Utah State University
DigitalCommons@USU

Foundations of Wave Phenomena | Open Textbooks

8-2014

14 Conservation of Energy

Charles G. Torre
Department of Physics, Utah State University, charles.torre@usu.edu

14. Conservation of Energy.

After all of these developments it is nice to keep in mind the idea that the wave equation describes (a continuum limit of) a network of coupled oscillators. This raises an interesting question. Certainly you have seen by now how important energy and momentum — and their conservation — are for understanding the behavior of dynamical systems such as an oscillator. If a wave is essentially the collective motion of many oscillators, might not there be a notion of conserved energy and momentum for waves? If you've ever been to the beach and swam in the ocean you know that waves do indeed carry energy and momentum which can be transferred to other systems. How to see energy and momentum and their conservation laws emerge from the wave equation? One way to answer this question would be to go back to the system of coupled oscillators and try to add up the energy and momentum of each oscillator at a given time and take the continuum limit to get the total energy and momentum of the wave. Of course, the energy and momentum of each individual oscillator is not conserved (*exercise*). Indeed, the propagation of a wave depends upon the fact that the oscillators are coupled, *i.e.,* can exchange energy and momentum. What we want to do here, however, is to show how to keep track of this energy flow in a wave, directly from the continuum description we have been developing. This will allow us to define the energy (and momentum) densities of the wave as well as the total energy contained in a region.

14.1 The Continuity Equation

To begin, let us limit attention to one spatial dimension for simplicity. You can think of the system under study as a rope or string under tension. The transverse displacement of the rope from equilibrium at the position x at time t is denoted by $q(x,t)$ as before. We understand how to analyze wave phenomena in this sort of system and we now seek some kind of energy which is conserved.

As it happens, conservation of energy* in the dynamics of a continuous medium mathematically manifests itself in the existence of a *continuity equation.* With one spatial dimension, this is an equation of the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0, \tag{14.1}$$

where $\rho = \rho(x,t)$ and $j = j(x,t)$ are some functions built from $q(x,t)$ and its derivatives. We shall give some formulas for ρ and j in terms of q in a moment. Do not confuse ρ with the radial coordinate of the cylindrical coordinate system. The continuity equation (14.1) is to be satisfied whenever $q(x,t)$ satisfies the wave equation. The physical meaning of ρ and j are as follows. The function $\rho(x,t)$ represents the *energy density* (actually, energy

* Or for that matter, conservation of any other extensive quantity, *e.g.,* mass, charge, *etc.*

per unit length in our one-dimensional model) of the wave at the point x and time t. The function $j(x,t)$ represents the x-component of the *energy current density* of the wave.* The energy current density is a vector field which represents the flow of energy from point to point (*i.e.,* reflects the exchange of energy among the oscillators).

The continuity equation is an "energy balance" equation. To see this, view $\rho(x,t)dx$ as the energy contained in an infinitesimal region, $[x, x+dx]$ at time t. Eq. (14.1) says that the time rate of change of energy in an infinitesimal region, $\frac{\partial\rho}{\partial t}dx$, is given by (*exercise*)

$$\frac{\partial\rho}{\partial t}dx = -\frac{\partial j(x,t)}{\partial x}dx = j(x)dx - j(x+dx), \tag{14.2}$$

which should be viewed as the net flow of energy *into* the region per unit time.† This is precisely the familiar statement of the principle of conservation of energy: "energy cannot be created or destroyed".

To see this conservation of energy business from a non-infinitesimal point of view, consider the energy $E(t)$ contained in a region $a \le x \le b$ at time t (a and/or b can be infinite). We get this by adding up the energy per unit length with an integral:

$$E(t) = \int_a^b dx\, \rho(x,t). \tag{14.3}$$

From the continuity equation, the time rate of change of energy in this region is given by

$$\begin{aligned} \frac{dE(t)}{dt} &= \int_a^b dx\, \frac{\partial\rho(x,t)}{\partial t} \\ &= -\int_a^b dx\, \frac{\partial j(x,t)}{\partial x} \\ &= j(a,t) - j(b,t). \end{aligned} \tag{14.4}$$

Physically, the expression $j(a,t) - j(b,t)$ is the energy entering the region per unit time. (Keep in mind that if j is positive it indicates a flow of energy in the positive x direction.) Mathematically, $j(a,t) - j(b,t)$ is the "flux"‡ of the energy current vector field *into* the closed region $[a,b]$ (*exercise*!). In particular, if $j(a,t) - j(b,t)$ is positive (negative) this indicates a net flow of energy into (out of) the region (*exercise*). If the net flux is zero, then the energy in the region does not change in time. This can happen because the flux vanishes at the boundaries of the region, or because the inward flux in at one endpoint of

* Of course, in one spatial dimension the x component is the only component of a vector field.

† If $j(x)$ is positive (negative) then the energy current density vector points into (out of) the region $[x, x+dx]$. Similarly, if $j(x+dx)$ is positive (negative) the energy current density vector points out of (into) the region at $x+dx$.

‡ Here "flux" is defined just as in the context of the Gauss law of electrostatics. We shall define it formally a little later.

the region is balanced by the outward flux out at the other. If the net flux is non-zero, the system in the region $[a, b]$ is exchanging energy with its "environment" outside of $[a, b]$.†

By the way, while we have been using the word "energy" in the discussion above, it is clear that we have not really used any fact except that ρ and j satisfy a continuity equation. The conservation law (14.4) will arise no matter what is the physical interpretation of ρ and j. Indeed, we will see later how a continuity equation is used to describe electric charge conservation in electrodynamics. The connection with energy comes when we specify exactly how ρ and j are to be constructed from the wave displacement q. For the energy density we define

$$\rho = \frac{1}{2}\left[\left(\frac{\partial q}{\partial t}\right)^2 + v^2\left(\frac{\partial q}{\partial x}\right)^2\right]. \tag{14.5}$$

You can think of the first term in (14.5) as a kinetic energy (density) and the second term as a potential energy (density). Indeed, the form of the energy per unit length can be understood via the continuum limit of the total energy of the chain of oscillators. For the energy current density we have

$$j = -v^2\frac{\partial q}{\partial t}\frac{\partial q}{\partial x}. \tag{14.6}$$

You can think of j as being proportional to a momentum density for the wave.

Now we show that ρ and j satisfy the continuity equation when q satisfies the wave equation. The time derivative of ρ is

$$\frac{\partial \rho}{\partial t} = \frac{\partial q}{\partial t}\frac{\partial^2 q}{\partial t^2} + v^2\frac{\partial q}{\partial x}\frac{\partial^2 q}{\partial x \partial t}. \tag{14.7}$$

The space derivative of j is

$$\frac{\partial j}{\partial x} = -v^2\left[\frac{\partial q}{\partial t}\frac{\partial^2 q}{\partial x^2} + \frac{\partial q}{\partial x}\frac{\partial^2 q}{\partial t \partial x}\right]. \tag{14.8}$$

We then get for the left-hand side of the continuity equation (14.1)

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = \frac{\partial q}{\partial t}\left(\frac{\partial^2 q}{\partial t^2} - v^2\frac{\partial^2 q}{\partial x^2}\right),$$

which vanishes when q satisfies the wave equation.

† You may wonder how this way of thinking about conservation laws jibes with the conservation laws you have seen for mechanical systems, *e.g.,* coupled oscillators. Roughly speaking, the conserved total energy involves a sum of the individual energies (including any interaction energy) over the degrees of freedom of the system. In the continuum, such as we have here, the degrees of freedom are – in effect – being labeled by x and the sum over degrees of freedom is the integral over x.

Remarks:

(1) Given a (system of) partial differential equation(s), there are systematic mathematical ways to find functions ρ and j, that is, formulas like (14.5) and (14.6), satisfying a continuity equation (if such equations exist for the given differential equations). When such techniques are applied to the wave equation one can derive the formulas (14.5) and (14.6).*

(2) Our use of the words "energy" and "momentum" is slightly misleading because of dimensional considerations. Depending upon the units of the wave displacement $q(x,t)$, the quantities ρ and j/v^2 need not have dimensions of energy per unit length, or momentum per unit length. You can check as a nice *exercise* that if q has dimensions of square root of mass times square root of distance, then ρ has dimensions of energy per unit length and j/v^2 has dimensions of momentum per unit length. Of course, q need not have these units since it can represent variety of physical displacements (position, current, temperature, *etc.*). Thus in applications you will often see other dimensionful constants in the definition ρ and j. For example, suppose that q represents a spatial displacement, *e.g.,* in a vibrating string, or in a fluid or in an elastic medium (such as a continuum limit of a chain of oscillators), so that q has units of length. Then, we need a constant μ, with dimensions of mass per unit length, to appear in the energy density:

$$\rho = \frac{\mu}{2}\left[\left(\frac{\partial q}{\partial t}\right)^2 + v^2\left(\frac{\partial q}{\partial x}\right)^2\right].$$

Similarly, in this example we would define

$$j = -\mu v^2 \frac{\partial q}{\partial t}\frac{\partial q}{\partial x}.$$

The constant μ would represent, say, the mass density (really mass per unit length) of the one-dimensional medium. You can easily see that we could absorb μ into the definition of q: $q \to q/\sqrt{\mu}$, thus giving q units of square root of mass times square root of distance. To keep things looking reasonably simple, we assume that such a redefinition has been performed in what follows.

(3) If no boundary conditions are imposed then the waves are free to transmit energy to and from the "environment", that is, the region outside of $[a,b]$. The energy in the interval is not constant and one says that one has an "open system". As you can see from the formula for the energy current density, if the wave displacement vanishes at the endpoints of the interval,

$$q(a,t) = 0 = q(b,t),$$

* See, for example, *Applications of Lie Groups to Differential Equations*, by Peter Olver, (Graduate Texts in Mathematics, Springer, 2000).

or

$$q(x,t) = 0 \quad \text{for} \quad x \leq a, \quad x \geq b \tag{14.9}$$

then the energy in the interval is constant. This would occur, for example, if one considers a vibrating string under tension with fixed ends (think: guitar string).* One can also let the interval become infinite, in which case the conditions (14.9) are asymptotic conditions. If we consider periodic boundary conditions,

$$q(a,t) = q(b,t), \quad \frac{\partial q(a,t)}{\partial x} = \frac{\partial q(b,t)}{\partial x} \tag{14.10}$$

then the energy in the interval is constant, basically because energy that flows "out" at $x = b$ is also flowing back "in" at $x = a$. In both of these cases one can view the system as "closed", that is, not interacting with its environment.

I emphasize that the continuity equation, and the conservation of energy it represents, only holds for solutions to the wave equation. This is analogous to the situation in Newtonian mechanics where the energy of a particle,

$$E = \frac{1}{2} m \dot{\vec{r}}^2(t) + V(\vec{r}(t)), \tag{14.11}$$

is conserved,

$$\frac{dE}{dt} = 0, \tag{14.12}$$

provided the particle obeys its equations of motion, *i.e.,* Newton's second law:

$$\frac{d^2\vec{r}(t)}{dt^2} = -\frac{1}{m}\nabla V(\vec{r}(t)). \tag{14.13}$$

But the energy will in general not be conserved for motions of the system $\vec{r}(t)$ not obeying Newton's second law *(exercise)*.

14.2 Some simple examples

Let us look at a couple of examples. First, we consider a sinusoidal wave in one dimension:

$$q(x,t) = A\cos(kx - \omega t), \tag{14.14}$$

where $\omega/k = v$. The energy density is (*exercise*)

$$\rho(x,t) = (A\omega)^2 \sin^2(kx - \omega t). \tag{14.15}$$

* Of course, real vibrating strings eventually stop vibrating because the energy of the strings is transferred elsewhere by dissipative processes. We are obviously not including these effects here.

The energy current density is (*exercise*)

$$j(x,t) = (A\omega)^2 v \sin^2(kx - \omega t). \tag{14.16}$$

Note that for $k > 0$ the wave moves to the right, and the energy current is positive. This reflects a net transport of energy to the right. If $k < 0$, then the wave moves to the left and the current is correspondingly negative.

The total energy in a region $-L \le x \le L$ is (see the Problems)

$$E(t) = (A\omega)^2 \left[L - \frac{1}{4k}\sin\{2(kL - \omega t)\} - \frac{1}{4k}\sin\{2(kL + \omega t)\}\right]. \tag{14.17}$$

As an exercise you can check that E in (14.17) is positive as it should be. *(Exercise: Why would you expect $E(t)$ to be positive?)* The net flux of energy into the region is

$$j(-L,t) - j(L,t) = A^2\frac{\omega^3}{k}\left[\sin^2(kL + \omega t) - \sin^2(kL - \omega t)\right] = (A\omega)^2 v \sin(2kL)\sin(2\omega t). \tag{14.18}$$

It is interesting to examine the special case in which the size of the the region is an integral number of half-wavelengths: $2L = \frac{n}{2}\lambda$, $n = 1, 2, 3, \ldots$, *i.e.,* $kL = \frac{1}{2}\pi n$. In this case the energy (14.17) is time-independent (*exercise*):

$$E = \frac{n\lambda}{8}(A\omega)^2. \tag{14.19}$$

As you should expect, the net energy flux vanishes:

$$j(-L,t) - j(L,t) = 0, \tag{14.20}$$

although the energy current is not identically zero at either boundary point. This result can be interpreted as saying that when $2L = \frac{1}{2}n\lambda$, the energy leaving the box from the right is exactly matched by the energy entering the box from the left.

A similar phenomenon occurs with a standing wave (*exercise*). Let us consider a solution of the form

$$q(x,t) = \begin{cases} A\sin(\frac{n\pi}{L}x)\cos\omega t, & -L < x < L; \\ 0 & |x| \ge L, \end{cases} \tag{14.21}$$

where $n = 1, 2, \ldots$. You can easily check that this is a solution of the wave equation provided $\omega = \frac{n\pi}{L}v$ (*exercise*). Note in particular that $q(\pm L, t) = 0$. Equation (14.21) represents a standing wave. It can be obtained by superposing a left and right moving traveling wave in the region $-L < x < L$ (check this as an *exercise*). You can visualize it as a bit of taut string of length $2L$ held fixed at $x = \pm L$, *e.g.,* in a guitar. The different values of n label the different harmonics you can get when you pluck the string. By pinning down the string at $x = \pm L$ we prohibit energy from being transferred outside of that region. Instead, the energy of the wave is reflected, so the net transfer of energy out of the region is zero at each endpoint. Physically, then, we would expect that the total energy in the region $-L < x < L$ is conserved (up to dissipative effects that are not modeled by the wave equation). Let us check this. The energy density is (*exercise*)

$$\rho(x,t) = \frac{1}{2}A^2\omega^2\left[\sin^2(\frac{n\pi}{L}x)\sin^2\omega t + \cos^2(\frac{n\pi}{L}x)\cos^2\omega t\right], \quad -L < x < L, \tag{14.22}$$

and zero elsewhere. The total energy is

$$\begin{aligned} E(t) &= \int_{-\infty}^{\infty} dx\,\rho(x,t), \\ &= \int_{-L}^{L} dx\,\rho(x,t) \\ &= \frac{1}{2}A^2\omega^2 L, \end{aligned} \tag{14.23}$$

which is indeed a constant in time. Note that the energy varies as the square of the amplitude and frequency and is proportional to the size of the region where the wave is allowed to exist. Because the energy is time independent, from the continuity equation we expect that the flux of energy at $x = \pm L$ should vanish. We have

$$j(x,t) = \frac{A^2}{4}v\omega^2\sin(\frac{2n\pi}{L}x)\cos(2\omega t), \tag{14.24}$$

and from this expression you can see that, for each value of n,

$$j(\pm L, t) = 0. \tag{14.25}$$

We have seen that if the wave is confined to a finite region (with appropriate boundary conditions), the total energy in that region will be conserved. We can generalize this to arbitrarily large regions. In particular, suppose that $q(x,t)$ vanishes as $x \to \pm\infty$. Our Gaussian wave (7.29) is a good example of such a solution to the wave equation (*exercise*). In this case you can check that even though the energy in any finite region need not be constant in time, the energy contained in all of space is a constant in time. This is because j will vanish as $x \to \pm\infty$. So, while different parts of the medium (string, air, etc.) can exchange energy, the total energy cannot be lost (provided no waves are "at infinity" after finite time).

14.3 The Continuity Equation in 3 Dimensions

Let us now see how our preceding discussion generalizes to waves which are 3-dimensional in nature. To begin, we need to generalize the continuity equation. We note that the energy density $\rho = \rho(\vec{r}, t)$ is a *scalar field*, as is its time derivative: they both assign a number (a scalar) to each point of space (and instant of time). The energy current density is a *vector field*: it has a magnitude and direction at each point of space (and instant of time). We denote this vector field by $\vec{j} = \vec{j}(\vec{r}, t)$. The energy current density $\vec{j}$ represents the energy passing through any given surface per unit area per unit time. We will make this more precise below. Evidently, the continuity equation equates space derivatives of $\vec{j}$ with time derivatives of ρ, which means we need to make a scalar from the derivatives of $\vec{j}$. This is done using the divergence (see §9.1); the continuity equation in 3 dimensions takes the form:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0. \tag{14.26}$$

It is instructive to see how (14.26) leads to a notion of conservation of energy in this 3-dimensional setting. We begin with a simple example. Consider a cubical region defined by $0 \leq x \leq L$, $0 \leq y \leq L$, and $0 \leq z \leq L$. Now integrate the continuity equation over the volume of the box. Defining

$$E(t) = \int_0^L dz \int_0^L dy \int_0^L dx\, \rho(\vec{r}, t), \tag{14.27}$$

we find (good *exercise*)

$$\begin{aligned} \frac{dE(t)}{dt} = -\int_0^L dz \int_0^L dy\, & [j^x(L, y, z, t) - j^x(0, y, z, t)] \\ - \int_0^L dz \int_0^L dx\, & [j^y(x, L, z, t) - j^y(x, 0, z, t)] \\ - \int_0^L dy \int_0^L dx\, & [j^z(x, y, L, t) - j^z(x, y, 0, t)] . \end{aligned} \tag{14.28}$$

To get this result we wrote out the divergence and used the fundamental theorem of integral calculus. Compare this result with (14.4).

Again we interpret the formula for dE/dt as equating the time rate of change of energy contained in the volume of the box with the net flux of energy into the box through the walls. This is what we mean by conservation of energy. The first term on the right hand side of (14.28) is the flux through the faces at $x = 0$ and $x = L$. In particular, we see that the x component of $\vec{j}$ controls the flux in the x direction. You can easily see that we have corresponding statements for the y and z components of the energy current density and the flux. If the net flux through the walls in the box vanish, then the energy in the cube will be constant in time.

We now provide the generalizations to three dimensions of formulas for the energy density and energy current density in terms of a solution $q(\vec{r}, t)$ to the wave equation. The energy density is

$$\rho = \frac{1}{2}\left[\left(\frac{\partial q}{\partial t}\right)^2 + v^2 \nabla q \cdot \nabla q\right], \tag{14.29}$$

and the current density is

$$\vec{j}(\vec{r},t) = -v^2 \frac{\partial q}{\partial t} \nabla q. \tag{14.30}$$

You should compare these formulas with their one-dimensional counterparts (14.5) and (14.6).* The function $\rho(\vec{r},t)$ represents the energy per unit volume at the point $\vec{r}$ at the time t. The vector field $\vec{j}(\vec{r},t)$ represents the net energy transported across a surface, per unit area per unit time, at the point $\vec{r}$ at the time t. As an exercise you can verify, just as we did in 1 dimension, that ρ and $\vec{j}$ satisfy the 3-d continuity equation provided q satisfies the wave equation.

14.4 Divergence theorem

We do not have to use a rectangular box to compute the energy — any region will do. The continuity equation still guarantees conservation of energy. It is worth spelling this out in detail since it brings into play a very important integral theorem known as *the divergence theorem*, which you may have encountered in electrostatics (via Gauss's law).

To begin, here is the geometric setting. Let V be a three-dimensional region (volume) enclosed by a surface S. At each point of the surface we have the unit normal $\hat{n}$, which by definition points out of V. For example, V could be the volume contained in a sphere of radius R. In this case the unit normal, at the point with coordinates (x, y, z) on the sphere $x^2 + y^2 + z^2 = R^2$, can be written as (*exercise*)

$$\hat{n} = \frac{x}{R}\hat{x} + \frac{y}{R}\hat{y} + \frac{z}{R}\hat{z}, \tag{14.31}$$

where $\hat{x}$, $\hat{y}$, $\hat{z}$ are the usual x, y, z unit vectors. More generally, if the surface S is expressed as the level surface of a function,

$$f(x, y, z) = \text{constant}, \tag{14.32}$$

then the unit normal can be obtained by (i) taking the gradient ∇f, (ii) evaluating the vector field on the surface (14.32), and then (iii) normalizing the result so that you have a unit vector at each point of S. You can check this recipe for the sphere example above, where $f = x^2 + y^2 + z^2$ (*exercise*).

The *divergence theorem* states that the integral of the divergence of a vector field $\vec{A}$ over a volume V is the same as the *flux* of the vector field through the bounding surface S. In formulas:

$$\int_V dV\, \nabla \cdot \vec{A} = \oint_S dS\, \vec{A} \cdot \hat{n}, \tag{14.33}$$

* The same comments we made there concerning units will apply here.

where dV and dS are the volume and surface elements respectively. The little circle on the surface integral sign is there to remind us that the surface is *closed*, that is, it has no boundary.* Note that the default use of "flux" in the context of the divergence theorem is the "outward" flux.

The divergence theorem is essentially an elaborate use of the fundamental theorem of integral calculus (in the guise of repeated integration by parts — re-examine our cube example). Indeed, you can check that (14.33) gives the result we obtained by direct integration in (14.28) for dE/dt when the V is taken to be a cube of length L (see the homework problems).

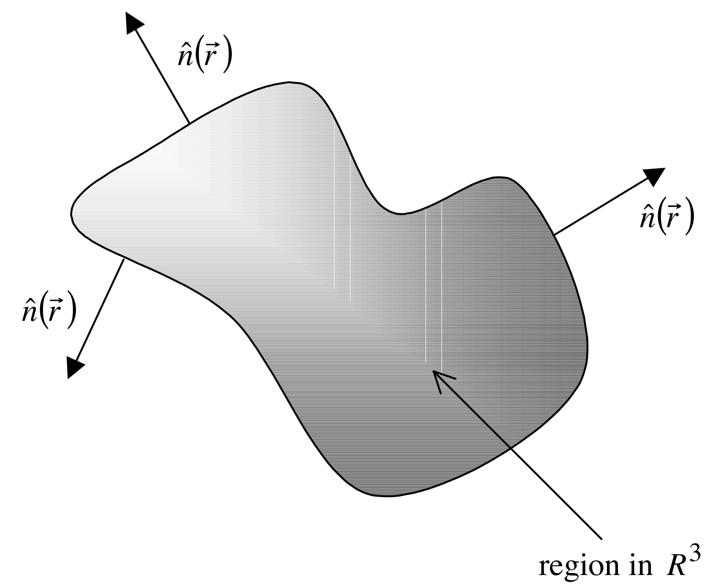

Figure 20. An enclosed region in R^3 illustrating several outwardly directed normal vectors associated with the surface enclosing the region.

With the divergence theorem in hand we can consider the integrated version of the continuity equation — *i.e.,* the energy conservation law. Integrate (14.26) over a volume V with boundary surface S, then use the divergence theorem to find (*exercise*)

$$\frac{d}{dt}\int_V dV\,\rho = -\oint_S dS\,\vec{j}\cdot\hat{n}, \qquad (14.34)$$

* For example, a sphere, *i.e.,* the surface of a ball in 3-d, is a closed surface. A hemisphere is not a closed surface. The boundary surface of a volume is always a closed surface.

showing that the change of energy in the region V is *minus* the energy flux through the boundary S, which is the net energy flow *into* the region. The conservation of energy formula we found for the one-dimensional wave equation can be viewed as a special case of this three-dimensional formula.

Incidentally, by using the identity (14.33) we can recover our previous interpretation of the divergence, presented in §9. As we stated in that section, the idea is simply that in a limit in which the volume V becomes arbitrarily small ("infinitesimal"), the integral on the left side of (14.33) becomes — to an arbitrarily good approximation — $(\nabla \cdot \vec{A})\, V$. Dividing both sides by V we see that, the divergence of $\vec{A}$ is the flux per unit volume through the boundary of the small volume V.

14.5 Applications of the 3-Dimensional Continuity Equation

Let us look at an illustrative example. Consider a spherical wave (see §13.3):

$$q(\vec{r}, t) = A\frac{\sin(kr)}{kr}\cos\omega t, \quad \omega = |k|v \tag{14.35}$$

where $\omega = kv$. The energy density associated with (14.35) is (*exercise*)

$$\rho = \frac{1}{2}A^2v^2\left(\sin^2\omega t\frac{\sin^2 kr}{r^2} + \cos^2\omega t\left[\frac{\cos kr}{r} - \frac{\sin kr}{kr^2}\right]^2\right). \tag{14.36}$$

Let us consider the time rate of change of energy $E(t)$ contained in a sphere of radius R centered at the origin:

$$\begin{aligned}\frac{d}{dt}E(t) &= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{R} dr\, r^2 \sin\theta\, \frac{\partial\rho(r,t)}{\partial t}\\ &= 4\pi\int_0^R dr\, r^2\frac{\partial\rho(r,t)}{\partial t}\\ &= -4\pi A^2v^3\sin\omega t\cos\omega t\sin kR\left(\cos kR - \frac{\sin kR}{kR}\right).\end{aligned} \tag{14.37}$$

The result of the integral appearing in (14.37) is probably not obvious to you, but can be obtained using the usual tricks. You might try checking the result using your favorite symbolic mathematics software. Note that if $R = n\pi/k$, *i.e.,* R is a multiple of half a wavelength, then the energy in the sphere is conserved, this can be viewed as a spherically symmetric version of the standing wave phenomenon (*exercise*).

According to our interpretation of the continuity equation via the divergence theorem, $\frac{dE}{dt}$ should be equal to minus the flux of the energy current through the sphere of radius

R. This is easily checked because of spherical symmetry. In detail, the energy current is (*exercise*)

$$\vec{j} = A^2\omega v^2 \frac{\sin kr}{kr} \sin\omega t \cos\omega t [\frac{\cos kr}{r} - \frac{\sin kr}{kr^2}] \frac{\vec{r}}{r}. \tag{14.38}$$

The normal to the sphere of radius r is $\hat{n} = \frac{\vec{r}}{r}$, so that, with S being the sphere of radius R,

$$\begin{aligned} \oint_S d^2S\, \vec{j} \cdot \hat{n} &= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, R^2 \sin\theta A^2 \omega v^2 \frac{\sin kR}{kR} \sin\omega t \cos\omega t [\frac{\cos kR}{R} - \frac{\sin kR}{kR^2}] \\ &= 4\pi A^2 v^3 \sin kR \sin\omega t \cos\omega t [\cos kR - \frac{\sin kR}{kR}]. \end{aligned} \tag{14.39}$$

where the integral was easily evaluated because the energy current was spherically symmetric! Comparing with the result (14.37) we obtained earlier for the time rate of change of energy in the sphere we see that the conservation law is satisfied.

Finally, let us consider the flux of energy at large distances, *i.e.,* we consider the limit of (14.39) as $R \to \infty$. More precisely, we consider the leading order behavior of the flux as the boundary of the region of interest is taken to be at a radius which is much larger than the wavelength of the wave. By inspecting this formula you will see that at large distances we have (*exercise*)

$$\oint_S d^2S\, \vec{j} \cdot \hat{n} \approx \pi A^2 v^3 \sin(2kR) \sin(2\omega t), \quad kR >> 1. \tag{14.40}$$

You see that despite the fact that the wave amplitude decreases with increasing r, the energy flux through a sphere does not in fact decrease aside from the sinusoidal variation present in the wave. This is because the spherical wave has an energy current density that decreases with radius as $1/r^2$ while the surface area of a sphere of radius r grows as r^2. This is the prototypical behavior of a *radiation field*, that is a wave disturbance produced by a bounded source (a point source at the origin in this example) that carries energy off "to infinity", *i.e.,* to arbitrarily large distances from the source.