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Nonlinear Pseudoparabolic Equations and Variational Inequalities

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This thesis is dedicated to my parents.

Abstract

The aim of this thesis is to prove existence and uniqueness of weak solutions for some types of quasilinear and nonlinear pseudoparabolic equations and for some types of quasilinear and nonlinear variational inequalities. The pseudoparabolic equations are characterized by the presence of mixed third order derivatives. Here the existence theory for degenerate parabolic equations is extended to the pseudoprabolic case, and degenerate pseudoparabolic equations with nonlinear integral operator are treated. Furthermore, quasilinear equations, posed on time intervals of the form $(-\infty, T]$, are considered. Some nonlinear pseudoparabolic equations are obtained as reduced form of systems of equations. To show existence, the Galerkin and Rothe methods are used. The system of the degenerate equations, where the term $\partial_t u$ is replace by $\partial_t b(u)$, is solved using the monotonicity and gradient assumptions on the nonlinear function b. The discretization along characteristics is applied to equations with convection. The existence of solutions of variational inequalities is proved by a penalty method; here an inequality is replaced by an equation with an added penalty operator. The uniqueness follows from the monotonicity of the differential operators. In the case of nonlinear pseudoparabolic equations, the uniqueness can be shown for regular solutions only. The needed regularity is shown for two dimensional domains.

Zusammenfassung

Thema dieser Arbeit sind sowohl quasilineare und nichtlineare pseudoparabolische Glei chungen als auch solche Variationsungleichungen. Pseudoparabolische Gleichungen sind durch Auftreten von gemischten Ableitungen von dritter Ordnung charakterisiert. Für einige Typen solcher Gleichungen bzw. Ungleichungen wird in dieser Arbeit die Lösbarkeit gezeigt. In fast allen Fällen kann auch die Eindeutigkeit bewiesen werden. Die Existenztheorie für entartete parabolische Gleichungen wird auf den Fall pseudoparabolischer Gleichungen erweitert. Entartete Gleichungen mit nichtlinearen Integraloperatoren werden ebenfalls behandelt. Außerdem werden quasilineare Gleichungen für Zeitintervalle der Form $(-\infty, T]$ betrachtet. Einige nichtlineare pseudoparabolische Gleichungen erhält man durch Reduktion von Systemen. Für den Beweis der Existenz werden die Rothe- und Galerkin-Methoden benutzt. Die Existenz von Lösungen des Systems entarteter Gleichungen ist unter Annahme der Monotonie und der Rotationsfreiheit der nichtlineare Funktion gezeigt; genauer, die nichtlineare Funktion ist ein Gradient. Die Gleichungen mit Konvektion werden hier entlang der Charakteristiken diskretisiert. Die Existenz von Lösungen für Variationsungleichungen ist mit Hilfe der Strafterm-Methode gezeigt. Die Eindeutigkeit der Lösung folgt aus der Monotonie der Operatoren. Die Eindeutigkeit der Lösung der nichtlinearen Gleichungen ist nur für reguläre Lösungen bewiesen, wobei schwache Lösungen in zwei Dimension schon diese Regularität besitzen.

Contents

Introduction

The pseudoparabolic equations are characterized by the occurrence of mixed third order derivatives, more precisely, second order in space and first order in time, for example

$$u_t - \Delta u_t - \Delta u = f(u).$$

Such equations are used to model fluid flow in fissured porous media (Barenblatt, Entov, and Ryzhik 1990), two phase flow in porous media with dynamical capillary pressure (Cuesta, van Duijn, and Hulshof 1999), heat conduction in two-temperature systems (Gurtin 1968a), and flow of some non-Newtonian fluids (Ting 1963). Pseudoparabolic equations can be used also as regularization of ill-posed transport problems (Barenblatt, Bertsch, Passo, and Ughii 1993; Novick-Cohen and Pego 1991).

The aim of this thesis is to show existence and uniqueness of weak solutions for some types of quasilinear and fully nonlinear pseudoparabolic equations and variational inequalities.

To prove the existence of solutions for pseudoparabolic equations Galerkin's and Rothe's methods are used. Using Galerkin's method, we approximate an infinite dimensional Banach space by a sequence of finite dimensional spaces and obtain an ordinary differential equation. From the theory of ordinary differential equations, i.e. the theorems of Peano and Carathéodory, we get the existence of continuously differentiable or uniformly continuous solutions. A priori estimates and compactness arguments give us the convergence of approximate solutions to a solution of the original problem.

Rothe's method is especially useful if it is applied to equations which are quasilinear in the time derivative. Kacur describes this method and its implementation in his book. By discretization in time, the evolution equation is reduced to a family of stationary equations. These elliptic equations can be solved directly by applying the theory of monotone operators. Alternatively, by Galerkin's method, we obtain nonlinear functional equations in finite dimensional spaces. In both cases, the convergence of the nonlinear monotone terms is shown by applying the Minty-Browder method. For some equations even strong convergence of approximate solutions is obtained.

Pseudoparabolic variational inequalities appear in obstacle problems (Scarpini 1987), and free boundary problems (DiBenedetto and Showalter 1982).

To prove existence of solutions of inequalities a penalty method is used, i.e. an inequality is replaced by an equation with an added penalty operator. The penalty operator is basically defined to be the difference of the identity and a projection on a closed and convex set, such that the unpenalized elements are exactly the elements of the convex set. Increasing the coefficient of the penalty operator yields a sequence of approximate solutions to the inequality. It will be shown that this sequence converges to a solution of the inequality.

The uniqueness follows by the strong monotonicity of the involved operators. In the case of degenerate equations, uniqueness can only be shown for a linear or Lipschitz continuous elliptic part. In the case of fully nonlinear equations, uniqueness can be shown for regular solutions, though the needed regularity can only be proved in two dimensions.

Various properties of solutions of pseudoparabolic equations are known.

Ralph Showalter uses the theory of semigroups to show existence, (Showalter 1969; 1970; 1972; 1975a; Brill 1977), and regularity, (Showalter 1975b; 1975c), of solutions.

Sequences of solutions of appropriately chosen pseudoparabolic equations can be used to approximate a solution of a parabolic equation, see (Showalter and Ting 1970) and (Ting 1969).

The question of regularity of solutions of linear and quasilinear pseudoparabolic equations is considered in (Showalter 1983) and in (Boehm and Showalter 1985a; 1985b; Boehm 1987a; 1987b). Here it is shown that regularity or singularity of the initial data is preserved. The Yosida approximation of a solution of a parabolic equation is actually the solution of a pseudoparabolic equation. For such equations it is shown that local singularities are stationary. More precisely, in (Boehm and Showalter 1985b) the existence of a solution is shown for initial conditions in $W_0^{s,p}(\Omega)$ and exterior forces in $C(0,T;W_0^{s,p}(\Omega))$. This solution is found in $C^1(0,T;W_0^{s,p}(\Omega))$. For forces r- integrable in time it lies in $W^{1,r}(0,T;W_0^{s,p}(\Omega))$, provided $r \in [1,\infty]$, $p > 1$, and s is sufficiently small. For initial conditions and forces of bounded variation in space it is in $W^{1,r}(0,T;BV(\Omega))$. In two dimensions the spatial regularity can be improved. The solution can be found in $W^{1,r}(0,T;W^{2,p})$ for $r \in [1,\infty]$ and $p \geqslant 2$. In (Boehm 1987a; 1987b) the existence and Hölder-continuity of a solution of the nonlinear pseudoparabolic equation is shown. The solution is an element of $C^{0,\delta}(0,T;C^{1,\lambda}(\bar{\Omega}))$ and of $C^{0,\delta}(0,T;W^{1,p}(\Omega))$ for all $p > N$.

In special cases, the differential operator acting upon the time derivative of the solution is invertible and dominates the elliptic operator. Therefore, the pseudoparabolic equation is equivalent to a Banach-space valued ordinary differential equation. In this manner, Gajewski and Zacharias prove strong convergence of a Galerkin approximation in (Gajewski and Zacharias 1970; 1971; 1973).

To discretize a partial differential equation, Crank-Nicolson approximation in time combined with finite element or finite difference scheme can be used, see for example (Ewing 1975a; 1975b; 1978; Ford and Ting 1974; Ford 1976; Wahlbin 1975), and (Gilbert and Lundin 1983). The approximation scheme for pseudoparabolic equations obtained in such a way, has the same order of convergence as for parabolic equations, $k^2 + h^2$. A predictor-corrector-Galerkin approximation is considered in (Ford 1976). The Euler-Galerkin method for quasilinear pseudoparabolic equation in $(0, T) \times \mathbb{R}$ with a periodic boundary condition is presented in (Arnold, Douglas, Jr.Thomée, and Thomée 1981).

There is no classical maximum principle for pseudoparabolic equations. For the nonnegativity of a solution not only nonnegative initial data, but also an extra condition on the elliptic operator is needed, see (Rundell and Stecher 1979; DiBenedetto and Pierre 1981; Showalter 1983), and (Boehm and Showalter 1985a).

Degenerate pseudoparabolic equations of the form $\frac{d}{dt}A(u) + B(u) \ni f$, where A and B are maximal monotone operators from a Hilbert space V to its dual V^*, are considered in (DiBenedetto and Showalter 1981). Existence of a solution is proved by Yosida's approximation, if one of the operator is strongly monotone and the other is a subgradient, $A : V \to V^*$ is compact, one of the operators is coercive, $B : V \to V^*$ is bounded, and $A : V \to W^*$ is bounded, where W is a reflexive Banach space such that V is densely and compactly embedded in W. Nonlinear pseudoparabolic variational inequalities of the form $\left(\frac{d}{dt}A(u(t)) + B(u(t)) - f, v - u(t)\right) \geqslant 0$ were studied in (DiBenedetto and Showalter 1982). To prove existence of a solution the penalty method, described above, is used. The existence is shown if $A : V \to V$ is compact perturbation of the identity and $B : V \to V$ is bounded. The uniqueness of the solution is known if the operator A is linear and self-adjoint and the operator B is strictly monotone.

In (Scarpini 1987) existence of a solution of a degenerate linear pseudoparabolic variational inequality is proved by a regularization method combined with Galerkin's method. The iteration to solve the inequality numerically is introduced. For pseudoparabolic problems with an internal obstacle a convergence rate is obtained, however this rate is slower than that already proved for parabolic inequalities. Regularization and Galerkin methods are also used to solve degenerate quasilinear variational inequalities, where the term $\partial_t u(t, x)$ is replace by $b(t, x)\partial_t u(t, x)$, see (Kenneth and Kuttler 1984).

Some remarks about nonlinear pseudoparabolic equations can be found in the book of Visintin (1996).

In the last years the existence of traveling waves for pseudoparabolic equations was considered, see (Cuesta and Hulshof 2001; van Duijn and Hulshof 2001; Cuesta, van Duijn, and Hulshof 1999), and (Hulshof and King 1998).

The main contribution of this thesis is the generalization of existence and uniqueness results, known for pseudoparabolic equations, to systems of more general degenerate pseudoparabolic equations, which may contain a nonlinear integral operator, to doubly nonlinear equations, and to fully nonlinear equations. Furthermore, more general forms of degenerate variational inequalities are studied. Without assumptions on the behavior of the solution at $-\infty$ the existence and uniqueness of solutions of quasilinear pseudoparabolic equations and variational inequalities in unbounded time intervals of the form $(-\infty, T]$ were not considered before.

This work consists of four chapters.

In Chapter 1 physical applications are presented. A model for fluid flow in fissured porous media, (Barenblatt, Entov, and Ryzhik 1990), and a model for two phase flow in porous media with a non-static relationship between pressure difference and saturation of wetting phase are considered. Reasons for this non-static capillary condition can be found in (Barenblatt, Garcia-Azorero, De Pablo, and Vazquez 1997; Hassanizadeh and Gray 1993; Beliaev and Hassanizadeh 1993).

The next three chapters deal with existence and uniqueness of solutions of nonlinear pseudoparabolic equations and variational inequalities.

In Chapter 2 Rothe's method is used. In Section 2.1 a system of degenerate pseudoparabolic equations, where the term $\partial_t u$ is replace by $\partial_t b(u)$, is considered. We consider fluid flow in a material with memory. Here the memory effect is described by a nonlinear integral operator. Its kernel may be weakly singular, i.e. it is dominated by a term of order $t^{-\gamma}$ for some $0 < \gamma < 1/p$, where $p \geqslant 2$. The existence of a solution is obtained by applying Rothe-Galerkin's method (Theorem 2.1.3). The crucial assumptions to guarantee existence and uniqueness are monotonicity and potentiality (Alt and Luckhaus 1983) of the nonlinear function b, that means, the nonlinear function is a gradient of a convex, continuously differentiable function. Due to these assumptions, the integration by parts formula is valid, see Lemma A.1.3. The discretization of integral operators is used similarly to (Kacur 1999), i.e. we define an approximation as a function, piecewise constant on a partition of the time interval. Using the strong monotonicity of the elliptic part, we prove the strong convergence of the approximate solutions. For uniqueness we need to assume linearity, Theorem 2.1.9, or Lipschitz-continuity, Theorem 2.1.8, of the function defining the diffusion operator.

In Section 2.2 a variational inequality is considered. The existence of solutions of quasilinear variational inequalities is proved under stronger assumptions, namely the nonlinear function defining the elliptic part is assumed to be a gradient and the nonlinear function b is Lipschitz continuous, Theorem 2.2.4.

In Section 2.3, the existence of a solution of a doubly nonlinear pseudoparabolic equation by the Rothe-Galerkin method is shown, Theorem 2.3.3. The doubly nonlinear parabolic equations are considered in (Jäger and Kacur 1995; Kacur 1998). Here the integration by parts formula from (Jäger and Kacur 1995) for parabolic equations is generalized to the case of pseudoparabolic equations. The uniqueness is shown for the Lipschitz-continuous function defining the diffusion operator, Theorem 2.3.7.

The method of characteristics for parabolic equations with convection is used in (Douglas and Russell 1982; Kacur 2001; Kacur and Keer 2001). This method can also be applied in the case of pseudoparabolic equations with convection. This is done in Section 2.4. In this case an approximate solution is obtained as a solution to a discretized differential equation along the approximated characteristics. The convergence of the family of approximate solutions to a solution of the pseudoparabolic equation is shown in Theorem 2.4.3. The uniqueness is proved for the linear elliptic parts and for the space dimension $N \leq 4$, Theorem 2.4.7.

In Chapter 3 existence and uniqueness of solutions of quasilinear equations and inequalities without initial conditions in $C((-\infty, T]; H^1(\Omega)) \cap L^p_{\text{loc}}((-\infty, T]; H^{1,p}(\Omega))$ is proved. Similar to the case of parabolic equations, see (Bokalo 1989), we get the uniqueness independent of an additional assumption on the behavior of the solution at $-\infty$, Theorem 3.1.4. For the proof we use the Pankov Lemma, Lemma A.2.2. We obtain the existence of a weak solution in the sense of Definition 3.1.2 using Galerkin's method. At first, we solve the problem in a bounded time interval with zero initial condition, Theorem 3.1.9. Assuming strong monotonicity of the nonlinear elliptic part provides us with strong convergence of a sequence of approximate solutions. The sequence of approximate solutions is constructed as solutions of the problem with vanishing initial conditions on a sequence of monotonically increasing family of time-intervals, exhausting $(-\infty, T)$. We use cut-off functions to show the strong convergence of this sequence to a solution of the original problem, Theorem 3.1.6. These results are already published in (Lavrenyuk and Ptashnik 2000).

The corresponding variational inequality is considered in Section 3.2. Under additional restrictions on the nonlinear functions and on the right hand side, we show existence of solutions of the pseudoparabolic variational inequality posed in unbounded time intervals, see Theorem 3.2.2. The uniqueness of solution of the variational inequality is proved in Theorem 3.2.5.

For $p = 2$, existence and uniqueness of a solution in the class of functions of at most exponential growth is proved in (Lavrenyuk and Ptashnik 1998; Ptashnyk 2002).

In Chapter 4 fully nonlinear pseudoparabolic equations and variational inequalities are considered. It is proved that the solution of a nonlinear pseudoparabolic equation is the quasistationary state of a system with cross diffusion, modeling the reaction and the diffusion of two biological, chemical, or physical substances if one of them does not diffuse. Since one of the equations is an ordinary differential equation, for a special kinetic function the system is reduced to the pseudohyperbolic equation. At first, the existence of a solution of a quasilinear pseudohyperbolic equation is shown using Galerkin's approximation in Theorem 4.1.5. For a priori estimates and convergence the monotonicity and the growth assumptions on nonlinear functions are used. For the strong convergence the strong monotonicity of the nonlinear functions is needed. Secondly, the convergence of the sequence of solutions to a solution of the pseudoparabolic equation is shown in Theorem 4.1.7. The uniqueness follows from the strong monotonicity and can only be shown for sufficiently regular solutions, Theorem 4.1.11. The needed regularity is proved in two dimensions, Theorems 4.1.8, 4.1.9. The existence of a solution of the pseudoparabolic variational inequality is proved in Theorem 4.2.2. The uniqueness of the solution of the inequality can be shown only by additional regularity assumption on the solution, Theorem 4.2.3.

In the Appendix, some theorems and lemmata, which are used in the thesis, are collected for convenience.

For additional reference on heat conduction problems in materials with memory and parabolic integro-differential equations see (Crandall, Londen, and Nohel 1978; Dafermos and Nohel 1979; Heard 1982; Engler 1984), and (Engler 1996).

For additional reference on quasilinear and doubly nonlinear parabolic equations see (Raviart 1970; Grance and Mignot 1972; Bamberger 1977; Kröner and Rodrigues 1985), and (Blanchard and Francfort 1988).

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1

Models Leading to Pseudoparabolic Equations

In this chapter two models for physical processes leading to pseudoparabolic equation are presented.

1.1 Fluid Flow in Fissured Porous Media

The first model describes the flow of fluids in a fissured porous medium (Barenblatt, Entov, and Ryzhik 1990). Fissured porous media consist of porous permeable blocks separated by a system of fissures. Here, both components, porous blocks and fissures, have nonzero porosity and permeability. An example of such media is limestone.

We assume that the blocks are large in comparison with the size of the pores, but small in comparison with the size of the whole reservoir, the volume of the fissures is very small in comparison with the total volume of the solid matrix or the volume of porous blocks, and the permeability of porous blocks is very low. It is characteristic for a fissured porous medium that the fluid flows through the fissures, even though their total volume is small. Because flow in the fissures is much more rapid than inside the porous blocks, the fluid does not flow directly from one block to another. Rather, it first flows into the fissure system, and then it can pass into a block or remain in the fissures.

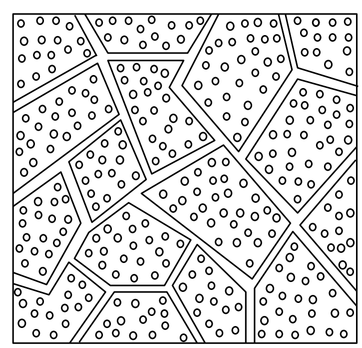

Figure 1.1.1.
Fissured porous medium

For the fissured porous media we have three different scales: the size of pores, as a microscopic scale, the size of fissures, as a mesoscopic scale, and the size of the medium, as a macroscopic scale. The macroscopic model for the double-porosity problem can also be obtained by applying the homogenization theory (Hornung 1996).

The essential point in the construction of the macroscopic fissured-porous-medium model in (Barenblatt, Entov, and Ryzhik 1990) is to introduce at each point in space two fluid pressures, the pressure in the fissures p_1 and the pressure in the porous blocks p_2. Both pressures are actually mean pressure values, averaged over scales, large in comparison with the scales of blocks, but small in comparison with the size of the flow region. Hence, the seepage flux for the blocks is very small, the equations for the mass conservation law in the fissures and in the blocks are

$$\partial_t(S_1\rho) + \nabla\cdot(\rho v) - q = 0, \qquad \partial_t(S_2\rho) + q = 0,$$

where S_1 is the saturation in the fissure, S_2 is the saturation in the blocks, v is the velocity in the fissures, q is the in-flow rate of fissures, the amount of fluid that flows per unit time and per unit volume of the medium from the porous blocks to the fissures, and ρ is the fluid density.

Since the saturation in the fissures is very small, the first term in the first equation is very small in comparison to the second, and we obtain

$$\nabla\cdot(\rho v) - q = 0.$$

In order to obtain a closed system of equations, we need the relations for the flow rate q and for the increment of the saturation S_2. The flow rate q can be governed by the pressures p_1 and p_2, the size l, the permeability k_2 of the blocks, and the fluid viscosity μ and density ρ. Furthermore, since q should vanish when $p_1 = p_2$, we obtain

$$q = \alpha\frac{\rho k_2}{\mu\, l^2}(p_2 - p_1),$$

where α is a dimensionless constant, characterizing the geometry. We can assume that the increment of the saturation of the blocks is a linear function of the pressure increment, since the influence of fissures is very small, this is

$$\partial_t S_2 = \beta\partial_t p_2.$$

The velocity is given by Darcy's law,

$$v = -\frac{k_1}{\mu}\nabla p_1,$$

where k_1 is the permeability of the fissures. Thus, we obtain two equations for the pressures

$$\nabla\cdot(k_1\nabla p_1) + \frac{\alpha k_2}{l^2}(p_2 - p_1) = 0,$$
$$\partial_t p_2 + \frac{\alpha k_2}{\beta l^2\mu}(p_2 - p_1) = 0.$$

In this system p_2 can be eliminated. Hence, p_1 satisfies

$$\partial_t p_1 - \eta\,\partial_t\Delta p_1 = k\Delta p_1,$$

where $\eta = k/A = (k_1 l^2)/(\alpha k_2)$. An analogous equation is obtained for the pressure p_2.

1.2 The Two-Phase Flow in Porous Media with Dynamical Capillary Pressure

In the second model the two-phase flow in porous media with dynamical capillary pressure is considered, (Cuesta, van Duijn, and Hulshof 1999; Hassanizadeh and Gray 1993).

Capillary pressure is an essential characteristic of two-phase flow in porous media. In the standard approach, capillary pressure is expressed as a monotone function of the wetting phase saturation and is equal to the difference of the pressures in the wetting and non-wetting phases. Here, the model of Gray and Hassanisadeh is presented. They propose to include the dynamical effects of the system in the relation between the pressures difference and the saturation.

The two phases in this model are water (wetting phase) and air (non-wetting phase). For water in a homogeneous and isotropic porous medium, we have the momentum balance equation (Darcy's law)

$$q = -K(S)(\nabla p_w + \rho g) \tag{1.2.1}$$

and the mass balance equation

$$\phi \partial_t(\rho S) + \nabla \cdot (\rho q) = 0. \tag{1.2.2}$$

Here q denotes the volumetric water flux, S water saturation, $K(S)$ hydraulic conductivity, p_w water pressure, ρ water density, ϕ porosity, and g a gravity constant. To solve these equations, an additional relation between p_w and S is needed. For this relation it is assumed that the air pressure p_a is constant and the static conditions hold, see (Bear 1988),

$$p_a - p_w = p_c(S), \tag{1.2.3}$$

where p_c denotes the capillary pressure.

For the processes with slowly monotonically varying water saturation the equilibrium condition (1.2.3) can be accepted. For the fast processes, for example capillary imbibition, the hysteresis and dynamical effect are important and the capillary equation (1.2.3) has to be modified.

To derive the dynamical relation between saturation S and pressures difference $p_a - p_w$, Hassanizadeh and Gray (1993) gave a definition of the capillary pressure $p_c(S)$ as a thermodynamic parameter in terms of the free energy functions of the phases, independent of $p_a - p_w$, more precisely, they set

$$p_c = -S\frac{\partial A_w}{\partial S} - (1-S)\frac{\partial A_n}{\partial S} - \frac{1}{\varepsilon}\sum_{\alpha\beta}\frac{A_{\alpha\beta}}{\partial S},$$

where A_w is the Helmholtz free energy of the wetting phase per unit volume of the phase, A_n is the Helmholtz free energy of the non-wetting phase, $A_{\alpha\beta}$ is the Helmholtz free energy of the interface per unit volume of the porous medium. Then we have the entropy inequality of the form

$$-\partial_t S[(p_a - p_w) - p_c] \geqslant 0. \tag{1.2.4}$$

This inequality requires that $\partial_t S$ must be negative, i.e. the system will undergo drainage, if $p_a - p_w > p_c$, and $\partial_t S$ must be positive, i.e. imbibition occurs, if $p_a - p_w < p_c$. Only in equilibrium, $\partial_t S = 0$ and no change of saturation is occurring, $p_a - p_w = p_c$. Though the relation (1.2.3) is not the definition of $p_c(S)$, but a constitutive approximation valid only in equilibrium.

From the thermodynamic conditions, we obtain the equation

$$p_a - p_w = p_c(S) - \phi L(S)\partial_t S, \tag{1.2.5}$$

where $L(S)$ is a nonlinear damping term.

The general inequality (1.2.4) and the approximation (1.2.5) suggest that, at a given point in the system and at a given time, saturation will change locally in order to restore the equilibrium and the equivalence between $p_a - p_w$ and p_c.

In the models with hysteresis we obtain the relation $p_a - p_w \in p_c(S)$ with multivalued function $p_c(S)$ if $\partial_t S = 0$. The equilibrium value of p_c as a function of S depends on the direction of the process, i.e. drainage or imbibition.

Now, from the equations (1.2.1), (1.2.2), and (1.2.5), a single equation can be obtained for the water saturation S,

$$\phi\partial_t(\rho S) = \nabla \cdot \{\rho K(S)\rho g + \rho K(S)\nabla(-p_c(S) + \phi L(S)\partial_t S)\}. \tag{1.2.6}$$

By choosing a new variable $\sigma = \mathcal{L}(S)$, where $\mathcal{L}(S) = \int\limits_0^S L(r)dr$ and $L(S)\partial_t S = \partial_t \mathcal{L}(S)$, the equation (1.2.6) can be rewritten in the form

$$\partial_t b(\sigma) = \nabla \cdot d(\sigma, \nabla\sigma) + \Delta\partial_t \sigma.$$

Equations of such structure with memory terms will be solved in the Section 2.1. The memory operator is obtained by modeling the two-phase flow in elastic porous media. For a medium with memory, Darcy's law has the form

$$q(t,x) = -k(t,x)\nabla p(t,x) - \int\limits_0^t K(t,s)\nabla p(s,x)ds.$$

The models for heat transport in material with memory can be found in (Gurtin and Pipkin 1968; Gurtin 1968b; Nunziato 1971; Miller 1978; Heard 1982).

The generalized Darcy's law in the integral form

$$v(t,x) = v_0(x) + \int\limits_0^t A(t-s)(f - \nabla p)(s,x)\,ds$$

is obtained also by the homogenization of unsteady Stokes problem, see (Hornung 1996).

2

Rothe's Method for Quasilinear and Nonlinear Equations and Inequalities

The existence of solutions of degenerate quasilinear equations with memory terms, quasilinear variational inequalities, doubly nonlinear equations, and equations with convection is shown. In the last case time discretization is used along characteristics. The existence of solutions of degenerate quasilinear equations is proved under the assumption that the nonlinear function b is monotone and a gradient of a convex, continuously differentiable function. The existence of solutions of quasilinear variational inequalities is proved under stronger assumptions, namely, the nonlinear function defining the elliptic part is assumed to be a gradient and the function b to be Lipschitz continuous. The uniqueness of the solution of a degenerate quasilinear or doubly nonlinear equation is proved for linear or Lipschitz-continuous elliptic parts. The uniqueness of the solution of equations with convection is proved for linear elliptic parts and for space dimensions $N \leq 4$.

2.1 Degenerate Quasilinear Pseudoparabolic Equations with Memory Terms

In this section a system of degenerate quasilinear pseudoparabolic equations with memory term is considered. Such equations describe the two-phase fluid flow in porous media with dynamical capillary pressure, as introduced in section 1.2.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary. The initial boundary value problem is given by

$$\begin{cases} \partial_t b^j(u) - \nabla \cdot (a(x)\nabla u_t)^j - \nabla \cdot d^j(t,x,u,\nabla u) + M^j(u) = f^j(u) & \text{in } Q_T = (0,T)\times\Omega, \\ u^j = 0 & \text{on } (0,T)\times\partial\Omega, \\ b^j(u(0,x)) = b^j(u_0(x)) & \text{in } \Omega, \end{cases} \tag{2.1.1}$$

where the memory operator M is defined by

$$\langle M^j(t)(u), v^j\rangle = \int_\Omega \int_0^t K^j(t,s)\, g^j(s,x,\nabla u(s,x))\, ds\, \nabla v^j(t,x)\, dx$$

for all functions $u,\ v\ \in L^p(0,T;H_0^{1,p}(\Omega)^l)$, for almost all $t \in (0,T)$.

The existence of a solution will be ensured by the following assumptions.

Assumption 2.1.1.

$(A1)$ The vector field $b : \mathbb{R}^l \to \mathbb{R}^l$ is monotone nondecreasing and a continuous gradient, i.e. there exists a convex C^1 function $\Phi : \mathbb{R}^l \to \mathbb{R}$ such that $b = \nabla\Phi$, and $b(0) = 0$.

$(A2)$ The tensor field $a \in (L^\infty(\Omega))^{N\times l\times N\times l}$, considered as a linear mapping on $L^\infty(\Omega)^{N\times l}$, is symmetric and elliptic, i.e. for some $0 < a_0 \le a^0 < \infty$, a satisfies $a_0|\xi|^2 \le a(x)\xi\,\xi \le a^0|\xi|^2$ for $\xi \in \mathbb{R}^{N\times l}$ and for almost all $x \in \Omega$.

$(A3)$ The diffusivity $d : (0,T)\times\Omega\times\mathbb{R}^l\times\mathbb{R}^{N\times l} \to \mathbb{R}^{N\times l}$ is continuous, elliptic, i.e. $d(t,x,\eta,\xi)\,\xi \geqslant d_0|\xi|^p$ for every $\xi \in \mathbb{R}^{N\times l}$, $d_0 > 0$, $p \geqslant 2$, strongly monotone, i.e. $(d(t,x,\eta,\xi_1) - d(t,x,\eta,\xi_2))\,(\xi_1 - \xi_2) \geqslant d_1|\xi_1-\xi_2|^p$ for $\xi_1,\xi_2 \in \mathbb{R}^{N\times l}$, $d_1 > 0$, and satisfies the growth assumption $|d(t,x,\eta,\xi)| \le C(1+|\eta|^{p-1}+|\xi|^{p-1})$ for $\eta \in \mathbb{R}^l$.

$(A4)$ The function $f : (0,T)\times\Omega\times\mathbb{R}^l \to \mathbb{R}^l$ is continuous and sublinear, i.e. $|f(t,x,\eta)| \le C(1+|\eta|)$ for $\eta \in \mathbb{R}^l$ and for almost all $(t,x) \in Q_T$.

$(A5)$ The matrix field $g : (0,T)\times\Omega\times\mathbb{R}^{N\times l} \to \mathbb{R}^{N\times l}$ is continuous, satisfies the growth assumption $|g(t,x,\xi)| \le C(1+|\xi|^{p-1})$, and $|g(t,x,\xi_1) - g(t,x,\xi_2)| \le C|\xi_1-\xi_2|^{p-1}$.

$(A6)$ The kernel $K : (0,T)\times(0,T) \to \mathbb{R}^l$ is weakly singular, i.e. $|K(t,s)| \le |t-s|^{-\gamma}\omega(t,s)$, for some $0 \le \gamma < 1/p$ and continuous $\omega : [0,T]\times[0,T] \to \mathbb{R}$.

$(A7)$ The initial condition u_0 is in $H_0^1(\Omega)^l$, and $b(u_0)$ is in $L^1(\Omega)^l$ and in $H^{-1}(\Omega)^l$.

The notion of a solution of the problem introduced above, will be given now. It is appropriate to show global existence.

Definition 2.1.2. A function $u : Q_T \to \mathbb{R}^l$ is called *a weak solution* of the problem (2.1.1) if it satisfies the following:

1) $u \in L^p(0,T;H_0^{1,p}(\Omega)^l)$, $u \in L^\infty(0,T;H_0^1(\Omega)^l)$, and $b(u) \in L^\infty(0,T;L^1(\Omega)^l)$, $\partial_t(b(u) - \nabla\cdot(a(x)\nabla u)) \in L^q(0,T;H^{-1,q}(\Omega)^l)$,

2) u satisfies the equality

$$-\int_0^T\int_\Omega \Big(b(u)\,v_t + a(x)\nabla u\,\nabla v_t\Big)\,dx\,dt + \int_0^T\int_\Omega \Big(b(u_0)\,v_t + a(x)\nabla u_0\,\nabla v_t\Big)\,dx\,dt$$
$$+\int_0^T\int_\Omega d(t,x,u,\nabla u)\,\nabla v\,dx\,dt + \int_0^T \langle M(u),v\rangle\,dt = \int_0^T\int_\Omega f(t,x,u)\,v\,dx\,dt \qquad (2.1.2)$$

for all test functions $v \in L^p(0,T;H_0^{1,p}(\Omega)^l)$, such that $v_t \in L^2(0,T;H_0^1(\Omega)^l) \cap L^1(0,T;L^\infty(\Omega)^l)$ and $v(T) = 0$.

We define the function

$$B(z) := b(z) \cdot z - \Phi(z) - \Phi(0) = \int_0^1 (b(z) - b(sz)) \cdot z \, ds = \int_0^z (b(z) - b(s)) \, ds.$$

The properties of the function B can be found in Lemma A.1.1. The function B will be used in the integration by parts formula, Lemma A.1.2 and Lemma A.1.3.

At first we formulate the existence result.

Theorem 2.1.3 (Existence).
Suppose Assumption 2.1.1 is satisfied. Then there exists a weak solution of the problem (2.1.1).

We approximate the differential equation by the time discretization, $h = T/n$, $t_i = ih$, $i = 0, ..., n$, and obtain the discrete problem

$$\begin{aligned} \frac{1}{h}(b(u_i) - b(u_{i-1})) - \frac{1}{h}\nabla \cdot (a(x)(\nabla u_i - \nabla u_{i-1})) & & & \\ -\nabla \cdot d(t_i, x, u_{i-1}, \nabla u_i) + M(\hat{u}_{i-1}) - f(t_i, x, u_{i-1}) &= 0 \quad && \text{in } \Omega, \\ u_i(x) &= 0 \quad && \text{on } \partial\Omega, \end{aligned} \tag{2.1.3}$$

where the function $\hat{u}_{i-1}$ is defined by

$$\hat{u}_{i-1} = \begin{cases} u_{j-1}, & t \in [t_{j-1}, t_j), \ j = 1, ..., i-1, \\ u_{i-1}, & t \in [t_{i-1}, T]. \end{cases}$$

Thus, we obtain elliptic problems, which can be solved by Galerkin's procedure. Let $\{e_k\}_{k=1}^{\infty}$ be a basis of $H_0^{1,p}(\Omega)^l$ and $e_k \in L^\infty(\Omega)^l$. We are looking for functions $\{u_i^m\}_{i=1}^n$ in the subspace H_m, spanned by $\{e_1, ..., e_m\}$,

$$u_i^m = \sum_{k=1}^m \alpha_{ik}^m e_k,$$

such that

$$\begin{aligned} &\int_\Omega \frac{1}{h}(b(u_i^m) - b(u_{i-1}^m)) \, \xi \, dx + \int_\Omega \frac{1}{h} a(x)(\nabla u_i^m - \nabla u_{i-1}^m) \, \nabla \xi \, dx \\ &+ \int_\Omega d(t_i, x, u_{i-1}^m, \nabla u_i^m) \, \nabla \xi \, dx + \langle M(\hat{u}_{i-1}^m), \xi \rangle - \int_\Omega f(t_i, x, u_{i-1}^m) \, \xi \, dx = 0 \end{aligned} \tag{2.1.4}$$

holds for all $\xi \in H_m$. Here $u_0^m \in H_m$ is an approximation of u_0 in $H_0^1(\Omega)^l$.

Lemma 2.1.4. *There exists a solution u_i^m in H_m of the family of discretized equations (2.1.4).*

Proof. The existence will be shown by induction. Since u_0^m is given, u_{i-1}^m can be assumed to be known. The left-hand side of (2.1.4) defines a continuous mapping $J_{hm} : \mathbb{R}^m \to \mathbb{R}^m$ given by

$$\begin{aligned} J_{hm}^j(r) = \; & \frac{1}{h} \int_\Omega (b(v)e_j + a(x)\nabla v \, \nabla e_j) \, dx - \frac{1}{h} \int_\Omega (b(u_{i-1}^m)e_j + a(x)\nabla u_{i-1}^m \, \nabla e_j) \, dx \\ & + \int_\Omega d(t_i, x, u_{i-1}^m, \nabla v) \, \nabla e_j \, dx + \langle M(\hat{u}_{i-1}^m), e_j \rangle - \int_\Omega f(t_i, x, u_{i-1}^m) \, e_j \, dx, \end{aligned}$$

where $v = \sum_{j=1}^{m} r_j\, e_j$. This mapping satisfies the following estimates:

$$
\begin{aligned}
J_{hm}(r)r \;&\geqslant\; d_0 \int_\Omega |\nabla v|^p\,dx + \frac{1}{h}\int_\Omega B(v)\,dx + \frac{a_0}{2h}\int_\Omega |\nabla v|^2\,dx - \frac{1}{h}\int_\Omega B(u^m_{i-1})\,dx - \frac{a^0}{2h}\int_\Omega |\nabla u^m_{i-1}|^2\,dx \\
&\quad -c_1\delta\int_\Omega |\nabla v|^p\,dx - c_2(\delta)c(\gamma)\sum_{k=1}^{i} h\int_\Omega |\nabla u^m_{k-1}|^p\,dx - c_3\int_\Omega |f(t_i,x,u^m_{i-1})|^2\,dx \\
&\geqslant\; c_4\int_\Omega |\nabla v|^2\,dx + c_5\int_\Omega |\nabla v|^p\,dx - c_6,
\end{aligned}
$$

as will be shown now. From the assumption on b and the definition of B it follows that

$$
\frac{1}{h}\int_\Omega (b(v) - b(u^m_{i-1}))\,v\,dx \;\geqslant\; \frac{1}{h}\int_\Omega B(v)\,dx - \frac{1}{h}\int_\Omega B(u^m_{i-1})\,dx.
$$

The assumptions on a and d imply

$$
\begin{aligned}
\int_\Omega d(t_i,x,u^m_{i-1},\nabla v)\,\nabla v\,dx \;&\geqslant\; d_0\int_\Omega |\nabla v|^p\,dx, \\
\frac{1}{h}\int_\Omega a(x)(\nabla v - \nabla u^m_{i-1})\,\nabla v\,dx \;&\geqslant\; \frac{a_0}{2h}\int_\Omega |\nabla v|^2\,dx - \frac{a^0}{2h}\int_\Omega |\nabla u^m_{i-1}|^2\,dx.
\end{aligned}
$$

Applying Hölder's and Young's inequalities yields

$$
\langle M(\hat u^m_{i-1}), v\rangle \le c_1/\delta\int_\Omega \Big(\int_0^{t_i} K(t_i,s)\,g(s,x,\nabla\hat u^m_{i-1})\,ds\Big)^q\,dx + c_2\delta\int_\Omega |\nabla v|^p\,dx,
$$

where $1/q + 1/p = 1$. The first integral can be estimated by using the assumptions on K and g, and the boundedness of ω,

$$
\begin{aligned}
\Big|\int_0^{t_i} K(t_i,s)\,g(s,x,\nabla\hat u^m_{i-1})\,ds\Big| \;&\le\; \sum_{k=1}^{i}\int_{t_{k-1}}^{t_k} |K(t_i,s)||g(s,x,\nabla\hat u^m_{i-1})|\,ds \\
&\le\; c_1\sum_{k=1}^{i}(1+|\nabla u^m_{k-1}|^{p-1})\int_{t_{k-1}}^{t_k}(t_i-s)^{-\gamma}\,ds \\
&\le\; \Big(\sum_{k=1}^{i} h|\nabla u^m_{k-1}|^p\Big)^{1/q}\Big(\sum_{k=1}^{i} h(t_i-t_k)^{-\gamma p}\Big)^{1/p} + c_2.
\end{aligned}
$$

Since $\gamma < 1/p$,

$$
\sum_{k=1}^{i} h(t_i-t_k)^{-\gamma p} \;\le\; \frac{1}{1-p\gamma} =: c(\gamma).
$$

Due to sublinearity of f and Poincaré's inequality,

$$
\int_\Omega |f(t_i,x,u^m_{i-1})|^2\,dx \;\le\; c_1\int_\Omega |\nabla u^m_{i-1}|^2\,dx + c_2.
$$

Hence, for $|r|$ big enough, $J(r)r \geqslant 0$ for all such r. The continuity of J implies the existence of a zero of $J(r)$, i.e. a solution of the discretized equation (2.1.4), see (Showalter 1996, Proposition 2.1). ❐

Now convergence of u_i^m to the solution u of the problem (2.1.1) for $n, m \to \infty$ is shown. For the proof a priori estimates, compactness arguments, and an integration by parts formula from (Alt and Luckhaus 1983), adapted for pseudoparabolic equations, are used.

At first we obtain the estimates for u_i^m.

Lemma 2.1.5. *The estimates*

$$\begin{aligned} \max_{1\le j\le n} \int_\Omega B(u_j^m)\,dx &\le C, \\ \max_{1\le j\le n} \int_\Omega |\nabla u_j^m|^2\,dx &\le C, \\ \sum_{i=1}^n h \int_\Omega |\nabla u_i^m|^p\,dx &\le C \end{aligned} \tag{2.1.5}$$

hold uniformly in m and n.

Proof. Choosing u_i^m as a test function in (2.1.4) and summing over i yield

$$\begin{aligned} &\sum_{i=1}^j \int_\Omega \frac{b(u_i^m) - b(u_{i-1}^m)}{h}\, u_i^m\,dx + \sum_{i=1}^j \int_\Omega a(x)\nabla \frac{u_i^m - u_{i-1}^m}{h}\, \nabla u_i^m\,dx \\ &+ \sum_{i=1}^j \int_\Omega d(t_i, x, u_{i-1}^m, \nabla u_i^m)\,\nabla u_i^m\,dx + \sum_{i=1}^j \langle M(\hat u_{i-1}^m), u_i^m\rangle = \sum_{i=1}^j \int_\Omega f(t_i, x, u_{i-1}^m)\,u_i^m\,dx. \end{aligned} \tag{2.1.6}$$

Each term will be dealt separately. From the assumption on b and the definition of the function B it follows that

$$\sum_{i=1}^j \int_\Omega (b(u_i^m) - b(u_{i-1}^m))\,u_i^m dx \geqslant \int_\Omega B(u_j^m)\,dx - \int_\Omega B(u_0^m)\,dx.$$

By Abel's summation formula we obtain

$$\sum_{i=1}^j \int_\Omega a(x)\nabla(u_i^m - u_{i-1}^m)\,\nabla u_i^m\,dx \geqslant \frac{a_0}{2}\int_\Omega |\nabla u_j^m|^2\,dx - \frac{a^0}{2}\int_\Omega |\nabla u_0^m|^2\,dx.$$

The ellipticity assumption implies

$$\sum_{i=1}^j \int_\Omega d(t_i, x, u_{i-1}^m, \nabla u_i^m)\,\nabla u_i^m\,dx \geqslant d_0 \sum_{i=1}^j \int_\Omega |\nabla u_i^m|^p\,dx.$$

For the integral operator we have the estimate

$$\sum_{i=1}^j \langle M(\hat u_{i-1}^m), u_i^m\rangle \le c_1/\delta \sum_{i=1}^j \int_\Omega \Big(\int_0^{t_i} K(t_i, s)\,g(s, x, \nabla \hat u_{i-1}^m)\,ds\Big)^q dx + c_2\delta \sum_{i=1}^j \int_\Omega |\nabla u_i^m|^p\,dx.$$

By the assumptions on the function g and the kernel K we have

$$\begin{aligned}
\Big|\int_0^{t_i} K(t_i,s)\, g(s,x,\nabla \hat{u}_{i-1}^m)\, ds\Big| &\le \sum_{k=1}^{i} \int_{t_{k-1}}^{t_k} |K(s,t_i)||g(s,x,\nabla \hat{u}_{i-1}^m)|\, ds \\
&\le c_1\Big(\sum_{k=1}^{i} h|\nabla u_{k-1}^m|^p\Big)^{1/q}\Big(\sum_{k=1}^{i} h(t_i-t_k)^{-\gamma p}\Big)^{1/p} + c_2.
\end{aligned}$$

The last integral in (2.1.6), due to sublinearity of f and Poincaré's inequality, is estimated by

$$\begin{aligned}
\sum_{i=1}^{j}\int_\Omega f(x,t_i,u_{i-1}^m)\, u_i^m\, dx &\le \frac{1}{2}\sum_{i=1}^{j}\int_\Omega |f(x,t_i,u_{i-1}^m)|^2\, dx + \frac{1}{2}\sum_{i=1}^{j}\int_\Omega |u_i^m|^2\, dx \\
&\le c_3\sum_{i=1}^{j}\int_\Omega |\nabla u_i^m|^2\, dx + c_4.
\end{aligned}$$

By the estimates above, from equation (2.1.6) we obtain the inequality

$$\begin{aligned}
&\int_\Omega B(u_j^m)\, dx + \frac{a_0}{2}\int_\Omega |\nabla u_j^m|^2\, dx + (d_0 - c_1\delta)\sum_{i=1}^{j} h\int_\Omega |\nabla u_i^m|^p\, dx \\
\le\ &\int_\Omega B(u_0^m)\, dx + \frac{a^0}{2}\int_\Omega |\nabla u_0^m|^2\, dx + c_2 c(\gamma)\sum_{i=1}^{j} h\sum_{k=1}^{i} h\int_\Omega |\nabla u_k^m|^p\, dx \\
&+ c_3\sum_{i=1}^{j} h\int_\Omega |\nabla u_i^m|^2\, dx + c_3.
\end{aligned}$$

Using discrete Gronwall's lemma in the last inequality implies the estimates in Lemma 2.1.5. Gronwall's lemma can be applied for all sufficiently small h and δ that satisfy $c_3 h < a_0/2$ and $c_2 c(\gamma) h < (d_0 - c_1\delta)$. ❒

To show the strong convergence of the approximation and equicontinuity of u in time with respect to $L^2(Q_T)$ the following lemma is needed.

Lemma 2.1.6. *The estimates*

$$\begin{aligned}
\sum_{j=1}^{n-k} h\int_\Omega (b(u_{j+k}^m) - b(u_j^m))(u_{j+k}^m - u_j^m)\, dx &\le Ckh, \\
\sum_{j=1}^{n-k} h\int_\Omega |\nabla u_{j+k}^m - \nabla u_j^m|^2\, dx &\le Ckh \qquad (2.1.7)
\end{aligned}$$

hold uniformly with respect to m and n.

Proof. Summing up the equations (2.1.4) for $i = j+1, ..., j+k$, then choosing $u^m_{j+k} - u^m_j$ as a test function, and finally summing up over $j = 1, ..., n-k$ yields

$$\sum_{j=1}^{n-k} \int_\Omega \frac{1}{h}(b(u^m_{j+k}) - b(u^m_j))(u^m_{j+k} - u^m_j)\,dx + \sum_{j=1}^{n-k} \int_\Omega \frac{1}{h} a(x)(\nabla u^m_{j+k} - \nabla u^m_j)(\nabla u^m_{j+k} - \nabla u^m_j)\,dx$$
$$+ \sum_{j=1}^{n-k} \sum_{i=j+1}^{j+k} \int_\Omega d(t_i, x, u^m_{i-1}, \nabla u^m_i)(\nabla u^m_{j+k} - \nabla u^m_j)\,dx + \sum_{j=1}^{n-k} \sum_{i=j+1}^{j+k} \langle M(\hat{u}^m_{i-1}), u^m_{j+k} - u^m_j \rangle$$
$$= \sum_{j=1}^{n-k} \sum_{i=j+1}^{j+k} \int_\Omega f(t_i, x, u^m_{i-1})(u^m_{j+k} - u^m_j)\,dx.$$

Due to the growth assumption on d we have

$$\sum_{i=1}^{n} \int_\Omega |d(t_i, x, u^m_{i-1}, \nabla u^m_i)|^q dx \quad \leq \quad c_1 \sum_{i=1}^{n} \int_\Omega |\nabla u^m_i|^p\,dx + c_2 \sum_{i=1}^{n} \int_\Omega |u^m_{i-1}|^p\,dx + c_3.$$

The operator M and the function f can be estimated similarly to the last lemma. Then we obtain the following inequality

$$\sum_{j=1}^{n-k} \int_\Omega \frac{1}{h}(b(u^m_{j+k}) - b(u^m_j))(u^m_{j+k} - u^m_j)\,dx + a_0 \sum_{j=1}^{n-k} \int_\Omega \frac{1}{h} |\nabla u^m_{j+k} - \nabla u^m_j|^2\,dx$$
$$\leq \quad c_1 \sum_{i=1}^{n} \int_\Omega |\nabla u^m_i|^2\,dx + c_2(T)c(\gamma) \sum_{i=1}^{n} \int_\Omega |\nabla u^m_i|^p\,dx + \sum_{i=1}^{n} \int_\Omega |\nabla u^m_i|^p\,dx$$
$$+k \sum_{j=1}^{n-k} \int_\Omega (|\nabla u^m_{j+k}|^2 + |\nabla u^m_{j+k}|^p + |u^m_{j+k}|^2 + |\nabla u^m_j|^2 + |\nabla u^m_j|^p + |u^m_j|^2)\,dx.$$

This, by using Lemma 2.1.5, implies the asserted estimates. ❐

Proof of Theorem 2.1.3. We define for $t \in (t_{i-1}, t_i]$ and $x \in \Omega$ the step functions by

$$\bar{u}^m_n(t, x) \quad := \quad u^m(t_i, x),$$

where the initial conditions are $\bar{u}^m_n(0, x) = u^m_0(x)$. From (2.1.5) we obtain

$$\begin{aligned} \sup_{0 \leq t \leq T} \int_\Omega B(\bar{u}^m_n(t))\,dx \quad &\leq \quad C, \\ \sup_{0 \leq t \leq T} \int_\Omega |\nabla \bar{u}^m_n(t)|^2\,dx \quad &\leq \quad C, \\ \int_0^T \int_\Omega |\nabla \bar{u}^m_n|^p\,dx\,dt \quad &\leq \quad C. \end{aligned} \tag{2.1.8}$$

The growth assumptions on d, g, and f imply

$$\begin{aligned} ||d_n(t, x, \bar{u}^m_{n,h}, \nabla \bar{u}^m_n)||_{L^q(Q_T)^{N\times l}} \quad &\leq \quad C, \\ ||M(\hat{u}^m_{n-1})||_{L^q(0,T;H^{-1,q}(\Omega)^l)} \quad &\leq \quad C, \\ ||f_n(t, x, \bar{u}^m_{n,h})||_{L^2(Q_T)^l} \quad &\leq \quad C, \end{aligned} \tag{2.1.9}$$

where $\bar{u}^m_{n,h}(t,x) := \bar{u}^m_n(t-h,x)$ for $t \in [h,T]$ and $\bar{u}^m_{n,h}(t,x) := u^m_0(x)$ for $t \in [0,h]$, $d_n(t,x,s,z) := d(t_i,x,s,z)$ for $t \in (t_{i-1},t_i]$, for $i = 1,...,n$, and $d_n(0,x,s,z) := d(0,x,s,z)$. From (2.1.7) we have

$$\int_0^{T-\tau}\int_\Omega \big(b(\bar{u}^m_n(t+\tau,x)) - b(\bar{u}^m_n(t,x))\big)\big(\bar{u}^m_n(t+\tau,x) - \bar{u}^m_n(t,x)\big)\,dx\,dt \leq C\tau,$$

$$\int_0^{T-\tau}\int_\Omega |\nabla\bar{u}^m_n(t+\tau,x) - \nabla\bar{u}^m_n(t,x)|^2\,dx\,dt \leq C\tau, \tag{2.1.10}$$

where for $k \in \{0,\dots,n-1\}$, $kh \leq \tau \leq (k+1)h$. The second estimate in (2.1.10) and Poincaré's inequality imply

$$||\bar{u}^m_n - \bar{u}^m_{n,h}||_{L^2(0,T;H^1_0(\Omega)^l)} \leq \frac{C}{\sqrt{n}}.$$

From the equation (2.1.4) we obtain

$$||\partial_h\big(b(\bar{u}^m_n) - \nabla\cdot(a(x)\nabla\bar{u}^m_n)\big)||_{L^q(0,T;H^{-1,q}(\Omega)^l)} \leq C. \tag{2.1.11}$$

Then the estimates in (2.1.8), (2.1.9), and (2.1.11) imply convergence of a subsequence of $\{\bar{u}^m_n\}$, again denoted by $\{\bar{u}^m_n\}$

$$\begin{aligned}
\bar{u}^m_n &\to u && \text{weakly in } L^p(0,T;H^{1,p}_0(\Omega)^l),\\
\bar{u}^m_n &\to u && \text{weakly-}* \text{ in } L^\infty(0,T;H^1_0(\Omega)^l),\\
d(x,t,\bar{u}^m_{n,h},\nabla\bar{u}^m_n)) &\to \chi && \text{weakly in } (L^q(Q_T))^{N\times l},\\
\partial_h\big(b(\bar{u}^m_n) - \nabla\cdot(a(x)\nabla\bar{u}^m_n)\big) &\to \zeta && \text{weakly in } L^q(0,T;H^{-1,q}(\Omega)^l),\\
M(\hat{u}^m_{n-1}) &\to \mu && \text{weakly in } L^q(0,T;H^{-1,q}(\Omega)^l)
\end{aligned} \tag{2.1.12}$$

as $m,n \to \infty$. The weak convergence of $\{\bar{u}^m_n\}$ in $L^p(0,T;H^{1,p}_0(\Omega)^l)$ and the second estimate in (2.1.10) imply, by Kolmogorov's Theorem, (Necas 1967), the strong convergence of $\{\bar{u}^m_n\}$ in $L^2(Q_T)^l$, and also the convergence almost everywhere in Q_T. Then we have also $\bar{u}^m_{n,h} \to u$ strongly in $L^2(Q_T)^l$ and almost everywhere in Q_T. Thus, since

$$|b(\bar{u}^m_n)| \leq \delta B(\bar{u}^m_n) + \sup_{|\sigma|\leq\frac{1}{\delta}}|b(\sigma)|$$

and sublinearity of f, due to the Dominated Convergence Theorem, (Evans 1998), and continuity of b in u and of f in t and u, we obtain the convergences $b(\bar{u}^m_n) \to b(u)$ a.e. in Q_T, $b(\bar{u}^m_n(t)) \to b(u)$ in $L^1(0,T;L^1(\Omega)^l)$, and $f_n(t,x,u^m_{n,h}) \to f(t,x,u)$ in $L^2(Q_T)^l$. From the continuity of B follows $B(\bar{u}^m_n) \to B(u)$ a.e. in Q_T. Since $\{B(\bar{u}^m_n)\}^m_n$ is bounded in $L^\infty(0,T;L^1(\Omega))$ and $B(\bar{u}^m_n)$ is nonnegative we obtain, by Fatou's Lemma,

$$\frac{1}{\tau}\int_{t-\tau}^{t}\int_\Omega B(u)\,dx\,dt \leq \liminf_{m,n\to\infty}\frac{1}{\tau}\int_{t-\tau}^{t}\int_\Omega B(\bar{u}^m_n)\,dx\,dt \leq C \quad \text{for all } t, t-\tau \in [0,T] \text{ and small } \tau,$$

and, hereby, $B(u) \in L^\infty(0,T;L^1(\Omega))$. Due to the estimate $|b(u)| \leq \delta B(u) + \sup_{|\sigma|\leq \frac{1}{\delta}} |b(\sigma)|$ we have $b(u) \in L^\infty(0,T;L^1(\Omega)^l)$.

Passing to the limit for $m,n \to \infty$ in the equation (2.1.4) yields

$$\int_0^T \langle \zeta, v\rangle \, dt + \int_0^T \int_\Omega \chi \, \nabla v \, dx \, dt + \int_0^T \langle \mu, v \rangle \, dt \;=\; \int_0^T \int_\Omega f(t,x,u) \, v \, dx \, dt. \tag{2.1.13}$$

Since $b(\bar{u}_n^m(0)) = b(u_0^m)$ and $u_0^m \to u_0$ in $H_0^1(\Omega)^l$, we have that

$$\begin{aligned}
&\int_0^T \int_\Omega \partial_h b(\bar{u}_n^m(t)) \, v(t) \, dx \, dt + \int_0^T \int_\Omega a(x) \partial_h \nabla \bar{u}_n^m(t) \, \nabla v(t) \, dx \, dt \\
&= - \int_0^{T-h} \int_\Omega (b(\bar{u}_n^m(t)) - b(u_0^m)) \partial_{-h} v(t) \, dx \, dt - \int_0^{T-h} \int_\Omega a(x) (\nabla \bar{u}_n^m(t) - \nabla u_0^m) \partial_{-h} \nabla v(t) \, dx \, dt \\
&\to - \int_0^T \int_\Omega (b(u(t)) - b(u_0)) \, v_t(t) \, dx \, dt - \int_0^T \int_\Omega a(x) (\nabla u(t) - \nabla u_0) \, \nabla v_t(t) \, dx \, dt
\end{aligned}$$

as $m,n \to \infty$, for $v \in L^p(0,T;H_0^{1,p}(\Omega)^l) \cap L^\infty(Q_T)^l$, such that $v_t \in L^2(0,T;H_0^1(\Omega)^l)$ and $v_t \in L^\infty(0,T;L^\infty(\Omega)^l)$, and $v(T) = 0$. Since such v form a dense subspace of $L^p(0,T;H_0^{1,p}(\Omega)^l)$ and the uniform boundedness (2.1.11), we obtain $\partial_t \big(b(u) - \nabla \cdot (a(x) \nabla u)\big) = \zeta$ as functions in $L^q(0,T;H^{-1,q}(\Omega)^l)$.

Now we prove $\bar{u}_n^m \to u$ strongly in $L^p(0,T;H_0^{1,p}(\Omega)^l)$. We choose in the discretized equation $\xi = \bar{u}_n^m - v_n^m$ and integrate over the interval $(0,\tau)$, where v_n^m in $L^p(0,T;H_m)$ is the approximation of u in $L^p(0,T;H_0^{1,p}(\Omega)^l)$, constant in each interval $((k-1)h, kh)$.

$$\begin{aligned}
&\int_{Q_\tau} \partial_h b(\bar{u}_n^m)(\bar{u}_n^m - v_n^m) \, dx \, dt + \int_{Q_\tau} a(x) \nabla \partial_h \bar{u}_n^m (\nabla \bar{u}_n^m - \nabla v_n^m) \, dx \, dt \\
&+ \int_{Q_\tau} d_n(t,x,\bar{u}_{n,h}^m, \nabla \bar{u}_n^m)(\nabla \bar{u}_n^m - \nabla v_n^m) \, dx \, dt + \int_0^\tau \langle M(\hat{u}_{n-1}^m), \bar{u}_n^m - v_n^m \rangle \, dt \\
&= \int_{Q_\tau} f_n(t,x,\bar{u}_{n,h}^m)(\bar{u}_n^m - v_n^m) \, dx \, dt.
\end{aligned} \tag{2.1.14}$$

By the strong convergence of v_n^m, the weak convergence of $\partial_h\big(b(\bar{u}_m^n) - \nabla\cdot(a(x)\nabla\bar{u}_m^n)\big)$, and the integration by parts formula, Lemma A.1.3, yields

$$\begin{aligned}
&\int_0^\tau\int_\Omega \partial_h b(\bar{u}_n^m)\,(\bar{u}_n^m - v_n^m)\,dx\,dt + \int_0^\tau\int_\Omega a(x)\partial_h\nabla\bar{u}_n^m\,\nabla(\bar{u}_n^m - v_n^m)\,dx\,dt\\
\geq\ & \frac{1}{h}\int_{\tau-h}^\tau\int_\Omega B(\bar{u}_n^m)\,dxdt + \frac{1}{2h}\int_{\tau-h}^\tau\int_\Omega a(x)\nabla\bar{u}_n^m\nabla\bar{u}_n^m\,dx\,dt\\
&-\int_\Omega B(u(\tau))\,dx - \frac{1}{2}\int_\Omega a(x)\nabla u(\tau)\nabla u(\tau)\,dx + c\varepsilon.
\end{aligned}$$

Fatou's lemma implies

$$\liminf_{m,n\to\infty}\frac{1}{h}\int_{\tau-h}^\tau\int_\Omega\Big(B(\bar{u}_n^m) + \frac{1}{2}a(x)\nabla\bar{u}_n^m\nabla\bar{u}_n^m\Big)\,dx\,dt \geqslant \int_\Omega B(u(\tau))\,dx + \frac{1}{2}\int_\Omega a(x)\nabla u(\tau)\nabla u(\tau)\,dx.$$

Thus, we obtain

$$\liminf_{m,n\to\infty}\int_0^\tau\int_\Omega\Big(\partial_h b(\bar{u}_n^m)\,(\bar{u}_n^m - v_n^m) + a(x)\partial_h\nabla\bar{u}_n^m\,\nabla(\bar{u}_n^m - v_n^m)\Big)\,dx\,dt \geqslant 0.$$

Strong convergence of $\bar{u}_n^m$ to u in $L^2(Q_T)^l$ and of v_n^m to u in $L^p(0,T;H_0^{1,p}(\Omega))$, continuity of d, weak convergence of d in $(L^q(Q_T))^{N\times l}$, and the Dominated Convergence Theorem, (Evans 1998), imply $d_n(t,x,\bar{u}_{n,h}^m,\nabla u) \to d(t,x,u,\nabla u)$ strongly in $(L^q(Q_T))^{N\times l}$. Hence, strong monotonicity of d yields

$$\begin{aligned}
&\int_0^\tau\int_\Omega d_n(t,x,\bar{u}_{n,h}^m,\nabla\bar{u}_n^m)(\nabla\bar{u}_n^m - \nabla v_n^m)\,dx\,dt\\
&= \int_0^\tau\int_\Omega (d_n(t,x,\bar{u}_{n,h}^m,\nabla\bar{u}_n^m) - d_n(t,x,\bar{u}_{n,h}^m,\nabla v_n^m))(\nabla\bar{u}_n^m - \nabla v_n^m)\,dx\,dt\\
&+ \int_0^\tau\int_\Omega d_n(t,x,\bar{u}_{n,h}^m,\nabla v_n^m)\nabla(\bar{u}_n^m - v_n^m)\,dx\,dt \geqslant d_1\int_0^\tau\int_\Omega |\nabla(\bar{u}_n^m - v_n^m)|^p\,dx\,dt - c\,\varepsilon.
\end{aligned}$$

The integral operator satisfies the estimate

$$\begin{aligned}
&\int_0^\tau \langle M(\hat{u}_{n-1}^m), \bar{u}_n^m - v_n^m\rangle\,dt\\
=\ & \int_0^\tau\int_\Omega\int_0^t K(t,s)(g(s,x,\nabla\hat{u}_{n-1}^m(s)) - g(s,x,\nabla v_n^m(s)))\,ds\,\nabla(\bar{u}_n^m(t) - v_n^m(t))\,dx\,dt\\
\leq\ & \frac{c_1}{\delta}\int_0^\tau\int_\Omega\Big(\int_0^t K(t,s)(g(s,\nabla\hat{u}_{n-1}^m(s)) - g(s,\nabla v_n^m(s)))ds\Big)^q\,dx\,dt + c_2\delta||\bar{u}_n^m - v_n^m||^p_{L^p(0,T;H_0^{1,p}(\Omega)^l)}.
\end{aligned}$$

Because of the weak singularity of the kernel K and the assumption on g we have

$$\int_0^t K(t,s)(g(s,\nabla\hat{u}^m_{n-1}(s)) - g(s,\nabla v^m_n(s)))\,ds \le C\Big(\int_0^t |K(t,s)|^p ds\Big)^{\frac{1}{p}}\Big(\int_0^t |\nabla\hat{u}^m_{n-1}(s) - \nabla v^m_n(s)|^p\,ds\Big)^{\frac{1}{q}}.$$

Combining the last two estimates and the estimate $||\bar{u}^m_n - \bar{u}^m_{n-1}||_{L^2(0,T;H^1_0(\Omega)^l)} \le C/\sqrt{n}$ yields

$$\int_0^\tau \langle M(\hat{u}^m_{n-1}), \bar{u}^m_n - v^m_n\rangle\,dt \le C_\delta \int_0^\tau\int_\Omega\int_0^t |\nabla\bar{u}^m_n - \nabla v^m_n|^p\,ds\,dx\,dt + C\delta||\bar{u}^m_n - v^m_n||^p_{L^p(0,T;H^{1,p}_0(\Omega)^l)} + c\,\varepsilon.$$

The strong convergences of $\bar{u}^m_n$ and f_n imply

$$\int_0^\tau\int_\Omega f_n(t,x,\bar{u}^m_{n,h})(\bar{u}^m_n - v^m_n)\,dx\,dt \;\le\; c\,\varepsilon.$$

The estimates of all terms in the equation (2.1.14) give

$$(d_1 - C\delta)\int_0^\tau\int_\Omega |\nabla\bar{u}^m_n - \nabla v^m_n|^p\,dx\,dt \;\le\; C_1\int_0^\tau\int_\Omega\int_0^t |\nabla\bar{u}^m_n - \nabla v^m_n|^p\,ds\,dx\,dt + C_2\varepsilon.$$

By Gronwall's lemma

$$\int_0^\tau\int_\Omega |\nabla\bar{u}^m_n - \nabla v^m_n|^p\,dx\,dt \;\le\; C\,\varepsilon$$

holds. Thus, we have the strong convergence of u^m_n to u in $L^p(0,T;H^{1,p}_0(\Omega)^l)$. Continuity of d and g yield

$$d_n(t,x,\bar{u}^m_{n,h},\nabla\bar{u}^m_n) \to d(t,x,u,\nabla u) \;\text{ a.e. in }\; Q_T$$

and

$$g_n(t,x,\nabla\hat{u}^m_{n-1}) \to g(t,x,\nabla u) \;\text{ a.e. in }\; Q_T.$$

The weak convergences of $d_n(t,x,\bar{u}^m_{n,h},\nabla\bar{u}^m_n)$ and $M(\hat{u}^m_{n-1})$ and the almost everywhere convergences imply $\chi = d(t,x,u,\nabla u)$ and $\mu = M(u)$. Thus, u is the solution of the problem (2.1.1). ❒

Remark 2.1.7. We can also consider the linear integral operator

$$\langle M(t)(u), v(t)\rangle \;=\; \int_\Omega\int_0^t a(t-s)\nabla u(s,x)\,ds\,\nabla v(t,x)\,dx,$$

for $u,v \in L^2(0,T;H^1_0(\Omega)^l)$ with positive-definite and weakly singular kernel $|a(t)| \le C|t|^{-\gamma}$, $0 \le \gamma < 1$. The kernel a is positive-definite iff

$$\int_0^T\int_0^t a(t-s)\beta(s)\,ds\,\beta(t)\,dt \;\geqslant\; 0.$$

The positive-definiteness of kernel a is equivalent to the assumption

$$(-1)^j \partial_t^j a(t) \geqslant 0 \text{ for all } t > 0, \text{ where } j = 0,1,2, \text{ and } \partial_t a \neq 0.$$

This definiteness implies the monotonicity of the operator M

$$\int_0^T \langle M(t)(u_1) - M(t)(u_2), u_1(t) - u_2(t)\rangle \, dt$$
$$= \int_0^T \int_\Omega \int_0^t a(t-s)\, \nabla\big(u_1(s,x) - u_2(s,x)\big)\, ds\, \nabla\big(u_1(t,x) - u_2(t,x)\big)\, dx\, dt \geqslant 0.$$

In this case we can weaken the assumption on d to show existence. Only monotonicity, but not strong monotonicity, is needed to apply the Minty-Browder theorem.

The existence can also be proved in the case of memory term operators of first order, i.e.

$$\langle M(t)(u), v(t)\rangle = \int_\Omega \int_0^t a(t-s) \cdot \nabla u(s,x)\, ds\, v(t,x)\, dx.$$

It is sufficient to assume monotonicity of d and weak singularity of the kernel a, i.e. $|a(t)| \leq C|t|^{-\gamma}$, $0 \leq \gamma < 1$. The convergence of $d_n(t, x, \bar{u}^m_{n,h}, \nabla \bar{u}^m_n)$ to $d(t, x, u, \nabla u)$ follows from Minty-Browder theorem.

Though we considered the Dirichlet boundary conditions only, the results remain valid for other boundary conditions, that allow a Poincaré inequality. For more general boundary conditions we have to assume $B(u_0) \in L^1(\Omega)$, see Remark A.1.4 .

Now the uniqueness of a solution will be proved twice. In the first proof less assumptions are needed. Nevertheless, the second proof uses a variant of an interesting method, which was applied in (Alt and Luckhaus 1983) to parabolic equations.

Theorem 2.1.8 (Uniqueness). *Let Assumption 2.1.1, $p = 2$, and*

$$|d(t, x, \eta_1, \zeta_1) - d(t, x, \eta_2, \zeta_2)| \leq C(|\eta_1 - \eta_2| + |\zeta_1 - \zeta_2|),$$

$$|f(t, x, \eta_1) - f(t, x, \eta_2)| \leq C|\eta_1 - \eta_2|$$

be satisfied for $(t, x) \in Q_T$, $\eta_1, \eta_2 \in \mathbb{R}^l$, and $\zeta_1, \zeta_2 \in \mathbb{R}^{N\times l}$. Then there exists at most one weak solution of the problem (2.1.1).

Proof. Suppose, there are two solutions $u^1, u^2 \in L^2(0,T;H_0^1(\Omega)^l)$. Then their difference satisfies

$$\begin{aligned} &-\int_0^T\int_\Omega \Big((b(u^1)-b(u^2))\, v_t + a(x)\nabla(u^1-u^2)\,\nabla v_t\Big)\,dx\,dt \\ &+\int_0^T\int_\Omega (d(t,x,u^1,\nabla u^1)-d(t,x,u^2,\nabla u^2))\,\nabla v\,dx\,dt \qquad (2.1.15)\\ &+\int_0^T \langle M(u^1)-M(u^2), v\rangle\,dt = \int_0^T\int_\Omega (f(t,x,u^1)-f(t,x,u^2))\,v\,dx\,dt, \end{aligned}$$

because $b(u_0^1) = b(u_0^2)$ and $\nabla u_0^1 = \nabla u_0^2$. Since $\partial_t\big(b(u^i) - \nabla\cdot(a(x)\nabla u^i)\big) \in L^2(0,T,H^{-1}(\Omega)^l)$, we may assume $b(u^i) - \nabla\cdot(a(x)\nabla u^i) \in C(0,T;H^{-1}(\Omega)^l)$. Due to $u^i \in L^2(0,T;H_0^1(\Omega)^l)$ and $a \in L^\infty(\Omega)^l$ we obtain $\nabla\cdot(a(x)\nabla u^i) \in L^2(0,T,H^{-1}(\Omega)^l)$ and $b(u^i) \in L^2(0,T;H^{-1}(\Omega)^l)$. We choose for $s \le T$

$$v_s(t) = \begin{cases} \int_t^s (u^1(\tau)-u^2(\tau))\,d\tau, & t<s,\\ 0, & \text{otherwise}\end{cases}$$

and integrate by parts. Notice that $v_s(s) = 0$. Hereby we obtain

$$\begin{aligned} &\int_0^s \langle b(u^1)-b(u^2), u^1-u^2\rangle\,dt + a_0\int_0^s\int_\Omega |\nabla u^1-\nabla u^2|^2\,dx\,dt \qquad (2.1.16)\\ \le\;& \delta_0\int_0^s\int_\Omega |\nabla u^1-\nabla u^2|^2\,dx\,dt + \frac{c_1}{\delta_0}\int_0^s\int_\Omega |\nabla v_s(t)|^2\,dx\,dt, \end{aligned}$$

where the last term satisfies the following estimate

$$\begin{aligned} \int_0^s\int_\Omega |\nabla v_s(t)|^2\,dx\,dt &\le c_2\int_0^s\int_t^s\int_\Omega |\nabla(u^1(x,\tau)-u^2(x,\tau))|^2\,dx\,d\tau\,dt\\ &= c_2\int_0^s\int_0^t\int_\Omega |\nabla(u^1(x,\tau)-u^2(x,\tau))|^2\,dx\,d\tau\,dt. \end{aligned}$$

Using the monotonicity of the function b and Gronwall's lemma for the inequality (2.1.16) yields

$$\int_0^s\int_\Omega |\nabla u^1-\nabla u^2|^2\,dx\,dt = 0$$

and $u^1 = u^2$ almost everywhere in Q_T. ❐

Theorem 2.1.9 (Uniqueness). *Let Assumption 2.1.1 be satisfied, let $p = 2$, and let $d^j(t,x,u,\nabla u)$ be of the form $(d(t,x)\nabla u)^j$, $j = 1,\dots,l$, where $d \in (L^\infty(\Omega))^{N\times l\times N\times l}$, d is symmetric and strictly positive definite. Furthermore, let $d_t \in (L^\infty(Q_T))^{N\times l\times N\times l}$ and*

$$|f(t,x,\eta_1) - f(t,x,\eta_2)| \le C|\eta_1-\eta_2|$$

for $(t,x) \in Q_T$, $\eta_1, \eta_2 \in \mathbb{R}^l$, and $\zeta_1, \zeta_2 \in \mathbb{R}^{N\times l}$. Then there exists at most one weak solution of the problem (2.1.1).

Proof. Suppose, there are two solutions $u^1, u^2 \in L^2(0,T;H_0^1(\Omega)^l)$. Then their difference satisfies

$$-\int\limits_{Q_T}\Big[(b(u^1)-b(u^2))\,v_t + a(x)\nabla(u^1-u^2)\,\nabla v_t\Big]\,dx\,dt + \int\limits_{Q_T} d(t,x)\nabla(u^1-u^2)\,\nabla v\,dx\,dt$$
$$+\int\limits_0^T \langle M(u^1)-M(u^2), v\rangle\,dt = \int\limits_{Q_T}(f(t,x,u^1)-f(t,x,u^2))\,v\,dx\,dt, \tag{2.1.17}$$

since $b(u_0^1) = b(u_0^2)$ and $\nabla u_0^1 = \nabla u_0^2$. Since $\partial_t\big(b(u^i) - \nabla\cdot(a(x)\nabla u^i)\big) \in L^2(0,T,H^{-1}(\Omega)^l)$, we may assume $b(u^i) - \nabla\cdot(a(x)\nabla u^i) \in C(0,T;H^{-1}(\Omega)^l)$. Due to $u^i \in L^2(0,T;H_0^1(\Omega)^l)$ and $a \in L^\infty(\Omega)^l$ we obtain $\nabla\cdot(a(x)\nabla u^i) \in L^2(0,T,H^{-1}(\Omega)^l)$ and then $b(u^i) \in L^2(0,T;H^{-1}(\Omega)^l)$. We define $\beta = b(u^1) - b(u^2)$ and $v = u^1 - u^2$.

Due to Lax-Milgram theorem there exists the solution w in $L^2(0,T;H_0^1(\Omega)^l)$ of the equation

$$\int\limits_0^T\int\limits_\Omega d(t,x)\,\nabla w\,\nabla\xi\,dx\,dt = \int\limits_0^T\langle\beta,\xi\rangle\,dt + \int\limits_0^T\int\limits_\Omega a(x)\,\nabla v\,\nabla\xi\,dx\,dt \tag{2.1.18}$$

for all $\xi \in L^2(0,T;H_0^1(\Omega)^l)$. Now we will prove the equality

$$\int\limits_0^\tau\langle\partial_t\big(\beta-\nabla\cdot(a(x)\nabla v)\big), w\rangle\,dt = \frac{1}{2}\int\limits_\Omega d(\tau,x)\nabla w\,\nabla w\,dx + \frac{1}{2}\int\limits_{Q_\tau} d_t(t,x)\nabla w\,\nabla w\,dx\,dt. \tag{2.1.19}$$

For the difference quotient we have the equality

$$2\int\limits_h^{\tau+h}\langle\partial_h\beta, w\rangle\,dt + 2\int\limits_h^{\tau+h}\int\limits_\Omega a(x)\partial_h\nabla v\nabla w\,dx\,dt + \frac{1}{h}\int\limits_0^h\langle\beta,w\rangle\,dt + \frac{1}{h}\int\limits_0^h\int\limits_\Omega a(x)\nabla v\,\nabla w\,dx\,dt$$
$$= \int\limits_0^\tau\langle\partial_{-h}\beta, w(t+h)\rangle\,dt - \int\limits_0^\tau\langle\beta,\partial_{-h}w\rangle\,dt + \frac{1}{h}\int\limits_\tau^{\tau+h}\langle\beta,w\rangle\,dt + \int\limits_0^\tau\int\limits_\Omega a(x)\partial_{-h}\nabla v\nabla w(t+h)\,dx\,dt$$
$$-\int\limits_0^\tau\int\limits_\Omega a(x)\nabla v\,\partial_{-h}\nabla w\,dx\,dt + \frac{1}{h}\int\limits_\tau^{\tau+h}\int\limits_\Omega a(x)\nabla v\,\nabla w\,dx\,dt.$$

Using the equation (2.1.18) for the integrals on the right hand side yields

$$
\begin{aligned}
&2\int_h^{\tau+h}\langle \partial_h\beta, w\rangle\,dt + 2\int_h^{\tau+h}\int_\Omega a(x)\partial_h\nabla v\,\nabla w\,dx\,dt + \frac{1}{h}\int_0^h\langle\beta,w\rangle\,dt + \frac{1}{h}\int_0^h\int_\Omega a(x)\nabla v\,\nabla w\,dx\,dt\\
&= \int_{Q_\tau} d(x,t)\nabla w\,\partial_{-h}\nabla w\,dx\,dt - \int_{Q_\tau} d(x,t)\partial_{-h}\nabla w\,\nabla w(t+h)\,dx\,dt\\
&+\frac{1}{h}\int_h^{\tau+h}\int_\Omega \partial_h d(t,x)\nabla w\,\nabla w\,dx\,dt + \frac{1}{h}\int_\tau^{\tau+h}\int_\Omega d(x,t)\nabla w\,\nabla w\,dx\,dt. \qquad (2.1.20)
\end{aligned}
$$

Due to the symmetry of d, we obtain

$$
\begin{aligned}
&\frac{1}{h}\int_{Q_\tau} d(t,x)(\nabla w(t+h)-\nabla w(t))(\nabla w(t+h)-\nabla w(t))\,dx\,dt\\
&= \int_{Q_\tau} d(t,x)\partial_{-h}\nabla w\,\nabla w(t+h)\,dx\,dt - \int_{Q_\tau} d(t,x)\nabla w\;\partial_{-h}\nabla w\,dx\,dt.
\end{aligned}
$$

The left hand side of the last equality converges to 0 as $h\to 0$. Passing to the limit in (2.1.20) as $h\to 0$ and using the convergences

$$
\int_h^{\tau+h}\langle \partial_h\big(\beta-\nabla\cdot(a(x)\nabla v)\big), w\rangle\,dt \;\to \int_0^\tau\langle \partial_t\big(\beta-\nabla\cdot(a(x)\nabla v)\big), w\rangle\,dt
$$

and

$$
\frac{1}{h}\int_0^h\langle\beta,w\rangle\,dt + \frac{1}{h}\int_0^h\int_\Omega a(x)\nabla v\,\nabla w\,dx\,dt \;\to 0,
$$

imply the equality (2.1.19). The last convergence holds true since $\beta(0)=0$ and $v(0)=0$.

The equation (2.1.18) for $\xi = u^1-u^2$ for $t\in(0,\tau)$ and $\xi=0$ for $t\in(\tau,T)$ has the form

$$
\begin{aligned}
\int_{Q_\tau} d(t,x)\,\nabla w\,\nabla(u^1-u^2)\,dx\,dt \;&=\; \int_0^\tau\langle b(u^1)-b(u^2), u^1-u^2\rangle\,dt\\
&+\;\int_{Q_\tau} a(x)\nabla(u^1-u^2)\,\nabla(u^1-u^2)\,dx\,dt. \qquad (2.1.21)
\end{aligned}
$$

Now in the equation (2.1.17) we choose $v=w$ for $t\in(0,\tau)$ and $v=0$ for $t\in(\tau,T)$ and obtain

$$
\begin{aligned}
&\int_0^\tau\langle \partial_t\big((b(u^1)-b(u^2))-\nabla\cdot(a(x)\,\nabla(u^1-u^2))\big), w\rangle\,dt + \int_{Q_\tau} d(t,x)\nabla(u^1-u^2)\,\nabla w\,dx\,dt\\
&+\int_0^\tau\langle M(u^1)-M(u^2), w\rangle dt = \int_{Q_\tau}\big(f(t,x,u^1)-f(t,x,u^2)\big)\,w\,dx\,dt.
\end{aligned}
$$

Due to (2.1.19), (2.1.21), and the sublinearity assumptions on f and g , the last equality implies

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega} d(\tau,x)\nabla w \nabla w\,dx + \int_0^{\tau}\langle b(u^1)-b(u^2),u^1-u^2\rangle\,dt + a_0\int_{Q_\tau}|\nabla u^1-\nabla u^2|^2\,dx\,dt\\
\leq\;& C_1\delta_0\int_{Q_\tau}|\nabla u^1-\nabla u^2|^2\,dx\,dt + \int_{Q_\tau}(C_2/\delta_0+||d_t||)|\nabla w|^2\,dx\,dt\\
&+C_3c(\gamma)\delta_0\int_{Q_\tau}|\nabla u^1-\nabla u^2|^2\,dx\,dt + C_4/\delta_0\int_{Q_\tau}|\nabla w|^2\,dx\,dt.
\end{aligned}$$

Using the monotonicity of b and Gronwall's lemma yields

$$\int_{\Omega}|\nabla w(\tau)|^2\,dx = 0 \quad \text{for all } \ \tau\in(0,T)$$

and

$$\int_{Q_\tau}|\nabla u^1-\nabla u^2|^2\,dx\,dt \;\leq\; 0.$$

Thus, $u^1 = u^2$ almost everywhere in Q_T. ❒

Remark 2.1.10. The classical examples for the functions b and d are given by

$$\begin{aligned}
&b(\eta) = |\eta|^{\alpha-2}\eta && \text{for } \ \alpha>1,\\
&d(t,x,\eta,\xi) = h(t,x,\eta)|\xi|^{p-2}\xi && \text{for } \ p\geqslant 2 \quad \text{and} \quad 0<h_0\leq h(t,x,\eta)\leq h_1<\infty\\
& && \qquad\qquad \text{for all } \ \eta\in\mathbb{R} \text{ and } (t,x)\in Q_T.
\end{aligned}$$

2.2 Quasilinear Pseudoparabolic Inequalities

Variational inequalities model free boundary value problems and problems with obstacles, and describe a minimization of an energy type functional on a convex set.

The latter is obtained as follows. Let Ω be a bounded domain in $\mathbb{R}^N$, V be a Hilbert space, and K be a convex and closed subset in V. Provided $\mathcal{A}$ is a continuous, symmetric, and elliptic bilinear form on V, L is a linear functional on V, the minimum of the functional J, given by

$$J(v) = \mathcal{A}(v, v) - 2L(v),$$

on K, is attained at an unique $u \in K$, which is in fact the solution of the inequality

$$\mathcal{A}(u, v-u) \geqslant L(v-u) \quad \text{for all } v \in K.$$

The solution of a parabolic obstacle problem is a function $u \in K$ that satisfies a variational inequality

$$(u_t, v-u) + \mathcal{A}(u, v-u) \geqslant (f, v-u) \text{ for a. a. } t \in (0,T), \text{ for all } v \in K,$$

where

$$K = \{v \in H^1((0,T) \times \Omega),\ v = g \text{ on } (0,T) \times \partial\Omega,\ v \geqslant \phi \text{ a.e. in } (0,T) \times \Omega\}$$

and ϕ describes the obstacle.

For a strong solution $u \in H^1(0,T; H^2(\Omega))$ this is equivalent to

$$\begin{aligned}
(u_t - Au - f)(u - \phi) = 0 &\quad \text{in} \quad (0,T) \times \Omega,\\
u_t - Au - f \geqslant 0 &\quad \text{in} \quad (0,T) \times \Omega,\\
u - \phi \geqslant 0 &\quad \text{in} \quad (0,T) \times \Omega,\\
u = g &\quad \text{on} \quad (0,T) \times \partial\Omega.
\end{aligned}$$

A special free boundary value problem is the Stefan Problem, describing the melting of ice. In complex materials the energy, entropy, heat flux, and thermodynamic temperature may depend on the conductive temperature. Nevertheless, the heat satisfies a pseudoparabolic inequality (DiBenedetto and Showalter 1982). The convex subset is given by

$$K = \{v \in H^1((0,T) \times \Omega),\ v = g \text{ on } \partial\Omega,\ v \geqslant 0 \text{ a.e. in } (0,T) \times \Omega\}.$$

In all this cases a pseudoparabolic inequality

$$\begin{aligned}
&\int\limits_{Q_T} \Big[\partial_t b(u)(v-u) + a(x)\nabla u_t \nabla(v-u) + d(t,x,\nabla u)\nabla(v-u)\Big]\,dx\,dt\\
&\geqslant \int\limits_{Q_T} f(t,x,u)(v-u)\,dx\,dt
\end{aligned} \tag{2.2.1}$$

is considered on a set $Q_T := (0,T) \times \Omega$ with an initial condition

$$b(u(0,x)) = b(u_0(x)). \tag{2.2.2}$$

In this section we prove an existence theorem for this quasilinear pseudoparabolic inequality.

The initial value problem is completed by posing spatial boundary conditions. An intermediate subspace V, $H_0^{1,p}(\Omega)^l \subset V \subset H^{1,p}(\Omega)^l$, is chosen such that it is densely and continuously embedded in $L^2(\Omega)^l$, is densely and continuously embedded into a closed subspace $V_0 \subset H^1(\Omega)^l$. The spaces V and V_0 should satisfy Poincaré inequalities, i.e.

$$||v||_{L^p(\Omega)^l} \leq C||\nabla v||_{L^p(\Omega)^l} \text{ for } v \in V$$

and

$$||v||_{L^2(\Omega)^l} \leq C||\nabla v||_{L^2(\Omega)^l} \text{ for } v \in V_0.$$

The constraint on u is given by the requirement $u \in K$, where K is chosen to be a closed and convex subset of V containing 0.

The following assumptions ensure the existence of a solution of the variational inequality.

Assumption 2.2.1.

$(A1)$ The vector field $b : \mathbb{R}^l \to \mathbb{R}^l$ is monotone nondecreasing, Lipschitz continuous, a continuous gradient, i.e. there exists a convex C^1 function $\Phi : \mathbb{R}^l \to \mathbb{R}$, such that $b = \nabla\Phi$, and $b(0) = 0$.

$(A2)$ The tensor field $a \in (L^\infty(\Omega))^{N\times l\times N\times l}$, considered as a linear mapping on $L^\infty(\Omega)^{N\times l}$, is symmetric and elliptic, i.e. for some $0 < a_0 \leq a^0 < \infty$, a satisfies $a_0|\xi|^2 \leq a(x)\xi\,\xi \leq a^0|\xi|^2$, for $\xi \in \mathbb{R}^{N\times l}$, for almost all $x \in \Omega$.

$(A3)$ The diffusivity $d : (0,T) \times \Omega \times \mathbb{R}^{N\times l} \to \mathbb{R}^{N\times l}$ is continuous, elliptic, i.e. there exists some $d_0 > 0$, such that $d(t,x,\xi)\,\xi \geqslant d_0|\xi|^p$, for all $\xi \in \mathbb{R}^{N\times l}$, $p \geqslant 2$, and monotone, i.e. $(d(t,x,\xi_1) - d(t,x,\xi_2))\,(\xi_1 - \xi_2) \geqslant 0$ for all $\xi_1, \xi_2 \in \mathbb{R}^{N\times l}$, and satisfies the growth assumption $|d(t,x,\xi)| \leq C(1 + |\xi|^{p-1})$ for $\xi \in \mathbb{R}^{N\times l}$, and is a gradient, i.e. there is a continuous function $D(t,x,\xi)$ such that $\nabla_\xi D = d$ and $|D(t,x,0)| \leq C$, $|\partial_t D(t,x,\xi)| \leq C(1 + |\xi|^p)$ for $\xi \in \mathbb{R}^{N\times l}$, for almost all $(t,x) \in Q_T$.

$(A4)$ The function $f : (0,T) \times \Omega \times \mathbb{R}^l \to \mathbb{R}^l$ is continuous and sublinear, i.e. $|f(t,x,\eta)| \leq C(1 + |\eta|)$ for $\eta \in \mathbb{R}^l$, for almost all $(t,x) \in Q_T$.

$(A5)$ The initial condition u_0 is in K.

Definition 2.2.2. A function $u : Q_T \to \mathbb{R}^l$ is called *a weak solution* of the inequality (2.2.1) if

1) $u \in L^p(0,T;V)$, $u_t \in L^2(0,T;V_0)$, $\partial_t b(u) \in L^2(Q_T)^l$, and $u(t) \in K$ for almost all $t \in (0,T)$,
2) u satisfies the inequality (2.2.1) for all test functions $v \in L^p(0,T;V)$ and $v(t) \in K$ for almost all t,
3) u satisfies the initial condition (2.2.2), i.e. $b(u(t,x)) \to b(u_0(x))$ in $L^2(\Omega)^l$ as $t \to 0$.

Remark 2.2.3. A partial integration of the time derivative in (2.2.1) would allow a seemingly weaker notion of a weak solution, provided v is differentiable in time. However, this is only possible if the generality of K is restricted, (Alt and Luckhaus 1983). The assumptions on b and d, posed in this section, are needed to show the existence of a solution.

Theorem 2.2.4. *Let Assumption 2.2.1 be satisfied. Then there exists a weak solution of the variational inequality (2.2.1) with the initial condition (2.2.2).*

For positive α we consider the penalized equation

$$\partial_t b(u) - \nabla \cdot (a(x)\nabla u_t) - \nabla \cdot d(t,x,\nabla u) + \alpha \mathcal{B}(u) \;=\; f(t,x,u),$$

where $\mathcal{B} : L^p(0,T;V) \to L^q(0,T;V^*)$ is the penalty operator, introduced in Definition A.1.10.

In the proof of this theorem some of the estimates of section 2.1 are reused. By the Rothe-Galerkin approximation we obtain the family of functions $u^m_{\alpha,i}$, satisfying

$$\int_\Omega \frac{1}{h}(b(u^m_{\alpha,i}) - b(u^m_{\alpha,i-1}))\,\xi\,dx + \int_\Omega a(x)\frac{1}{h}\nabla(u^m_{\alpha,i} - u^m_{\alpha,i-1})\,\nabla\xi\,dx + \int_\Omega d(t_i,x,\nabla u^m_{\alpha,i})\,\nabla\xi\,dx$$
$$+\alpha\langle \mathcal{B}(u^m_{\alpha,i}),\xi\rangle = \int_\Omega f(t_i,x,u^m_{\alpha,i-1})\,\xi\,dx \tag{2.2.3}$$

for $\xi \in H_m = \operatorname{span}\{e_1,...,e_m\}$, where $\{e_j\}_{j=1}^\infty$ is a basis of V and $e_j \in L^\infty(\Omega)^l$, and $\{u^m_{\alpha,0}\}_m$ is an approximation of u_0 in V. Since the operator $\mathcal{B}$ is monotone, the existence of $u^m_{\alpha,i}$ can be proved in the same manner as in section 2.1.

Similar to the proof of Lemma 2.1.5, using Assumption 2.2.1 and the monotonicity of $\mathcal{B}$ yields the estimates

$$\begin{aligned}
\max_{0\le j\le n}\int_\Omega B(u^m_{\alpha,j})\,dx &\le C,\\
\max_{0\le j\le n}\int_\Omega |\nabla u^m_{\alpha,j}|^2\,dx &\le C,\\
\sum_{i=1}^{j} h\int_\Omega |\nabla u^m_{\alpha,i}|^p\,dx &\le C,\\
\alpha\sum_{i=1}^{j} h\langle \mathcal{B}(u^m_{\alpha,i}),u^m_{\alpha,i}\rangle &\le C.
\end{aligned} \tag{2.2.4}$$

For the proof of existence of a solution of the variational inequality we have to show that $\partial_t b(u) \in L^2(Q_T)^l$ and $u_t \in L^2(0,T;V_0)$.

Since d is a gradient it is possible to prove the following.

Lemma 2.2.5. *The estimates*

$$\sum_{i=1}^{n} h \int_\Omega \frac{1}{h^2}(b(u^m_{\alpha,i}) - b(u^m_{\alpha,i-1}))(u^m_{\alpha,i} - u^m_{\alpha,i-1})\,dx \le C,$$
$$\sum_{i=1}^{n} h \int_\Omega \Big|\frac{\nabla u^m_{\alpha,i} - \nabla u^m_{\alpha,i-1}}{h}\Big|^2 dx \le C, \tag{2.2.5}$$
$$\max_{1\le j\le n} \int_\Omega |\nabla u^m_{\alpha,j}|^p\,dx \le C$$

hold uniformly with respect to m, n, and α.

Proof. Choosing $\xi = (u^m_{\alpha,i} - u^m_{\alpha,i-1})$ as a test function in (2.2.3) and summing over i yields

$$\sum_{i=1}^{j} \int_\Omega \frac{1}{h}(b(u^m_{\alpha,i}) - b(u^m_{\alpha,i-1}))(u^m_{\alpha,i} - u^m_{\alpha,i-1})\,dx + \sum_{i=1}^{j} \int_\Omega \frac{1}{h} a(x)\nabla(u^m_{\alpha,i} - u^m_{\alpha,i-1})\nabla(u^m_{\alpha,i} - u^m_{\alpha,i-1})\,dx$$
$$+ \sum_{i=1}^{j} \int_\Omega d(t_i, x, \nabla u^m_{\alpha,i})\nabla(u^m_{\alpha,i} - u^m_{\alpha,i-1})\,dx + \alpha \sum_{i=1}^{j} \langle \mathcal{B}(u^m_{\alpha,i}), u^m_{\alpha,i} - u^m_{\alpha,i-1}\rangle$$
$$= \sum_{i=1}^{j} \int_\Omega f(t_i, x, u^m_{\alpha,i-1})(u^m_{\alpha,i} - u^m_{\alpha,i-1})\,dx. \tag{2.2.6}$$

Due to the assumption on d the third integral on the left can be rewritten in the form

$$\begin{aligned} I &= \sum_{i=1}^{j} \int_\Omega d(t_i, x, \nabla u^m_{\alpha,i})(\nabla u^m_{\alpha,i} - \nabla u^m_{\alpha,i-1})\,dx \\ &= \sum_{i=1}^{j} \int_\Omega D(t_i, x, \nabla u^m_{\alpha,i}) - D(t_i, x, \nabla u^m_{\alpha,i-1})\,dx \\ &= \int_\Omega D(t_j, x, \nabla u^m_{\alpha,j})\,dx - \int_\Omega D(0, x, \nabla u^m_0)\,dx \\ &\quad - \sum_{i=1}^{j} \int_\Omega (D(t_i, x, \nabla u^m_{\alpha,i-1}) - D(t_{i-1}, x, \nabla u^m_{\alpha,i-1}))\,dx. \end{aligned}$$

Applying the assumed growth bounds of d to $|D(t,x,z)| \le \int_0^z |d(t,x,\xi)|\,d\xi + |D(t,x,0)|$ yields $|D(0,x,\nabla u^m_{\alpha,0})| \le C(1 + |\nabla u^m_{\alpha,0}|^p)$. The ellipticity assumption on d implies

$$\begin{aligned} D(t,x,\xi) - D(t,x,0) &= \int_0^1 \nabla D(t,x,s\xi)\,\xi\,ds = \int_0^1 d(t,x,s\xi)\,s\,\xi\,s^{-1}\,ds \\ &\geqslant d_0|\xi|^p \int_0^1 s^{p-1}\,ds = \frac{d_0}{p}|\xi|^p. \end{aligned}$$

Since $|D(t,x,0)| \le C$, we obtain

$$D(t_i, x, \nabla u^m_{\alpha,j}) \geqslant \frac{d_0}{p} |\nabla u^m_{\alpha,j}|^p - C.$$

Then, due to $|\partial_t D(t,x,z)| \le C(1+|z|^p)$, we have

$$I \geqslant \frac{d_0}{p} \int_\Omega |\nabla u^m_{\alpha,j}|^p \, dx - c_1 \int_\Omega |\nabla u^m_0|^p \, dx - c_2 \sum_{i=1}^j h \int_\Omega |\nabla u^m_{\alpha,i-1}|^p \, dx - c_3.$$

The penalty operator can be estimated by

$$\begin{aligned}
\sum_{i=1}^j \langle \mathcal{B}(u^m_{\alpha,i}), u^m_{\alpha,i} - u^m_{\alpha,i-1} \rangle &= \sum_{i=1}^j \langle J(u^m_{\alpha,i} - P_K u^m_{\alpha,i}), u^m_{\alpha,i} - u^m_{\alpha,i-1} \rangle \\
&= \sum_{i=1}^j \langle J(u^m_{\alpha,i} - P_K u^m_{\alpha,i}), P_K u^m_{\alpha,i} - P_K u^m_{\alpha,i-1} \rangle \\
&\quad + \sum_{i=1}^j \langle J(u^m_{\alpha,i} - P_K u^m_{\alpha,i}), (u^m_{\alpha,i} - P_K u^m_{\alpha,i}) - (u^m_{\alpha,i-1} - P_K u^m_{\alpha,i-1}) \rangle \\
&\geqslant \frac{1}{p} \sum_{i=1}^j \Big(||u^m_{\alpha,i} - P_K u^m_{\alpha,i}||^p_V - ||u^m_{\alpha,i-1} - P_K u^m_{\alpha,i-1}||^p_V \Big) \\
&= \frac{1}{p} ||u^m_{\alpha,j} - P_K u^m_{\alpha,j}||^p_V \geqslant 0.
\end{aligned}$$

Here $u_0 \in K$ and the property of P_K are used. Due to estimates for d and $\mathcal{B}$ from (2.2.6) we obtain the inequality

$$\begin{aligned}
&\sum_{i=1}^j \int_\Omega \frac{1}{h} (b(u^m_{\alpha,i}) - b(u^m_{\alpha,i-1}))(u^m_{\alpha,i} - u^m_{\alpha,i-1}) \, dx \\
&+ \sum_{i=1}^j \int_\Omega \frac{a_0 - \delta}{h} |\nabla(u^m_{\alpha,i} - u^m_{\alpha,i-1})|^2 dx + \frac{d_0}{p} \int_\Omega |\nabla u^m_{\alpha,j}|^p \, dx \\
\le \quad & c_\delta \sum_{i=1}^j h \int_\Omega |f(t_i, x, u^m_{\alpha,i-1})|^2 \, dx + \sum_{i=1}^j h \int_\Omega |\nabla u^m_{\alpha,i}|^p \, dx + \int_\Omega |\nabla u^m_0|^p \, dx.
\end{aligned}$$

Using the bounds (2.2.4) and sublinearity of f in the last inequality implies the assertion of the Lemma. ❒

Since b is Lipschitz continuous we have

$$\sum_{i=1}^j h \int_\Omega \left| \partial_h b(u^m_{\alpha,i}) \right|^2 dx \le C. \tag{2.2.7}$$

Proof of Theorem 2.2.4. We define the Rothe functions piecewise for $t \in (t_{i-1}, t_i]$ and $x \in \Omega$ by

$$u^m_{\alpha,n}(t,x) = u^m_\alpha(t_{i-1}, x) + (t - t_{i-1}) \frac{u^m_\alpha(t_i, x) - u^m_\alpha(t_{i-1}, x)}{h}$$

and the step functions by

$$\bar{u}^m_{\alpha,n}(t,x) = u^m_\alpha(t_i,x),$$

where the initial conditions are $u^m_{\alpha,n}(0,x) = u^m_0(x)$ and $\bar{u}^m_{\alpha,n}(0,x) = u^m_0(x)$.

From (2.2.4), (2.2.5), and (2.2.7) we obtain

$$\begin{aligned}
\sup_{0\le t\le T} \int_\Omega B(\bar{u}^m_{\alpha,n})\,dx &\le C,\\
\sup_{0\le t\le T} \int_\Omega |\nabla \bar{u}^m_{\alpha,n}|^2\,dx &\le C,\\
\sup_{0\le t\le T} \int_\Omega |\nabla \bar{u}^m_{\alpha,n}|^p\,dx &\le C,\\
\int_0^T \int_\Omega |\partial_h \nabla u^m_{\alpha,n}|^2\,dx\,dt &\le C,\\
\int_0^T \int_\Omega |\partial_h b(\bar{u}^m_{\alpha,n})|^2\,dx\,dt &\le C.
\end{aligned} \tag{2.2.8}$$

The growth assumption on d implies

$$||d_n(t,x,\nabla \bar{u}^m_{\alpha,n})||_{L^q(Q_T)^{N\times l}} \le C,$$

where $d_n(t,x,z) := d(t_i,x,z)$ for $t \in (t_{i-1},t_i]$ for $i = 1,...,n$, and $d_n(0,x,z) := d(0,x,z)$.

The penalty operator is bounded, hence

$$||\mathcal{B}(\bar{u}^m_{\alpha,n})||_{L^q(0,T;V^*)} \le C.$$

The fourth assumptions in (2.2.8) and the Poincaré inequality imply

$$||\bar{u}^m_{\alpha,n} - \bar{u}^m_{\alpha,n,h}||_{L^2(0,T;V_0)} \le \frac{C}{n}, \tag{2.2.9}$$

where $\bar{u}^m_{\alpha,n,h}(t,x) := \bar{u}^m_{\alpha,n}(t-h,x)$ for $t \in [h,T]$ and $\bar{u}^m_{\alpha,n,h}(t,x) := u^m_{\alpha,0}(x)$ for $t \in [0,h]$.

From (2.2.8) follows the existence of a subsequence of $\{\bar{u}^m_{\alpha,n}\}$ and of $\{u^m_{\alpha,n}\}$, resp., again denoted by $\{\bar{u}^m_{\alpha,n}\}$ and $\{u^m_{\alpha,n}\}$, resp., such that

$$\begin{aligned}
\bar{u}^m_{\alpha,n} &\to u_\alpha && \text{weakly in } L^p(0,T;V),\\
\bar{u}^m_{\alpha,n} &\to u_\alpha && \text{weakly} - * \text{ in } L^\infty(0,T;V_0),\\
\partial_h u^m_{\alpha,n} &\to \partial_t u_\alpha && \text{weakly in } L^2(0,T;V_0),\\
\partial_h b(\bar{u}^m_{\alpha,n}) &\to \eta_\alpha && \text{weakly in } L^2(Q_T)^l\\
d_n(t,x,\nabla \bar{u}^m_{\alpha,n})) &\to \chi_\alpha && \text{weakly in } (L^q(Q_T))^{N\times l},\\
\mathcal{B}(\bar{u}^m_{\alpha,n}) &\to \theta && \text{weakly in } L^q(0,T;V^*),
\end{aligned}$$

as $m,n \to \infty$. The strong convergence of $\{u^m_{\alpha,n}\}$ in $L^2(Q_T)^l$ follows from the Compactness Lions-Aubin Lemma, (Lions 1969). This and the estimate (2.2.9) imply the strong convergence of $\{u^m_{\alpha,n,h}\}$ in $L^2(Q_T)^l$. From the strong convergence of $\{u^m_{\alpha,n}\}$ in $L^2(Q_T)^l$ and the continuity of b follows $b(\bar{u}^m_{\alpha,n}) \to b(u_\alpha)$ a.e. in Q_T. The Lipschitz-continuity of b and $b(0)=0$ imply

$$||b(\bar{u}^m_{\alpha,n})||_{L^2(Q_T)^l} \le c||\bar{u}^m_{\alpha,n}||_{L^2(Q_T)^l} \le C.$$

Due to the Dominated Convergence Theorem, (Evans 1998), yields $b(\bar{u}^m_{\alpha,n}) \to b(u_\alpha)$ in $L^2(Q_T)^l$ and $\eta_\alpha = \partial_t b(u_\alpha)$. From the strong convergence of $\{u^m_{\alpha,n,h}\}$ and the continuity of f follows $f_n(t,x,\bar{u}^m_{\alpha,n,h}) \to f(t,x,u_\alpha)$ a.e. in Q_T. The sublinearity of f yields

$$||f_n(t,x,\bar{u}^m_{\alpha,n,h})||_{L^2(Q_T)^l} \le C_1(1+||\bar{u}^m_{\alpha,n,h}||_{L^2(Q_T)^l}) \le C.$$

Then the Dominated Convergence Theorem implies $f_n(t,x,\bar{u}^m_{\alpha,n,h}) \to f(t,x,u_\alpha)$ strongly in $L^2(Q_T)^l$. From the continuity of B follows $B(\bar{u}^m_{\alpha,n}) \to B(u_\alpha)$ a.e. in Q_T. Since $\{B(\bar{u}^m_{\alpha,n})\}$ is bounded in $L^\infty(0,T;L^1(\Omega))$ and $B(\bar{u}^m_{\alpha,n})$ is nonnegative we obtain, by Fatou's Lemma,

$$\frac{1}{\tau}\int_{t-\tau}^{t}\int_\Omega B(u_\alpha)\,dx\,dt \le \liminf_{m,n\to\infty}\frac{1}{\tau}\int_{t-\tau}^{t}\int_\Omega B(\bar{u}^m_{\alpha,n})\,dx\,dt \le C \quad \text{for all } t, t-\tau \in [0,T] \text{ and small } \tau,$$

and, hereby, $B(u_\alpha) \in L^\infty(0,T;L^1(\Omega))$.

Using $u_\alpha \in L^p(0,T;V)$, $\partial_t u_\alpha \in L^2(0,T;V_0)$ and (Evans 1998, Theorem 5.9.2), imply $u_\alpha \in C([0,T];V_0)$ and $u_\alpha(0) = u_0$. Due to the Lipschitz-continuity of b we obtain

$$\int_\Omega |b(u_\alpha(t)) - b(u_\alpha(s))|^2\,dx \le c_1||u_\alpha(t)-u_\alpha(s)||^2_{V_0} \quad \text{for all } t,s \in [0,T].$$

This implies $b \in C([0,T];L^2(\Omega)^l)$ and $b(u_\alpha(0)) = b(u_0)$ in $L^2(\Omega)^l$.

Passing to the limit as $m,n \to \infty$ in the discretized equation (2.2.3) yields

$$\int_{Q_T}\partial_t b(u_\alpha)\,v\,dx\,dt + \int_{Q_T} a(x)\partial_t\nabla u_\alpha\,\nabla v\,dx\,dt + \int_{Q_T}\chi_\alpha\,\nabla v\,dx\,dt + \alpha\int_0^T\langle\theta,v\rangle\,dt$$
$$= \int_{Q_T} f(t,x,u_\alpha)\,v\,dx\,dt. \tag{2.2.10}$$

Due to the monotonicity of d and $\mathcal{B}$ we will show

$$\int_{Q_T}\chi_\alpha\,\nabla v\,dx\,dt + \alpha\int_0^T\langle\theta,v\rangle\,dt = \int_{Q_T} d(t,x,\nabla u_\alpha)\,\nabla v\,dx\,dt + \alpha\int_0^T\langle\mathcal{B}(u_\alpha),v\rangle\,dt \tag{2.2.11}$$

for all functions $v \in L^p(0,T;V)$. Fatou's lemma implies

$$\liminf_{m,n\to\infty}\int_{Q_\tau} a(x)\partial_t\nabla u^m_{\alpha,n}\,\nabla u^m_{\alpha,n}\,dx\,dt = \liminf_{m,n\to\infty}\frac{1}{2}\int_\Omega a(x)\nabla u^m_{\alpha,n}\,\nabla u^m_{\alpha,n}\,dx - \frac{1}{2}\int_\Omega a(x)\nabla u_0\,\nabla u_0\,dx$$
$$\geqslant \int_{Q_\tau} a(x)\partial_t\nabla u_\alpha\,\nabla u_\alpha\,dx\,dt.$$

Then from equation (2.2.3), convergence of $\bar{u}^m_{\alpha,n}$, and equation (2.2.10) we have

$$\begin{aligned}
&\limsup_{m,n\to\infty}\Big(\int_{Q_T} d_n(t,x,\nabla\bar{u}^m_{\alpha,n})\,\nabla\bar{u}^m_{\alpha,n}\,dx\,dt+\alpha\int_0^T\langle\mathcal{B}(\bar{u}^m_{\alpha,n}),\bar{u}^m_{\alpha,n}\rangle\,dt\Big)\\
&\le\int_{Q_T} f(t,x,u_\alpha)\,u_\alpha\,dx\,dt-\int_{Q_T}\partial_t b(u_\alpha)\,u_\alpha\,dx\,dt-\liminf_{m,n\to\infty}\int_{Q_T} a(x)\partial_t\nabla u^m_{\alpha,n}\,\nabla u^m_{\alpha,n}\,dx\,dt\\
&\le\int_0^T\langle\chi_\alpha,\nabla u_\alpha\rangle\,dt+\alpha\int_0^T\langle\theta,u_\alpha\rangle\,dt.
\end{aligned}$$

Since d and $\mathcal{B}$ are monotone, we have

$$\int_{Q_T}\big(d_n(t,x,\nabla u^m_{\alpha,n})-d(t,x,\nabla w)\big)\,\big(\nabla u^m_{\alpha,n}-\nabla w\big)\,dx\,dt+\alpha\int_0^T\langle\mathcal{B}(u^m_{\alpha,n})-\mathcal{B}(w),u^m_{\alpha,n}-w\rangle\,dt\;\geqslant\;0.$$

Passing to the limit as $m,n\to\infty$ yields

$$\int_{Q_T}\big(\chi_\alpha-d(t,x,\nabla w)\big)\,\big(\nabla u_\alpha-\nabla w\big)\,dx\,dt+\alpha\int_0^T\langle\theta-\mathcal{B}(w),u_\alpha-w\rangle\,dt\;\geqslant\;0.$$

Choosing $w=u_\alpha-\lambda v$ for $v\in L^p(0,T;V)$ and $\lambda>0$, continuity of d and hemicontinuity of $\mathcal{B}$ imply the equality (2.2.11) by Minty-Browder's argument.

Then for every α the function u_α satisfies the equation

$$\begin{aligned}
&\int_{Q_T}\partial_t b(u_\alpha)\,v\,dx\,dt+\int_{Q_T} a(x)\partial_t\nabla u_\alpha\,\nabla v\,dx\,dt+\int_{Q_T} d(t,x,\nabla u_\alpha)\,\nabla v\,dx\,dt\\
&+\alpha\int_0^T\langle\mathcal{B}(u_\alpha),v\rangle\,dt=\int_{Q_T} f(t,x,u_\alpha)\,v\,dx\,dt.
\end{aligned}\tag{2.2.12}$$

Analogously as for $u^m_{\alpha,n}$, we obtain the estimates for u_α

$$\begin{aligned}
&\sup_{0\le t\le T}\int_\Omega B(u_\alpha)\,dx\le C, && \sup_{0\le t\le T}\int_\Omega|\nabla u_\alpha|^2\,dx\le C,\\
&\int_{Q_T}|\nabla u_\alpha|^p\,dx\,dt\le C, && \int_{Q_T}|\partial_t\nabla u_\alpha|^2\,dx\,dt\le C,\\
&\int_{Q_T}|\partial_t b(u_\alpha)|^2\,dx\,dt\le C, && \int_{Q_T}|d(t,x,\nabla u_\alpha)|^q\,dx\,dt\le C,\\
&\alpha\int_0^T\langle\mathcal{B}(u_\alpha),u_\alpha\rangle\,dt\le C.
\end{aligned}$$

Then there exists a subsequence of $\{u_\alpha\}$, again denoted by $\{u_\alpha\}$, such that

$$\begin{aligned}
u_\alpha \to u \quad & \text{weakly in } L^p(0,T;V), \\
u_\alpha \to u \quad & \text{weakly}-* \text{ in } L^\infty(0,T;V_0), \\
\partial_t u_\alpha \to \partial_t u \quad & \text{weakly in } L^2(0,T;V_0), \\
\partial_t b(u_\alpha) \to \eta \quad & \text{weakly in } L^2(Q_T)^l, \\
d(t,x,\nabla u_\alpha) \to \chi \quad & \text{weakly in } (L^q(Q_T))^{N\times l}.
\end{aligned}$$

Due to the similar argumentation as for $u^m_{\alpha,n}$, we obtain the strong convergences $u_\alpha \to u$, $f(t,x,u_\alpha) \to f(t,x,u)$, $b(u_\alpha) \to b(u)$ in $L^2(Q_T)^l$ and $\eta = \partial_t b(u)$.

Since $u \in L^p(0,T;V)$ and $u_t \in L^2(0,T;V_0)$ we have $u \in C([0,T];V_0)$ by (Evans 1998, Theorem 5.9.2). Then $u_\alpha(0) \to u(0)$ weakly in V_0 yields $u(0) = u_0$. Since b is Lipschitz continuous, we obtain $b \in C([0,T];L^2(\Omega)^l)$ and $b(u(0)) = b(u_0)$ in $L^2(\Omega)^l$, and thus the validity of the initial condition (2.2.2).

From equation (2.2.12) we obtain

$$\int_0^T \langle \mathcal{B}(u_\alpha), v\rangle \, dt = \frac{1}{\alpha} \int_{Q_T} \Big(f(t,x,u_\alpha)\, v - \partial_t b(u_\alpha)\, v - a(x)\, \partial_t \nabla u_\alpha \, \nabla v - d(t,x,\nabla u_\alpha)\, \nabla v \Big)\, dx\, dt$$

for all $v \in L^p(0,T;V)$. Since all the terms on the right hand side are bounded in $L^q(0,T;V^*)$,

$$\mathcal{B}(u_\alpha) \to 0 \;\text{ in }\; L^q(0,T;V^*) \;\text{ as }\; \alpha \to \infty.$$

Applying the monotonicity of $\mathcal{B}$ to the sequence $\{u_\alpha\}$ yields

$$\int_0^T \langle \mathcal{B}(v), u_\alpha - v\rangle \, dt \;\le\; \int_0^T \langle \mathcal{B}(u_\alpha), u_\alpha - v\rangle \, dt.$$

Together with the estimate $\int_0^T \langle \mathcal{B}(u_\alpha), u_\alpha\rangle \, dt \;\le\; C/\alpha$ and the convergence of $\mathcal{B}(u_\alpha) \to 0$ in $L^q(0,T;V^*)$ we obtain for $\alpha \to \infty$

$$\int_0^T \langle \mathcal{B}(v), u - v\rangle \, dt \;\le\; 0.$$

We take $v = u - \lambda w$ for $\lambda > 0$ and $w \in L^p(0,T;V)$. Passing to the limit as $\lambda \to 0$ and using the hemicontinuity of $\mathcal{B}$ imply

$$\int_0^T \langle \mathcal{B}(u), w\rangle \, dt \;\le\; 0 \quad \text{for all} \quad w \in L^p(0,T;V).$$

Thus, $\mathcal{B}(u) = 0$ and $u \in K$ for almost all $t \in (0,T)$.

Now we show that u satisfies the inequality (2.2.1). We choose $u_\alpha - u$ as a test function in the equation (2.2.12) and obtain

$$\int_{Q_T} \partial_t b(u_\alpha)\,(u_\alpha - u)\,dx\,dt + \int_{Q_T} a(x)\partial_t \nabla u_\alpha\,\nabla(u_\alpha - u)\,dx\,dt + \int_{Q_T} d(t,x,\nabla u_\alpha)\,\nabla(u_\alpha - u)\,dx\,dt$$
$$- \int_{Q_T} f(t,x,u_\alpha)\,(u_\alpha - u)\,dx\,dt = -\alpha \int_0^T \langle \mathcal{B}(u_\alpha) - \mathcal{B}(u), u_\alpha - u\rangle\,dt \le 0,$$

since $\mathcal{B}(u) = 0$. Due to Fatou's lemma and integration by parts

$$\liminf_{\alpha\to\infty} \int_{Q_T} a(x)\partial_t \nabla u_\alpha\,\nabla(u_\alpha - u)\,dx\,dt \;\geqslant\; 0.$$

Then, by using the convergence of u_α, we obtain

$$\limsup_{\alpha\to\infty} \int_{Q_T} d(t,x,\nabla u_\alpha)\,\nabla(u_\alpha - u)\,dx\,dt$$
$$\le \lim_{\alpha\to\infty} \int_{Q_T} f(t,x,u_\alpha)\,(u_\alpha - u)\,dx\,dt - \lim_{\alpha\to\infty} \int_{Q_T} \partial_t b(u_\alpha)\,(u_\alpha - u)\,dx\,dt \;=\; 0.$$

The monotonicity of d implies

$$\limsup_{\alpha\to\infty} \int_{Q_T} d(t,x,\nabla u_\alpha)\,\nabla(u_\alpha - u)\,dx\,dt \;\geqslant\; \lim_{\alpha\to\infty} \int_{Q_T} d(t,x,\nabla u)\,\nabla(u_\alpha - u)\,dx\,dt \;=\; 0.$$

Thus, we have

$$\lim_{\alpha\to\infty} \int_{Q_T} d(t,x,\nabla u_\alpha)\,\nabla(u_\alpha - u)\,dx\,dt \;=\; 0. \tag{2.2.13}$$

For the function $w = (1-\lambda)u + \lambda v$, where $v \in L^p(0,T;V)$ and $\lambda > 0$, the monotonicity of d implies

$$\begin{aligned} 0 \;&\le\; \int_{Q_T} (d(t,x,\nabla u_\alpha) - d(t,x,\nabla w))\,\nabla(u_\alpha - w)\,dx\,dt \\ &=\; \int_{Q_T} (d(t,x,\nabla u_\alpha) - d(t,x,\nabla w))\,\nabla(u_\alpha - u)\,dx\,dt \\ &\quad + \lambda \int_{Q_T} (d(t,x,\nabla u_\alpha) - d(t,x,\nabla w))\,\nabla(u - v)\,dx\,dt. \end{aligned}$$

The first integral on the right hand side converges to zero for $\alpha \to \infty$, due to the convergence of $\{u_\alpha\}$ and (2.2.13). Then we divide this inequality by λ, pass to the limits as $\alpha \to \infty$ and $\lambda \to 0$, and, due to continuity of d, obtain

$$\lim_{\alpha\to\infty} \int_{Q_T} d(t,x,\nabla u_\alpha)\nabla(u - v)\,dx\,dt \;\geqslant\; \int_{Q_T} d(t,x,\nabla u)\nabla(u - v)\,dx\,dt. \tag{2.2.14}$$

Now we choose $v - u_\alpha$ as a test function in the equation (2.2.12), where $v \in L^p(0,T;V)$ and $v(t) \in K$ for almost all $t \in (0,T)$, use the monotonicity of $\mathcal{B}$ and obtain

$$\int_{Q_T} \Big[\partial_t b(u_\alpha)\,(v-u_\alpha) + a(x)\partial_t \nabla u_\alpha\,\nabla(v-u_\alpha) + d(t,x,\nabla u_\alpha)\,\nabla(v-u_\alpha)\Big]\,dx\,dt$$

$$- \int_{Q_T} f(t,x,u_\alpha)\,(v-u_\alpha)\,dx\,dt = \alpha \int_0^T \langle \mathcal{B}(v) - \mathcal{B}(u_\alpha), v - u_\alpha\rangle\,dt \;\geqslant\; 0, \tag{2.2.15}$$

since $\mathcal{B}(v) = 0$. By Fatou's lemma we have

$$\liminf_{\alpha\to\infty} \int_{Q_T} a(x)\partial_t \nabla u_\alpha\,\nabla u_\alpha\,dx\,dt \;\geqslant\; \int_{Q_T} a(x)\partial_t \nabla u\,\nabla u\,dx\,dt.$$

The equality (2.2.13) and the inequality (2.2.14) yields

$$\begin{aligned}
&\lim_{\alpha\to\infty} \int_{Q_T} d(t,x,\nabla u_\alpha)\,\nabla(v-u_\alpha)\,dx\,dt \\
&= \lim_{\alpha\to\infty} \int_{Q_T} d(t,x,\nabla u_\alpha)\,\nabla(v-u)\,dx\,dt + \lim_{\alpha\to\infty} \int_{Q_T} d(t,x,\nabla u_\alpha)\,\nabla(u-u_\alpha)\,dx\,dt \\
&\leq \int_{Q_T} d(t,x,\nabla u)\nabla(v-u)\,dx\,dt.
\end{aligned}$$

Then taking the limit as $\alpha \to \infty$ in (2.2.15) and using the convergence of u_α imply that u satisfies the inequality (2.2.1). ❒

Remark 2.2.6. Assuming the strong monotonicity of d, i.e.

$$(d(t,x,\xi_1) - d(t,x,\xi_2))(\xi_1 - \xi_2) \geqslant d_1|\xi_1 - \xi_2|^p \text{ for } d_1 > 0, \;\; \xi_1,\, \xi_2 \in \mathbb{R}^{N\times l}$$

ensures the strong convergence of $u_\alpha \to u$ in $L^p(0,T;V)$.

2.3 Doubly Nonlinear Equations

The initial boundary value problem for doubly nonlinear pseudoparabolic equation is given by

$$\begin{cases} \partial_t b(u) - \partial_t \Delta a(u) - \nabla \cdot d(t,x,u,\nabla a(u)) = f(t,x,u), & (t,x) \in Q_T = (0,T) \times \Omega, \\ b(u(0,x)) = b(u_0(x)), & x \in \Omega, \\ u(t,x) = 0, & (t,x) \in (0,T) \times \partial\Omega. \end{cases} \tag{2.3.1}$$

We define the function $\tilde{B}$ by

$$\tilde{B}(s) := b(a^{-1}(s))\, s - \int_0^s b(a^{-1}(z))\, dz \qquad \text{for} \quad s \in \{y \in \mathbb{R},\ y = a(z),\ z \in \mathbb{R}\}. \tag{2.3.2}$$

The existence of a solution will be ensured by the following assumptions.

Assumption 2.3.1.

$(A1)$ The function $b : \mathbb{R} \to \mathbb{R}$ is strictly monotone increasing, continuous, $b(0) = 0$, and satisfies the growth assumption $|b(s)|^2 \leq C_1 \tilde{B}(a(s)) + C_2$ for all $s \in \mathbb{R}$.

$(A2)$ The function $a : \mathbb{R} \to \mathbb{R}$ is strictly monotone increasing, continuous, $a(0) = 0$.

$(A3)$ The diffusivity $d : (0,T) \times \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is continuous, elliptic, i.e. there exists some $d_0 > 0$, such that $d(t,x,z,\xi)\,\xi \geqslant d_0 |\xi|^p$ for $\xi \in \mathbb{R}^N, p \geqslant 2$, and monotone, i.e. $(d(t,x,z,\xi_1) - d(t,x,z,\xi_2))\,(\xi_1 - \xi_2) \geqslant 0$ for $\xi_1,\ \xi_2 \in \mathbb{R}^N$, and satisfies the growth assumption $|d(t,x,z,\xi)| \leq C\big(1 + |\xi|^{p-1} + (\tilde{B}(a(z)))^{\frac{p-1}{p}}\big)$ for almost all $(t,x) \in Q_T$ and for $z \in \mathbb{R},\ \xi \in \mathbb{R}^N$.

$(A4)$ The function $f : (0,T) \times \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the growth assumption, i.e. $|f(t,x,z)| \leq C\big(1 + (\tilde{B}(a(z)))^{\frac{p-1}{p}}\big)$ for almost all $(t,x) \in Q_T$ and for $z \in \mathbb{R}$.

$(A5)$ The initial condition $b(u_0)$ is in $L^2(\Omega)$ and $a(u_0)$ is in $H_0^1(\Omega)$.

The notion of a solution of the problem introduced above, will be given now.

Definition 2.3.2. A function $u : Q_T \to \mathbb{R}$ is called *a weak solution* of (2.3.1) if

1) $b(u) \in L^2(Q_T)$, $\partial_t(b(u) - \Delta a(u)) \in L^q(0,T; H^{-1,q}(\Omega))$, $a(u) \in L^p(0,T; H_0^{1,p}(\Omega))$, $a(u) \in L^\infty(0,T; H_0^1(\Omega))$, and

2)
$$- \int_{Q_T} b(u)\, \partial_t v \, dx\, dt - \int_{Q_T} \nabla a(u)\, \nabla \partial_t v\, dx\, dt + \int_{Q_T} d(t,x,u,\nabla a(u))\, \nabla v\, dx\, dt$$
$$+ \int_{Q_T} b(u_0)\, v_t\, dx\, dt + \int_{Q_T} \nabla a(u_0)\, \nabla v_t\, dx\, dt = \int_{Q_T} f(t,x,u)\, v\, dx\, dt, \tag{2.3.3}$$

for all functions $v \in L^p(0,T; H_0^{1,p}(\Omega))$, such that $v_t \in L^2(0,T; H_0^1(\Omega))$ and $v(T) = 0$.

Theorem 2.3.3. *Under Assumption 2.3.1 there exists a weak solution of the problem (2.3.1).*

We approximate the differential equation by the time discretization, $h = T/n$, $t_i = ih$, $i = 0, \dots, n$, and obtain the discrete problem

$$\begin{cases} \frac{1}{h}(b(u_i) - b(u_{i-1})) - \frac{1}{h}\Delta(a(u_i) - a(u_{i-1})) - \nabla \cdot d(t_i, x, u_{i-1}, \nabla a(u_i)) = f(t_i, x, u_{i-1}), \\ u_i(x) = 0 \qquad\qquad \text{for } x \in \partial\Omega. \end{cases}$$

The elliptic problems can be solved by Galerkin's approximation. Let $\{e_k\}_{k=1}^{\infty}$ be a basis in $H_0^{1,p}(\Omega)$. We are looking for functions $\{u_i^m\}_{i=1}^n$ in the subspace H_m, spanned by $\{e_1, ..., e_m\}$,

$$a(u_i^m) = \sum_{k=1}^{m} \alpha_{ik}^m e_k,$$

such that

$$\begin{aligned} &\int_\Omega \frac{1}{h}(b(u_i^m) - b(u_{i-1}^m))\,\xi\,dx + \int_\Omega \frac{1}{h}(\nabla a(u_i^m) - \nabla a(u_{i-1}^m))\,\nabla\xi\,dx \\ &+ \int_\Omega d(t_i, x, u_{i-1}^m, \nabla a(u_i^m))\,\nabla\xi\,dx - \int_\Omega f(t_i, x, u_{i-1}^m)\,\xi\,dx = 0 \end{aligned} \tag{2.3.4}$$

holds for all $\xi \in H_m$. Here $a(u_0^m) \in H_m$ is an approximation of $a(u_0)$ in $H_0^1(\Omega)$. The strict monotonicity of a yields $u_0^m \to u_0$ a.e. in Q_T.

Lemma 2.3.4. *There exists a solution u_i^m in H_m of the family of discretized equations (2.3.4).*

Proof. The existence will be shown by induction. Since u_0^m is given, u_{i-1}^m can be assumed to be known. The left-hand side of (2.3.4) defines a continuous mapping $J_{hm} : \mathbb{R}^m \to \mathbb{R}^m$ given by

$$\begin{aligned} J_{hm}^j(r) = \ & \frac{1}{h}\int_\Omega (b(v)e_j + \nabla a(v)\nabla e_j)\,dx - \frac{1}{h}\int_\Omega (b(u_{i-1}^m)\,e_j + \nabla a(u_{i-1}^m)\,\nabla e_j)\,dx \\ & + \int_\Omega d(t_i, x, u_{i-1}^m, \nabla a(v))\,\nabla e_j\,dx - \int_\Omega f(t_i, x, u_{i-1}^m)\,e_j\,dx, \end{aligned}$$

where $a(v) = \sum_{j=1}^{m} r_j e_j$. Due to Assumption 2.3.1, this mapping satisfies the estimate

$$\begin{aligned} J_{hm}(r)r \ \geqslant\ & \frac{1}{h}\int_\Omega \left(\tilde{B}(a(v)) + \frac{1}{2}|\nabla a(v)|^2\right) dx - \frac{1}{h}\int_\Omega \left(\tilde{B}(a(u_{i-1}^m)) + \frac{1}{2}|\nabla a(u_{i-1}^m)|^2\right) dx \\ & + d_0 \int_\Omega |\nabla a(v)|^p\,dx - c_1\delta_0 \int_\Omega |\nabla a(v)|^p\,dx - c_2(\delta_0)\int_\Omega |f(t_i, x, u_{i-1}^m)|^q\,dx \\ \geqslant\ & c_3 \int_\Omega |\nabla a(v)|^2\,dx + c_4 \int_\Omega |\nabla a(v)|^p\,dx - c_5. \end{aligned}$$

Hence, for $|r|$ big enough, $J(r)r \geqslant 0$ for all such r. The continuity of J implies the existence of a zero of J, see (Showalter 1996, Proposition 2.1). Due to the strict monotonicity of a there exists $v = u_i^m$ a solution of (2.3.4). ❐

We construct approximative solutions as piecewise constant interpolation of the u_i's to $[0,T]$. For the proof of the Theorem 2.3.3 we use an a priori estimate, a compactness argument, and an integration by parts formula from (Jäger and Kacur 1995), adapted for pseudoparabolic equations, Lemma A.1.6.

At first we obtain the estimates for u_i^m.

Lemma 2.3.5. *The estimates*

$$\begin{aligned}\max_{1\le i\le n}\int_\Omega \tilde{B}(a(u_i^m))\,dx &\le C,\\ \max_{1\le i\le n}\int_\Omega |\nabla a(u_i^m)|^2\,dx &\le C,\\ \sum_{i=1}^n h\int_\Omega |\nabla a(u_i^m)|^p\,dx &\le C\end{aligned}\tag{2.3.5}$$

hold uniformly with respect to m and n.

Proof. Choosing $a(u_i^m)$ as a test function in (2.3.4) and summing over i yields

$$\sum_{i=1}^j\int_\Omega \frac{1}{h}(b(u_i^m)-b(u_{i-1}^m))\,a(u_i^m)\,dx+\sum_{i=1}^j\int_\Omega \frac{1}{h}(\nabla a(u_i^m)-\nabla a(u_{i-1}^m))\,\nabla a(u_i^m)\,dx$$
$$+\sum_{i=1}^j\int_\Omega d(t_i,x,u_{i-1}^m,\nabla a(u_i^m))\,\nabla a(u_i^m)\,dx-\sum_{i=1}^j\int_\Omega f(t_i,x,u_{i-1}^m)\,a(u_i^m)\,dx=0.$$

By using Assumption 2.3.1 each term will be estimated separately. From the definition of the function $\tilde{B}$ it follows that

$$I_1 := \sum_{i=1}^j\int_\Omega (b(u_i^m)-b(u_{i-1}^m))\,a(u_i^m)\,dx \geqslant \int_\Omega \tilde{B}(u_j^m)\,dx-\int_\Omega \tilde{B}(u_0^m)\,dx.$$

By Abel's summation formula we obtain

$$I_2 := \sum_{i=1}^j\int_\Omega (\nabla a(u_i^m)-\nabla a(u_{i-1}^m))\,\nabla a(u_i^m)\,dx \geqslant \frac{1}{2}\int_\Omega |\nabla a(u_j^m)|^2\,dx-\frac{1}{2}\int_\Omega |\nabla a(u_0^m)|^2\,dx.$$

The ellipticity assumption implies

$$I_3 := \sum_{i=1}^j\int_\Omega d(t_i,x,u_{i-1}^m,\nabla a(u_i^m))\,\nabla a(u_i^m)\,dx \geqslant d_0\sum_{i=1}^j\int_\Omega |\nabla a(u_i^m)|^p\,dx.$$

By the growth assumption on f and the Poincaré inequality we have

$$\begin{aligned}I_4 &:= \sum_{i=1}^j\int_\Omega f(t_i,x,u_{i-1}^m)\,a(u_i^m)\,dx \le c_1/\delta_0\sum_{i=1}^j\int_\Omega |f(t_i,x,u_{i-1}^m)|^q\,dx+c_2\delta_0\sum_{i=1}^j\int_\Omega |a(u_i^m)|^p\,dx\\ &\le c_3(\delta_0)\sum_{i=1}^j\int_\Omega \tilde{B}(u_i^m)\,dx+c_4\delta_0\sum_{i=1}^j\int_\Omega |\nabla a(u_i^m)|^p\,dx+c_4.\end{aligned}$$

Collecting the estimates of integrals I_1, I_2, I_3, and I_4 implies

$$\int_\Omega \tilde{B}(u_j^m)\,dx + \frac{1}{2}\int_\Omega |\nabla a(u_j^m)|^2\,dx + (d_0 - c_1\delta_0)\sum_{i=1}^{j} h \int_\Omega |\nabla a(u_i^m)|^p\,dx$$
$$\leq \int_\Omega \tilde{B}(u_0^m)\,dx + \frac{1}{2}\int_\Omega |\nabla a(u_0^m)|^2\,dx + c_2(\delta_0)\sum_{i=1}^{j} h \int_\Omega \tilde{B}(u_i^m)\,dx + c_3.$$

The discrete version of Gronwall's Lemma implies the estimates in Lemma 2.3.5. The Gronwall Lemma is applicable for sufficiently small h and δ_0 that satisfy $d_0 > c_1\delta_0$ and $c_2(\delta_0)h < 1$. ❒

To show the strong convergence of the approximations in $L^2(Q_T)$ the following lemma is essential.

Lemma 2.3.6. *The estimates*

$$\sum_{j=1}^{n-k} h \int_\Omega (b(u_{j+k}^m) - b(u_j^m))\,(a(u_{j+k}^m) - a(u_j^m))\,dx \leq Ckh,$$
$$\sum_{j=1}^{n-k} h \int_\Omega |\nabla a(u_{j+k}^m) - \nabla a(u_j^m)|^2\,dx \leq Ckh \tag{2.3.6}$$

hold uniformly with respect to m and n.

Proof. Summing up the equations (2.3.4) for $j = j+1, ..., j+k$, choosing $a(u_{j+k}^m) - a(u_j^m)$ as a test function, and finally summing up over $j = 1, ..., n-k$ yields

$$\sum_{j=1}^{n-k}\int_\Omega \frac{1}{h}(b(u_{j+k}^m) - b(u_j^m))(a(u_{j+k}^m) - a(u_j^m))\,dx + \sum_{j=1}^{n-k}\int_\Omega \frac{1}{h}|\nabla a(u_{j+k}^m) - \nabla a(u_j^m)|^2\,dx$$
$$+\sum_{j=1}^{n-k}\sum_{i=j+1}^{j+k}\int_\Omega d(t_i, x, u_{i-1}^m, \nabla a(u_i^m))(\nabla a(u_{j+k}^m) - \nabla a(u_j^m))\,dx$$
$$-\sum_{j=1}^{n-k}\sum_{i=j+1}^{j+k}\int_\Omega f(t_i, x, u_{i-1}^m)(a(u_{j+k}^m) - a(u_j^m))\,dx = 0.$$

The third and fourth integral can be estimated by

$$\sum_{j=1}^{n-k}\sum_{i=j+1}^{j+k}\int_\Omega d(t_i, x, u_{i-1}^m, \nabla a(u_i^m))(\nabla a(u_{j+k}^m) - \nabla a(u_j^m))\,dx$$
$$\leq c_1\sum_{i=1}^{n}\int_\Omega |d(t_i, x, u_{i-1}^m, \nabla a(u_i^m))|^q\,dx + c_2 k\sum_{j=1}^{n-k}\int_\Omega (|\nabla a(u_{j+k}^m)|^p + |\nabla a(u_j^m)|^p)\,dx$$

and

$$\sum_{j=1}^{n-k}\sum_{i=j+1}^{j+k}\int_\Omega f(t_i, x, u_{i-1}^m)(a(u_i^m) - a(u_{i-1}^m))\,dx \leq c_3\sum_{i=1}^{n}\int_\Omega |f(t_i, x, u_{i-1}^m)|^q\,dx$$
$$+c_4 k\sum_{j=1}^{n-k}\int_\Omega (|a(u_{j+k}^m)|^p + |a(u_j^m)|^p)\,dx.$$

Due to the growth assumptions on d and f, we have

$$\sum_{i=1}^{n}\int_{\Omega}|d(x,t_i,u_{i-1}^m,\nabla a(u_i^m))|^q\,dx \;\leq\; c_5+c_6\sum_{i=1}^{n}\int_{\Omega}|\nabla a(u_i^m)|^p\,dx+c_7\sum_{i=1}^{n}\int_{\Omega}\tilde{B}(a(u_i^m))\,dx,$$

$$\sum_{i=1}^{n}\int_{\Omega}|f(x,t_i,u_{i-1}^m)|^q\,dx \;\leq\; c_8+c_9\sum_{i=1}^{n}\int_{\Omega}\tilde{B}(a(u_i^m))\,dx.$$

Then we obtain the following inequality

$$\sum_{j=1}^{n-k}\int_{\Omega}\frac{1}{h}(b(u_{j+k}^m)-b(u_j^m))(a(u_{j+k}^m)-a(u_j^m))\,dx+\sum_{j=1}^{n-k}\int_{\Omega}\frac{1}{h}|\nabla a(u_{j+k}^m)-\nabla a(u_j^m)|^2\,dx$$

$$\leq\; \sum_{i=1}^{n}\int_{\Omega}|\nabla a(u_i^m)|^p\,dx+c_1\sum_{i=1}^{n}\int_{\Omega}\tilde{B}(a(u_i^m))\,dx+c_2k\sum_{j=1}^{n-k}\int_{\Omega}(|a(u_{j+k}^m)|^p+|a(u_j^m)|^p)\,dx+c_3.$$

This, by using the estimates in Lemma 2.3.5, implies the asserted estimates. ❐

Proof of Theorem 2.3.3. We define the step functions for $t\in(t_{i-1},t_i]$, $x\in\Omega$ by

$$\bar{u}_n^m(t,x)\;:=\;u^m(t_i,x),$$

where the initial conditions are $\bar{u}_n^m(0,x)=u_0^m(x)$.

From estimates (2.3.5) and (2.3.6) we obtain

$$\sup_{0\leq t\leq T}\int_{\Omega}\tilde{B}(a(\bar{u}_n^m(t,x)))\,dx\;\leq\;C, \qquad \sup_{0\leq t\leq T}\int_{\Omega}|\nabla a(\bar{u}_n^m(t,x))|^2\,dx\;\leq\;C,$$

$$\int_{Q_T}|\nabla a(\bar{u}_n^m(t,x))|^p\,dx\,dt\;\leq\;C, \qquad \int_0^{T-\tau}\int_{\Omega}|\nabla a(\bar{u}_n^m(t+\tau,x))-\nabla a(\bar{u}_n^m(t,x))|^2\,dx\,dt\;\leq\;C\tau,$$

$$\int_0^{T-\tau}\int_{\Omega}(b(a(\bar{u}_n^m(t+\tau)))-b(a(\bar{u}_n^m(t)))(a(\bar{u}_n^m(t+\tau))-a(\bar{u}_n^m(t)))\,dx\,dt\;\leq\;C\tau, \tag{2.3.7}$$

where for $k\in\{0,\dots,n-1\}$, $\tau\in(kh,(k+1)h)$. The growth assumption on d and estimates (2.3.7) imply

$$||d_n(t,x,\bar{u}_{n,h}^m,\nabla a(\bar{u}_n^m))||_{L^q(Q_T)^N}\;\leq\;c_1||\nabla a(\bar{u}_n^m)||_{L^p(Q_T)^N}^{\frac{p}{q}}+c_2||\tilde{B}(a(\bar{u}_n^m))||_{L^1(Q_T)}^{\frac{1}{q}}+c_3\;\leq\;C,$$

where $\bar{u}_{n,h}^m(t):=\bar{u}_n^m(t-h)$ for $t\in[h,T]$ and $\bar{u}_{n,h}^m(t,x):=u_0^m(x)$ for $t\in[0,h]$, $d_n(t,x,s,z):=d(t_i,x,s,z)$ for $t\in(t_{i-1},t_i]$, for $i=1,...,n$, and $d_n(0,x,s,z):=d(0,x,s,z)$. From the equation (2.3.4) we obtain

$$||\partial_h\big(b(\bar{u}_n^m)-\Delta a(\bar{u}_n^m)\big)||_{L^q(0,T;H^{-1,q}(\Omega))}\;\leq\;C. \tag{2.3.8}$$

The growth assumption $|b(s)|^2\leq C_1\tilde{B}(a(s))+C_2$ and the estimate for $\tilde{B}$ in (2.3.7) imply

$$||b(\bar{u}_n^m)||_{L^2(Q_T)}\;\leq\;C.$$

Then there exists a subsequence in m and n of $\{\bar{u}_n^m\}$, again denoted by $\{\bar{u}_n^m\}$, such that

$$\begin{aligned} a(\bar{u}_n^m) &\to \alpha && \text{weakly} - * \text{ in } L^\infty(0,T;H_0^1(\Omega)),\\ a(\bar{u}_n^m) &\to \alpha && \text{weakly in } L^p(0,T;H_0^{1,p}(\Omega)),\\ b(\bar{u}_n^m) &\to \beta && \text{weakly in } L^2(Q_T),\\ \partial_h(b(\bar{u}_n^m) - \Delta a(\bar{u}_n^m)) &\to z && \text{weakly in } L^q(0,T;H^{-1,q}(\Omega)),\\ d(t,x,\bar{u}_{n,h}^m, \nabla a(\bar{u}_n^m)) &\to \chi && \text{weakly in } (L^q(Q_T))^N. \end{aligned} \tag{2.3.9}$$

The third and fourth estimates in (2.3.7) imply, by Kolmogorov's compactness criterium, (Necas 1967), the strong convergence of $\{a(\bar{u}_n^m)\}$ in $L^2(Q_T)$. Due to strict monotonicity of a we obtain convergence $\bar{u}_n^m \to u$ a.e. in Q_T. Since a is continuous, $a(\bar{u}_n^m) \to a(u)$ a. e. in Q_T and $\alpha = a(u)$. Thus, by the Dominated Convergence Theorem, (Evans 1998), $a(\bar{u}_n^m) \to a(u)$ strongly in $L^2(Q_T)$. The fourth estimate in (2.3.7) and the Poincaré inequality imply

$$||a(\bar{u}_n^m) - a(\bar{u}_{n,h}^m)||_{L^2(0,T;H_0^1(\Omega))} \le \frac{C}{\sqrt{n}}.$$

Then the strict monotonicity of a and $\bar{u}_n^m \to u$ a.e. in Q_T imply $\bar{u}_{n,h}^m \to u$ a.e. in Q_T. The continuity of b implies $b(\bar{u}_n^m) \to b(u)$ a. e. in Q_T. The Dominated Convergence Theorem yields $b(\bar{u}_n^m) \to b(u)$ strongly in $L^2(Q_T)$. Since $f(\bar{u}_{n,h}^m)$ is continuous and bounded in $L^q(Q_T)$,

$$||f_n(t,x,\bar{u}_{n,h}^m)||_{L^q(Q_T)} \le c_1 ||\tilde{B}(a(\bar{u}_n^m))||_{L^1(Q_T)}^{\frac{1}{q}} + c_2 \le C,$$

we obtain $f(\bar{u}_{n,h}^m) \to f(u)$ strongly in $L^q(Q_T)$. From the continuity of $\tilde{B}$ follows that $\tilde{B}(\bar{u}_n^m) \to \tilde{B}(u)$ a.e. in Q_T. Since $\{\tilde{B}(\bar{u}_n^m)\}$ is bounded in $L^\infty(0,T;L^1(\Omega))$ and $\tilde{B}(\bar{u}_n^m)$ is non-negative we obtain, by Fatou's Lemma, $\tilde{B}(u) \in L^\infty(0,T;L^1(\Omega))$.

Integrating the equation (2.3.4) over $(0,T)$ and passing to the limit as $m,n \to \infty$ imply

$$\int_0^T \langle z, v\rangle\, dt + \int_{Q_T} \chi\, \nabla v\, dx\, dt \;=\; \int_{Q_T} f(t,x,u)\, v\, dx\, dt.$$

The discrete time derivative can be rewritten in the form

$$\begin{aligned} &\int_{Q_T} \frac{1}{h}\big(b(\bar{u}_n^m(t)) - b(\bar{u}_n^m(t-h))\big)\, v\, dx\, dt + \int_{Q_T} \frac{1}{h}\big(\nabla a(\bar{u}_n^m(t)) - \nabla a(\bar{u}_n^m(t-h))\big)\nabla v\, dx\, dt = \\ &-\int_{Q_T} \Big(b(\bar{u}_n^m)\partial_{-h} v + \nabla a(\bar{u}_n^m)\partial_{-h}\nabla v\Big)\, dx\, dt + \int_{\Omega} \Big(b(\bar{u}_n^m(0))v(0) + \nabla a(\bar{u}_n^m(0))\nabla v(0)\Big)\, dx \end{aligned}$$

for all $v \in L^p(0,T;H_0^{1,p}(\Omega))$, such that $v_t \in L^2(0,T;H_0^1(\Omega))$ and $v(T) = 0$. Due to $a(\bar{u}_n^m(0)) = a(u_0^m)$, $a(u_0^m) \to a(u_0)$ in $H_0^1(\Omega)$, and $u_0^m \to u_0$ a.e. in Q_T for $m \to \infty$, we obtain

$$\begin{aligned} \int_0^T \langle z, v\rangle\, dt &= \lim_{m,n\to\infty} \int_{Q_T} \frac{1}{h}\Big(\big(b(\bar{u}_n^m(t)) - b(\bar{u}_n^m(t-h))\big)\, v + \nabla\big(a(\bar{u}_n^m(t)) - a(\bar{u}_n^m(t-h))\big)\nabla v\Big)\, dx\, dt \\ &= -\int_{Q_T} b(u) v_t\, dx\, dt - \int_{Q_T} \nabla a(u)\nabla v_t\, dx\, dt + \int_{Q_T} b(u_0) v_t\, dx\, dt + \int_{Q_T} \nabla a(u_0)\nabla v_t\, dx\, dt \end{aligned}$$

for all $v \in L^p(0,T;H_0^{1,p}(\Omega))$, such that $v_t \in L^2(0,T;H_0^1(\Omega))$ and $v(T) = 0$. Since such v form a dense subspace of $L^p(0,T;H_0^{1,p}(\Omega))$ and the boundedness in (2.3.8), we obtain that $z = \partial_t(b(u) - \Delta a(u))$ as functions in $L^q(0,T;H^{-1,q}(\Omega))$.

Now we show $\chi(t,x) = d(t,x,u,\nabla a(u))$ by using the monotonicity of d and Minty-Browder argument. The integration by parts formula, Lemma A.1.6, and Fatou's Lemma yield

$$\begin{aligned}
&\liminf_{m,n\to\infty} \int_0^T \langle \partial_t(b(\bar{u}_n^m) - \Delta a(\bar{u}_n^m)), a(\bar{u}_n^m)\rangle \, dt \\
&\geqslant \liminf_{m,n\to\infty} \Big(\int_\Omega \tilde{B}(a(\bar{u}_n^m))\, dx + \frac{1}{2}\int_\Omega |\nabla a(\bar{u}_n^m)|^2\, dx \Big) - \int_\Omega \Big(\tilde{B}(a(u_0)) + \frac{1}{2}|\nabla a(u_0)|^2 \Big)\, dx \\
&\geqslant \int_\Omega \tilde{B}(a(u))\, dx + \frac{1}{2}\int_\Omega |\nabla a(u)|^2\, dx - \int_\Omega \tilde{B}(a(u_0))\, dx - \frac{1}{2}\int_\Omega |\nabla a(u_0)|^2\, dx.
\end{aligned}$$

Then

$$\begin{aligned}
&\limsup_{m,n\to\infty} \int_0^T \int_\Omega d(t,x,\bar{u}_{nh}^m, \nabla a(\bar{u}_n^m)) \nabla a(\bar{u}_n^m)\, dx\, dt \\
&\leq \int_0^T \int_\Omega f(t,x,u)\, u\, dx\, dt - \liminf_{m,n\to\infty} \int_0^T \langle \partial_t(b(\bar{u}_n^m) - \Delta a(\bar{u}_n^m)), a(\bar{u}_n^m)\rangle\, dt \\
&\leq \int_0^T \int_\Omega \chi\, \nabla a(u)\, dx\, dt.
\end{aligned}$$

The monotonicity of d implies

$$\int_0^T \int_\Omega \big(d_n(t,x,\bar{u}_{nh}^m, \nabla a(\bar{u}_n^m)) - d_n(t,x,\bar{u}_{nh}^m, w) \big) \big(\nabla a(\bar{u}_n^m) - w \big)\, dx\, dt \;\geqslant\; 0.$$

Since $\bar{u}_{nh}^m \to u$ a.e. in Q_T, d_n is continuous, and $d_n(t,x,\bar{u}_{nh}^m, w)$ is uniformly bounded in $L^q(Q_T)^N$, we have $d_n(t,x,\bar{u}_{nh}^m, w) \to d(t,x,u,w)$ in $L^q(Q_T)^N$, by the Dominated Convergence Theorem. Taking the limit as $m,n \to \infty$ yields

$$\int_0^T \int_\Omega \big(\chi - d(t,x,u,w) \big) \big(\nabla a(u) - w \big)\, dx\, dt \quad \geqslant \quad 0.$$

Using the Minty-Browder Theorem implies $\chi(t,x) = d(t,x,u,\nabla a(u))$. Thus, we obtain that the function u satisfies the equation (2.3.3). ❒

Theorem 2.3.7 (Uniqueness). *Let Assumption 2.3.1, $p = 2$,*

$$|d(t,x,\xi_1,\zeta_1) - d(t,x,\xi_2,\zeta_2)| \;\leq\; C(|a(\xi_1) - a(\xi_2)| + |\zeta_1 - \zeta_2|), \;\; \text{and}$$

$$|f(t,x,\xi_1) - f(t,x,\xi_2)| \;\leq\; C|a(\xi_1) - a(\xi_2)|$$

for $t \in (0,T)$, $x \in \Omega$, $\xi_1, \xi_2 \in \mathbb{R}$, $\zeta_1, \zeta_2 \in \mathbb{R}^N$ be satisfied. Then there exists at most one weak solution of the problem (2.3.1).

Proof. Suppose, there are two solutions $u^1, u^2 \in L^2(Q_T)$. Then they satisfy

$$-\int_{Q_T} \Big((b(u^1) - b(u^2))\, v_t + \nabla(a(u^1) - a(u^2))\, \nabla v_t \Big)\, dx\, dt$$
$$+ \int_{Q_T} \big(d(t,x,u^1,\nabla a(u^1)) - d(t,x,u^2,\nabla a(u^2)) \big)\, \nabla v\, dx\, dt = \int_{Q_T} (f(t,x,u^1) - f(t,x,u^2))\, v\, dx\, dt,$$

using $b(u_0^1) = b(u_0^2)$ and $\nabla a(u_0^1) = \nabla a(u_0^2)$. Since $\partial_t(b(u^i) - \Delta\, a(u^i)) \in L^2(0,T,H^{-1}(\Omega))$, we can assume $b(u^i) - \Delta\, a(u^i) \in C(0,T;H^{-1}(\Omega))$. Due to $a(u^i) \in L^2(0,T;H_0^1(\Omega))$ we obtain at first $\Delta\, a(u^i) \in L^2(0,T,H^{-1}(\Omega))$ and thereby $b(u^i) \in L^2(0,T;H^{-1}(\Omega))$. We choose for $s \le T$

$$v_s(t) = \begin{cases} \int_t^s (a(u^1(\tau)) - a(u^2(\tau)))d\tau, & t < s, \\ 0, & \text{otherwise} \end{cases}$$

and integrate by parts. Notice that $v_s(s) = 0$. Then we obtain

$$\int_0^s \langle b(u^1) - b(u^2), a(u^1) - a(u^2)\rangle\, dt + \int_0^s \int_\Omega |\nabla a(u^1) - \nabla a(u^2)|^2\, dx\, dt$$
$$\le \delta_0 \int_0^s \int_\Omega |\nabla a(u^1) - \nabla a(u^2)|^2\, dx\, dt + \frac{c_1}{\delta_0} \int_0^s \int_\Omega |\nabla v_s(t)|^2\, dx\, dt.$$

The last integral satisfies the following estimate

$$\begin{aligned} \int_0^s \int_\Omega |\nabla v_s(t)|^2\, dx\, dt &\le c_2 \int_0^s \int_t^s \int_\Omega |\nabla a(u^1(x,\tau)) - \nabla a(u^2(x,\tau))|^2\, dx\, d\tau\, dt \\ &= c_2 \int_0^s \int_0^t \int_\Omega |\nabla a(u^1(x,\tau)) - \nabla a(u^2(x,\tau))|^2\, dx\, d\tau\, dt. \end{aligned}$$

Using the monotonicity of the functions b and a, and Gronwall's lemma in the last inequality yields

$$\int_0^s \int_\Omega |\nabla a(u^1) - \nabla a(u^2)|^2\, dx\, dt = 0$$

and, since the function a is strictly monotone increasing, $u^1 = u^2$ almost everywhere in Q_T. ❒

Remark 2.3.8. We considered the Dirichlet boundary conditions. The results remain valid for other boundary conditions, that allow the use of a Poincaré inequality.

2.4 Pseudoparabolic Equations with Convection

In this section, a better approximative solution of diffusion with convection is obtained by using discretization along characteristics. Such a convective term arises in the two-phase flow model described earlier. It is due to gravitational force.

In $Q_T = (0,T) \times \Omega$ consider the initial boundary value problem

$$\begin{cases} \partial_t u - \nabla \cdot (a(x)\partial_t \nabla u) + c(t,x,u)\nabla u - \nabla \cdot (d(t,x,u)\nabla u) = & f(t,x,u), \\ u(0,x) = u_0(x) & x \in \Omega, \\ u(t,x) = 0, & (t,x) \in (0,T) \times \partial\Omega. \end{cases} \tag{2.4.1}$$

The existence of a solution will be ensured by the following assumptions.

Assumption 2.4.1.

$(A1)$ The matrix field $a \in L^\infty(\Omega)^{N\times N}$ is symmetric and elliptic, i.e. for some a_0 and a^0, $0 < a_0 \leq a^0 < \infty,$ a satisfies $a_0|\xi|^2 \leq a(x)\xi\,\xi \leq a^0|\xi|^2$ for a.a. $x \in \Omega$ and for $\xi \in \mathbb{R}^N$.

$(A2)$ The function $c : (0,T) \times \Omega \times \mathbb{R} \to \mathbb{R}^N$ is continuous and bounded $|c(t,x,z)| \leq c^0 < \infty$.

$(A3)$ The matrix field $d : (0,T) \times \Omega \times \mathbb{R} \to \mathbb{R}^{N\times N}$ is continuous, elliptic, i.e. there exists some $d_0 > 0,$ such that d satisfies $d(t,x,z)\xi\,\xi \geqslant d_0|\xi|^2$ for $\xi \in \mathbb{R}^N$, and bounded, i.e. for some $d^0 < \infty$, $|d(t,x,z)| \leq d^0$ for almost all $(t,x) \in Q_T$ and $z \in \mathbb{R}$.

$(A4)$ The function $f : (0,T) \times \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and sublinear, i.e. $|f(t,x,z)| \leq C(1+|z|)$ for almost all $(t,x) \in Q_T$ and for $z \in \mathbb{R}$.

$(A5)$ The initial condition u_0 is in $H_0^1(\Omega)$.

The notion of a solution of the problem (2.4.1) will be given now.

Definition 2.4.2. A function $u : Q_T \to \mathbb{R}$ is called *a weak solution* of (2.4.1) if

1) $u \in H^1(0,T;H_0^1(\Omega))$,

2) u satisfies the initial condition, i.e. $u(t) \to u_0$ in $H_0^1(\Omega)$ as $t \to 0$,

3) u satisfies the equality

$$\int_{Q_T} u_t\, v\, dx\, dt + \int_{Q_T} a(x)\, \nabla u_t\, \nabla v\, dx\, dt + \int_{Q_T} c(t,x,u)\, \nabla u\, v\, dx\, dt$$
$$+ \int_{Q_T} d(t,x,u) \nabla u\, \nabla v\, dx\, dt = \int_{Q_T} f(t,x,u)\, v\, dx\, dt \tag{2.4.2}$$

for all test functions $v \in L^2(0,T;H_0^1(\Omega))$.

The main theorem of this section contains the existence of such a solution.

Theorem 2.4.3. *Under Assumption 2.4.1 there exists a weak solution of the problem (2.4.1).*

The equation (2.4.1) is of the form

$$\begin{cases} \partial_t u + v \cdot \nabla u - A(u) = f(t,x,u) & \text{in} \quad Q_T, \\ u = 0 & \text{on} \quad (0,T) \times \partial\Omega, \\ u(0,x) = u_0 & \text{in} \quad \Omega, \end{cases}$$

where $v(t,x) = c(t,x,u(t,x))$. Due to the characteristic method, we obtain the equation

$$\partial_t u(t, X(t,s,x)) - A(u) = f(t,x,u),$$

where X satisfies

$$\partial_t X(t,s,x) = v(t, X(t,s,x)), \;\; X(s,s,x) = x.$$

The basic structure of the in time discretized equation reads

$$\frac{u_i - u_{i-1} \circ \phi^i}{h} - A(u_i) = f(t_i, x, u_{i-1}),$$

where $\phi^i(x) = x - hv(t_i, x)$ is an approximation of $X(t_{i-1}, t_i, x)$ for $h = T/n, \;\; t_i = ih$, $i = 0, \dots, n$. To make this idea work there are some subtleties to be considered.

It is substantial that the characteristics X do not intersect; otherwise, neither the backward transport $X(t_{i-1}, t_i, x)$ nor $\phi^i(x)$ can be shown to exist. Provided

$$||\nabla v(t)||_{L^\infty(\Omega)} \leq c \;\text{ for all }\; t \in (0,T),$$

and therefore $det(\phi^i(x)) \geqslant 1 - hc > 0$, the backward transport exists. However, this estimate may not be satisfied. To circumvent this problem, we consider for $\tau = h^\omega$, $0 < \omega < 1$, the smoothed version of $v_i(x) := v(t_i, x)$ by $v_i^\tau := w_\tau * v_i$, where $w_\tau(x) = \frac{1}{\tau^N} w_1(\frac{x}{\tau})$,

$$w_1(x) = \begin{cases} \kappa \exp(\frac{|x|^2}{|x|^2-1}) & \text{for } |x| \leq 1, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \int\limits_{\mathbb{R}^N} w_1(x)dx = 1.$$

This concept will guarantee that $||\nabla v_i^\tau||_{L^\infty(\Omega)}$ will be uniformly bounded in $i = 1, \dots, n$ for each fixed τ. Choose

$$\Omega_h = \{x \in \mathbb{R}^N, \, \text{dist}(x, \Omega) < h||v||_{L^\infty(Q_T)}\}.$$

Then $\overline{\Omega} = \cap_{h>0} \Omega_h$. Fix some $h^* > 0$ and $\Omega^* = \Omega_{h^*}$. Let $\Omega_i = \phi^i(\Omega)$. The boundedness of v yields $\Omega_i \subset \Omega_h \subset \Omega^*$ for $h \leq h^*$. Since $h^* > 0$, there exists an extension $\tilde{u}_{i-1}$ of u_{i-1} from Ω to Ω^*, satisfying $||\tilde{u}_{i-1}||_{H^1(\Omega^*)} \leq c||u_{i-1}||_{H^1(\Omega)}$ uniformly in u. The function u_{i-1} from $H_0^1(\Omega)$ can be extended by zero to a function $\tilde{u}_{i-1} \in H_0^1(\Omega^*)$ and $||\tilde{u}_{i-1}||_{H_0^1(\Omega^*)} \leq ||u_{i-1}||_{H_0^1(\Omega)}$. This construction allows us to assume that $\tilde{u}_{i-1}$ is defined on all Ω_i. Especially, $\tilde{u}_{i-1} \circ \phi^i$ is well defined.

We approximate the differential equation (2.4.1) by the time discretization, $h = T/n$, $t_i = ih,\ i = 0, \ldots, n$, and obtain

$$\frac{1}{h}\left(u_i - \tilde{u}_{i-1} \circ \phi^i\right) - \nabla \cdot \left(a(x)\frac{1}{h}\nabla(u_i - u_{i-1})\right) - \nabla \cdot (d(t_i, x, u_{i-1})\nabla u_i) = f(t_i, x, u_{i-1}),$$
$$u_i(x) = 0 \qquad \text{on } \partial\Omega, \qquad (2.4.3)$$

where $\phi^i(x) := x - h\, v_i^\tau(x)$ and $v_i(x) = c(t_i, x, u_{i-1})$. It is equivalent to

$$-\nabla \cdot \left((a(x)\frac{1}{h} + d(t_i, x, u_{i-1}))\nabla u_i\right) + \frac{1}{h}u_i = f(t_i, x, u_{i-1}) + \frac{1}{h}\tilde{u}_{i-1} \circ \phi^i - \frac{1}{h}\nabla \cdot (a(x)\nabla u_{i-1})$$

The existence and uniqueness of the solution u_i of elliptic problems (2.4.3) follows from Lax-Milgram Theorem, (Evans 1998).

In the proof of the a priori estimates we use the following lemma.

Lemma 2.4.4. *(Kacur 2001) There exists $h_0 > 0$, such that ϕ^i is one to one and*

$$\frac{1}{2}|x - y| \quad \le \quad |\phi^i(x) - \phi^i(y)| \quad \le \quad 2|x - y|, \ \text{ for all } x, y \in \Omega$$

uniformly in n, $i = 1, \ldots, n$, and $h \le h_0$.

Proof. Due to $||v_i||_{L^\infty(\Omega)} \le C < \infty$, we have

$$||v_i^\tau||_{L^\infty(\Omega)} \le C$$

and

$$||\nabla v_i^\tau||_{L^\infty(\Omega)} \le C/\tau.$$

Since $\tau = h^\omega$ and $0 < \omega < 1$, we obtain for ϕ^i

$$(1 - h^{1-\omega}C)|x - y| \ \le\ |\phi^i(x) - \phi^i(y)| \ \le\ (1 + h^{1-\omega}C)|x - y|.$$

❒

Now we prove a priori estimates for u_i.

Lemma 2.4.5. *The estimates*

$$\max_{1 \le j \le n} \int_\Omega \left(|u_j|^2 + |\nabla u_j|^2\right) dx \quad \le \quad C,$$
$$\sum_{i=1}^{n} h \int_\Omega |\nabla u_i|^2\, dx \quad \le \quad C \qquad (2.4.4)$$

hold uniformly in n.

Proof. Testing the equation (2.4.3) with u_i and summing over i yield

$$\sum_{i=1}^{j} \frac{1}{h} \int_\Omega (u_i - u_{i-1})\, u_i\, dx + \sum_{i=1}^{j} \frac{1}{h} \int_\Omega (u_{i-1} - \tilde{u}_{i-1} \circ \phi^i)\, u_i\, dx + \sum_{i=1}^{j} \frac{1}{h} \int_\Omega a(x)\, \nabla(u_i - u_{i-1})\, \nabla u_i\, dx$$
$$+ \sum_{i=1}^{j} \int_\Omega d(t_i, x, u_{i-1})\, \nabla u_i\, \nabla u_i\, dx = \sum_{i=1}^{j} \int_\Omega f(t_i, x, u_{i-1})\, u_i\, dx.$$

Due to Assumption 2.4.1, Abel's summation formula, and multiplication with h, we obtain

$$\int_\Omega |u_j|^2\,dx + a_0 \int_\Omega |\nabla u_j|^2\,dx + d_0 \sum_{i=1}^{j} h \int_\Omega |\nabla u_i|^2\,dx \tag{2.4.5}$$

$$\leq \int_\Omega |u_0|^2\,dx + a^0 \int_\Omega |\nabla u_0|^2\,dx + \sum_{i=1}^{j} \int_\Omega |(u_{i-1} - \tilde{u}_{i-1} \circ \phi^i) u_i|\,dx + c_1 \sum_{i=1}^{j} h \int_\Omega |u_i|^2\,dx + c_2.$$

To estimate the third integral on the right hand side we use the equality

$$u_{i-1} - \tilde{u}_{i-1} \circ \phi^i \;=\; \int_0^1 \nabla \tilde{u}_{i-1}(x + s(\phi^i(x) - x))\,ds\, v_i^\tau(x)\,h.$$

Integration over Ω and boundedness of v_i^τ yields

$$\int_\Omega |u_{i-1} - \tilde{u}_{i-1} \circ \phi^i|^2\,dx \;\leq\; C \int_0^1 \int_\Omega |\nabla \tilde{u}_{i-1}(x + s(\phi^i(x) - x))|^2\,dx\,ds\,h^2.$$

Changing to the new variable $y = x + s(\phi^i(x) - x)$, using $y \in \Omega_i \subset \Omega^*$ and the monotonicity of the integral, and applying the estimate $|\det D\phi(x)| \geqslant \frac{1}{2^N}$ yields

$$\int_\Omega |u_{i-1} - \tilde{u}_{i-1} \circ \phi^i|^2\,dx \;\leq\; C\,h^2 \int_0^1 \int_{\Omega^*} |\nabla \tilde{u}_{i-1}(y)|^2\,dy\,ds.$$

From the boundedness of the extension operator it follows that

$$||u_{i-1} - \tilde{u}_{i-1} \circ \phi^i||_{L^2(\Omega)} \;\leq\; Ch||\nabla \tilde{u}_{i-1}||_{L^2(\Omega^*)} \;\leq\; C_1\,h||\nabla u_{i-1}||_{L^2(\Omega)}.$$

Using this estimate in the estimates in (2.4.5) yields

$$\sum_{i=1}^{j} \int_\Omega |(u_{i-1} - \tilde{u}_{i-1} \circ \phi^i)\,u_i|\,dx \;\leq\; c_1 \sum_{i=1}^{j} h \int_\Omega |\nabla u_i|^2\,dx + c_2 \sum_{i=1}^{j} h \int_\Omega |u_i|^2\,dx.$$

Then we obtain the inequality

$$\int_\Omega |u_j|^2\,dx + a_0 \int_\Omega |\nabla u_j|^2\,dx + d_0 \sum_{i=1}^{j} h \int_\Omega |\nabla u_i|^2\,dx \;\leq\; c_3 + c_4 \sum_{i=1}^{j} h \int_\Omega \left(|u_i|^2 + |\nabla u_i|^2\right) dx.$$

Due to the discrete Gronwall lemma we obtain the estimates (2.4.4). ❒

Lemma 2.4.6. *The estimate*

$$\sum_{i=1}^{n} h \int_\Omega \left(|\partial_h u_i|^2 + |\partial_h \nabla u_i|^2\right) dx \quad\leq\quad C \tag{2.4.6}$$

holds uniformly in n, where $\partial_h u_i := \frac{u_i - u_{i-1}}{h}$.

Proof. We test the equation (2.4.3) with $u_i - u_{i-1}$, sum up over i, and obtain the equality

$$\sum_{i=1}^{j} h \int_\Omega \frac{u_i - \tilde{u}_{i-1} \circ \phi^i}{h} \, \partial_h u_i \, dx + \sum_{i=1}^{j} h \int_\Omega a(x) \, \nabla \partial_h u_i \, \nabla \partial_h u_i \, dx$$
$$+ \sum_{i=1}^{j} h \int_\Omega d(t_i, x, u_{i-1}) \, \nabla u_i \, \nabla \partial_h u_i \, dx = \sum_{i=1}^{j} h \int_\Omega f(t_i, x, u_{i-1}) \, \partial_h u_i \, dx.$$

By Assumption 2.4.1 we have the inequality

$$\sum_{i=1}^{j} h \int_\Omega |\partial_h u_i|^2 \, dx + a_0 \sum_{i=1}^{j} h \int_\Omega |\nabla \partial_h u_i|^2 \, dx \ \le \ c_1 \delta \sum_{i=1}^{j} h \int_\Omega |\nabla \partial_h u_i|^2 \, dx + \frac{c_2 d^0}{\delta} \sum_{i=1}^{j} h \int_\Omega |\nabla u_i|^2 \, dx$$
$$+ c_3 \delta \sum_{i=1}^{j} h \int_\Omega |\partial_h u_i|^2 \, dx + \frac{c_4}{\delta} \sum_{i=1}^{j} h \int_\Omega |f(t_i, x, u_{i-1})|^2 \, dx + \frac{c_5}{\delta} \sum_{i=1}^{j} h \int_\Omega \left| \frac{u_{i-1} - \tilde{u}_{i-1} \circ \phi^i}{h} \right|^2 \, dx.$$

Similarly to Lemma 2.4.5 we obtain

$$\frac{u_{i-1}(x) - \tilde{u}_{i-1} \circ \phi^i(x)}{h} \ = \ \int_0^1 \nabla \tilde{u}_{i-1}(x + s(\phi^i(x) - x)) \, ds \, v_i^\tau$$

and

$$\left|\left| \frac{u_{i-1} - \tilde{u}_{i-1} \circ \phi^i}{h} \right|\right|_{L^2(\Omega)} \ \le \ C ||\nabla \tilde{u}_{i-1}||_{L^2(\Omega^*)} \ \le \ C ||\nabla u_{i-1}||_{L^2(\Omega)}.$$

Then we have the inequality

$$\sum_{i=1}^{j} h \int_\Omega |\partial_h u_i|^2 \, dx + \sum_{i=1}^{j} h \int_\Omega |\nabla \partial_h u_i|^2 \, dx \ \le \ C_1 \sum_{i=1}^{j} h \int_\Omega |u_i|^2 \, dx + C_2 \sum_{i=1}^{j} h \int_\Omega |\nabla u_i|^2 \, dx.$$

Due to the estimates in Lemma 2.4.5, this inequality implies the estimate for the discrete time derivative $\partial_h u_i$. ❐

Proof of Theorem 2.4.3. By using the a priori estimates for u_i and $\partial_h u_i$ we will show the convergence of an appropriate subsequence of the approximate solutions to a solution of the original problem (2.4.1).

Therefor, we define the Rothe functions piecewise for $t \in (t_{i-1}, t_i]$ and for $x \in \Omega$ by

$$u^n(t,x) \ \ := \ \ u(t_{i-1}, x) + (t - t_{i-1}) \frac{u(t_i, x) - u(t_{i-1}, x)}{h}$$

and the step functions by

$$\bar{u}^n(t,x) \ \ := \ \ u(t_i, x),$$

where the initial conditions are $u^n(0,x) = u_0(x)$ and $\bar{u}^n(0,x) = u_0(x)$. From (2.4.4) and (2.4.6) we have the estimates

$$\sup_{0 \le t \le T} \int_\Omega \left(|\bar{u}^n|^2 + |\nabla \bar{u}^n|^2 \right) dx \ \le \ C, \qquad \int_{Q_T} |\nabla \bar{u}^n|^2 \, dx \, dt \ \le C, \tag{2.4.7}$$
$$\int_{Q_T} \left(|\partial_h u^n|^2 + |\nabla \partial_h u^n|^2 \right) dx \, dt \ \le \ C, \quad \int_{Q_T} \left(|u^n - \bar{u}^n|^2 + |\nabla u^n - \nabla \bar{u}^n|^2 \right) dx \, dt \ \le \ \frac{C}{n^2}.$$

These estimates imply the existence of subsequences of $\{u^n\}$ and of $\{\bar{u}^n\}$, resp., again denoted by $\{u^n\}$ and $\{\bar{u}^n\}$, resp., such that

$$\begin{aligned} \bar{u}^n &\to u && \text{weakly} - * \text{ in } L^\infty(0,T;H_0^1(\Omega)), \\ \bar{u}^n &\to u && \text{weakly in } L^2(0,T;H_0^1(\Omega)), \\ \partial_h u^n &\to \partial_t u && \text{weakly in } L^2(0,T;H_0^1(\Omega)), \end{aligned} \tag{2.4.8}$$

where $\partial_h u^n(t) := \frac{u^n(t)-u^n(t-h)}{h}$ and $u^n(t-h) = u_0$ for $t \in [0,h]$. Using the Compactness Aubin-Lions Lemma, see (Lions 1969) or (Showalter 1996), implies $\bar{u}^n \to u$ strongly in $L^2(Q_T)$. Due to (Evans 1998, Theorem 5.9.2) and $u \in H^1(0,T;H_0^1(\Omega))$, we obtain $u \in C([0,T];H_0^1(\Omega))$ and $u(0) = u_0$.

Testing the discrete equation (2.4.3) with $v \in L^2(0,T;H_0^1(\Omega))$ yields

$$\begin{aligned} &\int\limits_{Q_T} \partial_h u^n \, v \, dx\, dt + \int\limits_{Q_T} \partial_h \nabla u^n \, \nabla v \, dx\, dt + \int\limits_{Q_T} d_n(t,x,\bar{u}_h^n)\, \nabla \bar{u}^n \, \nabla v \, dx\, dt \\ &+ \int\limits_{Q_T} \frac{1}{h}(\bar{u}_h^n - \widetilde{u}_h^n \circ \phi^n)\, v \, dx\, dt = \int\limits_{Q_T} f_n(t,x,\bar{u}_h^n)\, v\, dx\, dt, \end{aligned} \tag{2.4.9}$$

where $\phi^n(t,x) = x - h\, w_\tau * c_n(t,x,\bar{u}_h^n)$, $\bar{u}_h^n(t,x) = \bar{u}^n(t-h,x)$, $c_n(t,x,z) = c(t_i,x,z)$, $d_n(t,x,z) = d(t_i,x,z)$ for $t \in (t_{i-1},t_i]$, for $i = 1,\dots,n$, and $c_n(0,x,z) = c(0,x,z)$, $d_n(0,x,z) = d(0,x,z)$. The strong convergence of $\bar{u}^n$ and the last estimate in (2.4.7) imply $\bar{u}_h^n \to u$ strongly in $L^2(Q_T)$ and $\bar{u}_h^n \to u$ a. e. in Q_T. The continuity of $d(t,x,z)$ in t and z and the convergence of $\bar{u}_h^n$ a. e. in Q_T imply $d_n(t,x,\bar{u}_h^n) \to d(t,x,u)$ a. e. in Q_T. The boundedness of $d_n(t,x,\bar{u}_h^n)$ and $d(t,x,u)$, and the Egorov Theorem, (Alt 2002), imply the uniform convergence of $d_n(t,x,\bar{u}_h^n)$ to $d(t,x,u)$ a.e. in Q_T. Due to $\bar{u}^n \to u$ weakly in $L^2(0,T;H_0^1(\Omega))$, we obtain

$$\int\limits_{Q_T} d_n(t,x,\bar{u}_h^n)\, \nabla \bar{u}^n \, \nabla v \, dx\, dt \to \int\limits_{Q_T} d(t,x,u)\, \nabla u \, \nabla v \, dx\, dt.$$

The convergence of $f_n(t,x,\bar{u}_h^n) \to f(t,x,u)$ a.e. in Q_T follows from the continuity of f and the a.e. convergence of $\bar{u}_h^n$ in Q_T. Due to the sublinearity of f and the Dominated Convergence Theorem, (Evans 1998), we obtain $f_n(t,x,\bar{u}_h^n) \to f(t,x,u)$ in $L^2(Q_T)$. The continuity of c implies $c_n(t,x,\bar{u}_h^n) \to c(t,x,u)$ a. e. in Q_T. From the boundedness of $c_n(t,x,\bar{u}_h^n)$ and $c(t,x,u)$ in $L^\infty(Q_T)$ and the Egorov Theorem follows $c_n(t,x,\bar{u}_h^n) \to c(t,x,u)$ uniformly a. e. in Q_T.

Now we have to prove

$$\int\limits_{Q_T} \frac{1}{h}(\bar{u}_h^n - \widetilde{u}_h^n \circ \phi^n)\, v\, dx\, dt \quad \to \quad \int\limits_{Q_T} c(t,x,u)\, \nabla u \, v\, dx\, dt$$

for $n \to \infty$, where $h = \frac{T}{n}$. The equality

$$\int\limits_{Q_T} \frac{1}{h}(\bar{u}_h^n - \widetilde{u}_h^n \circ \phi^n)\, v\, dx\, dt \quad = \quad \int\limits_{Q_T} \int\limits_0^1 \nabla \widetilde{u}_h^n(x + s(\phi^n(t,x) - x))\, ds\, w_\tau * c_n(t,x,\bar{u}_h^n)\, v\, dx\, dt$$

holds. Since $c_n(t,x,\bar{u}_h^n) \to c(t,x,u)$ a. e. in Q_T, we have $w_\tau * c_n(t,x,\bar{u}_h^n) \to c(t,x,u)$ a. e. in Q_T as $n \to \infty$. The assumed boundedness of c yields

$$||w_\tau * c_n(t,x,\bar{u}_h^n)||_{L^\infty(Q_T)} \le c^0.$$

We need to show that $\nabla z_n \to \nabla u$ weakly in $L^2(Q_T)$, where

$$\nabla z_n(t,x) := \int_0^1 \nabla \widetilde{\bar{u}}_h^n(t, x + s(\phi^n(t,x) - x))\, ds.$$

Due to

$$\int_{Q_T} |\nabla z_n|^2 \, dx\, dt \le C_1,$$

there exists $\chi \in L^2(Q_T)$, such that $\nabla z_n \to \chi$ weakly in $L^2(Q_T)$. Now we show $z_n \to u$ in $L^2(Q_T)$. Integrating the difference

$$\begin{aligned} z_n(t,x) - \bar{u}_h^n(t,x) &= \int_0^1 \Big(\widetilde{\bar{u}}_h^n(t, x + s(\phi^n(t,x) - x)) - \bar{u}_h^n(t,x) \Big)\, ds \\ &= \int_0^1 \int_0^1 \nabla \widetilde{\bar{u}}_h^n(t, x + sr(\phi^n(t,x) - x))\, ds\, dr \;\; w_\tau * c_n(t,x,\bar{u}_h^n)\, h \end{aligned}$$

over Q_T and using the boundedness of c_n imply

$$\int_{Q_T} |z_n(t,x) - \bar{u}_h^n(t,x)|^2 \, dx\, dt \le c^0 h^2 \int_0^1 \int_0^1 \int_{Q_T} |\nabla \widetilde{\bar{u}}_h^n(t, x + sr(\phi^n(t,x) - x))|^2 \, dx\, dt\, ds\, dr.$$

From the boundedness of the extension operator and the a priori estimates for $\bar{u}_h^n$ it follows that

$$||\nabla \widetilde{\bar{u}}_h^n||_{L^2((0,T)\times\Omega^*)} \le C_2 ||\nabla \bar{u}_n^h||_{L^2(Q_T)} \le C_3.$$

Then we have

$$\int_0^T \int_\Omega |z_n(t,x) - \bar{u}_h^n(t,x)|^2 \, dx\, dt \le Ch^2.$$

Due to the fact that $\bar{u}_h^n \to u$ in $L^2(Q_T)$, we obtain $z_n \to u$ in $L^2(Q_T)$. Then, $\nabla z_n \to \chi$ weakly in $L^2(Q_T)$ implies that $\chi = \nabla u$. Passing in the equation (2.4.9) to the limit as $n \to \infty$, it follows that the function u is a solution of the problem (2.4.1). ❐

Theorem 2.4.7 (Uniqueness).

Let Assumption 2.4.1 be satisfied, where d depends only on time and space. Let $N \le 4$ and

$$|f(t,x,z^1) - f(t,x,z^2)| \le C|z^1 - z^2|, \quad |c(t,x,z^1) - c(t,x,z^2)| \le C|z^1 - z^2|$$

for $z^1, z^2 \in \mathbb{R}$, $(t,x) \in Q_T$. Then there exists at most one weak solution of (2.4.1).

Proof. Suppose u_1 and u_2 solve the problem (2.4.1). Then the difference $u = u_1 - u_2$ satisfies the equality

$$\int_{Q_\tau} u_t\, v\, dx\, dt + \int_{Q_\tau} a(x)\, \nabla u_t\, \nabla v\, dx\, dt + \int_{Q_\tau} \big(c(t,x,u_1)\nabla u_1 - c(t,x,u_2)\nabla u_2\big)\, v\, dx\, dt$$
$$+ \int_{Q_\tau} d(t,x)\, \nabla u\, \nabla v\, dx\, dt = \int_{Q_\tau} \big(f(t,x,u_1) - f(t,x,u_2)\big)\, v\, dx\, dt. \tag{2.4.10}$$

We choose the test function $v = u$. The third integral in the last equality is estimated by

$$\begin{aligned}
&\int_{Q_\tau} \big(c(t,x,u_1)\nabla u_1 - c(t,x,u_2)\nabla u_2\big)\, u\, dx\, dt \\
=\ & \int_{Q_\tau} c(t,x,u_1)\nabla u\, u\, dx\, dt + \int_{Q_\tau} \big(c(t,x,u_1) - c(t,x,u_2)\big)\nabla u_2\, u\, dx\, dt \\
\le\ & c_1 \int_{Q_\tau} |u|^2\, dx\, dt + c_2 \int_{Q_\tau} |\nabla u|^2\, dx\, dt + c_3 \Big(\int_{Q_\tau} |u|^4\, dx\, dt\Big)^{\frac{1}{2}} \Big(\int_{Q_\tau} |\nabla u_2|^2\, dx\, dt\Big)^{\frac{1}{2}}.
\end{aligned}$$

Sobolev's embedding theorem yields

$$\Big(\int_{Q_\tau} |u|^4\, dx\, dt\Big)^{\frac{1}{2}} \le c_4 \int_{Q_\tau} |u|^2\, dx\, dt + c_5 \int_{Q_\tau} |\nabla u|^2\, dx\, dt,$$

since $u \in L^\infty(0,T; H_0^1(\Omega))$ and $N \le 4$. Applying these estimates, ellipticity of a and d, and Lipschitz continuity of f to equation (2.4.10) implies

$$\int_\Omega \Big(|u(\tau)|^2 + |\nabla u(\tau)|^2\Big)\, dx \le C \int_{Q_\tau} \Big(|u|^2 + |\nabla u|^2\Big)\, dx\, dt.$$

Due to Gronwall's Lemma, we obtain

$$\int_\Omega \Big(|u(\tau)|^2 + |\nabla u(\tau)|^2\Big)\, dx \le 0$$

and $u_1 = u_2$ almost everywhere in Q_T. ❐

Remark 2.4.8. In this section the zero Dirichlet boundary conditions were considered. This restriction is not essential and the results can be obtain also for other boundary conditions.

3

Quasilinear Equations and Variational Inequalities in Unbounded Time Intervals

In this chapter the question of existence and uniqueness of solutions of quasilinear pseudoparabolic equations and variational inequalities without initial conditions is studied. In section 3.1 a pseudoparabolic equation with a monotone, bounded, hemicontinuous operator is solved. Such equations are used in the second section to approximate the solution of a variational inequality. The monotone operator is used as a penalty operator. The uniqueness will be proved using the strong monotonicity of the operators and Pankov's Lemma, see Lemma A.2.2 in the appendix.

3.1 Quasilinear Equations in Unbounded Time Intervals

Here we consider a boundary value problem in the time interval $(-\infty, T)$. Quasilinear pseudoparabolic equations model the fluid flow in fissured porous media with nonlinear diffusion. The model of Barenblatt, introduced in the Chapter 1, is based upon a linear diffusion. The existence and uniqueness of a solution in the whole time interval $(-\infty, T)$ is interesting, since such a solution represents the evolution far-off of initial perturbations. We prove the uniqueness of the solution of the nonlinear equation stated below without posing additional assumptions on the behavior of a solution at $-\infty$. In the linear case it is well known that extra assumptions at $-\infty$ are needed, (Lavrenyuk and Ptashnik 1998).

Consider a space V that satisfies $H_0^{1,p}(\Omega) \subset V \subset H^{1,p}(\Omega)$ for some $p \in (2, +\infty)$, is compactly and continuously embedded in $L^2(\Omega)$, and is densely and continuously embedded into a closed subspace $V_0 \subset H^1(\Omega)$. In $Q_T = (-\infty, T) \times \Omega$ the equation

$$u_t - \nabla \cdot (a(x)\nabla u_t) - \nabla \cdot d(t, x, \nabla u) + g(t, x, u) + \mathcal{B}(u) = f(t, x) \tag{3.1.1}$$

is considered, where $\mathcal{B} : V \to V^*$ is a monotone, bounded, and hemicontinuous operator that satisfies $\mathcal{B}(0) = 0$. The norm in V is the norm of $H^{1,p}(\Omega)$ and the norm in V_0 is the norm of $H^1(\Omega)$.

Here we assume the uniform ellipticity of pseudoparabolic and elliptic parts and the strong monotonicity and polynomial growth in ξ and in z of the nonlinear functions $d(t, x, \xi)$ and $g(t, x, z)$.

Assumption 3.1.1.

$(A1)$ The matrix field $a \in L^\infty(\Omega)^{N\times N}$, considered as a linear mapping on $L^\infty(\Omega)^N$, is symmetric and elliptic, i.e. for some $0 < a_0 \le a^0 < \infty$, $a_0|\xi|^2 \le a(x)\xi\,\xi \le a^0|\xi|^2$ for $\xi \in \mathbb{R}^N$ and for almost all $x \in \Omega$.

$(A2)$ The diffusivity $d : (0,T)\times\Omega\times\mathbb{R}^N \to \mathbb{R}^N$ is measurable in x, continuous in t and ξ, elliptic, i.e. for some $d_0 > 0$, $d(t,x,\xi)\,\xi \geqslant d_0|\xi|^p$ for $\xi \in \mathbb{R}^N$, strongly monotone, i.e. for some $d_1 > 0$, $(d(t,x,\xi_1) - d(t,x,\xi_2))(\xi_1 - \xi_2) \geqslant d_1|\xi_1 - \xi_2|^p$ for $\xi_1, \xi_2 \in \mathbb{R}^N$, and satisfies the growth assumption, i.e. for some $d^0 < \infty$, $|d(t,x,\xi)| \le d^0(1 + |\xi|^{p-1})$.

$(A3)$ The function $g : (0,T)\times\Omega\times\mathbb{R} \to \mathbb{R}$ is measurable in x, continuous in t and z, elliptic, i.e. for some $g_0 > 0$, $g(t,x,z)\,z \geqslant g_0|z|^p$, strongly monotone, i.e. for some $g_1 > 0$, $(g(t,x,z_1) - g(t,x,z_2))(z_1 - z_2) \geqslant g_1|z_1 - z_2|^p$ for $z_1, z_2 \in \mathbb{R}$, and satisfies the growth assumption, i.e. for some $g^0 < \infty$, $|g(t,x,z)| \le g^0(1 + |z|^{p-1})$.

$(A4)$ The external force $f \in C((-\infty,T];L^2(\Omega))$.

Now we define our notion of a weak solution.

Definition 3.1.2. A function $u : Q_T \to \mathbb{R}$ is called *a weak solution* of the equation (3.1.1) if

1) $u \in C((-\infty,T];V_0) \cap L^p_{\text{loc}}\big((-\infty,T];V\big)$ and

2) $$-\int_{t_1}^{t_2}\int_\Omega \Big(u\,w_t + a(x)\,\nabla u\,\nabla w_t\Big)\,dx\,dt + \int_{t_1}^{t_2}\int_\Omega \Big(d(t,x,\nabla u)\,\nabla w + g(t,x,u)\,w\Big)\,dx\,dt$$
$$+\int_{t_1}^{t_2} \langle \mathcal{B}(u), w\rangle\,dt + \int_\Omega \Big(u\,w + a(x)\,\nabla u\,\nabla w\Big)\,dx\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2}\int_\Omega f(t,x)\,w\,dx\,dt \tag{3.1.2}$$

for all $w \in L^p_{\text{loc}}((-\infty,T];V)$, such that $w_t \in L^2_{\text{loc}}\big((-\infty,T];V_0\big)$ and $w \in C((-\infty,T];V_0)$, and for all t_1 and t_2, such that $-\infty < t_1 < t_2 \le T$.

Remark 3.1.3. In the definition of the weak solution no information concerning the time derivative was assumed. In this class existence and uniqueness will be shown without posing assumptions at $-\infty$. However, in a different class, which consists of functions of at most exponential growth, existence and uniqueness can be shown also, (Lavrenyuk and Ptashnyk 1999).

Theorem 3.1.4 (Uniqueness). *Let Assumption 3.1.1 be satisfied. Then there exists at most one weak solution of the equation (3.1.1).*

Proof. Suppose u^1 and u^2 are two solutions of equation (3.1.1). Then the function $u = u^1 - u^2$ satisfies

$$\int_{Q_{t_1,t_2}} \Big[-u\, w_t - a(x)\nabla u\, \nabla w_t + (d(t,x,\nabla u^1) - d(t,x,\nabla u^2))\, \nabla w + (g(t,x,u^1) - g(t,x,u^2))\, w \Big]\, dx\, dt$$

$$+ \int_{t_1}^{t_2} \langle \mathcal{B}(u^1) - \mathcal{B}(u^2), w\rangle\, dt + \int_{\Omega} \Big[u\, w + b(x)\nabla u\, \nabla w \Big]\, dx \Big|_{t_1}^{t_2} = 0 \tag{3.1.3}$$

for every function $w \in L^p_{\mathrm{loc}}((-\infty, T]; V)$, such that $w_t \in L^2_{\mathrm{loc}}((-\infty, T]; V_0)$, $w \in C((-\infty, T]; V_0)$, and for all t_1 and t_2, such that $-\infty < t_1 < t_2 \le T$. Because of the lack of regularity in the time variable, the function u cannot be chosen as a test function in the equality (3.1.3). Hence, a function of the form

$$w = \Big((\gamma_m u) * \rho_k * \rho_k \Big) \gamma_m$$

for $k > 2m$ is used. Here γ_m are continuous, piecewise linear cut-off functions on $(-\infty, T]$, given by

$$\gamma_m(t) = \begin{cases} 1 & \text{for } t_1 + \frac{2}{m} < t < t_2 - \frac{2}{m}, \\ m(t - t_1) - 1 & \text{for } t_1 + \frac{1}{m} \le t \le t_1 + \frac{2}{m}, \\ m(t_2 - t) - 1 & \text{for } t_2 - \frac{2}{m} \le t \le t_2 - \frac{1}{m}, \\ 0 & \text{for } t < t_1 + \frac{1}{m} \text{ and } t > t_2 - \frac{1}{m}. \end{cases}$$

The sequence $\{\rho_k\} \subset \mathcal{D}(R)$ satisfies $\rho_k(t) = \rho_k(-t)$, $\int_{-\infty}^{\infty} \rho_k(t) dt = 1$, $\operatorname{supp} \rho_k \in \Big[-\frac{1}{k}, \frac{1}{k} \Big]$, and can be constructed in the form $\rho_k = \frac{1}{k^n} \rho_1(kx)$, where $\rho_1(x) = C \exp(-\frac{|x|^2}{1-|x|^2})$. The function u is extended by zero on $Q_{T,\infty}$. The function w is smooth in time and belongs to the space V.

By using the properties of convolution, we modify the first integral in (3.1.3) and obtain

$$\begin{aligned} I_1 \;&:=\; -\int_{Q_{t_1,t_2}} \Big[u\, w_t + a(x)\nabla u\, \nabla w_t \Big]\, dx\, dt \\ &= -\int_{Q_{t_1,t_2}} \Big[u\, ((\gamma_m u) * \rho_k * \rho_k) + a(x)\nabla u\, ((\gamma_m \nabla u) * \rho_k * \rho_k) \Big]\, \gamma_m'\, dx\, dt \\ &\quad - \int_{Q_{t_1,t_2}} \Big[u\gamma_m\, ((\gamma_m u) * \rho_k * \rho_k)_t + a(x)\nabla u \gamma_m\, ((\gamma_m \nabla u) * \rho_k * \rho_k)_t \Big]\, dx\, dt. \end{aligned}$$

The second integral on the right hand side vanishes because of

$$\begin{aligned} &- \int_{Q_{t_1,t_2}} \Big[u\, \gamma_m\, ((\gamma_m u) * \rho_k * \rho_k)_t + a(x)\, \nabla u\, \gamma_m\, ((\gamma_m \nabla u) * \rho_k * \rho_k)_t \Big]\, dx\, dt \\ &= \int_{Q_{t_1,t_2}} \Big[(\gamma_m u) * \rho_k\, ((\gamma_m u) * \rho_k)_t + a(x)(\gamma_m \nabla u) * \rho_k\, ((\gamma_m \nabla u) * \rho_k)_t \Big]\, dx\, dt \\ &= \frac{1}{2} \int_{\Omega} \Big[|(u\gamma_m) * \rho_k|^2 + a(x)(\gamma_m \nabla u) * \rho_k\, (\gamma_m \nabla u) * \rho_k \Big]\, dx \,\Big|_{t_1}^{t_2} = 0, \end{aligned}$$

since $\gamma_m(t) = 0$ at $t = t_1$ and $t = t_2$. So we obtain

$$\begin{aligned} I_1 &= \int_{Q_{t_1,t_2}} \Big[u\gamma_m' \, (\gamma_m u) * \rho_k * \rho_k + a(x)\nabla u \gamma_m' \, (\gamma_m \nabla u) * \rho_k * \rho_k \Big] \, dx\, dt \\ &\to \int_{Q_{t_1,t_2}} \Big[\gamma_m \gamma_m' |u|^2 + \gamma_m \gamma_m' a(x) \nabla u \, \nabla u \Big] \, dx \, dt \end{aligned}$$

as $k \to \infty$. Furthermore, the regularity $u \in C((-\infty, T]; V_0)$ implies

$$I_1 \to \frac{1}{2} \int_\Omega \Big[|u(t_2)|^2 + a(x) \nabla u(t_2) \, \nabla u(t_2) \Big] \, dx - \frac{1}{2} \int_\Omega \Big[|u(t_1)|^2 + a(x) \nabla u(t_1) \, \nabla u(t_1) \Big] \, dx$$

as $m \to \infty$. Passing to the limits in (3.1.3) with the function w, at first as $k \to \infty$, and afterwards as $m \to \infty$, yields

$$\begin{aligned} &\int_\Omega \Big[|u|^2 + a(x) \nabla u \, \nabla u \Big] \, dx \, \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_\Omega (d(t,x,\nabla u^1) - d(t,x,\nabla u^2)) \, \nabla u \, dx \, dt \\ &+ \int_{t_1}^{t_2} \int_\Omega (g(t,x,u^1) - g(t,x,u^2)) \, u \, dx \, dt + \int_{t_1}^{t_2} \langle \mathcal{B}(u^1) - \mathcal{B}(u^2), u \rangle \, dt = 0. \end{aligned} \tag{3.1.4}$$

The assumptions on d and g and the monotonicity of $\mathcal{B}$ give the estimates for the integrals in (3.1.4)

$$\begin{aligned} I_2 &:= \int_{Q_{t_1,t_2}} \Big[(d(t,x,\nabla u^1) - d(t,x,\nabla u^2)) \, \nabla u + (g(t,x,u^1) - g(t,x,u^2)) \, u \Big] \, dx \, dt \\ &\geq \int_{Q_{t_1,t_2}} \Big(d_1 |u|^p + g_1 |\nabla u|^p \Big) \, dx \, dt, \\ I_3 &:= \int_{t_1}^{t_2} \langle \mathcal{B}(u^1) - \mathcal{B}(u^2), u^1 - u^2 \rangle \, dt \; \geq \; 0 \, . \end{aligned}$$

For $p > 2$ we have the estimate

$$\int_\Omega \Big(|u(t)|^p + |\nabla u(t)|^p \Big) \, dx \; \geq \; c_1 \Big(\int_\Omega \Big[|u(t)|^2 + a(x) \nabla u(t) \, \nabla u(t) \Big] \, dx \Big)^{p/2},$$

where c_1 depends on p, n, and a^0.

Due to the estimates of I_2, I_3 and the last inequality, the equality (3.1.4) implies

$$y^2(t) \Big|_{t_1}^{t_2} + c_2 \int_{t_1}^{t_2} y^p(t) \, dt \; \leq \; 0$$

for all $t_1, t_2 \leq \tau_0$, where

$$y^2(t) = \int_\Omega \Big[|u(t)|^2 + a(x) \nabla u(t) \, \nabla u(t) \Big] \, dx \, .$$

Using Pankov's Lemma A.2.2, it follows from the last inequality that $y(t) = 0$ for all $t \in (-\infty, \tau_0]$. Therefore, $u = 0$ almost everywhere in Q_{τ_0}. To show $u = 0$ almost everywhere in $Q_{\tau_0,T}$, we consider the equation (3.1.4) for $t_1 = \tau_0$ and $t_2 = \tau$, $\tau_0 < \tau \leq T$, and obtain the inequality

$$\int_\Omega \Big(|u(\tau)|^2 + |\nabla u(\tau)|^2 \Big)\, dx \leq 0.$$

Thus, $u = 0$ almost everywhere in $Q_{\tau_0,T}$ and the theorem is proved. ❒

Remark 3.1.5. As it can be seen from the proof, the weak solution of the equation (3.1.1) satisfies the equality

$$\int_\Omega \Big(|u|^2 + a(x)\nabla u\, \nabla u \Big)\, dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_\Omega \Big(d(t,x,\nabla u)\, \nabla u + g(t,x,u)\, u \Big)\, dx\, dt + \int_{t_1}^{t_2} \langle \mathcal{B}(u), u \rangle\, dt$$
$$= \int_{t_1}^{t_2} \int_\Omega f(t,x)\, u\, dx\, dt \qquad \text{for all} \quad t_1,\, t_2 \in (-\infty, T].$$

Now we will prove the existence of the solution in two steps. At first we show the existence of a solution in any bounded time interval of the form (t_0, T) with zero initial condition. Secondly, we choose $T - k$ as lower bounds t_0 and obtain a sequence of solutions, which is shown to converge to a solution of the original problem.

Theorem 3.1.6 (Existence). *Let Assumption 3.1.1 be satisfied. Then there exists a weak solution u of the equation (3.1.1) that satisfies the estimates*

$$\begin{aligned} \int_\Omega \Big(|u(t)|^2 + |\nabla u(t)|^2 \Big)\, dx &\leq C\,, \quad t \in [\tau, T]\,, \\ \int_{Q_{\tau,T}} \Big(|u|^p + |\nabla u|^p \Big)\, dx\, dt &\leq C\,, \quad \tau \in (-\infty, T]\,, \end{aligned} \tag{3.1.5}$$

where C depends on τ.

Now we formulate the problem with an initial condition. In $\Omega \times (t_0, T)$, $t_0 \in (-\infty, T)$ the equation

$$u_t - \nabla \cdot (a(x)\nabla u_t) - \nabla \cdot d(t,x,\nabla u) + g(t,x,u) + \mathcal{B}(u) = f_{t_0}(t,x) \tag{3.1.6}$$

with the initial condition

$$u(t_0) = 0, \tag{3.1.7}$$

where

$$f_{t_0}(t,x) = \begin{cases} f(t,x), & \text{if } (x,t) \in Q_{t_0,T}\,, \\ 0, & \text{if } (x,t) \in Q_{t_0}\,, \end{cases}$$

is considered. Similarly to the solution in an unbounded interval, we define the solution in a bounded domain.

Definition 3.1.7. A function $u : Q_T \to \mathbb{R}$ is called *a weak solution* of the problem (3.1.6), (3.1.7) if

1) $u \in C([t_0, T]; V_0) \cap L^p(t_0, T; V)$,

2) u satisfies the initial condition (3.1.7), i.e. $u(t) \to 0$ in V_0 for $t \to t_0$, and

3)
$$-\int_{t_0}^{T}\int_{\Omega} \Big(u\, v_t + a(x)\nabla u\, \nabla v_t\Big)\, dx\, dt + \int_{\Omega} \Big(u(T)\, v(T) + a(x)\nabla u(T)\, \nabla v(T)\Big)\, dx \tag{3.1.8}$$
$$+ \int_{t_0}^{T}\int_{\Omega} d(t,x,\nabla u)\, \nabla v\, dx\, dt + \int_{t_0}^{T}\int_{\Omega} g(t,x,u)\, v\, dx\, dt + \int_{t_0}^{T} \langle \mathcal{B}(u), v\rangle\, dt = \int_{t_0}^{T}\int_{\Omega} f_{t_0}(t,x)\, v\, dx\, dt$$

for all functions $v \in L^p(t_0, T; V)$, such that $v_t \in L^2(t_0, T; V_0)$ and $v \in C([t_0, T]; V_0)$.

Remark 3.1.8. Because of the choice of the test functions, (3.1.8) holds true if and only if it holds true on any subinterval $[t_1, t_2] \subseteq [t_0, T]$.

We have the following existence result.

Theorem 3.1.9. *Under Assumption 3.1.1, there exists a weak solution of the problem (3.1.6), (3.1.7).*

The existence of the solution is proved using Galerkin's method. Let $\{\varphi^k\}_{k=1}^{\infty}$ be a basis of V. We are looking for a function u^m of the form

$$u^m(t,x) = \sum_{k=1}^{m} z_k^m(t)\varphi^k(x)\,, \quad l = 1, 2, \dots, \tag{3.1.9}$$

such that u^m solves the Cauchy problem

$$\int_{\Omega} \Big(u_t^m\, \varphi^k + a(x)\nabla u_t^m\, \nabla\varphi^k + d(t,x,\nabla u^m)\, \nabla\varphi^k + g(t,x,u^m)\, \varphi^k\Big)\, dx + \langle \mathcal{B}(u^m), \varphi^k\rangle$$
$$= \int_{\Omega} f_{t_0}(t,x)\, \varphi^k\, dx, \qquad \text{for } k = 1, \dots, m, \tag{3.1.10}$$
$$u^m(t_0) = 0. \tag{3.1.11}$$

By Peano's theorem, see (Amann 1995), there exists a continuously differentiable local solution of the problem (3.1.10), (3.1.11) in the time interval $[t_0, t_0 + \sigma]$, for some $\sigma > 0$. Due to the a priori estimate (3.1.13) this solution can be extended to the whole interval $[t_0, T]$.

Lemma 3.1.10. *The a priori estimates*

$$\int_{\Omega} \Big(|u^m(\tau)|^2 + |\nabla u^m(\tau)|^2\Big)\, dx \;\leq\; C\,, \quad \tau \in [t_0, T]\,, \tag{3.1.12}$$
$$\int_{Q_{t_0,T}} \Big(|u^m|^p + |\nabla u^m|^p\Big)\, dx\, dt \;\leq\; C\,, \tag{3.1.13}$$

hold uniformly with respect to m.

Proof. We multiply the equation (3.1.10) for $k = 1, \dots, m$ by the corresponding function z_k^m, sum up over k from 1 to m, and integrate over $[t_0, \tau] \subset [t_0, T]$. Hereby, we obtain

$$\int\limits_{Q_{t_0,\tau}} \Big[u_t^m\, u^m + a(x)\nabla u_t^m\, \nabla u^m + d(t,x,\nabla u^m)\, \nabla u^m + g(t,x,u^m)\, u^m\Big]\, dx\, dt$$

$$+ \int\limits_{t_0}^{\tau} \langle \mathcal{B}(u^m), u^m\rangle\, dt = \int\limits_{Q_{t_0,\tau}} f_{t_0}(t,x)\, u^m\, dx\, dt\,. \tag{3.1.14}$$

Now we estimate all terms in the last equality separately. Assumption 3.1.1 yields

$$\begin{aligned} I_1 &:= \int\limits_{Q_{t_0,\tau}} \Big[u_t^m\, u^m + a(x)\nabla u_t^m\, \nabla u^m\Big]\, dx\, dt \;\geq\; \frac{1}{2}\int\limits_{\Omega} \Big[|u^m(\tau)|^2 + a_0|\nabla u^m(\tau)|^2\Big]\, dx\,, \\ I_2 &:= \int\limits_{Q_{t_0,\tau}} \Big[d(t,x,\nabla u^m)\, \nabla u^m + g(t,x,u^m)\, u^m\Big]\, dx\, dt \;\geqslant \int\limits_{Q_{t_0,\tau}} \Big[d_0|\nabla u^m|^p + g_0|u^m|^p\Big]\, dx\, dt\,, \\ I_3 &:= \int\limits_{Q_{t_0,\tau}} f_{t_0}(t,x)\, u^m\, dx\, dt \;\leq\; \frac{1}{2}\delta_0 \int\limits_{Q_{t_0,\tau}} |u^m|^2\, dx\, dt + \frac{1}{2\delta_0}\int\limits_{Q_{t_0,\tau}} |f_{t_0}(t,x)|^2\, dx\, dt\,. \end{aligned}$$

Finally, the monotonicity of the operator $\mathcal{B}$, the estimates of the integrals I_1, I_2, and I_3, and Gronwall's lemma, imply the inequality

$$\int\limits_{\Omega} \Big(|u^m(\tau)|^2 + |\nabla u^m(\tau)|^2\Big)\, dx + \int\limits_{Q_{t_0,\tau}} \Big(|u^m|^p + |\nabla u^m|^p\Big)\, dx\, dt \;\leq\; C.$$

Hence, the estimates hold. ❒

Lemma 3.1.11. *The inequality*

$$\int\limits_{\Omega} \Big(|u^m(t+\delta,x) - u^m(t,x)|^2 + |\nabla u^m(t+\delta,x) - \nabla u^m(t,x)|^2\Big)\, dx \;\leq\; C\delta \tag{3.1.15}$$

holds uniformly with respect to m.

Proof. The monotonicity of d, g and $\mathcal{B}$, the estimate (3.1.12), and the continuity of the function f in time yield

$$\int\limits_{\Omega} \Big(|u^m(\tau)|^2 + a(x)\, \nabla u^m(\tau)\, \nabla u^m(\tau)\Big)\, dx \;\leq\; C(\tau - t_0), \tag{3.1.16}$$

where C is independent of m. Analogously to (3.1.14) we obtain for $\delta > 0$ the equation

$$\int\limits_{Q_{t_0,\tau}} \Big[u_t^m(t)\, u^m(t+\delta) + a(x)\nabla u_t^m(t)\, \nabla u^m(t+\delta)\Big]\, dx\, dt + \int\limits_{Q_{t_0,\tau}} d(t,x,\nabla u^m(t))\, \nabla u^m(t+\delta)\, dx\, dt +$$

$$\int\limits_{Q_{t_0,\tau}} g(t,x,u^m(t))\, u^m(t+\delta)\, dx\, dt + \int\limits_{t_0}^{\tau} \langle \mathcal{B}(u^m(t)), u^m(t+\delta)\rangle\, dt = \int\limits_{Q_{t_0,\tau}} f_{t_0}(t,x)\, u^m(t+\delta)\, dx\, dt.$$

We subtract (3.1.14) from the last equation and obtain

$$\int\limits_{Q_{t_0,\tau}} \Big[u_t^m(t)\, v^m(t) + a(x)\nabla u_t^m(t)\, \nabla v^m(t) + d(t,x,\nabla u^m(t))\, \nabla v^m(t) + g(t,x,u^m(t))\, v^m(t)\Big]\, dxdt$$

$$+ \int\limits_{t_0}^{\tau} \langle \mathcal{B}(u^m(t)), v^m(t)\rangle\, dt = \int\limits_{Q_{t_0,\tau}} f_{t_0}(t,x)\, v^m(t)\, dx\, dt\,, \tag{3.1.17}$$

where $v^m(t,x) = u^m(t+\delta,x) - u^m(t,x)$. From (3.1.10) we have the equation

$$\int\limits_{Q_{t_0,\tau}} \Big[u_t^m(t+\delta)\, v^m(t) + a(x)\nabla u_t^m(t+\delta)\, \nabla v^m(t)\Big]\, dx\, dt$$

$$+ \int\limits_{Q_{t_0,\tau}} \Big[d(t+\delta,x,\nabla u^m(t+\delta))\, \nabla v^m(t) + g(t+\delta,x,u^m(t+\delta))\, v^m(t)\Big]\, dx\, dt$$

$$+ \int\limits_{t_0}^{\tau} \langle \mathcal{B}(u^m(t+\delta)), v^m(t)\rangle\, dt = \int\limits_{Q_{t_0,\tau}} f_{t_0}(t+\delta,x)\, v^m(t)\, dx\, dt\,.$$

We subtract the last equality from (3.1.17), integrate by parts the first two terms, and obtain

$$\frac{1}{2}\int\limits_{\Omega} \Big[|v^m(\tau)|^2 + a(x)\nabla v^m(\tau)\, \nabla v^m(\tau)\Big]\, dx = \frac{1}{2}\int\limits_{\Omega} \Big[|u^m(t_0+\delta)|^2 + a(x)\nabla u^m(t_0+\delta)\, \nabla u^m(t_0+\delta)\Big]\, dx$$

$$- \int\limits_{Q_{t_0,\tau}} \Big[\big(d(t+\delta,\nabla u^m(t+\delta)) - d(t,\nabla u^m(t))\big)\, \nabla v^m(t) + \big(g(t+\delta,u^m(t+\delta)) - g(t,u^m(t))\big)\, v^m(t)\Big]\, dx\, dt$$

$$- \int\limits_{t_0}^{\tau} \langle \mathcal{B}(u^m(t+\delta)) - \mathcal{B}(u^m(t)), v^m(t)\rangle\, dt - \int\limits_{Q_{t_0,\tau}} (f_{t_0}(t+\delta,x) - f_{t_0}(t,x))\, v^m(t)\, dx\, dt.$$

Because of the monotonicity of d, g and $\mathcal{B}$, the estimates (3.1.12), (3.1.16), the continuity in time of d, g and f, and Gronwall's Lemma, the claimed estimate follows from the last equality. ❒

Proof of Theorem 3.1.9. Now, by using the a priori estimates, we will show the convergence of the approximate solutions to the solution in a bounded domain.

We denote by $\Gamma\,:\, L^p(t_0,T;V) \to L^q(t_0,T;V^*)$ the operator given by

$$\int\limits_{t_0}^{T} \langle \Gamma(u), v\rangle\, dt \;=\; \int\limits_{t_0}^{T}\int\limits_{\Omega} \Big(d(t,x,\nabla u)\, \nabla v + g(t,x,u)\, v\Big)\, dx\, dt + \int\limits_{t_0}^{T} \langle \mathcal{B}(u), v\rangle\, dt \;\text{ for } v \in L^p(t_0,T;V)\,.$$

From the boundedness of the operator $\mathcal{B}$ and the estimate (3.1.13) it follows that

$$\begin{aligned}\Big|\int\limits_{t_0}^{T} \langle \Gamma(u^m), v\rangle\, dt\Big| \;&\leq\; \int\limits_{Q_{t_0,T}} \Big(d^0(1+|\nabla u^m|^{p-1})|\nabla v| + g^0(1+|u^m|^{p-1})|v|\Big)\, dx\, dt \\ &\quad + \int\limits_{t_0}^{T} ||\mathcal{B}(u^m)||_{V^*}||v||_V\, dt \\ &\leq\; C||v||_{L^p(t_0,T;V)},\end{aligned}$$

where C is independent of m. Therefore,

$$||\Gamma(u^m)||_{L^q(t_0,T;V^*)} \leq C.$$

The estimates (3.1.12), (3.1.13), and (3.1.15), and the boundedness of Γ imply the existence of a subsequence of $\{u^m\}$, again denoted by $\{u^m\}$, such that

$$\begin{aligned} u^m &\to u && \text{weakly}-* \text{ in } L^\infty(t_0,T;V_0),\\ u^m &\to u && \text{weakly in } L^p(t_0,T;V),\\ u^m &\to u && \text{in } C([t_0,T];V_{\text{weak}}),\\ \Gamma(u^m) &\to Z && \text{weakly in } L^q(t_0,T;V^*) \end{aligned}$$

as $m \to \infty$, where V_{weak} is the space V_0 endowed with its weak topology.

We multiply the equations (3.1.10) by $\beta_s \in C^1([t_0,T])$, sum up over s from 1 to m_0, where $m_0 < m$ is an arbitrary positive integer, integrate over $[t_0,\tau]$, for some $\tau \in (t_0,T]$, and obtain

$$\int\limits_{Q_{t_0,\tau}} \left[u_t^m\, v^{m_0} + a(x)\nabla u_t^m\, \nabla v^{m_0}\right] dx\, dt + \int\limits_{t_0}^{\tau} \langle \Gamma(u^m), v^{m_0}\rangle\, dt \;=\; \int\limits_{Q_{t_0,\tau}} f_{t_0}(t,x)\, v^{m_0}\, dx\, dt\,,$$

where $v^{m_0}(x,t) = \sum\limits_{s=1}^{m_0} \beta_s(t)\phi^s(x)$. After integrating by parts, passing to the limit $m \to \infty$, and using the fact that the set of all functions of the form $\sum\limits_{s<\infty} \beta_s\, \phi^s$ is dense in each of the spaces $C([t_0,T];V_0)$, $L^p(t_0,T;V)$, and $H^1(t_0,T;V_0)$, we obtain the equality

$$\begin{aligned} &\int\limits_{\Omega} \left[u(\tau)\, v(\tau) + a(x)\nabla u(\tau)\, \nabla v(\tau)\right] dx - \int\limits_{Q_{t_0,\tau}} \left[u\, v_t + a(x)\nabla u\, \nabla v_t\right] dx\, dt\\ &+ \int\limits_{t_0}^{\tau} \langle Z, v\rangle\, dt \;=\; \int\limits_{Q_{t_0,\tau}} f_{t_0}(t,x)\, v\, dx\, dt \end{aligned} \tag{3.1.18}$$

for all functions $v \in L^p(t_0,T;V)$, such that $v_t \in L^2(t_0,T;V_0)$ and $v \in C([t_0,T];V_0)$, and for all $\tau \in (t_0,T]$.

Now we have to show $Z = \Gamma(u)$. At first we show the strong convergence of u^m. Consider the equation (3.1.10) with the test function $u^m - w^m$, where $w^m = \Big((\gamma_n v^m) * \rho_k * \rho_k\Big)\gamma_n$ as in Theorem 3.1.4 and $v^m \to u$ strongly in $L^p(t_0,T;V)$,

$$\begin{aligned} &\int\limits_{Q_{t_0,\tau}} \left[u_t^m\, (u^m - w^m) + a(x)\nabla u_t^m\, (\nabla u^m - \nabla w^m)\right] dx\, dt + \int\limits_{Q_{t_0,\tau}} d(t,x,\nabla u^m)\, (\nabla u^m - \nabla w^m)\, dx\, dt\\ &+ \int\limits_{Q_{t_0,\tau}} g(t,x,u^m)\, (u^m - w^m)\, dx\, dt + \int\limits_{t_0}^{\tau} \langle \mathcal{B}(u^m), u^m - w^m\rangle\, dt = \int\limits_{Q_{t_0,\tau}} f_{t_0}(t,x)\, (u^m - w^m)\, dx\, dt. \end{aligned}$$

By Fatou's Lemma, the weak convergence of $\{u^m\}$ and the calculation similar to Uniqueness Theorem 3.1.4, we obtain

$$\liminf_{m\to\infty} \int\limits_{Q_{t_0,\tau}} \left[u_t^m\,(u^m - w^m) + a(x)\nabla u_t^m\,(\nabla u^m - \nabla w^m)\right] dx\,dt$$
$$= \frac{1}{2}\liminf_{m\to\infty} \int\limits_{\Omega} \Big(|u^m|^2 + a(x)\nabla u^m \nabla u^m\Big)\,dx - \frac{1}{2}\int\limits_{\Omega}\Big(|u|^2 + a(x)\nabla u\nabla u\Big)\,dx \;\geqslant\; 0.$$

Then we obtain the inequality

$$\int\limits_{Q_{t_0,\tau}} \Big[(d(t,x,\nabla u^m) - d(t,x,\nabla v^m))\,\nabla(u^m - v^m) + (g(t,x,u^m) - g(t,x,v^m))(u^m - v^m)\Big]\,dx\,dt$$
$$+ \int\limits_{t_0}^{\tau} \langle \mathcal{B}(u^m) - \mathcal{B}(v^m), u^m - v^m\rangle\,dt$$
$$\leq\; - \int\limits_{Q_{t_0,\tau}} \Big[d(t,x,\nabla v^m)\,\nabla(u^m - v^m) + g(t,x,v^m)\,(u^m - v^m)\Big]\,dx\,dt - \int\limits_{t_0}^{\tau} \langle \mathcal{B}(v^m), u^m - v^m\rangle\,dt$$
$$+ \int\limits_{Q_{t_0,\tau}} f_{t_0}(t,x)\,(u^m - v^m)\,dx\,dt.$$

Due to the strong monotonicity of d and g, the monotonicity of $\mathcal{B}$, the weak convergence of u^m in $L^p(t_0,T;V)$, and the strong convergence of v^m in $L^p(t_0,T;V)$ yields

$$\int\limits_{t_0}^{T}\int\limits_{\Omega} \Big(|u^m - u|^p + |\nabla u^m - \nabla u|^p\Big)\,dx\,dt \;\leq\; C\varepsilon.$$

This implies

$$u^m \to u \;\text{ strongly in }\; L^p(t_0,T;V)$$

as $m \to \infty$. Then, since d and g are continuous, and because of the weak convergence of $\Gamma(u^m)$ to Z, we have $Z = \Gamma(u)$.

From the equation (3.1.18) it follows that u is the solution of (3.1.6). Due to $u \in C([t_0,T],V_0)$ and $u^m(t_0) = 0$, we obtain $u(t_0) = 0$. ❐

Remark 3.1.12. Consider the equation (3.1.18) for $\tau = t_2$ with test functions of the form $v(x,t)\gamma_n(t)$, where $v \in L^p(t_0,T;V)$, such that $v_t \in L^2(t_0,T;V_0)$, and γ_n as defined in Theorem 3.1.4. Then we obtain the equality

$$\int\limits_{Q_{t_1,t_2}} \Big[-u\,v_t - a(x)\nabla u\,\nabla v_t + d(t,x,\nabla u)\,\nabla v + g(t,x,u)\,v\Big]\,\gamma_n\,dx\,dt$$
$$+ \int\limits_{t_1}^{t_2} \langle \mathcal{B}(u), v\rangle\,\gamma_n\,dt - \int\limits_{Q_{t_1,t_2}} [u\,v + a(x)\,\nabla u\,\nabla v]\gamma_n'\,dx\,dt \;=\; \int\limits_{Q_{t_1,t_2}} f_{t_0}(t,x)\,v\,\gamma_n\,dx\,dt\,.$$

Passing in the last equality to the limit as $n \to \infty$ implies

$$- \int_{Q_{t_1,t_2}} \Big[u\, v_t + a(x) \nabla u \, \nabla v_t \Big] \, dx\, dt + \int_{Q_{t_1,t_2}} \Big[d(t,x,\nabla u) \, \nabla v + g(t,x,u)\, v \Big] \, dx\, dt$$
$$+ \int_{t_1}^{t_2} \langle \mathcal{B}(u), v \rangle \, dt + \int_{\Omega} \Big[u\, v + a(x) \nabla u \, \nabla v \Big] \, dx \Big|_{t_1}^{t_2} = \int_{Q_{t_1,t_2}} f_{t_0}(t,x)\, v \, dx\, dt$$

for all functions $v \in L^p(t_0,T;V)$, such that $v_t \in L^2(t_0,T;V_0)$ and $v \in C([t_0,T];V_0)$, and t_1, $t_2 \in [t_0,T]$, $t_1 < t_2$.

Proof of Theorem 3.1.6. Now we prove the existence of a solution in an unbounded interval. The key idea of this proof is to use cut-off functions. We choose for positive integer k lower bounds $t_0 = T - k$ and obtain a sequence of solutions u^k of problem (3.1.6), (3.1.7), which we extend by zero to all Q_{T-k}. To be able to pass to the limit in the nonlinear terms we have to show the strong convergence of the sequence $\{u^k\}$. It suffices to show that $\{u^k\}$ is a Cauchy sequence. Due to (3.1.8), for all positive integers k and s, and for all $t_0 > -\min\{k,s\}$, the functions $u^{k,s} = u^k - u^s$ satisfy the equation

$$- \int_{Q_{t_0,T}} \Big[u^{k,s}\, v_t + a(x) \nabla u^{k,s} \, \nabla v_t \Big] \, dx\, dt + \int_{Q_{t_0,T}} (d(t,x,\nabla u^k) - d(t,x,\nabla u^s)) \nabla u^{k,s} \, dx\, dt$$
$$+ \int_{Q_{t_0,T}} (g(t,x,u^k) - g(t,x,u^s)) u^{k,s} \, dx\, dt + \int_{t_0}^{T} \langle \mathcal{B}(u^k) - \mathcal{B}(u^s), u^{k,s} \rangle \, dt \tag{3.1.19}$$
$$+ \int_{\Omega} \Big[u^{k,s}(T)\, v(T) + a(x) \nabla u^{k,s}(T) \, \nabla v(T) \Big] \, dx = 0$$

for $v \in L^p(t_0,T;V)$, such that $v_t \in L^2(t_0,T;V_0)$ and $v(t_0) = 0$. Let us choose in (3.1.19) for $n > 2m$

$$v = \big((\psi_m \, u^{k,s}) * \rho_n * \rho_n \big) \psi_m,$$

where ψ_m is a cut-off function in the time variable, such that

$$\psi_m(t) = \begin{cases} 0, & \tau - \frac{1}{m} \le t \le T, \\ m(\tau - t) - 1, & \tau - \frac{2}{m} < t < \tau - \frac{1}{m}, \\ \left(\frac{t - t_0}{\tau - \frac{2}{m} - t_0} \right)^{\alpha_0}, & t_0 \le t \le \tau - \frac{2}{m}, \;\; \alpha_0 > 0, \\ 0, & t < t_0. \end{cases}$$

The sequence $\{\rho_n\} \subset \mathcal{D}(\mathbb{R})$ is used for mollification like in the proof of Theorem 3.1.4.

Using the bounded support of ψ_m and

$$\begin{aligned}
&\int\limits_{Q_{t_0,\tau}} u^{k,s}\,(u^{k,s}\psi_m)*\rho_n*\rho_{nt}\psi_m\,dx\,dt\\
&=\int\limits_{\Omega}\Big(\int\limits_{-\infty}^{\infty} u^{k,s}(t,x)\,\psi_m(t)\int\limits_{-\infty}^{\infty}\big(\int\limits_{-\infty}^{\infty} u^{k,s}(\sigma,x)\,\psi_m(\sigma)\rho_n(\tau-\sigma)\,d\sigma\big)\,\rho_{nt}(t-\tau)\,d\tau\,dt\Big)\,dx\\
&=-\int\limits_{Q_{t_0,\tau}}\frac{\partial}{\partial t}\Big(\int\limits_{-\infty}^{\infty} u^{k,s}(\tau,x)\psi_m(\tau)\rho_n(t-\tau)d\tau\Big)\int\limits_{-\infty}^{\infty} u^{k,s}(\tau,x)\,\psi_m(\tau)\rho_n(t-\tau)\,d\tau\,dt\,dx\;=\;0,
\end{aligned}$$

we obtain for one of the terms in equation (3.1.19)

$$\begin{aligned}
\int\limits_{Q_{t_0,\tau}} u^{k,s}\,v_t\,dx\,dt \;&=\; \int\limits_{Q_{\tau_0,\tau}} u^{k,s}\,(u^{k,s}\psi_m)*\rho_n*\rho_n\frac{d}{dt}\psi_m\,dx\,dt\\
&\quad+\int\limits_{Q_{t_0,\tau}} u^{k,s}\,(u^{k,s}\psi_m)*\rho_n*\rho_{nt}\psi_m\,dx\,dt \;\;\to\;\; \int\limits_{Q_{t_0,\tau}}|u^{s,k}|^2\psi_m\frac{d}{dt}\psi_m\,dx\,dt,
\end{aligned}$$

as $n\to\infty$. Convergence of the second term involving the time derivative is shown similarly. Hence, taking limits in equation (3.1.19) yields

$$-\int\limits_{Q_{t_0,\tau}}\Big(|u^{k,s}|^2+a(x)\nabla u^{k,s}\,\nabla u^{k,s}\Big)\,\psi_m\,\frac{d}{dt}\psi_m\,dx\,dt+\int\limits_{t_0}^{\tau}\langle\Gamma(u^k)-\Gamma(u^s),u^{k,s}\rangle\,\psi_m^2\,dt\;=\;0\,.$$

Splitting the integral

$$\begin{aligned}
&\int\limits_{Q_{t_0,\tau}}\Big[|u^{k,s}|^2+a(x)\nabla u^{k,s}\,\nabla u^{k,s}\Big]\,\psi_m\frac{d}{dt}\psi_m\,dx\,dt\;=\\
&\int\limits_{t_0}^{\tau-\frac{2}{m}}\int\limits_{\Omega}\Big[|u^{k,s}|^2+a(x)\nabla u^{k,s}\,\nabla u^{k,s}\Big]\,\psi_m\frac{d}{dt}\psi_m\,dx\,dt-m\int\limits_{\tau-\frac{2}{m}}^{\tau-\frac{1}{m}}\int\limits_{\Omega}\Big[|u^{k,s}|^2+a(x)\nabla u^{k,s}\,\nabla u^{k,s}\Big]\,\psi_m\,dx\,dt
\end{aligned}$$

and passing to the limit as $m\to\infty$ in each term separately yields

$$\begin{aligned}
&\int\limits_{\Omega}\Big[|u^{k,s}(\tau)|^2+a(x)\nabla u^{k,s}(\tau)\,\nabla u^{k,s}(\tau)\Big]\psi^2(\tau)\,dx-\int\limits_{Q_{t_0,\tau}}\Big(|u^{k,s}|^2+a(x)\nabla u^{k,s}\,\nabla u^{k,s}\Big)\psi\frac{d}{dt}\psi\,dx\,dt\\
&+\int\limits_{t_0}^{\tau}\langle\Gamma(u^k)-\Gamma(u^s),u^{k,s}\rangle\psi^2\,dt\;=\;0\,,
\end{aligned}\tag{3.1.20}$$

where $\psi(t)=\big(\frac{t-t_0}{\tau-t_0}\big)^{\alpha_0}$ for $t\in[t_0,\tau]$.

We estimate (3.1.20) term by term. For this, we use the boundedness of a, the strong monotonicity of d and g, and the monotonicity of the operator $\mathcal{B}$.

$$\begin{aligned}
I_1 &:= \int\limits_{Q_{t_0,\tau}} \Big[u^{k,s}\, u^{k,s} + a(x)\nabla u^{k,s}\, \nabla u^{k,s}\Big]\psi\, \psi'\, dx\, dt \\
&\le \int\limits_{Q_{t_0,\tau}} \Big[|u^{k,s}|^2 + a^0|\nabla u^{k,s}|^2\Big]\psi\, \psi'\, dx\, dt \\
&\le \delta_1 \int\limits_{Q_{t_0,\tau}} \Big(|u^{k,s}|^p + |\nabla u^{k,s}|^p\Big)\, \psi^2\, dx\, dt + c_1(\delta_1) \int\limits_{Q_{\tau_0,\tau}} \psi^{\frac{p-4}{p-2}} (\psi')^{\frac{p}{p-2}}\, dx\, dt,
\end{aligned}$$

$$\begin{aligned}
I_2 &:= \int\limits_{t_0}^{\tau} \langle \Gamma(u^k) - \Gamma(u^s), u^{k,s}\rangle \psi^2\, dt = \int\limits_{Q_{t_0,\tau}} (d(t,x,\nabla u^k) - d(t,x,\nabla u^s))\, \nabla u^{k,s}\, \psi^2\, dx\, dt \\
&+ \int\limits_{Q_{t_0,\tau}} (g(t,x,u^k) - g(t,x,u^s))\, u^{k,s}\, \psi^2\, dx\, dt + \int\limits_{t_0}^{\tau} \langle \mathcal{B}(u^k) - \mathcal{B}(u^s), u^{k,s}\rangle\, \psi^2\, dt \\
&\geqslant C \int\limits_{Q_{t_0,\tau}} \Big(d_0|\nabla u^{k,s}|^p + g_0|u^{k,s}|^p\Big)\, \psi^2\, dx\, dt.
\end{aligned}$$

Due to the estimates of I_1 and I_2, and the equation (3.1.20) we obtain

$$\begin{aligned}
&\int\limits_{\Omega} \Big(|u^{k,s}(\tau)|^2 + |\nabla u^{k,s}(\tau)|^2\Big)\psi^2(\tau)\, dx + \int\limits_{Q_{t_0,\tau}} \Big(|u^{k,s}|^p + |\nabla u^{k,s}|^p\Big)\psi^2(t)\, dx\, dt \\
&\le c_2 \int\limits_{Q_{t_0,\tau}} \psi^{\frac{p-4}{p-2}} (\psi')^{\frac{p}{p-2}}\, dx\, dt \;\le\; c_3(\tau - t_0)^{2\alpha_0+1-\frac{p}{p-2}},
\end{aligned}$$

for $\alpha_0 > \frac{1}{p-2}$. Let $t_1 \in (t_0, \tau)$ be an arbitrarily fixed number. Then the last inequality implies

$$\begin{aligned}
&\int\limits_{\Omega} \Big(|u^{k,s}(t_2)|^2 + |\nabla u^{k,s}(t_2)|^2\Big)\, dx + \int\limits_{Q_{t_1,t_2}} \Big(|u^{k,s}|^p + |\nabla u^{k,s}|^p\Big)\, dx\, dt \\
&\le\; c_4 \frac{(t_2 - t_0)^{2\alpha_0+1-\frac{p}{p-2}}}{(t_1 - t_0)^{2\alpha_0}} \;=\; c_5 \Big(\frac{t_2 - t_0}{t_1 - t_0}\Big)^{2\alpha_0} (t_2 - t_0)^{1-\frac{p}{p-2}},
\end{aligned}$$

where c_5 is independent of t_0 and $t_2 \in [t_0, T]$. For sufficiently large $|t_0|$ the right hand side of the last inequality is small. Therefore, the sequence $\{u^k\}$ converges uniformly in $C([t_1, T]; V_0)$ and strongly in $L^p(t_1, T; V)$ to a function u for all $t_1 \in (-\infty, T)$.

Now we show that the function u is a weak solution of the original problem. Because of (3.1.8), the function u^k satisfies

$$\begin{aligned}
&\int\limits_{\Omega} \Big[u^k\, v + a(x)\nabla u^k\, \nabla v\Big]\, dx\Big|_{t_1}^{t_2} - \int\limits_{Q_{t_1,t_2}} \Big[u^k\, v_t + a(x)\nabla u^k\, \nabla v_t\Big]\, dx\, dt \qquad (3.1.21) \\
&+ \int\limits_{Q_{t_1,t_2}} \Big[d(t,x,\nabla u^k)\, \nabla v + g(t,x,u^k)\, v\Big]\, dx\, dt + \int\limits_{t_1}^{t_2} \langle \mathcal{B}(u^k), v\rangle\, dt = \int\limits_{Q_{t_1,t_2}} f_k(t,x)\, v\, dx\, dt
\end{aligned}$$

for all $v \in L^p_{\mathrm{loc}}((-\infty, T]; V)$, such that $v_t \in L^2_{\mathrm{loc}}((-\infty, T]; V_0)$ and $v \in C((-\infty, T]; V_0)$, and for all t_1 and t_2 that satisfy $\infty < t_1 < t_2 \leq T$. The strong convergence of $\{u^k\}$ in $L^p_{\mathrm{loc}}((-\infty, T]; V)$ and the continuity of d and g yield

$$\int\limits_{Q_{t_1,t_2}} \Big(d(t, x, \nabla u^k) \nabla v + g(t, x, u^k) v \Big)\, dx\, dt \;\to \int\limits_{Q_{t_1,t_2}} \Big(d(t, x, \nabla u) \nabla v + g(t, x, u) v \Big)\, dx\, dt$$

as $k \to \infty$. The operator $\mathcal{B}$ is monotone, bounded, and hemicontinuous. Therefore, the convergence

$$\int\limits_{t_1}^{t_2} \langle \mathcal{B}(u^k), v \rangle\, dt \;\to \int\limits_{t_1}^{t_2} \langle \mathcal{B}(u), v \rangle\, dt$$

as $k \to \infty$ holds true. Passing to the limit as $k \to \infty$ in (3.1.21) implies that u satisfies the equation (3.1.1). The estimates for u follow from the estimates for u^m. ❒

3.2 Quasilinear Pseudoparabolic Variational Inequalities in Unbounded Time Intervals

The penalty method and the existence result of the section 3.1 are used to show the existence of a solution of an inequality. The functions d and g are assumed to be independent of the time variable. The uniqueness follows from the monotonicity of the operators and from Pankov's lemma.

In $Q_T = (-\infty, T) \times \Omega$ we consider the inequality

$$\int_{Q_{t_1,t_2}} \Big[v_t(v-u) + a(x)\nabla v_t\, \nabla(v-u) + d(x,\nabla u)\, \nabla(v-u) + g(x,u)(v-u) \Big]\, dx\, dt$$
$$- \int_{Q_{t_1,t_2}} f(t,x)\,(v-u)\, dx\, dt \geqslant \frac{1}{2} \int_{\Omega} \Big[|v-u|^2 + a(x)\nabla(v-u)\,\nabla(v-u) \Big]\, dx \Big|_{t_1}^{t_2}. \quad (3.2.1)$$

The constraint on u is given by the requirement $u \in K$, where K is chosen to be a closed and convex subset of V containing 0.

Definition 3.2.1. A function $u : Q_T \to \mathbb{R}$ is called *a weak solution* of the inequality (3.2.1) if

1) $u \in C((-\infty, T]; V_0) \cap L^p_{\text{loc}}((-\infty, T]; V), u(t) \in K$ for almost all $t \in (-\infty, T)$,
2) u satisfies the inequality for all t_1 and t_2, such that $-\infty < t_1 < t_2 \leq T$, and for all functions $v \in L^p_{\text{loc}}((-\infty, T]; V)$, such that $v_t \in L^2_{\text{loc}}((-\infty, T]; V_0)$, $v \in C((-\infty, T]; V_0)$, and $v(t) \in K$ for almost all $t \in (-\infty, T)$.

Theorem 3.2.2 (Existence). *Let Assumption 3.1.1, $d(x,\cdot) \in C^1(\mathbb{R}^N)$, $g(x,\cdot) \in C^1(\mathbb{R})$, and $\partial_t f \in L^2_{loc}((-\infty, T]; L^2(\Omega))$ be satisfied. Then there exists a weak solution of the inequality (3.2.1).*

At first we show the existence of a solution in any bounded time interval (t_0, T) with zero initial condition. For every $t_0 \in (-\infty, T)$ we define the space

$$W_{t_0} = \{v \in L^p(t_0, T; V) \cap C([t_0, T]; V_0), \text{ s. t. } v_t \in L^2(t_0, T; V_0) \text{ and } v(t) \in K \text{ for a.a. } t \in (t_0, T)\}.$$

Theorem 3.2.3. *Let the assumptions of Theorem 3.2.2 be satisfied. Then there exists a function $u \in W_{t_0}$ that satisfies the inequality*

$$\int_{Q_{t_1,t_2}} \Big[v_t\,(v-u) + a(x)\nabla v_t\, \nabla(v-u) \Big]\, \varphi\, dx\, dt + \frac{1}{2} \int_{Q_{t_1,t_2}} \Big[|v-u|^2 + a(x)\nabla(v-u)\,\nabla(v-u) \Big]\, \frac{d}{dt}\varphi\, dx\, dt$$
$$+ \int_{Q_{t_1,t_2}} \Big[d(x,\nabla u)\, \nabla(v-u) + g(x,u)\,(v-u) \Big]\, \varphi\, dx\, dt - \int_{Q_{t_1,t_2}} f(t,x)\,(v-u)\, \varphi\, dx\, dt$$
$$\geqslant \frac{1}{2} \int_{\Omega} \Big[|v-u|^2 + a(x)\nabla(v-u)\,\nabla(v-u) \Big]\, \varphi\, dx \Big|_{t_1}^{t_2} \quad (3.2.2)$$

for all functions $v \in W_{t_0}$ and all $t_1, t_2 \in [t_0, T], t_1 < t_2$, where $\varphi \in C^1((-\infty, T])$ and $\varphi(t) \geqslant 0$ for all $t \in (-\infty, T]$, and initial condition $u(t_0, x) = 0$.

Here the existence result Theorem 3.1.9 is used. We choose a family of penalty operators of the form $\alpha\mathcal{B}$ with a parameter $\alpha > 0$ and obtain similarly to (3.1.6)

$$u_t - \nabla \cdot (a(x)\nabla u_t) - \nabla \cdot d(x, \nabla u) + g(x,u) + \alpha\mathcal{B}(u) = f_{t_0}(t,x).$$

The definition and the properties of the penalty operator $\mathcal{B}$ can be found in Definition A.1.10 of the appendix. To show the convergence of a subsequence of the approximative solutions given by (3.1.9) to a solution of the inequality (3.2.2) an additional estimate is needed.

Lemma 3.2.4. *For the Galerkin approximation $u^{m,\alpha}$ in (3.1.9) the estimate*

$$\int\limits_{\Omega} \Big[|u_t^{m,\alpha}(\tau)|^2 + a(x)\nabla u_t^{m,\alpha}(\tau)\,\nabla u_t^{m,\alpha}(\tau)\Big]dx \quad \leq \quad C, \qquad \tau \in [t_0, T], \tag{3.2.3}$$

holds uniformly with respect to m and α.

Proof. We derive the equation (3.1.10) for k with respect to the time variable t, multiply by the corresponding function z_{kt}^m, sum over k from 1 to m, and integrate over $[t_0, \tau]$, $t_0 < \tau \leq T$. Hereby, we obtain

$$\int\limits_{Q_{t_0,\tau}} \Big[u_{tt}^{m,\alpha}\, u_t^{m,\alpha} + a(x)\nabla u_{tt}^{m,\alpha}\, \nabla u_t^{m,\alpha}\Big]\, dx\, dt + \int\limits_{Q_{t_0,\tau}} \Big[\partial_t d(x, \nabla u^{m,\alpha})\, \nabla u_t^{m,\alpha} + \partial_t g(x, u^{m,\alpha})\, u_t^{m,\alpha}\Big]\, dx\, dt$$

$$+\alpha \int\limits_{t_0}^{\tau} \langle [B(u^{m,\alpha})]_t, u_t^{m,\alpha}\rangle\, dt = \int\limits_{Q_{t_0,\tau}} \partial_t f(t,x)\, u_t^{m,\alpha}\, dx\, dt. \tag{3.2.4}$$

Now we estimate all terms in (3.2.4) separately. From the assumptions on the functions a, d, and g it follows that

$$\begin{aligned} I_1 \;&:=\; \int\limits_{Q_{t_0,\tau}} \big(u_{tt}^{m,\alpha}\, u_t^{m,\alpha} + a(x)\nabla u_{tt}^{m,\alpha}\, \nabla u_t^{m,\alpha}\big)\, dx\, dt \;\geq\; \frac{1}{2}\int\limits_{\Omega} \Big(|u_t^{m,\alpha}(\tau)|^2 + a_0|\nabla u_t^{m,\alpha}(\tau)|^2\Big)\, dx \\ &\qquad -\frac{1}{2}\int\limits_{\Omega} \Big(|u_t^{m,\alpha}(t_0)|^2 + a(x)\nabla u_t^{m,\alpha}(t_0)\, \nabla u_t^{m,\alpha}(t_0)\Big)\, dx\,, \\ I_2 \;&:=\; \int\limits_{Q_{t_0,\tau}} \partial_t d(x, \nabla u^{m,\alpha}) \nabla u_t^{m,\alpha}\, dx\, dt \;=\; \int\limits_{Q_{t_0,\tau}} \partial_\xi d(x, \nabla u^{m,\alpha})\, \nabla u_t^{m,\alpha}\, \nabla u_t^{m,\alpha}\, dx\, dt \;\geqslant\; 0, \\ I_3 \;&:=\; \int\limits_{Q_{t_0,\tau}} \partial_t g(x, u^{m,\alpha})\, u_t^{m,\alpha}\, dx\, dt = \int\limits_{Q_{t_0,\tau}} \partial_\zeta g(x, u^{m,\alpha})\, |u_t^{m,\alpha}|^2\, dx\, dt \;\geqslant\; 0. \end{aligned}$$

Here we used the following calculations. For $d(x,\cdot) \in C^1(\mathbb{R}^N)$ and

$$(d(x,\xi_1) - d(x,\xi_2))(\xi_1 - \xi_2) \geqslant d_1|\xi_1 - \xi_2|^p,\ \ p > 2$$

yield

$$(d(x, \tilde{\xi} + s\eta) - d(x, \tilde{\xi}))\, s\eta \;\geqslant\; d_1 s^p |\eta|^p \ \text{ for all }\ \eta \in \mathbb{R}^N \ \text{ and }\ |s| < \varepsilon$$

or

$$\frac{(d(x,\tilde{\xi}+s\eta)-d(x,\tilde{\xi}))\eta}{s} \geqslant d_1 s^{p-2}|\eta|^p \text{ for all } \eta \in \mathbb{R}^N \text{ and } |s|<\varepsilon,$$

respectively. Passing to the limit as $s \to 0$ implies

$$\partial_\xi d(x,\tilde{\xi})\eta\,\eta \;\geqslant\; 0 \quad \text{for all } \eta \in \mathbb{R}^N,\, \tilde{\xi} \in \mathbb{R}^N.$$

Similarly we obtain

$$\partial_\zeta g(x,\tilde{\zeta})\,|\tilde{\eta}|^2 \;\geqslant\; 0 \text{ for all } \tilde{\eta} \in \mathbb{R},\, \tilde{\zeta} \in \mathbb{R}.$$

Due to $f_t \in L^2_{\text{loc}}((-\infty,T];L^2(\Omega))$, we have

$$I_4 := \int\limits_{Q_{t_0,\tau}} f_t(t,x)\,u_t^{m,\alpha}\,dxdt \;\leq\; \frac{\delta}{2}\int\limits_{Q_{t_0,\tau}} |u_t^{m,\alpha}|^2\,dx\,dt + \frac{1}{2\delta}\int\limits_{Q_{t_0,\tau}} |f_t(t,x)|^2\,dx\,dt.$$

The monotonicity of the penalty operator implies

$$I_5 := \int\limits_{t_0}^{\tau} \langle [\mathcal{B}(u^{m,\alpha})]_t, u_t^{m,\alpha}\rangle\,dt \;\geqslant\; 0.$$

From the equality (3.1.10) and the initial condition (3.1.11) it follows that

$$I_6 := \int\limits_\Omega \Big(|u_t^{m,\alpha}(t_0)|^2 + a(x)\nabla u_t^{m,\alpha}(t_0)\,\nabla u_t^{m,\alpha}(t_0)\Big)\,dx \;=\; \int\limits_\Omega f(t_0,x)\,u_t^{m,\alpha}(t_0)\,dx.$$

This inequality, due to the assumptions on a, implies

$$\int\limits_\Omega \Big(|u_t^{m,\alpha}(t_0)|^2 + a(x)\nabla u_t^{m,\alpha}(t_0)\,\nabla u_t^{m,\alpha}(t_0)\Big)\,dx \;\leq\; C(a_0)\int\limits_\Omega |f(t_0,x)|^2\,dx.$$

By using the estimates of the integrals $I_1, ..., I_6$ in the equality (3.2.4) we obtain the claimed estimate. ❐

Proof of Theorem 3.2.3. Now we prove the existence of a solution of the inequality (3.2.2). We denote by $\Gamma : L^p(t_0,T;V) \to L^q(t_0,T;V^*)$ the operator given by

$$\int\limits_{t_0}^{T} \langle \Gamma(u), v\rangle\,dt \;=\; \int\limits_{t_0}^{T}\int\limits_\Omega \Big[d(x,\nabla u)\,\nabla v + g(x,u)\,v \Big]\,dx\,dt \text{ for } v \in L^p(t_0,T;V). \tag{3.2.5}$$

Due to estimate (3.1.13) and the growth assumptions on d and g, we obtain

$$\begin{aligned} \Big|\int\limits_{t_0}^{T} \langle \Gamma(u^{m,\alpha}), v\rangle\,dt\Big| \;&\leq\; \int\limits_{Q_{t_0,T}} \Big(d^0(1+|\nabla u^{m,\alpha}|^{p-1})|\nabla v| + g^0(1+|u^{m,\alpha}|^{p-1})|v|\Big)\,dx\,dt \\ &\leq\; C||v||_{L^p(t_0,T;V)}, \end{aligned}$$

where C is independent of m and α. Therefore,

$$||\Gamma(u^{m,\alpha})||_{L^q(t_0,T;V^*)} \;\leq\; C.$$

The estimates (3.1.12), (3.1.13), and (3.2.3), and the boundedness of the operators Γ and $\mathcal{B}$ imply the existence of a subsequence of $\{u^{m,\alpha}\}$, again denoted by $\{u^{m,\alpha}\}$, such that

$$\begin{aligned} u^{m,\alpha} &\to u^{\alpha} && \text{weakly}-* \ \text{ in } L^{\infty}(t_0,T;V_0),\\ u_t^{m,\alpha} &\to u_t^{\alpha} && \text{weakly}-* \ \text{ in } L^{\infty}(t_0,T;V_0),\\ u^{m,\alpha} &\to u^{\alpha} && \text{weakly in } L^{p}(t_0,T;V),\\ \Gamma(u^{m,\alpha}) &\to \Gamma(u^{\alpha}) && \text{weakly in } L^{q}(t_0,T;V^*),\\ \mathcal{B}(u^{m,\alpha}) &\to \mathcal{B}(u^{\alpha}) && \text{weakly in } L^{q}(t_0,T;V^*), \end{aligned}$$

as $m \to \infty$. The last two convergences follow from the strong convergence of $u^{m,\alpha}$ to u^{α} in $L^p(t_0,T;V)$, which was proved in Theorem 3.1.9. Due to $u^{\alpha} \in L^p(t_0,T;V)$, $u_t^{\alpha} \in L^2(t_0,T;V_0)$, and (Evans 1998, Theorem 5.9.2), we obtain $u^{\alpha} \in C([t_0,T];V_0)$. Then, because of $u^{m,\alpha}(t_0) = 0$, we have $u^{\alpha}(t_0) = 0$.

The function u^{α} satisfies the equation

$$\int_{Q_{t_0,\tau}} \Big[u_t^{\alpha}\, v + a(x)\nabla u_t^{\alpha}\, \nabla v + d(x,\nabla u^{\alpha})\,\nabla v + g(x,u^{\alpha})\, v\Big]\, dx\, dt$$
$$+\, \alpha \int_{t_0}^{\tau} \langle \mathcal{B}(u^{\alpha}), v\rangle\, dt = \int_{Q_{t_0,\tau}} f(t,x)\, v\, dx\, dt \tag{3.2.6}$$

for every $\alpha \in \mathbb{R}^+$, for all functions $v \in L^p(t_0,T;V)$, and all $\tau \in [t_0,T]$. Analogously as for $u^{m,\alpha}$, we obtain the estimates for u^{α}

$$\begin{aligned} \int_{\Omega} \Big(|u^{\alpha}(\tau)|^2 + |\nabla u^{\alpha}(\tau)|^2\Big)\, dx &\le C, \quad \tau \in [t_0,T],\\ \int_{\Omega} \Big(|u_t^{\alpha}(\tau)|^2 + |\nabla u_t^{\alpha}(\tau)|^2\Big)\, dx &\le C, \quad \tau \in [t_0,T],\\ \int_{Q_{t_0,T}} \Big(|u^{\alpha}|^p + |\nabla u^{\alpha}|^p\Big)\, dx\, dt &\le C,\\ ||\Gamma(u^{\alpha})||_{L^q(t_0,T;V^*)} &\le C,\\ \alpha \int_{t_0}^{T} \langle \mathcal{B}(u^{\alpha}), u^{\alpha}\rangle\, dt &\le C. \end{aligned} \tag{3.2.7}$$

Due to the estimates (3.2.7) there exists a subsequence of $\{u^{\alpha}\}$, again denoted by $\{u^{\alpha}\}$, such that

$$\begin{aligned} u^{\alpha} &\to u && \text{weakly in } H^1(t_0,T;V_0),\\ u^{\alpha} &\to u && \text{weakly in } L^p(t_0,T;V),\\ \Gamma(u^{\alpha}) &\to \chi && \text{weakly in } L^q(t_0,T;V^*),\\ \mathcal{B}(u^{\alpha}) &\to \beta && \text{weakly in } L^q(t_0,T;V^*) \end{aligned}$$

as $\alpha \to \infty$. For $v \in L^p(t_0, T; V)$, such that $v_t \in L^2(t_0, T; V_0)$, the equation (3.2.6) can be rewritten in the form

$$\int\limits_{Q_{t_0,\tau}} \Big[-u^\alpha\, v_t - a(x)\nabla u^\alpha\, \nabla v_t + d(x, \nabla u^\alpha)\, \nabla v + g(x, u^\alpha)\, v\Big]\, dx\, dt + \alpha \int\limits_{t_0}^{\tau} \langle \mathcal{B}(u^\alpha), v\rangle\, dt$$
$$+ \int\limits_{\Omega} \Big[u^\alpha(\tau)\, v(\tau) + a(x)\nabla u^\alpha(\tau)\, \nabla v(\tau)\Big]\, dx = \int\limits_{Q_{t_0,\tau}} f(t,x)\, v\, dx\, dt. \tag{3.2.8}$$

Applying the estimates (3.2.7) to (3.2.6) we obtain

$$\begin{aligned}\Big|\int\limits_{t_0}^{T} \langle \mathcal{B}(u^\alpha), v\rangle\, dt\Big| \;&\le\; \frac{C}{\alpha}||u^\alpha_t||_{L^2(t_0,T;V_0)}||v||_{L^2(t_0,T;V_0)} + \frac{C}{\alpha}||\Gamma(u^\alpha)||_{L^q(t_0,T;V^*)}||v||_{L^p(t_0,T;V)}\\ &\quad + \frac{C}{\alpha}||f||_{L^2(Q_{t_0,T})}||v||_{L^2(Q_{t_0,T})}\\ &\le\; \frac{C}{\alpha}||v||_{L^p(t_0,T;V)}\end{aligned}$$

for all functions $v \in L^p(t_0, T; V)$. Therefore,

$$||\mathcal{B}(u^\alpha)||_{L^q(t_0,T;V^*)} \;\le\; \frac{C}{\alpha}$$

holds true. Using the monotonicity of $\mathcal{B}$ we get

$$\int\limits_{t_0}^{T} \langle \mathcal{B}(v), u^\alpha - v\rangle\, dt \;\le\; \int\limits_{t_0}^{T} \langle \mathcal{B}(u^\alpha), u^\alpha - v\rangle\, dt.$$

Together with the estimate $\int\limits_{t_0}^{T} \langle \mathcal{B}(u^\alpha), u^\alpha\rangle\, dt \;\le\; C/\alpha$ and the convergence of $\mathcal{B}(u^\alpha) \to 0$ in $L^q(t_0, T; V^*)$ we obtain for $\alpha \to \infty$

$$\int\limits_{t_0}^{T} \langle \mathcal{B}(v), u - v\rangle\, dt \;\le\; 0.$$

We choose functions v of the form $u - \lambda w$ for $\lambda > 0$ and $w \in L^p(t_0, T; V)$. Passing to the limit as $\lambda \to 0$ and using the hemicontinuity of $\mathcal{B}$ imply

$$\int\limits_{t_0}^{T} \langle \mathcal{B}(u), w\rangle\, dt \;\le\; 0 \quad \text{for all} \quad w \in L^p(t_0, T; V).$$

Thus, $\mathcal{B}(u) = 0$ and, due to the definition of the penalty operator, $u(t) \in K$ for almost all $t \in (t_0, T)$.

Now we show that u satisfies the inequality (3.2.2). Due to $u^\alpha \in L^p(t_0,T;V)$ and $u^\alpha \in H^1(t_0,T;V_0)$, we can choose u^α as the test function in the equation (3.2.8) and obtain

$$\int\limits_{Q_{t_0,\tau}} \Big[d(x,\nabla u^\alpha)\,\nabla u^\alpha + g(x,u^\alpha)\,u^\alpha\Big]\,dx\,dt + \alpha\int\limits_{t_0}^{\tau}\langle\mathcal{B}(u^\alpha),u^\alpha\rangle\,dt$$
$$-\int\limits_{Q_{t_0,\tau}} f(t,x)\,u^\alpha\,dx\,dt + \frac{1}{2}\int\limits_{\Omega}\Big[|u^\alpha(\tau)|^2 + a(x)\nabla u^\alpha(\tau)\,\nabla u^\alpha(\tau)\Big]\,dx = 0$$

for $\tau \in [t_0,T]$. The integration by parts formula for $v \in W_{t_0}$ implies

$$\int\limits_{Q_{t_0,\tau}} \Big[v_t\,v + a(x)\nabla v_t\,\nabla v\Big]\,dx\,dt = \frac{1}{2}\int\limits_{\Omega}\Big[|v|^2 + a(x)\nabla v\,\nabla v\Big]\,dx\,\Big|_{t_0}^{\tau}.$$

We add the last two equalities, subtract (3.2.8), and obtain

$$\int\limits_{Q_{t_0,\tau}} \Big[v_t\,(v-u^\alpha) + a(x)\nabla v_t\,\nabla(v-u^\alpha) + d(x,\nabla u^\alpha)\,\nabla(v-u^\alpha) + g(x,u^\alpha)\,(v-u^\alpha)\Big]\,dx\,dt$$
$$+\alpha\int\limits_{t_0}^{\tau}\langle\mathcal{B}(u^\alpha),v-u^\alpha\rangle\,dt - \int\limits_{Q_{t_0,\tau}} f(t,x)(v-u^\alpha)\,dx\,dt$$
$$=\frac{1}{2}\int\limits_{\Omega}\Big[|v-u^\alpha|^2 + a(x)\nabla(v-u^\alpha)\nabla(v-u^\alpha)\Big]\,dx\,\Big|_{t_0}^{\tau}.$$

Due to the monotonicity of $\mathcal{B}$, the last equality implies

$$\int\limits_{Q_{t_0,\tau}} \Big[v_t\,(v-u^\alpha) + a(x)\nabla v_t\,\nabla(v-u^\alpha) + d(x,\nabla u^\alpha)\,\nabla(v-u^\alpha) + g(x,u^\alpha)\,(v-u^\alpha)\Big]\,dx\,dt$$
$$-\int\limits_{Q_{t_0,\tau}} f(t,x)\,(v-u^\alpha)\,dx\,dt \geqslant \frac{1}{2}\int\limits_{\Omega}\Big(|v-u^\alpha|^2 + a(x)\nabla(v-u^\alpha)\,\nabla(v-u^\alpha)\Big)\,dx\,\Big|_{t_0}^{\tau}. \quad (3.2.9)$$

Since $u \in W_{t_0}$, the inequality (3.2.9) also holds for $v = u$. Then we obtain

$$\frac{1}{2}\int\limits_{\Omega}\Big(|u-u^\alpha|^2 + a(x)\nabla(u-u^\alpha)\,\nabla(u-u^\alpha)\Big)\,dx\,\Big|_{t_0}^{\tau} + \int\limits_{Q_{t_0,\tau}} (d(x,\nabla u) - d(x,\nabla u^\alpha))\,\nabla(u-u^\alpha)\,dx\,dt$$
$$+\int\limits_{Q_{t_0,\tau}} (g(x,u) - g(x,u^\alpha))\,(u-u^\alpha)\,dx\,dt \le \int\limits_{Q_{t_0,\tau}} \Big(u_t\,(u-u^\alpha) + a(x)\nabla u_t\,\nabla(u-u^\alpha)\Big)\,dx\,dt$$
$$+\int\limits_{Q_{t_0,\tau}} \Big(d(x,\nabla u)\,\nabla(u-u^\alpha) + g(x,u)\,(u-u^\alpha)\Big)\,dx\,dt - \int\limits_{Q_{t_0,\tau}} f(t,x)\,(u-u^\alpha)\,dx\,dt.$$

Using the ellipticity of a, the strong monotonicity of d and g, and the weak convergence of $\{u^\alpha\}$ in $L^p(t_0,T;V)$ in the last inequality, yields

$$\int\limits_{\Omega}\Big(|u(\tau)-u^\alpha(\tau)|^2 + |\nabla u(\tau) - \nabla u^\alpha(\tau)|^2\Big)\,dx + \int\limits_{Q_{t_0,\tau}}\Big(|u-u^\alpha|^p + |\nabla u - \nabla u^\alpha|^p\Big)\,dx\,dt \le c_1\varepsilon$$

and

$$u^\alpha \to u \quad \text{uniformly in} \quad C([t_0,T];V_0),$$
$$u^\alpha \to u \quad \text{strongly in} \quad L^p(t_0,T;V),$$

as $\alpha \to \infty$. The strong convergence yields $\chi = \Gamma(u)$. Also, due to (Evans 1998, Theorem 5.9.2), $u \in L^p(t_0,T;V)$, and $u_t \in L^\infty(t_0,T;V_0)$ we obtain $u \in C([t_0,T];V_0)$ and $u(t_0,x) = 0$. Analogously to the inequality (3.2.9), we obtain the inequality

$$\begin{aligned}
&\int\limits_{Q_{t_1,t_2}} \Big[v_t\,(v-u^\alpha) + a(x)\nabla v_t\,\nabla(v-u^\alpha)\Big]\,\varphi\,dx\,dt \\
&+\frac{1}{2}\int\limits_{Q_{t_1,t_2}} \Big[|v-u^\alpha|^2 + a(x)\nabla(v-u^\alpha)\,\nabla(v-u^\alpha)\Big]\,\frac{d}{dt}\varphi\,dx\,dt \\
&+\int\limits_{Q_{t_1,t_2}} \Big[d(x,\nabla u^\alpha)\,\nabla(v-u^\alpha) + g(x,u^\alpha)\,(v-u^\alpha) - f(t,x)\,(v-u^\alpha)\Big]\,\varphi\,dx\,dt \\
&\geqslant \frac{1}{2}\int\limits_{\Omega} \Big[|v-u^\alpha|^2 + a(x)\nabla(v-u^\alpha)\,\nabla(v-u^\alpha)\Big]\,\varphi\,dx\,\Big|_{t_1}^{t_2}
\end{aligned} \tag{3.2.10}$$

for every $v \in W_{t_0}$ and all $t_1, t_2 \in [t_0,T]$, $t_1 < t_2$, where $\varphi \in C^1((-\infty,T])$, such that $\varphi(t) \geqslant 0$ for $t \in (-\infty,T]$. Passing to the limit as $\alpha \to \infty$ in (3.2.10) implies the inequality (3.2.2). ❒

Proof of Theorem 3.2.2. Now we prove the existence of a solution on the whole interval $(-\infty,T]$. We choose for positive integer k lower bounds $t_0 = T-k$ and hereby obtain a sequence of solutions u^k of the inequality (3.2.2), due to Theorem 3.2.3, which we extend by zero to all Q_{T-k}. To show that $\{u^k\}$ converges to a solution of the inequality without initial condition (3.2.1), the strong convergence of $\{u^k\}$ is needed.

Due to (3.2.2), for all positive integers k and l, such that $k \leq l$, and for all $t_1 > T-k$ the functions $u^{(i)}$ for $i = 1,2$ satisfy the inequalities

$$\begin{aligned}
&\int\limits_{Q_{t_1,t_2}} \Big[v_t\,(v-u^{(i)}) + a(x)\nabla v_t\,\nabla(v-u^{(i)})\Big]\,\varphi(t)\,dx\,dt \\
&+\frac{1}{2}\int\limits_{Q_{t_1,t_2}} \Big[|v-u^{(i)}|^2 + a(x)\nabla(v-u^{(i)})\,\nabla(v-u^{(i)})\Big]\,\frac{d}{dt}\varphi\,dx\,dt \\
&+\int\limits_{Q_{t_1,t_2}} \Big[d(x,\nabla u^{(i)})\,\nabla(v-u^{(i)}) + g(x,u^{(i)})\,(v-u^{(i)})\Big]\,\varphi\,dx\,dt - \int\limits_{Q_{t_1,t_2}} f_i(t,x)\,(v-u^{(i)})\,\varphi\,dx\,dt \\
&\geqslant \frac{1}{2}\int\limits_{\Omega} \Big[|v-u^{(i)}|^2\,\varphi + a(x)\nabla(v-u^{(i)})\,\nabla(v-u^{(i)})\Big]\,\varphi(t)\,dx\,\Big|_{t_1}^{t_2}
\end{aligned} \tag{3.2.11}$$

for all functions $v \in W_{T-l}$, where $u^{(1)} = u^k, u^{(2)} = u^l$, $f_1(t,x) = f_k(t,x)$, $f_2(t,x) = f_l(t,x)$,

$$f_k(t,x) = \begin{cases} f(t,x), & (t,x) \in Q_{T-k,T}, \\ 0, & (t,x) \in Q_{T-k}, \end{cases}$$

and φ is a cut-off function in the time variable

$$\varphi(t) = \begin{cases} (t-t_1)^\gamma, & t_1 \le t \le T, \quad \gamma > 1, \\ 0, & t < t_1. \end{cases}$$

Since K is convex, $v = \frac{1}{2}(u^k + u^l) \in K$ for almost all $t \in (-\infty, T]$. Choosing this function as the test function in (3.2.11) and adding the inequalities implies

$$\begin{aligned} &\frac{1}{2}\int_\Omega \left[|u^{k,l}(t_2)|^2 + a(x)\nabla u^{k,l}(t_2)\,\nabla u^{k,l}(t_2)\right]\,\varphi(t_2)\,dx \\ &+ \int_{Q_{t_1,t_2}} \left[(d(x,\nabla u^k) - d(x,\nabla u^l))\,\nabla u^{k,l} + (g(x,u^k) - g(x,u^l))\,u^{k,l}\right]\varphi\,dx\,dt \\ &\le \frac{1}{2}\int_{Q_{t_1,t_2}} \left[|u^{k,l}|^2 + a(x)\nabla u^{k,l}\,\nabla u^{k,l}\right]\frac{d}{dt}\varphi\,dx\,dt, \end{aligned} \tag{3.2.12}$$

where $u^{k,l} = u^k - u^l$. Due to the assumption $(A1)$, we obtain

$$I_1 := \int_\Omega a(x)\nabla u^{k,l}(t_2)\,\nabla u^{k,l}(t_2)\,\varphi(t_2)\,dx \geqslant a_0\int_\Omega |\nabla u^{k,l}(t_2)|^2\,\varphi(t_2)\,dx.$$

From the strong monotonicity of d and g follows the estimate

$$\begin{aligned} I_2 &:= \int_{Q_{t_1,t_2}} \left[(d(x,\nabla u^k) - d(x,\nabla u^l))\,\nabla u^{k,l} + (g(x,u^k) - g(x,u^l))\,u^{k,l}\right]\varphi\,dx\,dt \\ &\geqslant \int_{Q_{t_1,t_2}} \left[d_1|\nabla u^{k,l}|^p + g_1|u^{k,l}|^p\right]\varphi\,dx\,dt\,. \end{aligned}$$

The right hand side of (3.2.12) is estimated by

$$\begin{aligned} I_3 &:= \int_{Q_{t_1,t_2}} \left[|u^{k,l}|^2 + a(x)\nabla u^{k,l}\,\nabla u^{k,l}\right]\frac{d}{dt}\varphi\,dx\,dt \\ &= \int_{Q_{t_1,t_2}} \left[|u^{k,l}|^2 + a(x)\nabla u^{k,l}\,\nabla u^{k,l}\right]\varphi^{\frac{2}{p}}\varphi^{-\frac{2}{p}}\frac{d}{dt}\varphi\,dx\,dt \\ &\le c_1\delta\int_{Q_{t_1,t_2}} \left[|u^{k,l}|^p + |\nabla u^{k,l}|^p\right]\varphi\,dx\,dt + c_2/\delta\int_{t_1}^{t_2} \varphi^{-\frac{2}{p-2}}(t)\Big(\frac{d}{dt}\varphi(t)\Big)^{\frac{p}{p-2}}\,dt, \end{aligned}$$

where c_1 depends on p, Ω, and a^0. Using the estimates of the integrals I_1, I_2, and I_3, and the equality

$$\int_{t_1}^{t_2} \varphi^{-\frac{2}{p-2}}(t)\Big(\frac{d}{dt}\varphi(t)\Big)^{\frac{p}{p-2}}\,dt \;=\; \frac{\gamma(p-2)}{(\gamma+1)(p-2)-p}(t_2-t_1)^{\gamma+1-\frac{p}{p-2}}$$

in (3.2.12) implies

$$\int_\Omega \Big(|u^{k,l}(t_2)|^2 + |\nabla u^{k,l}(t_2)|^2\Big)\,dx + \int_{Q_{t_0,t_2}} \Big(|u^{k,l}|^p + |\nabla u^{k,l}|^p\Big)\,dx\,dt \;\le\; C\Big(\frac{t_2-t_1}{t_0-t_1}\Big)^{\gamma}(t_2-t_1)^{1-\frac{p}{p-2}},$$

where C is independent of k, l, and t_1, and t_0 is a fixed number, such that $t_1 < t_0 \le t_2$. We notice that $1-\frac{p}{p-2} < 0$ because of $p > 2$. For sufficiently large $|t_1|$ the right hand side of the last inequality is small. Therefore, the sequence $\{u^k\}$ is a Cauchy sequence in the spaces $C((-\infty,T];V_0)$ and $L^p_{\rm loc}((-\infty,T];V)$, and there exists a function u, such that

$$\begin{aligned} u^k &\to u \quad \text{uniformly in} \quad C((-\infty,T];V_0),\\ u^k &\to u \quad \text{strongly in} \quad L^p_{\rm loc}((-\infty,T];V) \end{aligned}$$

as $k\to\infty$. Due to Theorem 3.2.3, for every integer k, the function u^k satisfies the inequality

$$\int_{Q_{t_1,t_2}} \Big[v_t\,(v-u^k) + a(x)\nabla v_t\,\nabla(v-u^k) + d(x,\nabla u^k)\,\nabla(v-u^k) + g(x,u^k)\,(v-u^k)\Big]\,dx\,dt$$
$$-\int_{Q_{t_1,t_2}} f(t,x)\,(v-u^k)\,dx\,dt \geqslant \frac12\int_\Omega \Big[|v-u^k|^2 + a(x)\nabla(v-u^k)\,\nabla(v-u^k)\Big]\,dx\,\Big|_{t_1}^{t_2}$$

for all functions $v\in L^p_{\rm loc}((-\infty,T];V)$, such that $v_t\in L^2_{\rm loc}((-\infty,T];V_0)$, $v\in C((-\infty,T];V_0)$, and $v(t)\in K$ for almost all $t\in(-\infty,T]$, and $t_1,\,t_2\in(-\infty,T]$, $t_1<t_2$. Passing in this inequality to the limit as $k\to\infty$ and using the strong convergence we obtain that u is a solution of the inequality (3.2.1). ❐

Theorem 3.2.5 (Uniqueness). *Let Assumption 3.1.1 be satisfied. Then there exists at most one weak solution of the inequality (3.2.1).*

Proof. Suppose, $u^{(1)}$ and $u^{(2)}$ are two weak solutions of (3.2.1), i.e.

$$\int_{Q_{t_1,\tau}} \Big[v_t\,(v-u^{(i)}) + a(x)\nabla v_t\,\nabla(v-u^{(i)}) + d(x,\nabla u^{(i)})\,\nabla(v-u^{(i)}) + g(x,u^{(i)})\,(v-u^{(i)})\Big]\,dx\,dt$$
$$-\int_{Q_{t_1,\tau}} f(t,x)\,(v-u^{(i)})\,dx\,dt \geqslant \frac12\int_\Omega \Big[|v-u^{(i)}|^2 + a(x)\nabla(v-u^{(i)})\,\nabla(v-u^{(i)})\Big]\,dx\,\Big|_{t_1}^{\tau} \qquad (3.2.13)$$

for all functions $v\in L^p_{\rm loc}((-\infty,T];V)$, such that $v_t\in L^2_{\rm loc}((-\infty,T];V_0)$, $v\in C((-\infty,T];V_0)$, and $v(t)\in K$ for almost all $t\in(-\infty,T)$, and for all t_1,τ, such that $t_1<\tau\le T$, where $i=1,2$. In the inequalities (3.2.13) we choose $v=v_\xi$, where for all $\xi>0$ v_ξ is the solution of the Cauchy problem

$$\begin{cases} \xi\,v_t + v &= w,\\ v(t_1) &= w(t_1), \end{cases} \qquad (3.2.14)$$

and $w=\frac12(u^{(1)}+u^{(2)})$. Since $w\in L^p_{\rm loc}((-\infty,T];V)$ there exists a set $J\subset(-\infty,T]$, such that $w(t)\in V$ for all $t\in J$ and $(-\infty,T]\setminus J$ is a set of measure zero. We assume $t_1\in J$. For all ξ

there exists a solution of (3.2.14) of the form

$$v_\xi(t,x) = v(t_1)e^{-\frac{t-t_1}{\xi}} + \int_{t_1}^{t} e^{-\frac{t-s}{\xi}} \frac{w(s,x)}{\xi}\, ds.$$

From the properties of approximation of the Dirac function, it follows that $v_\xi \to w$ almost everywhere in $Q_{t_1,T}$. To show that $v_\xi \to w$ weakly in $L^p(t_1,T;V)$ we need the estimate $||v_\xi||_{L^p(t_1,T;V)} \le C$, where C is independent of ξ.

$$\begin{aligned} ||v_\xi||^p_{L^p(t_1,T;V)} &= ||w(t_1)||^p_V + \int_{Q_{t_1,T}} \Big(\int_{t_1}^{t} e^{-\frac{t-s}{\xi}} \frac{w(s,x)}{\xi}\, ds\Big)^p dx\, dt + \int_{Q_{t_1,T}} \Big(\int_{t_1}^{t} e^{-\frac{t-s}{\xi}} \frac{\nabla w(s,x)}{\xi}\, ds\Big)^p dx\, dt \\ &\le c_1 + \int_{Q_{t_1,T}} \Big(\int_{t_1}^{t} (e^{-\frac{t-s}{\xi}})^{p'/p'}\, ds\Big)^{\frac{p}{p'}} \Big(\int_{t_1}^{t} e^{-\frac{t-s}{\xi}} \frac{1}{\xi^p}(|w|^p + |\nabla w|^p)\, ds\Big)\, dx\, dt \\ &\le c_1 + c_2\Big(||u^{(1)}||^p_{L^p(t_1,T;V)} + ||u^{(2)}||^p_{L^p(t_1,T;V)}\Big)\xi^{\frac{p}{p'}}\xi\xi^{-p} \le C \end{aligned}$$

for $u^{(1)}, u^{(2)} \in L^p(t_1,T;V)$.

Summing up the inequalities (3.2.13) for $i = 1,2$ yields

$$\begin{aligned} &2\int_{Q_{t_1,\tau}} \Big(v_{\xi t}\,(v_\xi - w) + a(x)\nabla v_{\xi t}\,\nabla(v_\xi - w)\Big)\, dx\, dt \\ &+ \int_{t_1}^{\tau} \Big(\langle \Gamma(u^{(1)}), v_\xi - u^{(1)}\rangle + \langle \Gamma(u^{(2)}), v_\xi - u^{(2)}\rangle\Big)\, dt - 2\int_{Q_{t_1,\tau}} f(t,x)\,(v_\xi - w)\, dx\, dt \\ \geqslant\; &\frac{1}{2}\int_\Omega \Big(|v_\xi(\tau) - u^{(1)}(\tau)|^2 + a(x)\nabla(v_\xi(\tau) - u^{(1)}(\tau))\,\nabla(v_\xi(\tau) - u^{(1)}(\tau))\Big)\, dx \\ &+\frac{1}{2}\int_\Omega \Big(|v_\xi(\tau) - u^{(2)}(\tau)|^2 + a(x)\nabla(v_\xi(\tau) - u^{(2)}(\tau))\,\nabla(v_\xi(\tau) - u^{(2)}(\tau))\Big)\, dx \\ &-\frac{1}{4}\int_\Omega \Big(|u(t_1)|^2 + a(x)\nabla u(t_1)\,\nabla u(t_1)\Big)\, dx, \end{aligned}$$

where $u = u^{(1)} - u^{(2)}$ and the operator Γ is as defined in (3.2.5). Since

$$\begin{aligned} &\int_{Q_{t_1,\tau}} \Big(v_{\xi t}\,(v_\xi - w) + a(x)\nabla v_{\xi t}\,\nabla(v_\xi - w)\Big)\, dx\, dt = \\ &-\frac{1}{\xi}\int_{Q_{t_1,\tau}} \Big(|v_\xi - w|^2 + a(x)\nabla(v_\xi - w)\,\nabla(v_\xi - w)\Big)\, dx\, dt \le 0, \end{aligned}$$

we obtain from the last inequality that

$$2\int\limits_{Q_{t_1,\tau}} f(t)(v_\xi - w)\,dx\,dt \le$$

$$\int\limits_{t_1}^{\tau}\Big(\langle\Gamma(u^{(1)}),v_\xi-u^{(1)}\rangle+\langle\Gamma(u^{(2)}),v_\xi-u^{(2)}\rangle\Big)\,dt+\frac{1}{4}\int\limits_{\Omega}\Big(|u(t_1)|^2+a(x)\nabla u(t_1)\,\nabla u(t_1)\Big)\,dx$$

$$-\frac{1}{2}\int\limits_{\Omega}\Big(|v_\xi(\tau)-u^{(1)}(\tau)|^2+a(x)\nabla(v_\xi(\tau)-u^{(1)}(\tau))\,\nabla(v_\xi(\tau)-u^{(1)}(\tau))\Big)\,dx$$

$$-\frac{1}{2}\int\limits_{\Omega}\Big(|v_\xi(\tau)-u^{(2)}(\tau)|^2+a(x)\nabla(v_\xi(\tau)-u^{(2)}(\tau))\,\nabla(v_\xi(\tau)-u^{(2)}(\tau))\Big)\,dx.$$

We multiply the last inequality by $\psi_m(\tau)$, where $\psi_m \in C([t_1,t_2])$, $\int\limits_{t_1}^{t_2}\psi_m(t)dt = 1$, $\psi_m(t) \geqslant 0$ on $[t_1,t_2]$, supp $\psi_m \in [t_2-(t_2-t_1)/m, t_2]$, and integrate over $[t_1,t_2]$. Hereby, we obtain

$$2\int\limits_{t_1}^{t_2}\psi_m(\tau)\int\limits_{Q_{t_1,\tau}} f(t,x)\,(v_\xi-w)\,dx\,dt\,d\tau \;\le\; \frac{1}{4}\int\limits_{\Omega}\Big(|u(t_1)|^2+a(x)\nabla u(t_1)\,\nabla u(t_1)\Big)\,dx$$

$$+\int\limits_{t_1}^{t_2}\psi_m(\tau)\int\limits_{t_1}^{\tau}\Big(\langle\Gamma(u^{(1)}),v_\xi-u^{(1)}\rangle+\langle\Gamma(u^{(2)}),v_\xi-u^{(2)}\rangle\Big)\,dt\,d\tau$$

$$-\frac{1}{2}\int\limits_{t_1}^{t_2}\psi_m\int\limits_{\Omega}\Big(|v_\xi(\tau)-u^{(1)}(\tau)|^2+a(x)\nabla(v_\xi(\tau)-u^{(1)}(\tau))\,\nabla(v_\xi(\tau)-u^{(1)}(\tau))\Big)\,dx\,d\tau$$

$$-\frac{1}{2}\int\limits_{t_1}^{t_2}\psi_m\int\limits_{\Omega}\Big(|v_\xi(\tau)-u^{(2)}(\tau)|^2+a(x)\nabla(v_\xi(\tau)-u^{(2)}(\tau))\,\nabla(v_\xi(\tau)-u^{(2)}(\tau))\Big)\,dx\,d\tau.$$

Due to the weak convergence of $\{v_\xi\}$, passing in the last inequality to the limit as $\xi \to 0$ implies

$$\frac{1}{2}\int\limits_{t_1}^{t_2}\psi_m(\tau)\int\limits_{\Omega}\Big(|u|^2+a(x)\nabla u\,\nabla u\Big)\,dx\,\Big|_{t_1}^{\tau}\,d\tau$$

$$+\int\limits_{t_1}^{t_2}\psi_m(\tau)\int\limits_{t_1}^{\tau}\langle\Gamma(u^{(1)})-\Gamma(u^{(2)}),u^{(1)}-u^{(2)}\rangle\,dt\,d\tau \quad\le\quad 0.$$

Letting $m\to\infty$ in this inequality and using the strong monotonicity of d and g yields

$$\frac{1}{2}\int\limits_{\Omega}\Big(|u|^2+a(x)\nabla u\,\nabla u\Big)\,dx\,\Big|_{t_1}^{t_2}+\int\limits_{Q_{t_1,t_2}}\Big(|u|^p+|\nabla u|^p\Big)\,dx\,dt\;\le\;0.$$

The last inequality can be rewritten in the form

$$y^2(t)\Big|_{t_1}^{t_2}+\mu\int\limits_{t_1}^{t_2}y^p(t)\,dt\;\le\;0,$$

where

$$y^2(t) = \int_{\Omega} \Big(|u(t)|^2 + a(x) \nabla u(t) \, \nabla u(t) \Big) \, dx,$$

for all t_1, t_2, such that $-\infty < t_1 < t_2 \leq T$ and $t_1 \in J$. Then, due to Pankov's Lemma A.2.2, $y(t) = 0$ almost everywhere in $(-\infty, T]$. Therefore, $u = 0$ almost everywhere in Q_T and the uniqueness of the solution of the inequality (3.2.1) is proved. ❐

4

Nonlinear Pseudoparabolic Equations and Variational Inequalities

In the first section of this chapter it will be shown that a solution of a nonlinear pseudoparabolic equation can be obtained as a singular limit of solutions of quasilinear hyperbolic equations. If a system with cross diffusion, modeling the reaction and the diffusion of two biological, chemical, or physical substances, is reduced, such an hyperbolic equation is obtained. For regular solutions even uniqueness can be shown, though the needed regularity can only be proved in two dimensions.

The second section deals with existence and uniqueness of weak solutions of nonlinear pseudoparabolic variational inequalities. Again, uniqueness can only be shown for sufficiently regular solutions.

4.1 Nonlinear Pseudoparabolic Equations

We consider a reaction system with diffusion of one of the substances

$$\left\{ \begin{array}{lcl} \varepsilon\partial_t v & = & \nabla \cdot a(t,x,\nabla v) + \nabla \cdot (d(t,x)\nabla w) + \tilde{f}(t,x,w) - b(t,x,v), \\ \partial_t w & = & h(w)v, \end{array} \right.$$

where the function h may be of the form

$$h(w) = \frac{h_1 w}{w + h_2} + h_0.$$

This function satisfies

$$0 < h_0 \le h(w) \le h_1.$$

After a change to a new variable $u = H(w)$, where $H(w) = \int\limits_0^w \frac{1}{h(s)}\, ds$, we obtain

$$\partial_t u = v.$$

Hereby the system is reduced to the single equation

$$\varepsilon\, u_{tt} = \nabla \cdot a(t,x,\nabla u_t) + \nabla \cdot (d(t,x)h(u)\nabla u) + \tilde{f}(t,x,H^{-1}(u)) - b(t,x,u_t).$$

In this section we show at first the existence of a weak solution of the problem

$$\begin{cases} \varepsilon\, u_{tt} + b(t,x,u_t) - \nabla\cdot a(t,x,\nabla u_t) - \nabla\cdot(d(t,x)h(u)\nabla u) = f(t,x,u) & \text{in } Q_T, \\ u(0) = u_0 & \text{in } \Omega, \\ u_t(0) = 0 & \text{in } \Omega. \end{cases} \tag{4.1.1}$$

Secondly, we prove the convergence of the sequence of solutions $\{u^\varepsilon\}$ as $\varepsilon \to 0$ to a solution of the pseudoparabolic equation

$$\begin{cases} b(t,x,u_t) - \nabla\cdot a(t,x,\nabla u_t) - \nabla\cdot(d(t,x)h(u)\nabla u) = f(t,x,u) & \text{in } Q_T, \\ u(0) = u_0 & \text{in } \Omega. \end{cases} \tag{4.1.2}$$

Both initial value problems are completed by posing spatial boundary conditions. Herefore, we choose a closed subspace V_0, $H_0^1(\Omega) \subset V_0 \subset H^1(\Omega)$, densely and continuously embedded in $L^2(\Omega)$. The existence of a solution will be ensured by the following assumption.

Assumption 4.1.1.

$(A1)$ The function $b : (0,T)\times\Omega\times\mathbb{R} \to \mathbb{R}$ is measurable in t and x, continuous in ξ, elliptic, i.e. for some $b_0 > 0$, $b(t,x,\xi)\,\xi \geqslant b_0|\xi|^p$ for $\xi \in \mathbb{R}$, and strongly monotone, i.e. for some $b_1 > 0$, $(b(t,x,\xi_1) - b(t,x,\xi_2))\,(\xi_1 - \xi_2) \geqslant b_1|\xi_1 - \xi_2|^p$, for $\xi_1, \xi_2 \in \mathbb{R}$, $p \geqslant 2$, and satisfies a growth assumption, i.e. for some $b^0 < \infty$, $|b(t,x,\xi)| \leq b^0(1 + |\xi|^{p-1})$.

$(A2)$ The function $a : (0,T)\times\Omega\times\mathbb{R}^N \to \mathbb{R}^N$ is measurable in t and x, continuous in η, elliptic, i.e. for some $a_0 > 0$, $a(t,x,\eta)\,\eta \geqslant a_0|\eta|^2$ for $\eta \in \mathbb{R}^N$, and strongly monotone, i.e. for some $a_1 > 0$, $(a(t,x,\eta_1) - a(t,x,\eta_2))\,(\eta_1 - \eta_2) \geqslant a_1|\eta_1 - \eta_2|^2$ for $\eta_1, \eta_2 \in \mathbb{R}^N$, and satisfies a growth assumption, i.e. for some $a^0 < \infty$, $|a(t,x,\eta)| \leq a^0(1 + |\eta|)$ for $\eta \in \mathbb{R}^N$.

$(A3)$ The matrix field $d \in L^\infty(Q_T)^{N\times N}$, i.e. $|d(t,x)| \leq d_1$ for a. a. $(t,x) \in Q_T$, the function $h : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies $0 < h_0 \leq h(\xi) \leq h_1 < \infty$ for all $\xi \in \mathbb{R}$.

$(A4)$ The function $f : (0,T)\times\Omega\times\mathbb{R} \to \mathbb{R}$ is measurable in t and x, continuous in ξ, and sublinear, i.e. for some $c_1 < \infty$, $|f(t,x,\xi)| \leq c_1(1 + |\xi|^{p-1})$ for $\xi \in \mathbb{R}$.

$(A5)$ The initial condition u_0 is in V_0.

Now we define the notion of a weak solution of the problem (4.1.1).

Definition 4.1.2. A function $u : Q_T \to \mathbb{R}$ is called *a weak solution* of the problem (4.1.1) if

1) $\varepsilon u_t \in C([0,T];L^2(\Omega))$, $u_t \in L^p(Q_T) \cap L^2(0,T;V_0)$, $u \in C([0,T];V_0)$,

2) satisfies the initial condition, i.e. $u(t) \to u_0$ in V_0, $u_t(t) \to 0$ in $L^2(\Omega)$ for $t \to 0$, and

3) $$\int_{Q_T} \Big[-\varepsilon u_t\, v_t + b(t,x,u_t)\, v + a(t,x,\nabla u_t)\, \nabla v + d(t,x)h(u)\nabla u\, \nabla v\Big]\, dx\, dt$$
$$+\varepsilon \int_\Omega u_t(T)v(T)\, dx = \int_{Q_T} f(t,x,u)\, v\, dx\, dt \tag{4.1.3}$$

for all functions $v \in L^p(Q_T) \cap L^2(0,T;V_0)$, such that $v_t \in L^2(Q_T), v \in C([0,T];L^2(\Omega))$.

The existence of a solution of (4.1.1) is proved using Galerkin's method. Herefore, let $\{\phi^k\}_{k=1}^{\infty} \subset V_0 \cap L^p(\Omega)$ be a basis of the spaces V_0 and $L^p(\Omega)$. We consider the sequence of the functions $\{u^m\}$ of the form

$$u^m(t,x) = \sum_{k=1}^{m} z_k^m(t)\phi^{(k)}(x), \quad m = 1,2,...,$$

such that u^m is a solution of the Cauchy problem

$$\begin{aligned} &\varepsilon \int_\Omega u_{tt}^m \, \phi^{(k)} \, dx + \int_\Omega b(t,x,u_t^m) \, \phi^{(k)} \, dx + \int_\Omega a(t,x,\nabla u_t^m) \, \nabla \phi^{(k)} \, dx \\ &+ \int_\Omega d(t,x) h(u^m) \nabla u^m \, \nabla \phi^{(k)} \, dx = \int_\Omega f(t,x,u^m) \, \phi^{(k)} \, dx, \qquad (4.1.4) \\ &u^m(0,x) = u_0^m(x), \qquad u_t^m(0,x) = 0, \qquad (4.1.5) \end{aligned}$$

where $\{u_0^m\}$ is an approximation of u_0 in the space V_0. Due to the generalization of Peano's theorem for Carathéodory functions, (Deimling 1992), there exists a local solution of this problem in $[0, t_{0m}]$. The following lemma allows an extension of the solutions to the whole interval $[0, T]$.

Lemma 4.1.3. *The estimates*

$$\begin{aligned} &\varepsilon \int_\Omega |u_t^m(t)|^2 \, dx \le C, \qquad t \in [0, t_{0m}], \\ &\int_{Q_{t_{0m}}} |u_t^m|^p \, dx \, dt \le C, \qquad \int_{Q_{t_{0m}}} |\nabla u_t^m|^2 \, dx \, dt \le C \qquad (4.1.6) \end{aligned}$$

hold uniformly with respect to m and ε.

Proof. We multiply the equation (4.1.4) by z_{kt}^m, sum up over k from 1 to m, and integrate over $[0, \tau]$, where $0 < \tau \le t_{0m}$

$$\begin{aligned} &\int_{Q_\tau} \Big[\varepsilon u_{tt}^m \, u_t^m + b(t,x,u_t^m) \, u_t^m + a(t,x,\nabla u_t^m) \, \nabla u_t^m + d(t,x) \, h(u^m) \, \nabla u^m \, \nabla u_t^m \Big] \, dx \, dt \\ &= \int_{Q_\tau} f(t,x,u^m) \, u_t^m \, dx \, dt. \qquad (4.1.7) \end{aligned}$$

Now we estimate the integrals in (4.1.7) separately. Due to the second initial condition in (4.1.5) we obtain

$$I_1 = \varepsilon \int_{Q_\tau} u_{tt}^m \, u_t^m \, dx \, dt = \frac{\varepsilon}{2} \int_\Omega |u_t^m(\tau)|^2 \, dx.$$

Assumption 4.1.1 implies

$$
\begin{aligned}
I_2 &:= \int\limits_{Q_\tau} b(t,x,u_t^m)\, u_t^m \,dx\,dt \;\geqslant\; b_0 \int\limits_{Q_\tau} |u_t^m|^p \,dx\,dt,\\
I_3 &:= \int\limits_{Q_\tau} a(t,x,\nabla u_t^m)\, \nabla u_t^m \,dx\,dt \;\geqslant\; a_0 \int\limits_{Q_\tau} |\nabla u_t^m|^2 \,dx\,dt,\\
I_4 &:= \int\limits_{Q_\tau} d(t,x)h(u^m)\nabla u^m\, \nabla u_t^m \,dx\,dt \;\leq\; \frac{d_1^2 h_1^2}{2\delta} \int\limits_{Q_\tau} |\nabla u^m|^2 \,dx\,dt + \frac{\delta}{2} \int\limits_{Q_\tau} |\nabla u_t^m|^2 \,dx\,dt,\\
I_5 &:= \int\limits_{Q_\tau} f(t,x,u^m)\, u_t^m \,dx\,dt \;\leq\; \frac{\delta}{p} \int\limits_{Q_\tau} |u_t^m|^p \,dx\,dt + \frac{1}{q\delta^{q/p}} \int\limits_{Q_\tau} |f(t,x,u^m)|^q \,dx\,dt.
\end{aligned}
$$

Due to the assumption on f we have

$$
\int\limits_{Q_\tau} |f(t,x,u^m)|^q \,dx\,dt \;\leq\; c_1 \int\limits_{Q_\tau} |u^m|^p \,dx\,dt + c_2 \;\leq\; c_3 \int\limits_0^\tau \int\limits_{Q_t} |u_t^m|^p \,dx\,ds\,dt + c_4.
$$

Using the inequality

$$
\int\limits_{Q_\tau} |\nabla u^m|^2 \,dx\,dt \;\leq\; c_5 + \int\limits_0^\tau \int\limits_{Q_t} |\nabla u_t^m|^2 \,dx\,ds\,dt,
$$

the estimates of integrals $I_1, ..., I_5$, and Gronwall's lemma in the equation (4.1.7) implies the assertion. ❐

Remark 4.1.4. Since the constant C is independent of t_{0m}, the solution u^m may be assumed to be the maximal solution, i.e. the one that exists for all $t \in [0,T]$. Furthermore, since the estimates of the last Lemma are independent of m, they are satisfied by every u^m for all $t \in [0,T]$.

From the estimates for u_t^m we obtain the estimate for u^m. Due to (4.1.6) and $p \geqslant 2$, the second term on the right hand side of the equality

$$
\int\limits_\Omega \Big(|u^m|^2 + |\nabla u^m|^2\Big)\,dx = \int\limits_\Omega \Big(|u_0^m|^2 + |\nabla u_0^m|^2\Big)\,dx + 2\int\limits_{Q_\tau} \Big(u_t^m\, u^m \;+ \nabla u_t^m\, \nabla u^m\Big)\,dx\,dt,
$$

can be estimated by

$$
\begin{aligned}
2\int\limits_{Q_\tau} \Big(u_t^m\, u^m + \nabla u_t^m\, \nabla u^m\Big)\,dx\,dt \;&\leq\; \int\limits_{Q_\tau} \Big(|u_t^m|^2 + |\nabla u_t^m|^2\Big)\,dx\,dt + \int\limits_{Q_\tau} \Big(|u^m|^2 + |\nabla u^m|^2\Big)\,dx\,dt\\
&\leq\; c_1 + \int\limits_{Q_\tau} \Big(|u_t^m|^p + |\nabla u_t^m|^2\Big)\,dx\,dt + \int\limits_{Q_\tau} \Big(|u^m|^2 + |\nabla u^m|^2\Big)\,dx\,dt\\
&\leq\; c_2 + \int\limits_{Q_\tau} \Big(|u^m|^2 + |\nabla u^m|^2\Big)\,dx\,dt.
\end{aligned}
$$

Hence we obtain

$$\int_\Omega \left(|u^m|^2 + |\nabla u^m|^2\right) dx \;\le\; c_3 + \int_{Q_\tau} \left(|u^m|^2 + |\nabla u^m|^2\right) dx\, dt.$$

Gronwall's Lemma implies

$$\int_\Omega \left(|u^m|^2 + |\nabla u^m|^2\right) dx \;\le\; C.$$

Theorem 4.1.5 (Existence). *Let Assumption 4.1.1 be satisfied. Then there exists a weak solution u^ε of the problem (4.1.1).*

Proof. The growth assumptions on a and b imply

$$\Big|\int_{Q_T} b(t,x,u_t^m)\, v\, dx\, dt\Big| \;\le\; \int_{Q_T} b^0(1+|u_t^m|^{p-1})\,|v|\,dx\,dt \;\le\; C(1+||u_t^m||^{p/q}_{L^p(Q_T)})||v||_{L^p(Q_T)}$$

and

$$\Big|\int_{Q_T} a(t,x,\nabla u_t^m)\,\nabla v\, dx\, dt\Big| \;\le\; \int_{Q_T} a^0(1+|\nabla u_t^m|)|\nabla v|\,dx\,dt \;\le\; C(1+||u_t^m||_{L^2(0,T;V_0)})||v||_{L^2(0,T;V_0)}$$

for all $v \in L^p(Q_T)\cap L^2(0,T;V_0)$. From this and from the estimates (4.1.6) follows the existence of a subsequence of $\{u^m\}$, again denoted by $\{u^m\}$, such that

$$\begin{aligned}
u^m &\to u^\varepsilon && \text{weakly}-* \text{ in } L^\infty(0,T;V_0),\\
u_t^m &\to u_t^\varepsilon && \text{weakly in } L^p(Q_T)\cap L^2(0,T;V_0),\\
u_t^m &\to u_t^\varepsilon && \text{weakly}-* \text{ in } L^\infty(0,T;L^2(\Omega)),\\
b(t,x,u_t^m) &\to \beta^\varepsilon && \text{weakly in } L^q(Q_T),\\
a(t,x,\nabla u_t^m) &\to \eta^\varepsilon && \text{weakly in } L^2(Q_T)^N,
\end{aligned}$$

as $m \to \infty$. Using the Aubin-Lions Compactness Lemma, (Lions 1969), yields $u^m \to u^\varepsilon$ strongly in $L^2(Q_T)$. From the strong convergence of u^m follows $u^m \to u^\varepsilon$ a.e. in Q_T. The continuity of h and f implies $h(u^m) \to h(u^\varepsilon)$ and $f(t,x,u^m) \to f(t,x,u^\varepsilon)$ a.e. in Q_T. From the assumptions it follows that $h(u^m), h(u^\varepsilon) \in L^\infty(Q_T)$ and $f(t,x,u^m), f(t,x,u^\varepsilon) \in L^q(Q_T)$. Then by the Egorov Theorem (Alt 2002) we obtain $h(u^m) \to h(u^\varepsilon)$ uniformly a.e. in Q_T and by the Dominated Convergence Theorem (Evans 1998) we have that $f(t,x,u^m) \to f(t,x,u^\varepsilon)$ strongly in $L^q(Q_T)$.

All terms but the first of the equation (4.1.4) are uniformly bounded in $L^q(Q_T) + L^2(0,T;V_0^*)$ in m. Hence, there exists a bounded functional $w \in L^q(Q_T) + L^2(0,T;V_0^*)$ that satisfies

$$\varepsilon\langle w, \tilde v\rangle \;=\; \int_\Omega f(t,x,u^\varepsilon)\,\tilde v\, dx - \int_\Omega \beta^\varepsilon\,\tilde v\, dx - \int_\Omega \eta^\varepsilon\,\nabla\tilde v\, dx - \int_\Omega d(t,x)h(u^\varepsilon)\nabla u^\varepsilon\,\nabla\tilde v\, dx$$

in $L^q(0,T) + L^2(0,T)$ for all functions $\tilde v \in L^p(\Omega)\cap V_0$.

Since $u_t^m \to u_t^\varepsilon$ weakly in $L^p(Q_T)$, we obtain

$$\langle u_{tt}^m, \tilde{v}\rangle = \frac{d}{dt}\langle u_t^m, \tilde{v}\rangle \to \langle u_{tt}^\varepsilon, \tilde{v}\rangle \text{ in } \mathcal{D}'(0,T)$$

as $m \to \infty$ for $\tilde{v} \in L^p(\Omega)$. Hence, $w = u_{tt}^\varepsilon$ in $\mathcal{D}'(0,T,L^q(\Omega)+V_0^*)$. Since $w \in L^q(Q_T) + L^2(0,T;V_0^*)$ we may assume $u_{tt}^\varepsilon \in L^q(Q_T) + L^2(0,T;V_0^*)$. From the convergence of the u_t^m we have that $u_t^\varepsilon \in L^p(Q_T) \cap L^2(0,T;V_0)$. Thus, due to (Gajewski et al. 1974, Theorem IV.1.17) or (Showalter 1996, Proposition III.1.2), it may be assumed that $u_t^\varepsilon \in C([0,T];L^2(\Omega))$ and the integration by parts formula

$$\int_{t_1}^{t_2} \langle u_{tt}^\varepsilon, u_t^\varepsilon\rangle\, dt = \frac{1}{2}\int_\Omega |u_t^\varepsilon(t_2)|^2\, dx - \frac{1}{2}\int_\Omega |u_t^\varepsilon(t_1)|^2\, dx$$

holds for all $t_1, t_2 \in [0,T]$, such that $t_1 < t_2$. Now we will show that the function u^ε satisfies the initial condition. Since all u_t^m and u_t^ε are in $C([0,T];L^2(\Omega))$, and $u_t^m \to u_t^\varepsilon$ weakly–$*$ in $L^\infty(0,T,L^2(\Omega))$, we obtain

$$\int_\Omega u_t^m(0)\,\tilde{v}\,dx \to \int_\Omega u_t^\varepsilon(0)\,\tilde{v}\,dx \quad\text{and}\quad \int_\Omega u_t^m(T)\,\tilde{v}\,dx \to \int_\Omega u_t^\varepsilon(T)\,\tilde{v}\,dx,$$

as $m \to \infty$ for $\tilde{v} \in L^p(\Omega)$. Then we have $u_t^\varepsilon(0) = 0$ in $L^2(\Omega)$, because of $u_t^m(0) = 0$ in $L^2(\Omega)$. Since $u^\varepsilon \in L^\infty(0,T;V_0)$ and $u_t^\varepsilon \in L^2(0,T;V_0)$, due to (Evans 1998, Theorem 5.9.2), it may be assumed that $u^\varepsilon \in C([0,T];V_0)$ and

$$u^m(0) \to u^\varepsilon(0) \text{ strongly in } L^2(\Omega)$$

as $m \to \infty$. Thus, $u^\varepsilon(0) = u_0$.

Integrating in the equation (4.1.4) the first term by part, passing to the limit as $m \to \infty$ and using the fact, that the set of all functions of the form $\sum_{l<\infty} d_l\,\phi^l$, where $d_l \in C^1([0,T])$, is dense in $L^p(Q_T)$, in $L^2(0,T;V_0)$, in $C([0,T];L^2(\Omega))$, and in $H^1(0,T;L^2(\Omega))$, yields

$$\begin{aligned}
&-\varepsilon\int_{Q_T} u_t^\varepsilon\, v_t\,dx\,dt + \int_{Q_T} (\beta^\varepsilon\, v + \eta^\varepsilon\,\nabla v)\ dx\,dt + \int_{Q_T} d(t,x)\,h(u^\varepsilon)\,\nabla u^\varepsilon\,\nabla v\,dx\,dt \\
&+\varepsilon\int_\Omega u_t^\varepsilon(T)v(T)\,dx = \int_{Q_T} f(t,x,u^\varepsilon)\,v\,dx\,dt
\end{aligned}$$

for all functions $v \in L^p(Q_T) \cap L^2(0,T;V_0)$, such that $v_t \in L^2(Q_T)$ and $v \in C([0,T];L^2(\Omega))$.

To complete the proof we have to show that $\beta^\varepsilon = b(t,x,u_t^\varepsilon)$ and $\eta^\varepsilon = a(t,x,\nabla u_t^\varepsilon)$. For this we show the strong convergence of $\{u_t^m\}$ to u_t^ε in $L^p(Q_T) \cap L^2(0,T;V_0)$. We choose $u_t^m - u_t^\varepsilon$ as a test function in the equation (4.1.4), integrate this equation over $[0,\tau]$, and obtain

$$\begin{aligned}
&\varepsilon\int_0^\tau \langle u_{tt}^m, u_t^m - u_t^\varepsilon\rangle\,dt + \int_{Q_\tau} b(t,x,u_t^m)\,(u_t^m - u_t^\varepsilon)\,dx\,dt + \int_{Q_\tau} a(t,x,\nabla u_t^m)\,\nabla(u_t^m - u_t^\varepsilon)\,dx\,dt \\
&+\int_{Q_\tau} d(t,x)\,h(u^m)\,\nabla u^m\,\nabla(u_t^m - u_t^\varepsilon)\,dx\,dt = \int_{Q_\tau} f(t,x,u^m)\,(u_t^m - u_t^\varepsilon)\,dx\,dt.
\end{aligned}$$

By Fatou's lemma and weak convergence of u_{tt}^m in $L^q(Q_T) + L^2(0, T; V_0^*)$, we obtain for the first integral

$$\liminf_{m\to\infty} \int_0^\tau \langle u_{tt}^m, u_t^m - u_t^\varepsilon \rangle \, dt \;\geqslant\; \frac{1}{2} \liminf_{m\to\infty} \int_\Omega |u_t^m(\tau, x)|^2 \, dx - \frac{1}{2} \int_\Omega |u_t^\varepsilon(\tau, x)|^2 \, dx \;\geqslant\; 0.$$

Then, we have the inequality

$$\begin{aligned} &\int_{Q_\tau} (b(t,x,u_t^m) - b(t,x,u_t^\varepsilon))(u_t^m - u_t^\varepsilon) \, dx \, dt + \int_{Q_\tau} (a(t,x,\nabla u_t^m) - a(t,x,\nabla u_t^\varepsilon)) \nabla (u_t^m - u_t^\varepsilon) \, dx \, dt \\ \leq\; &\int_{Q_\tau} \Big[b(t,x,u_t^\varepsilon)\,(u_t^\varepsilon - u_t^m) + a(t,x,\nabla u_t^\varepsilon) \nabla (u_t^\varepsilon - u_t^m) \Big] \, dx \, dt + \int_{Q_\tau} d(t,x)\, h(u^m)\, \nabla u^\varepsilon \nabla (u_t^\varepsilon - u_t^m) \, dx \, dt \\ &+ \int_{Q_\tau} d(t,x)\, h(u^m)\, \nabla (u^m - u^\varepsilon) \nabla (u_t^\varepsilon - u_t^m) \, dx \, dt + \int_{Q_\tau} f(t,x,u^m)\,(u_t^m - u_t^\varepsilon) \, dx \, dt. \end{aligned}$$

Due to the weak convergence of $\{u_t^m\}$, the uniform convergence of $h(u^m) \to h(u^\varepsilon)$ a.e. in Q_T, $h(u^m)$, $h(u^\varepsilon) \in L^\infty(Q_T)$, and the strong convergence of $f(t,x,u^m) \to f(t,x,u^\varepsilon)$ in $L^q(Q_T)$, the first, the second, and the fourth integral on the right hand side converge to zero as $m \to \infty$. The third integral on the right hand side can be estimated by

$$\begin{aligned} &\int_{Q_\tau} d(t,x)\, h(u^m)\, \nabla (u^m - u^\varepsilon) \nabla (u_t^m - u_t^\varepsilon) \, dx \, dt \\ \leq\; &\frac{d_1^2 h_1^2}{2\delta} \int_{Q_\tau} |\nabla (u^m - u^\varepsilon)|^2 \, dx \, dt + \frac{\delta}{2} \int_{Q_\tau} |\nabla (u_t^m - u_t^\varepsilon)|^2 \, dx \, dt \\ \leq\; &c_1 \int_0^\tau \int_{Q_s} |\nabla u_t^m - \nabla u_t^\varepsilon|^2 \, dx \, dt \, ds + \frac{\delta}{2} \int_{Q_\tau} |\nabla (u_t^m - u_t^\varepsilon)|^2 \, dx \, dt. \end{aligned}$$

The monotonicity of b and a, and the convergence of $\{u^m\}$ and $\{u_t^m\}$ imply

$$b_1 \int_{Q_\tau} |u_t^m - u_t^\varepsilon|^p \, dxdt + (a_1 - \frac{\delta}{2}) \int_{Q_\tau} |\nabla (u_t^m - u_t^\varepsilon)|^2 \, dxdt \leq \sigma(\frac{1}{m}) + c_2 \int_0^\tau \int_{Q_s} |\nabla (u_t^m - u_t^\varepsilon)|^2 \, dxdtds.$$

Using Gronwall's lemma in the last inequality yields

$$||u_t^m - u_t^\varepsilon||_{L^p(Q_T)} + ||\nabla u_t^m - \nabla u_t^\varepsilon||_{L^2(Q_T)} \;\leq\; C\sigma(\frac{1}{m}).$$

Thus, $u_t^m \to u_t^\varepsilon$ strongly in $L^p(Q_T) \cap L^2(0, T; V_0)$ as $m \to \infty$. The strong convergence of $\{u_t^m\}$ and the weak convergence of $\{b(t,x,u_t^m)\}$ and $\{a(t,x,\nabla u_t^m)\}$ imply that $\beta^\varepsilon = b(t,x,u_t^\varepsilon)$ and $\eta^\varepsilon = a(t,x,\nabla u_t^\varepsilon)$, and the theorem is proved. ❒

Now we show that the sequence of solutions $\{u^\varepsilon\}$ converges as $\varepsilon \to 0$ to a solution of the initial boundary value problem for the nonlinear pseudoparabolic equation (4.1.2).

We consider a weak solution of the problem (4.1.2).

Definition 4.1.6. A function $u : Q_T \to \mathbb{R}$ is called *a weak solution* of the problem (4.1.2) if

1) $u \in C([0,T];V_0)$, $u_t \in L^p(Q_T) \cap L^2(0,T;V_0)$,
2) it satisfies the initial condition in the sense that $u(t) \to u_0$ in V_0 for $t \to 0$, and
3) $$\int\limits_{Q_T} \Big[b(t,x,u_t)\, v + a(t,x,\nabla u_t)\, \nabla v + d(t,x) h(u) \nabla u\, \nabla v \Big]\, dx\, dt = \int\limits_{Q_T} f(t,x,u)\, v\, dx\, dt \quad (4.1.8)$$
for all functions $v \in L^p(Q_T) \cap L^2(0,T;V_0)$.

Theorem 4.1.7 (Existence).
Let Assumption 4.1.1 be satisfied. Then, there exists a weak solution of the problem (4.1.2).

Proof. We rewrite the equation (4.1.3) for $v = u_t^\varepsilon$ and obtain

$$-\varepsilon \int\limits_{Q_T} u_t^\varepsilon\, u_t^\varepsilon\, dx\, dt + \int\limits_{Q_T} \Big[b(t,x,u_t^\varepsilon)\, u_t^\varepsilon + a(x,t,\nabla u_t^\varepsilon)\, \nabla u_t^\varepsilon + d(t,x) h(u^\varepsilon) \nabla u^\varepsilon\, \nabla u_t^\varepsilon \Big]\, dx\, dt$$
$$+\varepsilon \int\limits_{\Omega} u_t^\varepsilon(T)\, u_t^\varepsilon(T)\, dx = \int\limits_{Q_T} f(x,t,u^\varepsilon)\, u_t^\varepsilon\, dx\, dt. \quad (4.1.9)$$

We estimate all integrals in (4.1.9) analogously to (4.1.7) and have

$$\begin{aligned} \varepsilon \int\limits_{\Omega} |u_t^\varepsilon(t)|^2\, dx &\leq C, \quad t \in [0,T], \\ \int\limits_{Q_T} |u_t^\varepsilon|^p\, dx\, dt &\leq C, \\ \int\limits_{Q_T} |\nabla u_t^\varepsilon|^2\, dx\, dt &\leq C, \end{aligned}$$

where C is independent of ε. Due to the growth assumptions on b and a, and estimates for u_t^ε, we obtain

$$\begin{aligned} ||b(t,x,u_t^\varepsilon)||_{L^q(Q_T)} &\leq C, \\ ||a(t,x,\nabla u_t^\varepsilon)||_{L^2(Q_T)^N} &\leq C. \end{aligned}$$

Similarly as for u^m we obtain

$$\int\limits_{\Omega} \Big(|u^\varepsilon(t)|^2 + |\nabla u^\varepsilon(t)|^2 \Big)\, dx \leq C, \quad t \in [0,T].$$

Then there exists a subsequence of $\{u^\varepsilon\}$, again denoted by $\{u^\varepsilon\}$, such that

$$\begin{aligned} u^\varepsilon \to u \quad & \text{weakly-}* \text{ in } L^\infty(0,T;V_0), \\ u_t^\varepsilon \to u_t \quad & \text{weakly in } L^p(Q_T) \cap L^2(0,T;V_0), \\ b(t,x,u_t^\varepsilon) \to \beta \quad & \text{weakly in } L^q(Q_T), \\ a(t,x,\nabla u_t^\varepsilon) \to \eta \quad & \text{weakly in } L^2(Q_T)^N, \\ \varepsilon u_t^\varepsilon \to 0 \quad & \text{weakly in } L^2(0,T;L^2(\Omega)), \\ \varepsilon u_t^\varepsilon(\cdot,T) \to 0 \quad & \text{weakly in } L^2(\Omega), \end{aligned}$$

as $\varepsilon \to 0$. Using the Aubin-Lions Compactness Lemma, (Lions 1969), yields $u^\varepsilon \to u$ strongly in $L^2(Q_T)$. From the strong convergence of u^ε follows $u^\varepsilon \to u$ a.e. in Q_T. The continuity of h and f implies $h(u^\varepsilon) \to h(u)$ and $f(t,x,u^\varepsilon) \to f(t,x,u)$ a.e. in Q_T. From the assumptions it follows that $h(u^\varepsilon), h(u) \in L^\infty(Q_T)$ and $f(t,x,u^\varepsilon), f(t,x,u) \in L^q(Q_T)$. Then by the Egorov Theorem (Alt 2002) we obtain $h(u^\varepsilon) \to h(u)$ uniformly a.e. in Q_T and by the Dominated Convergence Theorem (Evans 1998) we have that $f(t,x,u^\varepsilon) \to f(t,x,u)$ strongly in $L^q(Q_T)$. Passing to the limit as $\varepsilon \to 0$ in (4.1.3) yields

$$\int_{Q_T} \Big(\beta\, v + \eta\, \nabla v\Big)\, dx\, dt + \int_{Q_T} d(t,x) h(u) \nabla u\, \nabla v\, dx\, dt = \int_{Q_T} f(t,x,u)\, v\, dx\, dt$$

for all $v \in L^p(Q_T) \cap L^2(0,T;V_0)$. Similarly as for $\{u_t^m\}$ we prove the strong convergence of $\{u_t^\varepsilon\}$ and obtain $\beta = b(t,x,u_t)$, $\eta = a(t,x,\nabla u_t)$. Using $u \in L^\infty(0,T;V_0)$, $u_t \in L^2(0,T;V_0)$, and (Evans 1998, Theorem 5.9.2), implies that $u : [0,T] \to V_0$ is continuous. Due to $u^\varepsilon(0) = u_0$, we obtain $u(0) = u_0$ in V_0. Thus, u is a solution of the problem (4.1.2). ❐

Theorem 4.1.8 (Regularity). *Let Assumption 4.1.1 be satisfied, Ω be a C^2-domain, $V_0 = H_0^1(\Omega)$, $u_0 \in H^2(\Omega)$, $a(t,\cdot,\cdot) \in C^1(\Omega \times \mathbb{R}^N)$, $d(t,\cdot) \in C^1(\Omega)^{N\times N}$ for $t \in (0,T)$, $h \in C^1(\mathbb{R})$, $N = 2$, $p = 2$, and for $\eta \in \mathbb{R}^N$, $\xi \in \mathbb{R}$*

$$\begin{array}{lclclcl} |\partial_\eta a(t,x,\eta)| & \leq & C, & |\nabla_x a(t,x,\eta)| & \leq & a_2(1+|\eta|), \\ |\partial_\xi h(\xi)| & \leq & C, & |\nabla_x d(t,x)| & \leq & C. \end{array}$$

Then the solution u^ε of the problem (4.1.1) is in $H^1(0,T;H_0^1(\Omega))$, in $H^1(0,T;H^2(\Omega))$, and satisfies $\varepsilon\, u_{tt}^\varepsilon \in L^2(Q_T)$.

Proof. First we show the local regularity. We fix any open set U, and choose an open set W, such that $U \subset\subset W \subset\subset \Omega$. We choose the basis functions ϕ^k as solutions of

$$\begin{array}{rcll} \Delta\phi^k & = & \lambda\phi^k & \text{in } \Omega, \\ \phi^k & = & 0 & \text{on } \partial\Omega. \end{array}$$

We choose $v = -\partial_{x_l}(\zeta_1^2 \partial_{x_l} u_t^m)$ as a test function in the equation (4.1.4), where the cut-off function ζ_1 is smooth and satisfies

$$\begin{cases} \zeta_1 = 1 & \text{in } \quad U, \\ \zeta_1 = 0 & \text{in } \quad \Omega \setminus W, \\ 0 \leq \zeta_1 \leq 1, & \end{cases}$$

and integrate over t. Due to the regularity of ϕ^k, we have that $v \in L^2(0,T;H_0^1(\Omega))$. Integrating by parts and summing over l from 1 to N implies

$$\varepsilon \int_{Q_T} \nabla u_{tt}^m \, \nabla u_t^m \, \zeta_1^2 \, dx\, dt - \sum_{l=1}^N \int_{Q_T} b(t,x,u_t^m) \, \partial_{x_l}(\zeta_1^2 \partial_{x_l} u_t^m) \, dx\, dt$$
$$+ \sum_{l=1}^N \sum_{i,j=1}^N \int_{Q_T} \partial_{\eta_j} a^i(t,x,\nabla u_t^m) \partial_{x_j} \partial_{x_l} u_t^m \, \partial_{x_i}(\zeta_1^2 \partial_{x_l} u_t^m) \, dx\, dt + \sum_{l=1}^N \int_{Q_T} \partial_{x_l} a(t,x,\nabla u_t^m) \nabla(\zeta_1^2 \partial_{x_l} u_t^m) \, dx\, dt$$
$$+ \sum_{l=1}^N \int_{Q_T} \partial_{x_l}(d(t,x) h(u^m) \nabla u^m) \, \nabla(\zeta_1^2 \partial_{x_l} u_t^m) \, dx\, dt = \sum_{l=1}^N \int_{Q_T} f(t,x,u^m) \, \partial_{x_l}(\zeta_1^2 \partial_{x_l} u_t^m) \, dx\, dt. \quad (4.1.10)$$

The strong monotonicity of a implies

$$\frac{1}{\sigma}\Big(a(t,x,\tilde{\eta}+\sigma\xi) - a(\tilde{\eta})\Big)\xi \geqslant a_1 |\xi|^2$$

for $\eta_1 = \tilde{\eta} + \sigma\xi$, $\sigma > 0$, and $\eta_2 = \tilde{\eta}$. Taking the limit as $\sigma \to 0$ yields

$$\nabla_\eta a(t,x,\tilde{\eta})\xi\,\xi \;\geqslant\; a_1|\xi|^2 \;\text{ for }\; \tilde{\eta}, \xi \in \mathbb{R}^N.$$

Then we have the estimate

$$\sum_{l=1}^N \sum_{i,j=1}^N \int_{Q_T} \partial_{\eta_j} a^i(t,x,\nabla u_t^m) \partial_{x_j} \partial_{x_l} u_t^m \, \partial_{x_i} \partial_{x_l} u_t^m \, \zeta_1^2 \, dx\, dt \;\geqslant\; a_1 \sum_{l=1}^N \sum_{i=1}^N \int_{Q_T} |\partial_{x_i} \partial_{x_l} u_t^m|^2 \, \zeta_1^2 \, dx\, dt.$$

From the equation (4.1.10), using Young's inequality, we obtain

$$\begin{aligned}
&\frac{\varepsilon}{2} \int_\Omega |\nabla u_t^m(T)|^2 \, \zeta_1^2 \, dx + a_1 \int_{Q_T} |\nabla^2 u_t^m|^2 \, \zeta_1^2 \, dx\, dt \\
\leq\;& \delta_0 \int_{Q_T} |\nabla^2 u_t^m|^2 \, \zeta_1^2 \, dx\, dt + \delta_0 \int_{Q_T} |\nabla_\eta a(t,x,\nabla u_t^m)|^2 |\nabla^2 u_t^m|^2 \, \zeta_1^2 \, dx\, dt + c_1(\delta_0) \int_{Q_T} |\zeta_1|^2 |\nabla \zeta_1|^2 |\nabla u_t^m|^2 \, dx\, dt \\
&+ c_2(\delta_0) \int_{Q_T} |\nabla \zeta_1|^2 |\nabla u_t^m|^2 \, dx\, dt + c_3(\delta_0) \int_{Q_T} |\nabla_x a(t,x,\nabla u_t^m)|^2 \, dx\, dt + c_4(\delta_0) \int_{Q_T} |b(t,x,u_t^m)|^2 \, dx\, dt \\
&+ c_5(\delta_0) \int_{Q_T} |\nabla d(t,x)|^2 |h(u^m)|^2 |\nabla u^m|^2 \, dx\, dt + c_6(\delta_0) \int_{Q_T} |d(t,x)|^2 |h(u^m)|^2 |\nabla^2 u^m|^2 \, \zeta_1^2 \, dx\, dt \\
&+ c_7(\delta_0) \int_{Q_T} |d(t,x)|^2 |\partial_\xi h(u^m)|^2 |\nabla u^m|^4 \, \zeta_1^2 \, dx\, dt + c_8(\delta_0) \int_{Q_T} |f(t,x,u_t^m)|^2 \, dx\, dt. \qquad (4.1.11)
\end{aligned}$$

For $N = 2$, due to the embedding theorem, we have $\nabla u^m \in L^4(Q_T)$ and the Gagliardo-Nirenberg inequality

$$\int_{Q_T} |\nabla u^m|^4 \, \zeta_1^2 \, dx\, dt \;\leq\; C\Big(\int_{Q_T} |\nabla^2 u^m|^2 \, \zeta_1^2 \, dx\, dt + \int_{Q_T} |\nabla u^m|^2 \, \zeta_1^2 \, |\nabla \zeta_1|^2 \, dx\, dt\Big) \int_{Q_T} |\nabla u^m|^2 \, dx\, dt.$$

The estimate for ∇u^m and the assumption $u_0 \in H^2(\Omega)$ imply

$$\int_{Q_T} |\nabla u^m|^4 \,\zeta_1^2 \,dx\,dt \;\le\; c_1 \int_{Q_T} |\nabla^2 u^m|^2 \,\zeta_1^2 \,dx\,dt + c_2 \;\le\; c_3 + c_4 \int_0^T \int_{Q_t} |\nabla^2 u_\tau^m|^2 \zeta_1^2 \,dx\,d\tau dt.$$

Due to the estimates (4.1.6) for u_t^m and the assumptions in the theorem, we obtain from (4.1.11) the inequality

$$\int_{Q_T} \zeta_1^2 |\nabla^2 u_t^m|^2 \,dx\,dt \le C_1 + C_2 \int_0^T \int_{Q_t} \zeta_1^2 |\nabla^2 u_\tau^m|^2 \,dx\,d\tau dt.$$

Then Gronwall's lemma implies the estimate

$$\int_{Q_T} \zeta_1^2 |\nabla^2 u_t^m|^2 \,dx\,dt \;\le\; C. \tag{4.1.12}$$

From (4.1.11) we obtain also

$$\sup_{0 \le t \le T} \int_{\Omega} |\nabla u_t^m|^2 \,\zeta_1^2 \,dx \;\le\; C.$$

Using these extra estimates in the proof of Theorem 4.1.5 yields a subsequence and a limit-function $u^\varepsilon \in H^1(0,T;H_0^1(\Omega))$, with satisfies $u_t^\varepsilon \in L^2(0,T;H^2_{\mathrm{loc}}(\Omega))$ and $u_t^\varepsilon \in L^\infty(0,T;H^1_{\mathrm{loc}}(\Omega))$ also.

To show the regularity of u^ε up to the boundary we need an estimate for $\nabla^2 u_t^m$ close to $\partial\Omega$. Here we use $\phi^k = 0$ and $\Delta\phi^k = 0$ on $\partial\Omega$.

We can cover $\partial\Omega$ with a finite number of balls, due to compactness. In local coordinates near the boundary Ω is of the form $B_1(0) \cap \{\mathbb{R} \times \mathbb{R}_+\}$. Hence, we consider the case $\Omega = B_1(0) \cap \{\mathbb{R} \times \mathbb{R}_+\}$ first. We choose $v = -\partial_{x_1}(\zeta^2 \partial_{x_1} u_t^m)$ as a test function in the equation (4.1.4), where the smooth cut-off function ζ is defined by

$$\begin{cases} \zeta = 1 & \text{in} \quad B_{\frac{1}{2}}(0), \\ \zeta = 0 & \text{in} \quad \mathbb{R}^2 \setminus B_1(0), \\ 0 \le \zeta \le 1, \end{cases}$$

and ζ vanishes near the curved part of $\partial\Omega$. Integrating over t and integrating by parts imply

$$\begin{aligned}
&\varepsilon \int\limits_{Q_T} \partial_{x_1} u^m_{tt}\, \partial_{x_1} u^m_t\, \zeta^2\, dx\, dt - \int\limits_{Q_T} b(t,x,u^m_t)\, (\partial^2_{x_1} u^m_t\, \zeta^2 + 2\zeta \partial_{x_1}\zeta\, \partial_{x_1} u^m_t)\, dx\, dt \\
&+ \int\limits_{Q_T} \nabla_\eta a(t,x,\nabla u^m_t) \partial_{x_1} \nabla u^m_t\, \partial_{x_1} \nabla u^m_t\, \zeta^2\, dx\, dt + 2\int\limits_{Q_T} \nabla_\eta a(t,x,\nabla u^m_t)\partial_{x_1}\nabla u^m_t\, \zeta \nabla\zeta\, \partial_{x_1} u^m_t\, dx\, dt \\
&+ \int\limits_{Q_T} \partial_{x_1} a(t,x,\nabla u^m_t)\partial_{x_1}\nabla u^m_t\, \zeta^2\, dx\, dt + 2\int\limits_{Q_T} \partial_{x_1} a(t,x,\nabla u^m_t)\zeta\nabla\zeta\, \partial_{x_1} u^m_t\, dx\, dt \\
&+ \int\limits_{Q_T} \partial_{x_1}(d(t,x)h(u^m)\nabla u^m)\, \partial_{x_1}\nabla u^m_t\, \zeta^2\, dx\, dt + 2\int\limits_{Q_T} \partial_{x_1}(d(t,x)h(u^m)\nabla u^m)\, \zeta\nabla\zeta\, \partial_{x_1} u^m_t\, dx\, dt \\
&= \int\limits_{Q_T} f(t,x,u^m)\, (\partial^2_{x_1} u^m_t\, \zeta^2 + 2\zeta\partial_{x_1}\zeta\, \partial_{x_1} u^m_t)\, dx\, dt. \qquad (4.1.13)
\end{aligned}$$

The boundary integrals vanish, since ζ vanishes near the curved part of $\partial\Omega$, and $\partial_{x_1} u^m = 0$ and $\partial^2_{x_1} u^m = 0$ vanish on $\{x_2 = 0\}$, because $u^m = \sum_{k=1}^m c^k_m \phi^k$ is zero on $\{x_2 = 0\}$ and the normal vector to this part of the boundary is $\nu = (0,-1)$. Furthermore, $v \in L^2(0,T; H^1_0(\Omega))$.

From the strong monotonicity of a we have the estimate

$$\int\limits_{Q_T} \nabla_\eta a(t,x,\nabla u^m_t)\nabla\partial_{x_1} u^m_t\, \nabla\partial_{x_1} u^m_t\, \zeta^2\, dx\, dt \;\geqslant\; a_1 \int\limits_{Q_T} |\nabla\partial_{x_1} u^m_t|^2\, \zeta^2\, dx\, dt.$$

Then from the equation (4.1.13) by using Young's inequality we obtain

$$\begin{aligned}
&\frac{\varepsilon}{2}\int\limits_{\Omega} |\partial_{x_1} u^m_t(T)|^2\, \zeta^2\, dx + a_1 \int\limits_{Q_T} |\partial_{x_1}\nabla u^m_t|^2\, \zeta^2\, dx\, dt \\
&\le \delta_0 \int\limits_{Q_T} |\partial_{x_1}\nabla u^m_t|^2\, \zeta^2\, dx\, dt + c_1(\delta_0)\int\limits_{Q_T} |\partial_{x_1} a(t,x,\nabla u^m_t)|^2\, dx\, dt + c_2(\delta_0)\int\limits_{Q_T} |b(t,x,u^m_t)|^2\, dx\, dt \\
&+ c_3(\delta_0)\int\limits_{Q_T} |\partial_{x_1} d(t,x)|^2 |h(u^m)|^2 |\nabla u^m|^2\, dx\, dt + c_4(\delta_0)\int\limits_{Q_T} |d(t,x)|^2 |h(u^m)|^2 |\partial_{x_1}\nabla u^m|^2\, \zeta^2\, dx\, dt \\
&+ c_5(\delta_0)\int\limits_{Q_T} |d(t,x)|^2 |\partial_\xi h(u^m)|^2 |\nabla u^m|^4\, \zeta^2\, dx\, dt + \int\limits_{Q_T} |f(t,x,u^m_t)|^2\, dx\, dt \\
&+ c_1 \int\limits_{Q_T} \zeta^2 |\nabla\zeta|^2 |\nabla u^m_t|^2\, dx\, dt + c_2\int\limits_{Q_T} |\nabla\zeta|^2 |\nabla u^m_t|^2\, dx\, dt. \qquad (4.1.14)
\end{aligned}$$

Using again the inequality

$$\int\limits_{Q_T} |\nabla u^m|^4\, \zeta^2\, dx\, dt \;\le\; C\Big(\int\limits_{Q_T} |\nabla^2 u^m|^2\, \zeta^2\, dx\, dt + \int\limits_{Q_T} |\nabla u^m|^2\, \zeta^2\, |\nabla\zeta|^2\, dx\, dt\Big) \int\limits_{Q_T} |\nabla u^m|^2\, dx\, dt,$$

the estimate for ∇u^m, and the assumption $u_0 \in H^2(\Omega)$ implies

$$\int\limits_{Q_T} |\nabla u^m|^4\, \zeta^2\, dx\, dt \;\le\; c_1 \int\limits_{Q_T} |\nabla^2 u^m|^2\, \zeta^2\, dx\, dt + c_2 \;\le\; c_3 + c_4 \int_0^T \int\limits_{Q_t} |\nabla^2 u^m_\tau|^2\, \zeta^2\, dx\, d\tau\, dt.$$

Due to the estimates (4.1.6) for u_t^m and the assumptions in the theorem, we obtain from (4.1.14) the inequality

$$\int\limits_{Q_T} |\partial_{x_1} \nabla u_t^m|^2 \zeta^2 \, dx\, dt \le C_1 + \int\limits_0^T \int\limits_{Q_t} |\partial_{x_2}^2 u_\tau^m|^2 \zeta^2 \, dx\, d\tau\, dt + C_2 \int\limits_0^T \int\limits_{Q_t} |\partial_{x_1} \nabla u_\tau^m|^2 \zeta^2 \, dx\, d\tau\, dt.$$

Then Gronwall's lemma implies the estimate

$$\int\limits_{Q_T} |\partial_{x_1} \nabla u_t^m|^2 \zeta^2 \, dx\, dt \;\le\; C\Big(1 + \int\limits_0^T \int\limits_{Q_t} |\partial_{x_2}^2 u_\tau^m|^2 \zeta^2 \, dx\, d\tau dt\Big). \tag{4.1.15}$$

Now we choose $v = -\partial_{x_2}(\zeta^2 \partial_{x_2} u_t^m)$ as a test function in the equation (4.1.4) and obtain

$$\begin{aligned}
&-\varepsilon \int\limits_{Q_T} u_{tt}^m \, \partial_{x_2}(\zeta^2 \partial_{x_2} u_t^m) \; dx\, dt - \int\limits_{Q_T} b(t,x,u_t^m)\, \partial_{x_2}(\zeta^2 \partial_{x_2} u_t^m)\, dx\, dt \\
&- \int\limits_{Q_T} a(t,x,\nabla u_t^m)\, \nabla \partial_{x_2}(\zeta^2 \partial_{x_2} u_t^m)\, dx\, dt - \int\limits_{Q_T} d(t,x) h(u^m) \nabla u^m \, \nabla \partial_{x_2}(\zeta^2 \partial_{x_2} u_t^m)\, dx\, dt \\
&= - \int\limits_{Q_T} f(t,x,u^m)\, \partial_{x_2}(\zeta^2 \partial_{x_2} u_t^m)\, dx\, dt.
\end{aligned}$$

Integrating by parts implies

$$\begin{aligned}
&\varepsilon \int\limits_{Q_T} \partial_{x_2} u_{tt}^m \, \partial_{x_2} u_t^m \, \zeta^2 \, dx\, dt - \int\limits_{Q_T} b(t,x,u_t^m)\, (\partial_{x_2}^2 u_t^m \zeta^2 + 2\zeta \partial_{x_2} \zeta \partial_{x_2} u_t^m)\, dx\, dt \\
&+ \int\limits_{Q_T} \nabla \cdot a(t,x,\nabla u_t^m)\, \partial_{x_2}^2 u_t^m \, \zeta^2 \, dx\, dt + 2 \int\limits_{Q_T} \nabla \cdot a(t,x,\nabla u_t^m)\, \zeta \partial_{x_2} \zeta \partial_{x_2} u_t^m \, dx\, dt \\
&+ \int\limits_{Q_T} \nabla \cdot (d(t,x) h(u^m) \nabla u^m)\, \partial_{x_2}^2 u_t^m \, \zeta^2 \, dx\, dt + 2 \int\limits_{Q_T} \nabla \cdot (d(t,x) h(u^m) \nabla u^m)\, \zeta \partial_{x_2} \zeta \partial_{x_2} u_t^m \, dx\, dt \\
&= - \int\limits_{Q_T} f(t,x,u^m)\, (\partial_{x_2}^2 u_t^m \zeta^2 + 2\zeta \partial_{x_2} \zeta \partial_{x_2} u_t^m)\, dx\, dt.
\end{aligned}$$

The boundary integrals vanish, since $\Delta u^m = 0$ and $\partial_{x_1}^2 u^m = 0$ on $\{x_2 = 0\}$ implies that $\partial_{x_2}^2 u^m = 0$ on $\{x_2 = 0\}$, and $\partial_{x_2} \zeta = 0$ on $\{x_2 = 0\}$, and ζ vanishes near the curved part of $\partial\Omega$. Furthermore, $v \in L^2(0,T; H_0^1(\Omega))$. In the last equality we use Young's inequality and obtain

$$\frac{\varepsilon}{2}\int\limits_{\Omega} |\partial_{x_2} u_t^m(T)|^2\,\zeta^2\,dx + \int\limits_{Q_T} \partial_{\eta_2} a^2 |\partial^2_{x_2} u_t^m|^2\,\zeta^2\,dx\,dt \;\le\; \delta \int\limits_{Q_T} |\partial^2_{x_2} u_t^m|^2\,\zeta^2\,dx\,dt$$
$$+ \int\limits_{Q_T} |\partial_{\eta_1} a^2|^2 |\partial_{x_2}\partial_{x_1} u_t^m|^2\,\zeta^2\,dx\,dt + \int\limits_{Q_T} |\partial_{\eta_1} a^1|^2 |\partial^2_{x_1} u_t^m|^2\,\zeta^2\,dx\,dt + \int\limits_{Q_T} |\partial_{\eta_2} a^1|^2 |\partial_{x_1}\partial_{x_2} u_t^m|^2\,\zeta^2\,dx\,dt$$
$$+ \int\limits_{Q_T} |d_{22}(t,x)|^2 |h(u^m)|^2 |\partial^2_{x_2} u^m|^2\,\zeta^2\,dx\,dt + \int\limits_{Q_T} |d(t,x)|^2 |h(u^m)|^2 |\partial_{x_1}\nabla u^m|^2\,\zeta^2\,dx\,dt$$
$$+ \int\limits_{Q_T} |\nabla d(t,x)|^2 |h(u^m)|^2 |\nabla u^m|^2\,dx\,dt + \int\limits_{Q_T} |\nabla u_t^m|^2\,|\partial_{x_2}\zeta|^2\,dx\,dt$$
$$+ \int\limits_{Q_T} |d(t,x)|^2 |\partial_\xi h(u^m)|^2 |\nabla u^m|^4\,\zeta^2\,dx\,dt + c(\delta)\int\limits_{Q_T} (|b(t,x,u_t^m)|^2 + |f(t,x,u^m)|^2)\,dx\,dt.$$

From the strong monotonicity of a for $\xi = (0,1)$ we obtain that $a^2_{\eta_2} \geqslant a_1$. Now, due to the inequality

$$\int\limits_{Q_T} |\nabla u^m|^4\,\zeta^2\,dx\,dt \le C + \int\limits_0^T\int\limits_{Q_t} |\partial_{x_1}\nabla u_\tau^m|^2\,\zeta^2\,dx\,d\tau\,dt + \int\limits_0^T\int\limits_{Q_t} |\partial^2_{x_2} u_\tau^m|^2\,\zeta^2\,dx\,d\tau\,dt,$$

the estimate (4.1.15), and the estimates for u^m, yields

$$\int\limits_{Q_T} |\partial^2_{x_2} u_t^m|^2\,\zeta^2\,dx\,dt \quad\le\quad C_1 + C_2\int\limits_0^T\int\limits_{Q_t} |\partial^2_{x_2} u_\tau^m|^2\,\zeta^2\,dx\,d\tau\,dt.$$

The Gronwall lemma implies

$$\int\limits_{Q_T} |\partial^2_{x_2} u_t^m|^2\,\zeta^2\,dx\,dt \quad\le\quad C.$$

From this and (4.1.15) it follows that

$$\int\limits_{Q_T} |\nabla^2 u_t^m|^2\,\zeta^2\,dx\,dt \quad\le\quad C.$$

Using this estimate and the local estimate (4.1.12) in the proof of Theorem 4.1.5 yields that $u_t^\varepsilon \in L^2(0,T;H_0^1(\Omega))$, $u_t^\varepsilon \in L^2(0,T;H^2(\Omega))$, and $\varepsilon\, u_t^\varepsilon \in L^\infty(0,T;H_0^1(\Omega))$.

From $u_t^\varepsilon \in L^2(0,T;H^2(\Omega))$ and the equation (4.1.1) it follows that $\varepsilon\, u_{tt}^\varepsilon \in L^2(Q_T)$.

All the above calculations are true for a general C^2 domain: for any point $x^0 \in \partial\Omega$, since $\partial\Omega$ is C^2, we may assume that

$$\Omega \cap B(x^0,r) = \{x \in B(x^0,r),\ x_N > \gamma(x_1,\dots,x_{N-1})\}$$

for some $r > 0$ and some C^2 function $\gamma : \mathbb{R}^{N-1} \to \mathbb{R}$. We change variables to $y = \Phi(x)$, $x = \Psi(y)$ and choose $s > 0$ so small that the half-ball $\Omega' := B(0,s) \cap \{y_N > 0\}$ lies in

$\Phi(\Omega \cap B(x^0, r))$. Define $\bar{u}^\varepsilon(t, y) := u^\varepsilon(t, \Psi(y))$ for $(t, y) \in (0, T) \times \Omega'$. Since $\nabla\Phi = (\nabla\Psi)^{-1}$, $|\det \nabla\Phi| = 1$, and $\Phi, \Psi \in C^2$, we obtain that $\bar{u}^\varepsilon$ is a solution of the equation

$$\varepsilon\, \bar{u}^\varepsilon_{tt} + \bar{b}(t, y, \bar{u}^\varepsilon_t) - \nabla \cdot \bar{a}(t, y, \nabla \bar{u}^\varepsilon_t) - \nabla \cdot (\bar{d}(t, y)\bar{h}(\bar{u}^\varepsilon)\nabla \bar{u}^\varepsilon) = \bar{f}(t, y, \bar{u}^\varepsilon)$$

for

$$\begin{aligned} &\bar{f}(t, y, \bar{u}^\varepsilon(t, y)) = f(t, \Psi(y), u^\varepsilon(t, \Psi(y))), && \bar{b}(t, y, \bar{u}^\varepsilon_t(t, y)) = b(t, \Psi(y), u^\varepsilon_t(t, \Psi(y))),\\ &\bar{h}(\bar{u}^\varepsilon(t, y)) = h(u^\varepsilon(t, \Psi(y))), && \bar{d}(t, y) = d(t, \Psi(y))\nabla_x\Phi(\Psi(y))\nabla_x\Phi(\Psi(y)),\\ &\bar{a}(t, y, \nabla \bar{u}^\varepsilon_t(t, y)) = a(t, \Psi(y), \nabla_x\Phi(\Psi(y))\nabla u^\varepsilon_t(t, \Psi(y)))\, \nabla_x\Phi(\Psi(y)). \end{aligned}$$

The ellipticity, boundedness, and regularity assumptions for a, b, d, h, and f translate into assumptions for $\bar{a}$, $\bar{b}$, $\bar{d}$, $\bar{h}$, and $\bar{f}$. From the calculations above we obtain the estimate for $\bar{u}^\varepsilon$ and consequently for u^ε. ❐

By using the regularity of u^ε we prove the regularity of a solution of the pseudoparabolic equation.

Theorem 4.1.9 (Regularity). *Let the assumptions of Theorem 4.1.8 be satisfied. Then the solution of the problem (4.1.2) is in $H^1(0, T; H^1_0(\Omega))$ and in $H^1(0, T; H^2(\Omega))$.*

Proof. For the proof of the local regularity we choose $v = -\nabla \cdot (\zeta_1^2 D^\sigma u^\varepsilon_t)$ as a test function in the equation (4.1.3), where $D^\sigma_i v(x) = \frac{1}{h}(u(x + he_i) - u(x))$, $i = 1, \dots N$, and $D^\sigma v := \left(D^\sigma_1 v, \dots, D^\sigma_N v\right)$, and the cut-off function ζ_1 is defined in Theorem 4.1.8, and obtain

$$\begin{aligned} &-\varepsilon \int_{Q_T} u^\varepsilon_{tt} \nabla \cdot (\zeta_1^2 D^\sigma u^\varepsilon_t)\, dx\, dt - \int_{Q_T} b(t, x, u^\varepsilon_t)\, \nabla \cdot (\zeta_1^2 D^\sigma u^\varepsilon_t)\, dx\, dt - \int_{Q_T} a(t, x, \nabla u^\varepsilon_t)\, \nabla\nabla \cdot (\zeta_1^2 D^\sigma u^\varepsilon_t)\, dx\, dt\\ &- \int_{Q_T} d(t, x)h(u^\varepsilon)\nabla u^\varepsilon\, \nabla\nabla \cdot (\zeta_1^2 D^\sigma u^\varepsilon_t)\, dx\, dt = -\int_{Q_T} f(t, x, u^\varepsilon)\, \nabla \cdot (\zeta_1^2 D^\sigma u^\varepsilon_t)\, dx\, dt. \end{aligned}$$

Integrating by parts implies

$$\begin{aligned} &\varepsilon \int_{Q_T} \nabla u^\varepsilon_{tt} D^\sigma u^\varepsilon_t\, \zeta_1^2\, dx\, dt - \int_{Q_T} b(t, x, u^\varepsilon_t)\, \nabla \cdot (\zeta_1^2 D^\sigma u^\varepsilon_t)\, dx\, dt + \int_{Q_T} \nabla a(t, x, \nabla u^\varepsilon_t)\, \nabla(\zeta_1^2 D^\sigma u^\varepsilon_t)\, dx\, dt\\ &+ \int_{Q_T} \nabla\big(d(t, x)h(u^\varepsilon)\nabla u^\varepsilon\big)\, \nabla(\zeta_1^2 D^\sigma u^\varepsilon_t)\, dx\, dt = -\int_{Q_T} f(t, x, u^\varepsilon)\, \nabla \cdot (\zeta_1^2 D^\sigma u^\varepsilon_t)\, dx\, dt. \end{aligned}$$

All integrands are integrable and uniformly bounded in σ by $L^1(Q_T)$ functions, because $u^\varepsilon_t \in L^2(0, T; H^2(\Omega))$. Then, due to the Dominated Convergence Theorem, (Evans 1998), we can take limits as $\sigma \to 0$ and obtain

$$\begin{aligned}
&\frac{\varepsilon}{2}\int_{\Omega} |\nabla u_t^\varepsilon|^2\,\zeta_1^2\,dx + \int_{Q_T} \nabla a(t,x,\nabla u_t^\varepsilon)\,\nabla(\zeta_1^2\nabla u_t^\varepsilon)\,dx\,dt \\
&= -\int_{Q_T} \nabla\big(d(t,x)h(u^\varepsilon)\nabla u^\varepsilon\big)\,\nabla(\zeta_1^2\nabla u_t^\varepsilon)\,dx\,dt \\
&- \int_{Q_T} f(t,x,u^\varepsilon)\,\nabla(\zeta_1^2\nabla u_t^\varepsilon)\,dx\,dt + \int_{Q_T} b(t,x,u_t^\varepsilon)\,\nabla(\zeta_1^2\nabla u_t^\varepsilon)\,dx\,dt.
\end{aligned}$$

After some calculations we obtain

$$\begin{aligned}
&\frac{\varepsilon}{2}\int_{\Omega} |\nabla u_t^\varepsilon|^2\,\zeta_1^2\,dx + \int_{Q_T} \nabla_\eta a(t,x,\nabla u_t^\varepsilon)\nabla^2 u_t^\varepsilon\,\nabla^2 u_t^\varepsilon\,\zeta_1^2\,dx\,dt + 2\int_{Q_T} \nabla_\eta a(t,x,\nabla u_t^\varepsilon)\nabla^2 u_t^\varepsilon\,\zeta_1\nabla\zeta_1\,\nabla u_t^\varepsilon\,dx\,dt \\
&+ \int_{Q_T} \nabla_x a(t,x,\nabla u_t^\varepsilon)\,\nabla^2 u_t^\varepsilon\,\zeta_1^2\,dx\,dt + 2\int_{Q_T} \nabla_x a(t,x,\nabla u_t^\varepsilon)\,\zeta_1\nabla\zeta_1\,\nabla u_t^\varepsilon\,dx\,dt = \\
&- \int_{Q_T} \Big(\nabla\cdot d(t,x)h(u^\varepsilon)\nabla u^\varepsilon + d(t,x)\partial_\xi h(u^\varepsilon)\nabla u^\varepsilon\nabla u^\varepsilon\Big)\Big(\nabla^2 u_t^\varepsilon\,\zeta_1^2 + 2\zeta_1\nabla\zeta_1\,\nabla u_t^\varepsilon\Big)\,dx\,dt \\
&+ \int_{Q_T} d(t,x)h(u^\varepsilon)\nabla^2 u^\varepsilon\,\Big(\nabla^2 u_t^\varepsilon\,\zeta_1^2 + 2\zeta_1\nabla\zeta_1\,\nabla u_t^\varepsilon\Big)\,dx\,dt \\
&- \int_{Q_T} f(t,x,u^\varepsilon)\,\Big(\Delta u_t^\varepsilon\,\zeta_1^2 + 2\zeta_1\nabla\zeta_1\,\nabla u_t^\varepsilon\Big)\,dx\,dt + \int_{Q_T} b(t,x,u_t^\varepsilon)\,\Big(\Delta u_t^\varepsilon\,\zeta_1^2 + 2\zeta_1\nabla\zeta_1\,\nabla u_t^\varepsilon\Big)\,dx\,dt.
\end{aligned}$$

From the strong monotonicity of a we have the estimate

$$\int_{Q_T} \nabla_\eta a(t,x,\nabla u_t^\varepsilon)\nabla^2 u_t^\varepsilon\,\nabla^2 u_t^\varepsilon\,\zeta_1^2\,dx\,dt \;\geqslant\; a_1\int_{Q_T} |\nabla^2 u_t^\varepsilon|^2\,\zeta_1^2\,dx\,dt.$$

Then we have

$$\begin{aligned}
\frac{\varepsilon}{2}\int_{\Omega} |\nabla u_t^\varepsilon|^2\,\zeta_1^2\,dx + a_1\int_{Q_T} |\nabla^2 u_t^\varepsilon|^2\,\zeta_1^2\,dx\,dt &\le \delta\int_{Q_T} |\nabla^2 u_t^\varepsilon|^2\,\zeta_1^2\,dx\,dt + \int_{Q_T} |\nabla^2 u^\varepsilon|^2\,\zeta_1^2\,dx\,dt \quad (4.1.16)\\
&+ \int_{Q_T} |\nabla u^\varepsilon|^4\,\zeta_1^2\,dx\,dt + c\int_{Q_T} (|\nabla u_t^\varepsilon|^2 + |u_t^\varepsilon|^2 + |\nabla u^\varepsilon|^2 + |u^\varepsilon|^2)\,dx\,dt.
\end{aligned}$$

Using the Gagliardo-Nirenberg inequality, the estimate for ∇u^ε, and the assumption that $u_0 \in H^2(\Omega)$ implies

$$\int_{Q_T} \zeta_1^2|\nabla u^\varepsilon|^4\,dx\,dt \;\le\; c_1\int_{Q_T} \zeta_1^2|\nabla^2 u^\varepsilon|^2\,dx\,dt + c_2 \;\le\; c_3 + c_4\int_0^T\int_{Q_t} \zeta_1^2|\nabla^2 u_\tau^\varepsilon|^2\,dx\,d\tau dt.$$

Due to the estimates for u_t^ε and the assumptions in the theorem, we obtain from (4.1.16)

$$\int_{Q_T} \zeta_1^2|\nabla^2 u_t^\varepsilon|^2\,dx\,dt \le C_1 + C_2\int_0^T\int_{Q_t} \zeta_1^2|\nabla^2 u_\tau^\varepsilon|^2\,dx\,d\tau\,dt.$$

Then the Gronwall lemma implies

$$\int_{Q_T} \zeta_1^2 |\nabla^2 u_t^\varepsilon|^2 \, dx\, dt \le C. \tag{4.1.17}$$

Using this estimate in the proof of Theorem 4.1.7 yields a subsequence and a limit-function such that $u_t \in L^2(0,T; H_0^1(\Omega)$ and $u_t \in L^2(0,T; H^2_{\text{loc}}(\Omega))$.

For the estimate near the boundary we use the same argument as for the hyperbolic equation. We can consider the equation in the half-ball, i.e. $\Omega = B_1(0) \cap \{\mathbb{R} \times \mathbb{R}_+\}$ with a straight boundary. Then we choose $v = -\partial_{x_1}(\zeta^2 D_1^\sigma u_t^\varepsilon)$ as a test function in the equation (4.1.3), where the cut-off function ζ is as defined in Theorem 4.1.8, and, after integrating by parts and taking limits as $\sigma \to 0$ as above, we obtain

$$\frac{\varepsilon}{2}\int_\Omega |\partial_{x_1} u_t^\varepsilon|^2 \zeta^2 \, dx - \int_{Q_T} b(t,x,u_t^\varepsilon)\partial_{x_1}(\zeta^2 \partial_{x_1} u_t^\varepsilon)\, dx\, dt + \int_{Q_T} \partial_{x_1} a(t,x,\nabla u_t^\varepsilon)\nabla(\zeta^2 \partial_{x_1} u_t^\varepsilon)\, dx\, dt$$
$$+ \int_{Q_T} \partial_{x_1}(d(t,x)h(u^\varepsilon)\nabla u^\varepsilon)\nabla(\zeta^2 \partial_{x_1} u_t^\varepsilon)\, dx\, dt = \int_{Q_T} f(t,x,u^\varepsilon)\partial_{x_1}(\zeta^2 \partial_{x_1} u_t^\varepsilon)\, dx\, dt.$$

The boundary integrals vanish since ζ vanishes near the curved part of $\partial\Omega$, and $\partial_{x_1} u^\varepsilon = 0$ and $\partial^2_{x_1} u^\varepsilon = 0$ vanish on $\{x_2 = 0\}$, because the normal vector to this part of the boundary is $\nu = (0,-1)$. Furthermore, $v \in L^2(0,T; H_0^1(\Omega))$, since $u_t^\varepsilon \in L^2(0,T; H^2(\Omega))$. Similarly, as for the hyperbolic equation, we obtain

$$\int_{Q_T} |\partial_{x_1} \nabla u_t^\varepsilon|^2 \, \zeta^2 \, dx\, dt \;\le\; C\Big(1 + \int_0^T \int_{Q_t} |\partial^2_{x_2} u_\tau^\varepsilon|^2 \, \zeta^2 \, dx\, d\tau dt\Big). \tag{4.1.18}$$

The estimate for $\partial^2_{x_2} u_t^\varepsilon$ is obtained from the equation in (4.1.1). Since $u_t^\varepsilon \in L^2(0,T; H^2(\Omega))$, $u_t^\varepsilon \in L^2(0,T; H_0^1(\Omega))$, and $\varepsilon\, u_{tt}^\varepsilon \in L^2(Q_T)$, we have that u^ε satisfies the equation in (4.1.1) almost everywhere. Then we obtain

$$\begin{aligned}\partial_{x_2} a^2(t,x,\nabla u_t^\varepsilon) + \partial_{x_2}(d^2(t,x)h(u^\varepsilon)\nabla u^\varepsilon) \;&=\; -\partial_{x_1} a^1(t,x,\nabla u_t^\varepsilon) - \partial_{x_1}(d^1(t,x)h(u^\varepsilon)\nabla u^\varepsilon) \\ &\quad + b(t,x,u_t^\varepsilon) + \varepsilon u_{tt}^\varepsilon - f(t,x,u^\varepsilon),\end{aligned}$$

or

$$\begin{aligned}\partial_{\eta_2} a^2(t,x,\nabla u_t^\varepsilon)\partial^2_{x_2} u_t^\varepsilon \;&=\; -d_{22}(t,x)h(u^\varepsilon)\partial^2_{x_2} u^\varepsilon - \partial_{x_1} a^1(t,x,\nabla u_t^\varepsilon) - \nabla_\eta a^1(t,x,\nabla u_t^\varepsilon)\partial_{x_1}\nabla u_t^\varepsilon \\ &\quad - \partial_{\eta_1} a^2(t,x,\nabla u_t^\varepsilon)\partial_{x_1}\nabla u_t^\varepsilon - \partial_{x_1}(d^1(t,x)h(u^\varepsilon)\nabla u^\varepsilon) \\ &\quad - \partial_{x_2} d^2(t,x)h(u^\varepsilon)\nabla u^\varepsilon - d(t,x)\partial_\xi h(u^\varepsilon)\nabla u^\varepsilon \nabla u^\varepsilon \\ &\quad + b(t,x,u_t^\varepsilon) + \varepsilon u_{tt}^\varepsilon - f(t,x,u^\varepsilon).\end{aligned}$$

From the strict monotonicity of a it follows that $\partial_{\eta_2} a^2 \;\geqslant\; a_1$. Then

$$\int_{Q_T} |\partial^2_{x_2} u_t^\varepsilon|^2 \, \zeta^2 \, dx\, dt \le C + \int_0^T \int_{Q_t} |\partial^2_{x_2} u_\tau^\varepsilon|^2 \, \zeta^2 \, dx\, d\tau.$$

The Gronwall's lemma implies the estimate

$$\int\limits_{Q_T} |\partial^2_{x_2} u^\varepsilon_t|^2 \, \zeta^2 \, dx \, dt \;\le\; C,$$

where C is independent of ε. This, together with (4.1.18), implies

$$\int\limits_{Q_T} |\nabla^2 u^\varepsilon_t|^2 \, \zeta^2 \, dx \, dt \;\le\; C.$$

Using the last estimate and the local estimate (4.1.17) in the proof of Theorem 4.1.7 yields a subsequence and a limit-function such that $u \in H^1(0,T;H^1_0(\Omega))$ and $u \in H^1(0,T;H^2(\Omega))$. ❐

Remark 4.1.10. For the linear function a and for the function d, the proof of Theorem 4.1.8 can be simplified. We can choose the basis functions ϕ^k as solutions of

$$\begin{aligned} \Delta\phi^k &= \lambda\phi^k \quad \text{in } \Omega, \\ \phi^k &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Then, choosing $-\Delta u^m_t$ as a test function in the equation (4.1.4) and integrating by parts, imply

$$\begin{aligned} &\varepsilon \int\limits_{Q_T} \nabla u^m_{tt} \, \nabla u^m_t \, dx \, dt - \int\limits_{Q_T} b(t,x,u^m_t) \, \Delta u^m_t \, dx \, dt + \int\limits_{Q_T} \Delta u^m_t \, \Delta u^m_t \, dx \, dt \\ &+ \int\limits_{Q_T} \nabla d(t,x) h(u^m) \nabla u^m \, \Delta u^m_t \, dx \, dt + \int\limits_{Q_T} d(t,x) \partial_\xi h(u^m) \, \nabla u^m \nabla u^m \, \Delta u^m_t \, dx \, dt \\ &+ \int\limits_{Q_T} d(t,x) h(u^m) \Delta u^m \, \Delta u^m_t \, dx \, dt = - \int\limits_{Q_T} f(t,x,u^m) \, \Delta u^m_t \, dx \, dt. \end{aligned}$$

For the estimates we use the Gagliardo-Nirenberg inequality

$$\int\limits_{Q_T} |\nabla u^m|^4 \, dx \, dt \;\le\; \int\limits_{Q_T} |\nabla^2 u^m|^2 \, dx \, dt \int\limits_{Q_T} |\nabla u^m|^2 \, dx \, dt$$

and the fact that for $u^m \in L^2(0,T;H^2(\Omega))$ and $u^m \in L^2(0,T;H^1_0(\Omega))$ we have

$$\int\limits_{Q_T} |\nabla^2 u^m|^2 \, dx \, dt \;\le\; C \int\limits_{Q_T} |\Delta u^m|^2 \, dx \, dt.$$

Theorem 4.1.11 (Uniqueness). *Let the assumptions of the Theorem 4.1.8 and*

$$|f(t,x,\xi_1) - f(t,x,\xi_2)| \le C|\xi_1 - \xi_2|,$$

$$|h(\xi_1) - h(\xi_2)| \le C|\xi_1 - \xi_2|$$

for $(t,x) \in Q_T$, $\xi_1, \xi_2 \in \mathbb{R}$, *be satisfied. Then there exists at most one weak solution of (4.1.2).*

Proof. Suppose u^1 and u^2 are two solutions of the problem (4.1.2). Then for $u = u^1 - u^2$ and the test function $v = u_t$ we obtain the equation

$$\int_{Q_T} (b(t,x,u_t^1) - b(t,x,u_t^2))\, u_t \,dx\,dt + \int_{Q_T} (a(t,x,\nabla u_t^1) - a(t,x,\nabla u_t^2))\, \nabla u_t \,dx\,dt$$
$$+ \int_{Q_T} d(t,x)(h(u^1)\nabla u^1 - h(u^2)\nabla u^2)\, \nabla u_t \,dx\,dt = \int_{Q_T} (f(t,x,u^1) - f(t,x,u^2))\, u_t \,dx\,dt.$$

Now we estimate all terms in the last equation separately

$$\begin{aligned}
I_1 &:= \int_{Q_T} (b(t,x,u_t^1) - b(t,x,u_t^2)) u_t \,dx\,dt \geq b_1 \int_{Q_T} |u_t|^2 \,dx\,dt, \\
I_2 &:= \int_{Q_T} (a(t,x,\nabla u_t^1) - a(t,x,\nabla u_t^2)) \nabla u_t \,dx\,dt \;\geq\; a_1 \int_{Q_T} |\nabla u_t|^2 \,dx\,dt, \\
I_3 &:= \int_{Q_T} d(t,x)(h(u^1)\nabla u^1 - h(u^2)\nabla u^2) \nabla u_t \,dx\,dt \;=\; \int_{Q_T} d(t,x) h(u^1) \nabla u \nabla u_t \,dx\,dt \\
&\quad + \int_{Q_T} d(t,x)(h(u^1) - h(u^2)) \nabla u^2 \nabla u_t \,dx\,dt.
\end{aligned}$$

The first integral in I_3 can be estimated by

$$\int_{Q_T} d(t,x) h(u^1) \nabla u \nabla u_t \,dx\,dt \;\leq\; \frac{c_1}{2\delta} \int_{Q_T} |\nabla u|^2 \,dx\,dt + \frac{\delta}{2} \int_{Q_T} |\nabla u_t|^2 \,dx\,dt$$

From the embedding theorem we have that $v \in H^1(0,T;H^2(\Omega))$ implies $\nabla v \in L^4(Q_T)$ even for Ω of the dimension $N \leq 4$. Then, due to the regularity of u^1 and u^2, we obtain $u^1, u^2 \in L^4(Q_T)$ and $\nabla u^2 \in L^4(Q_T)$, and the second integral in I_3 can be estimated by

$$\begin{aligned}
&\int_{Q_T} d(t,x)(h(u^1) - h(u^2)) \nabla u^2 \nabla u_t \,dx\,dt \\
\leq\;& c_2 \Big(\int_{Q_T} |u|^4 \,dx\,dt \Big)^{\frac{1}{4}} \Big(\int_{Q_T} |\nabla u^2|^4 \,dx\,dt \Big)^{\frac{1}{4}} \Big(\int_{Q_T} |\nabla u_t|^2 \,dx\,dt \Big)^{\frac{1}{2}} \\
\leq\;& c_3 \Big(\int_{Q_T} |u|^4 \,dx\,dt \Big)^{\frac{1}{4}} \Big(\int_{Q_T} |\nabla u_t|^2 \,dx\,dt \Big)^{\frac{1}{2}} \\
\leq\;& \frac{c_4}{2\delta} \int_{Q_T} \Big(|u|^2 + |\nabla u|^2 \Big) \,dx\,dt + \frac{\delta}{2} \int_{Q_T} |\nabla u_t|^2 \,dx\,dt.
\end{aligned}$$

The right hand side is estimated by

$$\begin{aligned} I_4 &:= \int\limits_{Q_T} (f(t,x,u^1) - f(t,x,u^2))\, u_t \,dx\,dt \\ &\le \frac{c_5}{\delta_0} \int\limits_{Q_T} |u|^2 \,dx\,dt + \delta_0 \int\limits_{Q_T} |u_t|^2 \,dx\,dt \\ &\le \frac{c_6}{\delta_0} \int\limits_0^T \int\limits_{Q_\tau} |u_t|^2 \,dx\,dt\,d\tau + \delta_0 \int\limits_{Q_T} |u_t|^2 \,dx\,dt, \end{aligned}$$

since $u^1(0) = u^2(0)$. Thus, due to estimates of the integrals I_1, I_2, I_3, and I_4, we obtain the inequality

$$(b_1 - \delta_0) \int\limits_{Q_T} |u_t|^2 \,dx\,dt + (a_1 - \delta) \int\limits_{Q_T} |\nabla u_t|^2 \,dx\,dt \le C \int\limits_0^T \int\limits_{Q_\tau} \Big(|u_t|^2 + |\nabla u_t|^2 \Big) \,dx\,dt\,d\tau.$$

Using Gronwall's lemma in the last inequality implies $u_t = 0$ almost everywhere in Q_T and, since $u^1(0) = u^2(0)$, we obtain that $u^1 = u^2$ almost everywhere in Q_T. ❐

4.2 Nonlinear Pseudoparabolic Variational Inequalities

In this section we consider the nonlinear pseudoparabolic variational inequality. The existence result for a nonlinear equation and the penalty method imply the existence of a solution of an inequality.

In $(0,T) \times \Omega$ we consider the inequality

$$\begin{aligned} &\int\limits_{Q_T} \Big(b(t,x,u_t)\,(v - u_t) + a(t,x,\nabla u_t)\,\nabla(v - u_t) + d(t,x)\,h(u)\,\nabla u\,\nabla(v - u_t) \Big) \,dx\,dt \\ &\qquad\qquad \geqslant \int\limits_{Q_T} f(t,x,u)\,(v - u_t) \,dx\,dt \end{aligned} \tag{4.2.1}$$

with initial condition

$$u(0) = u_0. \tag{4.2.2}$$

The constraint on u is given by the requirement $u_t(t) \in K$ for a.a. $t \in (0,T)$, where K is chosen to be a closed and convex subset of $L^p(\Omega) \cap V_0$ containing 0.

Definition 4.2.1. A function $u : Q_T \to \mathbb{R}$ is called *a weak solution* of (4.2.1), (4.2.2) if

1) $u \in C([0,T]; V_0)$, $u_t \in L^p(Q_T) \cap L^2(0,T;V_0)$, $u_t(t) \in K$ for almost all $t \in (0,T)$,
2) u satisfies (4.2.1) for all functions $v \in L^p(Q_T) \cap L^2(0,T;V_0)$, such that $v(t) \in K$ for almost all $t \in (0,T)$,
3) u satisfies the initial condition (4.2.2) in the sense that $u(t) \to u_0$ in V_0 for $t \to 0$.

Theorem 4.2.2 (Existence). *Let Assumption 4.1.1 be satisfied. Then there exists a weak solution of the problem (4.2.1), (4.2.2).*

Proof. We consider the equation with the penalty operator $\mathcal{B}$ for a positive parameter α

$$\int_{Q_T} \Big[b(t,x,u_t^\alpha)\, v + a(t,x,\nabla u_t^\alpha)\, \nabla v + d(t,x) h(u^\alpha) \nabla u^\alpha\, \nabla v \Big]\, dx\, dt$$

$$+\alpha \int_0^T \langle \mathcal{B}(u_t^\alpha), v\rangle\, dt = \int_{Q_T} f(t,x,u^\alpha)\, v\, dx\, dt\,. \tag{4.2.3}$$

Due to Theorem 4.1.7, since the operator $\mathcal{B} : L^p(Q_T) \cap L^2(0,T;V_0) \to L^q(Q_T) + L^2(0,T;V_0^*)$ is monotone, bounded and hemicontinuous, there exists for all α a solution $u^\alpha \in C([0,T];V_0)$ of the equation (4.2.3), such that $u_t^\alpha \in L^p(Q_T) \cap L^2(0,T;V_0)$. Similarly as in Theorem 4.1.7, using Assumption 4.1.1 and the last equation with the test function $v = u_t^\alpha$ implies the estimates

$$\begin{aligned}
\int_\Omega \Big(|u^\alpha(t)|^2 + |\nabla u^\alpha(t)|^2 \Big)\, dx &\le C \qquad \text{for } t \in [0,T],\\
\int_{Q_T} \Big(|u_t^\alpha|^p + |\nabla u_t^\alpha|^2 \Big)\, dx\, dt &\le C,\\
||b(t,x,u_t^\alpha)||_{L^q(Q_T)} &\le C,\\
||a(t,x,\nabla u_t^\alpha)||_{L^2(Q_T)^N} &\le C,\\
\int_0^T \langle \mathcal{B}(u_t^\alpha), u_t^\alpha\rangle\, dt &\le \frac{C}{\alpha},
\end{aligned}$$

uniformly in α. There exists a subsequence of $\{u^\alpha\}$, again denoted by $\{u^\alpha\}$, such that

$$\begin{aligned}
u^\alpha &\to u && \text{weakly}-* \text{ in } L^\infty(0,T;V_0),\\
u_t^\alpha &\to u_t && \text{weakly in } L^p(Q_T) \cap L^2(0,T;V_0),\\
b(t,x,u_t^\alpha) &\to \beta && \text{weakly in } L^q(Q_T),\\
a(t,x,\nabla u_t^\alpha) &\to \gamma && \text{weakly in } L^2(Q_T)^N
\end{aligned}$$

as $\alpha \to \infty$. From the equation (4.2.3) we obtain the equality

$$\begin{aligned}
\int_0^T \langle \mathcal{B}(u_t^\alpha), v\rangle\, dt &= \frac{1}{\alpha}\Big(\int_{Q_T} f(t,x,u^\alpha)\, v\, dx\, dt - \int_{Q_T} b(t,x,u_t^\alpha)\, v\, dx\, dt\\
&\quad - \int_{Q_T} a(t,x,\nabla u_t^\alpha)\, \nabla v\, dx\, dt - \int_{Q_T} d(t,x) h(u^\alpha) \nabla u^\alpha \nabla v\, dx\, dt \Big)
\end{aligned}$$

for all $v \in L^p(Q_T) \cap L^2(0,T;V_0)$. Since all terms on the right hand side in the last equality are bounded in $L^q(Q_T) + L^2(0,T;V_0^*)$ we obtain

$$\mathcal{B}(u_t^\alpha) \to 0 \quad \text{in} \quad L^q(Q_T) + L^2(0,T;V_0^*)$$

as $\alpha \to \infty$. Applying the monotonicity of $\mathcal{B}$ to u_t^α implies

$$\langle \mathcal{B}(u_t^\alpha) - \mathcal{B}(v), u_t^\alpha - v \rangle \geqslant 0$$

for all $v \in L^p(Q_T) \cap L^2(0,T;V_0)$. Using the estimate $\int_0^T \langle \mathcal{B}(u_t^\alpha), u_t^\alpha \rangle \, dt \le c/\alpha$ and passing to the limit as $\alpha \to \infty$ yields

$$\int_0^T \langle \mathcal{B}(v), u_t^\alpha - v \rangle \, dt \le 0.$$

Choosing in this inequality $v = u_t - \lambda w$, where $w \in L^p(Q_T) \cap L^2((0,T);V_0)$ and $\lambda > 0$, taking the limit as $\lambda \to +0$, and using the hemicontinuity of $\mathcal{B}$ implies

$$\int_0^T \langle \mathcal{B}(u_t), w \rangle \, dt \le 0$$

for all $w \in L^p(Q_T) \cap L^2(0,T;V_0)$. Thus, $\mathcal{B}(u_t) = 0$ and $u_t(t) \in K$ for almost all $t \in [0,T]$. Using $u \in L^2(0,T;V_0)$ and $u_t \in L^2(0,T;V_0)$ implies $u \in C([0,T];V_0)$ and $u^\alpha(0) \to u(0)$ weakly in V_0. Since $u^\alpha(0) = u_0$, we obtain $u(0) = u_0$, i.e. u satisfies the initial condition (4.2.2).

Now we show the strong convergence of $\{u_t^\alpha\}$ in $L^p(Q_T) \cap L^2(0,T;V_0)$. We choose $v = (u_t^\alpha - u_t)$ as a test function in the equation (4.2.3). Due to $u_t(t) \in K$ and $\mathcal{B}(u_t) = 0$, we obtain the equality

$$\begin{aligned}
&\int_{Q_T} \Big[b(t,x,u_t^\alpha)\,(u_t^\alpha - u_t) + a(t,x,\nabla u_t^\alpha)\,\nabla(u_t^\alpha - u_t) \Big] \, dx\,dt \\
&+ \int_{Q_T} d(t,x) h(u^\alpha) \nabla u^\alpha \,\nabla(u_t^\alpha - u_t)\, dx\,dt + \alpha \int_0^T \langle \mathcal{B}(u_t^\alpha) - \mathcal{B}(u_t), u_t^\alpha - u_t \rangle \, dt \\
&= \int_{Q_T} f(t,x,u^\alpha)\,(u_t^\alpha - u_t)\, dx\,dt.
\end{aligned}$$

Using the monotonicity of b, a, and $\mathcal{B}$ implies

$$\begin{aligned}
&b_1 \int_{Q_T} |u_t^\alpha - u_t|^p \, dx\,dt + a_1 \int_{Q_T} |\nabla(u_t^\alpha - u_t)|^2 \, dx\,dt \le \int_{Q_T} b(t,x,u_t)\,(u_t - u_t^\alpha)\, dx\,dt \\
&+ \int_{Q_T} a(t,x,\nabla u_t)\,\nabla(u_t - u_t^\alpha)\, dx\,dt + \int_{Q_T} d(t,x) h(u^\alpha)\,\nabla u\,\nabla(u_t - u_t^\alpha)\, dx\,dt \\
&+ \int_{Q_T} d(t,x) h(u^\alpha) \nabla(u^\alpha - u) \nabla(u_t - u_t^\alpha)\, dx\,dt + \int_{Q_T} f(t,x,u^\alpha)\,(u_t^\alpha - u_t)\, dx\,dt.
\end{aligned}$$

From the strong convergence of u^α in $L^2(Q_T)$ it follows that $u^\alpha \to u$ a. e. in Q_T. Thus, since h and f are continuous, $h(u^\alpha) \to h(u)$, $f(t,x,u^\alpha) \to f(t,x,u)$ a.e. in Q_T. Due to the assumptions we have that $h(u^\alpha)$, $h(u) \in L^\infty(Q_T)$ and $f(t,x,u^\alpha)$, $f(t,x,u) \in L^q(Q_T)$. From the Egorov Theorem it follows that $h(u^\alpha) \to h(u)$ uniformly a.e. in Q_T. From the Dominated Convergence Theorem it follows that $f(t,x,u^\alpha) \to f(t,x,u)$ strongly in $L^q(Q_T)$. The fourth integral on the right hand side can be estimated analogously as in Theorem 4.1.7. Then, using the weak convergence of u_t^α, the strong convergence of u^α, and Gronwall's lemma in the last inequality implies

$$||u_t^\alpha - u_t||_{L^p(Q_T)} + ||\nabla u_t^\alpha - \nabla u_t||_{L^2(Q_T)} \le C\,\sigma(\frac{1}{\alpha}).$$

From the last estimate we have $u_t^\alpha \to u_t$ strongly in $L^p(Q_T) \cap L^2(0,T;V_0)$ as $\alpha \to \infty$.

Due to monotonicity of $\mathcal{B}$, from equation (4.2.3) it follows that the inequality

$$\int\limits_{Q_T} \Big[b(t,x,u_t^\alpha)\,(v-u_t^\alpha) + a(t,x,\nabla u_t^\alpha)\,\nabla(v-u_t^\alpha) + d(t,x)h(u^\alpha)\nabla u^\alpha\,\nabla(v-u_t^\alpha)\Big]\,dx\,dt$$
$$- \int\limits_{Q_T} f(t,x,u^\alpha)\,(v-u_t^\alpha)\,dx\,dt = \alpha \int\limits_0^T \langle \mathcal{B}(v) - \mathcal{B}(u_t^\alpha), v-u_t^\alpha\rangle\,dt \;\geqslant\; 0$$

holds true for all functions $v \in L^p(Q_T) \cap L^2(0,T;V_0)$, such that $v(t) \in K$ for a.a. $t \in (0,T)$. The strong convergence of $\{u^\alpha\}$ and $\{u_t^\alpha\}$ for $\alpha \to \infty$ now implies the inequality (4.2.1). ❐

Theorem 4.2.3 (Uniqueness). *Let Assumption 4.1.1, $u \in L^4(0,T;H^{1,4}(\Omega))$, and*

$$|f(t,x,\xi_1) - f(t,x,\xi_2)| \le C|\xi_1 - \xi_2|, \qquad |h(\xi_1) - h(\xi_2)| \le C|\xi_1 - \xi_2|$$

for $(t,x) \in Q_T$ and ξ_1, $\xi_2 \in \mathbb{R}$ be satisfied. Then there exists at most one solution of (4.2.1), (4.2.2).

Proof. We assume there exist two solutions $u^{(1)}$ and $u^{(2)}$ of the problem (4.2.1), (4.2.2), i.e.

$$\int\limits_{Q_T} \Big[b(t,x,u_t^{(i)})\,(v-u_t^{(i)}) + a(t,x,\nabla u_t^{(i)})\,\nabla(v-u_t^{(i)}) + d(t,x)h(u^{(i)})\nabla u^{(i)}\,\nabla(v-u_t^{(i)})\Big]\,dx\,dt$$
$$\geqslant \int\limits_{Q_T} f(t,x,u^{(i)})\,(v-u_t^{(i)})\,dx\,dt \tag{4.2.4}$$

for all $v \in L^p(Q_T) \cap L^2(0,T;V_0)$, where $i = 1,2$. The function $u_t^{(1)} + u_t^{(2)}$ is in the space $L^p(Q_T) \cap L^2(0,T;V_0)$, and, since K is convex, the function $v = \frac{1}{2}(u_t^{(1)} + u_t^{(2)})$ may be taken as the test function in the inequalities (4.2.4). Adding these inequalities implies

$$\int\limits_{Q_T} \Big[\big(b(t,x,u_t^{(1)}) - b(t,x,u_t^{(2)})\big)\,u_t + \big(a(t,x,\nabla u_t^{(1)}) - a(t,x,\nabla u_t^{(2)})\big)\,\nabla u_t\Big]\,dx\,dt$$
$$+ \int\limits_{Q_T} d(t,x)\big(h(u^{(1)})\nabla u^{(1)} - h(u^{(2)})\nabla u^{(2)}\big)\,\nabla u_t\,dx\,dt$$
$$\le \int\limits_{Q_T} \big(f(t,x,u^{(1)}) - f(t,x,u^{(2)})\big)\,u_t\,dx\,dt,$$

where $u = u^{(1)} - u^{(2)}$. Now we estimate the integrals of the following equality

$$\begin{aligned} &\int_{Q_T} d(t,x)\big(h(u^{(1)})\nabla u^{(1)} - h(u^{(2)})\nabla u^{(2)}\big)\nabla u_t\,dx\,dt \\ = &\int_{Q_T} d(t,x)h(u^{(1)})\nabla u\nabla u_t\,dx\,dt + \int_{Q_T} d(t,x)\big(h(u^{(1)}) - h(u^{(2)})\big)\nabla u^{(2)}\nabla u_t\,dx\,dt. \end{aligned}$$

For the first integral in the last equality we have the estimate

$$\int_{Q_T} d(t,x)h(u^{(1)})\nabla u\nabla u_t\,dx\,dt \;\le\; \frac{c_1}{2\delta}\int_{Q_T} |\nabla u|^2\,dx\,dt + \frac{\delta}{2}\int_{Q_T} |\nabla u_t|^2\,dx\,dt.$$

Due to the regularity of $u^{(1)}$ and $u^{(2)}$, i.e. $u^{(1)}, u^{(2)} \in L^4(Q_T)$ and $\nabla u^{(2)} \in L^4(Q_T)$, the second integral in the last equality can be estimated by

$$\begin{aligned} &\int_{Q_T} d(t,x)\big(h(u^{(1)}) - h(u^{(2)})\big)\nabla u^{(2)}\nabla u_t\,dx\,dt \\ \le\;& c_2\Big(\int_{Q_T} |u|^4\,dx\,dt\Big)^{\frac14}\Big(\int_{Q_T} |\nabla u^{(2)}|^4\,dx\,dt\Big)^{\frac14}\Big(\int_{Q_T} |\nabla u_t|^2\,dx\,dt\Big)^{\frac12} \\ \le\;& c_3\Big(\int_{Q_T} |u|^4\,dx\,dt\Big)^{\frac14}\Big(\int_{Q_T} |\nabla u_t|^2\,dx\,dt\Big)^{\frac12} \\ \le\;& \frac{c_4}{2\delta}\int_{Q_T}\Big(|u|^2 + |\nabla u|^2\Big)\,dx\,dt + \frac{\delta}{2}\int_{Q_T} |\nabla u_t|^2\,dx\,dt. \end{aligned}$$

The right hand side is estimated by

$$\begin{aligned} \int_{Q_T}\big(f(t,x,u^{(1)}) - f(t,x,u^{(2)})\big)\,u_t\,dx\,dt \;&\le\; \frac{c_5}{\delta_0}\int_{Q_T} |u|^2\,dx\,dt + \delta_0\int_{Q_T} |u_t|^2\,dx\,dt \\ &\le\; \frac{c_6}{\delta_0}\int_0^T\int_{Q_\tau} |u_t|^2\,dx\,dt\,d\tau + \delta_0\int_{Q_T} |u_t|^2\,dx\,dt, \end{aligned}$$

since $u^{(1)}(0) = u^{(2)}(0)$. Then, we obtain the inequality

$$(b_1 - \delta_0)\int_{Q_T} |u_t|^2\,dx\,dt + (a_1 - \delta)\int_{Q_T} |\nabla u_t|^2\,dx\,dt \;\le\; C\int_0^T\int_{Q_\tau}\Big(|u_t|^2 + |\nabla u_t|^2\Big)\,dx\,dt\,d\tau.$$

This, using Gronwall's Lemma, implies that $u_t = 0$ a.e. in Q_T and, since $u^{(1)}(0) = u^{(2)}(0)$, that $u^{(1)} = u^{(2)}$ a.e. in Q_T. ❐

Conclusion

In this thesis we considered pseudoparabolic equations and variational inequalities. We proved the existence and uniqueness for degenerate quasilinear equations, Theorem 2.1.3, Theorem 2.1.8, Theorem 2.1.9, the existence for degenerate quasilinear variational inequalities, Theorem 2.2.4, the existence and uniqueness for doubly nonlinear equations, Theorem 2.3.3, Theorem 2.3.7, the existence and uniqueness for equations with convection, Theorem 2.4.3, the existence and uniqueness for quasilinear equations and variational inequalities on the time interval $(-\infty, T]$, Theorem 3.1.4, Theorem 3.1.6, Theorem 3.2.2, Theorem 3.2.5, and the existence and uniqueness for nonlinear pseudoparabolic equations and variational inequalities, Theorem 4.1.7, Theorem 4.2.2.

Here Rothe's and Galerkin's methods were applied. To show the convergence of approximate solutions we used Minty–Browder Theorem, strong convergence, Aubin-Lions Compactness Lemma, and Kolmogorov's compactness criterium.

Degenerate equations, where the term $\partial_t u$ is replaced by $\partial_t b(u)$, were solved using the monotonicity and the gradient assumptions on the nonlinear function b. To prove higher regularity in time, we needed linearity or a gradient assumption on the flux and Lipschitz continuity of the function b. These assumptions were used to prove the existence of a solution of the degenerate quasilinear variational inequalities. As we saw, the memory operator may be linear, Lipschitz continuous, or of first order. However, it is not clear how to prove similar results for the degenerate quasilinear equations containing the nonlinear function of the gradient in the pseudoparabolic term, i.e. the mixed third-order derivative. The uniqueness was proved by using the monotonicity of the operators and the linearity or the Lipschitz continuity of the function defining the elliptic part. The Kruzhkov method, which was applied in (Otto 1996) for degenerate parabolic equations, implies the uniqueness making fewer assumptions. But, it is not clear if this method can be applied to prove the uniqueness for degenerate pseudoparabolic equations. The problem is due to the lack of estimates for the mixed third-order derivative.

For doubly nonlinear equations it is essential to have the same nonlinearity in the elliptic part as in the pseudoparabolic term, i.e. the mixed third-order derivative. Here we extended the integration by parts formula given by (Jäger and Kacur 1995) for parabolic doubly nonlinear equations.

We showed that the characteristic method is useful for the pseudoparabolic equations with convection. These equations can also be considered as a degenerate equation with nonlinear function b.

The existence and uniqueness in an unbounded time interval without the restriction at $-\infty$ was shown for quasilinear pseudoparabolic equations. Here we used Pankov's Lemma and the cut-off function method. For variational inequalities we assumed, additionally, that the nonlinear functions were independent of the time variable. For an appropriate set K and a corresponding penalty operator $\mathcal{B}$, which satisfies $\int_0^T \langle \mathcal{B}(u), u_t \rangle \, dt \geqslant 0$, the existence could be proved without this restriction.

In the last chapter we considered the solution of a nonlinear pseudoparabolic equation as a quasistationary state of a system with cross diffusion

$$\begin{cases} \varepsilon \partial_t v &= \nabla \cdot a(t, x, \nabla v) + \nabla \cdot (d(t, x, v) \nabla u) + f(t, x, u) - b(t, x, v), \\ \partial_t u &= g(u, v). \end{cases}$$

Here we choose a function g of the form $g(u, v) = h(u)v$. For arbitrary g a similar result is unknown. We also considered a function d, which depended on time and space, but was independent of v. Otherwise, if d depends on v, we can not prove the convergence of the sequence of solutions $\{u^\varepsilon\}$ to a solution of the pseudoparabolic equation. The problem is the uncertainty of the strong convergence of $\{u^\varepsilon_t\}$. For a regular solution the uniqueness can be shown; however, the existence of such regular solutions is proved in two dimensions only.

The existence proofs in this thesis are constructive, in the sense that they are based on Rothe's and Galerkin's methods. Hence, the corresponding numerical approximation schemes should converge. For real computations the order of convergence and error estimates for a particular finite element space are needed. The relaxation method for degenerate parabolic equations introduced in (Jäger and Kacur 1995), (Kacur 1998), and (Kacur 2001) to control the degenerate term, can be applied to degenerate pseudoparabolic equations. We used the characteristic method for equations with convection. The change of solutions of problems with convection along the characteristic lines is small compared to the change in time. Thus, the standard Rothe-Galerkin's method is only applicable for small time steps. On the other hand, the discretization along characteristics is allowed for rather large time steps in the time discretization.

Pseudoparabolic equations often arise if the dimension of a system of partial differential equations is reduced. The decreased number of equations facilitates the analysis and numerical approximation of the original system, albeit these equations with the term of mixed third-order derivatives are un-canonical. But known results may be adapted to include pseudoparabolic equations.

A

Appendix

A.1 Auxiliary Lemmata and Theorems for Chapter 2

Let $b : \mathbb{R}^l \to \mathbb{R}^l$ be a monotone vector field and a continuous gradient, i.e. there exists a convex C^1 function $\Phi : \mathbb{R}^l \to \mathbb{R}$ such that $b = \nabla\Phi$. Then we define the function Ψ by

$$\Psi(z) = \Psi_b(z) := \sup_{\sigma\in\mathbb{R}^l} \int_0^1 (z - b(s\sigma))\,\sigma\,ds = \sup_{\sigma\in\mathbb{R}^l} (z\cdot\sigma - \Phi(\sigma) - \Phi(0)).$$

The convexity of Φ implies that

$$B(z) := \Psi(b(z)) = b(z)\cdot z - \Phi(z) - \Phi(0) = \int_0^1 (b(z) - b(sz))\cdot z\,ds = \int_0^z (b(z) - b(s))\,ds.$$

Lemma A.1.1. *(Kacur 1985) The estimates*

$$\begin{aligned}
&1) \qquad B(z) = \int_0^1 (b(z) - b(\sigma z))\,z\,d\sigma \geqslant 0,\\
&2) \qquad B(z) - B(z_0) \geqslant (b(z) - b(z_0))\,z_0,\\
&3) \qquad b(z) - \Phi(z) + \Phi(0) = B(z) \leq b(z)\,z,\\
&4) \qquad |b(z)| \leq \delta B(z) + \sup_{|\sigma|\leq\frac{1}{\delta}} |b(\sigma)|,
\end{aligned}$$

hold true for all $z, z_0 \in \mathbb{R}^l$, and for positive δ.

We define the space $V := \{v \in H^{1,p}(\Omega)^l, v = 0 \text{ on } \Gamma\}$, where $\Gamma \subset \partial\Omega$ is measurable with $\mathcal{H}^{N-1}(\Gamma) > 0$.

Lemma A.1.2. *(Alt and Luckhaus 1983, Lemma 1.5, Integration by parts)*
Suppose $u \in u^D + L^p(0,T;V)$, $u^D \in L^p(0,T;H^{1,p}(\Omega)^l) \cap L^\infty(Q_T)^l$, $\partial_t u^D \in L^1(0,T;L^\infty(\Omega)^l)$, $b(u) \in L^\infty(0,T;L^1(\Omega)^l)$, $\partial_t b(u) \in L^q(0,T;V^*)$, $\Psi(b^0) \in L^1(\Omega)$, b^0 maps into the rang of b, and

$$\int_0^T \langle \partial_t b(u), \xi\rangle\,dt + \int_0^T\int_\Omega (b(u) - b^0)\partial_t\xi\,dx\,dt = 0$$

for all test functions $\xi \in L^p(0,T;V) \cap H^{1,1}(0,T;L^\infty(\Omega)^l)$ with $\xi(T) = 0$.

Then $B(u) \in L^\infty(0,T;L^1(\Omega))$ and for almost all t the following formula holds

$$\int\limits_\Omega B(u(t))\,dx - \int\limits_\Omega B(u_0)\,dx = \int\limits_0^t \langle \partial_t b(u), u - u^D\rangle\,dt - \int\limits_0^t\int\limits_\Omega (b(u) - b(u^0))\partial_t u^D\,dx\,dt + \int\limits_\Omega (b(u(t)) - b(u^0))u^D(t)\,dx.$$

For pseudoparabolic equations we have adapted version of Lemma A.1.2.

Lemma A.1.3. *Suppose $u \in L^p(0,T;H_0^{1,p}(\Omega)^l)$, $u \in L^\infty(0,T;H_0^1(\Omega)^l)$, $b(u) \in L^\infty(0,T;L^1(\Omega)^l)$, $B(u) \in L^\infty(0,T;L^1(\Omega))$, $\partial_t(b(u) - \nabla\cdot(a(x)\nabla u)) \in L^q(0,T;H^{-1,q}(\Omega)^l)$, $u_0 \in H_0^1(\Omega)$, $b(u_0) \in L^1(\Omega)$, and $b(u_0) \in H^{-1}(\Omega)$. Then for almost all t the following formula holds*

$$\int\limits_0^t \langle \partial_t(b(u) - \nabla\cdot(a(x)\nabla u), u\rangle\,dt = \int\limits_\Omega B(u(t))\,dx + \frac{1}{2}\int\limits_\Omega a(x)\nabla u(t)\nabla u(t)\,dx - \int\limits_\Omega B(u_0)\,dx - \frac{1}{2}\int\limits_\Omega a(x)\nabla u_0\nabla u_0\,dx.$$

Remark A.1.4. As was shown by Brezis and Browder, (1978), the assumptions $u_0 \in H_0^1(\Omega)$, $b(u_0) \in L^1(\Omega)$, $b(u_0) \in H^{-1}(\Omega)$, and $b(u_0)u_0 \geqslant 0$ yield $B(u_0) \in L^1(\Omega)$.

Proof of Lemma. For almost all $(t,x) \in (h,T)\times\Omega$, where $u(t-h,x) = u_0(x)$ for $t \in (0,h)$ and $b(u(t-h,x)) = b(u_0(x))$ for $t \in (0,h)$, we have the inequalities

$$\begin{aligned} & B(u(t,x)) - B(u(t-h,x)) + \frac{1}{2}a(x)\nabla u(t,x)\nabla u(t,x) - \frac{1}{2}a(x)\nabla u(t-h,x)\nabla u(t-h,x) \\ \leq\; & (b(u(t,x)) - b(u(t-h,x)))u(t,x) + a(x)(\nabla u(t,x) - \nabla u(t-h,x))\nabla u(t,x) \end{aligned}$$

and

$$\begin{aligned} & B(u(t,x)) - B(u(t-h,x)) + \frac{1}{2}a(x)\nabla u(t,x)\nabla u(t,x) - \frac{1}{2}a(x)\nabla u(t-h,x)\nabla u(t-h,x) \\ \geqslant\; & (b(u(t,x)) - b(u(t-h,x)))u(t-h) + a(x)(\nabla u(t,x) - \nabla u(t-h,x))\nabla u(t-h,x). \end{aligned}$$

Now we multiply the first inequality by h^{-1} and integrate over $(0,\tau)\times\Omega$. Due to $u \in L^p(0,T;H_0^{1,p}(\Omega)^l)$ and $b(u) \in L^q(0,T;H^{-1,q}(\Omega)^l)$, we obtain

$$\begin{aligned} & \frac{1}{h}\int\limits_{\tau-h}^{\tau}\int\limits_\Omega \Big(B(u(t,x)) + \frac{1}{2}a(x)\nabla u(t,x)\nabla u(t,x)\Big)\,dx\,dt - \int\limits_\Omega \Big(B(u_0) + \frac{1}{2}a(x)\nabla u_0\nabla u_0\Big)\,dx \\ \leq\; & \int\limits_0^\tau \frac{1}{h}\langle b(u(t,x)) - b(u(t-h,x)), u(t,x)\rangle\,dt + \int\limits_0^\tau\int\limits_\Omega \frac{1}{h}a(x)\nabla(u(t,x) - u(t-h,x))\nabla u(t,x)\,dx\,dt. \end{aligned}$$

Multiplying the second inequality by h^{-1} and integrating over $(h,\tau)\times\Omega$ implies

$$\begin{aligned} & \frac{1}{h}\int\limits_{\tau-h}^{\tau}\int\limits_\Omega \Big(B(u(t,x)) + \frac{1}{2}a(x)\nabla u(t,x)\nabla u(t,x)\Big)\,dx\,dt - \frac{1}{h}\int\limits_0^h\int\limits_\Omega \Big(B(u(t,x)) + \frac{1}{2}a(x)\nabla u(t,x)\nabla u(t,x)\Big)\,dx\,dt \\ \geqslant\; & \int\limits_0^{\tau-h} \frac{1}{h}\langle b(u(t+h,x)) - b(u(t,x)), u(t,x)\rangle\,dt + \int\limits_0^{\tau-h}\int\limits_\Omega \frac{1}{h}a(x)\nabla(u(t+h,x) - u(t,x))\nabla u(t,x)\,dx\,dt. \end{aligned}$$

Since $\partial_h(b(u) - \nabla\cdot(a(x)\nabla u)) = \frac{1}{h}\int\limits_{\tau-h}^{\tau} \frac{d}{dt}(b(u) - \nabla\cdot(a(x)\nabla u))\,dt$, we obtain

$$\begin{aligned}
&\int\limits_h^T ||\partial_h(b(u) - \nabla\cdot(a(x)\nabla u))||^q_{H^{-1,q}(\Omega)}\,dt \le \int\limits_h^T \frac{1}{h}\int\limits_{\tau-h}^{\tau} ||\frac{d}{dt}(b(u) - \nabla\cdot(a(x)\nabla u))||^q_{H^{-1,q}(\Omega)}\,dt\,d\tau \\
\le\ & \int\limits_h^T \frac{1}{h}\int\limits_0^h ||\frac{d}{dz}(b(u(z+t-h)) - \nabla\cdot(a(x)\nabla u(z+t-h)))||^q_{H^{-1,q}(\Omega)}\,dz\,dt \\
\le\ & \frac{1}{h}\int\limits_0^h\int\limits_0^T ||\frac{d}{dt}(b(u) - \nabla\cdot(a(x)\nabla u))||^q_{H^{-1,q}(\Omega)}\,dt\,ds \le C.
\end{aligned}$$

Then $\partial_h(b(u) - \nabla\cdot(a(x)\nabla u)) \to \chi$ in $L^q(0,T;H^{-1,q}(\Omega)^l)$.

Due to $\partial_t(b(u) - \nabla\cdot(a(x)\nabla u)) \in L^q(0,T;H^{-1,q}(\Omega)^l)$ and

$$\int\limits_0^T\int\limits_\Omega \partial_h(b(u) - \nabla\cdot(a(x)\nabla u))v\,dx\,dt = -\int\limits_0^T\int\limits_\Omega (b(u) - \nabla\cdot(a(x)\nabla u))\partial_{-h}v\,dx\,dt$$

for $v \in L^p(0,T;H^{1,p}(\Omega)^l) \cap L^\infty(Q_T)^l$, $v_t \in L^2(0,T;H^1_0(\Omega)^l)$ and $v(t,x) = 0$ for $t \in (0,\delta)$ and $t \in (T-\delta,T)$, $0 < \delta < T$, $x \in \Omega$, we have

$$\partial_h(b(u) - \nabla\cdot(a(x)\nabla u)) \to \partial_t(b(u) - \nabla\cdot(a(x)\nabla u))$$

in $L^q(0,T;H^{-1,q}(\Omega)^l)$.

Since $u \in L^p(0,T;H^{1,p}(\Omega)^l)$ we can take the limit as $h \to 0$ and obtain

$$\begin{aligned}
&\int\limits_\Omega \Big(B(u(\tau)) + \frac{1}{2}a(x)\nabla u(\tau)\nabla u(\tau)\Big)\,dx - \int\limits_\Omega \Big(B(u_0) + \frac{1}{2}a(x)\nabla u_0\nabla u_0\Big)\,dx \\
\le\ & \int\limits_0^\tau \langle\partial_t\big(b(u(t)) - \nabla\cdot(a(x)\nabla u(t))\big), u(t)\rangle\,dt
\end{aligned}$$

and

$$\begin{aligned}
&\int\limits_\Omega \Big(B(u(\tau)) + \frac{1}{2}a(x)\nabla u(\tau)\nabla u(\tau)\Big)\,dx - \int\limits_\Omega \Big(B(u_0) + \frac{1}{2}a(x)\nabla u_0\nabla u_0\Big)\,dx \\
\geqslant\ & \int\limits_0^\tau \langle\partial_t\big(b(u(t)) - \nabla\cdot(a(x)\nabla u(t))\big), u(t)\rangle\,dt.
\end{aligned}$$

These two inequalities imply the assertion of Lemma. By passing to the limit in the second inequality we used that

$$\liminf_{h\to 0}\frac{1}{h}\int\limits_0^h\int\limits_\Omega \Big(B(u(t)) + \frac{1}{2}a(x)\nabla u(t)\nabla u(t)\Big)\,dx\,dt \geqslant \int\limits_\Omega \Big(B(u_0) + \frac{1}{2}a(x)\nabla u_0\nabla u_0\Big)dx. \quad (1.1.1)$$

Due to $\partial_t(b(u) - \nabla \cdot (a(x)\nabla u)) \in L^q(0,T;H^{-1,q}(\Omega)^l)$ and the second estimate in Lemma A.1.1, we obtain

$$\begin{aligned}
&\liminf_{h\to 0} \frac{1}{h}\int_0^h \int_\Omega \Big(B(u(t)) + \frac{1}{2}a(x)\nabla u(t)\nabla u(t) - B(u_0) - \frac{1}{2}a(x)\nabla u_0 \nabla u_0\Big)\, dx\, dt \\
\geqslant\ &\liminf_{h\to 0} h \int_0^1 \langle \partial_t(b(u) - \nabla\cdot(a(x)\nabla u)), u_0\rangle\, dt \ = \ 0.
\end{aligned}$$

Thus, we have the inequality (1.1.1). ❐

Lemma A.1.5. *(Jäger and Kacur 1995, Lemma 3.25)*
Let $u : Q_T \to \mathbb{R}$ with $\beta(u) \in L^p(0,T;V)$, $b(x,u) \in L^\infty(0,T;L^2(\Omega))$, $\partial_t b(x,u) \in L^q(0,T;V^*)$, $b(u_0) \in L^2(\Omega)$, $\beta(u_0) \in V$. Then for almost all $t \in (0,T)$

$$\int_0^t \langle \partial_t b(x,u), \beta(u)\rangle\, dt = \int_\Omega B^*(x,\beta(u(t)))\, dx - \int_\Omega B^*(x,\beta(u_0))\, dx$$

holds true, where for $s \in \{y \in \mathbb{R} : y = \beta(z)\}$

$$B^*(x,s) := b(x,\beta^{-1}(s))\, s - \int_0^s b(x,\beta^{-1}(z))\, dz.$$

For pseudoparabolic equations we have adapted version of this Lemma.

Lemma A.1.6. *Suppose $\partial_t(b(u) - \Delta a(u)) \in L^q(0,T;H^{-1,q}(\Omega))$, $a(u) \in L^\infty(0,T;H^1_0(\Omega))$, $a(u) \in L^p(0,T;H^{1,p}_0(\Omega))$, $\tilde{B} \in L^\infty(0,T,L^1(\Omega))$, $b(u_0) \in L^2(\Omega)$, and $a(u_0) \in H^1_0(\Omega)$. Then for almost all $t \in (0,T)$ the integration by parts formula*

$$\begin{aligned}
&\int_0^t \langle \partial_t(b(u) - \Delta a(u)), a(u)\rangle\, dt \\
&= \int_\Omega \tilde{B}(a(u(t)))\, dx + \frac{1}{2}\int_\Omega |\nabla a(u(t))|^2\, dx - \int_\Omega \tilde{B}(a(u_0))\, dx - \frac{1}{2}\int_\Omega |\nabla a(u_0)|^2\, dx
\end{aligned}$$

holds, where for $s \in \{y \in \mathbb{R} : y = a(z)\}$

$$\tilde{B}(s) := b(a^{-1}(s))\, s - \int_0^s b(a^{-1}(z))\, dz.$$

Proof. For $(t,x) \in (0,T-h)\times\Omega$, where $u(t) = u_0$ for $t \in [-h,0]$, we have the inequalities

$$\begin{aligned}
&\tilde{B}(a(u(t))) - \tilde{B}(a(u(t-h))) + \frac{1}{2}|\nabla a(u(t))|^2 - \frac{1}{2}|\nabla a(u(t-h))|^2 \\
\leq\ &(b(u(t)) - b(u(t-h)))a(u(t)) + \nabla(a(u(t)) - a(u(t-h)))\nabla a(u(t)) \qquad (1.1.2)
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{B}(a(u(t))) - \tilde{B}(a(u(t-h))) + \frac{1}{2}|\nabla a(u(t))|^2 - \frac{1}{2}|\nabla a(u(t-h))|^2 \\
\geqslant\ &(b(u(t)) - b(u(t-h)))a(u(t-h)) + \nabla(a(u(t)) - a(u(t-h)))\nabla a(u(t-h)). \quad (1.1.3)
\end{aligned}$$

Now we multiply (1.1.2) by h^{-1} and integrate it over $(0,t)\times\Omega$. Due to $a(u) \in L^p(0,T;H_0^{1,p}(\Omega))$, $p \geqslant 2$, and $b(u) \in L^2(Q_T)$, we obtain

$$\begin{aligned}
&\frac{1}{h}\int_{\tau-h}^{\tau}\int_{\Omega}\Big(\tilde{B}(a(u(t))) + \frac{1}{2}|\nabla a(u(t))|^2\Big)\,dx\,dt - \int_{\Omega}\Big(\tilde{B}(a(u_0)) + \frac{1}{2}|\nabla a(u_0)|^2\Big)\,dx \\
&\leq \int_0^{\tau}\int_{\Omega}\frac{1}{h}\Big[(b(u(t)) - b(u(t-h)))a(u(t)) + (\nabla a(u(t)) - \nabla a(u(t-h)))\nabla a(u(t))\Big]\,dx\,dt.
\end{aligned}$$

Since $\partial_t(b(u) - \Delta a(u)) \in L^q(0,T;H^{-1,q}(\Omega))$, we have $\partial_h(b(u) - \Delta a(u)) \to \partial_t(b(u) - \Delta a(u))$ in $L^q(0,T;H^{-1,q}(\Omega))$.

Due to $a(u) \in L^p(0,T;H^{1,p}(\Omega))$ we can take the limit as $h \to 0$ and obtain

$$\begin{aligned}
&\int_{\Omega}\Big(\tilde{B}(a(u(\tau))) + \frac{1}{2}|\nabla a(u(\tau))|^2\Big)\,dx - \int_{\Omega}\Big(\tilde{B}(a(u_0)) + \frac{1}{2}|\nabla a(u_0)|^2\Big)\,dx \\
&\leq \int_0^{\tau}\langle \partial_t(b(u) - \Delta a(u)), a(u)\rangle\,dt.
\end{aligned}$$

To prove the reverse inequality we integrate (1.1.3) over $(h,t)\times\Omega$ and proceed analogously to the above. For the reverse inequality we use

$$\int_{\Omega}\Big(\tilde{B}(a(u_0)) + \frac{1}{2}|\nabla a(u_0)|^2\Big)\,dx \ \leq\ \liminf_{h\to 0}\frac{1}{h}\int_0^h\int_{\Omega}\Big(\tilde{B}(a(u(t))) + \frac{1}{2}|\nabla a(u(t))|^2\Big)\,dx\,dt.$$

This estimate can be obtained similarly to that in Lemma A.1.3. ❒

Lemma A.1.7. *(Gajewski, Gröger, and Zacharias 1974; Showalter 1996)*
Let $f : \mathbb{R}^N \to \mathbb{R}^N$ be continuous and, for any $R > 0$, $(f(x), x) \geqslant 0$ for every $|x| = R$. Then there exists an $a \in \mathbb{R}^N$, such that $|a| \leq R$ and $f(a) = 0$.

Theorem A.1.8. *(Necas 1967) [Kolmogorov's theorem]*
Let Ω be a bounded domain in $\mathbb{R}^N$. A set M of functions $f \in L^p(\Omega)$ is precompact iff M is bounded and equicontinuous, i.e for any $\varepsilon > 0$ there is a $\delta > 0$ such that for all $f \in M$

$$\int_{\Omega}|f(x+y) - f(x)|^p dx < \varepsilon \quad \textit{for} \quad |y| < \delta.$$

Theorem A.1.9. *(Evans 1998)[Minty-Browder Theorem]*
Let $d : (0,T)\times\Omega\times\mathbb{R}^N \to \mathbb{R}^N$ be monotone in the last variable, i.e.

$$(d(t,x,z_1) - d(t,x,z_2))(z_1 - z_2) \geqslant 0 \ \textit{ for } \ z_1, z_2 \in \mathbb{R}^N,$$

and

$$\begin{aligned} u_n \to u \quad & \text{weakly in} \quad L^p(Q_T)^N, \\ d(t,x,u_n) \to \chi \quad & \text{weakly in} \quad L^q(Q_T)^N, \\ \limsup_{n\to\infty} \int_\Omega d(t,x,u_n)\, u_n \, dx \quad & \leq \quad \int_\Omega \chi\, u \, dx. \end{aligned}$$

Then $\chi = d(t,x,u)$.

Definition A.1.10. (Lions 1969)[Definition of penalty operator]
Let V be a reflexive Banach space and K a closed convex subset in V. Then *a penalty operator* $\mathcal{B} : V \to V^*$ related to K is a monotone, bounded and hemicontinuous operator such that

$$\{v \mid v \in V,\ \mathcal{B}(v) = 0\} = K.$$

Such an operator $\mathcal{B}$ is given by

$$\mathcal{B} = J(I - P_K),$$

where $J : V \to V^*$ a dual mapping and $P_K : V \to K$ is the projection operator on K.

In the case $V = H^{1,p}$, for some $p > 1$, the dual mapping J can be chosen as

$$\langle J(u), v\rangle = \int_\Omega (|u|^{p-2}uv + |\nabla u|^{p-2}\nabla u \nabla v)\, dx$$

and the projection operator P_K satisfies

$$\langle J(u - P_K u), P_K u - v\rangle \geqslant 0 \quad \text{for} \quad v \in K.$$

A.2 Auxiliary Lemmata and Theorems for Chapters 3 and 4

Theorem A.2.1. *(Deimling 1992) [Generalization for Carathéodory functions]*
Let $X = \mathbb{R}^n$, $D \subset X$, and suppose the function $f : (0,T) \times D \to \mathbb{R}$ satisfies the Carathéodory conditions, i.e.

(1) $f(\cdot, x)$ *is measurable for each* $x \in D$,

(2) $f(t, \cdot)$ *is continuous for almost all* $t \in (0,T)$,

and

$$|f(t,x)| \leq c(t)(1 + |x|) \text{ on } (0,T) \times D, \text{ with } c \in L^1(0,T).$$

Then

$$x' = f(t,x), \qquad x(0) = x_0$$

has an absolutely continuous solution for every $x_0 \in D$.

Lemma A.2.2. *(Pankov 1990, Lemma 1.3, p.47)*
Assume $\lambda \in C(J)$, where J is an half-line or $J = \mathbb{R}$, $\lambda \geqslant 0$ and

$$\lambda^2(t)\Big|_{t_1}^{t_2} + \alpha \int_{t_1}^{t_2} \lambda^p(t)\,dt \leq 0, \quad t_1, t_2 \in J, \;\; t_1 \leq t_2$$

for some $\alpha > 0$ and $p \geqslant 2$. Then we have

$$\lambda(t) \underset{(\geqslant)}{\leq} \left[\frac{c}{c_1\lambda(t_0)^{p-2}(t-t_0)+c}\right]^{1/(p-2)} \lambda(t_0),$$

for $t \geq t_0$, ($t \leq t_0$, respectively), $\;t_0 \in J$, $\;$if $p > 2$, and

$$\lambda(t) \underset{(\leq)}{\geq} e^{-c(t-t_0)}\lambda(t_0)$$

for $t \geq t_0$, ($t \leq t_0$, respectively), $\;$if $p = 2$. Here $c > 0$ and $c_1 > 0$ depend on α and p.

Lemma A.2.3. *(Gajewski et al. 1974; Showalter 1996) [Integration by parts formula]*
Let the Banach space V be dense and continuously embedded in the Hilbert space H, $H^* = H$, so that $V \subset H \subset V^*$ and $X = L^p(0,T;V) \cap L^{p_0}(0,T;H)$. Then the Banach space

$$W = \{u | u \in X,\, u' \in X^*\}$$

is continuously embedded in $C([0,T];H)$ and for every $u, v \in W$ the integration by parts formula

$$(u(t),v(t)) - (u(s),v(s)) = \int_s^t \Big[\langle u'(\tau), v(\tau)\rangle + \langle u(\tau), v'(\tau)\rangle\Big] d\tau,\; s,t \in [0,T]$$

holds, where $(\cdot,\cdot)$ denotes the scalar product in H .

Lemma A.2.4. *(Lions 1969; Showalter 1996)[Lions-Aubin Compactness Lemma]*
Let B_0, B, B_1 be Banach spaces with $B_0 \subset B \subset B_1$; assume $B_0 \hookrightarrow B$ is compact and $B \hookrightarrow B_1$ is continuous. Let $1 < p < \infty$, $1 < q < \infty$, let B_0 and B_1 be reflexive, and define

$$W = \{v | v \in L^p(0,T;B_0), v_t \in L^q(0,T;B_1)\}.$$

Then the inclusion $W \hookrightarrow L^p(0,T;B)$ is compact.

Let Banach spaces X and Y be subspaces of linear space V. Then $X \cap Y$ is a Banach space with the norm $||x||_{X\cap Y} = ||x||_X + ||x||_Y$

Let X, Y be Banach spaces, continuously embedded in the locally convex space V. Then

$$X + Y = \{x + y |\, x \in X, y \in Y\}$$

is a Banach space with the norm $||z||_{X+Y} = \inf\limits_{x\in X, y\in Y, x+y=z} \max\{||x||_X, ||y||_Y\}$.

Theorem A.2.5.*(Gajewski et al. 1974, Theorem I.5.13) Let X, Y be Banach spaces, continuously embedded in the locally convex space V, $X \cap Y$ is dense in X and in Y. Then*

$$X^* + Y^* = (X \cap Y)^*$$

and

$$(X + Y)^* = X^* \cap Y^*.$$

Notations

$\mathbb{R}^N$ is Euclidean N- dimensional space

Ω is a bounded domain in $\mathbb{R}^N$ with a Lipschitz continuous boundary $\partial\Omega$

$U \subset\subset \Omega$ is if $U \subset \overline{U} \subset \Omega$ and $\overline{U}$ is compact, i.e. U is compactly contained in Ω

$Q_{t_1,t_2} = (t_1, t_2) \times \Omega, \;\; t_1 < t_2$

$Q_T = (0, T) \times \Omega \quad \text{or} \quad Q_T = (-\infty, T) \times \Omega$

$H^{1,p}(\Omega)$ is a Banach space, $H^{1,p}(\Omega) = \{u \in L^p(\Omega), u_{x_i} \in L^p(\Omega), \, i = 1, \ldots, N\}$

$H^1(\Omega) = H^{1,2}(\Omega)$

$L^p(0, T; H^{1,s}(\Omega))$ is the set of measurable functions $u : t \in (0, T) \mapsto u(t) \in H^{1,s}(\Omega)$, such that $||u(\cdot)||_{H^{1,s}(\Omega)} \in L^p(0, T)$

$\mathcal{D}'(\Omega)$ distributions defined on Ω

X^* is the dual space to the Banach space X

$\langle\cdot,\cdot\rangle$ is a duality product between X and X^*, $\langle\cdot,\cdot\rangle : X^* \times X \to \mathbb{R}$

$L^r_{\text{loc}}((-\infty, T]; B)$, $1 \le r \le +\infty$ is the space of functions z, such that $z \in L^r(t_1, T; B)$ for all $t_1 < T$, where B is a Banach space

The Banach space X is reflexive if $i(X) = X^{**}$

The topological space X is separable if X contains a countable dense subset

The set M is dense in the topological space X if $\overline{M} = X$

Let Banach spaces X and Y be subspaces of linear space V. Then $X \cap Y$ is a Banach space with the norm $||x||_{X\cap Y} = ||x||_X + ||x||_Y$

Let X, Y be Banach spaces, continuously embedded in the locally convex space V. Then

$$X + Y = \{x + y | \, x \in X, y \in Y\}$$

is a Banach space with the norm $||z||_{X+Y} = \inf_{x \in X, y \in Y, x+y=z} \max\{||x||_X, ||y||_Y\}$

The generic constants will be denoted by C, C_i, c, and c_i

$\frac{1}{p} + \frac{1}{q} = 1$ for $p > 1, q > 1$.

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