\varepsilon}$ in Definition 6.3 and the set–valued maps (whose reachable sets are candidates for sleek transitions) in Definition 6.6.

Definition 6.2 *Let $C \subset \mathbb{R}^N$ be a nonempty closed set.*
A vector $\eta \in \mathbb{R}^N$, $\eta \neq 0$, is said to be a proximal normal vector *to C at $x \in C$ if there exists $\rho > 0$ with* $\mathbb{B}_\rho(x + \rho\,\frac{\eta}{|\eta|}) \,\cap\, C \,=\, \{x\}$.
The supremum of all ρ with this property is called proximal radius *of C at x in direction η. The cone of all these proximal normal vectors is called the* proximal normal cone *to C at x and is abbreviated as $N_C^P(x)$.*



For any $\rho > 0$, the set $N_{C,\rho}^P(x) \subset \mathbb{R}^N$ consists of all vectors $\eta \in N_C^P(x) \setminus \{0\}$ with the proximal radius $\geq \rho$ (and thus might be empty). Furthermore ${}^\flat N_{C,\rho}^P(x) \,:=\, N_{C,\rho}^P(x) \cap \mathbb{B}$.

The so–called limiting normal cone $N_C(x)$ *to C at x consists of all vectors $\eta \in \mathbb{R}^N$ that can be approximated by sequences $(\eta_n)_{n\in\mathbb{N}}$, $(x_n)_{n\in\mathbb{N}}$ satisfying*

$$\begin{array}{llll} x_n \longrightarrow x, & & x_n \in C, \\ \eta_n \longrightarrow \eta, & & \eta_n \in N_C^P(x_n), \end{array}$$

i.e. $\quad N_C(x) \overset{\text{Def.}}{=} \operatorname{Limsup}_{\substack{y \longrightarrow x \\ y \in C}} N_C^P(y).$

Definition 6.3 *Set* $\widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N) \,:=\, \mathbb{R} \times \{1\} \times \mathcal{K}(\mathbb{R}^N)$, $\widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N) \,:=\, \mathbb{R} \times \{0\} \times \mathcal{K}(\mathbb{R}^N)$.
For $\varepsilon, \kappa \in [0,1]$, define $\widetilde{q}_{\mathcal{K},\varepsilon,\kappa} : (\widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N) \cup \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)) \times (\widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N) \cup \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)) \longrightarrow [0,\infty[$,

$$\begin{aligned} \widetilde{q}_{\mathcal{K},\varepsilon,\kappa}((s,\mu,C),\ (t,\nu,D)) \ :=\ & d\!l(C,D) \,+ \\ & \int_\varepsilon^\infty \psi(\rho{+}\kappa{+}200\,\Lambda\,|t-s|) \cdot \operatorname{dist}\Big(\text{Graph } {}^\flat N_{D,\,(\rho+\kappa+200\,\Lambda\,|t-s|)}^P, \\ & \qquad\qquad \text{Graph } {}^\flat N_{C,\,\rho}^P\Big)\ d\rho \end{aligned}$$

with a fixed nonincreasing weight function $\psi \in C_0^\infty([0,2[)$, $\psi \geq 0$, and set

$$\begin{aligned} \widetilde{q}_{\mathcal{K},\varepsilon}((s,\mu,C),\ (t,\nu,D)) \ :=\ & \sup_{\kappa \in\,]0,1] \,\cap\, \mathbb{Q}} \ \widetilde{q}_{\mathcal{K},\varepsilon,\kappa}((s,\mu,C),\ (t,\nu,D)) \\ =\ & \limsup_{\kappa \downarrow 0} \ \widetilde{q}_{\mathcal{K},\varepsilon,\kappa}((s,\mu,C),\ (t,\nu,D)). \end{aligned}$$

In fact, the second component (being either 0 or 1) does not have any influence on $\widetilde{q}_{\mathcal{K},\varepsilon}$ and $\widetilde{q}_{\mathcal{K},\varepsilon,\kappa}$. Its purpose will only be to determine the evolution of time components for "test elements" and "normal" elements in a different way (as specified in Definition 6.7).

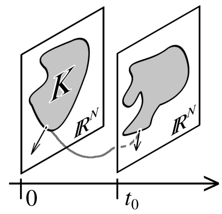
Lemma 6.4 *For each $\varepsilon \in [0,1]$, the function $\widetilde{q}_{\mathcal{K},\varepsilon}$ is reflexive and satisfies the timed triangle inequality on $\widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N) \cup \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$. Moreover, $\widetilde{q}_{\mathcal{K},\varepsilon,\kappa}$ satisfies the triangle hypothesis of Proposition 5.4.*

Proof. Reflexivity (in the sense of Definition 3.1) is obvious. For verifying the timed triangle inequality, choose any (t_1,μ_1,K_1), (t_2,μ_2,K_2), $(t_3,\mu_3,K_3) \in \mathbb{R} \times \{0,1\} \times \mathcal{K}(\mathbb{R}^N)$ with $t_1 \leq t_2 \leq t_3$. Then, we obtain for every $\kappa, \kappa' > 0$

$$\begin{aligned} & \operatorname{dist}\Big(\text{Graph } {}^\flat N_{K_3,\,(\rho+\kappa+\kappa'+200\,\Lambda\,(t_3-t_1))}^P,\ \ \text{Graph } {}^\flat N_{K_1,\,\rho}^P\Big) \\ \leq\ & \operatorname{dist}\Big(\text{Graph } {}^\flat N_{K_3,\,(\rho+\kappa+\kappa'+200\,\Lambda\,(t_3-t_1))}^P,\ \ \text{Graph } {}^\flat N_{K_2,\,(\rho+\kappa+200\,\Lambda\,(t_2-t_1))}^P\Big) \\ & + \ \operatorname{dist}\Big(\text{Graph } {}^\flat N_{K_2,\,(\rho+\kappa+200\,\Lambda\,(t_2-t_1))}^P,\ \ \text{Graph } {}^\flat N_{K_1,\,\rho}^P\Big). \end{aligned}$$

With regard to the weighted integral occurring in $\widetilde{q}_{\mathcal{K},\varepsilon,\,\kappa+\kappa'}((t_1,\mu_1,K_1),\ (t_3,\mu_3,K_3))$, a simple translation of coordinates (for the first distance term) and the monotonicity of ψ (related with the second distance term) imply

$$\begin{aligned} &\widetilde{q}_{\mathcal{K},\varepsilon,\,\kappa+\kappa'}\,((t_1,\mu_1,K_1),\ (t_3,\mu_3,K_3)) \ \leq \\ &\leq\ \widetilde{q}_{\mathcal{K},\varepsilon,\kappa'}((t_1,\mu_1,K_1),\ (t_2,\mu_2,K_2)) \ +\ \widetilde{q}_{\mathcal{K},\varepsilon,\kappa}((t_2,\mu_2,K_2),\ (t_3,\mu_3,K_3)) \\ &\leq\ \ \widetilde{q}_{\mathcal{K},\varepsilon}((t_1,\mu_1,K_1),\ (t_2,\mu_2,K_2)) \ +\ \ \widetilde{q}_{\mathcal{K},\varepsilon}((t_2,\mu_2,K_2),\ (t_3,\mu_3,K_3)). \end{aligned}$$ □



Now we focus on the evolution of limiting normal cones at the topological boundary and use the *Hamilton condition* as a key tool. It implies that roughly speaking, every boundary point x_0 of $\vartheta_F(t_0,K)$ and normal vector $\nu \in N_{\vartheta_F(t_0,K)}(x_0)$ have a solution and an adjoint arc linking x_0 to some $z \in \partial K$ and ν to $N_K(z)$, respectively.

Although the Hamilton condition is known in much more general forms (consider e.g. [31, Theorem 7.7.1] applied to proximal balls), we use only the well–known "smooth" version — due to later regularity conditions on F.

Proposition 6.5 *Suppose for the set–valued map* $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$

1. $F(\cdot)$ *has nonempty convex compact values,*
2. $\mathcal{H}_F(x,p) := \sup\limits_{v \in F(x)} \ p \cdot v$ *is continuously differentiable in* $\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\})$,
3. *the derivative of* $\mathcal{H}_F(\cdot,\cdot)$ *has linear growth in* $\mathbb{R}^N \times (\mathbb{R}^N \setminus \mathbb{B}_1)$,
   i.e. $\|D\mathcal{H}_F(x,p)\| \ \leq\ const \cdot (1 + |x| + |p|)$ *for all* $x,p \in \mathbb{R}^N,\ |p| > 1$.

Let $K \in \mathcal{K}(\mathbb{R}^N)$ *be any initial set and* $t_0 > 0$.

For every boundary point $x_0 \in \partial\,\vartheta_F(t_0,K)$ *and normal vector* $\nu \in N_{\vartheta_F(t_0,K)}(x_0)$, *there exist a solution* $x(\cdot) \in C^1([0,t_0],\mathbb{R}^N)$ *and its adjoint* $p(\cdot) \in C^1([0,t_0],\mathbb{R}^N)$ *satisfying*

$$\left\{ \begin{array}{lll} \dot{x}(t) \ =\ \ \frac{\partial}{\partial p}\ \mathcal{H}_F(x(t),\,p(t)) \ \in\ F(x(t)), & x(t_0) \ =\ x_0, & x(0) \ \in\ \partial K, \\ \dot{p}(t) \ =\ -\,\frac{\partial}{\partial x}\ \mathcal{H}_F(x(t),\,p(t)), & p(t_0) \ =\ \nu, & p(0) \ \in\ N_K(x(0)). \end{array} \right.$$

Definition 6.6 *For* $\Lambda > 0$ *fixed, the set* $\mathrm{LIP}^{(C^2)}_{\Lambda}(\mathbb{R}^N,\mathbb{R}^N)$ *consists of all set–valued maps* $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ *satisfying*

1. $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ *has nonempty compact convex values,*
2. $\mathcal{H}_F(x,p) := \sup\limits_{v \in F(x)} \ p \cdot v$ *is twice continuously differentiable in* $\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\})$,
3. $\|\mathcal{H}_F\|_{C^2(\mathbb{R}^N \times\, \partial\mathbb{B}_1)} \ <\ \Lambda\,.$

These set–valued maps of $\mathrm{LIP}^{(C^2)}_{\Lambda}(\mathbb{R}^N,\mathbb{R}^N)$ induce the candidates for timed sleek transitions on $(\widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N),\ \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N),\ (\widetilde{q}_{\mathcal{K},\varepsilon})_{\varepsilon \in\,]0,1]})$ in the following sense:

Definition 6.7 *For any set–valued map* $F \in \mathrm{LIP}_\Lambda^{(C^2)}(\mathbb{R}^N, \mathbb{R}^N)$, *element* $(t, \mu, K) \in \mathbb{R} \times \{0,1\} \times \mathcal{K}(\mathbb{R}^N) = \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N) \cup \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$ *and time* $h > 0$, *set*

$$\widetilde{\vartheta}_F(h,\ (t,\mu,K)) \ := \ (t + \mu\, h,\ \ \mu,\ \ \vartheta_F(h,K))$$

with the reachable set $\vartheta_F(h,K) \subset \mathbb{R}^N$ *of the differential inclusion* $\dot{x}(\cdot) \in F(x(\cdot))$ *a.e.*

Lemma 6.8 *For every set–valued map* $F \in \mathrm{LIP}_\Lambda^{(C^2)}(\mathbb{R}^N, \mathbb{R}^N)$, *initial element* $\widetilde{K} = (b, 1, K) \in \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$ *and any times* $0 \le s < t \le 1$,

$$\widetilde{q}_{\mathcal{K},\varepsilon}\left(\widetilde{\vartheta}_F(s,\ \widetilde{K}),\ \ \widetilde{\vartheta}_F(t,\ \widetilde{K})\right) \ \le \ \Lambda\,(1 + \|\psi\|_{L^1}\ \ (e^\Lambda + 1)) \ \cdot\ |t-s|.$$

Proof. Obviously, the Pompeiu–Hausdorff distance satisfies for every $s, t \ge 0$

$$d\!\!l\left(\vartheta_F(s,K),\ \ \vartheta_F(t,K)\right) \ \le\ \sup_{\mathbb{R}^N}\ \|F(\cdot)\|_\infty \ \cdot\ (t-s) \ \le\ \Lambda\ (t-s).$$

Let $\tau(\varepsilon,\Lambda) > 0$ denote the time period mentioned in Corollary A.2. Without loss of generality, we can now assume $0 < t - s < \frac{1}{200\,\Lambda}\ \tau(\varepsilon,\Lambda)$ as a consequence of the timed triangle inequality. For any $(x,p) \in \mathrm{Graph}\ {}^\flat N^P_{\vartheta_F(t,K),\,(\rho + 200\,\Lambda\,(t-s))}$ and $\rho \ \ge\ \varepsilon$ with $\rho + 200\,\Lambda\,(t-s) \ \le\ 2$, Corollary A.2 and Proposition 6.5 provide a solution $x(\cdot) \in C^1([s,t], \mathbb{R}^N)$ and its adjoint arc $p(\cdot) \in C^1([s,t], \mathbb{R}^N)$ satisfying

$$\begin{cases} \dot{x}(\sigma) \ = \ \ \frac{\partial}{\partial p}\ \mathcal{H}_F(x(\sigma),\, p(\sigma)) \ \in\ F(x(\sigma)), & x(t) \ = \ x, \qquad x(s) \ \in\ \partial\vartheta_F(s,K), \\ \dot{p}(\sigma) \ = \ -\frac{\partial}{\partial x}\ \mathcal{H}_F(x(\sigma),\, p(\sigma)), & p(t) \ = \ p, \qquad p(s) \ \in\ N^P_{\vartheta_F(s,K)}(x(s)) \end{cases}$$

and, $p(s)$ has proximal radius $\ge\ \rho + 200\,\Lambda\,(t-s)\ -\ 81\,\Lambda\,(t-s)\ >\ \rho$.
Obviously, $\mathcal{H}_F$ is (positively) homogeneous with respect to its second argument and thus, its definition implies $|\dot{p}(\sigma)| \ \le\ \Lambda\ |p(\sigma)|$ for all σ. Moreover $|p| \le 1$ implies that the projection of p on any cone is also contained in $\mathbb{B}_1$. So finally, we obtain

$$\begin{aligned} \mathrm{dist}\Big((x,p),\ \ \mathrm{Graph}\ {}^\flat N^P_{\vartheta_F(s,K),\,\rho}\Big) \ &\le\ |x - x(s)| \ +\ |p - p(s)| \\ &\le\ \sup_{s \le \sigma \le t}\ \Big(|\tfrac{\partial}{\partial x}\,\mathcal{H}_F| \ +\ |\tfrac{\partial}{\partial p}\,\mathcal{H}_F|\Big)\Big|_{(x(\sigma),p(\sigma))} \cdot (t-s) \\ &\le\ \Big(\Lambda\ e^{\Lambda\,t} \ +\ \Lambda\Big) \cdot (t-s). \end{aligned}$$

□

Lemma 6.9 *For any* $\varepsilon \in\ ]0,1]$, *let* $\tau(\varepsilon,\Lambda) > 0$ *denote the time period mentioned in Corollary A.2. Choose any set–valued maps* $F,\ G \in \mathrm{LIP}_\Lambda^{(C^2)}(\mathbb{R}^N, \mathbb{R}^N)$, *initial elements* $\widetilde{K}_1 = (t_1, 0, K_1) \in \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N)$, $\widetilde{K}_2 = (t_2, 1, K_2) \in \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$ *with* $t_1 \le t_2$.
Then for all $h \in [0, \tau(\varepsilon,\Lambda)[$,

$$\begin{aligned} &\widetilde{q}_{\mathcal{K},\varepsilon}\Big(\widetilde{\vartheta}_F(h, \widetilde{K}_1),\ \widetilde{\vartheta}_G(h, \widetilde{K}_2)\Big) \ \le \\ &\le\ e^{(\lambda_{\mathcal{H}} + \Lambda)\ h}\ \cdot\ \Big(\widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{K}_1,\ \widetilde{K}_2) \ +\ (1 + 6\,N\,\|\psi\|_{L^1})\ h\ \|\mathcal{H}_F - \mathcal{H}_G\|_{C^1(\mathbb{R}^N \times\, \partial\mathbb{B}_1)}\Big)\,. \end{aligned}$$

with the abbreviation $\lambda_{\mathcal{H}} \ :=\ 9\,\Lambda\, e^{2\,\Lambda\,\cdot\,\tau(\varepsilon,\Lambda)}$.

Proof. As presented in [1], Proposition 3.7.3, the well–known Theorem of Filippov provides the estimate of the Pompeiu–Hausdorff distance

$$
\begin{aligned}
d\!\!l\Big(\vartheta_F(h,K_1),\ \vartheta_G(h,K_2)\Big) \;&\le\; d\!\!l(K_1,K_2)\cdot e^{\Lambda h} \;+\; \sup_{\mathbb{R}^N} d\!\!l\Big(F(\cdot),G(\cdot)\Big)\;\cdot \tfrac{e^{\Lambda h}-1}{\Lambda}\\
&\le\; d\!\!l(K_1,K_2)\cdot e^{\Lambda h} \;+\; \sup_{\mathbb{R}^N\times\partial\mathbb{B}_1} |\mathcal{H}_F-\mathcal{H}_G|\;\cdot\, h\ e^{\Lambda h}.
\end{aligned}
$$

According to Definition 6.7, $\widetilde{\vartheta}_F(h,\widetilde{K}_1)\in\{t_1\}\times\{0\}\times\mathcal{K}(\mathbb{R}^N)\subset\widetilde{\mathcal{K}}^{\Upsilon}(\mathbb{R}^N)$ and $\widetilde{\vartheta}_G(h,\widetilde{K}_2)\in\{t_2+h\}\times\{1\}\times\mathcal{K}(\mathbb{R}^N)\subset\widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$.
So for any $\kappa\in\,]0,1]\cap\mathbb{Q}$ and $\rho\ge\varepsilon$ with $\rho+\kappa+200\,\Lambda\,(t_2-t_1+h)\le 2$, we need an upper bound of $\operatorname{dist}\Big(\operatorname{Graph}{}^{\flat}N^P_{\vartheta_G(h,K_2),\,(\rho+\kappa+200\,\Lambda\,(t_2-t_1+h))},\ \operatorname{Graph}{}^{\flat}N^P_{\vartheta_F(h,K_1),\,\rho}\Big)$.

Choose $\delta>0$, $x\in\partial\,\vartheta_G(h,K_2)$ and $p\in N^P_{\vartheta_G(h,K_2)}(x)\cap\partial\mathbb{B}_1$ with proximal radius $\ge\rho+\kappa+200\,\Lambda\,(t_2-t_1+h)$ arbitrarily. According to Corollary A.2 and Proposition 6.5, there exist a solution $x(\cdot)\in C^1([0,h],\mathbb{R}^N)$ and its adjoint arc $p(\cdot)\in C^1([0,h],\mathbb{R}^N)$ fulfilling

$$
\begin{array}{rclrcl}
\dot{x}(\cdot) &=& \frac{\partial}{\partial p}\,\mathcal{H}_G(x(\cdot),\,p(\cdot))\in G(x(\cdot)), & \dot{p}(\cdot) &=& -\frac{\partial}{\partial x}\,\mathcal{H}_G(x(\cdot),\,p(\cdot))\in\Lambda\,|p(\cdot)|\cdot\mathbb{B}\\
x(0) &\in& \partial K_2, & p(0) &\in& N^P_{K_2}(x(0)),\\
x(h) &=& x, & p(h) &=& p,
\end{array}
$$

and the proximal radius at $x(0)$ in direction $p(0)$ is $\ge\ \rho+\kappa+200\,\Lambda\,(t_2-t_1+h)-81\,\Lambda\,h$ $>\ \rho+\kappa+100\,\Lambda\,h+200\,\Lambda\,(t_2-t_1)$. Gronwall's Lemma guarantees $e^{-\Lambda h}\le|p(\cdot)|\le e^{\Lambda h}$ and so, $p(0)\,e^{-\Lambda h}\ \in\ {}^{\flat}N^P_{K_2}(x(0))\setminus\{0\}$.
Now let $(y_0,\widehat{q}_0)$ denote an element of $\operatorname{Graph}{}^{\flat}N^P_{K_1,\,(\rho+100\,\Lambda\,h)}$ with $\widehat{q}_0\neq 0$ and

$$
\begin{aligned}
&\Big|(y_0,\widehat{q}_0)\;-\;\big(x(0),\ p(0)\,e^{-\Lambda h}\big)\Big|\;\le\\
&\le\ \operatorname{dist}\Big(\operatorname{Graph}{}^{\flat}N^P_{K_2,\,(\rho+\kappa+100\,\Lambda\,h+200\,\Lambda\,(t_2-t_1))},\ \operatorname{Graph}{}^{\flat}N^P_{K_1,\,(\rho+100\,\Lambda\,h)}\Big)\;+\;\delta.
\end{aligned}
$$

As a further consequence of Corollary A.2, we obtain a solution $y(\cdot)\in C^1([0,h],\mathbb{R}^N)$ and its adjoint arc $q(\cdot)$ satisfying

$$
\begin{array}{rclrcl}
\dot{y}(\cdot) &=& \frac{\partial}{\partial p}\,\mathcal{H}_F(y(\cdot),\,q(\cdot)), & \dot{q}(\cdot) &=& -\frac{\partial}{\partial y}\,\mathcal{H}_F(y(\cdot),\,q(\cdot))\in\Lambda\,|q(\cdot)|\cdot\mathbb{B}\\
y(0) &=& y_0, & q(0) &=& \widehat{q}_0\ e^{\Lambda h}\ \neq\ 0,\\
y(h) &\in& \partial\,\vartheta_F(h,K_1), & q(h) &\in& N^P_{\vartheta_F(h,K_1)}(y(h))
\end{array}
$$

and the proximal radius at $y(h)$ in direction $q(h)$ is $\ge\ \rho+100\,\Lambda\,h\ -\ 81\,\Lambda\,h\ >\ \rho$.

$\mathcal{H}_F$ is assumed to be *twice* continuously differentiable with $\|\mathcal{H}_F\|_{C^2(\mathbb{R}^N\times\,\partial\mathbb{B}_1)}<\Lambda$. Moreover, $\mathcal{H}_F(x,p)$ is positively homogeneous with respect to p and thus, the derivative of $\mathcal{H}_F$ proves to be $\lambda_{\mathcal{H}}$–Lipschitz continuous on $\mathbb{R}^N\times\big(\mathbb{B}_{e^{\Lambda\cdot\tau(\varepsilon,\Lambda)}}\setminus\overset{\circ}{\mathbb{B}}_{e^{-\Lambda\cdot\tau(\varepsilon,\Lambda)}}\big)$ with the abbreviation $\lambda_{\mathcal{H}}:=9\,\Lambda\,e^{2\,\Lambda\cdot\tau(\varepsilon,\Lambda)}$ ([23], Lemma 4.4.24). The Theorem of Cauchy–Lipschitz leads to

$$
\begin{aligned}
&\operatorname{dist}\Big((x,p),\ \operatorname{Graph}\ {}^{\flat}N^P_{\vartheta_F(h,K_1),\,\rho}\Big)\qquad\le\ \Big|(x,\,p)\ -\ (y(h),\,q(h))\Big|\\
&\le\ e^{\lambda_{\mathcal{H}}\cdot h}\cdot\Big|(x(0),\,p(0))\ -\ (y_0,\,\widehat{q}_0\,e^{\Lambda h})\Big|\ +\ \tfrac{e^{\lambda_{\mathcal{H}}\cdot h}-1}{\lambda_{\mathcal{H}}}\cdot\sup_{0\le s\le h}\ |D\,\mathcal{H}_F-D\,\mathcal{H}_G|\Big|_{(x(s),\,p(s))}.
\end{aligned}
$$

$\mathcal{H}_F$ and $\mathcal{H}_G$ are positively homogeneous with respect to the second argument and thus,

$$
\begin{aligned}
\Big|\tfrac{\partial}{\partial x_j}\,(\mathcal{H}_F-\mathcal{H}_G)|_{(x(s),\,p(s))}\Big|\ &\le\ e^{\Lambda h}\quad\|D\mathcal{H}_F-D\mathcal{H}_G\|_{C^0(\mathbb{R}^N\times\partial\mathbb{B}_1)},\\
\Big|\tfrac{\partial}{\partial p_j}\,(\mathcal{H}_F-\mathcal{H}_G)|_{(x(s),\,p(s))}\Big|\ &\le\quad 3\cdot\ \ \|\mathcal{H}_F-\mathcal{H}_G\|_{C^1(\mathbb{R}^N\times\partial\mathbb{B}_1)}.
\end{aligned}
$$

So we obtain

$$\begin{aligned}&\mathrm{dist}\Big((x,p),\ \text{Graph }{}^\flat N^P_{\vartheta_F(h,K_1),\,\rho}\Big)\\ \le\ & e^{(\lambda_{\mathcal{H}}+\Lambda)\ h}\ \ \Big|(x(0),\,p(0)\,e^{-\Lambda\,h})\ -\ (y_0,\widehat{q}_0)\Big|\ +\ e^{\lambda_{\mathcal{H}}\,h}\,h\cdot 6\,N\,e^{\Lambda\,h}\ \ \|\mathcal{H}_F-\mathcal{H}_G\|_{C^1(\mathbb{R}^N\times\partial\mathbb{B}_1)}\end{aligned}$$

and, since $\delta>0$ is arbitrarily small and $|p|=1$,

$$\begin{aligned}&\mathrm{dist}\Big(\text{Graph }{}^\flat N^P_{\vartheta_G(h,K_2),\,(\rho+\kappa+200\,\Lambda\,(t_2-t_1+h))},\ \ \text{Graph }{}^\flat N^P_{\vartheta_F(h,K_1),\,\rho}\Big)\\ \le\ & e^{(\lambda_{\mathcal{H}}+\Lambda)\ h}\ \cdot\ \Big\{\mathrm{dist}\Big(\text{Graph }{}^\flat N^P_{K_2,\,(\rho+\kappa+100\,\Lambda\,h+200\,\Lambda\,(t_2-t_1))},\ \ \text{Graph }{}^\flat N^P_{K_1,\,(\rho+100\,\Lambda\,h)}\Big)\ +\\ &\qquad\qquad +\ 6\,N\,h\ \cdot\ \|\mathcal{H}_F-\mathcal{H}_G\|_{C^1(\mathbb{R}^N\times\partial\mathbb{B}_1)}\Big\}.\end{aligned}$$

With regard to $\widetilde{q}_{\mathcal{K},\varepsilon,\kappa}\Big(\widetilde{\vartheta}_F(h,\widetilde{K}_1),\ \widetilde{\vartheta}_G(h,\widetilde{K}_2)\Big)$, integrating over ρ and the monotonicity of the weight function ψ (supposed in Definition 6.2) leads to the claimed estimate for all $h\in[0,\tau(\varepsilon,\Lambda)[$. □

Corollary 6.10 *Under the assumptions of Lemma 6.9,*

$$\begin{aligned}&\widetilde{q}_{\mathcal{K},\varepsilon}\Big(\widetilde{\vartheta}_F(t{+}h,\widetilde{K}_1),\ \widetilde{\vartheta}_G(h,\widetilde{K}_2)\Big)\ \le\\ \le\ & e^{(\lambda_{\mathcal{H}}+\Lambda)\ h}\ \cdot\ \Big(\widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{\vartheta}_F(t,\widetilde{K}_1),\ \widetilde{K}_2)\ +\ (1+6\,N\,\|\psi\|_{L^1})\ h\ \|\mathcal{H}_F-\mathcal{H}_G\|_{C^1(\mathbb{R}^N\times\,\partial\mathbb{B}_1)}\Big)\,.\end{aligned}$$

for all $h,\,t\ge 0$ *with* $t+h<\tau(\varepsilon,\Lambda)$ *and*
$\widetilde{K}_1=(t_1,0,K_1)\in\widetilde{\mathcal{K}}^{\Upsilon}(\mathbb{R}^N)$, $\widetilde{K}_2=(t_2,1,K_2)\in\widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$ *with* $t_1\le t_2$.

Proof results directly from Lemma 6.9 since

$$\begin{aligned}\widetilde{\vartheta}_F(t{+}h,\ \widetilde{K}_1)\ &=\ \{t_1\}\times\{0\}\times\vartheta_F(t{+}h,K_1)\ =\ \widetilde{\vartheta}_F(h,\ \widetilde{\vartheta}_F(t,\widetilde{K}_1)),\\ \widetilde{\vartheta}_F(t,\ \widetilde{K}_1)\ &=\ \{t_1\}\times\{0\}\times\vartheta_F(t,K_1)\quad\ \in\ \widetilde{\mathcal{K}}^{\Upsilon}(\mathbb{R}^N).\end{aligned}$$

□

Proposition 6.11 *The maps* $\widetilde{\vartheta}_F$ *of all set–valued* $F\in\mathrm{LIP}^{(C^2)}_{\Lambda}(\mathbb{R}^N,\mathbb{R}^N)$ *introduced in Definition 6.7 induce timed sleek transitions on* $(\widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N),\ \widetilde{\mathcal{K}}^{\Upsilon}(\mathbb{R}^N),\ (\widetilde{q}_{\mathcal{K},\varepsilon})_{\varepsilon\in\,]0,1]\cap\mathbb{Q}})$ *with*

$$\begin{aligned}\alpha_\varepsilon(\widetilde{\vartheta}_F,\,\cdot\,)\ &\stackrel{\text{Def.}}{=}\ 10\ \Lambda\ \ e^{2\,\Lambda\,\cdot\,\tau(\varepsilon,\Lambda)},\\ \beta_\varepsilon(\widetilde{\vartheta}_F)\ &\stackrel{\text{Def.}}{=}\ \ \Lambda\ \ (1+\|\psi\|_{L^1}\ \ (e^{\Lambda}+1)),\\ \mathbb{T}_\varepsilon(\widetilde{\vartheta}_F,\,\cdot\,)\ &\stackrel{\text{Def.}}{=}\ \min\{\tau(\varepsilon,\Lambda),\ 1\}\quad(\textit{mentioned in Corollary A.2}),\\ \widetilde{Q}_\varepsilon(\widetilde{\vartheta}_F,\widetilde{\vartheta}_G;\,\cdot\,)\ &\le\ (1+6\,N\,\|\psi\|_{L^1})\ \cdot\ \|\mathcal{H}_F-\mathcal{H}_G\|_{C^1(\mathbb{R}^N\times\,\partial\mathbb{B}_1)}\,.\end{aligned}$$

Proof. The semigroup property of reachable sets implies

$$\begin{aligned}\widetilde{q}_{\mathcal{K},\varepsilon}\Big(\widetilde{\vartheta}_F(h,\ \widetilde{\vartheta}_F(t,\widetilde{K})),\quad \widetilde{\vartheta}_F(t+h,\ \widetilde{K})\Big)\ &=\ 0,\\ \widetilde{q}_{\mathcal{K},\varepsilon}\Big(\widetilde{\vartheta}_F(t+h,\ \widetilde{K}),\quad \widetilde{\vartheta}_F(h,\ \widetilde{\vartheta}_F(t,\widetilde{K}))\Big)\ &=\ 0\end{aligned}$$

for all $F\in\mathrm{LIP}^{(C^2)}_{\lambda}(\mathbb{R}^N,\mathbb{R}^N)$, $\widetilde{K}\in\widetilde{\mathcal{K}}^{\Upsilon}(\mathbb{R}^N)\cup\widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$, $h,t\ge 0$, $\varepsilon\in\,]0,1]$ since $\widetilde{q}_{\mathcal{K},\varepsilon}$ is reflexive. Thus, condition (2.) on timed sleek transitions (in Definition 4.1) is satisfied.

As an obvious choice of $i_{\widetilde{\mathcal{D}}}:\widetilde{\mathcal{K}}^{\Upsilon}(\mathbb{R}^N)\longrightarrow\widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$, define $i_{\widetilde{\mathcal{D}}}((t,0,K)):=(t,1,K)$. In particular, it fulfills $\widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{Z},i_{\widetilde{\mathcal{D}}}\,\widetilde{Z})=0$ and $\pi_1\,\widetilde{Z}=\pi_1\,i_{\widetilde{\mathcal{D}}}\,\widetilde{Z}$ for all $\widetilde{Z}\in\widetilde{\mathcal{K}}^{\Upsilon}(\mathbb{R}^N)$. Definition 6.7 has the immediate consequences

$$\begin{array}{llll} \widetilde{\vartheta}_F(0,\widetilde{K}) &=& \widetilde{K} & \text{for all } \widetilde{K} \in \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N) \cup \widetilde{\mathcal{K}}^{\to}(\mathbb{R}^N), \\ \widetilde{\vartheta}_F(h,\widetilde{Z}) &\in& \{\pi_1\,\widetilde{Z}\}\times\{0\}\times\mathcal{K}(\mathbb{R}^N) \subset \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N) & \text{for all } \widetilde{Z} \in \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N),\quad h\in[0,1], \\ \widetilde{\vartheta}_F(h,\widetilde{K}) &\in& \{h+\pi_1\,\widetilde{K}\}\times\{1\}\times\mathcal{K}(\mathbb{R}^N) \subset \widetilde{\mathcal{K}}^{\to}(\mathbb{R}^N) & \text{for all } \widetilde{K} \in \widetilde{\mathcal{K}}^{\to}(\mathbb{R}^N),\quad h\in[0,1], \\ \multicolumn{3}{l}{\widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{\vartheta}_F(h,\,\widetilde{\vartheta}(t,i_{\widetilde{\mathcal{D}}}\widetilde{Z})),\ \widetilde{\vartheta}_F(h,\,\widetilde{\vartheta}(t,\widetilde{Z}))) \;=\; 0} & \text{for all } \widetilde{Z} \in \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N), t,h\in[0,1], \end{array}$$

i.e. conditions (1.), (5.), (7'.), (8'.) of Definition 4.1 hold.

Set $\mathbb{T}_\varepsilon(\widetilde{\vartheta}_F,\cdot) \stackrel{\text{Def.}}{=} \min\{\tau(\varepsilon,\Lambda),1\}$ with the time parameter $\tau(\varepsilon,\Lambda)>0$ mentioned in Corollary A.2. Then, Corollary 6.10 guarantees for all $\widetilde{Z}\in\widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N)$, $\widetilde{K}\in\widetilde{\mathcal{K}}^{\to}(\mathbb{R}^N)$, $t\in[0,\mathbb{T}_\varepsilon(\widetilde{\vartheta}_F,\widetilde{Z})[$ with $t+\pi_1\,\widetilde{Z} \le \pi_1\,\widetilde{K}$

$$\limsup_{h\downarrow 0}\left(\frac{\widetilde{q}_{\mathcal{K},\varepsilon}\big(\widetilde{\vartheta}_F(t+h,\widetilde{Z}),\,\widetilde{\vartheta}_F(h,\widetilde{K})\big) - \widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{\vartheta}_F(t,\widetilde{Z}),\widetilde{K})}{h\quad \widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{\vartheta}_F(t,\widetilde{Z}),\widetilde{K})}\right)^{+} \le\; \lambda_{\mathcal{H}}+\Lambda \;\le\; 10\,\Lambda\,e^{2\,\Lambda\cdot\tau(\varepsilon,\Lambda)}.$$

Furthermore Lemma 6.8 implies condition (4.) of Definition 4.1 with the Lipschitz constant

$$\beta_\varepsilon(\widetilde{\vartheta}_F) \stackrel{\text{Def.}}{=} \Lambda\,(1+\|\psi\|_{L^1}\ (e^{\Lambda}+1))$$

and, we obtain for all $\widetilde{Z}\in\widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N)$, $F,G\in \mathrm{LIP}^{(C^2)}_{\Lambda}(\mathbb{R}^N,\mathbb{R}^N)$

$$\widetilde{Q}_\varepsilon(\widetilde{\vartheta}_F,\,\widetilde{\vartheta}_G;\,\widetilde{Z}) \;\le\; (1+6\,N\,\|\psi\|_{L^1})\ \|\mathcal{H}_F-\mathcal{H}_G\|_{C^1(\mathbb{R}^N\times\partial\mathbb{B}_1)}\,.$$

Finally condition (6.) of Definition 4.1 has to be verified, i.e.

$$\limsup_{h\downarrow 0}\ \widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{\vartheta}_F(t-h,\,\widetilde{Z}),\,\widetilde{K}) \quad\ge\quad \widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{\vartheta}_F(t,\widetilde{Z}),\,\widetilde{K})$$

for all $\widetilde{Z}\in\widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N)$, $\widetilde{K}\in\widetilde{\mathcal{K}}^{\to}(\mathbb{R}^N)$, $t\in[0,\mathbb{T}_\varepsilon(\widetilde{\vartheta}_F,\widetilde{Z})]$ with $t+\pi_1\,\widetilde{Z}\le\pi_1\,\widetilde{K}$.
Indeed, the well-known Theorem of Filippov guarantees $d\!\!l\Big(\vartheta_F(t-h,Z),\ \vartheta_F(t,Z)\Big) \longrightarrow 0$ for $h\downarrow 0$ and any set $Z\in\mathcal{K}(\mathbb{R}^N)$. So according to subsequent Proposition B.1 (1.),

$$\mathrm{Limsup}_{h\downarrow 0}\ \ \mathrm{Graph}\ {}^\flat N^P_{\vartheta_F(t-h,Z),\,\rho} \ \subset\ \mathrm{Graph}\ {}^\flat N^P_{\vartheta_F(t,Z),\,\rho}$$

and thus, we obtain for every $\widetilde{Z}=(a,0,Z)\in\widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N)$, $\widetilde{K}=(b,1,K)\in\widetilde{\mathcal{K}}^{\to}(\mathbb{R}^N)$, $\rho>0$, $\kappa\in\,]0,1]$ and $t\in[0,\mathbb{T}_\varepsilon(\widetilde{\vartheta}_F,\widetilde{Z})]$ with $a+t\le b$

$$\begin{array}{ll} \limsup_{h\downarrow 0} & \mathrm{dist}\Big(\mathrm{Graph}\ {}^\flat N^P_{K,\,(\rho+\kappa+200\,\Lambda\,|b-a|)},\ \ \mathrm{Graph}\ {}^\flat N^P_{\vartheta_F(t-h,Z),\,\rho}\Big) \\ \ge & \mathrm{dist}\Big(\mathrm{Graph}\ {}^\flat N^P_{K,\,(\rho+\kappa+200\,\Lambda\,|b-a|)},\ \ \mathrm{Graph}\ {}^\flat N^P_{\vartheta_F(t,Z),\,\rho}\Big)\,. \end{array}$$

Due to $\pi_1\,\widetilde{\vartheta}_F(t-h,\,\widetilde{Z}) \,=\, a \,=\, \pi_1\,\widetilde{\vartheta}_F(t,\widetilde{Z})$, this inequality implies the wanted condition (6.) of Definition 4.1 with respect to $\widetilde{q}_{\mathcal{K},\varepsilon}$. □

In §§ 4, 5, the results about existence of timed right–hand sleek solutions are based on appropriate forms of (transitional) compactness (see Definitions 4.12, 5.3). Considering a converging sequence of compact sets, some features of their proximal cones are summarized in Appendix B. In particular, $\mathrm{Graph}\ N^P_{K,\rho} \ \subset\ \mathrm{Limsup}_{n\to\infty}\ \mathrm{Graph}\ N^P_{K_n,\rho}$ does not hold for every radius $\rho>0$ in general. For this reason, we now prefer the second approach (of § 5) using "right-weakly transitionally compact" and Proposition 5.4.

Proposition 6.12 $(\widetilde{\mathcal{K}}^{\to}(\mathbb{R}^N),\ (\widetilde{q}_{\mathcal{K},\varepsilon})_{\varepsilon\in\,]0,1]\,\cap\,\mathbb{Q}},\ (\widetilde{q}_{\mathcal{K},\varepsilon,\kappa})_{\varepsilon,\kappa\in\,]0,1]\,\cap\,\mathbb{Q}},\ \mathrm{LIP}^{(C^2)}_{\Lambda}(\mathbb{R}^N,\mathbb{R}^N))$ *is right-weakly transitionally compact* (*in the sense of Definition 5.3*).

Proof. Applying Definition 5.3 to this tuple, the situation is the following: Let $(\widetilde{K}_n = (t_n,1,K_n))_{n\in\mathbb{N}}$, $(h_j)_{j\in\mathbb{N}}$ be sequences in $\widetilde{\mathcal{K}}^{\to}(\mathbb{R}^N)$ and $]0,1[$, respectively, with $h_j\downarrow 0$

and $\sup_n |t_n| < \infty$, $\sup_n \widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{K}_1, \widetilde{K}_n) < \infty$. Furthermore suppose each $G_n : [0,1] \longrightarrow \mathrm{LIP}^{(C^2)}_{\Lambda}(\mathbb{R}^N, \mathbb{R}^N)$ to be piecewise constant $(n \in \mathbb{N})$ and set

$$\begin{array}{l} \widetilde{G}_n \;:\; [0,1] \times \mathbb{R}^N \rightsquigarrow \mathbb{R}^N, \quad (t,x) \longmapsto G_n(t)(x), \\ \widetilde{K}_n(h) \;:=\; \{t_n + h\} \times \{1\} \times \vartheta_{\widetilde{G}_n}(h, K_n) \;\in\; \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N) \qquad \text{for } h \geq 0. \end{array}$$

We have to prove the existence of a sequence $n_k \nearrow \infty$ of indices and an element $\widetilde{K} = (t, 1, K) \in \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$ satisfying $t_{n_k} \longrightarrow t \;(k \longrightarrow \infty)$ and for every $\varepsilon, \kappa \in \,]0,1] \cap \mathbb{Q}$

$$\begin{array}{rl} \limsup\limits_{k \longrightarrow \infty} \widetilde{q}_{\mathcal{K},\varepsilon,\kappa}(\widetilde{K}_{n_k}(0), \;\; \widetilde{K}) &= 0, \\ \limsup\limits_{j \longrightarrow \infty} \; \sup\limits_{k \geq j} \; \widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{K}, \; \widetilde{K}_{n_k}(h_j)) &= 0. \end{array}$$

Closed bounded balls in $(\mathbb{R}, |\cdot|)$ and $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$ are known to be compact. So there are a subsequence (again denoted by) $(\widetilde{K}_n = (t_n, 1, K_n))_{n \in \mathbb{N}}$ and $\widetilde{K} = (t, 1, K) \in \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$ with $d\!\!l(K_n, K) \leq \frac{1}{n}$ and $t_n \longrightarrow t \;(n \longrightarrow \infty)$. Proposition B.1 (3.) ensures for all $\rho, \kappa > 0$

$$\mathrm{dist}\Big(\mathrm{Graph}\; {}^\flat N^P_{K,\,\rho+\kappa}, \;\mathrm{Graph}\; {}^\flat N^P_{K_n,\,\rho}\Big) \longrightarrow 0 \qquad (n \longrightarrow \infty)$$

and thus, $\widetilde{q}_{\mathcal{K},\varepsilon,\kappa}(\widetilde{K}_n, \; \widetilde{K}) \longrightarrow 0$ for every $\varepsilon, \kappa \in \,]0,1] \cap \mathbb{Q}$.

Now we prove for an appropriate subsequence that $\sup_{n > j} \widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{K}, \; \widetilde{K}_n(h_j)) \longrightarrow 0$ for $j \to \infty$, i.e. the convergence is uniform in κ. (Exploiting the timed triangle inequality directly, however, is prevented by lacking information about whether the time components of $\widetilde{K}$, $\widetilde{K}_n$, $\widetilde{K}_n(h_j)$ are ordered or not.)

Indeed,
$$\mathrm{dist}\Big(\mathrm{Graph}\; {}^\flat N^P_{K_n,\,\rho}, \;\mathrm{Graph}\; {}^\flat N^P_{K,\,\rho}\Big) \longrightarrow 0 \qquad (n \longrightarrow \infty)$$
results from Proposition B.1 (1.) for every $\rho > 0$ and so, Lebesgue's Dominated Convergence Theorem guarantees

$$\int_0^2 \mathrm{dist}\Big(\mathrm{Graph}\; {}^\flat N^P_{K_n,\,\rho}, \;\mathrm{Graph}\; {}^\flat N^P_{K,\,\rho}\Big)\, d\rho \longrightarrow 0 \qquad (n \longrightarrow \infty).$$

In particular, we can choose a subsequence (again denoted by) $(\widetilde{K}_n = (t_n, 1, K_n))_{n \in \mathbb{N}}$ with the additional properties $|t - t_n| < \frac{h_j}{2}$ for all $n > j$ and

$$\int_0^2 \mathrm{dist}\Big(\mathrm{Graph}\; {}^\flat N^P_{K_n,\,\rho}, \;\mathrm{Graph}\; {}^\flat N^P_{K,\,\rho}\Big)\, d\rho \;\leq\; \frac{1}{n \cdot \|\psi\|_{L^\infty}} \qquad \text{for all } n \in \mathbb{N}.$$

Similarly to the preceding Lemma 6.8, the Hamilton condition (of Proposition 6.5) provides the following upper bound of $\widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{K}, \; \widetilde{K}_n(h_j))$ for every $j \in \mathbb{N}$ and all $n > j$

$$\begin{array}{rl} & d\!\!l(K, \;\vartheta_{\widetilde{G}_n}(h_j, K_n)) \\ + & \sup\limits_{\kappa > 0} \displaystyle\int_\varepsilon^\infty \psi(\rho{+}\kappa{+}200\,\Lambda\,|t - t_n + h_j|) \,\cdot \\ & \qquad \mathrm{dist}\Big(\mathrm{Graph}\; {}^\flat N^P_{K_n(h_j),\,(\rho+\kappa+200\,\Lambda\,|t-t_n+h_j|)}, \;\; \mathrm{Graph}\; {}^\flat N^P_{K,\,\rho}\Big)\; d\rho \\ \leq & d\!\!l(K, \; K_n) + \Lambda\, h_j \\ + & \sup\limits_{\kappa > 0} \displaystyle\int_\varepsilon^\infty \psi(\rho{+}\kappa{+}200\,\Lambda\,|t - t_n + h_j|) \,\cdot \\ & \qquad \mathrm{dist}\Big(\mathrm{Graph}\; {}^\flat N^P_{K_n(h_j),\,(\rho+\kappa+100\,\Lambda\,h_j)}, \;\; \mathrm{Graph}\; {}^\flat N^P_{K_n,\,\rho}\Big)\; d\rho \;+\; \frac{1}{n} \\ \leq & \frac{1}{n} \quad + \Lambda\, h_j \quad + \sup\limits_{\kappa > 0} \; \Lambda\,(e^\Lambda + 1)\, \|\psi\|_{L^1} \cdot h_j \;+\; \frac{1}{n}\,, \end{array}$$

i.e. $\sup_{n > j} \widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{K}, \; \widetilde{K}_n(h_j)) \longrightarrow 0$ for $j \longrightarrow \infty$. □

Applying Proposition 5.4 to this tuple provides existence of timed right–hand sleek solutions:

Proposition 6.13
Regard the maps $\widetilde{\vartheta}_F$ *of all set–valued* $F \in \mathrm{LIP}^{(C^2)}_\Lambda(\mathbb{R}^N, \mathbb{R}^N)$ *(defined in Definitions 6.6, 6.7) as timed sleek transitions on* $(\widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N),\ \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N),\ (\widetilde{q}_{\mathcal{K},\varepsilon})_{\varepsilon \in]0,1] \cap \mathbb{Q}})$ *according to Proposition 6.11.*
For $\widetilde{f} : \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N) \times [0,T] \longrightarrow \mathrm{LIP}^{(C^2)}_\Lambda(\mathbb{R}^N, \mathbb{R}^N)$, *suppose*

$$\|\mathcal{H}_{\widetilde{f}(\widetilde{K},t)} - \mathcal{H}_{\widetilde{f}(\widetilde{K}_m,t_m)}\|_{C^1(\mathbb{R}^N \times \partial\mathbb{B}_1)} \stackrel{m\to\infty}{\longrightarrow} 0$$

whenever $0 \le t_m - t \longrightarrow 0$ *and* $\widetilde{q}_{\mathcal{K},0}(\widetilde{K}, \widetilde{K}_m) \longrightarrow 0 \quad (\widetilde{K}, \widetilde{K}_m \in \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N),\ \pi_1\,\widetilde{K} \le \pi_1\,\widetilde{K}_m)$.

Then for every initial element $\widetilde{K}_0 \in \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$, *there exists a timed right–hand sleek solution* $\widetilde{K} : [0,T[\longrightarrow \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$ *of the generalized mutational equation* $\overset{\circ}{\widetilde{K}}(\cdot) \ni \widetilde{f}(\widetilde{K}(\cdot), \cdot)$ *with* $\widetilde{K}(0) = \widetilde{K}_0$. *In particular,* $\limsup_{h \downarrow 0}\ \frac{1}{h} \cdot d\!\!l\Big(\vartheta_{\widetilde{f}(\widetilde{K}(t),\,t)}(h,\ K(t)),\ K(t+h)\Big) = 0$ *for all* t.

Proof results directly from Proposition 5.4. Indeed, $\widetilde{q}_{\mathcal{K},0}$ and $\widetilde{q}_{\mathcal{K},\varepsilon}$ $(\varepsilon > 0)$ satisfy
$d\!\!l(K_1, K_2) \le \widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{K}_1, \widetilde{K}_2) \le \widetilde{q}_{\mathcal{K},0}(\widetilde{K}_1, \widetilde{K}_2) \le \widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{K}_1, \widetilde{K}_2) + \|\psi\|_{L^\infty}\,(\|K_1\|_\infty + \|K_2\|_\infty + 2)\,\varepsilon$
for all $\widetilde{K}_j = (t_j, \mu_j, K_j) \in \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N) \cup \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N)$ (abbreviating $\|K_1\|_\infty \stackrel{\text{Def.}}{=} \sup_{x \in K_1} |x|$).

For any sequence $\Big(\widetilde{K}_m = (t_m, 1, K_m)\Big)_{m \in \mathbb{N}}$ in $\widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$ and $\widetilde{K} = (t, 1, K) \in \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$ suppose $t_m \downarrow t$ and $\widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{K}, \widetilde{K}_m) \longrightarrow 0$ $(m \to \infty)$ for all $\varepsilon \in]0,1] \cap \mathbb{Q}$. Then,

$$\widetilde{q}_{\mathcal{K},0}(\widetilde{K}, \widetilde{K}_m) = \limsup_{\varepsilon \downarrow 0}\ \widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{K}, \widetilde{K}_m) \stackrel{m\to\infty}{\longrightarrow} 0$$

and finally $\|\mathcal{H}_{\widetilde{f}(\widetilde{K},t)} - \mathcal{H}_{\widetilde{f}(\widetilde{K}_m,t_m)}\|_{C^1(\mathbb{R}^N \times \partial\mathbb{B}_1)} \stackrel{m\to\infty}{\longrightarrow} 0$ – as required for Proposition 5.4. □

In comparison with previous results in [20, 23] (mentioned in Remark 6.1), an essential advantage of *sleek* solutions is that Proposition 5.6 specifies sufficient conditions (on the right-hand side $\widetilde{f}$) for uniqueness:

Proposition 6.14 *For* $\widetilde{f} : (\widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N) \cup \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N)) \times [0,T] \longrightarrow \mathrm{LIP}^{(C^2)}_\Lambda(\mathbb{R}^N, \mathbb{R}^N)$, *suppose that there exist a modulus* $\widehat{\omega}(\cdot)$ *of continuity and a constant* $L \ge 0$ *satisfying*

$$\|\mathcal{H}_{\widetilde{f}(\widetilde{Z},s)} - \mathcal{H}_{\widetilde{f}(\widetilde{K},t)}\|_{C^1(\mathbb{R}^N \times \partial\mathbb{B}_1)} \le L \cdot \widetilde{q}_{\mathcal{K},0}(\widetilde{Z}, \widetilde{K}) + \widehat{\omega}(t-s)$$

for all $0 \le s \le t \le T$ *and* $\widetilde{Z} \in \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N)$, $\widetilde{K} \in \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$ $(\pi_1\,\widetilde{Z} \le \pi_1\,\widetilde{K})$.

Then for every initial element $\widetilde{K}_0 \in \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$, *the timed right–hand sleek solution* $\widetilde{K} : [0,T[\longrightarrow \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$ *of the generalized mutational equation* $\overset{\circ}{\widetilde{K}}(\cdot) \ni \widetilde{f}(\widetilde{K}(\cdot), \cdot)$ *with* $\widetilde{K}(0) = \widetilde{K}_0$ *is unique.*

Proof results from Proposition 5.6. For any element $\widetilde{K}_0 = (t_0, 1, K_0) \in \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$ fixed, let $\widetilde{K}_1(\cdot) = (t_0 + \cdot, 1, K_1(\cdot))$ and $\widetilde{K}_2(\cdot) = (t_0 + \cdot, 1, K_2(\cdot))$ denote two timed right–hand sleek solutions $[0,T[\longrightarrow \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$ of the generalized mutational equation $\overset{\circ}{\widetilde{K}}_j(\cdot) \ni \widetilde{f}(\widetilde{K}_j(\cdot), \cdot)$ with $\widetilde{K}_1(0) = \widetilde{K}_0 = \widetilde{K}_2(0)$.
Then the continuity of $\widetilde{K}_1(\cdot), \widetilde{K}_2(\cdot)$ with respect to each $\widetilde{q}_{\mathcal{K},\varepsilon}$ (in forward time direction) implies the continuity of $K_1(\cdot), K_2(\cdot) : [0,T[\longrightarrow \mathcal{K}(\mathbb{R}^N)$ w.r.t. Pompeiu-Hausdorff distance $d\!\!l$.
In particular, $R > 1$ can be chosen sufficiently large with

$$K_1(t) \cup K_2(t) \subset \mathbb{B}_{R-1}(0) \subset \mathbb{R}^N \qquad \text{for all } t \in [0,T[.$$

Set $\widehat{R} := 4\,(R+1)\,(\|\psi\|_{L^1}+1) > R$ as an additional abbreviation.
So without loss of generality, we can restrict our considerations to compact subsets M_1, M_2 of the closed ball $\mathbb{B}_{\widehat{R}}(0) \subset \mathbb{R}^N$. In particular, for all $\varepsilon \in \mathcal{J} \stackrel{\text{Def.}}{=}]0,1] \cap \mathbb{Q}$, we obtain

$$\widetilde{q}_{\mathcal{K},0}\,((t_1,0,M_1),\ (t_2,1,M_2)) \ \leq\ \widetilde{q}_{\mathcal{K},\varepsilon}\,((t_1,0,M_1),\ (t_2,1,M_2)) \ +\ \|\psi\|_{L^\infty}\,2\,(\widehat{R}+1)\,\varepsilon$$

implying

$$\|\mathcal{H}_{\widetilde{f}(\widetilde{Z},s)} - \mathcal{H}_{\widetilde{f}(\widetilde{K},t)}\|_{C^1(\mathbb{R}^N\times\partial\mathbb{B}_1)} \ \leq\ L\,\|\psi\|_{L^\infty}\,2\,(\widehat{R}+1)\cdot\varepsilon \ +\ L\cdot\widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{Z},\widetilde{K}) \ +\ \widehat{\omega}(t-s)$$

for all $s \leq t \leq T$, $\widetilde{Z} \in \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N)$, $\widetilde{K} \in \widetilde{\mathcal{K}}^{\rightarrow}(\mathbb{R}^N)$ with $\pi_1\,\widetilde{Z} \leq \pi_1\,\widetilde{K}$, $Z, K \subset \mathbb{B}_{\widehat{R}}(0) \subset \mathbb{R}^N$.
Seizing now the notion of Proposition 5.6, the auxiliary function $\varphi_\varepsilon : [0,T[\longrightarrow [0,\infty[$

$$\varphi_\varepsilon(t) \ :=\ \inf_{\widetilde{Z}\,\in\,\widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N),\ \pi_1\,\widetilde{Z}\,<\,t_0+t} \Big(\widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{Z},\ \widetilde{K}_1(t)) \ +\ \widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{Z},\ \widetilde{K}_2(t))\Big)$$

has obviously the upper bound $d\!\!l(K_1(t), K_2(t)) + \|\psi\|_{L^1}\,(2\,R+2) \ <\ \frac{1}{2}\,\widehat{R}$ (as the "test set" $\widetilde{Z} := (t_0+t-\delta,\ 0,\ K_1(t))$ with arbitrarily small $\delta > 0$ shows). Thus, $\varphi_\varepsilon(t)$ can be described as infimum of "test sets" $\widetilde{Z} = (s,0,Z) \in \widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N)$ satisfying $Z \subset \mathbb{B}_{\widehat{R}}(0) \subset \mathbb{R}^N$ additionally:

$$\varphi_\varepsilon(t) \ =\ \inf_{\substack{\widetilde{Z}\,\in\,\widetilde{\mathcal{K}}^{\curlyvee}(\mathbb{R}^N),\\ \pi_1\,\widetilde{Z}\,<\,t_0+t,\ \|Z\|_\infty\leq\widehat{R}}} \Big(\widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{Z},\ \widetilde{K}_1(t)) \ +\ \widetilde{q}_{\mathcal{K},\varepsilon}(\widetilde{Z},\ \widetilde{K}_2(t))\Big).$$

Furthermore, the time parameter $\mathbb{T}_\varepsilon(\cdot,\ \cdot)$ (specified in Proposition 6.11) depends only on ε and Λ. So due to $\widetilde{K}_1(0) = \widetilde{K}_2(0)$, Proposition 5.6 ensures for each $t \in [0,T[$, $\varepsilon \in]0,1] \cap \mathbb{Q}$

$$\varphi_\varepsilon(t) \ \leq\ 2\ \cdot\ L\,\|\psi\|_{L^\infty}\,2\,(R+1)\ \varepsilon\ \cdot\ t\ \ e^{(L+10\,\Lambda\,e^{2\,\Lambda})\,\cdot\,t} \ \stackrel{\varepsilon\downarrow 0}{\longrightarrow}\ 0.$$

Finally, the triangle inequality of the Pompeiu-Hausdorff distance $d\!\!l$ implies

$$d\!\!l(K_1(t),\ K_2(t)) \ \leq\ \inf_{\varepsilon>0}\ \varphi_\varepsilon(t) \ =\ 0 \qquad \square$$

A Tools of reachable sets of differential inclusions

In this appendix, we investigate the proximal radius of boundary points while sets are evolving along differential inclusions. Compact balls and their complements exemplify the key features for short times (as stated in Proposition A.1). So they lead to the main results about proximal radii in both forward and backward time direction as a corollary.

Proposition A.1 *Let F be any set–valued map of $\mathrm{LIP}^{(C^2)}_\Lambda(\mathbb{R}^N,\mathbb{R}^N)$ (according to Definition 6.6) and $B := \mathbb{B}_r(x_0) \subset \mathbb{R}^N$ a compact ball of positive radius r.*
Then there exists a time $\tau = \tau(r,\Lambda) > 0$ such that for all times $t \in [0, \tau(r,\Lambda)[$,

1. *$\vartheta_F(t,B)$ is convex and has radius of curvature $\geq\ r - 9\,\Lambda\,(1+r)^2\,t$,*
2. *$\vartheta_F(t,\ \mathbb{R}^N \setminus B)$ is concave and has radius of curvature $\geq\ r - 9\,\Lambda\,(1+r)^2\,t$,*

Restricting ourselves to $0 < r \leq 2$, the time $\tau(r,\Lambda) > 0$ can be chosen as an increasing function of r. The claim of Proposition A.1 does not include, however, that $r - 9\,\Lambda\,(1+r)^2\,t\ \geq\ 0$ for all $t \in [0,\tau(r,\Lambda)[$ (because then it is not immediately clear how to choose $\tau(r,\Lambda) > 0$ as increasing with respect to all $r \in]0,2]$).
As an equivalent formulation of statement (1.), the convex set $\vartheta_F(t,B)$ has *positive erosion* of radius $\rho(t) \geq r - 9\,\Lambda\,(1+r)^2\,t$, i.e. there is some $K_t \subset \mathbb{R}^N$ with $\vartheta_F(t,B) = \mathbb{B}_{\rho(t)}(K_t)$ (as defined e.g. in [21, 23]). The question of preserving positive erosion or interior balls has already been investigated in [21] and in [7] under different assumptions.

So strictly speaking, statement (2.) is of more interest here. It ensures that $\vartheta_F(t, \mathbb{R}^N \setminus B)$ has *positive reach* of radius $\rho(t) \geq r - 9\,\Lambda\,(1+r)^2\,t$ (in the sense of Federer [16]), i.e. for each point $y \in \partial\,\vartheta_F(t, \mathbb{R}^N \setminus B)$, there exists an exterior ball $\mathbb{B}_{\rho(t)}(y_0) \subset \mathbb{R}^N$ with $y \in \partial\,\mathbb{B}_{\rho(t)}(y_0)$ and $\vartheta_F(t, \mathbb{R}^N \setminus B) \cap \overset{\circ}{\mathbb{B}}_{\rho(t)}(y_0) = \emptyset$. Roughly speaking, the proofs of these two statements just differ in a sign and thus, both of them are mentioned here.

Applying Proposition A.1 to adequate proximal balls, the inclusion principle of reachable sets and Proposition 6.5 have the immediate consequence:

Corollary A.2 *For every set–valued map $F \in \mathrm{LIP}^{(C^2)}_\Lambda(\mathbb{R}^N, \mathbb{R}^N)$ and radius $r_0 \in]0, 2]$, there exists some $\tau = \tau(r_0, \Lambda) > 0$ such that for any $K \in \mathcal{K}(\mathbb{R}^N)$, $r \in [r_0, 2]$ and $t \in [0, \tau[$,*

1. *each $x_1 \in \partial\vartheta_F(t, K)$ and $\nu_1 \in N^P_{\vartheta_F(t,K)}(x_1)$ with proximal radius r are linked to some $x_0 \in \partial K$ and $\nu_0 \in N^P_K(x_0)$ with proximal radius $\geq r - 81\,\Lambda\,t$ by a solution to $\dot{x}(\cdot) \in F(x(\cdot))$ and its adjoint arc, respectively.*
2. *each $x_0 \in \partial K$ and $\nu_0 \in N^P_K(x_0)$ with proximal radius r are linked to some $x_1 \in \partial\vartheta_F(t, K)$ and $\nu_1 \in N^P_{\vartheta_F(t,K)}(x_1)$ with proximal radius $\geq r - 81\,\Lambda\,t$ by a solution to $\dot{x}(\cdot) \in F(x(\cdot))$ and its adjoint arc, respectively.*

For describing the time–dependent limiting normals, we use adjoint arcs and benefit from the Hamiltonian system they are satisfying together with the solutions (as quoted in preceding Proposition 6.5). In short, the graph of normal cones at time t, Graph $N_{\vartheta_F(t,K)}(\cdot)|_{\partial\,\vartheta_F(t,K)}$, can be traced back to the beginning by means of the Hamiltonian system with $\mathcal{H}_F$. Roughly speaking, we now take the next order into consideration and, the matrix Riccati equation provides an analytical access to geometric properties like curvature. The next lemma motivates the assumption $\mathcal{H}_F \in C^2(\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}))$ for all maps $F \in \mathrm{LIP}^{(C^2)}_\Lambda(\mathbb{R}^N, \mathbb{R}^N)$.

Lemma A.3
Suppose for $H : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}$, $\psi : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ and the Hamiltonian system

$$\left\{ \begin{array}{rclcrcl} \dot{y}(t) &=& \frac{\partial}{\partial q}\,H(t,\,y(t),\,q(t)), & & y(0) &=& y_0 \\ \dot{q}(t) &=& -\frac{\partial}{\partial y}\,H(t,\,y(t),\,q(t)), & & q(0) &=& \psi(y_0) \end{array} \right. \qquad (*)$$

1. *$H(t,\cdot,\cdot)$ is* twice *continuously differentiable for every $t \in [0,T]$.*
2. *for every $R > 0$, there exists $k_R \in L^1([0,T])$ such that the derivative of $H(t,\cdot,\cdot)$ is $k_R(t)$–Lipschitz continuous on $\mathbb{B}_R \times \mathbb{B}_R$ for almost every t,*
3. *ψ is locally Lipschitz continuous,*
4. *every solution $(y(\cdot), q(\cdot))$ of the Hamiltonian system $(*)$ can be extended to $[0,T]$ and depends continuously on the initial data in the following sense :*
   Let each $(y_n(\cdot), q_n(\cdot))$ be a solution satisfying $y_n(t_n) \longrightarrow z_0$, $q_n(t_n) \longrightarrow q_0$ for some $t_n \longrightarrow t_0$, $z_0, q_0 \in \mathbb{R}^N$. Then $(y_n(\cdot), q_n(\cdot))_{n\in\mathbb{N}}$ converges uniformly to a solution $(y(\cdot), q(\cdot))$ of the Hamiltonian system with $y(t_0) = z_0$, $q(t_0) = q_0$.

Then for every initial set $K \in \mathcal{K}(\mathbb{R}^N)$, the following statements are equivalent :

(i) *For all* $t \in [0,T]$,
$M_t^{\mapsto}(K) := \left\{ (y(t),\ q(t)) \,\middle|\, (y(\cdot),\ q(\cdot)) \text{ solves system } (*),\ \ y_0 \in K \right\}$
is the graph of a locally Lipschitz continuous function,

(ii) *For any solution* $(y(\cdot), q(\cdot)) : [0,T] \longrightarrow \mathbb{R}^N \times \mathbb{R}^N$ *of the initial value problem* $(*)$ *and each cluster point* $Q_0 \in \operatorname{Limsup}_{z \to y_0} \{\nabla \psi(z)\}$, *the following matrix Riccati equation has a solution* $Q(\cdot)$ *on* $[0,T]$

$$\begin{cases} \partial_t Q + \frac{\partial^2 H}{\partial p\, \partial x}(t,\ y(t),\ q(t))\ Q + Q\ \frac{\partial^2 H}{\partial x\, \partial p}(t,\ y(t),\ q(t)) \\ \quad + Q\ \frac{\partial^2 H}{\partial p^2}(t,\ y(t),\ q(t))\ Q + \frac{\partial^2 H}{\partial x^2}(t,\ y(t),\ q(t)) = 0, \\ \quad Q(0) = Q_0. \end{cases}$$

If one of these equivalent properties is satisfied and if ψ *is (continuously) differentiable, then* $M_t^{\mapsto}(K)$ *is even the graph of a (continuously) differentiable function.*

Proof is presented in [17, Theorem 5.3], for the same Hamiltonian system but with $y(T) = y_T$, $q(T) = q_T$ given (see also [10]). So this lemma is an immediate consequence considering $-H(T - \cdot, \cdot, \cdot)$ and $(y(T - \cdot),\ q(T - \cdot))$. □

Remark A.4 In addition to the final statement of Lemma A.3, well–known properties of variational equations (see e.g. [17]) imply that $Q(t)$ is the derivative of the C^1 function with graph $M_t^{\mapsto}(K)$ at the point $y(t)$.

For preventing singularities of $Q(\cdot)$, the following comparison principle provides a bridge to solutions of a *scalar* Riccati equation.

Lemma A.5 (Comparison theorem for matrix Riccati equation, [30, Theorem 2]**)**
Let $A_j, B_j, C_j : [0,T[\, \longrightarrow \mathbb{R}^{N,N}$ $(j = 0,1,2)$ *be bounded continuous matrix–valued functions such that each* $M_j(t) := \begin{pmatrix} A_j(t) & B_j(t) \\ B_j(t)^T & C_j(t) \end{pmatrix}$ *is symmetric.*
Assume that $U_0,\ U_2 : [0,T[\, \longrightarrow \mathbb{R}^{N,N}$ *are solutions of the matrix Riccati equation*

$$\tfrac{d}{dt} U_j = A_j + B_j\, U_j + U_j\, B_j^T + U_j\, C_j\, U_j$$

with $M_2(\cdot) \geq M_0(\cdot)$ *(i.e.* $M_2(t) - M_0(t)$ *is positive semi–definite for every* t*).*
Then, given symmetric $U_1(0) \in \mathbb{R}^{N,N}$ *with*

$$U_2(0) \geq U_1(0) \geq U_0(0), \qquad M_2(\cdot) \geq M_1(\cdot) \geq M_0(\cdot),$$

there exists a solution $U_1 : [0,T[\, \longrightarrow \mathbb{R}^{N,N}$ *of the corresponding Riccati equation with matrix* $M_1(\cdot)$*. Moreover,* $U_2(t) \geq U_1(t) \geq U_0(t)$ *for all* $t \in [0,T[$. □

Proof of Proposition A.1 (1) is based on applying Lemma A.3 to the boundary $K := \partial\, \mathbb{B}_r(0)$ and its exterior unit normals, i.e. $\psi(x) := \frac{x}{r}$, after assuming $B = \mathbb{B}_r(0)$ without loss of generality. Obviously, ψ can be extended to $\psi \in C^1(\mathbb{R}^N, \mathbb{R}^N)$.
(Statement (2.) of Proposition A.1 is shown in the same way – just with inverse signs, i.e. $\widehat{\psi}(x) := -\frac{x}{r}$ instead. So we do not formulate this part in detail.)

For every point $y_0 \in \partial\, \mathbb{B}_r$, there exist a solution $y(\cdot) \in C^1([0,\infty[, \mathbb{R}^N)$ and its adjoint $q(\cdot) \in C^1([0,\infty[, \mathbb{R}^N)$ satisfying

$$\begin{cases} \dot{y}(t) = \frac{\partial}{\partial q} \mathcal{H}_F(y(t), q(t)) \in F(y(t)), & y(0) = y_0, \\ \dot{q}(t) = -\frac{\partial}{\partial y} \mathcal{H}_F(y(t), q(t)), & q(0) = \psi(y_0) \end{cases} \qquad (*)$$

and, $F \in \mathrm{LIP}^{(C^2)}_\Lambda(\mathbb{R}^N, \mathbb{R}^N)$ implies the a priori bounds $|y(t) - y_0| \le \Lambda\, t$, $e^{-\Lambda t} \le |q(t)| \le e^{\Lambda t}$. So after restricting to the finite time interval $I_r = [0, t_r[$ (specified explicitly later), a simple cut-off function provides a twice continuously differentiable extension $\mathcal{H} : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}$ of $\mathcal{H}_F|_{\mathbb{R}^N \times (\mathbb{R}^N \setminus \mathbb{B}^\circ_{\exp(-\Lambda t_r)}(0))}$ and finally, Lemma A.3 can be applied to $\partial \mathbb{B}_r$, ψ and $\mathcal{H}_F$.
Furthermore $\mathcal{H}_F(x,p) \overset{\text{Def.}}{=} \sup_{v \in F(x)} p \cdot v$ is positively homogeneous with respect to p and thus, the second derivatives of $\mathcal{H}_F$ are bounded by $9\,\Lambda\, R^2$ on $\mathbb{R}^N \times (\mathbb{B}_R \setminus \overset{\circ}{\mathbb{B}}_{\frac{1}{R}})$ (according to [23], Lemma 4.4.24). Together with the preceding a priori bounds, we obtain

$$\| D^2 \mathcal{H}_F(y(t), q(t)) \|_{\mathrm{Lin}(\mathbb{R}^{2N}, \mathbb{R}^{2N})} \le 9\,\Lambda\, e^{2\Lambda t}.$$

Let $Q(\cdot)$ denote the solution of the matrix Riccati equation

$$\begin{cases} \partial_t Q + \frac{\partial^2 \mathcal{H}_F}{\partial p\, \partial x}(y(t), q(t))\; Q + Q\; \frac{\partial^2 \mathcal{H}_F}{\partial x\, \partial p}(y(t), q(t)) \\ \qquad + Q\; \frac{\partial^2 \mathcal{H}_F}{\partial p^2}(y(t), q(t))\; Q + \frac{\partial^2 \mathcal{H}_F}{\partial x^2}(y(t), q(t)) = 0, \\ \qquad Q(0) = \nabla \psi(y_0) = \frac{1}{r} \cdot \mathrm{Id}_{\mathbb{R}^N}. \end{cases}$$

Due to the comparison principle of Lemma A.5, $Q(\cdot)$ exists (at least) as long as the two scalar Riccati equations $\quad \partial_t u_\pm = \pm 9\,\Lambda\, e^{2\Lambda t} \pm 9\,\Lambda\, e^{2\Lambda t}\, u_\pm^2, \quad u_\pm(0) = \frac{1}{r}$
have finite solutions and within this period, they fulfill $u_-(t) \cdot \mathrm{Id}^N_{\mathbb{R}} \le Q(t) \le u_+(t) \cdot \mathrm{Id}^N_{\mathbb{R}}$.
In fact, we get the explicit solutions on $I_r := [0, \frac{1}{2\Lambda} \cdot \log(1 + \frac{\pi}{9} - \frac{2}{9} \cdot \arctan \frac{1}{r})[$, namely

$$u_\pm(t) = \tan\left(\pm \tfrac{9}{2}(e^{2\Lambda t} - 1) + \arctan \tfrac{1}{r}\right),$$

So $Q(t)$ is positive definite with eigenvalues $\ge u_-(t)$ for every time t of the (maybe smaller) interval $I'_r := I_r \cap [0, \frac{1}{2\Lambda} \cdot \log(1 + \frac{2}{9} \cdot \arctan \frac{1}{r})[$.

Now we focus on the geometric interpretation of $Q(\cdot)$.
Due to Lemma A.3, $M_t^{\mapsto}(\partial \mathbb{B}_r) := \{ (y(t), q(t)) \mid (y(\cdot), q(\cdot)) \text{ solves system } (*),\ |y_0| = r \}$ is graph of a continuously differentiable function and, $Q(t)$ is its derivative at $y(t)$ (due to Remark A.4). Furthermore the Hamilton condition of Proposition 6.5 ensures

$$\text{Graph } N_{\vartheta_F(t,\mathbb{B}_r)}(\cdot) \subset \left\{ (y(t), \lambda\, q(t)) \,\middle|\, (y(\cdot), q(\cdot)) \text{ solves system } (*),\ |y_0| = r,\ \lambda \ge 0 \right\}$$

and thus, the graph property of $M_t^{\mapsto}(\partial \mathbb{B}_r)$ implies that each $q(t)$ is normal vector to the smooth reachable set $\vartheta_F(t, \mathbb{B}_r)$ at $y(t)$.
As $q(t) \ne 0$ need not have norm 1, the eigenvalues of $Q(t)$ are not always identical to the principal curvatures $(\kappa_j)_{j=1\ldots N}$ of $\vartheta_F(t, \mathbb{B}_r)$ at $y(t)$, but they provide bounds:

$$e^{-\Lambda t} \cdot u_-(t) \le \kappa_j \le e^{\Lambda t} \cdot u_+(t) \qquad \text{(due to } e^{-\Lambda t} \le |q(t)| \le e^{\Lambda t}).$$

Thus, $\vartheta_F(t, \mathbb{B}_r)$ is convex for all times $t \in I'_r$ and, so the *local* properties of principal curvatures have the *nonlocal* consequence that $\vartheta_F(t, \mathbb{B}_r)$ has positive erosion of radius

$$\rho(t) \ge \frac{1}{e^{\Lambda t} \cdot u_+(t)} \ge r - 9\,\Lambda\,(1+r)^2\, t \qquad \text{for all } t \in I'_r$$

Indeed, the linear estimate at the end is shown by means of the auxiliary function $t \mapsto \frac{1}{e^{\Lambda t} \cdot u_+(t)} - r + 9\,\Lambda\,(1+r)^2\, t$ that is 0 at $t = 0$, has positive derivative at $t = 0$ and is convex (due to nonnegative second derivative in I'_r).
Finally, the time $\tau(r, \Lambda) > 0$ is chosen as minimum of $\frac{1}{2\Lambda} \cdot \log(1 + \frac{\pi}{9} - \frac{2}{9} \cdot \arctan \frac{1}{r})$, $\frac{1}{2\Lambda} \cdot \log(1 + \frac{2}{9} \cdot \arctan \frac{1}{r})$. The linear estimate need not be positive in $[0, \tau(r, \Lambda)[$ though. □

B Tools of proximal normals

Comparing the proximal normals of a converging sequence $(K_n)_{n \in \mathbb{N}}$ in $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$ with the normals of its limit $K \in \mathcal{K}(\mathbb{R}^N)$, the following inclusion is well known

$$\text{Graph } N_K^P \;\subset\; \text{Limsup}_{n\to\infty} \text{ Graph } N_{K_n}^P$$

(see e.g. [4, Theorem 8.4.6], or [11, Lemma 4.1]). Of course, the equality here is not fulfilled in general. A key advantage of the subset $N_{K,\rho}^P$ (for $\rho > 0$) now is that an inverse inclusion is satisfied. This feature is very useful for the preceding Propositions 6.11 and 6.12.

Proposition B.1 *Let $(K_n)_{n \in \mathbb{N}}$ be a converging sequence in $\mathcal{K}(\mathbb{R}^N)$ and K its limit. $\Pi_{K_n}, \Pi_K : \mathbb{R}^N \leadsto \mathbb{R}^N$ denote the projections on K_n, K $(n \in \mathbb{N})$ respectively. Then,*

1. $\text{Limsup}_{n\to\infty} \text{ Graph } {}^\flat N_{K_n,\rho}^P \;\subset\; \text{Graph } {}^\flat N_{K,\rho}^P$ *for any* $\rho > 0$,
2. $\text{Limsup}_{\substack{y\to x\\ n\to\infty}} \Pi_{K_n}(y) \;\subset\; \Pi_K(x)$ *for any* $x \in \mathbb{R}^N$,
3. $\text{Graph } {}^\flat N_{K,\rho}^P \;\subset\; \text{Liminf}_{n\to\infty} \text{ Graph } {}^\flat N_{K_n,r}^P$ *for any* $0 < r < \rho$.

Proof. (1.) Choose any converging sequence $((x_{n_j}, p_{n_j}))_{j\in\mathbb{N}}$ with $p_{n_j} \in N_{K_{n_j},\rho}^P(x_{n_j}) \cap \partial\mathbb{B}$ and set $x := \lim_{j\to\infty} x_{n_j} \in K$, $p := \lim_{j\to\infty} p_{n_j} \in \partial\mathbb{B}$. According to Definition 6.2, each K_{n_j} is contained in the complement of the open ball with center $x_{n_j} + \rho\, p_{n_j}$ and radius ρ,

$$K_{n_j} \;\subset\; \mathbb{R}^N \setminus \overset{\circ}{\mathbb{B}}_\rho\big(x_{n_j} + \rho\, p_{n_j}\big)\,.$$

As an indirect consequence, $j \longrightarrow \infty$ leads to $K \subset \mathbb{R}^N \setminus \overset{\circ}{\mathbb{B}}_\rho(x + \rho\, p)$, i.e. $p \in N_{K,\rho}^P(x)$.

(2.) Let $r > 0$ and $n \in \mathbb{N}$ be arbitrary. For $y \in \mathbb{B}_r(x)$ given, choose any $z \in \Pi_{K_n}(y)$ and $\xi \in \Pi_K(z)$. Then, $|\xi - z| \;\le\; d\!\!l(K_n, K)$ and

$$\begin{aligned} |x-\xi| &\le |x-y| + |y-z| + |z-\xi| \\ &\le |x-y| + \text{dist}(y,K) + d\!\!l(K,K_n) + |z-\xi| \\ &\le |x-y| + |y-x| + \text{dist}(x,K) + d\!\!l(K,K_n) + d\!\!l(K_n,K) \\ &\le 2\,r + \text{dist}(x,K) + 2\,d\!\!l(K_n,K). \end{aligned}$$

Thus, $\Pi_{K_n}(y) \;\subset\; \mathbb{B}_{d\!\!l(K_n,K)}\Big(K \cap \mathbb{B}_{2\,r + \text{dist}(x,K) + 2\,d\!\!l(K_n,K)}(x)\Big)$ for any $y \in \mathbb{B}_r(x)$.

The set–valued map $[0,\infty[\; \leadsto \mathbb{R}^N$, $r \longmapsto K \cap \mathbb{B}_r(x)$ is upper semicontinuous (due to [5, Corollary 1.4.10]) and in the closed interval $[\text{dist}(x,K), \infty[$, it has nonempty compact values. So for every $\eta > 0$, there exists $\rho = \rho(x,\eta) \in \,]0,\eta[$ such that

$$K \cap \mathbb{B}_r(x) \;\subset\; \mathbb{B}_\eta\Big(\Pi_K(x)\Big) \qquad \text{for all } r \in \Big[\text{dist}(x,K),\; \text{dist}(x,K) + \rho\Big].$$

Due to $d\!\!l(K_n,K) \longrightarrow 0$ $(n \longrightarrow \infty)$, there is an index $m \in \mathbb{N}$ with $d\!\!l(K_n,K) \le \frac{\rho}{4}$ for all $n \ge m$. Thus we obtain for every $y \in \mathbb{B}_{\rho/4}(x)$ and $n \ge m$

$$\begin{aligned} \Pi_{K_n}(y) &\subset \mathbb{B}_{\frac{\rho}{4}}\Big(K \cap \mathbb{B}_{2\,\frac{\rho}{4} + \text{dist}(x,K) + 2\,\frac{\rho}{4}}(x)\Big) &&= \mathbb{B}_{\frac{\rho}{4}}\Big(K \cap \mathbb{B}_{\text{dist}(x,K)+\rho}(x)\Big) \\ &\subset \mathbb{B}_{\frac{\rho}{4}}\Big(\mathbb{B}_\eta(\Pi_K(x))\Big) &&\subset \mathbb{B}_{2\,\eta}\Big(\Pi_K(x)\Big), \end{aligned}$$

i.e. $\text{Limsup}_{\substack{y\to x\\ n\to\infty}} \Pi_{K_n}(y) \;\subset\; \Pi_K(x)$.

(3.) Choose any $x \in \partial K$ and $p \in N^P_{K,\rho}(x) \neq \emptyset$ with $|p| = 1$. Then x is the unique projection of $x + \delta\ p$ on K for every $\delta \in]0, \rho[$. Considering now a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in \Pi_{K_n}(x + \delta\ p) \subset K_n$, the preceding statement (2.) implies $x_n \longrightarrow x$ and, the definition of proximal normal guarantees $p_n := \frac{x+\delta\ p - x_n}{|x+\delta\ p - x_n|} \in {}^\flat N^P_{K_n}(x_n)$ converging to p. Finally the proximal radius of p_n is $\geq |x + \delta\ p - x_n| \geq \delta - |x - x_n|$, and thus, $(x, p) \in \mathrm{Liminf}_{n\to\infty}$ Graph ${}^\flat N^P_{K_n,\, r}$ for every positive $r < \delta < \rho$. □

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