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Foundations of Wave Phenomena

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Problem Set 7

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PROBLEM SET 7

Problem 7.1

Recall the mechanical system consisting of two coupled oscillators. The kinetic energy T for the system is defined as usual ($T = \frac{1}{2}m(v_1^2 + v_2^2)$). The potential energy is denoted by $V(x_1, x_2)$ and is defined so that the force, F_i, on the i^{th} particle ($i = 1, 2$) is given by

$$F_i = -\frac{\partial V}{\partial x_i}.$$

Find the form of V, and prove that the total energy $E = T + V$ is conserved, that is, $\frac{dE}{dt} = 0$ for solutions of the equations of motion.

Problem 7.2

Solutions to the wave equation have a conserved momentum. The momentum density for a wave $q(x, t)$ is defined by

$$\rho = \frac{\partial q}{\partial t}\frac{\partial q}{\partial x}.$$

Find the corresponding momentum current density j for the wave. (*Hint:* Use the continuity equations.)

Problem 7.3

Recall the Gaussian wave

$$q(x, t) = A\left[e^{-(x-vt)^2} + e^{-(x+vt)^2}\right].$$

Compute the total energy contained in this wave by integrating the energy density $\rho(x, t)$ over all x and show that the result does not depend upon the time t.

Problem 7.4

In the previous problem, it is shown that the total energy of the Gaussian wave is time independent. Explain this result by showing that the energy current density j vanishes as $x \to \pm\infty$.

Problem 7.5

Verify that

$$\rho(\vec{r}, t) = \frac{1}{2}\left[\left(\frac{\partial q}{\partial t}\right)^2 + v^2(\nabla q \cdot \nabla q)\right],$$

and

$$\vec{j}(\vec{r},t) = -v^2 \frac{\partial q}{\partial t} \nabla q.$$

satisfy the continuity equation when q satisfies the (3-d) wave equation.

Problem 7.6

If $q(\vec{r},t)$ depends only upon x and t (*i.e.,* q is independent of y and z) show that the 3-dimensional forms for the energy density, energy current density, and continuity equation reduce to the 1-dimensional results.

Problem 7.7

Use the divergence theorem (14.33) to derive (14.28).

Problem 7.8

Verify (14.17). Show that the time rate of change of energy in the region is the net flux of energy into the region (14.18).

Problem 7.9

Derive the approximate formula (14.40).

Problem 7.10

Show that the quantity

$$\Pi(t) = \int_{-\infty}^{\infty} dx \, \frac{\partial q(x,t)}{\partial t}$$

is independent of t (*i.e.,* is a conserved quantity) for all solutions q of the one-dimensional wave equation whose first derivatives vanish at infinity,

$$\lim_{x\to\pm\infty} \frac{\partial q(x,t)}{\partial x} = 0.$$

Problem 7.11

Compute the energy contained in a cylindrically symmetric wave within a cylinder about the z axis of radius R as a function of time. Compute the flux of the energy current density through the cylinder and verify conservation of energy.