Evolution equations in ostensible metric spaces : Definitions and existence.

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Abstract

The primary aim is to unify the definition of "solution" for completely different types of evolutions. Such a common approach is to lay the foundations for solving systems like, for example, a semilinear evolution equation (of parabolic type) in combination with a first–order geometric evolution. In regard to geometric evolutions, this concept is to fulfill 3 conditions : First, consider nonempty compact subsets $K(t) \subset \mathbb{R}^N$ without a priori restrictions on the regularity of the boundary. Second, the evolution of $t \mapsto K(t)$ might depend on nonlocal properties of the set $K(t)$ and its normal cones. Last, but not least, no inclusion principle.
The approach here is based on generalizing the mutational equations of Aubin for metric spaces in two respects : Replacing the metric by a countable family of (possibly nonsymmetric) distances (called *ostensible metrics*) and extending the basic idea of distributions.

1. Introduction : Diverse evolutions come together under the same roof

Many applications consist of diverse components so that their mathematical description as functions often starts with long preliminaries (like general assumptions about regularity). However, *shapes and images are basically sets, not even smooth* (Aubin [2]).
So the question is posed how to specify models in which both (real– or vector–valued) functions and shapes are involved. Usually the components depend on time and have a huge amount of influence over each other.
Consider, for example, a bacterial colony growing in a nonhomogeneous nutrient broth, region growing methods in image segmentation and Lyapunov methods in shape optimization ([18, Demongeot, Kulesa, Murray 97], [26, Lorenz 2001] and [19, Demongeot, Leitner 96], [20, Doyen 95]).

The primary aim here is to unify the definition of "solution" for completely different types of evolutions. In particular, the motivation behind the generalizing process is given by the following model problem : For each point of time $t \in [0, T[$, we consider a pair $(u(t), K(t))$ whose first component $u(t)$ is an element of a reflexive Banach space X whereas the second component $K(t)$ is a nonempty compact subset of $\mathbb{R}^N$. Roughly speaking, the "rate of change with respect to time" of each component depends on time t, the vector $u(t) \in X$ and the compact set $K(t) \subset \mathbb{R}^N$ (including its limiting normal cones $N_{K(t)}(\cdot)$) :

$$\begin{cases} \partial_t\, u(t) = A\, u(t) + f\left(t,\ u(t),\ K(t),\ N_{K(t)}(\cdot)|_{\partial K(t)}\right) \\ \overset{\circ}{K}(t) \ni \qquad\qquad g\left(t,\ u(t),\ K(t),\ N_{K(t)}(\cdot)|_{\partial K(t)}\right) \end{cases}$$

with the generator A of a strongly continuous semigroup on X.

Considering the second component $K(t)$, it is not directly evident how to define the "rate of change" for a compact subset of $\mathbb{R}^N$. The widespread idea of prescribing the normal velocity has the disadvantage that much preparation is usually required for generalizing the speed in normal direction to arbitrary compact subsets (see [16, Chen, Giga, Goto 91], [32, Soner 93]. [8, Barles, Soner, Souganidis 93] [9, Barles, Souganidis 98], [1, Ambrosio 2000], [14, Cardaliaguet 2000], [13, Cardaliaguet 2001], for example). Many concepts start with basic assumptions that restrict applications to local effects on deformation.
So the aspect of geometric evolutions poses three additional challenges. They provide the main starting points for generalizing mutational equations of Aubin [2].

- *Extending the notion of derivative to time–dependent compact subsets* $K(t) \subset \mathbb{R}^N$ *without any regularity conditions on its boundary* $\partial\, K(t)$.

  As in Aubin's theory of mutational equations, the derivative of $K(\cdot)$ at time t is described by a set $\overset{\circ}{K}(t)$ of continuous maps of deformation that induce a first–order approximation of $K(t + \cdot)$ each. Thus, a distance between compact subsets (maybe in a generalized sense) is essential.
  So firstly, no regularity conditions on the topological boundaries are supposed a priori and secondly, no subsets of the boundaries have to be neglected as in geometric measure theory, for example (see [22, Federer 69], [12, Brakke 78]).

- *Evolution of* $K(t)$ *depending on nonlocal properties "up to* 1^{st} *order".*

  For the evolution of $K(\cdot)$ at time t, an element of the set $\overset{\circ}{K}(t)$ is prescribed as a function g of time t, the vector $u(t) \in X$ and the compact set $K(t) \subset \mathbb{R}^N$ (including its normal cones at the boundary). So on the one hand, we exclude boundary properties of second order (like mean curvature), but on the other hand nonlocal features of both $K(t)$ and the graph of normal cones $N_{K(t)}(\cdot)$ can be taken into consideration. In this respect, the concept here differs from many approaches, especially from level set methods (see [1, Ambrosio 2000] for a general survey).

- *No restricting to geometric evolutions with inclusion principle.*

  If a compact initial set is contained in another one, then the so–called *inclusion principle* states that this inclusion is preserved while the sets are evolving.
  Several approaches use it as a geometric starting point for extending analytical tools to nonsmooth subsets. An excellent example is De Giorgi's theory of barriers formulated in [17, De Giorgi 94] and elaborated in [11, Bellettini, Novaga 97], [10, Bellettini, Novaga 98]. Another widespread concept is based on the level set method using viscosity solutions. There the inclusion principle is closely related with the corresponding partial differential equation being degenerate parabolic and thus, it can be regarded as a geometric counterpart of the maximum principle (see e.g. [9, Barles, Souganidis 98], [1, Ambrosio 2000]).

An elegant approach to front propagation problems with nonlocal terms has been presented in [14, Cardaliaguet 2000], [13, Cardaliaguet 2001], [15, Cardaliaguet, Pasquignon 2001]. The inclusion principle again is the key for generalizing the evolution from $C^{1,1}$ submanifolds with boundary to nonsmooth subsets of $\mathbb{R}^N$.

As mentioned before, the primary aim of this paper consists in a unified concept for completely different types of evolutions and, geometric evolutions represent just a typical example. So we use only the properties of compact subsets with respect to a given generalized distance function (as presented in § 3). In comparison with earlier nonlocal approaches like [13], [14], it has the advantage of covering the very easy example that the normal velocity at the boundary is $\frac{1}{1 + \text{set diameter}}$.

Let us give a brief overview of this paper : Among previous approaches, C^0 semigroups have been a very successful concept for evolution equations in Banach spaces, but the two main pillars (i.e. exponential series and Cauchy integral formula) cannot be used beyond vector spaces.

In § 2, we sketch the mutational equations of Aubin ([2],[4],[5]). They extend ordinary differential equations even to metric spaces and thus provide our starting point for combining diverse types of evolutions. In [2], the primary geometric example is the set $\mathcal{K}(\mathbb{R}^N)$ of all nonempty compact subsets of $\mathbb{R}^N$ supplied with the Pompeiu–Hausdorff distance $d\!\!l$.
At the end of § 2, we provide a link between mild solutions of semilinear evolution equations and mutational equations. Indeed, considering the weak topology instead of the norm topology has the analytical interpretation that the metric is replaced by a family of pseudo–metrics. Then adequate assumptions about the reflexive Banach space X and the infinitesimal generator of the semigroup imply the existence of solutions for systems in both X and $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$ (see Proposition 4, the detailed proof will be presented in forthcoming part II).

However, *first–order* geometric evolutions have not been covered so far because the topological boundary and its normal cones are not taken into account. In § 3, the two main obstacles due to boundaries are sketched. They motivate both the definition of "ostensible metric" and extending the basic idea of distributions (in the figurative sense that an important property has to be satisfied merely by the elements of a given "test set" instead of all elements).

Then in § 4, this notion is formulated for a nonempty set with a countable family of ostensible metrics. This section is to point out the differences between Aubin's concept and our definitions of so–called *right–hand forward solutions*. At the end of § 4, we present two further aspects of generalizing mutational equations. In particular, the time direction is now taken into consideration, i.e. roughly speaking, a "later" element is always compared with an "earlier" one or — to be more precise — the arguments of ostensible metrics are always sorted by time. So the triangle inequality can be replaced by the weaker condition called *timed* triangle inequality.

Thus, right–hand forward solutions prove to be a special case of so–called *timed* right–hand forward solutions. Finally, the most general framework for mutational equations (discussed in this paper) is presented in § 5 providing all the definitions and the proofs in detail.

2. A previous approach : Mutational equations of Aubin

An approach to evolution problems in metric spaces is the *mutational analysis* of Jean–Pierre Aubin (presented in [4, Aubin 93], [2, Aubin 99]). It proves to be the more general background of "shape derivatives" introduced by Jean Céa and Jean–Paul Zolésio and has similarities to "quasi-differential equations" of Panasyuk (e.g. [29, Panasyuk 85]).

Roughly speaking, the starting point consists in extending the terms "direction" and "velocity" from vector spaces to metric spaces. Then the basic idea of first–order approximation leads to a definition of derivative for curves in a metric space and step by step, we can follow the same track as for ordinary differential equations.

Let us now describe the mutational approach in more detail : In a vector space like $\mathbb{R}^N$, each vector $v \neq 0.$ defines a continuous function

$$[0,\infty[\, \times \mathbb{R}^N \longrightarrow \mathbb{R}^N, \qquad (h,x) \longmapsto x + h\,v$$

mapping the time h and the initial point x to its final point — similar to the topological notion of a homotopy. This concept does not really require addition or scalar multiplication and thus can be applied to every metric space (M,d) instead : According to [2, Aubin 99], a map $\vartheta : [0,1] \times M \longrightarrow M$ is called *transition* on (M,d) if it satisfies

1. $\vartheta(0,x) = x \qquad \forall\ x \in M,$
2. $\limsup_{h \downarrow 0}\ \frac{1}{h} \cdot d\big(\vartheta(h,\ \vartheta(t,x)),\ \vartheta(t+h,\ x)\big) = 0 \qquad \forall\ x \in M,\ t < 1,$
3. $\alpha(\vartheta) := \sup_{x \neq y}\ \limsup_{h \downarrow 0} \left(\frac{d\big(\vartheta(h,x),\ \vartheta(h,y)\big)\ -\ d(x,y)}{h \quad d(x,y)}\right)^+ < \infty,$
4. $\beta(\vartheta) := \sup_{x \in M}\ \limsup_{h \downarrow 0} \frac{d\big(x,\ \vartheta(h,x)\big)}{h} < \infty$

with the abbreviation $(r)^+ := \max(0,r)$ for $r \in \mathbb{R}$.

Condition (1.) guarantees that the second argument x represents the initial point at time $t = 0$. Moreover condition (2.) can be regarded as a weakened form of the semigroup property. Finally the parameters $\alpha(\vartheta)$, $\beta(\vartheta)$ imply the continuity of ϑ with respect to both arguments. In particular, condition (4.) together with Gronwall's Lemma ensures the uniform Lipschitz continuity of ϑ with respect to time :

$$d\big(\vartheta(s,x),\ \vartheta(t,x)\big) \le \beta(\vartheta) \cdot |t-s| \qquad \text{for all } s,t \in [0,1],\ x \in M.$$

Obviously the function $[0,1] \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$, $(h,x) \longmapsto x + h\,v$ mentioned before fulfills the conditions on a transition on $(\mathbb{R}^N, |\cdot|)$. Let us give some further examples :

1. Leaving vector spaces like $\mathbb{R}^N$, we consider the set $\mathcal{K}(\mathbb{R}^N)$ of all nonempty compact subsets of $\mathbb{R}^N$ supplied with the so–called *Pompeiu–Hausdorff distance*
$$d\!\!l(K_1,K_2) := \max\left\{ \sup_{x \in K_1} \operatorname{dist}(x,K_2),\quad \sup_{y \in K_2} \operatorname{dist}(y,K_1)\right\}$$
It has the advantage that $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$ is compact (see e.g. [2] or [31]). Supposing $f : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ again to be bounded and Lipschitz, the transitions are defined as *reachable sets* of the vector field f, i.e.
$$\vartheta_f : [0,1] \times \mathcal{K}(\mathbb{R}^N) \longrightarrow \mathcal{K}(\mathbb{R}^N)$$
$$(t,\ K_0) \longmapsto \Big\{ x(t) \ \Big|\ \exists\ x(\cdot) \in C^1([0,t],\mathbb{R}^N) : \ \tfrac{d}{dt}\, x(\cdot) = f(x(\cdot)), \quad x(0) \in K_0 \Big\}.$$
The Theorem of Cauchy–Lipschitz ensures that ϑ_f is a transition on

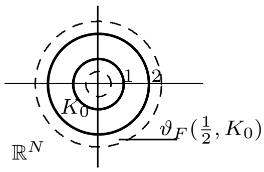
$(\mathcal{K}(\mathbb{R}^N), d\!\!l)$ and, $\alpha(\vartheta_f) \le \text{Lip } f, \ \beta(\vartheta_f) \le \|f\|_{L^\infty}$ (see [2], Prop. 3.5.2).

2. Now more than one velocity is admitted at every point of $\mathbb{R}^N$, i.e. strictly speaking, we consider the differential inclusion $\frac{d}{dt}\, x(\cdot) \in F(x(\cdot))$ (a.e.) with a set–valued map $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ instead of the ODE $\frac{d}{dt}\, x(\cdot) = f(x(\cdot))$. For every bounded Lipschitz map $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ with convex values in $\mathcal{K}(\mathbb{R}^N)$,

$$\vartheta_F : [0,1] \times \mathcal{K}(\mathbb{R}^N) \longrightarrow \mathcal{K}(\mathbb{R}^N)$$
$$(t,\ K_0) \longmapsto \big\{ x(t) \,\big|\, \exists\ x(\cdot) \in AC([0,t],\mathbb{R}^N) : \tfrac{d}{dt}\, x(\cdot) \in F(x(\cdot)) \text{ a.e.},\quad x(0) \in K_0 \big\}$$

is a transition on $(\mathcal{K}(\mathbb{R}^N),\ d\!\!l)$ — as a consequence of Filippov's Theorem (see [2, Aubin 99], Proposition 3.7.3). For any $\lambda > 0$, $\text{LIP}_\lambda(\mathbb{R}^N,\mathbb{R}^N)$ abbreviates the set of bounded λ–Lipschitz maps $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ with compact convex values.

In contrast to example (1.), the reachable set $\vartheta_F(t,K_0)$ of a set–valued map F might change its topological properties. $F(\cdot) := \mathbb{B}_1 \stackrel{\text{Def.}}{=} \{ v \in \mathbb{R}^N \,|\, |v| \le 1\}$, for example, leads to the expansion with constant speed 1 in all directions and makes the "hole" of the annulus $K_0 := \{ x \,|\, 1 \le |x| \le 2 \} \subset \mathbb{R}^N$ disappear at time 1.



This phenomenon cannot occur in the examples of ordinary differential equations (with Lipschitz right–hand side) since their evolutions are reversible in time.

A transition $\vartheta : [0,1]\times M \longrightarrow M$ provides a first–order approximation of a curve $x(\cdot) : [0,T[\longrightarrow M$ at time $t \in [0,T[$ if

$$\limsup_{h\downarrow 0}\ \tfrac{1}{h} \cdot d\big(\vartheta(h,\, x(t)),\quad x(t+h)\big) \;=\; 0.$$

Naturally ϑ need not be unique in general and so, all transitions fulfilling this condition form the so–called *mutation* of $x(\cdot)$ at time t, abbreviated as $\overset{\circ}{x}(t)$. A *mutational equation* is based on a given function f of time $t \in [0,T[$ and state $x \in M$ whose values are transitions on (M,d), i.e.

$$f : M \times [0,T[\longrightarrow \Theta(M,d),\ \ (x,t) \longmapsto f(x,t),$$

and we look for a Lipschitz curve $x(\cdot) : [0,T[\longrightarrow (M,d)$ such that $f(x(t),t)$ belongs to its mutation $\overset{\circ}{x}(t)$ for almost every time $t \in [0,T[$ (see [2], Definition 1.3.1).

The Theorem of Cauchy–Lipschitz and its proof suggest Euler method for constructing solutions of mutational equations. In this context we need an upper estimate of the distance between two points while evolving along two (different) transitions.
First of all, a distance between two transitions $\vartheta, \tau : [0,1] \times M \longrightarrow M$ has to be defined and, it is based on comparing the evolution of one and the same initial point

$$D(\vartheta,\tau) \;:=\; \sup_{x\in M}\ \ \limsup_{h\downarrow 0}\ \tfrac{1}{h} \cdot d\big(\vartheta(h,\,x),\ \ \tau(h,\,x)\big)$$

(see [2], Definition 1.1.2). Considering the preceding example of $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$ and reachable sets $\vartheta_F,\ \vartheta_G$ of bounded Lipschitz maps $F, G : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$, Filippov's Theorem implies $D(\vartheta_F,\vartheta_G) \le \sup_{x\in\mathbb{R}^N} d\!\!l(F(x),\ G(x))$ (see [2], Proposition 3.7.3).

These definitions lead to the substantial estimate

$$d\big(\vartheta(h,\,x),\ \ \tau(h,\,y)\big) \ \le\ d(x,y)\cdot e^{\alpha(\vartheta)\,h} \ +\ h\ D(\vartheta,\tau)\cdot \tfrac{e^{\alpha(\vartheta)\,h}-1}{\alpha(\vartheta)\ h} \qquad (*)$$

for arbitrary points $x, y \in M$ and time $h \in [0,1[$ ([2], Lemma 1.1.3). The proof of this inequality provides an excellent insight into the basic technique for drawing global conclusions from local properties : Due to the definition of transitions, the distance $\psi : \ [0,1] \longrightarrow [0,\infty[, \ \ h \longmapsto d\big(\vartheta(h,\,x),\,\tau(h,\,y)\big)$ is a Lipschitz continuous function of time and satisfies

$$\begin{aligned}
&\lim_{h\downarrow 0}\ \tfrac{\psi(t+h)\,-\,\psi(t)}{h} \ = \\
&= \lim_{h\downarrow 0}\ \tfrac{1}{h}\cdot\Big(d\Big(\vartheta(t+h,\,x),\ \ \tau(t+h,\,y)\Big) \ -\ d\Big(\vartheta(t,\,x),\ \tau(t,\,y)\Big)\Big) \\
&\le \limsup_{h\downarrow 0}\ \tfrac{1}{h}\cdot\Big(d\Big(\vartheta(t+h,\,x),\ \ \vartheta(h,\ \vartheta(t,x))\Big) \ + \\
&\qquad d\Big(\vartheta(h,\ \vartheta(t,x)),\ \vartheta(h,\ \tau(t,y))\Big) \ -\ d\Big(\vartheta(t,\,x),\ \tau(t,\,y)\Big) \ + \\
&\qquad d\Big(\vartheta(h,\ \tau(t,y)),\ \tau(h,\ \tau(t,y))\Big) \ + \\
&\qquad d\Big(\tau(h,\ \tau(t,y)),\ \tau(t+h,\ \ \ y\,)\Big)\ \Big) \\
&\le 0 \ +\ \alpha(\vartheta)\cdot\psi(t) \ +\ D(\vartheta,\tau) \ +\ 0
\end{aligned}$$

for almost every $t \in [0,1[$ (i.e. every t at which the limit on the left–hand side exists). So the estimate results from well–known Gronwall's Lemma about Lipschitz continuous functions. In fact, Gronwall's Lemma proves to be the key analytical tool for all these conclusions of mutational analysis and, its integral version holds even for continuous functions (see [2], Lemma 8.3.1).

Considering now mutational equations, estimate $(*)$ is laying the foundations for proving the convergence of Euler method. It leads to the following mutational counterpart of the Theorem of Cauchy–Lipschitz (quoted from Theorem 1.4.2 in [2, Aubin 99]).

Theorem 1. *Assume that the closed bounded balls of the metric space (M,d) are compact. Let f be a function from M to a set of transitions on (M,d) satisfying*

1. $\exists\ \lambda > 0 : \ \ D(f(x),\ f(y)) \ \le\ \lambda\cdot d(x,y) \qquad \forall\ x,y\in M$
2. $A := \sup_{x\in M}\ \alpha(f(x)) \ <\ \infty.$

Suppose for $y : [0,T[\longrightarrow M$ that its mutation $\overset{\circ}{y}(t)$ is nonempty for each t.

Then for every initial value $x_0 \in M$, there exists a unique solution $x(\cdot) : [0,T[\longrightarrow M$ of the mutational equation $\overset{\circ}{x}(t) \ni f(x(t))$, i.e. for almost every $t \in [0,T[$,

$$\limsup_{h\downarrow 0}\ \tfrac{1}{h}\cdot d\big(x(t+h),\ f(x(t))\,(h,x(t))\big) \ =\ 0,$$

satisfying $x(0) = x_0$ and the inequality (for every $t \in [0,T[$)

$$d\big(x(t),\ y(t)\big) \ \le\ d(x_0,\ y(0))\cdot e^{(A+\lambda)\,t} \ + \int_0^t e^{(A+\lambda)\,(t-s)}\cdot \inf_{\vartheta\,\in\,\overset{\circ}{y}(s)} D\big(f(y(s)),\ \vartheta\big)\ \ ds.$$

□

Linking semilinear evolution equations to mutational equations

Extending now the results of Aubin ([2]), *strongly continuous semigroups* on reflexive Banach spaces induce an interesting example of transitions in a slightly generalized sense. Basically, the metric is replaced by a family of distance functions. Here we just state some conclusions from the general results of §§ 4, 5 briefly and, the detailed verification is presented in forthcoming part II.

Let $A : D_A \longrightarrow X$ $(D_A \subset X)$ be a closed linear operator on a Banach space X generating a semigroup $(S(t))_{t\geq 0}$. Then for every $w \in X$ and initial point $u_0 \in X$, the inhomogeneous equation $\frac{d}{dt}\, u(t) = A\, u(t) + w$ has a unique solution $u : [0,\infty[\longrightarrow X$ with $u(0) = u_0$, namely

$$\varSigma_w(t,\ u_0) \ := \ u(t) \ = \ S(t)\, u_0 \ + \int_0^t S(t-s)\ w\ ds.$$

If $\varSigma_w(\cdot,\cdot)$ is a transition on $(X, \|\cdot\|_X)$, then the condition

$$\beta(\varSigma_w) \stackrel{\text{Def.}}{=} \sup_{u_0\in X}\ \limsup_{h\downarrow 0}\ \tfrac{1}{h}\cdot \big\|\, u_0 - \varSigma_w(h,\, u_0)\, \big\|_X \ < \ \infty$$

implies that the infinitesimal generator $A : X \longrightarrow X$ is bounded and so many important examples of semigroup theory are excluded.

For applying the mutational approach to C^0 semigroups, we prefer the weak topology on X to the norm $\|\cdot\|_X$ and define

$$q_{v'} : X\times X \longrightarrow [0,\infty[, \qquad (x,y) \longmapsto |\langle x-y,\ v'\rangle|$$

for every linear form $v' \in X'$ with $\|v'\|_{X'} \leq 1$. Each $q_{v'}$ is a so–called *pseudo–metric*, i.e. it is reflexive $(q_{v'}(x,x) = 0$ for all $x)$, symmetric $(q_{v'}(x,y) = q_{v'}(y,x)$ for all $x, y)$ and satisfies the triangle inequality. The family $\{q_{v'}\}$ induces the weak topology on X.

From now on, we suppose the Banach space X to be reflexive. This assumption has two advantages : Firstly, closed bounded balls of X are weakly compact (see e.g. [35, Yosida 78]). So any bounded sequence in X has a subsequence converging with respect to every $q_{v'}$ simultaneously. Secondly, the reflexivity of X guarantees that the adjoint operators $S(t)' : X' \longrightarrow X'$ $(t \geq 0)$ form a C^0 semigroup on X' with the infinitesimal generator A' (see [21, Engel,Nagel 2000], Prop. I.5.14). This useful consequence opens the possibility that $\varSigma_w(\cdot,\cdot)$ fulfills (slightly weakened) continuity conditions on transitions with respect to each $q_{v'}$ for $v' \in X'$ fixed :
In regard to time, we obtain

$$\begin{aligned} q_{v'}\big(\varSigma_w(t_1,u_0),\ \varSigma_w(t_2,u_0)\big) &= \big|\big\langle S(t_1)\, u_0 - S(t_2)\, u_0,\ \ v'\big\rangle\big| \\ &= \big|\big\langle u_0,\ \ \big(S(t_1)' - S(t_2)'\big)\ v'\big\rangle\big| \\ &\longrightarrow 0 \qquad\qquad\qquad\qquad \text{for } t_2 - t_1 \longrightarrow 0 \end{aligned}$$

uniformly for all $u_0 \in X$, $\|u_0\|_X \leq 1$. So for all $\rho > 0$ and $v' \in D_{A'} \subset X'$,

$$\sup_{\substack{\|u_0\|_X \,\leq\, \rho \\ 0 \,\leq\, t \,\leq\, 1}}\ \ \limsup_{h\downarrow 0}\ \ \tfrac{1}{h}\cdot q_{v'}\big(\varSigma_w(t,u_0),\ \ \varSigma_w(t+h,u_0)\big) \ \leq\ \rho\ \|A'\,v'\|_{X'},$$

i.e. restricting ourselves to a priori bounded subsets of X, we can follow the steps of mutational analysis using a finite parameter $\beta(\varSigma_w)$ w.r.t. $q_{v'}$. Similarly, all $u_0, u_1 \in X$ and every linear form $v' \in D_{A'} \subset X'$ satisfy

$$q_{v'}\big(\varSigma_w(h,u_0),\ \varSigma_w(h,u_1)\big) \ - \ q_{v'}(u_0,\ u_1) \ \leq\ \big|\big\langle u_0 - u_1, \big(S(h)' {-} \mathrm{Id}_{X'}\big) v'\big\rangle\big|$$

$$\limsup_{h\downarrow 0}\ \frac{q_{v'}\big(\varSigma_w(h,u_0),\ \varSigma_w(h,u_1)\big) \ - \ q_{v'}(u_0,\ u_1)}{h} \ \leq\ \big|\big\langle u_0 - u_1, \quad A'\ v'\big\rangle\big|.$$

If additionally $v' \in D_{A'}$ is an eigenvector of A' (and λ its eigenvalue), then it provides an upper estimate of the parameter $\alpha(\Sigma_w)$ w.r.t. $q_{v'}$

$$\limsup_{h \downarrow 0} \frac{q_{v'}\big(\Sigma_w(h,u_0),\ \Sigma_w(h,u_1)\big) - q_{v'}(u_0,\, u_1)}{h \qquad q_{v'}(u_0,\, u_1)} \quad \le \ |\lambda|$$

for all $u_0,\, u_1 \in X$ with $q_{v'}(u_0, u_1) > 0$.
These preliminaries form the basis for proving the existence of weak solutions by means of mutational analysis and according to [7, Ball 1977], weak solutions are *mild* solutions :

Proposition 2. *Suppose :*

1. *X is a reflexive Banach space.*
2. *The linear operator A generates a C^0 semigroup $(S(t))_{t \ge 0}$ on X.*
3. *The dual operator A' of A has countably many eigenvectors $\{v'_j\}_{j \in \mathcal{J}}$ $(\|v'_j\|_{X'} = 1)$ spanning the dual space X'. Set $q_j := q_{v'_j}$.*
4. *Let $f : X \times [0,T] \longrightarrow X$ satisfy $\|f\|_{L^\infty} < \infty$ and for each $j \in \mathcal{J}$, $q_j\big(f(x_1,t_1),\ f(x_2,t_2)\big) \ \le\ \omega_j\big(q_j(x_1,x_2) + |t_2 - t_1|\big)$ for all $x_k,\ t_k$ with a modulus $\omega_j(\cdot)$ of continuity.*

For each $x_0 \in X$, there exists a mild solution $x : [0,T[\longrightarrow X$ of the initial value problem $\frac{d}{dt}\, x(t) \ =\ A\, x(t) \ +\ f(x(t),t), \qquad x(0) = x_0,$

$$i.e. \qquad x(t) \ = \ S(t)\, x_0 \ + \int_0^t S(t-s)\ f(x(s),\, s)\ ds \qquad (\textit{by definition}).$$

Assumptions (1.)–(3.) are formulated in a quite general way for pointing out the key features. Basic results of functional analysis provide interesting examples like

- a compact symmetric operator $A : X \longrightarrow X$ on a separable Hilbert space X, e.g. some integral operators of Hilbert–Schmidt type on $L^2(O)$ $(O \subset \mathbb{R}^N$ open),

- an infinitesimal generator $A : D_A \longrightarrow X$ of a C^0 semigroup on a Hilbert space X whose resolvent is compact and normal, e.g. a strongly elliptic differential operator (of second order) in divergence form with smooth autonomous coefficients.

Assumption (4.) of Prop. 2 is very restrictive because $f : X \times [0,T] \longrightarrow X$ has to be continuous with respect to each linear form v'_j separately. Even easy examples of rotation might fail to satisfy this condition. Thus, we take more than one linear form v'_j $(j \in \mathcal{J} = \{j_1, j_2, j_3 \ldots\})$ into consideration simultaneously :

Proposition 3. *In addition to assumptions (1.)–(3.) of Proposition 2, let $f : X \times [0,T] \longrightarrow X$ fulfill $\|f\|_{L^\infty} < \infty$ and*

$$\sum_{k=1}^{\infty} 2^{-k}\ q_{j_k}\big(f(x,t_1),\, f(y,t_2)\big) \ \le\ \widehat{\omega}\Big(\sum_{k=1}^{\infty} 2^{-k}\ \frac{q_{j_k}(x,y)}{1 + q_{j_k}(x,y)} \ + |t_2 - t_1|\Big)$$

for all $x, y \in X$ and $t_1, t_2 \in [0,T]$ with a modulus $\widehat{\omega}(\cdot)$ of continuity.
For each $x_0 \in X$, there exists a mild solution $x : [0,T[\longrightarrow X$ of the semilinear equation $\frac{d}{dt}\, x(t) \ =\ A\, x(t) \ +\ f(x(t),t), \qquad x(0) \ =\ x_0,$

After replacing the metric by a family of distance functions, the main steps of mutational analysis have not changed so far. So in principle, we can already deal with systems of semilinear evolution equations in reflexive Banach spaces and mutational equations in $(\mathcal{K}(\mathbb{R}^N),\ d\!\!l)$.
Using the abbreviations for $x, y \in X$

$$p_\infty(x,y) \ := \ \sum_{k=1}^{\infty} 2^{-k}\ \frac{q_{j_k}(x,y)}{1 + q_{j_k}(x,y)}\,, \qquad P_\infty(x,y) \ := \ \sum_{k=1}^{\infty} 2^{-k}\ q_{j_k}(x,y),$$

Proposition 4.
In addition to assumptions (1.)–(3.) of Proposition 2, suppose for

$$f : X \times \mathcal{K}(\mathbb{R}^N) \times [0,T] \longrightarrow X$$
$$g : X \times \mathcal{K}(\mathbb{R}^N) \times [0,T] \longrightarrow \mathrm{LIP}_\Lambda(\mathbb{R}^N, \mathbb{R}^N) :$$

4. $\|f\|_{L^\infty} < \infty, \quad \Lambda < \infty$
5. $P_\infty\big(f(x_1,K_1,t_1),\ f(x_2,K_2,t_2)\big) \le \omega\big(|\xi_{1,2}|\big)$
6. $\sup_{z\in\mathbb{R}^N} d\!l\big(g(x_1,K_1,t_1)(z),\ g(x_2,K_2,t_2)(z)\big) \le \omega\big(|\xi_{1,2}|\big)$

using the abbreviation $|\xi_{1,2}| := p_\infty(x_1,x_2) + d\!l(K_1,K_2) + t_2 - t_1$ *for all* $x_1, x_2 \in X,\ K_1, K_2 \in \mathcal{K}(\mathbb{R}^N),\ 0 \le t_1 \le t_2 \le T$ *with a modulus* $\omega(\cdot)$ *of continuity.*

Then for every initial elements $x_0 \in X$ *and* $K_0 \in \mathcal{K}(\mathbb{R}^N)$, *there exists a solution* $(x,K) : [0,T[\longrightarrow X \times \mathcal{K}(\mathbb{R}^N)$ *of the following problem :*

a) $x : [0,T[\longrightarrow X$ *is a mild solution of the initial value problem*
$$\wedge \begin{cases} \frac{d}{dt}\, x(t) = A\, x(t) + f(x(t),\, K(t),\, t) \\ x(0) = x_0 \end{cases}$$
b) $K(\cdot) : [0,T[\leadsto \mathcal{K}(\mathbb{R}^N)$ *is Lipschitz w.r.t.* $d\!l$ *and,* $K(0) = K_0$.

c) $\limsup_{h\downarrow 0} \frac{1}{h} \cdot d\!l\big(\vartheta_{g(x(t),\,K(t),\,t)}(h,\, K(t)),\ K(t+h)\big) = 0$ *for a.e.* t.

3. Obstacles to first–order geometric evolutions due to boundaries

Applying the mutational analysis of Aubin to a metric space (M,d), obstacles are mostly related to the continuity parameters of a transition ϑ

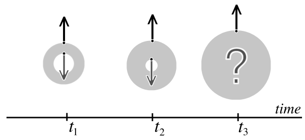
$$\alpha(\vartheta) \overset{\text{Def.}}{=} \sup_{x \neq y} \ \limsup_{h\downarrow 0} \ \left(\frac{d\big(\vartheta(h,x),\ \vartheta(h,y)\big) \ - \ d(x,y)}{h \quad d(x,y)}\right)^+ < \infty,$$
$$\beta(\vartheta) \overset{\text{Def.}}{=} \sup_{x\in M} \ \limsup_{h\downarrow 0} \ \frac{1}{h} \cdot d\big(x,\ \vartheta(h,\,x)\big) < \infty.$$

In regard to first–order geometric evolutions, these difficulties arise when incorporating normal cones into a distance function of compact subsets. We are going to use reachable sets $\vartheta_F(\cdot,\cdot)$ of differential inclusions $\dot{x}(\cdot) \in F(x(\cdot))$ a.e. as candidates for transitions on $\mathcal{K}(\mathbb{R}^N)$. So the topological properties of $\vartheta_F(t,K)$ may change in the course of time.

For the regularity in time : Ostensible metrics

Let us consider first the consequences of the boundary for the continuity of $\vartheta_F : [0,1] \times \mathcal{K}(\mathbb{R}^N) \longrightarrow \mathcal{K}(\mathbb{R}^N)$ with respect to time.

The key aspect is illustrated easily by an annulus $K_\circledcirc$ expanding isotropically at a constant speed. After a positive finite time t_3, the "hole" in the center has disappeared of course.



Every boundary point x_3 at time t_3 has close counterparts at earlier sets. To be more precise, $x_3 \in \partial\,\vartheta_F(t_3, K_\circledcirc)$ is final point of a trajectory $x(\cdot) : [0,t_3] \longrightarrow \mathbb{R}^2$ of $F(\cdot) := \mathbb{B}_1$ and, each $x(t)$ belongs to the boundary of $\vartheta_F(t, K_\circledcirc)$. Furthermore a so–called adjoint arc connects each normal vector at x_3 to a normal vector at $x(t)$. However, this tool of control theory works only in backward time direction. In particular, starting at a point $y \in \partial K_\circledcirc$ of the "hole", there is no trajectory belonging to each $\partial\,\vartheta_F(t, K_\circledcirc)$ up to time t_3.

In general, the topological boundary of $\vartheta_F(\cdot, K) : [0,\infty[\rightsquigarrow \mathbb{R}^N$ (with $K \in \mathcal{K}(\mathbb{R}^N)$) is not continuous with respect to $d\!\!l$. Furthermore, the normals of *later* sets find close counterparts among the normals of *earlier* sets, but usually not vice versa.
For this purpose, we dispense with the symmetry condition on a metric :

Definition 5. *Let E be a nonempty set. $q : E \times E \longrightarrow [0,\infty[$ is called* ostensible metric *on E if it satisfies the conditions :*

1. $\forall\ x \in E : \quad q(x,x) = 0$ (*reflexive*)
2. $\forall\ x,y,z \in E : \ q(x,z) \le q(x,y) + q(y,z)$ (*triangle inequality*).

Then (E,q) is called ostensible metric space.

In the literature on topology (e.g. [34, Wilson 31], [23, Kelly 63], [33, Stoltenberg 69], [24, Künzi 92]), a *quasi–metric* $p : E \times E \longrightarrow [0,\infty[$ on a set E satisfies the triangle inequality and is positive definite, i.e. $p(x,y) = 0 \iff x = y$ for every $x, y \in E$. A *pseudo–metric* $p : E \times E \longrightarrow [0,\infty[$ on a set $E \neq \emptyset$ is characterized by the properties : reflexive (i.e. $p(x,x) = 0$ for all x), symmetric (i.e. $p(x,y) = p(y,x)$ for all x,y) and the triangle inequality. So this generalized distance of Definition 5 is sometimes called *quasi–pseudo–metric* (see [23, Kelly 63], [24, Künzi 92], for example), but just for linguistic reasons we prefer the adjective "ostensible".

In regard to the first–order geometric evolution, we suggest the ostensible metric $q_{\mathcal{K},N} : \mathcal{K}(\mathbb{R}^N) \times \mathcal{K}(\mathbb{R}^N) \longrightarrow [0,\infty[$,

$$q_{\mathcal{K},N}(K_1,\ K_2) := d\!\!l(K_1, K_2) + \operatorname{dist}\big(\text{Graph } {}^\flat N_{K_2},\ \text{Graph } {}^\flat N_{K_1}\big)$$

with $N_K(x)$ denoting the limiting normal cone of $K \subset \mathbb{R}^N$ at $x \in \partial K$, ${}^\flat N_K(x) := N_K(x) \cap \mathbb{B}_1$.

So, $q_{\mathcal{K},N}(K_1,\ K_2) \ge 0$ takes the graphical distance from the limiting normal vectors ${}^\flat N_{K_2} \subset \mathbb{B}_1$ to ${}^\flat N_{K_1} \subset \mathbb{B}_1$ into account. Correspondingly to the example of an annulus $K_\circledcirc$ expanding isotropically, the first argument K_1 can be regarded as *earlier* set whereas the second argument K_2 represents the *later* set. In particular, it is easy to verify

$$q_{\mathcal{K},N}(\vartheta_F(s, K_\circledcirc),\ \vartheta_F(t, K_\circledcirc)) \le \text{const} \cdot (t-s) \qquad \text{for all } s \le t \le 1.$$

Applying now the steps of mutational analysis to an ostensible metric space (E,q), we encounter analytical obstacles soon. In particular,

$$[0,1] \longrightarrow [0,\infty[, \quad t \longmapsto q_{\mathcal{K},N}\big(\vartheta_F(t,K_1),\ \vartheta_F(t,K_2)\big)$$

need not be continuous. For example, the isotropic expansion at a speed of 1 ($F(\cdot) := \mathbb{B}_1$) and $K_1 := \mathbb{B}_2$, $K_2 := \{1 \le |x| \le 2\} \subset \mathbb{R}^N$ satisfy

$$q_{\mathcal{K},N}\big(\vartheta_F(t,K_1),\ \vartheta_F(t,K_2)\big) \begin{cases} \ge 1 & \text{for } 0 \le t < 1 \\ = 0 & \text{for } \quad\ t \ge 1 \end{cases}.$$

So we cannot apply the proof of key estimate $(*)$ (mentioned in § 2) to ostensible metric spaces immediately. A more general form of Gronwall's Lemma is needed instead — without supposing continuity.

Lemma 6 (Lemma of Gronwall for semicontinuous functions I). *Let $\psi : [a,b] \longrightarrow \mathbb{R}$, $f, g \in C^0([a,b[, \mathbb{R})$ satisfy $f(\cdot) \ge 0$ and*

$$\psi(t) \le \limsup_{h \downarrow 0}\ \psi(t-h), \qquad \forall\ t \in\ ]a,\ b],$$

$$\psi(t) \ge \limsup_{h \downarrow 0}\ \psi(t+h), \qquad \forall\ t \in [a,\ b[,$$

$$\limsup_{h \downarrow 0}\ \tfrac{\psi(t+h)-\psi(t)}{h} \le f(t) \cdot \limsup_{h \downarrow 0}\ \psi(t-h) + g(t) \qquad \forall\ t \in\ ]a,\ b[.$$

Then, for every $t \in [a,b]$, *the function* $\psi(\cdot)$ *fulfills the upper estimate*

$$\psi(t) \;\le\; \psi(a) \cdot e^{\mu(t)} \;+\; \int_a^t e^{\mu(t)-\mu(s)}\, g(s)\, ds \quad \textit{with } \mu(t) := \int_a^t f(s)\, ds.$$

Proof. Let $\delta > 0$ be arbitrarily small. The proof is based on comparing ψ with the auxiliary function $\varphi_\delta : [a,b] \longrightarrow \mathbb{R}$ that uses $\psi(a)+\delta$, $g(\cdot)+\delta$ instead of $\psi(a)$, $g(\cdot)$:

$$\varphi_\delta(t) \;:=\; \big(\psi(a)+\delta\big)\, e^{\mu(t)} \;+\; \int_a^t e^{\mu(t)-\mu(s)}\, (g(s)+\delta)\, ds.$$

Then,
$$\begin{aligned} \varphi_\delta'(t) &= f(t)\,\varphi_\delta(t) + g(t) + \delta && \text{on } [a,b[,\\ \varphi_\delta(t) &> \psi(t) && \text{for all } t \in [a,b[\text{ close to } a. \end{aligned}$$

Assume now that there is some $t_0 \in]a,b]$ with $\varphi_\delta(t_0) < \psi(t_0)$. Setting

$$t_1 \;:=\; \inf\,\big\{t \in [a,t_0] \;\big|\; \varphi_\delta(t) < \psi(t)\,\big\},$$

we obtain $\varphi_\delta(t_1) = \psi(t_1)$ and $a < t_1 < t_0$ because

$$\begin{aligned} \varphi_\delta(t_1) &= \lim_{h\downarrow 0} \;\varphi_\delta(t_1-h) \;\ge\; \limsup_{h\downarrow 0}\; \psi(t_1-h) \;\ge\; \psi(t_1),\\ \varphi_\delta(t_1) &= \lim_{\substack{h\to 0\\ h\ge 0}} \;\varphi_\delta(t_1+h) \;\le\; \limsup_{\substack{h\to 0\\ h\ge 0}}\; \psi(t_1+h) \;\le\; \psi(t_1). \end{aligned}$$

Thus, we conclude from the definition of t_1

$$\begin{aligned} \liminf_{h\downarrow 0}\; \tfrac{\varphi_\delta(t_1+h)-\varphi_\delta(t_1)}{h} &\le \limsup_{h\downarrow 0}\; \tfrac{\psi(t_1+h)-\psi(t_1)}{h}\\ \varphi_\delta'(t_1) &\le f(t_1)\cdot \limsup_{h\downarrow 0}\; \psi\,(t_1-h) \;+\; g(t_1)\\ f(t_1)\,\varphi_\delta(t_1) + g(t_1) + \delta &\le f(t_1)\cdot \limsup_{h\downarrow 0}\; \varphi_\delta(t_1-h) \;+\; g(t_1)\\ &\le f(t_1)\cdot \varphi_\delta(t_1) \qquad\qquad + \; g(t_1) \end{aligned}$$

— a contradiction. So $\varphi_\delta(\cdot) \ge \psi(\cdot)$ for any $\delta > 0$. □

Remark 7. (i) The condition $\limsup_{h\downarrow 0} \frac{\psi(t+h)-\psi(t)}{h} \le f(t)\cdot\psi(t) + g(t)$ (supposed in the widespread forms of Gronwall's Lemma) is stronger than the third assumption of this lemma due to the semicontinuity condition $\psi(t) \le \limsup_{h\downarrow 0}\; \psi(t-h)$.

(ii) This and the following subdifferential version of Gronwall's Lemma also hold if the functions $f, g : [a,b[\longrightarrow \mathbb{R}$ are only upper semicontinuous (instead of continuous). The proof is based on upper approximations of $f(\cdot)$, $g(\cdot)$ by continuous functions.

Corollary 8 (Lemma of Gronwall for semicontinuous functions II). *Let* $\psi : [a,b] \longrightarrow \mathbb{R}$, $f, g \in C^0([a,b[,\mathbb{R})$ *satisfy* $f(\cdot) \ge 0$ *and*

$$\begin{aligned} \psi(t) &\le \liminf_{h\downarrow 0}\; \psi(t-h), && \forall\; t \in]a,\, b],\\ \psi(t) &\ge \liminf_{h\downarrow 0}\; \psi(t+h), && \forall\; t \in [a,\, b[,\\ \liminf_{h\downarrow 0}\; \tfrac{\psi(t+h)-\psi(t)}{h} &\le f(t)\cdot \liminf_{h\downarrow 0}\; \psi(t-h) \;+\; g(t) && \forall\; t \in]a,\, b[. \end{aligned}$$

Then, for every $t \in [a,b]$, *the function* $\psi(\cdot)$ *fulfills the upper estimate*

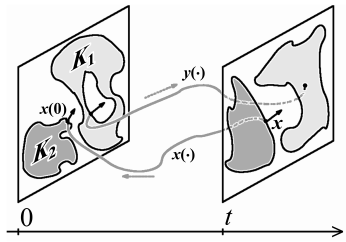
$$\psi(t) \;\le\; \psi(a) \cdot e^{\mu(t)} \;+\; \int_a^t e^{\mu(t)-\mu(s)}\, g(s)\, ds \quad \textit{with } \mu(t) := \int_a^t f(s)\, ds.$$

Proof follows the same track as in Lemma 6 – just using instead $t_1' \;:=\; \inf\,\big\{t \in [a,t_0] \;\big|\; \varphi_\delta(\cdot) < \psi(\cdot) \text{ in } [t,t_0]\,\big\} \;>\; a.$ □

When extending key estimate $(*)$ to transitions ϑ, τ on an *ostensible* metric space (E, q), the required semicontinuity of $t \longmapsto q(\vartheta(t,x),\ \tau(t,y))$ will be guaranteed by a further condition on generalized transitions.

For the regularity with respect to initial states : the distributional notion

Now we consider the consequences of the topological boundary for the continuity of $\vartheta_F : [0,1] \times \mathcal{K}(\mathbb{R}^N) \longrightarrow \mathcal{K}(\mathbb{R}^N)$ with respect to the second argument.
For any initial sets $K_1, K_2 \in \mathcal{K}(\mathbb{R}^N)$ and a given map $F \in \mathrm{LIP}_\lambda(\mathbb{R}^N, \mathbb{R}^N)$, the reachable sets of $\dot{x}(\cdot) \in F(x(\cdot))$ a.e. at time t are compared with respect to $q_{\mathcal{K},N}$. In particular, we need an estimate of the distance from any $x \in \partial\, \vartheta_F(t, K_2)$ to the boundary of $\vartheta_F(t, K_1)$.
x is reached by a trajectory $x(\cdot)$ of F starting in K_2 and, $x(0)$ belongs to the boundary of K_2. Moreover, each normal vector to $\vartheta_F(t, K_2)$ at x is connected to some $p_0 \in N_{K_2}(x(0))$ by an adjoint arc (due to Hamilton condition). Now $(x(0), \frac{p_0}{|p_0|})$ has its closest counterpart (y_0, q_0) at Graph $N_{K_1}|_{\partial K_1}$ and, its distance is bounded by $q_{\mathcal{K},N}(K_1, K_2)$.



However we cannot guarantee that any trajectory of F starting in y_0 stays in the boundary of $\vartheta_F(s, K_1)$ up to time t. Roughly speaking, y_0 might belong to a "hole" of K_1 disappearing with the course of time.
For excluding this phenomenon, additional assumptions about K_1 are needed. Suitable conditions on F guarantee, for example, that compact sets with $C^{1,1}$ boundary preserve this regularity for short times (see [25] or part II) and, their topological properties do not change.

Assuming further conditions on one of the sets $K_1, K_2 \in \mathcal{K}(\mathbb{R}^N)$ prevents us from applying the mutational analysis of Aubin. Thus, we use the basic idea of distributions.
In an ostensible metric space, there are no obvious generalizations of linear forms or partial integration and so, distributions in their widespread sense cannot be introduced. More generally speaking, *their basic idea is to select an important property and demand it for all elements of a given "test set".*

In the mutational analysis of a metric space (M, d), the estimate

$$d\big(\vartheta(h,x),\ \tau(h,y)\big) \ \le\ d(x,y) \cdot e^{\alpha(\vartheta)\, h} \ +\ h\ D(\vartheta,\tau) \cdot \frac{e^{\alpha(\vartheta)\ h} - 1}{\alpha(\vartheta)\ h} \qquad (*)$$

(for arbitrary $x, y \in M$ and $h \in [0,1[$) represents the probably most important tool for constructing solutions by means of Euler method. So it is our starting point for overcoming the recent obstacle in $(\mathcal{K}(\mathbb{R}^N), q_{\mathcal{K},N})$. In fact, forthcoming part II will verify in detail that under suitable assumptions about $F, G : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ (and their Hamiltonian functions $\mathcal{H}_F, \mathcal{H}_G$),

$$\begin{aligned} & q_{\mathcal{K},N}\big(\vartheta_F(h, K_1),\ \vartheta_G(h, K_2)\big) \\ \le\ & \big(q_{\mathcal{K},N}(K_1, K_2) \ +\ h \cdot 4\,N\ \|\mathcal{H}_F - \mathcal{H}_G\|_{C^1(\mathbb{R}^N \times \partial\mathbb{B}_1)}\big) \cdot e^{\Lambda\, h} \end{aligned}$$

holds for every $K_1 \in \mathcal{K}(\mathbb{R}^N)$ with $C^{1,1}$ boundary, all $K_2 \in \mathcal{K}(\mathbb{R}^N)$ and $h > 0$ sufficiently small (depending only on K_1, F).
So following the basic idea of distributions in an *ostensible* metric space (E, q), we are interested in how to realize the formal estimate

$$q\big(\vartheta(h,z),\ \tau(h,y)\big) \ \le\ \big(q(z,y) \ +\ h\ Q^{\mapsto}(\vartheta,\tau)\big) \cdot e^{\alpha^{\mapsto}\, h} \qquad (**)$$

for all points $y \in E$, every element z of a given "test set" $D \subset E$ and $h > 0$ sufficiently small (depending only on ϑ, z). In particular, the definitions of $Q^{\mapsto}(\vartheta,\ \tau)$ and the parameter $\alpha^{\mapsto}$ have to be adapted.

4. Right–hand forward solutions of mutational equations : Definitions

Seizing the motivation of first–order geometric evolutions, we now specify the approach for the more general situation of a nonempty set E (instead of $\mathcal{K}(\mathbb{R}^N)$) and a fixed nonempty "test set" $D \subset E$. In this section, the mutational analysis of Aubin is extended to so–called *right–hand forward solutions* and, the main definitions are stated. Finally we explain two further aspects of generalization that are leading to the so–called *timed* right–hand forward solutions in *timed* ostensible metric spaces presented in section 5 in detail.

As a consequence of § 2 (about linking semilinear evolution equations to mutational equations), we consider more than one distance on E. Thus, suppose $(q_\varepsilon)_{\varepsilon\in\mathcal{J}}$ to be a countable family of ostensible metrics on E. Assuming $\mathcal{J}$ to be countable makes the Cantor diagonal construction available for proofs of existence.

Definition 9. *Assume for* $\vartheta : [0,1] \times E \longrightarrow E$ *and each index* $\varepsilon \in \mathcal{J}$

1. $\vartheta(0,\cdot) = \mathrm{Id}_E$,
2. $\limsup_{h \downarrow 0} \frac{1}{h} \cdot q_\varepsilon\big(\vartheta(h,\ \vartheta(t,x)),\ \ \vartheta(t+h,\ x)\big) = 0 \quad \forall\ x\in E,\ t<1,$
   $\limsup_{h \downarrow 0} \frac{1}{h} \cdot q_\varepsilon\big(\vartheta(t+h,\ x),\ \ \vartheta(h,\ \vartheta(t,x))\big) = 0 \quad \forall\ x\in E,\ t<1,$
3. $\exists\ \alpha_\varepsilon^{\mapsto}(\vartheta) < \infty : \sup_{\substack{z\,\in\, D\\ y\,\in E}} \ \limsup_{h \downarrow 0} \left(\frac{q_\varepsilon\left(\vartheta(h,z),\ \vartheta(h,y)\right)\ -\ q_\varepsilon(z,y)}{h \quad q_\varepsilon(z,y)}\right)^+ \le \alpha_\varepsilon^{\mapsto}(\vartheta)$
4. $\exists\ \beta_\varepsilon(\vartheta) :\ ]0,1] \longrightarrow [0,\infty[\ :$ *nondecreasing*, $\quad \limsup_{h \downarrow 0} \beta_\varepsilon(\vartheta)(h) = 0,$
   $q_\varepsilon\big(\vartheta(s,x),\ \vartheta(t,x)\big) \ \le\ \beta_\varepsilon(\vartheta)(t-s) \qquad \forall\ s<t\le 1,\ \ x\in E,$
5. $\forall\ z \in D \qquad \exists\ \mathcal{T}_\Theta = \mathcal{T}_\Theta(\vartheta,z) \in\ ]0,1] : \qquad \vartheta(t,z) \in D \qquad \forall\ t \le \mathcal{T}_\Theta,$
6. $\limsup_{h \downarrow 0}\ q_\varepsilon\big(\vartheta(t-h,\ z),\ y\big) \ \ge\ q_\varepsilon\big(\vartheta(t,z),\ y\big) \quad \forall\ z\in D,\ y\in E,\ t \le \mathcal{T}_\Theta$

Then $\vartheta(\cdot,\cdot)$ *is a so–called* forward transition *on* $(E, D, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$.

Here the term "forward" and the symbol $\mapsto$ (representing the time axis) indicate that we usually compare the state at time t with the element at time $t+h$ for $h \downarrow 0$.
Conditions (1.)–(4.) are quite similar to the properties of Aubin's transitions on metric spaces (see § 2). Indeed, condition (1.) states that x is the initial value of $[0,1] \longrightarrow E,\ t \longmapsto \vartheta(t,x)$ and, condition (2.) can again be regarded as a weakened form of the semigroup property. It consists of two demands as q_ε need not be symmetric any longer.
Condition (3.) differs from its earlier counterpart in two respects : The first argument is restricted to elements z of the "test set" D and, $\alpha_\varepsilon^{\mapsto}(\vartheta)$ may be chosen larger than necessary. Thus, it is easier to define $\alpha_\varepsilon^{\mapsto}(\cdot) < \infty$ uniformly in some applications like the first–order geometric example. In condition (4.), the Lipschitz continuity of Aubin's transitions is replaced by equi–continuity with respect to time as this detail is used only for technical reasons in proofs.
Condition (5.) guarantees that every element $z \in D$ stays in the "test set" D for short times at least. Roughly speaking, it means in the preceding geometric example that smooth sets stay smooth shortly. This assumption is required because estimates using the parameter $\alpha^{\mapsto}(\cdot)$ can be ensured only within this period. Further conditions on $\mathcal{T}_\Theta(\vartheta,\cdot) > 0$ are avoidable for proving existence of solutions, but they are used for proving uniqueness (see §§ 5.3.4, 5.3.5).

Condition (6.) forms the basis for applying generalized Gronwall's Lemma 6. Indeed, every curve $y : [0,1] \longrightarrow E$ with $q_\varepsilon(y(t{-}h),\, y(t)) \longrightarrow 0$ (for $h \downarrow 0$ and each t) satisfies

$$q_\varepsilon\big(\vartheta(t,z),\ y(t)\big) \ \le\ \limsup_{h\,\downarrow\,0}\ q_\varepsilon\big(\vartheta(t-h,z),\ \ y(t-h)\big).$$

for all $z \in D$ and times $t \in \,]\,0,\ \mathcal{T}_\Theta(\vartheta,z)]$ (due to Lemma 20 in § 5.1).

In the preceding section, we mentioned the formal estimate $(**)$ as starting point and, its general counterpart in $(E,\ D,\ (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ is

$$q_\varepsilon\big(\vartheta(h,\,z),\ \ \tau(h,\,y)\big) \ \ \le\ \ \big(q_\varepsilon(z,y)\ \ +\ \ h\ \ Q_\varepsilon^{\mapsto}(\vartheta,\,\tau)\big)\ \cdot\ e^{\mathrm{const}\,\cdot\,h}$$

for all $z \in D$, $y \in E$, $\varepsilon \in \mathcal{J}$ and small $t > 0$. For realizing this formal inequality, we specify the distance between forward transitions on $(E, D, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ in the following way :

Definition 10.
$\Theta^{\mapsto}(E, D, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ denotes a set of forward transitions on $(E, D, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ supposing

$$Q_\varepsilon^{\mapsto}(\vartheta,\tau)\ :=\ \sup_{\substack{z\,\in\,D\\ y\,\in E}}\ \limsup_{h\,\downarrow\,0}\ \left(\frac{q_\varepsilon\big(\vartheta(h,z),\,\tau(h,y)\big)\ -\ q_\varepsilon(z,y)\,\cdot\,e^{\alpha_\varepsilon^{\mapsto}(\tau)\,h}}{h}\right)^{+} < \infty$$

for all $\vartheta,\tau \in \Theta^{\mapsto}(E, D, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$, $\varepsilon \in \mathcal{J}$.

Remark 11. Using here the parameter $\alpha_\varepsilon^{\mapsto}(\tau)$ of the second argument τ (instead of ϑ) is just for technical reasons. Indeed, it ensures the triangle inequality of $Q_\varepsilon^{\mapsto}$ immediately, i.e. $Q_\varepsilon^{\mapsto}(\vartheta_1,\vartheta_3) \ \le\ Q_\varepsilon^{\mapsto}(\vartheta_1,\vartheta_2) + Q_\varepsilon^{\mapsto}(\vartheta_2,\vartheta_3)$ for any transitions $\vartheta_1,\vartheta_2,\vartheta_3$ on $(E,\ D,\ (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ because for all $z{\in}D$, $y{\in}E$, $t \in [0,1]$, we conclude from $q_\varepsilon(z,z) = 0$ and the triangle inequality

$$\begin{aligned}
&\quad q_\varepsilon\big(\vartheta_1(h,z),\,\vartheta_3(h,y)\big) - q_\varepsilon(z,y)\ \cdot\ e^{\alpha_\varepsilon^{\mapsto}(\vartheta_3)\,h}\\
\le&\quad q_\varepsilon\big(\vartheta_1(h,z),\,\vartheta_2(h,z)\big) - q_\varepsilon(z,z)\ \cdot\ e^{\alpha_\varepsilon^{\mapsto}(\vartheta_2)\,h}\\
&\ + q_\varepsilon\big(\vartheta_2(h,z),\,\vartheta_3(h,y)\big) - q_\varepsilon(z,y)\ \cdot\ e^{\alpha_\varepsilon^{\mapsto}(\vartheta_3)\,h}.
\end{aligned}$$

Moreover, it usually does not impose serious restrictions on applications since the parameter $\alpha_\varepsilon^{\mapsto}(\vartheta)$ is often chosen as a global constant.

These definitions lay the foundations for concluding from generalized Gronwall's Lemma 6 (see Proposition 22) :

Proposition 12. *Let $\vartheta,\tau \in \Theta^{\mapsto}(E, D, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ be forward transitions, $\varepsilon \in \mathcal{J}$, $z \in D$, $y \in E$ and $0 \le h < \mathcal{T}_\Theta(\vartheta, z)$. Then,*

$$q_\varepsilon\big(\vartheta(h,z),\ \tau(h,y)\big)\ \ \le\ \ q_\varepsilon(z,\,y)\ \cdot\ e^{\alpha_\varepsilon^{\mapsto}(\tau)\ h}\ +\ h\ \ \ Q_\varepsilon^{\mapsto}(\vartheta,\tau)\ \ \ \frac{e^{\alpha_\varepsilon^{\mapsto}(\tau)\,h}-1}{\alpha_\varepsilon^{\mapsto}(\tau)\,h}\,.$$

The next step is to generalize the term "mutation" to $(E, D, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$. Considering a curve $x(\cdot) : [0,T[\ \longrightarrow M$ in a metric space (M,d), its mutation $\overset{\circ}{x}(t)$ at time $t{\in}[0,T[$ consists of all transitions ϑ on (M,d) with

$$\limsup_{h\,\downarrow\,0}\ \ \tfrac{1}{h}\ \cdot\ d\big(\vartheta(h,\ x(t)),\ \ \ x(t{+}h)\big)\ \ =\ \ 0$$

according to the definition of Aubin ([2], § 1.2). It reflects the idea of first–order approximation that most concepts of "derivative" start with. For $(E,\ D,\ (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ however, we prefer adapting the criterion to the key estimate of Proposition 12.
So firstly, only elements of D are used in the first argument of q_ε and secondly, a first–order approximation is to have the same effect, roughly speaking, as if the factor $Q_\varepsilon^{\mapsto}(\cdot,\cdot)$ was 0. Thus, a forward transition ϑ

on $(E, D, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ is regarded as a generalized derivative of a curve $x(\cdot) : [0,T[\longrightarrow E$ at time t if for each $\varepsilon \in \mathcal{J}$, there is a parameter $\widehat{\alpha}_\varepsilon^{\mapsto}(t) \geq 0$ with

$$\limsup_{h\downarrow 0} \ \tfrac{1}{h} \left(q_\varepsilon\big(\vartheta\big(h, z\big),\ x(t+h)\big) \ - \ q_\varepsilon(z,\ x(t)) \cdot e^{\widehat{\alpha}_\varepsilon^{\mapsto}(t)\cdot h} \right) \ \leq \ 0$$

for all "test elements" $z \in D$. To minimize the risk of confusion over Aubin's concept and its generalization here, we dispense with a new definition of "mutation" and introduce the term "primitive" instead (in accordance with the more general Definition 24).

Definition 13. *Let* $\vartheta(\cdot) : [0,T[\longrightarrow \Theta^{\mapsto}(E, D, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ *be a given function and, suppose for* $x(\cdot) : [0,T[\longrightarrow (E, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$

1. $\forall \ \ t \in [0,T[, \ \ \varepsilon \in \mathcal{J} \qquad \exists \ \widehat{\alpha}_\varepsilon^{\mapsto}(t) = \widehat{\alpha}_\varepsilon^{\mapsto}(t, x(\cdot), \vartheta(\cdot)) < \infty \ :$
   $$\limsup_{h\downarrow 0} \ \tfrac{1}{h} \left(q_\varepsilon\big(\vartheta(t)\big(h, z\big),\ x(t+h)\big) \ - \ q_\varepsilon(z,\ x(t)) \cdot e^{\widehat{\alpha}_\varepsilon^{\mapsto}(t)\cdot h} \right) \ \leq \ 0,$$
   for all $z \in D$ *and* $\widehat{\alpha}_\varepsilon^{\mapsto}(t) \ \geq \ \alpha_\varepsilon^{\mapsto}(\vartheta(t)) \ \geq \ 0,$
2. $x(\cdot)$ *is uniformly continuous in time direction w.r.t. each* q_ε, *i.e. there is* $\omega_\varepsilon(x, \cdot) :]0,T[\longrightarrow [0,\infty[$ *with* $\limsup_{h\downarrow 0} \ \omega_\varepsilon(x,h) \ = \ 0,$
   $q_\varepsilon\big(x(s),\ x(t)\big) \ \leq \ \omega_\varepsilon(x,\ t-s)$ *for* $0 \leq s < t < T.$

Then $x(\cdot)$ *is a so–called* right–hand forward primitive *of* $\vartheta(\cdot)$, *abbreviated to* $\overset{\circ}{x}(\cdot) \ni \vartheta(\cdot)$.

The additional term "right–hand" indicates that $x(\cdot)$ appears in the second argument of the distances q_ε $(\varepsilon \in \mathcal{J})$.
Forward transitions induce their own primitives. To be more precise, every constant function $\vartheta(\cdot) : [0,1[\longrightarrow \Theta^{\mapsto}(E, D, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ with $\vartheta(\cdot) = \vartheta_0$ has the right–hand forward primitives $[0,1[\longrightarrow E, \ \ t \longmapsto \vartheta_0(t,x)$ with any $x \in E$ — as a consequence of Proposition 12 in a slightly generalized form

$$\begin{aligned} & q_\varepsilon\big(\vartheta(t_1+h, x),\ \tau(t_2+h, y)\big) \\ \leq \ & \Big(q_\varepsilon\big(\vartheta(t_1, x),\ \tau(t_2, y)\big) \ + \ h \ \ Q_\varepsilon^{\mapsto}(\vartheta, \tau)\Big) \ \cdot \ e^{\alpha_\varepsilon^{\mapsto}(\tau)\ h} \end{aligned}$$

(see Proposition 22). This property is easy to extend to piecewise constant functions $[0,T[\longrightarrow \Theta^{\mapsto}(E, D, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ and so it will be the basis for Euler approximations later.

Let us apply now this concept to mutational equations in a generalized form. Correspondingly to ordinary differential equations, the definition of "solution" can be formulated by means of "primitives".

Definition 14. *For* $f : E \times [0,T[\longrightarrow \Theta^{\mapsto}(E, D, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ *given, a map* $x(\cdot) : [0,T[\longrightarrow E$ *is a so–called* right–hand forward solution *of the generalized mutational equation*

$$\overset{\circ}{x}(\cdot) \ \ni \ f(x(\cdot), \cdot)$$

if $x(\cdot)$ *is a right–hand forward primitive of the composition* $f(x(\cdot), \cdot) : [0,T[\longrightarrow \Theta^{\mapsto}(E,\ D,\ (q_\varepsilon))$, *i.e. for each* $\varepsilon \in \mathcal{J}$,

1. $\forall \ \ t \in [0,T[\qquad \exists \ \widehat{\alpha}_\varepsilon^{\mapsto}(t) \ \geq \ \alpha_\varepsilon^{\mapsto}(f(x(t),t)) \ : \qquad \forall \ z \in D$
   $$\limsup_{h\downarrow 0} \tfrac{1}{h} \ \Big(q_\varepsilon\big(f(x(t),t)\big(h, z\big),\ x(t+h)\big) \ - \ q_\varepsilon(z, x(t)) \cdot e^{\widehat{\alpha}_\varepsilon^{\mapsto}(t)\cdot h} \Big) \ \leq 0,$$
2. $x(\cdot)$ *is uniformly continuous in time direction w.r.t. each* q_ε.

Two further aspects of generalizing mutational equations

In the preceding sections, the same feature of an ostensible metric space (E, q) occurred for several times : Considering $q(x,y)$, the first argument x refers to the state at an *earlier* point of time whereas the second argument y represents the *later* element.
In fact, this rule can be extended to the entire concept of right–hand forward solutions. We only need the possibility of distinguishing between the "earlier" and "later" element of E. For this reason, $\widetilde{E} := \mathbb{R} \times E$ with an additional time component is regarded instead of the nonempty set E. The term "timed" and the tilde usually symbolize that the (forward) time direction is taken into consideration by means of a separate real component.

Definition 15. *Set* $\widetilde{E} := \mathbb{R} \times E$. $\widetilde{q} : \widetilde{E} \times \widetilde{E} \longrightarrow [0, \infty[$ *fulfills the so–called* timed triangle inequality *if for every* $(r,x),\ (s,y),\ (t,z) \in \widetilde{E}$ *with* $r \le s \le t$,

$$\widetilde{q}\big((r,x),\ (t,z)\big) \ \le\ \widetilde{q}\big((r,x),\ (s,y)\big) + \widetilde{q}\big((s,y),\ (t,z)\big).$$

$\widetilde{q} : \widetilde{E} \times \widetilde{E} \longrightarrow [0, \infty[$ *is called* timed ostensible metric *on* $\widetilde{E}$ *if it satisfies*

$$\widetilde{q}\big((t,z),\ (t,z)\big) \ =\ 0$$
$$\widetilde{q}\big((r,x),\ (t,z)\big) \ \le\ \widetilde{q}\big((r,x),\ (s,y)\big) + \widetilde{q}\big((s,y),\ (t,z)\big)$$

for all $(r,x),\ (s,y),\ (t,z) \in \widetilde{E}$ *with* $r \le s \le t$.
$(\widetilde{E}, \widetilde{q})$ *is then called* timed ostensible metric space.

Every ostensible metric q on E induces a *timed* ostensible metric $\widetilde{q}$ on $\widetilde{E} \overset{\text{Def.}}{=} \mathbb{R} \times E$ according to $\widetilde{q}\big((s,x),\ (t,y)\big) := |t-s| + q(x,y)$. Thus, all statements about ostensible metric spaces result immediately from their more general counterparts about timed ostensible metric spaces.
From the topological point of view, there is only one additional condition to suppose, i.e. the convergence with respect to the timed ostensible metric implies the convergence of the time components.

Definition 16. *Let E be a nonempty set,* $\widetilde{E} \overset{\text{Def.}}{=} \mathbb{R} \times E$, $\widetilde{q} : \widetilde{E} \times \widetilde{E} \longrightarrow [0, \infty[$. $(\widetilde{E}, \widetilde{q})$ *is called* time continuous *if every sequence* $(\widetilde{x}_n = (t_n, x_n))_{n \in \mathbb{N}}$ *in* $\widetilde{E}$ *and element* $\widetilde{x} = (t,x) \in \widetilde{E}$ *with* $\widetilde{q}(\widetilde{x}_n, \widetilde{x}) \longrightarrow 0 \ (n \longrightarrow \infty)$ *fulfill* $t_n \longrightarrow t \ \ (n \longrightarrow \infty)$.

The second aspect of generalization is related to the modified semigroup condition on transitions, i.e. condition (2.) of Definition 9. Using the Landau symbol $o(\cdot)$, it demands for every $\widetilde{x} \in \widetilde{E},\ t \in [0,1[$ and $\varepsilon \in \mathcal{J}$

$$\wedge \begin{cases} \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t, \widetilde{x})),\ \ \widetilde{\vartheta}(t+h,\ \widetilde{x})\big) \ =\ o(h) \\ \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t+h,\ \widetilde{x}),\ \ \widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t, \widetilde{x}))\big) \ =\ o(h) \end{cases} \qquad \text{for } h \downarrow 0.$$

In short, the main idea now is to replace $o(h)$ with the other Landau symbol $O(h)$. Strictly speaking, each $\widetilde{\vartheta}$ has a parameter $\gamma_\varepsilon(\widetilde{\vartheta}) \in [0, \infty[$ (depending only on ε) with

$$\wedge \begin{cases} \limsup\limits_{h \downarrow 0}\ \frac{1}{h} \cdot \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t, \widetilde{x})),\ \ \widetilde{\vartheta}(t+h,\ \widetilde{x})\big) \ \le\ \gamma_\varepsilon(\widetilde{\vartheta}) \\ \limsup\limits_{h \downarrow 0}\ \frac{1}{h} \cdot \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t+h,\ \widetilde{x}),\ \ \widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t, \widetilde{x}))\big) \ \le\ \gamma_\varepsilon(\widetilde{\vartheta}) \end{cases}$$

for all $\widetilde{x} \in \widetilde{E},\ t \in [0,1[$ and each $\varepsilon \in \mathcal{J}$. So the challenge is to incorporate this parameter in the concept of "timed" right–hand forward solutions.
The dependence of $\gamma_\varepsilon(\widetilde{\vartheta})$ on $\varepsilon \in \mathcal{J}$ exemplifies an additional feature for

characterizing $\widetilde{\vartheta}$. Assuming $0 \in \overline{\mathcal{J}}$, we choose the asymptotic behavior of $\gamma_\varepsilon(\widetilde{\vartheta})$ (for $\varepsilon \longrightarrow 0$) as a further criterion and specify "timed forward transitions" on $(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ in final Definition 17.

Analytically speaking, $\gamma_\varepsilon(\cdot) \geq 0$ gives the opportunity to introduce an additional limit process that follows the process of first–order approximation. This might be useful for multi–scale problems, for example, although they are not considered in this paper.
However, $\gamma_\varepsilon(\cdot)$ and its upper bounds are also of direct use for semilinear evolution equations mentioned in Proposition 3. Its continuity assumption

$$\sum_{k=1}^{\infty} 2^{-k} \; q_{j_k}\big(f(x,t_1),\ f(y,t_2)\big) \;\leq\; \widehat{\omega}\Big(\sum_{k=1}^{\infty} 2^{-k} \; \tfrac{q_{j_k}(x,y)}{1+q_{j_k}(x,y)} + |t_2 - t_1|\Big)$$

(for all $x, y \in X$ and $t_1, t_2 \in [0,T]$ with a modulus $\widehat{\omega}(\cdot)$ of continuity) was to take more than one pseudo–metric $q_j \stackrel{\text{Def.}}{=} q_{v'_j}$ $(j \in \mathcal{J} = \{j_1, j_2, j_3 \ldots\})$ into account.
The corresponding parameters $\alpha^{\hookrightarrow}(\cdot)$ are closely related with the eigenvalues of the infinitesimal generator A (as mentioned in § 2). For this technical reason, we consider only a finite number of pseudo–metrics q_j simultaneously and define for $x, y \in X$, $n \in \mathbb{N}$

$$p_n(x,y) \;:=\; \sum_{k=1}^{n} 2^{-k} \; \tfrac{q_{j_k}(x,y)}{1+q_{j_k}(x,y)}\,, \qquad P_n(x,y) \;:=\; \sum_{k=1}^{n} 2^{-k} \; q_{j_k}(x,y)$$

Obviously, each p_n is a pseudo–metric on the reflexive Banach space X, but the preceding continuity assumption (of Proposition 3) implies merely

$$\begin{aligned} P_n\big(f(x,t_1),\ f(y,t_2)\big) &\leq \widehat{\omega}\Big(\, p_n(x,y) + \sum_{k=n+1}^{\infty} 2^{-k} \; \tfrac{q_{j_k}(x,y)}{1+q_{j_k}(x,y)} + |t_2 - t_1|\,\Big) \\ &\leq \widehat{\omega}\big(\, p_n(x,y) + 2^{-n} + |t_2 - t_1|\,\big), \end{aligned}$$

i.e. the continuity of the right–hand side with respect to P_n, p_n is not really guaranteed in the way we need *without* introducing the parameter $\gamma_n(\cdot)$.

5. Timed right–hand forward solutions of mutational equations

Now so–called *timed forward transitions* $\widetilde{\vartheta}$ of order p on $(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ are defined precisely in § 5.1. In § 5.2, the definition of *timed right–hand forward primitive* is formulated and, we present three ways for estimating the distance between a transition and a primitive. § 5.3 deals with *timed right–hand forward solutions* of generalized mutational equations : definition, stability, existence and estimates.

General assumptions for § 5. Let E be a nonempty set, $D \subset E$, $p \in \mathbb{R}$ and set $\widetilde{E} := \mathbb{R} \times E$, $\widetilde{D} := \mathbb{R} \times D$, $\pi_1 : \widetilde{E} \longrightarrow \mathbb{R}$, $(t,x) \longmapsto t$. $\mathcal{J} \subset [0,1]^\kappa$ abbreviates a countable index set with $\kappa \in \mathbb{N}$, $0 \in \overline{\mathcal{J}}$.
Furthermore we assume for each $\widetilde{q}_\varepsilon : \widetilde{E} \times \widetilde{E} \longrightarrow [0, \infty[$ $(\varepsilon \in \mathcal{J})$:

1. timed triangle inequality,
2. time continuity, i.e. every sequence $(\widetilde{x}_n = (t_n, x_n))_{n \in \mathbb{N}}$ in $\widetilde{E}$ and $\widetilde{x} = (t,x) \in \widetilde{E}$ with $\widetilde{q}_\varepsilon(\widetilde{x}_n, \widetilde{x}) \longrightarrow 0$ $(n \longrightarrow \infty)$ fulfill $t_n \longrightarrow t$ $(n \longrightarrow \infty)$,
3. reflexivity on $\widetilde{D}$, i.e. $\widetilde{q}_\varepsilon(\widetilde{z}, \widetilde{z}) = 0$ for all $\widetilde{z} \in \widetilde{D}$.

5.1. Timed forward transitions

Definition 17. *A map* $\widetilde{\vartheta} : [0,1] \times \widetilde{E} \longrightarrow \widetilde{E}$ *is a so–called* timed forward transition of order p *on* $(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ *if it fulfills for each* $\varepsilon \in \mathcal{J}$

1. $\widetilde{\vartheta}(0,\cdot) = \mathrm{Id}_{\widetilde{E}}$,

2. $\exists\ \gamma_\varepsilon(\widetilde{\vartheta}) \geq 0 :\quad \limsup_{\varepsilon \longrightarrow 0} \varepsilon^p \cdot \gamma_\varepsilon(\widetilde{\vartheta}) = 0$ *and*

$$\limsup_{h\downarrow 0} \tfrac{1}{h} \cdot \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t,\widetilde{x})),\ \ \widetilde{\vartheta}(t+h,\ \widetilde{x})\big) \leq \gamma_\varepsilon(\widetilde{\vartheta}) \quad \forall\ \widetilde{x}\in\widetilde{E},\ t\in[0,1[,$$
$$\limsup_{h\downarrow 0} \tfrac{1}{h} \cdot \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t+h,\ \widetilde{x}),\ \ \widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t,\widetilde{x}))\big) \leq \gamma_\varepsilon(\widetilde{\vartheta}) \quad \forall\ \widetilde{x}\in\widetilde{E},\ t\in[0,1[,$$

3. $\exists\ \alpha_\varepsilon^{\mapsto}(\widetilde{\vartheta}) : \sup_{\substack{\widetilde{z}\,\in\,\widetilde{D},\ \widetilde{y}\,\in\widetilde{E} \\ \pi_1\,\widetilde{z}\,\leq\,\pi_1\,\widetilde{y}}} \limsup_{h\downarrow 0} \left(\frac{\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(h,\widetilde{z}),\ \widetilde{\vartheta}(h,\widetilde{y})\big) - \widetilde{q}_\varepsilon(\widetilde{z},\widetilde{y}) - \gamma_\varepsilon(\widetilde{\vartheta})\ h}{h\ \ \big(\widetilde{q}_\varepsilon(\widetilde{z},\widetilde{y}) + \gamma_\varepsilon(\widetilde{\vartheta})\ h\big)} \right)^+ \leq \alpha_\varepsilon^{\mapsto}(\widetilde{\vartheta})$

4. $\exists\ \beta_\varepsilon(\widetilde{\vartheta}) :\ ]0,1] \longrightarrow [0,\infty[:$ *nondecreasing,* $\lim_{h\downarrow 0} \beta_\varepsilon(\widetilde{\vartheta})(h) = 0$,

$$\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(s,\widetilde{x}),\ \widetilde{\vartheta}(t,\widetilde{x})\big) \leq \beta_\varepsilon(\widetilde{\vartheta})(t-s) \qquad \forall\ s<t\leq 1,\ \ \widetilde{x}\in\widetilde{E},$$

5. $\forall\ \widetilde{z} \in \widetilde{D}\ \ \exists\ \mathcal{T}_\Theta = \mathcal{T}_\Theta(\widetilde{\vartheta},\widetilde{z}) \in\ ]0,1] :\ \ \widetilde{\vartheta}(t,\widetilde{z})\in\widetilde{D} \quad \forall\ t\in[0,\mathcal{T}_\Theta]$,

6. $\limsup_{h\downarrow 0} \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t-h,\ \widetilde{z}),\ \widetilde{y}\big) \geq \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t,\widetilde{z}),\ \widetilde{y}\big) \quad \forall\ \widetilde{z}\in\widetilde{D},\ \widetilde{y}\in\widetilde{E},\ t\leq\mathcal{T}_\Theta$ $(t+\pi_1\,\widetilde{z} \leq \pi_1\,\widetilde{y})$,

7. $\widetilde{\vartheta}\big(h,\ (t,x)\big) \in \{t+h\}\times E \qquad \forall\ (t,x)\in\widetilde{E},\ h\in[0,1]$.

$\widetilde{\Theta}_p^{\mapsto}(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ *denotes a set of timed forward transitions on* $(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon))$ *assuming*

$$\widetilde{Q}_\varepsilon^{\mapsto}(\widetilde{\vartheta},\widetilde{\tau}) := \sup_{\substack{\widetilde{z}\,\in\,\widetilde{D},\ \widetilde{y}\,\in\widetilde{E} \\ \pi_1\,\widetilde{z}\,\leq\,\pi_1\,\widetilde{y}}} \limsup_{h\downarrow 0} \left(\frac{\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(h,\widetilde{z}),\ \widetilde{\tau}(h,\widetilde{y})\big) - \widetilde{q}_\varepsilon(\widetilde{z},\widetilde{y}) \cdot e^{\alpha_\varepsilon^{\mapsto}(\widetilde{\tau})\,h}}{h} \right)^+$$

to be finite for all $\widetilde{\vartheta},\widetilde{\tau} \in \widetilde{\Theta}_p^{\mapsto}(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}}),\ \ \varepsilon\in\mathcal{J}$.

Remark 18. (i) A set $\widetilde{E} \neq \emptyset$ supplied with only one function $\widetilde{q} : \widetilde{E}\times\widetilde{E} \longrightarrow [0,\infty[$ can be regarded as easy (but important) example by setting $\mathcal{J} := \{0\}$, $\widetilde{q}_0 := \widetilde{q}$.
Considering a timed forward transition $\widetilde{\vartheta} : [0,1]\times\widetilde{E} \longrightarrow \widetilde{E}$ of order 0, the condition $\limsup_{\varepsilon\longrightarrow 0} \varepsilon^0 \cdot \gamma_\varepsilon(\widetilde{\vartheta}) = 0$ means $0 = 0^0 \cdot \gamma_0(\widetilde{\vartheta}) = \gamma_0(\widetilde{\vartheta})$ — due to the definition $0^0 \overset{\text{Def.}}{=} 1$.
So it leads to the key property for all $\widetilde{x}\in\widetilde{E},\ t\in[0,1[$.

$$\wedge \begin{cases} \limsup_{h\downarrow 0} \frac{1}{h}\ \widetilde{q}\big(\widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t,\widetilde{x})),\ \ \widetilde{\vartheta}(t+h,\ \widetilde{x})\big) = 0 \\ \limsup_{h\downarrow 0} \frac{1}{h}\ \widetilde{q}\big(\widetilde{\vartheta}(t+h,\ \widetilde{x}),\ \ \widetilde{\vartheta}(h,\ \widetilde{\vartheta}(t,\widetilde{x}))\big) = 0 \end{cases}$$

Then many of the following results do not depend on ε or $\gamma_\varepsilon(\cdot)$ (and its upper bounds) explicitly. So we do not mention the index ε there any longer and abbreviate the corresponding set of timed transitions (of order 0) as $\widetilde{\Theta}^{\mapsto}(\widetilde{E},\widetilde{D},\widetilde{q})$. In particular, the transitions in metric spaces (introduced by Aubin in [2], [4]) prove to be a special case.

(ii) For a set $E\neq\emptyset$, a family $q_\varepsilon : E\times E \longrightarrow [0,\infty[\ \ (\varepsilon\in\mathcal{J})$ and $p\in\mathbb{R}$ given, let $\widetilde{q}_\varepsilon : \widetilde{E}\times\widetilde{E} \longrightarrow [0,\infty[$ be defined similarly to the remark after Definition 15, i.e.

$$\widetilde{q}_\varepsilon\big((s,x),\ (t,y)\big) := f(\varepsilon)\ |s-t| + q_\varepsilon(x,y) \qquad \text{for all } (s,x),\ (t,y)\in\widetilde{E}.$$

with a function $f(\varepsilon) = o(\varepsilon^p) > 0$ for $\varepsilon\downarrow 0$. Then every $\vartheta : [0,1]\times E \longrightarrow E$ satisfying conditions (1.)–(6.) for $(E,D,(q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ induces a *timed* forward transition $\widetilde{\vartheta} : [0,1]\times\widetilde{E} \longrightarrow \widetilde{E}$ of order p on $(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon))$ by

$$\widetilde{\vartheta}\big(h,\ (t,x)\big) := \big(t+h,\ \vartheta(h,x)\big) \qquad \text{for all } (t,x)\in\widetilde{E},\ \ h\in[0,1].$$

As a consequence, the following statements about $\widetilde{\Theta}_p^{\hookrightarrow}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ can be applied to their counterparts without separate time component very easily. Correspondingly these functions $\vartheta : [0,1] \times E \longrightarrow E$ are called *forward transitions of order* p on $(E, D, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$ and abbreviated as $\Theta_p^{\hookrightarrow}(E, D, (q_\varepsilon)_{\varepsilon\in\mathcal{J}})$.

(iii) Condition (6.), the timed triangle inequality and the continuity of $\widetilde{\vartheta}(\cdot, \widetilde{z})$ imply $\limsup_{h\downarrow 0} \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t-h, \widetilde{z}),\ \widetilde{y}\big) = \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t, \widetilde{z}),\ \widetilde{y}\big)$ for all $\widetilde{\vartheta} \in \widetilde{\Theta}_p^{\hookrightarrow}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$, $\widetilde{z} \in \widetilde{D}$, $\widetilde{y} \in \widetilde{E}$, $0 < t < T_\Theta(\widetilde{\vartheta}, \widetilde{z})$, $\varepsilon \in \mathcal{J}$ with $t + \pi_1\, \widetilde{z} \le \pi_1\, \widetilde{y}$.

(iv) $\widetilde{Q}_\varepsilon^{\hookrightarrow} : \widetilde{\Theta}_p^{\hookrightarrow}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)) \times \widetilde{\Theta}_p^{\hookrightarrow}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)) \longrightarrow [0, \infty[$ satisfies the triangle inequality for the same reasons as in Remark 11.

(v) As an alternative, we can follow exactly the track of this section with the condition $\pi_1\, \widetilde{z} < \pi_1\, \widetilde{y}$ on test elements $\widetilde{z} \in \widetilde{D}$ instead of $\pi_1\, \widetilde{z} \le \pi_1\, \widetilde{y}$ (for example, in the definitions of $\alpha_\varepsilon^{\hookrightarrow}(\widetilde{\vartheta})$, $\widetilde{Q}_\varepsilon^{\hookrightarrow}(\widetilde{\vartheta}, \widetilde{\tau})$). However the equivalence of these two modifications is not obvious in general.

Now an abbreviation for continuous functions of time is introduced. The symbol $\rightarrow$ is to remind us of considering the forward time direction :

Definition 19. *Let $J \subset \mathbb{R}$ be nonempty, $D \subset E \neq \emptyset$, $q : E \times E \longrightarrow [0, \infty[$.*
$UC^{\rightarrow}(J, E, q)$ abbreviates the set of uniformly continuous functions $f : J \longrightarrow E$ in the sense that there is a nondecreasing $\omega(f, \cdot) :]0, \infty[\longrightarrow [0, \infty[$ with

$$\limsup_{h\downarrow 0}\ \omega(f, h) = 0$$

and $\quad \forall\ s, t \in J : \quad s < t \implies q\big(f(s), f(t)\big) \le \omega(f, t-s).$
Such function $\omega(f, \cdot)$ is called modulus of continuity *of $f(\cdot)$.*

Now Gronwall's Lemma 6 for semicontinuous functions proves to be the main tool. For applying this idea to distances like $\psi_\varepsilon(t) := \widetilde{q}_\varepsilon(\widetilde{\vartheta}(t, \widetilde{z}), \widetilde{y}(t))$ with a function $\widetilde{y}(\cdot) : [0, T] \longrightarrow \widetilde{E}$, we have to ensure the semicontinuity property $\psi_\varepsilon(t) \le \limsup_{h\downarrow 0} \psi_\varepsilon(t-h)$. It is the key point for using condition (6.) of Definition 17.

Lemma 20. *Let $\widetilde{\vartheta} \in \widetilde{\Theta}_p^{\hookrightarrow}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$, $\varepsilon \in \mathcal{J}$, $\widetilde{z} \in \widetilde{D}$, $0 < t < T_\Theta(\widetilde{\vartheta}, \widetilde{z})$, $\widetilde{y}(\cdot) : [0, t] \longrightarrow \widetilde{E}$ satisfy $\widetilde{q}_\varepsilon\big(\widetilde{y}(t-h),\ \widetilde{y}(t)\big) \longrightarrow 0$ for $h \downarrow 0$ and $\pi_1\, \widetilde{\vartheta}(\cdot, \widetilde{z}) \le \pi_1\, \widetilde{y}(\cdot)$ increasing.*
Then, $\quad \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t, \widetilde{z}),\ \widetilde{y}(t)\big) \le \limsup_{h\downarrow 0}\ \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t-h, \widetilde{z}),\ \widetilde{y}(t-h)\big).$

Proof. Due to condition (6.) of Def. 17 and the timed triangle inequality,

$$\begin{aligned}
\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t, \widetilde{z}),\ \widetilde{y}(t)\big) &\le \limsup_{h\downarrow 0}\ \ \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t-h,\ \widetilde{z}),\ \widetilde{y}(t)\big) \\
&\le \limsup_{h\downarrow 0}\ \Big(\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t-h,\ \widetilde{z}),\ \widetilde{y}(t-h)\big) + \widetilde{q}_\varepsilon\big(\widetilde{y}(t-h),\ \widetilde{y}(t)\big)\Big) \\
&\le \limsup_{h\downarrow 0}\ \ \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t-h,\ \widetilde{z}),\ \widetilde{y}(t-h)\big) + 0. \qquad \square
\end{aligned}$$

As another application of Gronwall's Lemma 6, we consider $\widetilde{Q}_\varepsilon^{\mapsto}(\widetilde{\vartheta},\widetilde{\vartheta})$ for any $\widetilde{\vartheta}$. Although $\widetilde{Q}_\varepsilon^{\mapsto} : \widetilde{\Theta}_p^{\mapsto}(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon)) \times \widetilde{\Theta}_p^{\mapsto}(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon)) \longrightarrow [0,\infty[$ satisfies the triangle inequality, it need not be reflexive, i.e. we cannot expect $\widetilde{Q}_\varepsilon^{\mapsto}(\widetilde{\vartheta},\widetilde{\vartheta}) = 0$ for every $\widetilde{\vartheta} \in \widetilde{\Theta}_p^{\mapsto}(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ in general. The parameter $\gamma_\varepsilon(\widetilde{\vartheta})$ provides an upper bound :

Lemma 21. *Every timed transition $\widetilde{\vartheta} \in \widetilde{\Theta}_p^{\mapsto}(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ fulfills*
$$\widetilde{Q}_\varepsilon^{\mapsto}(\widetilde{\vartheta},\widetilde{\vartheta}) \;\le\; 3\,\gamma_\varepsilon(\widetilde{\vartheta}).$$

Proof is based on Gronwall's Lemma 6 applied to
$$\varphi_\varepsilon : \; [0,1] \longrightarrow [0,\infty[, \qquad h \longmapsto \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(h,\widetilde{z}),\; \widetilde{\vartheta}(h,\widetilde{y})\big)$$
with any $\widetilde{z}\in\widetilde{D},\ \widetilde{y}\in\widetilde{E}$ $(\pi_1\,\widetilde{z} \le \pi_1\,\widetilde{y})$. The preceding Lemma 20 guarantees
$$\varphi_\varepsilon(h) \;\le\; \limsup_{k\downarrow 0}\ \varphi_\varepsilon(h-k).$$
Now choose $h \in [0, \mathcal{T}_\Theta(\widetilde{\vartheta},\widetilde{z})[,\ \delta > 0$ arbitrarily and we obtain for any $k > 0$ small enough
$$\begin{aligned}
&\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(h{+}k,\widetilde{z}),\ \widetilde{\vartheta}(k,\ \widetilde{\vartheta}(h,\widetilde{z}))\big) && \le (\gamma_\varepsilon(\widetilde{\vartheta}) + \delta)\ k,\\
&\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(k,\ \widetilde{\vartheta}(h,\widetilde{y})),\ \widetilde{\vartheta}(h{+}k,\widetilde{y})\big) && \le (\gamma_\varepsilon(\widetilde{\vartheta}) + \delta)\ k,\\
&\frac{\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(k,\widetilde{\vartheta}(h,\widetilde{z})),\ \widetilde{\vartheta}(k,\widetilde{\vartheta}(h,\widetilde{y}))\big) - \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(h,\widetilde{z}),\ \widetilde{\vartheta}(h,\widetilde{y})\big) - \gamma_\varepsilon(\widetilde{\vartheta})\ k}{k\ \cdot\ \big\{\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(h,\widetilde{z}),\ \widetilde{\vartheta}(h,\widetilde{y})\big)\ +\ \gamma_\varepsilon(\widetilde{\vartheta})\ k\big\}} && \le\ \alpha_\varepsilon^{\mapsto}(\widetilde{\vartheta}) + \delta.
\end{aligned}$$
So the timed triangle inequality leads to
$$\begin{aligned}
\varphi_\varepsilon(h{+}k) =\ & \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(h{+}k,\widetilde{z}),\ \widetilde{\vartheta}(h{+}k,\widetilde{y})\big)\\
\le\ & \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(h{+}k,\widetilde{z}),\ \widetilde{\vartheta}(k,\ \widetilde{\vartheta}(h,\widetilde{z}))\big)\\
& + \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(k,\ \widetilde{\vartheta}(h,\widetilde{z})),\ \widetilde{\vartheta}(k,\ \widetilde{\vartheta}(h,\widetilde{y}))\big)\\
& + \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(k,\ \widetilde{\vartheta}(h,\widetilde{y})),\ \widetilde{\vartheta}(h{+}k,\widetilde{y})\big)\\
\le\ & 2\ (\gamma_\varepsilon(\widetilde{\vartheta}) + \delta)\ k\\
& +\ (\alpha_\varepsilon^{\mapsto}(\widetilde{\vartheta}) + \delta)\ k\ \big(\varphi_\varepsilon(h) + \gamma_\varepsilon(\widetilde{\vartheta})\cdot k\big)\ +\ \varphi_\varepsilon(h)\ +\ \gamma_\varepsilon(\widetilde{\vartheta})\cdot k,
\end{aligned}$$
i.e. $$\limsup_{k\downarrow 0}\ \frac{\varphi_\varepsilon(h{+}k) - \varphi_\varepsilon(h)}{k}\ \le\ (\alpha_\varepsilon^{\mapsto}(\widetilde{\vartheta}){+}\delta)\cdot\varphi_\varepsilon(h)\ +\ 3\ (\gamma_\varepsilon(\widetilde{\vartheta}){+}\delta).$$
Since $\delta > 0$ is arbitrarily small, we conclude from Gronwall's Lemma 6
$$\varphi_\varepsilon(h) \le \varphi_\varepsilon(0)\ \cdot\ e^{\alpha_\varepsilon^{\mapsto}(\widetilde{\vartheta})\cdot h} + 3\ \gamma_\varepsilon(\widetilde{\vartheta})\ \frac{e^{\alpha_\varepsilon^{\mapsto}(\widetilde{\vartheta})\cdot h} - 1}{\alpha_\varepsilon^{\mapsto}(\widetilde{\vartheta})}$$
$$\limsup_{h\downarrow 0}\ \frac{\varphi_\varepsilon(h) - \varphi_\varepsilon(0)\cdot e^{\alpha_\varepsilon^{\mapsto}(\widetilde{\vartheta})\cdot h}}{h} \le 3\ \gamma_\varepsilon(\widetilde{\vartheta}).$$
□

The final result of this subsection is the upper estimate of the distance between two points while evolving along different timed transitions. In comparison with transitions on metric spaces (M,d) (according to [2, Aubin 99]), it generalizes the key estimate $(*)$ mentioned § 2. So we continue this approach and use the inequality as a motivation for defining "primitives" and "solutions" in the next subsections.

Proposition 22. *Let $\widetilde{\vartheta},\widetilde{\tau} \in \widetilde{\Theta}_p^{\mapsto}(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ be timed forward transitions, $\varepsilon\in\mathcal{J}$, $\widetilde{z}\in\widetilde{D}$, $\widetilde{y}\in\widetilde{E}$ and $0 \le t_1 \le t_2 \le 1$, $h \ge 0$ (with $\pi_1\,\widetilde{z} \le \pi_1\,\widetilde{y}$, $t_1 + h < \mathcal{T}_\Theta(\widetilde{\vartheta},\widetilde{z})$). Then the following estimate holds*
$$\begin{aligned}
\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t_1{+}h,\widetilde{z}),\ \widetilde{\tau}(t_2{+}h,\widetilde{y})\big) \le\ & \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t_1,\widetilde{z}),\ \widetilde{\tau}(t_2,\widetilde{y})\big)\ \cdot e^{\alpha_\varepsilon^{\mapsto}(\widetilde{\tau})\ h}\\
& +\ h\ \big(\widetilde{Q}_\varepsilon^{\mapsto}(\widetilde{\vartheta},\widetilde{\tau}){+}\gamma_\varepsilon(\widetilde{\vartheta}){+}\gamma_\varepsilon(\widetilde{\tau})\big)\ \frac{e^{\alpha_\varepsilon^{\mapsto}(\widetilde{\tau})\,h}-1}{\alpha_\varepsilon^{\mapsto}(\widetilde{\tau})\,h}\,.
\end{aligned}$$

Proof. The auxiliary function $\varphi_\varepsilon : h \longmapsto \widetilde{q}_\varepsilon(\widetilde{\vartheta}(t_1+h, \widetilde{z}),\ \widetilde{\tau}(t_2+h, \widetilde{y}))$ has the semicontinuity property $\varphi_\varepsilon(h) \le \limsup_{k\downarrow 0} \varphi_\varepsilon(h-k)$ due to the assumptions of $\widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ and the preceding Lemma 20. Moreover it fulfills for any $h \in [0,1[$ with $t_1 + h < \mathcal{T}_\Theta(\widetilde{\vartheta}, \widetilde{z})$

$$\limsup_{k\downarrow 0} \frac{\varphi_\varepsilon(h+k) - \varphi_\varepsilon(h)}{k} \le \alpha_\varepsilon^{\mapsto}(\widetilde{\tau}) \cdot \varphi_\varepsilon(h) + \widetilde{Q}_\varepsilon^{\mapsto}(\widetilde{\vartheta}, \widetilde{\tau}) + \gamma_\varepsilon(\widetilde{\vartheta}) + \gamma_\varepsilon(\widetilde{\tau}).$$

Indeed, for all $k > 0$ sufficiently small, the timed triangle inequality leads to

$$\begin{aligned}
\varphi_\varepsilon(h+k) \le\ & \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t_1+h+k,\ \widetilde{z}),\ \ \widetilde{\vartheta}(k,\ \widetilde{\vartheta}(t_1+h,\ \widetilde{z}))\big) \\
& + \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(k,\ \widetilde{\vartheta}(t_1+h, \widetilde{z})),\ \ \widetilde{\tau}(k,\ \widetilde{\tau}(t_2+h,\ \widetilde{y}))\big) \\
& + \widetilde{q}_\varepsilon\big(\widetilde{\tau}(k,\ \widetilde{\tau}(t_2+h, \widetilde{y})),\ \ \widetilde{\tau}(t_2+h+k,\ \ \widetilde{y})\ \big) \\
\le\ & \gamma_\varepsilon(\widetilde{\vartheta})\ k + \widetilde{Q}_\varepsilon^{\mapsto}(\widetilde{\vartheta}, \widetilde{\tau}) \cdot k + \varphi_\varepsilon(h)\ e^{\alpha_\varepsilon^{\mapsto}(\widetilde{\tau})\ k} + \gamma_\varepsilon(\widetilde{\tau})\ k + o(k)
\end{aligned}$$

since $t_1 + h + k < \mathcal{T}_\Theta(\widetilde{\vartheta}, \widetilde{z})$ implies $\widetilde{\vartheta}(t_1+h, \widetilde{z}),\ \widetilde{\vartheta}(t_1+h+k, \widetilde{z}) \in \widetilde{D}$. Thus the claim results from Gronwall's Lemma 6. □

Remark 23. If $\alpha_\varepsilon^{\mapsto}(\widetilde{\tau}) = 0$, then the corresponding inequality is

$$\begin{aligned}
& \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t_1+h, \widetilde{z}),\ \widetilde{\tau}(t_2+h, \widetilde{y})\big) \\
\le\ & \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t_1, \widetilde{z}),\ \widetilde{\tau}(t_2, \widetilde{y})\big) + \big(\widetilde{Q}_\varepsilon^{\mapsto}(\widetilde{\vartheta}, \widetilde{\tau}) + \gamma_\varepsilon(\widetilde{\vartheta}) + \gamma_\varepsilon(\widetilde{\tau})\big) \cdot t.
\end{aligned}$$

5.2. Timed right–hand forward primitives

Definition 24. *The curve* $\widetilde{x} : [0,T[\longrightarrow (\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ *is called* timed right–hand forward primitive of *a map* $\widetilde{\vartheta} : [0,T[\longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$, *abbreviated to* $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{\vartheta}(\cdot)$, *if for each* $\varepsilon \in \mathcal{J}$,

1. $\forall\ \ t \in [0,T[\qquad \exists\ \widehat{\alpha}_\varepsilon^{\mapsto}(t) \ge 0,\ \widehat{\gamma}_\varepsilon(t) \ge 0\ :$

   $\widehat{\alpha}_\varepsilon^{\mapsto}(t) \ge \alpha_\varepsilon^{\mapsto}(\vartheta(t)), \quad \widehat{\gamma}_\varepsilon(t) \ge \gamma_\varepsilon(\vartheta(t)), \quad \limsup_{\varepsilon'\downarrow 0}\ \varepsilon'^p \cdot \widehat{\gamma}_{\varepsilon'}(t) = 0,$

   $\limsup_{h\downarrow 0} \frac{1}{h} \Big(\widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t)\,(h, \widetilde{z}),\ \widetilde{x}(t+h)\big) - \widetilde{q}_\varepsilon(\widetilde{z},\ \widetilde{x}(t)) \cdot e^{\widehat{\alpha}_\varepsilon^{\mapsto}(t)\cdot h}\Big) \le \widehat{\gamma}_\varepsilon(t),$

   for all $\widetilde{z} \in \widetilde{D}$ *with* $\pi_1\,\widetilde{z} \le \pi_1\,\widetilde{x}(t)$,

2. $\widetilde{x}(\cdot) \in UC^{\rightarrow}([0,T[, \widetilde{E}, \widetilde{q}_\varepsilon)$,

3. $\pi_1\,\widetilde{x}(t) = t + \pi_1\,\widetilde{x}(0) \qquad$ *for all* $t \in [0,T[$.

Remark 25. Let $\widetilde{x}(\cdot) : [0,T[\longrightarrow \widetilde{E}$ be a timed right–hand forward primitive of $\widetilde{\vartheta} : [0,T[\longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$. For any $t \in]0,T[$, the map $\widetilde{x}(t+\cdot) : [0, T-t[\longrightarrow \widetilde{E}$ is a timed right–hand forward primitive of $\widetilde{\vartheta}(t+\cdot)$. From now on we skip the attributes 'timed', 'right–hand', 'forward' of primitives in this subsection.

Remark 26. Timed transitions induce their own primitives — as a direct consequence of Definition 17 and Proposition 22. Correspondingly, each piecewise constant function $\widetilde{\vartheta} : [0,T[\longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ has a primitive that is defined piecewise as well.

Now three ideas are presented how to estimate the distance between a primitive and a point evolving along a timed transition. As an obstacle though, all preceding definitions have in common that only points of $\widetilde{D}$ usually appear in the first argument of $\widetilde{q}_\varepsilon$. So essentially, we have two possibilities : Either restricting ourselves to the comparison with elements of $\widetilde{D}$ (as in Proposition 27) or using auxiliary functions for the distance (as in Propositions 28, 30).

Proposition 27. *Suppose* $\widetilde{\psi} \in \widetilde{\Theta}^{\mapsto}_p(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$, $\widetilde{z} \in \widetilde{D}$, $t_1 \in [0,1[$, $t_2 \in [0,T[$. *Let the curve* $\widetilde{x}(\cdot) : [0,T[\longrightarrow \widetilde{E}$ *be a timed primitive of* $\widetilde{\vartheta}(\cdot) : [0,T[\longrightarrow \widetilde{\Theta}^{\mapsto}_p(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ *such that for each* $\varepsilon \in \mathcal{J}$,

$$\wedge \begin{cases} \widehat{\alpha}^{\mapsto}_\varepsilon(\cdot, \widetilde{x}, \widetilde{\vartheta}) & \le\ M_\varepsilon(\cdot), \\ \widehat{\gamma}_\varepsilon(\cdot, \widetilde{x}, \widetilde{\vartheta}) & \le\ R_\varepsilon(\cdot), \\ \widetilde{Q}^{\mapsto}_\varepsilon(\widetilde{\psi},\ \widetilde{\vartheta}(\cdot)) & \le\ c_\varepsilon(\cdot), \\ t_1 + \pi_1\, \widetilde{z} & \le\ \pi_1\, \widetilde{x}(t_2) \end{cases}$$

with upper semicontinuous $M_\varepsilon,\ R_\varepsilon,\ c_\varepsilon : [0,T[\longrightarrow [0,\infty[$.
Set $\mu_\varepsilon(h) := \displaystyle\int_{t_2}^{t_2+h} M_\varepsilon(s)\, ds$.
Then, for every $\varepsilon \in \mathcal{J}$ *and* $h \in]0,T[$ *with* $t_1 + h < \mathcal{T}_\Theta(\widetilde{\psi}, \widetilde{z})$,

$$\begin{aligned} &\widetilde{q}_\varepsilon\big(\widetilde{\psi}(t_1{+}h,\ \widetilde{z}),\ \ \widetilde{x}(t_2{+}h)\big) \\ &\le\ \widetilde{q}_\varepsilon\big(\widetilde{\psi}(t_1, \widetilde{z}),\ \widetilde{x}(t_2)\big)\ \cdot\ e^{\mu_\varepsilon(h)}\ + \\ &\quad +\ \int_0^h e^{\mu_\varepsilon(h)-\mu_\varepsilon(s)}\ \big(c_\varepsilon(t_2{+}s) + 2\, R_\varepsilon(t_2{+}s) + \gamma_\varepsilon(\widetilde{\psi})\big)\ ds. \end{aligned}$$

Proof. We follow the same track as in the proof of Proposition 22 and consider the function $\varphi_\varepsilon : h \longmapsto \widetilde{q}_\varepsilon(\widetilde{\psi}(t_1{+}h, \widetilde{z}),\ \widetilde{x}(t_2{+}h))$. The property $\varphi_\varepsilon(h) \le \limsup_{k\downarrow 0}\ \varphi_\varepsilon(h-k)$ results from Lemma 20.
Furthermore we prove for any $h \in [0,T[$ with $t_1 + h < \mathcal{T}_\Theta(\widetilde{\psi}, \widetilde{z})$,
$\limsup_{k\downarrow 0} \frac{\varphi_\varepsilon(h+k)\, -\, \varphi_\varepsilon(h)}{k} \le M_\varepsilon(t_1{+}h)\cdot\varphi_\varepsilon(h) + c_\varepsilon(t_2{+}h) + 2\, R_\varepsilon(t_2{+}h) + \gamma_\varepsilon(\widetilde{\psi})$.
In particular, this inequality implies $\varphi_\varepsilon(h) \ge \limsup_{k\downarrow 0}\ \varphi_\varepsilon(h{+}k)$ since its right–hand side is finite. Thus, the claim results from Gronwall's Lemma 6 and its Remark 7 (ii).

For all small $k > 0$, the timed triangle inequality and Prop. 22 lead to

$$\begin{aligned} &\varphi_\varepsilon(h+k) \\ \le\ \ & \widetilde{q}_\varepsilon\big(\widetilde{\psi}(t_1{+}h+k,\ \widetilde{z}), \qquad\qquad \widetilde{\vartheta}(t_2{+}h)\,\big(k,\ \ \widetilde{\psi}(t_1{+}h,\ \widetilde{z})\big)\big) \\ &+ \widetilde{q}_\varepsilon\big(\widetilde{\vartheta}(t_2{+}\,h)\,\big(k,\ \widetilde{\psi}(t_1{+}h, \widetilde{z})\big)\big),\quad \widetilde{x}(t_2 + h + k))\big) \\ \le\ \ & \big(\widetilde{Q}^{\mapsto}_\varepsilon(\widetilde{\psi},\ \widetilde{\vartheta}(t_2{+}h)) \quad + \gamma_\varepsilon(\widetilde{\psi})\ +\ \widehat{\gamma}_\varepsilon(t_2{+}h, \widetilde{x}, \widetilde{\vartheta})\,\big)\ \frac{e^{M_\varepsilon(t_2+h)\,\cdot\, k} - 1}{M_\varepsilon(t_2+h)} \\ &+ \varphi_\varepsilon(h)\ \cdot\ e^{\widehat{\alpha}^{\mapsto}_\varepsilon(t_2+h)\,\cdot\, k} + \widehat{\gamma}_\varepsilon(t_2{+}h, \widetilde{x}, \widetilde{\vartheta}) \cdot k \qquad\qquad +\ o(k) \\ \le\ \ & \varphi_\varepsilon(h)\ \cdot\ e^{M_\varepsilon(t_2+h)\,\cdot\, k} + \big|c_\varepsilon(t) + \gamma_\varepsilon(\widetilde{\psi}) + 2\ R_\varepsilon(t)\big|_{t\,=\,t_2+h} \cdot\ k +\ o(k) \end{aligned}$$

since $t_1 + h + k < \mathcal{T}_\Theta(\widetilde{\psi}, \widetilde{z})$ implies $\widetilde{\psi}(t_1{+}h, \widetilde{z}),\ \widetilde{\psi}(t_1{+}h{+}k, \widetilde{z}) \in \widetilde{D}$. □

The next proposition provides an upper bound of the auxiliary function

$$\varphi_\varepsilon(t)\ :=\ \inf_{\widetilde{z}\,\in\,\widetilde{D},\ \pi_1\,\widetilde{z}\,\le\, t}\ \big(\widetilde{q}_\varepsilon(\widetilde{z},\ \widetilde{\psi}(t, \widetilde{y}))\ +\ \widetilde{q}_\varepsilon(\widetilde{z},\ \widetilde{x}(t))\big)$$

for describing the distance between $\widetilde{\psi}(t, \widetilde{y})$ and a timed primitive $\widetilde{x}(t)$ without restricting to $\widetilde{\psi}(t, \widetilde{y}) \in \widetilde{D}$. The basic idea consists in estimating both $h \longmapsto \widetilde{q}_\varepsilon\big(\widetilde{\psi}(h, \widetilde{z}_m),\ \widetilde{\psi}(t{+}h, \widetilde{y})\big)$ and $h \longmapsto \widetilde{q}_\varepsilon\big(\widetilde{\psi}(h, \widetilde{z}_m),\ \widetilde{x}(t{+}h)\big)$ (for small $h > 0$) with a minimizing sequence $(\widetilde{z}_m)_{m\in\mathbb{N}}$ in $\widetilde{D}$. Here assumptions about the time parameter $\mathcal{T}_\Theta(\widetilde{\psi}, \cdot) > 0$ are required for the first time. Roughly speaking, we need lower bounds of $\mathcal{T}_\Theta(\widetilde{\psi}, \widetilde{z}_m) > 0$ for "preserving" the information while $m \longrightarrow \infty$.
If $\mathcal{T}_\Theta(\widetilde{\psi}, \widetilde{z}_m)$ vanishes too quickly, then the comparison with $\widetilde{\psi}(\cdot, \widetilde{z}_m)$ cannot be put into practice long enough for proving estimates.

Proposition 28. *Let a timed forward transition* $\widetilde{\psi} \in \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon))$, *a map* $\widetilde{\vartheta}(\cdot) : [0,1[\longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon))$, *a curve* $\widetilde{x} : [0,1[\longrightarrow \widetilde{E}$ *and* $\widetilde{y} \in \widetilde{E}$, $\lambda_\varepsilon > 0$ *satisfy*

1. $\widetilde{x}(\cdot)$ *is a timed primitive of* $\widetilde{\vartheta}(\cdot)$ *with* $\pi_1\, \widetilde{x}(0) = \pi_1\, \widetilde{y} = 0$,
2. $\alpha_\varepsilon^{\mapsto}(\widetilde{\psi}),\ \widehat{\alpha}_\varepsilon^{\mapsto}(\,\cdot\,, \widetilde{x}, \widetilde{\vartheta}) \ \leq\ M_\varepsilon \ <\ \infty$
   $\widehat{\gamma}_\varepsilon(\,\cdot\,, \widetilde{x}, \widetilde{\vartheta}) \ \leq\ R_\varepsilon(\cdot)$
   $\widetilde{Q}_\varepsilon^{\mapsto}(\widetilde{\psi}, \widetilde{\vartheta}(\cdot)) \ \leq\ c_\varepsilon(\cdot)$, *with upper semicontinuous* $R_\varepsilon(\cdot),\ c_\varepsilon(\cdot) \geq 0$,
3. *for each* $t \in [0,1[$, $\varphi_\varepsilon(t) := \inf_{\widetilde{z} \in \widetilde{D},\, \pi_1 \widetilde{z} \leq t} \big(\widetilde{q}_\varepsilon(\widetilde{z},\ \widetilde{\psi}(t, \widetilde{y})) + \widetilde{q}_\varepsilon(\widetilde{z},\ \widetilde{x}(t))\big)$
   can be approximated by a minimizing sequence $(\widetilde{z}_n)_{n \in \mathbb{N}}$ *in* $\widetilde{D}$ *and* $h_n \downarrow 0$ *with*
   $$\begin{aligned} \pi_1 \widetilde{z}_m \ &\leq\ \pi_1\, \widetilde{z}_n \ \leq\ t, \\ \widetilde{q}_\varepsilon(\widetilde{z}_m, \widetilde{z}_n) \ &\leq\ \lambda_\varepsilon \cdot h_m, \\ h_m \ &<\ \mathcal{T}_\Theta(\widetilde{\psi}, \widetilde{z}_m) \quad \text{for all } m < n, \end{aligned}$$

Then,

$$\varphi_\varepsilon(t) \ \leq\ \varphi_\varepsilon(0)\, e^{M_\varepsilon\, t} \ +\ \int_0^t e^{M_\varepsilon \cdot (t-s)} \big(c_\varepsilon(t) + 2\, R_\varepsilon(t) + 2\, \lambda_\varepsilon + 7\, \gamma_\varepsilon(\widetilde{\psi})\big)\, ds.$$

Remark 29. If this minimizing sequence $(\widetilde{z}_n)$ in $\widetilde{D}$ fulfills

$$\frac{\sup_{n > m}\ \widetilde{q}_\varepsilon(\widetilde{z}_m, \widetilde{z}_n)}{\mathcal{T}_\Theta(\widetilde{f}(\widetilde{z}_j, t), \widetilde{z}_j)} \ \longrightarrow\ 0 \qquad (m \longrightarrow \infty)$$

then the estimate is fulfilled with $\lambda_\varepsilon = 0$. This provides a way to uniqueness results in the case of $R_\varepsilon(\cdot) = 0,\ \gamma_\varepsilon(\widetilde{\psi}) = 0$.

Proof is based on the second version of Gronwall's Lemma (Cor. 8) : The timed triangle inequality implies for $t_1 \leq t_2 < 1$, $\widetilde{z} \in \widetilde{D}$ with $\pi_1\, \widetilde{z} \leq t_1$

$$\begin{aligned} \widetilde{q}_\varepsilon(\widetilde{z},\ \widetilde{\psi}(t_2, \widetilde{y})) \ &\leq\ \widetilde{q}_\varepsilon(\widetilde{z},\ \widetilde{\psi}(t_1, \widetilde{y})) + \beta_\varepsilon(\widetilde{\psi})\, (t_2 - t_1), \\ \widetilde{q}_\varepsilon(\widetilde{z},\ \widetilde{x}(t_2)) \ &\leq\ \widetilde{q}_\varepsilon(\widetilde{z},\ \widetilde{x}(t_1)) \ + \omega_\varepsilon(\widetilde{x}(\cdot),\ t_2 - t_1), \end{aligned}$$

As a consequence, $\varphi_\varepsilon(t) \ \leq\ \liminf_{h \downarrow 0}\ \varphi_\varepsilon(t-h)$ for every $t \in]0,1[$.

Now we prove for any $t \in [0,1[$

$$\liminf_{h \downarrow 0}\ \frac{\varphi_\varepsilon(t+h) - \varphi_\varepsilon(t)}{h} \ \leq\ M_\varepsilon\ \varphi_\varepsilon(t) \ +\ c_\varepsilon(t) + 2\, R_\varepsilon(t) + 2\, \lambda_\varepsilon + 7\, \gamma_\varepsilon(\widetilde{\psi}).$$

Let $(\widetilde{z}_n)$ denote a sequence in $\widetilde{D}$ and $h_n \downarrow 0$ according to cond. (3.), i.e.

$$\begin{cases} \pi_1 \widetilde{z}_m \ \leq\ \pi_1 \widetilde{z}_n \ \leq\ t,\ \ \widetilde{q}_\varepsilon(\widetilde{z}_m, \widetilde{z}_n) \leq \lambda_\varepsilon\, h_m,\ \ h_m < \mathcal{T}_\Theta(\widetilde{\psi}, \widetilde{z}_m)\ \ \forall\, m < n, \\ \widetilde{q}_\varepsilon(\widetilde{z}_n,\ \widetilde{\psi}(t, \widetilde{y})) \ +\ \widetilde{q}_\varepsilon(\widetilde{z}_n,\ \widetilde{x}(t)) \ \longrightarrow\ \varphi_\varepsilon(t) \qquad (n \longrightarrow \infty) \end{cases}$$

Due to Prop. 27 and Lemma 21, we obtain for $0 \ <\ h\ \leq\ h_m\ <\ \mathcal{T}_\Theta(\widetilde{\psi}, \widetilde{z}_m)$

$$\begin{aligned} &\widetilde{q}_\varepsilon\big(\widetilde{\psi}(h, \widetilde{z}_m),\ \ \widetilde{\psi}(t+h, \widetilde{y})\big) \\ &\leq \widetilde{q}_\varepsilon\big(\widetilde{z}_m,\ \widetilde{\psi}(t, \widetilde{y})\big) \cdot e^{M_\varepsilon\, h} + \int_0^h e^{M_\varepsilon\, (h-s)}\ \big(\widetilde{Q}_\varepsilon^{\mapsto}(\widetilde{\psi}, \widetilde{\psi}) + 3\, \gamma_\varepsilon(\widetilde{\psi})\big)\, ds \\ &\leq \widetilde{q}_\varepsilon\big(\widetilde{z}_m,\ \widetilde{\psi}(t, \widetilde{y})\big) \cdot e^{M_\varepsilon\, h} + \quad \frac{e^{M_\varepsilon\, h} - 1}{M_\varepsilon} \quad 6\, \gamma_\varepsilon(\widetilde{\psi}) \end{aligned}$$

and

$$\begin{aligned} &\widetilde{q}_\varepsilon\big(\widetilde{\psi}(h, \widetilde{z}_m),\ \ \widetilde{x}(t+h)\big) \\ &\leq \widetilde{q}_\varepsilon\big(\widetilde{z}_m,\ \widetilde{x}(t)\big) \quad \cdot e^{M_\varepsilon\, h} + \int_0^h e^{M_\varepsilon\, \cdot\, (h-s)}\ \big(c_\varepsilon(t+s) + 2\, R_\varepsilon(t+s) + \gamma_\varepsilon(\widetilde{\psi})\big)\, ds. \end{aligned}$$

Firstly, $\varphi_\varepsilon(t+h) \ \leq\ \widetilde{q}_\varepsilon\big(\widetilde{\psi}(h, \widetilde{z}_m),\ \ \widetilde{\psi}(t+h, \widetilde{y})\big) \ +\ \widetilde{q}_\varepsilon\big(\widetilde{\psi}(h, \widetilde{z}_m),\ \ \widetilde{x}(t+h)\big)$ results directly from its definition. Secondly, the timed triangle inequality implies for any $n > m$

$$\widetilde{q}_\varepsilon\big(\widetilde{z}_m,\ \widetilde{\psi}(t,\widetilde{y})\big) \le \widetilde{q}_\varepsilon(\widetilde{z}_m,\widetilde{z}_n) + \widetilde{q}_\varepsilon\big(\widetilde{z}_n,\ \widetilde{\psi}(t,\widetilde{y})\big) \le \lambda_\varepsilon\, h_m + \widetilde{q}_\varepsilon\big(\widetilde{z}_n,\ \widetilde{\psi}(t,\widetilde{y})\big)$$
$$\widetilde{q}_\varepsilon\big(\widetilde{z}_m,\ \widetilde{x}(t)\big) \le \widetilde{q}_\varepsilon(\widetilde{z}_m,\widetilde{z}_n) + \widetilde{q}_\varepsilon\big(\widetilde{z}_n,\ \widetilde{x}(t)\big) \le \lambda_\varepsilon\, h_m + \widetilde{q}_\varepsilon\big(\widetilde{z}_n,\ \widetilde{x}(t)\big)$$

and, $n \longrightarrow \infty$ leads to the estimate

$$\widetilde{q}_\varepsilon\big(\widetilde{z}_m,\ \widetilde{\psi}(t,\widetilde{y})\big) + \widetilde{q}_\varepsilon\big(\widetilde{z}_m,\ \widetilde{x}(t)\big) \le 2\,\lambda_\varepsilon\, h_m + \varphi_\varepsilon(t).$$

As a consequence,

$$\varphi_\varepsilon(t+h_m) \le \big(2\,\lambda_\varepsilon\, h_m + \varphi_\varepsilon(t)\big)\, e^{M_\varepsilon\, h_m} + \int_0^{h_m} e^{M_\varepsilon\cdot(h_m-s)}\, \Big|c_\varepsilon + 2\,R_\varepsilon + 7\,\gamma_\varepsilon(\widetilde{\psi})\Big|_{t+s}\, ds.$$

So finally,

$$\liminf_{h\downarrow 0} \frac{\varphi_\varepsilon(t+h) - \varphi_\varepsilon(t)}{h} \le M_\varepsilon\, \varphi_\varepsilon(t) + 2\,\lambda_\varepsilon + c_\varepsilon(t) + 2\,R_\varepsilon(t) + 7\,\gamma_\varepsilon(\widetilde{\psi}). \ \Box$$

Finally, the auxiliary function $\varphi_\varepsilon(\cdot)$ is modified with regard to the transition $\widetilde{\psi}(\cdot,\widetilde{y})$:

$$\varphi_\varepsilon(t) := \inf_{\substack{\widetilde{z}\,\in\,\widetilde{D},\\ \pi_1\,\widetilde{z}\,\le\,\pi_1\,\widetilde{x}(t)}} \big(\widetilde{p}_\varepsilon(\widetilde{z},\ \widetilde{\psi}(t,\widetilde{y})) + \widetilde{q}_\varepsilon(\widetilde{z},\ \widetilde{x}(t))\big)$$

Here $\widetilde{p}_\varepsilon : \widetilde{E}\times\widetilde{E} \longrightarrow [0,\infty[$ represents a generalized distance function on $\widetilde{E}$ that has the additional advantage of symmetry (by assumption) and satisfies the triangle inequality (not just the *timed* one).
Roughly speaking, $\widetilde{p}_\varepsilon$ might take not all the properties of elements $\widetilde{x},\widetilde{y} \in \widetilde{E}$ into consideration – compared with $\widetilde{q}_\varepsilon$. The compact subsets of $\mathbb{R}^N$ give an example with $\widetilde{p}_\varepsilon := d\!\!l$ (Pompeiu–Hausdorff distance) and $\widetilde{q}_\varepsilon := q_{\mathcal{K},N}$.
In regard to timed transitions, the assumptions about $\widetilde{p}_\varepsilon$ have the advantage that they do not consider the comparison of two transitions. Instead we suppose continuity properties for each transition $\widetilde{\psi}$, e.g. the distance $\widetilde{p}_\varepsilon(\widetilde{z}_1,\widetilde{z}_2)$ between arbitrary points $\widetilde{z}_1,\widetilde{z}_2 \in \widetilde{E}$ may grow exponentially at the most while evolving along $\widetilde{\psi}$.

Proposition 30. *Let* $\widetilde{p}_\varepsilon, \widetilde{q}_\varepsilon : \widetilde{E}\times\widetilde{E} \longrightarrow [0,\infty[$ $(\varepsilon\in\mathcal{J})$, $\widetilde{D} \stackrel{\text{Def.}}{=} \mathbb{R}\times D \subset \widetilde{E}$, $p\in\mathbb{R}$, *and* $\widetilde{\psi} \in \widetilde{\Theta}_p^{\mapsto}(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$, $\widetilde{\vartheta}(\cdot) : [0,1[\longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon))$, $\widetilde{x} : [0,1[\longrightarrow \widetilde{E}$, $\widetilde{y}\in\widetilde{E}$, $\lambda_\varepsilon > 0$ *satisfy the following conditions :*

1. *Each* $\widetilde{q}_\varepsilon$ *fulfills the timed triangle inequality,* $\widetilde{q}_\varepsilon(\widetilde{z},\widetilde{z}) = 0 \ \ \forall\, \widetilde{z}\in\widetilde{D}$,
2. $\widetilde{p}_\varepsilon$ *is symmetric and satisfies the triangle inequality,*
3. $\widetilde{x}(\cdot)$ *is a timed primitive of* $\widetilde{\vartheta}(\cdot)$ *with* $\pi_1\,\widetilde{x}(0) \ge \pi_1\,\widetilde{y}$,
4. $\exists\ M_\varepsilon < \infty :\ \alpha_\varepsilon^{\mapsto}(\widetilde{\psi}),\ \widehat{\alpha}_\varepsilon^{\mapsto}(\cdot,\widetilde{x},\widetilde{\vartheta}) \le M_\varepsilon$,
   $\widetilde{p}_\varepsilon\big(\widetilde{\psi}(h,\widetilde{v}_1),\ \widetilde{\psi}(h,\widetilde{v}_2)\big) \le \widetilde{p}_\varepsilon(\widetilde{v}_1,\widetilde{v}_2)\cdot e^{M_\varepsilon\, h} \quad \forall\ \widetilde{v}_1,\widetilde{v}_2,h$
   $\exists\ R_\varepsilon(\cdot)\ge 0 :\ \gamma_\varepsilon(\widetilde{\psi}),\ \widehat{\gamma}_\varepsilon(\cdot,\widetilde{x},\widetilde{\vartheta}) \le R_\varepsilon(\cdot)$,
   $\limsup_{h\downarrow 0} \frac{\widetilde{p}_\varepsilon\left(\widetilde{\psi}(h,\,\widetilde{\psi}(t,\widetilde{y})),\ \widetilde{\psi}(t+h,\,\widetilde{y})\right)}{h} \le R_\varepsilon(t)$,
   $\widetilde{p}_\varepsilon\big(\widetilde{\psi}(t-h,\widetilde{y}),\ \widetilde{\psi}(t,\widetilde{y})\big) \longrightarrow 0$ *for* $h\downarrow 0$,
   $\widetilde{Q}_\varepsilon^{\mapsto}(\widetilde{\psi},\widetilde{\vartheta}(\cdot)) \le c_\varepsilon(\cdot)$, *with upper semicontinuous* $R_\varepsilon(\cdot),\ c_\varepsilon(\cdot)$,
5. *for each* $t\in[0,1[$, $\varphi_\varepsilon(t) := \inf_{\substack{\widetilde{z}\,\in\,\widetilde{D},\\ \pi_1\,\widetilde{z}\,\le\,\pi_1\,\widetilde{x}(t)}} \big(\widetilde{p}_\varepsilon(\widetilde{\psi}(t,\widetilde{y}),\ \widetilde{z}) + \widetilde{q}_\varepsilon(\widetilde{z},\ \widetilde{x}(t))\big)$
   can be approximated by a minimizing sequence $(\widetilde{z}_n)_{n\in\mathbb{N}}$ *in* $\widetilde{D}$ *and* $h_n\downarrow 0$ *with*
   $\pi_1\widetilde{z}_m \le \pi_1\,\widetilde{z}_n \le \pi_1\,\widetilde{x}(t)$, $\quad \widetilde{p}_\varepsilon(\widetilde{z}_m,\widetilde{z}_n) \le \lambda_\varepsilon\cdot h_m$,
   $h_m < \mathcal{T}_\Theta(\widetilde{\psi},\widetilde{z}_m)$, $\quad \widetilde{q}_\varepsilon(\widetilde{z}_m,\widetilde{z}_n) \le \lambda_\varepsilon\cdot h_m$ *for all* $m<n$.

Then, $\displaystyle \varphi_\varepsilon(t) \le \varphi_\varepsilon(0)\, e^{M_\varepsilon\, t} + \int_0^t e^{M_\varepsilon\,\cdot\,(t-s)}\, \big(c_\varepsilon(t) + 4\,R_\varepsilon(t) + 2\,\lambda_\varepsilon\big)\, ds.$

Proof is based on the same version of Gronwall's Lemma as the preceding Prop. 28 : $\varphi_\varepsilon(t) \le \liminf_{h\downarrow 0} \varphi_\varepsilon(t-h)$ results from conditions (1.), (2.) because for any $\widetilde{z} \in \widetilde{D}$ with $\pi_1\,\widetilde{z} \le \pi_1\,\widetilde{x}(t-h)$,

$$\begin{aligned}
\widetilde{p}_\varepsilon\big(\widetilde{\psi}(t,\widetilde{y}),\ \widetilde{z}\big) &\le \widetilde{p}_\varepsilon\big(\widetilde{\psi}(t,\widetilde{y}),\ \widetilde{\psi}(t-h,\widetilde{y})\big) + \widetilde{p}_\varepsilon\big(\widetilde{\psi}(t-h,\widetilde{y}),\ \widetilde{z}\big)\\
&\le \widetilde{p}_\varepsilon\big(\widetilde{\psi}(t-h,\widetilde{y}),\ \widetilde{\psi}(t,\widetilde{y})\big) + \widetilde{p}_\varepsilon\big(\widetilde{\psi}(t-h,\widetilde{y}),\ \widetilde{z}\big)\\
\widetilde{q}_\varepsilon(\widetilde{z},\ \widetilde{x}(t)) &\le \widetilde{q}_\varepsilon(\widetilde{z},\ \widetilde{x}(t-h)) \qquad\qquad + \omega_\varepsilon(\widetilde{x}(\cdot),h).
\end{aligned}$$

For showing $\liminf_{h\downarrow 0} \frac{\varphi_\varepsilon(t+h)\,-\,\varphi_\varepsilon(t)}{h} \le M_\varepsilon\cdot\varphi_\varepsilon(t) + c_\varepsilon(t) + 4\,R_\varepsilon(t) + 2\,\lambda_\varepsilon$, let $(\widetilde{z}_n)_{n\in\mathbb{N}}$ denote a minimizing sequence in $\widetilde{D}$ and $h_n \downarrow 0$ such that

$$\begin{cases}
\pi_1\,\widetilde{z}_m \quad\le\quad \pi_1\,\widetilde{z}_n \quad\le \pi_1\,\widetilde{x}(t), & \\
\widetilde{p}_\varepsilon(\widetilde{z}_m,\ \widetilde{z}_n),\ \widetilde{q}_\varepsilon(\widetilde{z}_m,\ \widetilde{z}_n) \le \lambda_\varepsilon\cdot h_m, \quad h_m < \mathcal{T}_\Theta(\widetilde{\psi},\ \widetilde{z}_m) & \forall\ m<n,\\
\widetilde{p}_\varepsilon(\widetilde{\psi}(t,\widetilde{y}),\ \widetilde{z}_n) + \widetilde{q}_\varepsilon(\widetilde{z}_n,\ \widetilde{x}(t)) \longrightarrow \varphi_\varepsilon(t) & (n\longrightarrow\infty).
\end{cases}$$

According to conditions (2.), (4.), we obtain for all $m<n$, $0<h\le h_m$

$$\begin{aligned}
&\widetilde{p}_\varepsilon\big(\widetilde{\psi}(t+h,\ \widetilde{y}),\ \widetilde{\psi}(h,\widetilde{z}_m)\big)\\
&= \widetilde{p}_\varepsilon\big(\widetilde{\psi}(h,\widetilde{z}_m),\ \widetilde{\psi}(t+h,\ \widetilde{y})\big)\\
&\le \widetilde{p}_\varepsilon\big(\widetilde{\psi}(h,\widetilde{z}_m),\ \widetilde{\psi}(h,\ \widetilde{\psi}(t,\widetilde{y}))\big) \qquad + \widetilde{p}_\varepsilon\big(\widetilde{\psi}(h,\ \widetilde{\psi}(t,\widetilde{y})),\ \widetilde{\psi}(t+h,\ \widetilde{y})\big)\\
&\le \widetilde{p}_\varepsilon\big(\widetilde{z}_m,\ \widetilde{\psi}(t,\widetilde{y})\big) \qquad\qquad \cdot e^{M_\varepsilon\,h} + (R_\varepsilon(t)+o(1))\ h\\
&\le \big(\lambda_\varepsilon\,h_m + \widetilde{p}_\varepsilon(\widetilde{z}_n,\ \widetilde{\psi}(t,\widetilde{y}))\big)\ \cdot e^{M_\varepsilon\,h} + (R_\varepsilon(t)+o(1))\ h.
\end{aligned}$$

Furthermore Prop. 27 implies for any $0<h\le h_m<\mathcal{T}_\Theta(\widetilde{\psi},\ \widetilde{z}_m)$, $n>m$

$$\begin{aligned}
&\widetilde{q}_\varepsilon\big(\widetilde{\psi}(h,\widetilde{z}_m),\ \widetilde{x}(t+h)\big)\\
&\le \big(\lambda_\varepsilon\,h_m + \widetilde{q}_\varepsilon(\widetilde{z}_n,\ \widetilde{x}(t))\big)\cdot e^{M_\varepsilon\,h} + \int_0^h e^{M_\varepsilon\cdot(h-s)}\ \big(c_\varepsilon(t+s) + 3\,R_\varepsilon(t+s)\big)\,ds
\end{aligned}$$

and $n\longrightarrow\infty$ leads to

$$\begin{aligned}
\varphi_\varepsilon(t+h_m) \le\ & \varphi_\varepsilon(t)\cdot e^{M_\varepsilon\,h_m} + 2\,\lambda_\varepsilon\,e^{M_\varepsilon\,h_m}\,h_m + (R_\varepsilon(t)+o(1))\,h_m\\
&+ \int_0^{h_m} e^{M_\varepsilon\cdot(h_m-s)}\ \big(c_\varepsilon(t+s)+3\,R_\varepsilon(t+s)\big)\,ds.
\end{aligned}$$

So finally, $\liminf_{h\downarrow 0} \frac{\varphi_\varepsilon(t+h)\,-\,\varphi_\varepsilon(t)}{h} \le M_\varepsilon\cdot\varphi_\varepsilon(t) + c_\varepsilon(t) + 4\,R_\varepsilon(t) + 2\,\lambda_\varepsilon$. □

5.3. Timed right–hand forward solutions

5.3.1. Definition The term "primitive" of $\widetilde{\vartheta} : [0,T[\longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon))$ is closely related to the expression "solution" $\widetilde{x}(\cdot)$ of a generalized mutational equation $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot),\cdot)$.

Definition 31. *For* $\widetilde{f} : \widetilde{E}\times[0,T[\longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon))$ *given, a map* $\widetilde{x} : [0,T[\longrightarrow \widetilde{E}$ *is a* timed right–hand forward solution *of the generalized mutational equation* $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot),\cdot)$ *if* $\widetilde{x}(\cdot)$ *is timed right–hand forward primitive of* $\widetilde{f}(\widetilde{x}(\cdot),\cdot) : [0,T[\longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon))$, *i.e. for each* $\varepsilon\in\mathcal{J}$,

1. $\forall\ t\in[0,T[\quad \exists\ \widehat{\alpha}_\varepsilon^{\mapsto}(t) \ge \alpha_\varepsilon^{\mapsto}(\widetilde{f}(\widetilde{x}(t),t)),\quad \widehat{\gamma}_\varepsilon(t) \ge \gamma_\varepsilon(\widetilde{f}(\widetilde{x}(t),t)) :$

$$\limsup_{h\downarrow 0} \tfrac{1}{h}\Big(\widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(h,\widetilde{z}),\widetilde{x}(t+h)\big) - \widetilde{q}_\varepsilon(\widetilde{z},\ \widetilde{x}(t))\cdot e^{\widehat{\alpha}_\varepsilon^{\mapsto}(t)\,h}\Big) \le \widehat{\gamma}_\varepsilon(t),$$

for all $\widetilde{z}\in\widetilde{D}$ *with* $\pi_1\,\widetilde{z}\le\pi_1\,\widetilde{x}(t)$ *and* $\limsup_{\varepsilon'\downarrow 0}\ \varepsilon'^p\cdot\widehat{\gamma}_{\varepsilon'}(t) = 0$,

2. $\widetilde{x}(\cdot) \in UC^{\rightarrow}([0,T[,\widetilde{E},\widetilde{q}_\varepsilon)$,

3. $\pi_1\,\widetilde{x}(t) = t + \pi_1\,\widetilde{x}(0)$ *for all* $t\in[0,T[$.

5.3.2. Topological preliminaries Generally speaking, constructing solutions (of evolution systems) by approximation is usually based on compactness or completeness. In this subsection, we are adapting the term of sequential compactness to $(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ and distinguish between the order of arguments $\widetilde{x}_{n_j}, \widetilde{x}$ in the vanishing distance $\widetilde{q}_\varepsilon$:
$\widetilde{q}_\varepsilon(\widetilde{x}_{n_j}, \widetilde{x}) \longrightarrow 0$ is regarded as *right – convergence* of $(\widetilde{x}_{n_j})_{j \in \mathbb{N}}$ to $\widetilde{x}$,
$\widetilde{q}_\varepsilon(\widetilde{x}, \widetilde{x}_{n_j}) \longrightarrow 0$ as *left – convergence*.
The following definitions can be extended to tuples $(E, (q_\varepsilon)_{\varepsilon \in \mathcal{J}})$ without time component in a canonical way.

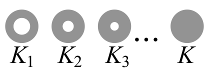
Definition 32. *Let E be a set, $\widetilde{E} \stackrel{\text{Def.}}{=} \mathbb{R} \times E$, $\widetilde{q}_\varepsilon : \widetilde{E} \times \widetilde{E} \longrightarrow [0, \infty[$ $(\varepsilon \in \mathcal{J})$.*
$(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ is called timed two–sided sequentially compact (uniformly with respect to ε) *if for every $\widetilde{v} \in \widetilde{E}$, $r_\varepsilon > 0$ $(\varepsilon \in \mathcal{J})$ and any sequences $(\widetilde{x}_n)_{n \in \mathbb{N}}$, $(\widetilde{y}_n)_{n \in \mathbb{N}}$ in E satisfying*

$$\begin{array}{lll} \widetilde{q}_\varepsilon(\widetilde{x}_n, \widetilde{y}_n) \longrightarrow 0 & \text{for } n \longrightarrow \infty & \forall\ \varepsilon \in \mathcal{J} \\ \widetilde{q}_\varepsilon(\widetilde{v}, \widetilde{x}_n),\ \widetilde{q}_\varepsilon(\widetilde{v}, \widetilde{y}_n) \ \le\ r_\varepsilon & \forall\ n \in \mathbb{N} & \forall\ \varepsilon \in \mathcal{J} \\ \pi_1\, \widetilde{x}_n \ <\ \pi_1\, \widetilde{y}_n & \forall\ n \in \mathbb{N} & \end{array}$$

there exist subsequences $(\widetilde{x}_{n_j})_{j \in \mathbb{N}}$, $(\widetilde{y}_{n_j})_{j \in \mathbb{N}}$ and some $\widetilde{x} \in \widetilde{E}$ such that

$$\left.\begin{array}{l} \widetilde{q}_\varepsilon(\widetilde{x}_{n_j}, \widetilde{x}) \longrightarrow 0 \\ \widetilde{q}_\varepsilon(\widetilde{x}, \widetilde{y}_{n_j}) \longrightarrow 0 \end{array}\right. \quad \text{for } j \longrightarrow \infty \quad \forall\ \varepsilon \in \mathcal{J}.$$

Some ostensible metric spaces have this compactness property in common like $(\mathcal{K}(\mathbb{R}^N), d\!\!l)$, but in general, it is too restrictive.

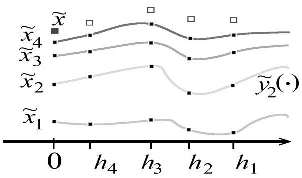


Indeed, $(\mathcal{K}(\mathbb{R}^N), q_{\mathcal{K},N})$ is not two–sided sequentially compact since, for example, $K_n := \{\frac{1}{n+1} \le |x| \le 1\}$ and $K := \mathbb{B}_1$ satisfy $d\!\!l(K_n, K) = q_{\mathcal{K},N}(K_n, K) \longrightarrow 0$ $(n \to \infty)$, but $q_{\mathcal{K},N}(K, K_n) \ge \frac{1}{2}$.
So for weakening this condition on $(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$, we coin a more general term of sequential compactness that is particularly adapted for a sequence of Euler approximations at a fixed point of time :

Definition 33. *Let $\widetilde{\Theta}$ denote a nonempty set of maps $[0,1] \times \widetilde{E} \longrightarrow \widetilde{E}$. $(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}}, \widetilde{\Theta})$ is called* timed transitionally compact *if it fulfills :*

Let $(\widetilde{x}_n)_{n \in \mathbb{N}}$, $(h_j)_{j \in \mathbb{N}}$ be any sequences in $\widetilde{E}$, $]0,1[$, respectively and $\widetilde{v} \in \widetilde{E}$ with $\sup_n \widetilde{q}_\varepsilon(\widetilde{v}, \widetilde{x}_n) < \infty$ for each $\varepsilon \in \mathcal{J}$, $h_j \longrightarrow 0$. Moreover suppose $\widetilde{\vartheta}_n : [0,1] \longrightarrow \widetilde{\Theta}$ to be piecewise constant $(n \in \mathbb{N})$ such that all curves $\widetilde{\vartheta}_n(t)(\cdot, \widetilde{x}) : [0,1] \longrightarrow \widetilde{E}$ have a common modulus of continuity $(n \in \mathbb{N}, t \in [0,1], \widetilde{x} \in \widetilde{E})$.
Each $\widetilde{\vartheta}_n$ induces a function $\widetilde{y}_n(\cdot) : [0,1] \longrightarrow \widetilde{E}$ with $\widetilde{y}_n(0) = \widetilde{x}_n$ in the same (piecewise) way as timed forward transitions induce their own primitives according to Remark 26 (i.e. using $\widetilde{\vartheta}_n(t_m)\,(\cdot, \widetilde{y}_n(t_m))$ in each interval $]t_m, t_{m+1}]$ in which $\widetilde{\vartheta}_n(\cdot)$ is constant).
Then there exist a sequence $n_k \nearrow \infty$ and $\widetilde{x} \in \widetilde{E}$ satisfying for each $\varepsilon \in \mathcal{J}$,

$$\begin{array}{rlcl} & \lim\limits_{k \to \infty} \ \pi_1\, \widetilde{x}_{n_k} & = & \pi_1\, \widetilde{x}, \\ & \limsup\limits_{k \longrightarrow \infty} \ \widetilde{q}_\varepsilon(\widetilde{x}_{n_k}, \widetilde{x}) & = & 0, \\ \limsup\limits_{j \longrightarrow \infty} & \sup\limits_{k \ge j} \ \widetilde{q}_\varepsilon(\widetilde{x}, \ \widetilde{y}_{n_k}(h_j)) & = & 0. \end{array}$$



A nonempty subset $\widetilde{F} \subset \widetilde{E}$ is called timed transitionally compact in *$(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}}, \widetilde{\Theta})$ if the same property holds for any sequence $(\widetilde{x}_n)_{n \in \mathbb{N}}$ in $\widetilde{F}$ (but $\widetilde{x} \in \widetilde{F}$ is* not *required).*

Remark 34. Suppose that $(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ is timed two–sided sequentially compact (uniformly with respect to ε). Then $(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}}, \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)))$ is timed transitionally compact since any sequences $(\widetilde{x}_n)$, (h_j), $(\vartheta_n(\cdot))$, $(\widetilde{y}_n)$ as in the preceding Definition 33 fulfill

$$\widetilde{q}_\varepsilon(\widetilde{x}_n,\, \widetilde{y}_n(h_n)) \;\le\; c_\varepsilon(h_n) \longrightarrow 0 \qquad \text{for } n \longrightarrow \infty \text{ and every } \varepsilon \in \mathcal{J}.$$

So there exist a sequence $n_k \nearrow \infty$ of indices and $\widetilde{x} \in \widetilde{E}$ with

$$\widetilde{q}_\varepsilon(\widetilde{x}_{n_k},\, \widetilde{x}) \longrightarrow 0, \quad \widetilde{q}_\varepsilon(\widetilde{x},\, \widetilde{y}_{n_k}(h_{n_k})) \longrightarrow 0 \qquad \text{for } k \longrightarrow \infty$$

and finally, $\widetilde{q}_\varepsilon(\widetilde{x},\, \widetilde{y}_{n_k}(h_j)) \;\le\; \widetilde{q}_\varepsilon(\widetilde{x},\, \widetilde{y}_{n_k}(h_{n_k})) \,+\, c_\varepsilon(h_j) \qquad$ for $h_{n_k} < h_j$.

5.3.3. Convergence theorem Generally speaking, the existence of a solution can often be concluded from approximation. Seizing this well–tried notion here, we use Euler method in the next subsection. As a first step in this direction, the relevant kind of convergence has to be specified. It is to guarantee that the limit function of approximating solutions is a solution (in other words, it is to preserve the solution property).

Assumptions (5.ii), (5.iii) of the next proposition formulate a suitable form of convergence that might be subsumed under the term "two–sided graphically convergent". Obviously, it is weaker than pointwise convergence (with respect to time) and consists of two conditions with the limit function appearing in both arguments of $\widetilde{q}_\varepsilon$. Admitting vanishing "time perturbations" $\delta_j, \delta'_j \ge 0$ exemplifies the basic idea that the first argument of $\widetilde{q}_\varepsilon$ usually refers to the earlier element whereas the second argument mostly represents the later point.

Proposition 35 (Convergence Theorem).
Suppose the following properties of

$$\begin{array}{ll} \widetilde{f}_m,\, \widetilde{f} : \widetilde{E} \times [0,T[\;\longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}}) & (m \in \mathbb{N}) \\ \widetilde{x}_m, \widetilde{x} : \quad [0,T[\;\longrightarrow \widetilde{E} \;: & \end{array}$$

1. $M_\varepsilon \;:=\; \sup_{m,t,\widetilde{y}} \{\, \alpha_\varepsilon^{\mapsto}(\widetilde{f}_m(\widetilde{y},t)) \,\} \;<\; \infty,$
   $R_\varepsilon \;\ge\; \sup_{m,t,\widetilde{y}} \{\, \widehat{\gamma}_\varepsilon(t,\, \widetilde{x}_m,\, \widetilde{f}_m(\widetilde{x}_m,\cdot)),\;\; \gamma_\varepsilon(\widetilde{f}_m(\widetilde{y},t)),\;\; \gamma_\varepsilon(\widetilde{f}(\widetilde{y},t)) \,\}$
   with $\limsup_{\varepsilon'\downarrow 0} \; \varepsilon'^p \cdot R_{\varepsilon'} \;=\; 0,$
2. $\limsup \;\; \widetilde{Q}_\varepsilon^{\mapsto}\big(\widetilde{f}_m(\widetilde{y}_1,t_1),\;\; \widetilde{f}_m(\widetilde{y}_2,t_2)\big) \;\le\; R_\varepsilon$
   for $m \longrightarrow \infty, \quad t_2 - t_1 \downarrow 0, \quad \widetilde{q}_\varepsilon(\widetilde{y}_1, \widetilde{y}_2) \longrightarrow 0 \;\; (\pi_1\, \widetilde{y}_1 \le \pi_1\, \widetilde{y}_2),$
3. $\overset{\circ}{\widetilde{x}}_m(\cdot) \;\ni\; \widetilde{f}_m(\widetilde{x}_m(\cdot),\,\cdot)$ *in* $[0,T[\,,$
4. $\widehat{\omega}_\varepsilon(h) \;:=\; \sup_m \; \omega_\varepsilon(\widetilde{x}_m, h) \;<\; \infty$ *(moduli of continuity w.r.t.* $\widetilde{q}_\varepsilon$*)*
   $\limsup_{h\downarrow 0} \;\; \widehat{\omega}_\varepsilon(h) \;=\; 0,$
5. $\forall\; t_1, t_2 \in [0,T[, \quad t_3 \in\,]0,T[\quad \exists\; (m_j)_{j\in\mathbb{N}}$ *with* $m_j \nearrow \infty$ *and*
   (i) $\limsup_{j\longrightarrow\infty} \;\; \widetilde{Q}_\varepsilon^{\mapsto}\big(\widetilde{f}(\widetilde{x}(t_1),t_1),\;\; \widetilde{f}_{m_j}(\widetilde{x}(t_1),t_1)\big) \;\le\; R_\varepsilon,$
   (ii) $\exists\; (\delta'_j)_{j\in\mathbb{N}} : \quad \delta'_j \searrow 0, \qquad \widetilde{q}_\varepsilon\big(\widetilde{x}(t_2),\;\; \widetilde{x}_{m_j}(t_2+\delta'_j)\big) \longrightarrow 0,$
   $\pi_1\, \widetilde{x}(t_2) \;\le\; \pi_1\, \widetilde{x}_{m_j}(t_2+\delta'_j).$
   (iii) $\exists\; (\delta_j)_{j\in\mathbb{N}} : \quad \delta_j \searrow 0, \qquad \widetilde{q}_\varepsilon\big(\widetilde{x}_{m_j}(t_3-\delta_j),\;\; \widetilde{x}(t_3)\big) \longrightarrow 0,$
   $\pi_1\, \widetilde{x}_{m_j}(t_3-\delta_j) \;\le\; \pi_1\, \widetilde{x}(t_3),$

for each $\varepsilon \in \mathcal{J}$.

Then, $\widetilde{x}(\cdot)$ *is a timed right–hand forward solution of* $\overset{\circ}{\widetilde{x}}(\cdot) \;\ni\; \widetilde{f}(\widetilde{x}(\cdot),\,\cdot)$ *in* $[0,T[$.

Proof. The uniform continuity of $\widetilde{x}(\cdot)$ results from assumption (4.) : Each $\widetilde{x}_m(\cdot)$ satisfies $\widetilde{q}_\varepsilon\big(\widetilde{x}_m(t_1),\ \widetilde{x}_m(t_2)\big) \le \widehat{\omega}_\varepsilon(t_2-t_1)$ for $t_1 < t_2 < T$. Let $\varepsilon \in \mathcal{J}$, $0 \le t_1 < t_2 < T$ be arbitrary and choose $(\delta'_j)_{j\in\mathbb{N}}$, $(\delta_j)_{j\in\mathbb{N}}$, for $t_1,\ t_2$ (according to condition (5.ii), (5.iii)). For all $j \in \mathbb{N}$ large enough, we obtain $t_1 + \delta'_j < t_2 - \delta_j$ and so,

$$\begin{aligned}\widetilde{q}_\varepsilon\big(\widetilde{x}(t_1),\ \widetilde{x}(t_2)\big) \le\ & \widetilde{q}_\varepsilon\big(\widetilde{x}(t_1),\ \widetilde{x}_{m_j}(t_1+\delta'_j)\big) + \widetilde{q}_\varepsilon\big(\widetilde{x}_{m_j}(t_1+\delta'_j),\ \widetilde{x}_{m_j}(t_2-\delta_j)\big) \\ & + \widetilde{q}_\varepsilon\big(\widetilde{x}_{m_j}(t_2-\delta_j),\ \widetilde{x}(t_2)\big) \\ \le\ & o(1) \qquad\qquad + \widehat{\omega}_\varepsilon(t_2-t_1) \qquad \text{for } j \longrightarrow \infty.\end{aligned}$$

Now let $\varepsilon \in \mathcal{J}$, $\widetilde{z} \in \widetilde{D}$ and $t \in [0,T[$, $0 < h < \mathcal{T}_\Theta(\widetilde{f}(\widetilde{x}(t),t),\ \widetilde{z})$ be chosen arbitrarily. Condition (6.) of Definition 17 ensures for all $k \in]0,h[$ sufficiently small

$$\widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(h,\widetilde{z}),\ \ \widetilde{x}(t+h)\big) \ \le\ \widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(h-k,\widetilde{z}),\ \ \widetilde{x}(t+h)\big) \ +\ h^2.$$

According to cond. (5.i) – (5.iii), there exist sequences $(m_j)_{j\in\mathbb{N}}$, $(\delta_j)_{j\in\mathbb{N}}$, $(\delta'_j)_{j\in\mathbb{N}}$ satisfying $m_j \nearrow \infty$, $\delta_j \downarrow 0$, $\delta'_j \downarrow 0$, $\delta_j + \delta'_j < k$ and

$$\begin{cases} \widetilde{Q}^{\mapsto}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t), & \widetilde{f}_{m_j}(\widetilde{x}(t),t)\big) \ \le\ R_\varepsilon + h^2, \\ \widetilde{q}_\varepsilon\big(\widetilde{x}_{m_j}(t+h-\delta_j),\ \widetilde{x}(t+h)\big) & \longrightarrow 0, \\ \widetilde{q}_\varepsilon\big(\widetilde{x}(t), & \widetilde{x}_{m_j}(t+\delta'_j)\big) \ \longrightarrow 0. \end{cases}$$

Thus, Proposition 27 implies for all large $j \in \mathbb{N}$ (depending on $\varepsilon, \widetilde{z}, t, h, k$),

$$\begin{aligned}
&\widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(h,\ \widetilde{z}),\ \ \widetilde{x}(t+h)\big) \\
\le\ & \widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(h-k,\ \widetilde{z}),\ \ \widetilde{x}_{m_j}(t+\delta'_j+h-k)\big) \\
& + \widetilde{q}_\varepsilon\big(\widetilde{x}_{m_j}(t+\delta'_j+h-k),\ \ \widetilde{x}_{m_j}(t+h-\delta_j)\big) \\
& + \widetilde{q}_\varepsilon\big(\widetilde{x}_{m_j}(t+h-\delta_j),\ \ \widetilde{x}(t+h)\big) \ +\ h^2 \\
\le\ & \widetilde{q}_\varepsilon\big(\widetilde{z},\ \widetilde{x}_{m_j}(t+\delta'_j)\big)\ \cdot\ e^{M_\varepsilon\cdot(h-k)} \ + \\
& + \int_0^{h-k} e^{M_\varepsilon\cdot(h-k-s)}\ \Big(\widetilde{Q}^{\mapsto}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t),\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j},\cdot)\big|_{t+\delta'_j+s}\big) \ +\ 3\,R_\varepsilon\Big)\ ds \\
& + \widehat{\omega}_\varepsilon(k-\delta_j-\delta'_j) \\
& + \widetilde{q}_\varepsilon\big(\widetilde{x}_{m_j}(t+h-\delta_j),\ \ \widetilde{x}(t+h)\big) \ +\ h^2 \\
\le\ & \Big(\widetilde{q}_\varepsilon(\widetilde{z},\ \widetilde{x}(t)) \ +\ \widetilde{q}_\varepsilon\big(\widetilde{x}(t),\ \widetilde{x}_{m_j}(t+\delta'_j)\big)\Big)\ \cdot\ e^{M_\varepsilon\cdot(h-k)} \ + \\
& + \int_0^{h} e^{M_\varepsilon\cdot(h-s)}\ \ \widetilde{Q}^{\mapsto}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t),\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j},\cdot)\big|_{t+\delta'_j+s}\big)\ ds \\
& + \widehat{\omega}_\varepsilon(k) \qquad\qquad + \ 2\,h^2 \ +\ \text{const}\cdot h\ R_\varepsilon \\
\le\ & \widetilde{q}_\varepsilon(\widetilde{z},\ \widetilde{x}(t))\ \cdot\ e^{M_\varepsilon\,h} \qquad + \ 3\,h^2 \ +\ \text{const}\cdot h\ R_\varepsilon \ +\ \widehat{\omega}_\varepsilon(k) \\
& + \int_0^{h} e^{M_\varepsilon\cdot(h-s)}\Big(R_\varepsilon + h^2 + \widetilde{Q}^{\mapsto}_\varepsilon\big(\widetilde{f}_{m_j}(\widetilde{x}(t),t),\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j},\cdot)\big|_{t+\delta'_j+s}\big)\Big)\ ds \\
\le\ & \widetilde{q}_\varepsilon(\widetilde{z},\ \widetilde{x}(t))\cdot e^{M_\varepsilon\,h} \ +\ \text{const}\cdot\ h\,(R_\varepsilon+h) \ +\ \widehat{\omega}_\varepsilon(k) \\
& + \ h\ \ e^{M_\varepsilon\,h}\ \ \ \widetilde{Q}^{\mapsto}_\varepsilon\big(\widetilde{f}_{m_j}(\widetilde{x}(t),t),\ \ \ \widetilde{f}_{m_j}(\widetilde{x}_{m_j}(\cdot),\cdot)\big|_{t+\delta'_j}\big) \\
& + \int_0^{h} e^{M_\varepsilon\cdot(h-s)}\ \ \widetilde{Q}^{\mapsto}_\varepsilon\big(\widetilde{f}_{m_j}(\widetilde{x}_{m_j}(\cdot),\cdot)\big|_{t+\delta'_j},\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j}(\cdot),\cdot)\big|_{t+\delta'_j+s}\big)\ ds.
\end{aligned}$$

Now $j \longrightarrow \infty$ and then $k \longrightarrow 0$ provide the estimate

$$\begin{aligned}
&\widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(h,\ \widetilde{z}),\ \ \widetilde{x}(t+h)\big) \\
\le\ & \widetilde{q}_\varepsilon(\widetilde{z},\ \widetilde{x}(t))\cdot e^{M_\varepsilon\,h} \ +\ \text{const}\cdot\ h\,(R_\varepsilon+h) \ +\ 0 \ +\ 0 \\
& + h\ \ e^{M_\varepsilon\,h}\ \ \limsup_{j\longrightarrow\infty}\ \sup_{0\le s\le h}\ \widetilde{Q}^{\mapsto}_\varepsilon\big(\widetilde{f}_{m_j}(\widetilde{x}_{m_j},\cdot)\big|_{t+\delta'_j},\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j},\cdot)\big|_{t+\delta'_j+s}\big).
\end{aligned}$$

Finally convergence assumption (2.) together with the equi-continuity of $(\widetilde{x}_m(\cdot))_{m\in\mathbb{N}}$ ensures

$$\limsup_{h\downarrow 0}\ \limsup_{j\longrightarrow\infty}\ \sup_{0\le s\le h}\ \widetilde{Q}_\varepsilon^{\mapsto}\big(\widetilde{f}_{m_j}(\widetilde{x}_{m_j},\cdot)\big|_{t+\delta'_j},\ \widetilde{f}_{m_j}(\widetilde{x}_{m_j},\cdot)\big|_{t+\delta'_j+s}\big)\ \le\ R_\varepsilon$$

and thus,

$$\limsup_{h\downarrow 0}\ \tfrac{1}{h}\ \Big(\widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\ \big(h,\widetilde{z}\big),\ \widetilde{x}(t{+}h)\big)\ -\ \widetilde{q}_\varepsilon(\widetilde{z},\ \widetilde{x}(t))\cdot e^{M_\varepsilon\,h}\Big)\ \le\ \text{const}\cdot R_\varepsilon.$$

□

5.3.4. Existence due to compactness Our intention is to construct a timed right–hand forward solution of a generalized mutational equation by means of Euler method. For considering the family of $(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}}$, we prefer some form of compactness to a version of completeness. Thus in view of Convergence Theorem (Proposition 35), the term “timed transitionally compact” (Definition 33) comes in useful.

Proposition 36. (Existence of timed right–hand forward solutions due to timed transitional compactness)

Assume that the tuple $(\widetilde{E},\ (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}},\ \widetilde{\Theta}_p^{\mapsto}(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon)))$ *is timed transitionally compact. Furthermore let* $\widetilde{f}:\widetilde{E}\times[0,T]\longrightarrow\widetilde{\Theta}_p^{\mapsto}(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}})$ *fulfill for every* $\varepsilon\in\mathcal{J}$

1. $M_\varepsilon\ :=\ \sup_{t,\widetilde{y}}\ \alpha_\varepsilon^{\mapsto}(\widetilde{f}(\widetilde{y},t))\ <\ \infty,$
2. $c_\varepsilon(h)\ :=\ \sup_{t,\widetilde{y}}\ \beta_\varepsilon(\widetilde{f}(\widetilde{y},t))(h)\ <\ \infty,\qquad c_\varepsilon(h)\ \stackrel{h\downarrow 0}{\longrightarrow}\ 0$
3. $\exists\ R_\varepsilon\ :\ \sup_{t,\widetilde{y}}\ \gamma_\varepsilon(\widetilde{f}(\widetilde{y},t))\ \le\ R_\varepsilon<\infty,\qquad \varepsilon'^p\ R_{\varepsilon'}\ \stackrel{\varepsilon'\downarrow 0}{\longrightarrow}\ 0$
4. $\exists\ \widehat{\omega}_\varepsilon(\cdot):\ \widetilde{Q}_\varepsilon^{\mapsto}\big(\widetilde{f}(\widetilde{y}_1,t_1),\ \widetilde{f}(\widetilde{y}_2,t_2)\big)\ \le\ R_\varepsilon\ +\ \widehat{\omega}_\varepsilon\big(\widetilde{q}_\varepsilon(\widetilde{y}_1,\widetilde{y}_2)+t_2-t_1\big)$ *for all* $0\le t_1\le t_2\le T$ *and* $\widetilde{y}_1,\widetilde{y}_2\in\widetilde{E}$ $(\pi_1\,\widetilde{y}_1\le\pi_1\,\widetilde{y}_2)$, $\widehat{\omega}_\varepsilon(\cdot)\ge 0$ *nondecreasing,* $\limsup_{s\downarrow 0}\ \widehat{\omega}_\varepsilon(s)\ =\ 0.$

Then for every $\widetilde{x}_0\in\widetilde{E}$, *there is a timed right–hand forward solution* $\widetilde{x}:[0,T[\longrightarrow\widetilde{E}$ *of the generalized mutational equation* $\overset{\circ}{\widetilde{x}}(\cdot)\ \ni\ \widetilde{f}(\widetilde{x}(\cdot),\cdot)$ *with* $\widetilde{x}(0)=\widetilde{x}_0$.

Proof is based on Euler method for an approximating sequence $(\widetilde{x}_n(\cdot))$ and Cantor diagonal construction for its limit $\widetilde{x}(\cdot)$. For $n\in\mathbb{N}$ $(2^n>T)$ set

$$\begin{array}{lll} h_n := \frac{T}{2^n}, & t_n^j := j\,h_n & \text{for } j=0\ldots 2^n,\\ \widetilde{x}_n(0) := \widetilde{x}_0, & \widetilde{x}_0(\cdot) := \widetilde{x}_0, & \\ \widetilde{x}_n(t) := \widetilde{f}(\widetilde{x}_n(t_n^j),\ t_n^j)\ \big(t-t_n^j,\ \widetilde{x}_n(t_n^j)\big) & & \text{for } t\in\,]t_n^j,\ t_n^{j+1}],\ j\le 2^n. \end{array}$$

The uniform modulus of continuity $c_\varepsilon(\cdot)$ can be replaced by a nondecreasing convex function $[0,T{+}1]\longrightarrow[0,\infty[$ such that all $\widetilde{x}_n(\cdot)$ are equi–continuous in the sense of

$$\widetilde{q}_\varepsilon\big(\widetilde{x}_n(s),\ \widetilde{x}_n(t)\big)\ \le\ c_\varepsilon(t-s)\qquad\text{for any } 0\le s<t<T+h_n \text{ and } \varepsilon\in\mathcal{J}.$$

Since $\mathcal{J}$ is countable there is a sequence $(j_k)_{k\in\mathbb{N}}$ with $\{j_1,j_2\ldots\}=\mathcal{J}\subset[0,1]^\kappa$. Now for every $t\in\,]0,T[$, choose a decreasing sequence $(\delta_k(t))_{k\in\mathbb{N}}$ in $\mathbb{Q}\cdot T$ satisfying

$$\begin{array}{ll} 0\ <\ \delta_k(t)\ <\ \frac{h_k}{2}, & t+\delta_k(t)\ <\ T,\\ c_{\varepsilon_j}(\delta_k(t))\ <\ h_k & \text{for any } j\in\{j_1\ldots j_k\}. \end{array}$$

Then, $\widetilde{q}_{\varepsilon_j}\big(\widetilde{x}_n(t),\ \widetilde{x}_n(t{+}\delta_k(t))\big)\ \le\ h_k$ for any $j\in\{j_1\ldots j_k\},\ k,n\in\mathbb{N}$

and so $\widetilde{q}_\varepsilon\big(\widetilde{x}_n(t),\ \widetilde{x}_n(t+\delta_k(t))\big) \longrightarrow 0 \quad (k \longrightarrow \infty)$ for every $\varepsilon \in \mathcal{J}$, uniformly in n.

Thus for each $t \in]0,T[$ and any fixed $\varepsilon \in \mathcal{J}$, the timed transitional compactness of $(\widetilde{E},\ (\widetilde{q}_\varepsilon),\ \widetilde{\Theta}_p^{\mapsto}(\widetilde{E},\widetilde{D},(\widetilde{q}_\varepsilon)))$ provides sequences $m_k \nearrow \infty$, $n_k \nearrow \infty$ $(m_k \le n_k)$ of indices and an element $\widetilde{x}(t) \in \widetilde{E}$ (independent of ε) satisfying for every $k \in \mathbb{N}$

$$\wedge \begin{cases} \sup\limits_{l \ge k} \ \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_l}(t),\ \widetilde{x}(t)\big) & \le \frac{1}{k}, \\ \sup\limits_{l \ge k} \ \widetilde{q}_\varepsilon\big(\widetilde{x}(t),\quad \widetilde{x}_{n_l}(t+\delta_{m_k}(t))\big) & \le \frac{1}{k}. \end{cases}$$

(In particular, each m_k, n_k may be replaced by larger indices preserving the properties.) For arbitrary $K \in \mathbb{N}$, these sequences $m_k,\ n_k \nearrow \infty$ can even be chosen in such a way that the estimates are fulfilled for the finite set of parameters $t \in Q_K :=]0,T[\cap \mathbb{N}{\cdot}h_K$ and $\varepsilon \in \mathcal{J}_K := \{\varepsilon_{j_1}, \varepsilon_{j_2} \dots \varepsilon_{j_K}\} \subset \mathcal{J}$ simultaneously.

Now the Cantor diagonal construction (with respect to the index K) provides subsequences (again denoted by) $m_k, n_k \nearrow \infty$ such that $m_k \le n_k$,

$$\wedge \begin{cases} \sup\limits_{l \ge k} \ \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_l}(t),\ \widetilde{x}(t)\big) & \le \frac{1}{k} \\ \sup\limits_{l \ge k} \ \widetilde{q}_\varepsilon\big(\widetilde{x}(s),\quad \widetilde{x}_{n_l}(s+\delta_{m_k}(s))\big) & \le \frac{1}{k} \end{cases}$$

for every $K \in \mathbb{N}$ and all $\varepsilon \in \mathcal{J}_K$, $s,t \in Q_K$, $k \ge K$.
In particular, $\widetilde{q}_\varepsilon(\widetilde{x}(s),\ \widetilde{x}(t)) \le c_\varepsilon(t-s)$ for any $s,t \in Q_{\mathbb{N}} := \bigcup_K Q_K$ with $s < t$ and every $\varepsilon \in \mathcal{J}$. Moreover, the sequence $(\widetilde{x}_{n_k}(\cdot))_{k \in \mathbb{N}}$ fulfills for all $\varepsilon \in \mathcal{J}$, $K \in \mathbb{N}$, $t \in Q_K$ and sufficiently large $k,l \in \mathbb{N}$ (depending merely on ε, K)

$$\widetilde{q}_\varepsilon\big(\widetilde{x}_{n_k}(t),\ \widetilde{x}_{n_l}(t+\delta_{m_l}(t))\big) \ \le \ \tfrac{1}{k} + \tfrac{1}{l}\,.$$

For extending $\widetilde{x}(\cdot)$ to $t \in]0,T[\setminus Q_{\mathbb{N}}$, we apply the timed transitional compactness to $((\widetilde{x}_{n_k}(t))_{k \in \mathbb{N}}$ and obtain a subsequence $n_{l_j} \nearrow \infty$ of indices (depending on t) and an element $\widetilde{x}(t) \in \widetilde{E}$ satisfying for every $\varepsilon \in \mathcal{J}$,

$$\wedge \begin{cases} \qquad \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_{l_j}}(t),\ \widetilde{x}(t)\big) & \longrightarrow 0, \\ \sup\limits_{i \ge j} \ \widetilde{q}_\varepsilon\big(\widetilde{x}(t),\quad \widetilde{x}_{n_{l_i}}(t+\delta_{m_j}(t))\big) & \longrightarrow 0 \end{cases} \quad \text{for } j \longrightarrow \infty.$$

This implies the following convergence even uniformly in t (but not necessarily in ε)

$$\wedge \begin{cases} \limsup\limits_{K \longrightarrow \infty} \ \limsup\limits_{k \longrightarrow \infty} \ \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_k}(t-2\,h_K),\quad \widetilde{x}(t)\,\big) = 0, \\ \limsup\limits_{K \longrightarrow \infty} \ \limsup\limits_{k \longrightarrow \infty} \ \widetilde{q}_\varepsilon\big(\widetilde{x}(t),\quad \widetilde{x}_{n_k}(t+2\,h_K)\,\big) = 0. \end{cases}$$

Indeed, for $K \in \mathbb{N}$ fixed arbitrarily, there are $s = s(t,K) \in Q_K$ and $K' = K'(\varepsilon,K) \in \mathbb{N}$ with $t - 2\,h_K < s \le t - h_K$, $K' \ge K$ and

$$\widetilde{q}_\varepsilon\big(\widetilde{x}_{n_k}(s),\ \widetilde{x}_{n_l}(s+\delta_{m_l}(s))\big) \ \le \ \tfrac{1}{k} + \tfrac{1}{l} \text{ for all } k,l \ge K'.$$

So for any $k, l_j \ge K'$, we conclude from $\delta_{m_{l_j}}(\cdot) < \frac{1}{2}\,h_{m_{l_j}} < \frac{1}{2}\,h_{l_j} \le \frac{1}{2}\,h_K$

$$\begin{aligned} \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_k}(t-2\,h_K),\ \widetilde{x}(t)\big) \le \quad & \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_k}\,(t-2\,h_K),\quad \widetilde{x}_{n_k}\,(s)\big) \\ & + \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_k}\,(s),\quad \widetilde{x}_{n_{l_j}}(s+\delta_{m_{l_j}}(s))\big) \\ & + \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_{l_j}}(s+\delta_{m_{l_j}}(s)),\ \widetilde{x}_{n_{l_j}}(t)\big) \\ & + \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_{l_j}}(t),\quad \widetilde{x}(t)\big) \\ \le \quad & c_\varepsilon(h_K) + \tfrac{1}{k} + \tfrac{1}{l_j} + c_\varepsilon(2\,h_K) + \widetilde{q}_\varepsilon\big(\widetilde{x}_{n_{l_j}}(t), \widetilde{x}(t)\big) \end{aligned}$$

and $j \longrightarrow \infty$ leads to the estimate $\widetilde{q}_\varepsilon\big(\widetilde{x}_{n_k}(t-2\,h_K),\ \widetilde{x}(t)\big) \le 2c_\varepsilon(2\,h_K) + \frac{2}{K}$.
The proof of $\limsup\limits_{K \longrightarrow \infty} \limsup\limits_{k \longrightarrow \infty} \widetilde{q}_\varepsilon\big(\widetilde{x}(t), \widetilde{x}_{n_k}(t{+}2\,h_K)\,\big) = 0$ is analogous (with $s' = s'(t,K) \in Q_K$ satisfying $t{+}h_K \le s' < t{+}2\,h_K$).

Now we summarize the construction of $\widetilde{x}(\cdot)$ in the following notation : For each $\varepsilon \in \mathcal{J}$ and $j \in \mathbb{N}$, there exist $K_j \in \mathbb{N}$ (depending on ε, j) and $N_j \in \mathbb{N}$ (depending on ε, j, K_j) such that $N_j > K_j > N_{j-1}$ and

$$\wedge \begin{cases} \widetilde{q}_\varepsilon\big(\widetilde{x}_{N_j}(s - 2\,h_{K_j}), \quad \widetilde{x}(s)\,\big) \;\le\; \frac{1}{j} \\ \widetilde{q}_\varepsilon\big(\widetilde{x}(t), \quad \widetilde{x}_{N_j}(t + 2\,h_{K_j})\,\big) \;\le\; \frac{1}{j} \end{cases}$$

for every $s, t \in [0, T[$.

Convergence Theorem (Prop. 35) states that $\widetilde{x}(\cdot)$ is a timed right–hand forward solution of the generalized mutational equation $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}, \cdot)$. Indeed, set $\widetilde{g}_j : (\widetilde{y}, t) \longmapsto \widetilde{f}\big(\widetilde{x}_{N_j}(t^{a+2}_{N_j}{+}2h_{K_j}), t^{a+2}_{N_j}{+}2h_{K_j}\big)$ for $t^a_{N_j} \le t < t^{a+1}_{N_j}$ and regard the sequence $t \longmapsto \widetilde{x}_{N_j}(t + 2\,h_{N_j} + 2\,h_{K_j})$ of solutions. Obviously conditions (1.), (3.), (4.) of Proposition 35 result from the assumptions here. Furthermore, we obtain for any $0 \le t < t' < T$ (with $t^a_{N_j} \le t < t^{a+1}_{N_j}$, $t^b_{N_j} \le t' < t^{b+1}_{N_j}$) and $j \in \mathbb{N}$, $\varepsilon \in \mathcal{J}$

$$\begin{aligned}
&\widetilde{Q}^{\mapsto}_\varepsilon\big(\widetilde{g}_j(\widetilde{y}, t), \;\; \widetilde{g}_j(\widetilde{y}', t')\big) \\
&= \widetilde{Q}^{\mapsto}_\varepsilon\Big(\widetilde{f}\Big(\widetilde{x}_{N_j}(t^{a+2}_{N_j} + 2\,h_{K_j}), \;\; t^{a+2}_{N_j} + 2\,h_{K_j}\Big), \\
&\qquad\quad \widetilde{f}\Big(\widetilde{x}_{N_j}(t^{b+2}_{N_j} + 2\,h_{K_j}), \;\; t^{b+2}_{N_j} + 2\,h_{K_j}\Big)\Big) \\
&\le R_\varepsilon \;+\; \widehat{\omega}_\varepsilon\big(\widetilde{q}_\varepsilon\big(\widetilde{x}_{N_j}(t^{a+2}_{N_j}{+}2\,h_{K_j}), \, \widetilde{x}_{N_j}(t^{b+2}_{N_j}{+}2\,h_{K_j})\big) + (b{-}a)\; h_{N_j}\big) \\
&\le R_\varepsilon \;+\; \widehat{\omega}_\varepsilon\big(c_\varepsilon(t'{-}t + 2\,h_{N_j}) \qquad\qquad\qquad\qquad + \; t'{-}t + 2\,h_{N_j}\big) \\
&\longrightarrow R_\varepsilon \qquad\qquad\qquad\qquad \text{for } j \longrightarrow \infty, \;\; t' - t \downarrow 0 \text{ and all } \widetilde{y}, \widetilde{y}',
\end{aligned}$$

i.e. condition (2.) of Proposition 35 is also satisfied by $(\widetilde{g}_j)_{j\in\mathbb{N}}$.
Finally for verifying assumption (5.) of Convergence Theorem, we benefit from the convergence properties of the subsequence $(\widetilde{x}_{N_j})_{j\,\in\,\mathbb{N}}$ mentioned before. It ensures that for every $t \in [0, T[$ (with $t^a_{N_j} \le t < t^{a+1}_{N_j}$),

$$\begin{aligned}
&\widetilde{Q}^{\mapsto}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t), \, t), \;\; \widetilde{g}_j(\widetilde{x}(t), \, t)\big) \\
&= \widetilde{Q}^{\mapsto}_\varepsilon\big(\widetilde{f}\,(\widetilde{x}(t), \, t)\,, \;\; \widetilde{f}\big(\widetilde{x}_{N_j}(t^{a+2}_{N_j}{+}2\,h_{K_j}), \; t^{a+2}_{N_j}{+}2\,h_{K_j}\big)\big) \\
&\le R_\varepsilon \;+\; \widehat{\omega}_\varepsilon\big(\,\widetilde{q}_\varepsilon\big(\widetilde{x}(t), \; \widetilde{x}_{N_j}(t^{a+2}_{N_j}{+}2\,h_{K_j})\big)\big) \qquad\quad + 2\,h_{K_j} \;+ t^{a+2}_{N_j}{-}t\big) \\
&\le R_\varepsilon \;+\; \widehat{\omega}_\varepsilon\big(\,\widetilde{q}_\varepsilon\big(\widetilde{x}(t), \; \widetilde{x}_{N_j}(t + 2\,h_{K_j})\big)\big) + c_\varepsilon(2\,h_{N_j}) + 2\,h_{K_j} \;+ 2\,h_{N_j}\big) \\
&\longrightarrow R_\varepsilon \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } j \longrightarrow \infty.
\end{aligned}$$

□

Remark 37. (i) Assumption (2.) is only to guarantee the uniform continuity of the Euler approximations $\widetilde{x}_n(\cdot)$. If this property results from other arguments, then we can dispense with this assumption and even with condition (4.) of Definition 17.

(ii) The proof shows that the compactness hypothesis can be weakened slightly. We only need that all $\widetilde{x}_n(t)$ $(0 < t < T, n \in \mathbb{N})$ are contained in a set $\widetilde{F} \subset \widetilde{E}$ that is transitionally compact in $(\widetilde{E}, \, (\widetilde{q}_\varepsilon), \, \widetilde{\Theta}^{\mapsto}_p(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)))$. This modification is useful if each transition $\widetilde{\vartheta} \in \widetilde{\Theta}^{\mapsto}_p(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon))$ has all values in $\widetilde{F}$ after any positive time, i.e. $\widetilde{\vartheta}(t, \widetilde{x}) \in \widetilde{F}$ for all $0 < t \le 1$, $\widetilde{x} \in \widetilde{E}$. In particular, it does not require additional assumptions about the initial value $\widetilde{x}_0 \in \widetilde{E}$.

Corollary 38. (Existence of timed right–hand forward solutions due to timed two–sided sequential compactness)
Suppose that $(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon \in \mathcal{J}})$ *is timed two–sided sequentially compact (uniformly with respect to* ε*). Moreover let* $\widetilde{f} : \widetilde{E} \times [0,T] \longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon))$ *satisfy the assumptions (1.)–(4.) of Proposition 36 for all* $\varepsilon \in \mathcal{J}$.

Then for every $\widetilde{x}_0 \in \widetilde{E}$, *there is a timed right–hand forward solution* $\widetilde{x} : [0,T[\longrightarrow \widetilde{E}$ *of* $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot), \cdot)$ *in* $[0,T[$ *with* $\widetilde{x}(0) = \widetilde{x}_0$.

Proof results directly from Proposition 36 and Remark 34. □

5.3.5. Estimates Finally we extend the estimates of § 5.2 to timed right–hand forward solutions. To be more precise, Propositions 27, 28 and 30 find their counterparts here and their proofs are based on the same notions. So the same obstacles as before keep us from estimates that are easy to apply : Due to the definitions, only elements of $\widetilde{D}$ usually appear in the first argument of $\widetilde{q}_\varepsilon$. Furthermore a solution $\widetilde{x}(\cdot)$ of $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot), \cdot)$ is required to fulfill the condition

$$\limsup_{h \downarrow 0} \ \tfrac{1}{h} \Big(\widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(h, \widetilde{z}),\ \widetilde{x}(t{+}h)\big) \ - \ \widetilde{q}_\varepsilon(\widetilde{z},\ \widetilde{x}(t)) \cdot e^{\widehat{\alpha}_\varepsilon^{\mapsto}(t)\cdot h} \Big) \ \le \ \widehat{\gamma}_\varepsilon(t)$$

with $\widetilde{x}(t+h)$ and $\widetilde{x}(t)$ merely in the second arguments of $\widetilde{q}_\varepsilon$. So we cannot expect an explicit estimate of $\widetilde{q}_\varepsilon(\widetilde{x}(t), \widetilde{y}(t))$ for timed right–hand forward solutions $\widetilde{x}(\cdot), \widetilde{y}(\cdot)$ in general.

Proposition 39. *Assume for* $\widetilde{f} : \widetilde{E} \times [0,T] \longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon))$ *and the curves* $\widetilde{x}, \widetilde{y} \in UC^{\rightarrow}([0,T[, \widetilde{E}, \widetilde{q}_\varepsilon)$

1. a) $\overset{\circ}{\widetilde{y}}(\cdot) \ni \widetilde{f}(\widetilde{y}(\cdot), \cdot)$ *in* $[0,T[$,
   b) $\widetilde{x}(t) \in \widetilde{D}$ *for all* $t \in [0,T[$,
   $\limsup_{h \downarrow 0} \ \frac{1}{h} \ \widetilde{q}_\varepsilon\big(\widetilde{x}(t+h),\ \widetilde{f}(\widetilde{x}(t),t)\,(h, \widetilde{x}(t))\big) \ \le \ \gamma_\varepsilon(\widetilde{f}(\widetilde{x}(t),t))$,
   c) $\widetilde{q}_\varepsilon\big(\widetilde{x}(t), \widetilde{y}(t)\big) \ \le \ \limsup_{h \downarrow 0} \ \widetilde{q}_\varepsilon\big(\widetilde{x}(t{-}h),\ \widetilde{y}(t{-}h)\big)$,
   d) $\pi_1\, \widetilde{x}(0) \ = \ \pi_1\, \widetilde{y}(0) \ = \ 0$,
2. $M_\varepsilon \ := \ \sup_{t, \widetilde{v}} \ \alpha_\varepsilon^{\mapsto}(\widetilde{f}(\widetilde{v},t)) \ < \ \infty$,
3. $\exists\ R_\varepsilon < \infty : \ \sup_{t,\widetilde{v}} \ \gamma_\varepsilon(\widetilde{f}(\widetilde{v},t)) \ \le \ R_\varepsilon, \qquad \varepsilon'^p\, R_{\varepsilon'} \overset{\varepsilon' \downarrow 0}{\longrightarrow} 0$,

4'. $\exists\ \widehat{\omega}_\varepsilon(\cdot), L_\varepsilon$:
   $\widetilde{Q}_\varepsilon^{\mapsto}\big(\widetilde{f}(\widetilde{v}_1,t_1),\ \widetilde{f}(\widetilde{v}_2,t_2)\big) \ \le R_\varepsilon + L_\varepsilon \cdot \widetilde{q}_\varepsilon(\widetilde{v}_1, \widetilde{v}_2) + \widehat{\omega}_\varepsilon(t_2 - t_1)$
   for all $0 \le t_1 \le t_2 \le T$ *and* $\widetilde{v}_1, \widetilde{v}_2 \in \widetilde{E}$ *with* $\pi_1\, \widetilde{v}_1 \le \pi_1\, \widetilde{v}_2$,
   $\widehat{\omega}_\varepsilon(\cdot) \ge 0$ *nondecreasing,* $\limsup_{s \downarrow 0} \ \widehat{\omega}_\varepsilon(s) \ = \ 0$.

Then, $\widetilde{q}_\varepsilon\big(\widetilde{x}(t),\ \widetilde{y}(t)\big) \ \le \ \widetilde{q}_\varepsilon\big(\widetilde{x}(0),\ \widetilde{y}(0)\big) \cdot e^{(L_\varepsilon + M_\varepsilon)\cdot t} \ + \ 5\, R_\varepsilon \ \frac{e^{(L_\varepsilon+M_\varepsilon)\cdot t} - 1}{L_\varepsilon + M_\varepsilon}$.

Proof is a consequence of Gronwall's Lemma 6 : $\varphi_\varepsilon(t) := \widetilde{q}_\varepsilon\big(\widetilde{x}(t), \widetilde{y}(t)\big)$ satisfies the semicontinuity property $\varphi_\varepsilon(t) \ \le \ \limsup_{h \downarrow 0} \ \varphi_\varepsilon(t{-}h)$ according to assumption (1.c).
Moreover, $\limsup_{h \downarrow 0} \ \frac{\varphi(t+h) - \varphi(t)}{h} \ \le \ (L_\varepsilon {+} M_\varepsilon)\ \varphi_\varepsilon(t) \ + \ 5\, R_\varepsilon$ for all t results from Proposition 27 and

$$\begin{aligned} \varphi(t+h) \ \le\ & \widetilde{q}_\varepsilon\big(\widetilde{x}(t{+}h),\ \widetilde{f}(\widetilde{x}(t),t)\,(h,\ \widetilde{x}(t))\big) \\ & + \widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(h,\ \widetilde{x}(t)),\ \widetilde{y}(t{+}h)\big) \\ \le\ & \widetilde{q}_\varepsilon\big(\widetilde{f}(\widetilde{x}(t),t)\,(h,\ \widetilde{x}(t)),\ \widetilde{y}(t{+}h)\big) \ + \ R_\varepsilon\, h \ + \ o(h). \end{aligned}$$

□

Proposition 40.
Assume for $\widetilde{f} : \widetilde{E} \times [0,T] \longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon))$ *and* $\widetilde{x}, \widetilde{y} : [0,T[\longrightarrow \widetilde{E}$

1. $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot), \cdot), \quad \overset{\circ}{\widetilde{y}}(\cdot) \ni \widetilde{f}(\widetilde{y}(\cdot), \cdot)$ *in* $[0,T[$,
   $\pi_1\, \widetilde{x}(0) = \pi_1\, \widetilde{y}(0) = 0$,
2. $M_\varepsilon := \sup_{t,\widetilde{v}} \alpha_\varepsilon^{\mapsto}(\widetilde{f}(\widetilde{v},t)) < \infty$,
3. $\exists\ R_\varepsilon < \infty : \quad \sup_{t,\widetilde{v}} \gamma_\varepsilon(\widetilde{f}(\widetilde{v},t)) \leq R_\varepsilon, \qquad \varepsilon'^p\, R_{\varepsilon'} \overset{\varepsilon'\downarrow 0}{\longrightarrow} 0$,

4'. $\exists\ \widehat{\omega}_\varepsilon(\cdot), L_\varepsilon :$
   $\widetilde{Q}_\varepsilon^{\mapsto}\big(\widetilde{f}(\widetilde{v}_1,t_1),\ \widetilde{f}(\widetilde{v}_2,t_2)\big) \leq R_\varepsilon + L_\varepsilon \cdot \widetilde{q}_\varepsilon(\widetilde{v}_1,\widetilde{v}_2) + \widehat{\omega}_\varepsilon(t_2-t_1)$
   for all $0 \leq t_1 \leq t_2 \leq T$ *and* $\widetilde{v}_1, \widetilde{v}_2 \in \widetilde{E}$ *with* $\pi_1\, \widetilde{v}_1 \leq \pi_1\, \widetilde{v}_2$,
   $\widehat{\omega}_\varepsilon(\cdot) \geq 0$ *nondecreasing,* $\limsup_{s\downarrow 0} \widehat{\omega}_\varepsilon(s) = 0$.

Furthermore suppose the existence of $\lambda_\varepsilon > 0$ *such that for each* $t \in [0,T[$, *the infimum*

$$\varphi_\varepsilon(t) := \inf_{\widetilde{z} \in \widetilde{D},\ \pi_1\, \widetilde{z} \leq t} \big(\widetilde{q}_\varepsilon(\widetilde{z}, \widetilde{x}(t)) + \widetilde{q}_\varepsilon(\widetilde{z}, \widetilde{y}(t))\big) < \infty$$

can be approximated by a sequence $(\widetilde{z}_j)_{j\in\mathbb{N}}$ *in* $\widetilde{D}$ *and* $h_j \downarrow 0$ *with*

$\pi_1 \widetilde{z}_j \leq \pi_1\, \widetilde{z}_k \leq t, \quad \widetilde{q}_\varepsilon(\widetilde{z}_j, \widetilde{z}_k) \leq \lambda_\varepsilon \cdot h_j, \quad h_j < \mathcal{T}_\Theta(\widetilde{f}(\widetilde{z}_j,t), \widetilde{z}_j) \qquad \forall\, j < k.$

Then, $\varphi_\varepsilon(t) \leq \varphi_\varepsilon(0)\, e^{(L_\varepsilon+M_\varepsilon)\cdot t} + 2\,\big((L_\varepsilon{+}1)\,\lambda_\varepsilon + 4\,R_\varepsilon\big) \cdot \frac{e^{(L_\varepsilon+M_\varepsilon)\cdot t} - 1}{L_\varepsilon+M_\varepsilon}$.

Proof follows exactly the same track as for Proposition 28 and is based on the second version of Gronwall's Lemma (i.e. Corollary 8). □

Remark 41. If the above–mentioned sequence $(\widetilde{z}_j)_{j\in\mathbb{N}}$ in $\widetilde{D}$ satisfies

$$\frac{\sup_{k>j}\ \widetilde{q}_\varepsilon(\widetilde{z}_j,\widetilde{z}_k)}{\mathcal{T}_\Theta(\widetilde{f}(\widetilde{z}_j,t), \widetilde{z}_j)} \longrightarrow 0 \qquad (j \longrightarrow \infty)$$

then, $\quad \varphi_\varepsilon(t) \leq \varphi_\varepsilon(0)\, e^{(L_\varepsilon+M_\varepsilon)\cdot t} + 8\,R_\varepsilon \cdot \frac{e^{(L_\varepsilon+M_\varepsilon)\cdot t} - 1}{L_\varepsilon+M_\varepsilon}$.

In the case of symmetric $\widetilde{q}_\varepsilon$ and $\widetilde{D}$ dense in $(\widetilde{E}, \widetilde{q}_\varepsilon)$, we obtain $\varphi_\varepsilon(t) = \widetilde{q}_\varepsilon(\widetilde{x}(t), \widetilde{y}(t))$.

In the following counterpart of Proposition 30, it is a relevant point that the assumptions about $\widetilde{p}_\varepsilon$ do not consist in the comparison of two transitions of $\widetilde{f}$, i.e. regularity condition (9.) on $\widetilde{f}(\widetilde{v}_1,t_1), \widetilde{f}(\widetilde{v}_2,t_2)$ is used only with $\widetilde{Q}_\varepsilon^{\mapsto}$ (induced by $\widetilde{q}_\varepsilon$).

Proposition 42. *Suppose for* $\widetilde{p}_\varepsilon, \widetilde{q}_\varepsilon : \widetilde{E} \times \widetilde{E} \longrightarrow [0,\infty[\ \ (\varepsilon \in \mathcal{J})$, $p \in \mathbb{R}, \lambda_\varepsilon \geq 0$ *and* $\widetilde{f} : \widetilde{E} \times [0,T] \longrightarrow \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon))$, $\widetilde{x}, \widetilde{y} : [0,T[\longrightarrow \widetilde{E}$ *the following properties :*

1. $(\widetilde{E}, (\widetilde{q}_\varepsilon)_{\varepsilon\in\mathcal{J}}, \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}, \widetilde{D}, (\widetilde{q}_\varepsilon)))$ *is timed transitionally compact,*
2. *each* $\widetilde{p}_\varepsilon$ *is symmetric and satisfies the triangle inequality,*
3. $\widetilde{\Delta}_\varepsilon(\widetilde{v}_1, \widetilde{v}_2) := \inf_{\substack{\widetilde{z} \in \widetilde{D},\\ \pi_1 \widetilde{z} \leq \pi_1 \widetilde{v}_2}} \big(\widetilde{p}_\varepsilon(\widetilde{v}_1, \widetilde{z}) + \widetilde{q}_\varepsilon(\widetilde{z}, \widetilde{v}_2)\big) < \infty$ *for* $\widetilde{v}_1, \widetilde{v}_2 \in \widetilde{E}$,
4. $\widetilde{x}(\cdot)$ *is a timed right–hand forward solution of* $\overset{\circ}{\widetilde{x}}(\cdot) \ni \widetilde{f}(\widetilde{x}(\cdot), \cdot)$ *constructed by Euler method according to the proof of Prop. 36,*
5. $\widetilde{y}(\cdot)$ *is a timed right–hand forward solution of* $\overset{\circ}{\widetilde{y}}(\cdot) \ni \widetilde{f}(\widetilde{y}(\cdot), \cdot)$ *in* $[0,T[$ *with* $\pi_1\, \widetilde{x}(0) = \pi_1\, \widetilde{y}(0) = 0$,

6. $\exists\ M_\varepsilon < \infty : \widehat{\alpha}_\varepsilon^{\mapsto}(\cdot, \widetilde{x}, \widetilde{f}(\widetilde{x}, \cdot)),\ \widehat{\alpha}_\varepsilon^{\mapsto}(\cdot, \widetilde{y}, \widetilde{f}(\widetilde{y}, \cdot)) \le M_\varepsilon,$
$$\widetilde{p}_\varepsilon\big(\widetilde{\psi}(h, \widetilde{z}_1),\ \widetilde{\psi}(h, \widetilde{z}_2)\big) \le \widetilde{p}_\varepsilon(\widetilde{z}_1, \widetilde{z}_2) \cdot e^{M_\varepsilon h}$$
$$\forall\ \widetilde{z}_1, \widetilde{z}_2 \in \widetilde{E},\ h \in]0,1[,\ \widetilde{\psi} \in \big\{\widetilde{f}(\widetilde{z}, s) \,\big|\, \widetilde{z} \in \widetilde{E}, s < T\big\},$$

7. $\exists\ R_\varepsilon < \infty : \widehat{\gamma}_\varepsilon(\cdot, \widetilde{x}, \widetilde{f}(\widetilde{x}, \cdot)),\ \widehat{\gamma}_\varepsilon(\cdot, \widetilde{y}, \widetilde{f}(\widetilde{y}, \cdot)) \le R_\varepsilon,$
$$\limsup_{h \downarrow 0} \frac{\widetilde{p}_\varepsilon\big(\widetilde{\psi}(h, \widetilde{\psi}(t, \widetilde{z})),\ \widetilde{\psi}(t+h, \widetilde{z})\big)}{h} \le R_\varepsilon$$
$$\forall\ \widetilde{z} \in \widetilde{E},\ t \in [0,1[,\ \widetilde{\psi} \in \big\{\widetilde{f}(\widetilde{z}, s) \,\big|\, \widetilde{z} \in \widetilde{E}, s < T\big\},$$

8. $\exists\ c_\varepsilon(\cdot) :$
$$\widetilde{p}_\varepsilon\big(\widetilde{\psi}(t, \widetilde{z}), \widetilde{\psi}(t+h, \widetilde{z})\big) + \beta_\varepsilon(\widetilde{\psi})(h) \le c_\varepsilon(h)$$
$$\forall\ \widetilde{z} \in \widetilde{E},\ t \in [0,1[,\ \widetilde{\psi} \in \big\{\widetilde{f}(\widetilde{z}, s) \,\big|\, \widetilde{z} \in \widetilde{E},\ s < T\big\},$$
$$c_\varepsilon(h) \longrightarrow 0 \quad \textit{for } h \downarrow 0,$$

9. $\exists\ \widehat{\omega}_\varepsilon(\cdot), L_\varepsilon :$
$$\widetilde{Q}_\varepsilon^{\mapsto}\big(\widetilde{f}(\widetilde{v}_1, t_1),\ \widetilde{f}(\widetilde{v}_2, t_2)\big) \le R_\varepsilon + L_\varepsilon \cdot \widetilde{\Delta}_\varepsilon(\widetilde{v}_1, \widetilde{v}_2) + \widehat{\omega}_\varepsilon(t_2 - t_1)$$
for all $0 \le t_1 \le t_2 \le T$ *and* $\widetilde{v}_1, \widetilde{v}_2 \in \widetilde{E}$ *with* $\pi_1 \widetilde{v}_1 \le \pi_1 \widetilde{v}_2$,
$\widehat{\omega}_\varepsilon(\cdot) \ge 0$ *nondecreasing,* $\limsup_{s \downarrow 0} \widehat{\omega}_\varepsilon(s) = 0,$

10. *for each* $\widetilde{v} \in \widetilde{E},\ \delta > 0,\ 0 \le s \le t < T,\ 0 < h < 1$ *with* $t+h+\delta < T$, *the infimum* $\widetilde{\Delta}_\varepsilon\big(\widetilde{f}(\widetilde{v}, s)\,(h, \widetilde{v}),\ \widetilde{y}(t+h+\delta)\big)$ *can be approximated by a sequence* $(\widetilde{z}_n)_{n \in \mathbb{N}}$ *in* $\widetilde{D}$ *and* $h_n \downarrow 0$ *such that for all* $m < n$,
$$\pi_1 \widetilde{z}_m \le \pi_1 \widetilde{z}_n \le \pi_1 \widetilde{y}(t+h+\delta), \qquad \widetilde{p}_\varepsilon(\widetilde{z}_m, \widetilde{z}_n) \le \lambda_\varepsilon \cdot h_m,$$
$$h_m < \mathcal{T}_\Theta(\widetilde{f}(\widetilde{v}, s),\ \widetilde{z}_m), \qquad \widetilde{q}_\varepsilon(\widetilde{z}_m, \widetilde{z}_n) \le \lambda_\varepsilon \cdot h_m.$$

Then, $\varphi_\varepsilon(t) := \limsup_{\delta \downarrow 0} \widetilde{\Delta}_\varepsilon\big(\widetilde{x}(t),\ \widetilde{y}(t+\delta)\big)$ *fulfills*
$$\varphi_\varepsilon(t) \le \big(\varphi_\varepsilon(0) + (5\,R_\varepsilon + 2\,\lambda_\varepsilon)\ t\big)\ (1 + L_\varepsilon\, t)\ e^{2\,M_\varepsilon\, t}.$$

Proof. Let $(\widetilde{x}_n(\cdot))_{n \in \mathbb{N}}$ denote the sequence of Euler approximations according to the proof of Proposition 36, i.e. for $n \in \mathbb{N}$ (with $2^n > T$) set
$$b_n := \tfrac{T}{2^n}, \qquad t_n^j := j\, b_n \qquad \text{for } j = 0 \dots 2^n,$$
$$\widetilde{x}_n(0) := \widetilde{x}_0, \qquad \widetilde{x}_0(\cdot) := \widetilde{x}_0,$$
$$\widetilde{x}_n(t) := \widetilde{f}(\widetilde{x}_n(t_n^j),\ t_n^j)\ \big(t - t_n^j,\ \widetilde{x}_n(t_n^j)\big) \quad \text{for } t \in]t_n^j,\ t_n^{j+1}],\ j \le 2^n.$$
Then the Cantor diagonal construction provided a subsequence $(\widetilde{x}_{n_k}(\cdot))$ with the additional property
$$\widetilde{q}_\varepsilon\big(\widetilde{x}(t),\ \widetilde{x}_{n_k}(t + 2\,b_k)\big) \longrightarrow 0 \quad (k \longrightarrow \infty) \qquad \text{for every } t \in [0, T[.$$
Proposition 30 and condition (9.) imply for any $\delta > 0$, $k \in \mathbb{N}$ (with $2\,b_k < \delta$)
$$\begin{aligned}
&\widetilde{\Delta}_\varepsilon\big(\widetilde{x}_{n_k}(t + 2\,b_k),\ \widetilde{y}(t+\delta)\big) \\
\le\ & \widetilde{\Delta}_\varepsilon\big(\widetilde{x}_{n_k}(2\,b_k), \widetilde{y}(\delta)\big) \cdot e^{M_\varepsilon\, t} \\
&+ \int_0^t e^{M_\varepsilon \cdot (t-s)} \Big(R_\varepsilon + L_\varepsilon \cdot \widetilde{\Delta}_\varepsilon\big(\widetilde{x}_{n_k}([\tfrac{s+2\,b_k}{b_{n_k}}]\, b_{n_k}),\ \widetilde{y}(s+\delta)\big) + \widehat{\omega}_\varepsilon(\delta) \\
&\qquad + 4\,R_\varepsilon + 2\,\lambda_\varepsilon\Big)\ ds.
\end{aligned}$$
The triangle inequality of $\widetilde{p}_\varepsilon$ ensures $\widetilde{\Delta}_\varepsilon(\widetilde{v}_1, \widetilde{v}_3) \le \widetilde{p}_\varepsilon(\widetilde{v}_1, \widetilde{v}_2) + \widetilde{\Delta}_\varepsilon(\widetilde{v}_2, \widetilde{v}_3)$ for any $\widetilde{v}_1, \widetilde{v}_2, \widetilde{v}_3 \in \widetilde{E}$ and thus,
$$\begin{aligned}
&\widetilde{\Delta}_\varepsilon\big(\widetilde{x}_{n_k}(t + 2\,b_k),\ \widetilde{y}(t+\delta)\big) \\
\le\ & \widetilde{\Delta}_\varepsilon\big(\widetilde{x}_{n_k}(2\,b_k), \widetilde{y}(\delta)\big) \cdot e^{M_\varepsilon t} \\
&+ \int_0^t e^{M_\varepsilon \cdot (t-s)} \Big(L_\varepsilon\ c_\varepsilon(b_{n_k}) + L_\varepsilon \cdot \widetilde{\Delta}_\varepsilon\big(\widetilde{x}_{n_k}(s+2\,b_k),\ \widetilde{y}(s+\delta)\big) \\
&\qquad + 5\,R_\varepsilon + 2\,\lambda_\varepsilon + \widehat{\omega}_\varepsilon(\delta)\Big)\ ds
\end{aligned}$$

$$\leq \quad \widetilde{\Delta}_\varepsilon\big(\widetilde{x}_{n_k}(2\,b_k),\ \widetilde{y}(\delta)\big)\cdot e^{M_\varepsilon t} + \big(5R_\varepsilon + 2\lambda_\varepsilon + \widehat{\omega}_\varepsilon(\delta) + L_\varepsilon\ c_\varepsilon(b_{n_k})\big)\ e^{M_\varepsilon t}\ t$$
$$+\ e^{M_\varepsilon t}\int_0^t e^{-M_\varepsilon s}\ L_\varepsilon\cdot\widetilde{\Delta}_\varepsilon\big(\widetilde{x}_{n_k}(s+2\,b_k),\ \widetilde{y}(s+\delta)\big)\ ds.$$

Now the well–known integral version of Gronwall's Lemma (strictly speaking, applied to a nondecreasing semicontinuous auxiliary function) provides an upper bound

$$\widetilde{\Delta}_\varepsilon\big(\widetilde{x}_{n_k}(t+2\,b_k),\ \widetilde{y}(t+\delta)\big)\ \cdot\ e^{-M_\varepsilon\,t}$$
$$\leq \quad \widetilde{\Delta}_\varepsilon\big(\widetilde{x}_{n_k}(2\,b_k),\ \widetilde{y}(\delta)\big) \qquad \big(1\ +\ L_\varepsilon\ \ t\ \ e^{M_\varepsilon\,t}\big)$$
$$+\ \big(5\,R_\varepsilon + 2\,\lambda_\varepsilon + \widehat{\omega}_\varepsilon(\delta) + L_\varepsilon\ c_\varepsilon(b_{n_k})\big)\ \big(t\ +\ L_\varepsilon\ \tfrac{t^2}{2}\ e^{M_\varepsilon\,t}\big).$$

So finally we obtain $\quad \widetilde{\Delta}_\varepsilon\big(\widetilde{x}(t),\ \widetilde{y}(t+\delta)\big)$

$$\leq \limsup_{k\longrightarrow\infty}\ \Big(\widetilde{p}_\varepsilon\big(\widetilde{x}(t),\ \widetilde{x}_{n_k}(t+2\,b_k)\big)\ +\ \widetilde{\Delta}_\varepsilon\big(\widetilde{x}_{n_k}(t+2\,b_k),\ \widetilde{y}(t+\delta)\big)\Big)$$
$$\leq \qquad 0 \qquad +\ \limsup_{k\longrightarrow\infty}\ \widetilde{\Delta}_\varepsilon\big(\widetilde{x}_{n_k}(2\,b_k),\ \widetilde{y}(\delta)\big)\ (1+L_\varepsilon\,t) \qquad e^{2\,M_\varepsilon\,t}$$
$$+\big(5\,R_\varepsilon + 2\,\lambda_\varepsilon + \widehat{\omega}_\varepsilon(\delta)\big) \qquad t \qquad (1+L_\varepsilon\,t) \qquad e^{2\,M_\varepsilon\,t}$$
$$\leq \Big(\ \widetilde{\Delta}_\varepsilon(\widetilde{x}(0),\ \widetilde{y}(\delta))\ +\ (5\,R_\varepsilon + 2\,\lambda_\varepsilon + \widehat{\omega}_\varepsilon(\delta)) \qquad t\ \Big)\ \cdot\ (1+L_\varepsilon\,t) \qquad e^{2\,M_\varepsilon\,t}$$

because $\widetilde{\Delta}_\varepsilon\big(\widetilde{x}_{n_k}(2\,b_k),\ \widetilde{y}(\delta)\big)\ \leq\ \widetilde{p}_\varepsilon\big(\widetilde{x}_{n_k}(2\,b_k),\ \widetilde{x}(0)\big) + \widetilde{\Delta}_\varepsilon\big(\widetilde{x}(0),\ \widetilde{y}(\delta)\big)$. □

5.4. Systems of generalized mutational equations

Generalizing mutational equations in the presented way has the useful advantage that components of a system can come from different applications – for example, a first–order geometric evolution and a C^0 semigroup on a reflexive Banach space. To be more precise now, let $(\widetilde{E}_1, \widetilde{D}_1, (\widetilde{q}_\varepsilon^{\,1})_{\varepsilon\in\mathcal{J}_1})$ and $(\widetilde{E}_2, \widetilde{D}_2, (\widetilde{q}_{\varepsilon'}^{\,2})_{\varepsilon'\in\mathcal{J}_2})$ satisfy the general assumptions of this section 5. Furthermore $\widetilde{\Theta}_p^{\mapsto}(\widetilde{E}_1, \widetilde{D}_1, (\widetilde{q}_\varepsilon^{\,1})_{\varepsilon\in\mathcal{J}_1})$ abbreviates timed right–hand forward transitions of order p and, $\widetilde{\Theta}_{p'}^{\mapsto}(\widetilde{E}_2, \widetilde{D}_2, (\widetilde{q}_{\varepsilon'}^{\,2})_{\varepsilon'\in\mathcal{J}_2})$ denotes timed right–hand forward transitions of order p'.

Convention in § 5.4. For the sake of simplicity, we always restrict ourselves to tuples $(\widetilde{x}_1, \widetilde{x}_2) \in \widetilde{E}_1 \times \widetilde{E}_2$ with $\pi_1\,\widetilde{x}_1 = \pi_1\,\widetilde{x}_2$, i.e. the components $\widetilde{x}_1 \in \widetilde{E}_1$, $\widetilde{x}_2 \in \widetilde{E}_2$ refer to the same point of time.
Strictly speaking, we consider elements $(t, x_1, x_2) \in \mathbb{R}\times E_1 \times E_2$ with sets $E_1, E_2 \neq \emptyset$ and prefer the notation $(\widetilde{x}_1,\ \widetilde{x}_2) \stackrel{\text{Def.}}{=} ((t,x_1),\ (t,x_2))$ in the style of preceding sections.

Definition 43. *For* $\widetilde{\vartheta}_1 \in \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}_1, \widetilde{D}_1, (\widetilde{q}_\varepsilon^{\,1}))$ *and* $\widetilde{\vartheta}_2 \in \widetilde{\Theta}_{p'}^{\mapsto}(\widetilde{E}_2, \widetilde{D}_2, (\widetilde{q}_{\varepsilon'}^{\,2}))$, *define*

$$\widetilde{\vartheta}_1 \times \widetilde{\vartheta}_2 : [0,1]\times\widetilde{E}_1\times\widetilde{E}_2 \longrightarrow \widetilde{E}_1\times\widetilde{E}_2,$$
$$(h,\ \widetilde{x}_1,\ \widetilde{x}_2) \longmapsto \big(\widetilde{\vartheta}_1(h,\widetilde{x}_1),\ \widetilde{\vartheta}_2(h,\widetilde{x}_2)\big).$$

These maps $\widetilde{\vartheta}_1\times\widetilde{\vartheta}_2$ induce timed forward transitions of order $\max\{p,p'\}$ on

$$\big(\widetilde{E}_1\times\widetilde{E}_2,\ \ \widetilde{D}_1\times\widetilde{D}_2,\ \ (\widetilde{q}_\varepsilon^{\,1} + \widetilde{q}_{\varepsilon'}^{\,2})_{\varepsilon\in\mathcal{J}_1,\ \varepsilon'\in\mathcal{J}_2}\big)$$

(as it is easy to verify in details). So assuming transitional compactness of both components and suitable conditions on

$$(\widetilde{f}_1,\ \widetilde{f}_2)\ :\ [0,T]\times\widetilde{E}_1\times\widetilde{E}_2\ \longrightarrow\ \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}_1, \widetilde{D}_1, (\widetilde{q}_\varepsilon^{\,1}))\times\widetilde{\Theta}_p^{\mapsto}(\widetilde{E}_2, \widetilde{D}_2, (\widetilde{q}_{\varepsilon'}^{\,2}))$$

the results of § 5.3.4 guarantee the existence of a timed right-hand forward solution $(\widetilde{x}_1, \widetilde{x}_2) : [0,T[\longrightarrow \widetilde{E}_1\times\widetilde{E}_2$ of the generalized mutational equation

$$(\widetilde{x}_1(\cdot),\ \widetilde{x}_2(\cdot))^\circ\ \ni\ \big(\widetilde{f}_1(\widetilde{x}_1(\cdot),\ \widetilde{x}_2(\cdot),\ \cdot),\ \widetilde{f}_2(\widetilde{x}_1(\cdot),\ \widetilde{x}_2(\cdot),\ \cdot)\big).$$

In this context, only *one* asymptotic demand (for $h \downarrow 0$) has to be fulfilled by both components $\widetilde{x}_1(\cdot), \widetilde{x}_2(\cdot)$ simultaneously. So it is not obvious that $(\widetilde{x}_1(\cdot), \widetilde{x}_2(\cdot))$ is a timed right–hand forward solution of the system

$$\wedge \begin{cases} \overset{\circ}{\widetilde{x}}_1(\cdot) \ni \widetilde{f}_1(\widetilde{x}_1(\cdot), \widetilde{x}_2(\cdot), \cdot) \\ \overset{\circ}{\widetilde{x}}_2(\cdot) \ni \widetilde{f}_2(\widetilde{x}_1(\cdot), \widetilde{x}_2(\cdot), \cdot) \end{cases}$$

(i.e. separately with respect to each component). Seizing the notion of Proposition 36, the Euler method can be applied to

$$\big(\widetilde{E}_1 \times \widetilde{E}_2, \ (\widetilde{q}_\varepsilon^{\,1} + \widetilde{q}_{\varepsilon'}^{\,2})_{\varepsilon \in \mathcal{J}_1, \, \varepsilon' \in \mathcal{J}_2}, \ \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}_1, \widetilde{D}_1, (\widetilde{q}_\varepsilon^{\,1})) \times \widetilde{\Theta}_p^{\mapsto}(\widetilde{E}_2, \widetilde{D}_2, (\widetilde{q}_{\varepsilon'}^{\,2}))\big).$$

immediately. In addition, each component of the Euler approximations solves its own 'approximated' mutational equation in $\widetilde{E}_1$ and $\widetilde{E}_2$, respectively. So the key point is to adapt Convergence Theorem 35 to each component of limit function $[0, T[\longrightarrow \widetilde{E}_1 \times \widetilde{E}_2$. (Its proof follows exactly the same track.) Finally Proposition 36 also holds for systems.

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