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Foundations of Wave Phenomena | Open Textbooks

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Problem Set 5

Charles G. Torre
charles.torre@usu.edu

PROBLEM SET 5

Problem 5.1

Let $\vec{A}$ be a given constant vector field (Cartesian components are constants) and let c a given constant scalar. Show that the equation $\vec{A} \cdot \vec{r} = c$ is the equation of a plane. If $\vec{A} = \hat{x} + \hat{y} + \hat{z}$, and $c = 0$, where is the plane?

Problem 5.2

Prove that the gradient of a function $f(\vec{r})$ is always orthogonal to the surfaces $f(\vec{r}) =$ constant. (Hint: This one is easy; think about the directional derivative of f along any direction tangent to the surface.)

Problem 5.3

Consider the sphere defined by $x^2 + y^2 + z^2 = 1$. Compute the gradient of the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

and check that it is everywhere orthogonal to the sphere. Consider a linear function

$$f(\vec{r}) = \vec{a} \cdot \vec{r},$$

where $\vec{a}$ is a fixed, constant vector field. Compute the gradient of f and check that the resulting vector field is perpendicular to the plane $f(\vec{r}) = 0$.

Problem 5.4

Compute the divergence of the following vector fields:

(a) $\vec{E}(\vec{r}) = \frac{\vec{r}}{r^3}, \quad r = \sqrt{x^2 + y^2 + z^2} > 0,$ (Coulomb electric field)

(b) $\vec{B}(\vec{r}) = -\frac{y}{x^2+y^2}\,\hat{x} + \frac{x}{x^2+y^2}\,\hat{y}, \quad x^2+y^2 > 0,$ (magnetic field outside a long straight wire)

(c) $\vec{D}(\vec{r}) = \vec{r}$ (electric field inside a uniform ball of charge).

Problem 5.5

Derive (9.15) and (9.20).

Problem 5.6

Consider a spherically symmetric function $f = f(r)$, $r = \sqrt{x^2 + y^2 + z^2}$. Show that its Fourier transform takes the following form:

$$h(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int_{\text{all space}} d^3x\, e^{-i\vec{k}\cdot\vec{r}} f(r) = \sqrt{\frac{2}{\pi}} \frac{1}{k} \int_0^\infty dr\, r f(r) \sin(kr).$$

(*Hint:* Use spherical polar coordinates, choosing your z axis along $\vec{k}$.) Note that the transform is spherically symmetric also in $\vec{k}$ space. Use this formula to compute the Fourier transform of a 3-dimensional Gaussian

$$f = e^{-a^{-2}(x^2+y^2+z^2)}.$$

Problem 5.7

Derive (9.25) from (9.21). In particular, express $c(\vec{k})$ in terms of the initial data.

Problem 5.8

Let f and g be two functions. We can take the gradients of f and g to get vector fields, ∇f and ∇g. We can multiply these vector fields by the functions f and g to get more vector fields, *e.g.,* $f\nabla g$. As with any vector field, we can make a function by taking a divergence, *e.g.,* $\nabla \cdot (f\nabla g)$. Using the definitions of gradient, divergence and Laplacian show that

$$\nabla \cdot (f\nabla g) = \nabla f \cdot \nabla g + f\nabla^2 g. \tag{10.3}$$

and

$$f\nabla^2 g - g\nabla^2 f = \nabla \cdot (f\nabla g - g\nabla f).$$