$$= -\boldsymbol{\Lambda}_{m-1}^{-1}\boldsymbol{f}_{m-1} + \ddot{\boldsymbol{x}}_{m-1_{\text{open}}} \quad \text{and,} \tag{49}$$

$$= -\boldsymbol{\Lambda}_{m}^{-1}\boldsymbol{f}_{m} + \ddot{\boldsymbol{x}}_{m_{\text{open}}}. \tag{50}$$

The previous result concerning the existence of solutions indicates that if $\boldsymbol{f}_m$ satisfies these equations, then $\boldsymbol{f}'_m = \boldsymbol{f}_m + \boldsymbol{v}$ also satisfies these equations where $\boldsymbol{v}$ is any vector in the null space of $\boldsymbol{A}$ as defined previously. By substituting $\boldsymbol{f}'_m$ into (50), we obtain the following equation:

$$\boldsymbol{a}'_r = -\boldsymbol{\Lambda}_m^{-1}\boldsymbol{f}'_m + \ddot{\boldsymbol{x}}_{m_{\text{open}}}, \tag{51}$$

$$= -\boldsymbol{\Lambda}_m^{-1}(\boldsymbol{f}_m + \boldsymbol{v}) + \ddot{\boldsymbol{x}}_{m_{\text{open}}}. \tag{52}$$

Since $\boldsymbol{\Lambda}_m^{-1}\boldsymbol{v} = \boldsymbol{0}$, then

$$\boldsymbol{a}'_r = -\boldsymbol{\Lambda}_m^{-1}\boldsymbol{f}_m + \ddot{\boldsymbol{x}}_{m_{\text{open}}}, \tag{53}$$

$$= \boldsymbol{a}_r. \tag{54}$$

Therefore, the reference member acceleration has not changed. To find the associated force at the tip of the other singular chain, $\boldsymbol{f}'_{m-1}$, this result is substituted into (48), which yields the following equalities:

$$\boldsymbol{a}_r = \boldsymbol{\Lambda}_*^{-1}(\boldsymbol{f}_* + \boldsymbol{f}_{m-1} + \boldsymbol{f}_m) = \boldsymbol{\Lambda}_*^{-1}(\boldsymbol{f}_* + \boldsymbol{f}'_{m-1} + \boldsymbol{f}_m + \boldsymbol{v}) \tag{55}$$

which implies that

$$\boldsymbol{f}'_{m-1} = \boldsymbol{f}_{m-1} - \boldsymbol{v}. \tag{56}$$

This result shows that any components of force added to the tip of one singular manipulator in the "direction" of the singularity, are compensated for by equal and opposite components of force at the tip of the other singular manipulator, and the acceleration of the reference member is still unique.

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Classification Using Set-Valued Kalman Filtering and Levi's Decision Theory

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***Abstract*—We consider the problem of using Levi's expected epistemic decision theory for classification when the hypotheses are of different informational values, conditioned on convex sets obtained from a set-valued Kalman filter. The background of epistemic utility decision theory with convex probabilities is outlined and a brief introduction to set-valued estimation is given. The decision theory is applied to a classifier in a multiple-target tracking (MTT) scenario. A new probability density, appropriate for classification using the ratio of intensities, is introduced.**

I. INTRODUCTION

Bayes theory, for all its historical significance and mature theoretical development [1], [2], continues to be re-examined with regard to the requirement for knowledge of *a priori* probability distributions. The minimax and Neyman-Pearson techniques are well-known methods [3] to eliminate the need for specific priors. In this paper we present an application of a new method promoted by Stirling and Morrell [4]–[6] that incorporates an information valuation into the decision making process. This information valuation provides the decision theory with the potential for more human-like response, as both the importance of the decision and its truth value are incorporated in the judgment.

The philosophical viewpoint is based on the work of Levi [7]–[9]. Under the Levi theory, two modifications are made to traditional

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Bayesian decision theory. In the first place, a convex set of prior probabilities is postulated, eliminating the need for restrictive and often unrealistic assumptions about initial conditions. An elegant estimation-theoretic using convex Bayesian theory is the set-valued Kalman filter, whose output is the set of conditional means based on the observations and a convex set of priors.

The other modification of the theory is the use of Levi's epistemic utility decision theory, in which decisions are made by trading off informational value vs. correctness. As in traditional Bayes theory, hypotheses are accepted if a criterion function exceeds some threshold. The criterion function depends not only on the perceived truth of the hypotheses, but also on their (subjective) importance. As in human decisionmaking, hypotheses of low importance might be rejected regardless of their truth, simply because the decision-maker is apathetic about low-priority decisions in the face of other higher-priority options. Tradeoff between importance and truth of the hypotheses is governed by a design parameter called the "boldness." All hypotheses for which the ratio of truth value to importance value exceeds the boldness are accepted, leading to another human-like attribute of suspended judgment. Suspended judgment is useful in scenarios where sequential data is available; the set of best decisions at one time may be refined upon successive observation, and a choice made at a later time is not inhibited by a short-sighted premature termination of an option at an earlier time.

Decision theory, according to Levi, has as its proximate aim the avoidance of error [9]. Suspension of judgment, which might be termed agnosticism, allows the avoidance of error by refusing to answer the question. Thus, we might accept both hypotheses X and Y, knowing that surely at least one of them must be correct. Such a decision, however, lacks boldness in a technical sense to be developed in this paper. By being more bold we avail ourselves of more decisive capability at the potential expense of more error. It is this tradeoff between agnosticism and error that forms the heart of the decision theory described in this paper, and that is different from the viewpoint provided by conventional Bayes theory. An acronym has been coined to describe the general philosophy entailed in Levi's methods: EUCLID, for Epistemic Utility for Computer Learning, Inference, and Decision Making.

It has been argued that decisionmaking based on trading off truth vs. importance is nothing more than weighted Bayes risk decision theory. There are, however, some subtle but important distinctions. In the first place, the choices made are not traded off against each other. That is, we do not make the choice based on the risk of one choice compared to another choice. The importance measure developed by Levi is in a sense an absolute measure, indicative of the relevance of the choice. Thus, as an explicit part of the development system designers are able to incorporate into their decision model effects that would otherwise be difficult. Operationally, decision costs in a Bayesian setting might be found that give performance identical to the EUCLID method, but the insight and explicit presentation of ideas makes the Levi method useful.

While convex sets of prior probabilities make sense from a decision-theoretic point of view, it may be difficult to conceive of how actually the sets of priors may be obtained in many problems. Indeed, other than the set-valued Kalman filter (SVKF), the use of convex sets has seen little application. It is the intent of this paper to marry the ideas of set-valued Kalman filtering and convex decision theory, using the output of the set-valued Kalman filter to form a convex set of likelihood functions. The approach may be generally applicable, but to reinforce the ideas and make them more concrete, an example of classification in a multiple-target tracking (MTT) scenario is presented. Further detail on the MTT problem can be found in the companion paper [10]. In the current paper, we develop a likelihood function formed from the ratio of two non-zero mean Gaussian densities is presented for target classification.

The remainder of this paper proceeds as follows. In Section II the salient features of EUCLID (Levi's) decision theory are outlined. The fundamental ideas of the set-valued Kalman filter are briefly discussed in Section III. In Section IV, the set-valued Kalman filter outputs are used as conditioning information for the credal probabilities. In Section V, an example of this joint estimation scheme in multiple-target tracking is presented. Finally, a discussion and conclusions are presented in Section VI.

II. LEVI DECISION THEORY

We present in this section a summary of the epistemological framework for decision making as originated by Levi [7] and propounded by Stirling and Morrell [4]. Under this epistemology, two measures are used in making decisions: the informational value of a hypothesis to the decisionmaker, and its truth value. For example, an agent, denoted by X, performing target classification would place much more informational value on a threat than on a benign target. A decision rule incorporating such a value would be reluctant to reject the hypothesis of a threat, even though the probability of a threat may be small.

To formalize this process, let U denote the set of all hypotheses under consideration by X at a particular juncture. U is called the ultimate partition for X. Let n be the number of hypotheses to choose from in U. X may select any subset of hypotheses from U—he is not restricted to taking one and only one. As X may select various subsets of U, there are 2^n potential answers available for X. Let g denote a set of potential answers under consideration by X, $g \subset U$.

To obtain information, X must reject certain hypotheses. The alternative of accepting all hypotheses in U conveys no information to X—it is like selecting all choices in a multiple-choice exam when only one should be selected to impart information. On the other hand, accepting all hypotheses avoids error; certainly one of the choices must be true. There is thus a tradeoff between information and error.

By writing $U = \{h_1, h_2, \cdots, h_n\}$ and $g \subset U$, we define the informational utility of rejecting the subset g by $M(g)$, where

$$M(g) = \sum_{h_i \in g} M(h_i)$$

and $M(h_i)$ is the utility of rejecting a single hypothesis. The informational utility is constrained so that $M(h_j) > 0$ and $\sum_{j=1}^{n} M(h_j) = 1$; M is often termed the information-determining probability. If $M(h_i) < M(h_j)$ then rejecting h_j is informationally more valuable than rejecting h_i; equivalently, accepting h_i is of more information value than accepting h_j.

The error-determining utility is represented by $Q(g)$, which is the so-called credal-probability. $Q(h_i)$ represents the agents subjective probability (analogous to the prior probability in classical Bayesian reasoning) that h_i is true.

The tradeoff between information value and credal value can be written as the expected utility function. This expected utility can be written as [4]

$$u(g) = Q(g) - bM(g) \tag{1}$$

where b is a parameter used to weight the importance of error-avoidance versus informational value to the agent X. The parameter b is termed the boldness, where $0 \leq b \leq 1$. If $b = 1$, then the informational value M is weighed heavily in comparison to avoidance of error (as determined by the credal probability Q). X's best policy for decision making, given his credal probability Q and his information-determining probability M, is to maximize $u(g)$. This can be accomplished if the agent takes all and only those $h_i \in g$ such that

$$u(h_i) = Q(h_i) - bM(h_i) > 0.$$

Reorganizing this, agent X accepts h_i iff

$$Q(h_i)/M(h_i) > b. \tag{2}$$

This ratio test is similar to ratio tests common in traditional Bayes theory. In the context of the classification problem discussed below, it will be called the classification likelihood ratio test (CLRT).

Further flexibility is introduced by the use of convex sets of credal probabilities. Suppose that agent X has two credal probabilities, Q and Q', and that X has compelling external reasons for accepting both of them and is thus unable to decide between them. This represents a state of ignorance for X. This ignorance, the inability to select a definite credal probability, is in contrast to uncertainty in the decisionmaking processing, which is already modeled by the density functions. Rather than forcing X to make an arbitrary decision of one particular density, one approach is to relax the requirement of a single credal probability and consider instead convex combinations of Q and Q', e.g.

$$\alpha Q + (1 - \alpha)Q'$$

for $\alpha \in [0, 1]$. This combination is called a convex credal combination. A set $\mathcal{B}_{X,t}$ denotes the credal state of X at time t. If $Q \in \mathcal{B}_{X,t}$ and $Q' \in \mathcal{B}_{X,t}$, then all convex combinations $Q^\alpha = \alpha Q + (1 - \alpha)Q' \in \mathcal{B}_{X,t}$ for all $\alpha \in [0, 1]$. As shown by Stirling [4], convex sets of credal probabilities are closed under conditioning. It is also possible, though not useful to us in the current context, to define convex sets of information-valuation functions.

III. Set–Valued Filtering

Related to the idea of convex sets of credal probabilities is the concept of set-valued estimation. Traditional Bayesian estimation propagates a density by means of conditional probability. The best-known instance of this is the Kalman filter, in which the mean and variance of a Gaussian density are propagated. One of the problems with traditional Bayesian estimation is the choice of the prior probabilities. Often priors must be chosen on the basis of subjective judgment. The improper selection of a prior results in biased or incorrect results for all stages of the estimation (although asymptotically the effect of the prior probability may vanish). Ignorance concerning the prior probability can be explicitly displayed using a convex set of priors and employing a set-valued estimation procedure. In set-valued estimation, rather than a single density, a whole family of densities is propagated. Since convex sets of credal probabilities are closed under conditioning, an initial convex set of densities is propagated as a set of densities under all conditional updates.

When applied to convex sets of Gaussian densities, the set-valued Kalman filter (SVKF) is produced. Rather than a point-valued mean, the set-valued Kalman filter propagates a convex set of conditional means. Rather elegantly, this set of means can be propagated in closed form, with a computational load only slightly greater than the point-valued Kalman filter [5]. This is rather remarkable, since the effect is the same as propagating an infinite number of Kalman filters, each with a different initial mean. A single common covariance matrix is associated with each mean to determine the Gaussian densities. To contrast the traditional (point-valued) Kalman filter with the SVKF, the sets produced by the set-valued Kalman filter give a spread of first moments, whereas the second-moment (covariance) computed for each estimate in the set is a measure of the spread of the true value relative to the estimate. Thus, the SVKF provides two measures of uncertainty that may be useful, rather than just one.

As the SVKF runs, the set of estimates converges asymptotically to a point estimate, provided that the system is observable. Lack of observability in a state variable leads to failure of convergence. The SVKF is thus well-suited to poorly observable systems. Those state variables which are observable will converge; those which are not will not. The "size" of the set is also an indication of the convergence of the filter. This may be more meaningful than examination of the posterior covariance matrix which is used in many cases to determine quality of estimate. Another viewpoint of the SVKF is that it provides all estimates that are consistent with the data, subject to the uncertainty of the initial condition. As more data are received, the initial uncertainty decreases and the sets decrease in size. Figure 1 illustrates the performance of the SVKF (and the association logic derived from the decision theory above) in a multiple-target tracking problem.

In what follows, we will use the SVKF as the estimator. The updated set of means at time t, using the measurement at time t will be denoted by $X_{t|t}$, and is given by [5]

$$X_{t|t} = \{x \in \Re^n : ||K_{tt}^{-1}(x - c_{tt})|| \leq 1\}$$

where the norm may be any convenient norm. If the L_2 norm is used, the set of means is an ellipsoid in n dimensions, n being the number of state variables. c_{tt} is the centroid of the ellipse of means and K_{tt} defines the extent of the ellipse about the centroid.

IV. SVKF and Epistemic Utility Classification

In this section we will combine the use of the convex epistemic decision theory of Section II and the set-valued Kalman filter of Section III in a decision-theoretic application. At time t an agent X has a sequence of observations from τ_t targets and desires to assign each of the τ_t targets to one of T target types. Let the observations at time t be denoted by z_{it}, $i = 1, 2, \cdots, s_t$, where s_t is the number of observations of the target. For simplicity in the following discussion, it is assumed that the number of targets τ_t is equal to the number of observations s_t. (This is for clarity of exposition with respect to the classification issue. In many circumstances, e.g. ballistic tracking, the number of targets actually present may be more or less than the number of observations due to target clustering, clutter, data dropout, etc.) From each observation, a convex set of means is formed using the set-valued Kalman filter. Let these sets of means be denoted X_{it}, $i = 1, 2, \cdots, s_t$.

Let the classification hypotheses be represented by h_{ijt}, where h_{ijt} is the hypothesis that target i (from observation i), $i = 1, 2, \cdots, s_t$, classifies as target type j, $j = 1, 2, \cdots, T$ at time t. Then for each target there is an ultimate partition

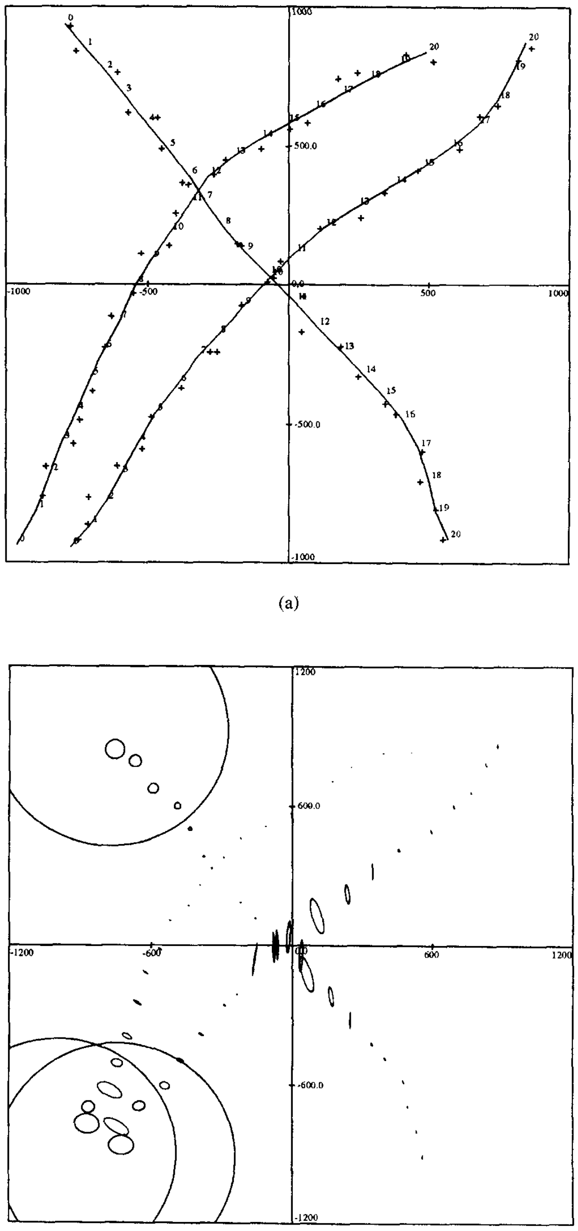
$$U_{it} = \{h_{i1t}, h_{i2t}, \cdots, h_{iCt}\}.$$

The ith target is classified as being of type j if we fail to reject the hypothesis h_{ijt}.

Traditionally, decisions about classification hypotheses would be made strictly on the basis of the observed data. That is, we would form a likelihood function

$$L(h_{ijt}|z_{it}) \tag{4}$$

and make our decision based on the likelihood function and the prior probability of each hypothesis. We are motivated to look beyond this traditional scheme for two reasons. First, the prior probabilities of the hypotheses may be unknown; indeed, it is not the probability of occurrence of a particular target type that may be of most interest to us but rather the importance of the target type. Second, in the presence



(a)

(b)

Fig. 1. Three crossing tracks: (a) Simulated trajectories and observations. (b) Filtered track sets.

of system and observation noise, we should be reluctant to use the simple observation z_{it} as the conditioning quantity in our likelihood function. Since we are classifying the target based on a history of observations, decision should, if possible, be based on the entire history (including prior assumptions), not just the most recent noisy observation. The convex set of means X_{it} incorporates information from all observations. The uncertainty due to a single observation is thus averaged out. In addition, the set explicitly displays, by its size, all uncertainty introduced by the uncertainty in the initial assumptions.

In light of this reasoning, it makes sense to consider an epistemic utility approach. The credal probability function is determined upon the set-valued estimate, and we write it as $Q(h_{ijt}|X_{it})$. By so doing, we have introduced a credal probability which is no longer unique. For each point $x \in X_{it}$, we have a different credal probability which we will write as $Q(h_{ijt}|x)$, each of which is consistent with all observations and prior assumptions. This set of credal probabilities is not convex, in the sense defined by Stirling. However, it is defined over a convex set of parameters (the set X_{it}).

By using the estimated sets X_{it} as the basis for our decision making, we are performing something similar to decision-directed estimation [11], but in reverse. In this case we are using estimates (which are actually decision directed themselves) to form likelihoods to be used in a decision process. Such involution makes claims of optimality difficult. The merit of the method, however, is the ability to explicitly indicate what is known about the data.

To complete the epistemic utility formulation, to each hypothesis we assign a utility of rejection, and denote that utility by $M(h_{ijt}|X_{it})$. The decision rule can be written conceptually as

$$Q(h_{ijt}|X_{it}) > bM(h_{ijt}|X_{it}) \tag{5}$$

where b is a boldness parameter as described above. $Q(h_{ijt}|X_{it})$ actually represents a whole family of credal probabilities and $M(h_{ijt}|X_{it})$ actually represents a whole family of information-determining probabilities. Well-defined comparisons must be made at specific points $x \in X_{it}$. We note four possible alternative strategies for making this decision:

1) Accept any hypothesis such that

$$Q(h_{ijt}|x) > bM(h_{ijt}|x) \text{ for some } x \in X_{it}. \tag{6}$$

From a theoretical point of view, this is most defensible, as all points in X_{it} are equally possible according to the set-valued Kalman filter. However, from a computational point of view it may be hard to determine the existence of such a point x.

2) Accept any hypothesis such that

$$\max_{x \in X_{it}} Q(h_{ijt}|x) > b \min_{x \in X_{it}} M(h_{itj}|x). \tag{7}$$

This strategy is reluctant to reject any hypotheses. It is the most cautious method, and would have the lowest probability of error, but gains information the slowest. Computationally, it involves both a constrained minimization and maximization, but these can often be performed using a gradient search method.

3) Accept any hypothesis such that

$$\min_{x \in X_{it}} Q(h_{ijt}|x) > b \max_{x \in X_{it}} M(h_{itj}|x). \tag{8}$$

This is the boldest strategy, acquiring information (in the Levi sense) fastest. The error would, on the other hand, be the highest of the strategies. Computationally, it is about the same as strategy 2.

4) Select any hypothesis such that

$$\max_{x \in X_{it}} Q(h_{ijt}|x) > bM(h_{itj}|x). \tag{9}$$

In other words, find the point $x \in X_{it}$ that maximizes Q, and evaluate M at that point. This is still quite cautious, but avoids the computation of another constrained optimization.

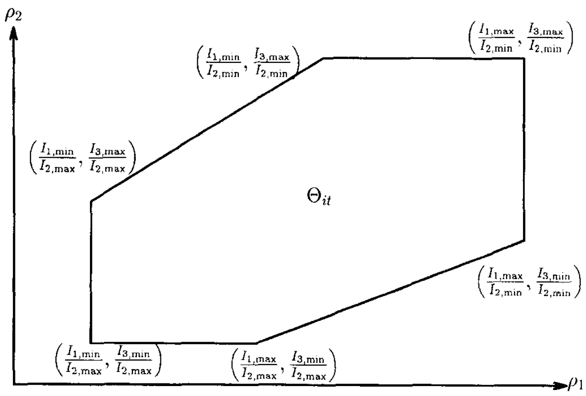


Fig. 2. Set formed by the ratios of means.

V. Application to Multiple-Target Classification

In this section, the ideas presented previously are applied to a multiple-target classification problem. At every time t, s_t observations in C different intensity spectral wavebands are made and associated with τ_t track sets representing the set of possible means from a set-valued Kalman filter. The observed values are position (azimuth and elevation) and intensity, giving a three-dimensional observation vector. The details of the tracking, dynamics, and association are presented in [10]. The tracks from the different intensity spectral wavebands are combined together. A track set consists of a set of means representing all possible values of the state variables (position, velocities, and each different intensity) consistent with the observations and prior assumptions.

Classification of targets in this system is performed using ratios of intensities. To make the example more complete, assume that there are three spectral bands, denoted as color C_1, color C_2, and color C_3. The set of colors will be written $\mathcal{C}$. The intensities in each of these bands are written as I_{C1}, I_{C2}, and I_{C3}. The classification ratios used are $\rho_1 = I_{C_1}/I_{C_2}$ and $\rho_2 = I_{C_3}/I_{C_2}$. These two ratios will be written as an ordered pair $\rho = (\rho_1, \rho_2) \in \Re^{C-1}$. In general with C colors, the set of ratios form a $(C-1)$-tuple.

It is assumed that there is a set of prior target ratios used for classification, denoted by $r_j = (r_{1j}, r_{2j})$, $j = 1, 2, \cdots, T$. These ratios represent the analytically or empirically derived values for the targets to be classified. Classification will be made based on the similarity of the observed ratios ρ to the target ratios r.

The "observations" of the intensities are taken from the set-valued Kalman filter. The intensities are thus assumed to have a Gaussian density (justifying the use of the Kalman filter) with mean i_{ict} and variance σ^2_{ict}, where the mean i_{it} comes from the set computed by the set-valued Kalman filter

$$i_{ict} \in [I_{ict,\min}, I_{ict,\max}] \triangleq I_{ict}$$

for $c \in \mathcal{C}$ and $i = 1, 2, \cdots, s_t$. The set of ratios of means consistent will all observations forms an irregular hexagon, as illustrated in Fig. 2. Any point in this region is a valid estimate for the ratio of means. This region of possible ratios for the ith target is denoted by the symbol $\Theta_{it} \subset \Re^{C-1}$ and the set of possible intensities for the ith target is denoted by $\mathcal{I}_{it} \subset \Re^{C}$. Any point in the set $\mathcal{I}_{it}$ has a unique corresponding point in Θ_i; the inverse mapping ¿ from Θ_{it} to $\mathcal{I}_{it}$ is not well defined for most elements in Θ_{it}.

A credal probability function is formed from the density function of a ratio of Gaussian random variables. The derivation of this density, a non-central Cauchy density, is outlined in the appendix. It can be written as

$$f_\rho(\rho|\mathbf{i}_{it}, \mathbf{R}_{it}) = \exp[(D^2 - C)/2]\frac{1}{2\pi\sqrt{|\mathbf{P}|}A^3}[1 + D^2] \quad (10)$$

where $\rho = [\rho_1, \rho_2]'$, $\mathbf{i}_{it} = [i_{ic_1t}, ii_{ic_2,t}, i_{ic_3t}]' \in \mathcal{I}_i$ and $A = A(\rho; \mathbf{i}_{it}, \mathbf{R}_{it})$, $B = B(\rho; \mathbf{i}_{it}, \mathbf{R}_{it})$, $C = C(\mathbf{i}_{it}, \mathbf{R}_{it})$ and $D = D(\rho; \mathbf{i}_{it}, \mathbf{R}_{it})$ are given in (15), (16), (17), and (18). $\mathbf{R}_{it}$ is the inverse covariance matrix of the intensity vector; typically the intensities are assumed independent so that

$$\mathbf{R}_{it} = \mathrm{diag}[1/\sigma^2_{ic_1t}, 1/\sigma^2_{ic_2t}, 1/\sigma^2_{ic_3t}].$$

The credal probability is formed by integrating this density function over the region of interest for classification. Let B_j be a ball formed in ratio space around the ratio $\mathbf{r}_j$. The credal probability that the ith track classifies to class j conditioned on the intensity $\mathbf{i}_{it} \in \mathcal{I}_{it}$ and the inverse covariance matrix $\mathbf{R}_{it}$ is

$$Q(B_j|\mathbf{i}_i, \mathbf{R}_{it}) = \int_{B_j} f_{\mathbf{r}_j}(\mathbf{r}_j|\mathbf{i}_i, \mathbf{R}_{it})d\rho. \quad (11)$$

An information-determining probability function can be formulated on the basis of the importance of the target, as mentioned above. For more important targets, there should be greater utility in not rejecting them. For a target ratio (r_1, r_2), the utility of rejecting it should also be a function of the distance of the target ratio from the set of possible ratios Θ_{it}. A reasonable utility function may be defined in a manner similar to [10]. Let $d(\mathbf{r}, \rho)$ denote the (Euclidean) distance from a target ratio $\mathbf{r}$ to a point $\rho \in \Theta_{it}$. Around each target ratio point define a seriously possible region, that is, a region outside of which no classification can be reasonably obtained. Denote the seriously possible region by Σ_{jt}. Now let the importance of a class be indicated by weighting the distance of the class target ratio from the set of possible mean ratios, Θ, normalized so that utility of rejection forms a probability. Let $m_{ijt}(\mathbf{r}_j, \rho_{it})$ represent the information-determining probability of rejecting the hypothesis that target i is of class j at time t. Then we can write

$$m_{ijt}(\mathbf{r}_j, \rho_{it}) = \begin{cases} \frac{\alpha_j d(\mathbf{r}_j, \rho_{it})}{N_m} & \rho_{it} \in \Sigma_{jt} \\ \text{undefined} & \rho_{it} \notin \Sigma_{jt} \end{cases} \quad (12)$$

where the normalizing factor N_m is such that

$$\sum_{j=1}^{T} \int_{\Sigma_{jt}} m_{ijt}(\mathbf{r}_j, \rho_{it})d\rho = 1.$$

The weighting function α_j is set so that there is less utility in rejecting important classes. Thus, more important classes have a smaller α_j. An illustration of this information-determining probability density is given in Figure 3. The assignment of the seriously possible regions Σ_{jt} are part of the design problem and should reflect the agent's priority assigned to the targets.

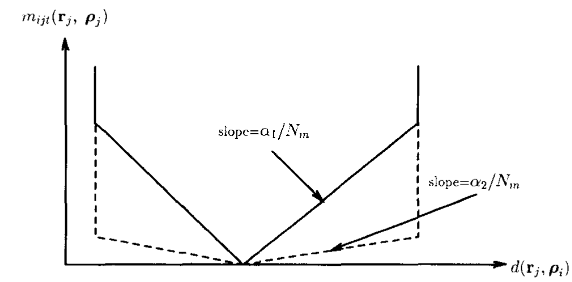


Fig. 3. Illustration of the utility function for targets of two different priorities and equal seriously possible regions. Solid line: utility of rejecting a target of low importance. Dashed line: utility of rejecting a target of high importance.

The information value of rejecting the ball B_j centered on the ratio $\mathbf{r}_j$ is

$$M_{it}(B_j|\rho_{it}) = \int_{B_j \cap \Sigma_{jt}} m_{itj}(\mathbf{r}_j, \rho_{it}) d\rho. \tag{13}$$

The credal probability and the information-determining probability can now be combined in the classification likelihood ratio test (CLRT). We will state this using (9) from above. Let b be a fixed boldness. Then the CLRT accepts class j if

$$\frac{\max_{\mathbf{i}_{it} \in \mathcal{I}_{it}} Q(B_j|\mathbf{i}_{it})}{M_{it}(B_j|\rho_{it})} > b. \tag{14}$$

By taking the ball around $\mathbf{r}_j$ infinitesimally small and using the continuity of the integrands, this CLRT can be written as

$$\frac{\max_{\mathbf{i}_{it} \in \mathcal{I}_{it}} f_{\mathbf{r}_j}(\mathbf{r}_j|\mathbf{i}_{it})}{m_{ijt}(\mathbf{r}_j, \rho_{it})} > b.$$

This formulations requires computation of

$$\max_{\mathbf{i}_{it} \in \mathcal{I}_{it}} f_{\mathbf{r}_j}(\mathbf{r}_j|\mathbf{i}_{it}).$$

This is a constrained optimization problem, but is not too difficult due to the linear constraints and the fact that all derivatives can be readily computed. Taking the logarithm of the density function results in a function that is nearly quadratic so that points near the maximum can be obtained in one Newton iteration.

VI. Discussion and Conclusion

The use of the set-valued Kalman filter as the observation in the likelihood function has several features that recommend it. In the first place, except in the case of high noise observations, the set-valued estimate will usually include the observation point. The use of the set is thus (usually) a generalization of the epistemic utility, which in turn is a generalization of Bayesian methods. In those circumstances where the set does not contain the observation, it is because information preceding the observation suggests that the observation is noisy. The filtered data thus avoid incorrect classification due to a single bad observation.

In addition, the approach described above should provide some robustness with respect to the set of prior target ratios. The ratios $\mathbf{r}_j$ determine how targets are classified. However, it is unlikely that they will be known precisely. The use of the set-valued estimate to compare with the ratios combined with a maximization routine should make it so the system is somewhat forgiving of prior target ratios that are slightly incorrect. Further robustness could be provided using a set of prior target ratios consistent with all believed values of the prior target ratios.

Because the classification decision is based on the CLRT for each of the hypotheses in the ultimate partition U, it is possible to arrive at a set of possible classifications at each observation of the targets. At the beginning of the observations, the set may contain several, if not all, of the possible classes. This is not disturbing, however, since it represents the ignorance about the targets contained in the initial conditions used to start the SVKF. As more observations are made, the size of the ratio set illustrated in Fig. 2 shrinks until all of the hypotheses in U are rejected except one.

The set-valued classification is very similar to the way that human decisions are made. As the human observer gathers more data, he rejects obviously bad classifications until he has made enough observations to settle on one classification for each target. In man-in-the-loop decision making, the classifier presented here allows for the human decisionmaker to hold off a commitment of resources until more information is obtained. For example, if the classification set includes only benign targets, no resources would be committed to interception. On the other hand, if the set contains both benign and threat targets, the human decisionmaker may elect to wait for more observations before allocating resources. Finally, if the set contains only threats, immediate action can be taken.

The use of the noncentral Cauchy distribution for classification of ratio intensities also appears to be new here. Further investigation into properties of the noncentral Cauchy are necessary to determine the merit, if any, of this distribution over others such as, for example, the ratio of the squares of the intensities. It does, however, have a convenient closed form expression.

Appendix: Derivation of the Noncentral Cauchy Distribution

In this section we derive the noncentral Cauchy distribution in two variables which is used as the credal probability in Section V. Extension to higher dimensions follows the same development. Let $\mu = [\mu_1, \mu_2, \mu_3]'$ denote a vector of means of a Gaussian random vector $\mathbf{x} = [x_1, x_2, x_3]$, $\mathbf{P}$ be the covariance matrix of the same and $\mathbf{R} = \mathbf{P}^{-1}$ be the inverse covariance matrix,

$$\mathbf{P} = \begin{bmatrix} \sigma_1^2 & p_2 & p_3 \\ p_2 & \sigma_2^2 & p_4 \\ p_3 & p_4 & \sigma_3^2 \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} q_1 & q_2 & q_3 \\ q_2 & q_4 & q_5 \\ q_3 & q_5 & q_6 \end{bmatrix}.$$

Let $w = x_1/x_2,$, $z = x_3/x_2$ and $y = x_2$. Then

$$f_{wzy}(w, z, y) = \frac{y^2}{(2\pi)^{3/2}\sqrt{|\mathbf{P}|}} \exp[-(\alpha - \mu)'\mathbf{R}(\alpha - \mu)/2]$$

where $\alpha = [wy, y, zy]'$. This density can be integrated to eliminated y by completing the square. The variables A, B, C, and D resulting from this are

$$\begin{aligned} A^2((w,z);\mu,\mathbf{R}) &= q_1 w^2 + 2q_2 w + 2q_3 wz \\ &\quad + 2q_6 z + q_6 z^2 + q_4 \end{aligned} \tag{15}$$

$$\begin{aligned} B((w,z);\mu,\mathbf{R}) &= -(2q_1\mu_1 w + 2q_2(\mu_1 + \mu_2 w) \\ &\quad + 2q_3(\mu_1 z + \mu_3 w) + 2q_4\mu_2 \\ &\quad + 2q_6(\mu_2 z + \mu_3) + 2q_6\mu_3 z) \end{aligned} \tag{16}$$

$$\begin{aligned} C(\mu,\mathbf{R}) &= q_1\mu_1^2 + 2q_2\mu_1\mu_2 + 2q_3\mu_1\mu_3 \\ &\quad + q_4\mu_2^2 + 2q_6\mu_2\mu_3 + q_6\mu_3^2 \end{aligned} \tag{17}$$

$$\begin{aligned} D((w,z);\mu,\mathbf{R}) &= B((w,z);\mu,\mathbf{R})/ \\ &\quad (2A((w,z)\mu,\mathbf{R})). \end{aligned} \tag{18}$$

Performing the integration yields

$$f_{w,z}(w,z|\mu,\mathbf{R}) = \exp[(D^2 - C)/2]\frac{1}{2\pi\sqrt{|\mathbf{P}|}A^3}[1 + D^2].$$

In the development above, the intensities may be assumed to be independent so that $\mathbf{P}$ and $\mathbf{R}$ are diagonal, greatly simplifying the computations.

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N-Learners Problem: Fusion of Concepts

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***Abstract*—Given *N* learners each capable of learning concepts (subsets) in the sense of Valiant, we are interested in combining them using a single *fuser*. We consider two cases. In *open fusion* the fuser is given the sample and the hypotheses of the individual learners; we show that a fusion rule can be obtained by formulating this problem as another learning problem. We show sufficiency conditions that ensure the composite system to be better than the best of the individual. Second, in *closed fusion* the fuser does not have an access to either the training sample or the hypotheses of the individual learners. By using a linear threshold fusion function (of the outputs of individual learners) we show that the composite system can be made better than the best of the statistically independent learners.**

I. INTRODUCTION

The N-Learners Problem is a special (abstracted) case of data fusion: we are given multiple learners of Valiant [29] kind that infer concepts, and the problem is to design a *fuser* that combines the outputs of the individual learners. The problems of designing individual learners under this framework have been extensively studied during the past decade [22], [29]. Potential applications of the N-learners problem include sensor fusion [11], [16], hybrid systems [13], information pooling and group decision models [14], [20], and majority systems [8].

Consider a system of N learners $L_1, L_2, \cdots, L_N$, where L_i learns concepts (subsets) of a domain X in the sense of Valiant [29]; i.e., given a sufficiently large sample of examples of $c \in C \subseteq 2^X$, a hypothesis h close to c will be produced with a high probability. The closeness of the hypothesis (learned concept) h to c is specified by a *precision* parameter ϵ, and the probability that this closeness is achieved is specified by a *confidence* parameter δ. Given two learners, the one with higher or equal confidence for the same value of precision is considered *better* (this notion is more precisely defined in Section IV). In this paper, we only consider the problem of designing a *fuser* such that the *composite system*, of the fuser with the N learners, can be made better[1] than best of the learners.

We first illustrate some simple cases where the composite system can be easily seen to be better than each of the learners (Section III), and then consider more general cases.

We consider two paradigms:

a) *Open Fusion:* In open fusion, the fuser is given the training examples and the hypotheses of the individual learners. We introduce a property called the *isolation*, and present sufficiency conditions that ensure the composite system to be better than the best of the learners. We show that the problem of designing the fuser can be solved by casting it as another learning problem that can be solved using known methods if the suitable isolation property is satisfied. We consider two cases: i) all learners are trained with the same sample, and ii) each learner is individually trained with a separate random sample. We derive sufficiency conditions for several formulations of the learnability problem such that the composite system is better than best of the learners. In both cases, the hypothesis class of the fuser must satisfy the isolation property of degree N; additionally, the condition in the first case is that the Vapnik-Chervonenkis dimension [7] (VC dimension) of the fuser be smaller than or equal to that of every learner. And in the second case the fuser can have much larger VC dimension (the exact bound is specific to the formulation of the learning mechanism of L_i's). In formulations such as learnability under fixed distributions [6], learning under metric spaces [15], we use the corresponding parameters to express sufficiency conditions.

b) *Closed Fusion:* In closed fusion, the fuser does not have access to either the examples or the hypotheses of the individual learners. We show that a linear threshold fuser can be designed such that the composite system is better than the best of the statistically independent learners. This result shows that that even if all individual learners are completely consistent with the sample (i.e., all of them have zero empirical error), we can still make the performance of the composite system better than that of any individual learner. Further work on closed fusion can be found in [27].

The organization of this paper is as follows: A precise formulation of the N-learners problem is presented in Section II. Specialized examples where a suitable fusion rule makes the overall system better than the best of the learners are given in Section III. A selection of existing learning formulations, and an approach to compare the learners are outlined in Section IV. The general problem is solved

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[1] There are other interesting criteria for designing a fuser. For example, we might be interested in making the composite system learn concepts that are not learnable by the individual learners. In [26] a system capable of learning Boolean combinations of halfspaces by utilizing a system of perceptrons is described; note that a single perceptron is incapable of learning such concepts [23].