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Tropical Arithmetics and Dot Product Representations of Graphs

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TROPICAL ARITHMETICS AND DOT PRODUCT REPRESENTATIONS OF GRAPHS

by

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ABSTRACT

Tropical Arithmetics and Dot Product

Representations of Graphs

by

Nicole Turner, Master of Science

Utah State University, 2015

Major Professor: Dr. David Brown
Department: Mathematics and Statistics

A *dot product representation* (DPR) of a graph is a function that maps each vertex to a vector and two vertices are adjacent if and only if the dot product of their function values is greater than a given threshold. A *tropical algebra* is the antinegative semiring on $\mathbb{R} \cup \{\infty, -\infty\}$ with either $\min\{a, b\}$ replacing $a+b$ and $a+b$ replacing $a \cdot b$ (*min-plus*), or $\max\{a, b\}$ replacing $a + b$ and $a + b$ replacing $a \cdot b$ (*max-plus*), and the symbol ∞ is the additive identity in min-plus while $-\infty$ is the additive identity in max-plus; the multiplicative identity is 0 in min-plus and in max-plus. Recall the *threshold dimension* of graph G is the minimum number of threshold graphs whose union is G.

We study DPRs in the context of tropical semi-rings, and discuss results on minimizing the dimension of the space from which vectors must come in order to represent certain classes of graphs. These results differ depending on whether min-plus or max-plus is used, but a relationship is shown between the min-plus and max-plus results. Finally we show that max-plus dot product dimension and the threshold dimension of a graph are the same.

(30 pages)

PUBLIC ABSTRACT

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In tropical algebras we substitute min or max for the typical addition and then substitute addition for multiplication. A dot product representation of a graph assigns each vertex of the graph a vector such that two edges are adjacent if and only if the dot product of their vectors is greater than some chosen threshold. The resultS of creating dot product representations of graphs using tropical algebras are examined. In particular we examine the tropical dot product dimensions of graphs and establish connections to threshold graphs and the threshold dimension of a graph.

(39 pages)

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Nicole Turner

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CHAPTER 1
INTRODUCTION

1.1 Introduction

Graph theory has applications in almost every field of study. In particular it is often used in social, natural, and computer science situations. What is the best assignment of workers to available jobs? How many colors are required when coloring a map to ensure that no two neighboring regions are colored the same color? What is the optimal route to take when making multiple stops on a round trip journey? How can airlines make most effective and efficient use of their fleet? Questions in social networks, train schedules, computer chip manufacturing, DNA decoding, and biological evolution all have solutions in graph theory. Many of these questions involve optimization. What is the best, fastest, most efficient way to do something. Concerned only with sums, minimums, and maximums, tropical numbers are a natural pair with graph theory to help answer these questions of optimization. In this work we will explore the effect of using tropical algebras when creating dot product representations of graphs and finding the dot product dimension. Uncited definitions given can be found in sources such as [8] or [5].

1.2 Basic Definitions

A *graph* G is an ordered pair (V, E) where $V = V(G)$ is a set of elements called *vertices*, and $E = E(G)$ is a set of 2-element subsets of V called *edges*. Each edge can be seen as a representation of some symmetric relation between it's two constituent vertices, these vertices are referred to as the edge's *endpoints*. Two vertices that constitute an edge are said to be *adjacent* and are *neighbors* of each other. If two vertices are adjacent this may be denoted with the juxtaposition of their names. So, if u and v are adjacent this may be denoted $uv \in E$ as opposed to $\{u, v\} \in E$. We say a vertex is *incident* to an edge if and only it is one of the endpoints of that edge.

A graph is *simple* if no endpoints have more than one edge between them, and no vertex is adjacent to itself. The graphs considered here are *undirected*; meaning the relation adjacent is symmetric.

It is convenient to have names for various structures found within a graph. To assist in understanding these consider the graph, G, in Figure 1.1.

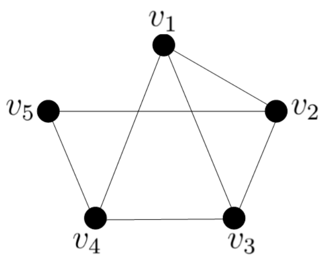

FIGURE 1.1: G_{ex}: An example graph

The graph G_{ex} is a simple undirected graph with vertex set $V = \{v_1, v_2, v_3, v_4, v_5\}$ and edge set $E = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_5, v_3v_4, v_4v_5\}$.

A *path* is a sequence of distinct vertices (except for possibly the first and last) with each two consecutive vertices in the sequence being adjacent. The *length* of a path is the number of edges used in the path. The *distance* between two vertices is the length of the shortest path between them. In G_{ex} (v_1, v_2, v_3, v_4) is a path of length 3, but the distance between v_1 and v_4 is 1. We will use the notation P_n for a graph that is a path on n vertices. A graph is *connected* if there is a path between every pair of vertices in the graph.

If the first and last vertices of a path are the same the path is called a *cycle*. A *n-cycle* is a cycle on n vertices, denoted as C_n. C_3 and C_4 are called a *triangle* and a *square* respectively. A *chord* is an edge that connects two nonadjacent vertices of a path or cycle. In G_{ex} $(v_2, v_3, v_4, v_1, v_2)$ is a cycle with a chord v_1v_3 and the cycle $(v_2, v_3, v_4, v_5, v_2)$ is *chordless*. A graph is called *chordal* if it has no chordless cycles.

Given two graphs, $G = (V, E)$ and $H = (W, F)$, we say H is a *subgraph* of G if $W \subseteq V$ and $F \subseteq E$. The paths and cycles described above are subgraphs of G_{ex}. An *induced subgraph* of a graph G is a graph that can be obtained from G by deletion of a selection of vertices and all edges incident to them. Some examples of induced subgraphs of G_{ex} are shown in Figure 1.2.

When every pair of vertices in a graph are adjacent we say that the graph is *complete*. A complete graph on n vertices is denoted K_n. Every graph is a subgraph of the complete graph on the same number of vertices, but not every graph is an induced

FIGURE 1.2: Two induced subgraphs of G_{ex}

subgraph of a complete graph. When a complete graph is a subgraph of a larger graph those vertices form a *clique*, a set of pairwise adjacent vertices. The opposite of a clique is an *independent set*, vertices which are pairwise nonadjacent. In Figure 1.1 $\{v_1, v_2, v_3\}$ form a clique, as do $\{v_1, v_3, v_4\}$. Vertices v_1 and v_5 form an independent set. The size of the largest independent set in a graph G is denoted as $\alpha(G)$.

The *complement* of a graph $G = (V, E)$ is the graph $\overline{G} = (V, F)$ such that for any $v_i, v_j \in V$ we have $v_i v_j \in F$ if and only if $v_i v_j \notin E$. The *union* of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. The *intersection* of G_1 and G_2 is the graph $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$. An example of each a union and an intersection of two graphs are shown in Figures 1.3 and 1.4.

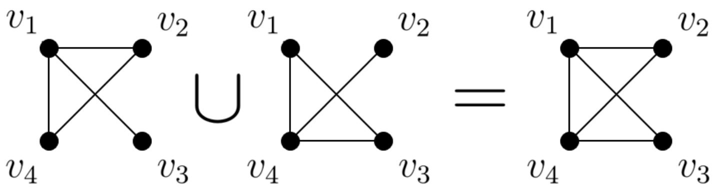

FIGURE 1.3: The union of two graphs

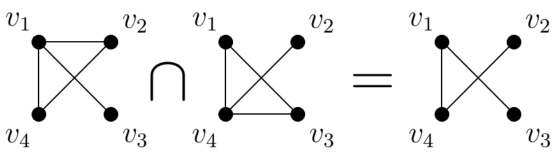

FIGURE 1.4: The intersection of two graphs

1.3 Graph Classes and Parameters

By analyzing structures in the graph we can classify graphs and introduce parameters that can be compared across different classes. Many classes of graphs deal with the connectivity of the graph, others with the presence or absence of certain structures. We shall review some of these classes and parameters that are relevant to the later discussion.

A *k-partite* graph is one in which the vertices can be partitioned into k independent (partite) sets such that all edges in G join vertices from different partite sets. A frequently studied class is bipartite graphs, i.e. 2-partite graphs. Because partite graphs require independent sets there can be no complete k-partite graphs, $k \geq 2$, using the classical definition of complete; therefore we adopt a modified concept of completeness when discussing them. A k-partite graph is considered complete when every pair of vertices not in the same partite set are adjacent. Complete k-partite graphs are denoted similarly to complete graphs, $K_{n_1,n_2,\ldots,n_k}$, where $n_1, n_2, \ldots, n_k$ are the sizes of the partite sets. For example, a $K_{3,2}$ is shown in Figure 1.5. A $K_{1,m}$, with $m \geq 1$, is called a *star*. If v is the vertex in the partite set of size 1 the star is centered at v.

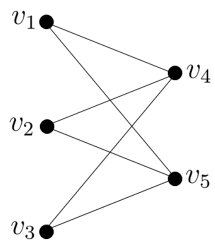

FIGURE 1.5: An example bipartite graph

A *tree* is a connected graph with no cycles. All trees are also bipartite graphs. A *caterpillar* is a tree with a single path containing at least one endpoint of every edge. A vertex not on this path is called a *leaf*. A *star* is a tree consisting of a central vertex and $n-1$ leaves adjacent to it. Examples of a tree and caterpillar can be seen in Figure 1.6. Since trees have no cycles they are chordal by default.

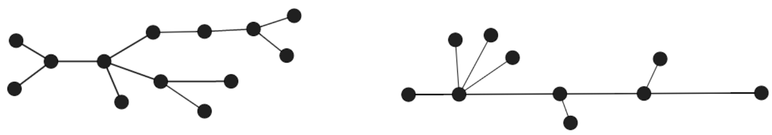

FIGURE 1.6: A tree that is not a caterpillar (left), and a caterpillar (right)

Threshold graphs are another class of graphs that are pertinent to the subject of this paper. While there are many characterizations and definitions of threshold graphs we include only three. For a graph $G = (V, E)$ the following are equivalent:

1. G is a threshold graph;

2. There are real weights $w(i)$, $i \in V$, and a threshold value t such that $w(i)+w(j) \geq t$ if and only if $ij \in E$ (We may refer to the function $w : V \to \mathbb{R}$ as the *threshold realization*);

3. G can be built sequentially from the empty graph by adding vertices one at a time where each new vertex, when added, is either isolated or adjacent to every existing vertex;

4. There is no induced subgraph C_4, P_4, or $2K_2$ (shown in Figure 1.7).

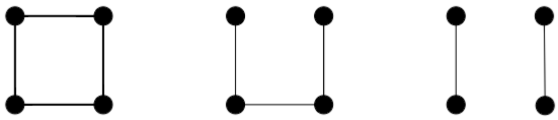

FIGURE 1.7: Forbidden subgraphs of threshold graphs: C_4, P_4, $2K_2$ respectively.

In this work we will be primarily concerned with the definition 2 given above. However, definition 4 gives us the result that the complement of a threshold graph is a threshold graph, since the set $\{C_4, P_4, 2K_2\}$ is closed under complementation [7].

The *threshold dimension* of G is the minimum number k of threshold subgraphs $T_1, \ldots, T_k$ of G such that the union of these graphs yields G and is denoted $\Theta(G)$. The threshold dimension is well defined, since a single edge along with isolated vertices is a threshold subgraph of G, and is bounded above by $|E|$. The problem of finding the threshold dimension of a graph has been shown to be NP-hard and has been of interest since the 80's, see [1]. Though generally finding the threshold dimension of a graph is hard it is known for many graphs and classes of graphs. When the threshold dimension is not known a bound may be established that is based on the structure of the graph, or a well-known parameter such as $\alpha(G)$. Here is on such from [1] whose proof is given as an example of the arguments made for such results.

Theorem 1.1 ([1])**.** *For every graph G on n vertices we have $\Theta(G) \leq n - \alpha(G)$. Furthermore, if G is triangle-free, then $\Theta(G) = n - \alpha(G)$.*

Proof. Let S be a largest independent set. For each vertex $u \in V - S$, consider the subgraph consisting of all the edges incident to u, i.e. the star centered at u. Each such star is a threshold graph and all the stars together cover G since S has no edges. Thus $\Theta(G) \leq |V - S| = n - \alpha(G)$. If, in addition, G is triangle free then every threshold subgraph is a star, and the centers of the stars that cover G form the complement of an independent set. It follows then that $\Theta(G) \geq n - \alpha(G)$ and hence $\Theta(G) = n - \alpha(G)$ for every triangle-free graph. $\blacksquare$

CHAPTER 2
BACKGROUND

2.1 Dot Product Representations

We will be examining a type of representation of graphs called a *dot product representation*. A *k-dot product representation* of $G = (V, E)$ is a function $f : V \to \mathbb{R}^k$ where for distinct vertices $u, v \in V$ we have $uv \in E$ if and only if $f(u) \cdot f(v) \geq t$ given $t \geq 0$ and where $\cdot$ is the standard inner product on $\mathbb{R}^k$. A graph with a k-dot product representation is called a *k-dot product graph*.

Proposition 2.0.1. *For every graph G there is an integer k so that G has a k-dot product representation.*

Proof. Let $G = (V, E)$ with $|E(G)| = k$, and choose $z \in \mathbb{R}$ such that $z^2 \geq t$. Arbitrarily assign each edge of G a unique label from 1 to k. Create the function $f : V \to R^k$ with the i^{th} component, with $1 \leq i \leq k$, of $f(v)$ equal to z if v is incident to edge i and equal to 0 otherwise. Then, for $v, u \in V$, $f(v)$ and $f(u)$ will have z in the same component if and only if they are incident to the same edge. Then we have $vu \in E$ if and only if $f(v) \cdot f(u) \geq t$. Note also that if $vu \notin E$ then $f(v) \cdot f(u) = 0$. Therefore G is a k-dot product graph. ■

2.1.1 Example Dot Product Representations

To better understand dot product representations of graphs consider the graph H in Figure 2.1.

Using the method described in the proof of Proposition 1 we can create a 5-dot product representation of H. A brief examination of the dot products of the following

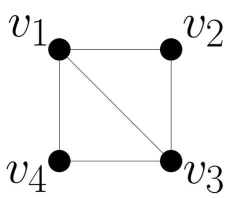

FIGURE 2.1: H: An example graph

vectors shows that they are a representation of H when we let $t = 1$:

$$f(v_1) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, f(v_2) = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, f(v_3) = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, f(v_4) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

This is not the only dot product representation of H. We can make a 2-dot product representation by following a similar procedure as above, only now we label the cliques of the graph, instead of the edges. Here we labeled the top triangle as clique 1 and the bottom triangle as clique 2. Again, let $t = 1$.

$$f(v_1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, f(v_2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, f(v_3) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, f(v_4) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

2.1.2 Dot Product Dimension

Since a primary motivation behind these representations is reducing the amount of information required to represent a graph uniquely in a computer the question arises of how small we can make these vectors. The minimum $k \in \mathbb{N}$ such that a k-dot product representation of G exists is called the *dot product dimension* of G, denoted $\rho(G)$. From the proof of Proposition 1 we obtain an upper bound for this value, $\rho(G) \leq |E(G)|$. Thus if G is edgeless we may define $\rho(G) = 0$. While several upper bounds are known, determining the value of this parameter for any given graph has been shown to be NP-hard [3].

That is not to say that finding the dot product dimension of a graph is always difficult. If we return to H in Figure 2.1 we have shown that $\rho(H) \leq 2$. With careful thought we can find a dot product representation of dimension 1, $f(v_1) = 2$, $f(v_2) =$

$\frac{1}{2}$, $f(v_3) = 2$, $f(v_4) = \frac{1}{2}$. Since only empty graphs have dot product representations of dimension 0, we can say that for our example graph $\rho(H) = 1$. Indeed it is the plight of many to determine which classes of graphs have easily computed ρ or which structures give a bound on ρ.

2.1.3 Relevant Results

Often local or global properties that are used to classify graphs result in whole classes of graphs having the same dot product dimension, some sort of formula for finding it, or bounds on the dimension's value. The following are some known results about dot product dimension from [7] and [2].

- $\rho(G) \leq 1$ if and only if no induced subgraph of G is either a $3K_2$, P_4, or C_4 if and only if G has at most two nontrivial components, both of which are threshold graphs. (Figure 2.2 shows a representation in one dimension of a graph with two nontrivial components that are threshold graphs.)

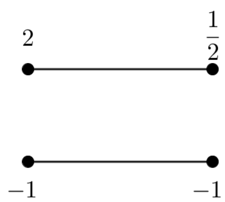

FIGURE 2.2: $\rho(2K_2) = 1$

- $\rho(C_n) = 2$ if $n \geq 4$
- $\rho(P_n) = 2$ if $n \geq 4$
- $\rho(G) \leq \Theta(G)$
- $\rho(G) \leq \rho(G - v) + 1$ for $v \in V$
- $\rho(G) \leq n - 1$
- If G is a tree, then $\rho(G) \leq 3$
- If G is a caterpillar, then $\rho(G) \leq 2$
- If $G = K_{n,m}$, then $\rho(G) = \min\{n, m\}$

- Let $n_1 \geq n_2 \geq \ldots \geq n_p \geq 1$ be integers. Then $\rho(K_{n_1,n_2,\ldots,n_p}) = n_2$.
- Let $G = (V, E)$, $A \subset V$, and K_A be the clique on A. Then $\rho(G \cup K_A) \leq \rho(G) + 1$.

2.1.4 As a Generalization of Intersection Representations

An *intersection representation* of a graph G assigns a set S_v to each vertex v so that $v_i v_j \in E(G)$ if and only if $S_{v_i} \cap S_{v_j} \neq \emptyset$. The smallest size of the union of the sets assigned to the vertices in an intersection representation of a graph is the *intersection number* of that graph. An intersection graph is one with an intersection representation. Every graph is an intersection graph. The class of threshold graphs defined in Chapter 1 is an example of a subclass of intersection graphs. Intersection graphs and their many subclasses have been the subject of research for over half a century because of the broadness of their applications. In the preface of [6] the authors state, "Intersection graphs provide *theory* to underlie much of graph theory. They epitomize graph-theoretic structure and have their own distinctive concepts and emphasis... They have real applications to topics like biology, computing, matrix analysis, and statistics (with many of these applications not well known)."

In fact, the concept of dot product representation is a generalization of intersection representation and dot product dimension of intersection number [2]. Thus our work, an expansion of the concept of dot product representation, is a contribution to the study of intersection graphs as well.

2.2 Tropical Arithmetics

In this work we will be using two algebra systems called the *tropcial semirings*. They are named so in honor of Imre Simon, a Brazilian mathematician and computer scientist, [4] and are defined here. The first is the *min-plus* tropical semring, denoted $\mathbb{T}$ we define as

$$(\mathbb{R} \cup \{\infty\}, \oplus, \otimes)$$

with operations

$$x \oplus y := min\{x, y\} \quad \text{and} \quad x \otimes y := x + y.$$

The second, the *max-plus* tropical semiring denoted $\widehat{\mathbb{T}}$, is defined similarly

$$(\mathbb{R} \cup \{-\infty\}, \widehat{\oplus}, \otimes)$$

with

$$x \mathbin{\widehat{\oplus}} y := max\{x, y\} \quad \text{and} \quad x \otimes y := x + y.$$

By letting the operator $\otimes$ take precedence when $\oplus, \widehat{\oplus}$ and $\otimes$ occur in the same expression it is easy to determine that associative, commutative, and distributive laws hold in tropical arithmetics. For example in $\mathbb{T}$ we have

$$4 \otimes (2 \oplus 5) = 4 \otimes 2 = 6 \quad \text{or} \quad 4 \otimes (2 \oplus 5) = (4 \otimes 2) \oplus (4 \otimes 5) = 6 \oplus 9 = 6.$$

For the same question in $\widehat{\mathbb{T}}$ we have

$$4 \otimes (2 \mathbin{\widehat{\oplus}} 5) = 4 \otimes 5 = 9 \quad \text{or} \quad 4 \otimes (2 \mathbin{\widehat{\oplus}} 5 = (4 \otimes 2) \mathbin{\widehat{\oplus}} (4 \otimes 5) = 6 \mathbin{\widehat{\oplus}} 9 = 9.$$

Both operations also have identity elements with

$$x \oplus \infty = x$$

$$x \mathbin{\widehat{\oplus}} - \infty = x$$

and

$$x \otimes 0 = x.$$

Tropical division is defined to be classical subtraction so we make no special notation for this operation. Since there is no real number, x, such that $7 \oplus x = 11$ has a solution there is no tropical subtraction. Thus $\mathbb{T}$ and $\widehat{\mathbb{T}}$ meet all of the ring axioms except for the existence of additive inverse, hence the name semiring. Perhaps not surprisingly $\mathbb{T}$ and $\widehat{\mathbb{T}}$ are isomorphic [4], meaning that there is a bijective homomorphism between them. That is, there exists some bijective function $f : \mathbb{T} \to \widehat{\mathbb{T}}$ (or the other way around) such that for $a, b \in \mathbb{T}$ we have $f(a + b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$. Essentially, this means the semirings are the same, up to labeling.

Other mathematical concepts translate naturally over to tropical arithmetic. Exponents can be treated as repeated multiplication (i.e. tropical exponentiation is classical multiplication), so in tropical algebras $\sqrt{-1} = -.5$ because $(-.5)^2 = -.5 \otimes -.5 = -1$. The linear algebra operations of vector addition, scalar multiplication, matrix multiplication, and the dot product of vectors all make perfect sense in the tropical semiring. This last operation is of special importance to this work. We define the *tropical dot* *product* of two vectors of dimension n as the tropical sum of the component wise tropical products. We will use $\odot$ to denote the operation of tropical dot product in $\mathbb{T}$ and $\widehat{\odot}$ in $\widehat{\mathbb{T}}$. Then given two vectors, $\vec{u} = [u_1, u_2, \ldots, u_n]$ and $\vec{v} = [v_1, v_2, \ldots, v_n]$, we have

$$\vec{u} \odot \vec{v} := min\{u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n\} \text{ in } \mathbb{T}$$
$$\vec{u} \,\widehat{\odot}\, \vec{v} := max\{u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n\} \text{ in } \widehat{\mathbb{T}}.$$

For example, given $\vec{v} = [1, 2, 3, 4]$ and $\vec{u} = [7, 5, 3, 2]$ we have

$$\vec{v} \odot \vec{u} = min\{8, 7, 6\} = 6$$
$$\vec{v} \,\widehat{\odot}\, \vec{u} = max\{8, 7, 6\} = 8.$$

Any operation not specifically designated as tropical $(\oplus, \widehat{\oplus}, \otimes, \odot, \widehat{\odot})$ should be considered as arithmetic in real, not tropical.

CHAPTER 3
TROPICAL DOT PRODUCT REPRESENTATIONS

Given the direct translation of the classical dot product to tropical arithmetics the definition of a DPR changes little when done in tropical arithmetics. A *tropical dot product representation of* G is a function $f : V \to \mathbb{R}^k$ for some k such that, for any $x, y \in V$, $xy \in E$ if and only if $f(x) \odot f(y) \geq t$ or $f(x) \mathbin{\widehat{\odot}} f(y) \geq t$. We will use the notation $\rho_T(G)$ and $\rho_{\widehat{T}}(G)$ to denote the tropical dot product dimension of G using min-plus and max-plus algebras respectively. Note that although $\mathbb{T}$ and $\widehat{\mathbb{T}}$ are isomorphic as rings $\rho_{_T}$ and $\rho_{_{\widehat{T}}}$ differ, but we ultimately show a connection between the two.

Theorem 3.1. *Every graph has a min-plus tropical dot product representation.*

Proof. Let G be a graph on n vertices and set $t > 0$. Assign an ordering to the vertices, $\{v_1, v_2, \ldots v_i, \ldots, v_n\}$. For $f : V \to \mathbb{R}^n$, build $f(v_i)$ component wise as follows:

$$\text{component } j \text{ of } f(v_i) = \begin{cases} \frac{t}{3} & \text{if } j = i \\ \infty & \text{if } j < i \\ t & \text{if } j > i \text{ and } v_iv_j \in E(G) \\ \frac{t}{2} & \text{if } j > i \text{ and } v_iv_j \notin E(G) \end{cases}$$

Now, for $v_iv_j \in E(G)$ we have $v_i \odot v_j = \frac{4t}{3} > t$, but if $v_iv_j \notin E(G)$ we have $v_i \odot v_j = \frac{5t}{6} < t$ as desired. ■

Corollary 3.1.1. *Let G be a graph and $v \in V(G)$. Then $\rho_{_T}(G) \leq \rho_{_T}(G - \{v\}) + 1$.*

Proof. Let $k = \rho_{_T}(G - \{v\})$ and choose $f : V \to T^k$ be a min-plus k-dot product representation of G with threshold $t > 0$. Now a min-plus $(k+1)$-dot product representation

$\hat{f}$ of G can be formed by adding an extra coordinate to the representation f. Let

$$\hat{f}(u) = \begin{cases} \begin{pmatrix} f(u) \\ t \end{pmatrix} & \text{when } u \neq v \text{ and } uv \in E(G) \\ \begin{pmatrix} f(u) \\ \frac{t}{2} \end{pmatrix} & \text{when } u \neq v \text{ and } uv \notin E(G) \\ \begin{pmatrix} \infty_k \\ \frac{t}{3} \end{pmatrix} & \text{when } u = v \end{cases}$$

where ∞_k is a vector of dimension k with all components equal to ∞. ■

Corollary 3.1.2. $\rho_{_T}(G) \leq n-1$

Proof. Let $u, v \in V(G)$ such that $uv \in E(G)$. Choose $t \geq 0$ and let $f : \{u, v\} \to \mathbb{R}$ be the mapping $f(u) = t$ and $f(v) = t$. Recursively apply Corollary 3.1.1 to build a $(n-1)$-dot product representation of G in $\mathbb{T}$. ■

Theorem 3.2. *Every graph has a max-plus tropical dot product representation.*

Proof. Let G be a graph on n vertices and let the threshold be $t > 0$. A n-dot product representation may be made in $\widehat{\mathbb{T}}$ by placing $\frac{t}{2}$ in the i^{th} component of $f(v_j)$ when $v_i v_j \in E(G)$ or $i = j$. ■

Given a particular set of vectors, the resulting graph is dependent on the choice of threshold, t. However, when given the graph and are left to determine the vectors we find that the threshold does not influence the value of the tropical threshold dimension.

Theorem 3.3. *The threshold, t, of any tropical dot product representation of a graph G can be any arbitrary $t > 0$.*

Proof. Let $f : V \to \mathbb{R}^k$ be a tropical k-dot product representation of G with threshold t^*. Then $\hat{f} : V \to \mathbb{R}^k$, $\hat{f}(u) = \frac{t}{t^*} f(u)$, is a tropical k-dot product representation of G with threshold t ■

Since our choice of threshold does not affect the tropical dot product dimension of a graph, from here on we will put the threshold $t = 1$ unless otherwise noted.

3.1 Results with $\Theta(G)$

Note that for vectors u, v in one dimension $v \odot u = v \mathbin{\widehat{\odot}} u$, since the minimum value of a set of size one is the same as the maximum value of that set. Hence if G can be represented in one dimension we have $\rho_{_T}(G) = \rho_{_{\widehat{T}}}(G)$. Thus the following theorem holds also for $\rho_{_{\widehat{T}}}(G)$.

Theorem 3.4. *Let G be a graph. Then $\rho_{_T}(G) = 1$ if and only if G is a threshold graph.*

Proof. $\rho_{_T}(G) = 1 \Leftrightarrow (f : V \to \mathbb{R}$ such that $f(v) \odot f(u) \geq t \Leftrightarrow vu \in E) \Leftrightarrow (f(v) \odot f(u) \geq t \Leftrightarrow f(v) + f(u) \geq t) \Leftrightarrow G$ is a threshold graph. ∎

This result implies that any graph with tropical dot product dimension 1 can have at most one non-trivial component. This is different than the result for $\rho(G) = 1$, where G could have 2 components that are non-trivial threshold graphs. Thus a $2K_2$ is an example where $\rho(G)$ differs from $\rho_{_T}(G)$ and $\rho_{_{\widehat{T}}}(G)$.

Due to the nature of $\oplus$ and $\widehat{\oplus}$ adding a dimension to a representation has limited effects on the tropical dot product of two vectors from the representation. Suppose a graph G has a dot product representation in $\mathbb{T}^k$ using a given threshold. Create a set of vectors of dimension $k+1$, by adding an additional component to the end of each of the vectors representing the vertices of G then consider the min-plus dot product graph of these vectors, using the same threshold. If $v \odot u < t$ in the k-dimensional representation then there is a component sum that is less than 1 which will remain less than one in the $(k+1)$-dimensional representation. So adding a dimension to a set of vectors can only yield a graph with an equal or smaller number of edges than the original. Conversely in $\widehat{\mathbb{T}}$ adding a component cannot remove edges. This quality, in $\widehat{\mathbb{T}}$, is reminiscent of the result of a union of graphs. The union of two graphs cannot have fewer edges than either of the original graphs had. This caused us to search for a connection to threshold dimension.

Theorem 3.5. $\rho_{\widehat{T}}(G) = \Theta(G)$

Proof. $\rho_{\widehat{T}}(G) \leq \Theta(G)$: Let $t = 1$ and G be a graph such that $\Theta(G) = m$. Let $\{G_1, G_2, \ldots, G_m\}$ be a set of threshold graphs that realize $\Theta(G)$. Let $g_i : \mathbb{R} \to \mathbb{R}$

be the function for the threshold representation of G_i. Create $f(v_i)$ as follows:

$$\text{component } j \text{ of } f(v_i) = \begin{cases} g_j(v_i) & \text{when } v_i \in V(G_i) \\ 0 & otherwise \end{cases}$$

Thus we have an m-dimensional max-plus dot product representation of G and $\rho_{\widehat{T}}(G) \leq \Theta(G)$.

$\rho_{\widehat{T}}(G) \geq \Theta(G)$: Let (G) be a graph such that $\rho_{\widehat{T}}(G) = k$. This means there is a k-dimensional representation of G, call it f. By the theorem above we see that if we consider the graph created by the dot product of vectors made of only the value of the j^{th} component of $f(v_i)$ for all $v_i \in G$ we have a threshold graph that is a subgraph of G. By this consideration we find k threshold graphs such that the union of these graphs gives us G. By the definition of threshold dimension we know that $\Theta(G) \leq k$ and hence $\Theta(G) \leq \rho_{\widehat{T}}(G)$
Thus $\rho_{\widehat{T}}(G) = \Theta(G)$. ■

This result gives us a wealth of free information about $\rho_{\widehat{T}}(G)$ and also gives us a new way to explore the well researched parameter Θ. Problems and results dealing with $\Theta(G)$ can now be rephrased in terms of $\rho_{\widehat{T}}(G)$. For example, from Theorem 1.1 we have:

Corollary 3.5.1. *For every graph G on n vertices we have $\rho_{\widehat{T}}(G) \leq n - \alpha(G)$. Furthermore, if G is triangle-free, then $\rho_{\widehat{T}}(G) = n - \alpha(G)$.*

Define the *threshold graph intersection number* as the minimum value for k such that there exists a set of k threshold graphs, $G_1, G_2, \ldots, G_k$, $G_i = (V, E(G_i))$, where $G = \cap_{i=1}^{k} G_i$. We denote the threshold graph intersection number as $\widehat{\Theta}(G)$ for lack of imagination. The intersection of two graphs cannot have more edges than either of the original graphs. This hints at a connection between our new parameter and $\rho_{T}(G)$.

Theorem 3.6. *For $G = (V, E)$, $\rho_{T}(G) = \widehat{\Theta}(G)$.*

Proof. $\rho_{T}(G) \geq \widehat{\Theta}(G)$: Suppose $\rho_{T}(G) = k$, and $f : V \to R^k$ is a k-tropical dot product representation of G. Let v_i represent the value of the i^{th} component of $f(v)$. Create a set of $G_i = (V, E(G_i))$ which are the threshold graphs formed by $f_i(v) = v_i$ for $v \in V$ and $t = 1$. If $xy \in E(G)$ then $x \odot y \geq 1$ so $x_i + y_i \geq 1$ for all $1 \leq i \leq k$. Hence $xy \in E(G_i)$

for $1 \leq i \leq k$ and $xy \in \cap_{i=1}^{k} E(G_i)$. If $xy \notin E(G)$ then $x \odot y < 1$ and there exists some value of i for which $x_i + y_i < 1$. For this i, $xy \notin E(G_i)$ and hence $xy \notin \cap_{i=1}^{k} G_i$. Thus $G = \cap_{i=1}^{k} G_i$ and $\rho_{_T}(G) \geq \widehat{\Theta}(G)$.

$\rho_{_T}(G) \leq \widehat{\Theta}(G)$: Let $X = \{G_1, G_2, \ldots, G_k\}$ be a set of threshold graphs on V such that $G = \cap_{i=1}^{k} G_i$. Since each G_i is a threshold graph, let the function $f_i : V \to R$ be a mapping that realizes G_i as a threshold graph. Create $f : V \to R^k$ where for $v \in V$ the i^{th} component of $f(v)$ is $f_i(v)$. Since $G = \cap_{i=1}^{k} G_i$, for $xy \in E(G)$ we know that $xy \in E(G_i)$ for $1 \leq i \leq k$ so $f_i(x) + f_i(y) \geq 1$ and by definition $f(x) \odot f(y) \geq 1$ as desired. If $xy \notin E(G)$ there is a G_i with $xy \notin E(G_i)$. Then $f_i(x) + f_i(y) < 1$ and $f(x) \odot f(y) < 1$. Thus f is a tropical k-dot product of G and $\rho_{_T} \leq \widehat{\Theta}(G)$.
Therefore $\rho_{_T}(G) = \widehat{\Theta}(G)$. ∎

Recall the following theorem from set theory.

Theorem 3.7 (de Morgan's Law). *Let $\cup$ represent "or", $\cap$ represent "and", and $'$ represent "not". Then for two logical units E and F, $(E \cup F)' = E' \cap F'$ and $(E \cap F)' = E' \cup F'$.*

Note that these laws also apply in set theory where $\cup$ denotes union, $\cap$ denotes intersection, and $'$ is complementation with respect to a superset of E and F. We are now able to make a connection between $\rho_{_T}(G)$ and $\rho_{_{\widehat{T}}}(G)$ using de Morgan's law and their connections to $\Theta(G)$ and $\widehat{\Theta}(G)$.

Theorem 3.8. $\widehat{\Theta}(G) = \rho_{_T}(G) = \rho_{_{\widehat{T}}}(\overline{G}) = \Theta(\overline{G})$

Proof. If G is a threshold graph then $\overline{G}$ is a threshold graph.[7] Thus $\rho_{_T}(G) = \rho_{_{\widehat{T}}}(G) = 1$. If G is not a threshold graph then suppose $\rho_{_T}(G) = k$. Then there exist graphs $G_1, G_2, \ldots, G_k$, $G_i = (V, E(G_i)$, such that $G = \cap_{i=1}^{k} G_i$. By de Morgan's Law $\overline{G} = \overline{\cap_{i=1}^{k} G_i} = \cup_{i=1}^{k} \overline{G_i}$ and so $\rho_{_T}(G) \geq \rho_{_{\widehat{T}}}(\overline{G})$.

For $\rho_{_T}(G) \leq \rho_{_{\widehat{T}}}(\overline{G})$ suppose $\rho_{_{\widehat{T}}}(\overline{G}) = k$ then $\overline{G} = \cup_{i=1}^{k} G_i$. By de Morgan's Law $G = \overline{\cup_{i=1}^{k} G_i} = \cap_{i=1}^{k}(\overline{G})$ so $\rho_{_T}(G) \leq \rho_{_{\widehat{T}}}(\overline{G})$. ∎

The following corollary is a direct result of this theorem.

Corollary 3.8.1. *If G is self complementary then $\rho_{_T}(G) = \rho_{_{\widehat{T}}}(G)$.*

3.2 Other Results

Beginning with the results above we can establish values and bounds on the tropical threshold dimensions for non-threshold graphs. For many of the cases below the bound or value of the max-plus dot product dimension has already been determined by those studying threshold dimension and can be found in resources such as [5]. As the purpose of this work is not to reiterate these findings, we predominantly give results about $\rho_{_T}(G)$.

Corollary 3.8.2. $\rho_{_T}(C_4) = 2$

Proof. The complement of a C_4 is a $2K_2$. This graph has threshold dimension 2, so $\rho_{\hat{T}}(\overline{C_4}) = 2$ by Theorem 3.5. Then by Theorem 3.8 $\rho_{_T}(C_4) = 2$. ∎

Corollary 3.8.3. $\rho_{_T}(2K_2) = 2$

Proof. The complement of a $2K_2$ is a C_4 which has threshold dimension 2. Thus by Theorem 3.5 $\rho_{\hat{T}}(\overline{2K_2}) = 2$ so $\rho_{_T}(2K_2) = 2$ by Theorem 3.8. ∎

Corollary 3.8.4. $\rho_{_T}(P_4) = 2$

Proof. A P_4 is self complementary and has threshold dimension 2. Thus by Theorem 3.5 and Corollary 3.8.1 $\rho_{\hat{T}}(P_4) = \rho_{_T}(P_4) = 2$. ∎

Though the threshold dimension of a graph, or its complement, generally may not be known we can still say some things about the min-plus dot product dimension of it. Many of these results echo the type of results that have been obtained regarding $\rho(G)$, some of which are listed in Section 2.1.3.

Corollary 3.8.5. *For any graph G on $n \geq 3$ vertices, let H be the largest induced subgraph of G that is a threshold graph. If H has k vertices then $\rho_{_T}(G) \leq n - k + 1$*

Proof. Since any graph on $n = 3$ vertices is a threshold graph, every graph has an induced subgraph that is a threshold graph on at least 3 vertices. Find the largest such graph, call it H. By Theorem 3.4 there exists a function f that is a 1-dimensional representation of this subgraph. Now apply Lemma 3.1.1 $n - (k - 1)$ times to create $\hat{f}$ an $n - k + 1$ tropical dot product representation of G. ∎

A result for $\rho(G)$ is that by adding edges to G to form a clique on some subset of vertices of G the dot product dimension is increased by at most 1. The *join* of two graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, is the graph $G_1 \vee G_2 = G_1 \cup G_2 \cup K_{V_1,V_2}$, where K_{V_1,V_2} is the bipartite graph with V_1 as one partite set and V_2 as the other. In other words, $G_1 \vee G_2$ adds an edge from every vertex of G_1 to every vertex of G_2. When G_2 is a complete graph this operation does not increase the tropical dot product dimension of a graph:

Lemma 3.8.1. *Let G be a graph and K_n a complete graph on n-vertices. Then $\rho_{_T}(G \vee K_n) = \rho_{_T}(G)$.*

Proof. Let $\rho_{_T}(G) = k$ and $f : V(G) \to \mathbb{R}^k$ be a tropical k-dot product representation of G. Let $H = G \vee K_n$ and let $g : V(H) \to \mathbb{R}^k$ be the mapping where

$$g(v_i) = \begin{cases} f(v_i) & \text{when } v_i \in V(G) \\ \vec{1}_k & \text{when } v_i \in K_n \end{cases}$$

where (1_k) is the vector of entries equal to 1 of dimension k. Now the vertices of the K_n are adjacent to all vertices in H, but the adjacencies between vertices originally in G remain as they were. ∎

Note that this same proof works to show that $\rho_{_{\widehat{T}}}(G \vee K_n) = \rho_{_{\widehat{T}}}(G)$.

Theorem 3.9. *Let G be a complete k-partite graph. Then $\rho_{_T}(G) \leq k$.*

Proof. Let $t = 1$. Consider the function $f : V \to \mathbb{R}^k$ that maps vertices in partite set i, $1 \leq i \leq k$, to the vector with entry i equal to 0 and all other entries equal to 1. Suppose v_p and v_q are in different partite sets; then $f(v_p) \odot f(v_q) = 1$. Suppose v_p and v_q are in the same partite set; then $f(v_p) \odot f(v_q) = 0$. Thus we have a tropical k-dot product representation of a complete k-partite graph. ∎

If given only a little more information about the sizes of the partite sets we improve the results.

Corollary 3.9.1. *Let G be a complete k-partite graph with the size of each partite set greater than 1. Then $\rho_{_T}(G) = k$.*

Proof. Consider $\overline{G}$ which consists of k components that are complete graphs. The threshold dimension of a disconnected graph must be at least the number of components, and exactly that number when each component is a threshold graph. Since a complete graph is a threshold graph we have $\Theta(\overline{G}) = k$ and hence $\rho_{_T}(G) = k$ by Theorem 3.8. ∎

Corollary 3.9.2. *Let G be a complete k-partite graph with m $(m < k)$ of the partite sets of cardinality 1. Then $\rho_{_T}(G) = k - m$.*

Proof. Let $\hat{G}$ be the graph induced on the $k - m$ partite sets of size greater than 1. By Corollary 3.9.1 $\rho_{_T}(\hat{G}) = k - m$. Now apply Lemma 3.8.1 m times to add m K_1s to $\hat{G}$ to have a tropical $(k - m)$-dot product representation of G. ∎

We saw above that $\rho(2K_2) < \rho_{_T}(2K_2)$. Consider now $K_{m,n}$ where $n, m > 2$. From Corollary 3.9.1 we know $\rho_{_T}(K_{m,n}) = 2$ but $\rho(K_{m,n}) = min\{m, n\} > 2$. Thus we have examples of $\rho(G) \leq \rho_{_T}(G)$ and $\rho(G) \geq \rho_{_T}(G)$ so we cannot use one as a bound of the other. In [7] it was shown that $\rho(G) \leq \Theta(G)$ so $\rho(G) \leq \rho_{_{\widehat{T}}}(G)$ for any graph.

It was also shown in [7] that if T is a tree then $\rho(T) \leq 3$. A bound on $\rho_{_T}$ for trees in general has not been established, but we have been able to establish the same bound for $\rho_{_T}$ of caterpillars as for ρ in [7]:

Theorem 3.10. *Let G be a caterpillar. Then $\rho_{_T}(G) \leq 2$.*

Proof. Proof by construction: First we will construct a min-plus dot product representation of a path of longest length. Find such a path and label the vertices $p_1, p_2, \ldots, p_m$ beginning at one end and proceeding down the path. Let v_{i_j} be the j^{th} leaf attached at vertex p_i. A caterpillar labeled in this way is shown in Figure 3.1. Choose $k \geq 2$ and let $d_i = \lfloor \frac{i-1}{2} \rfloor$ for vertex p_i, $1 \leq i \leq m$. Then give each vertex a function value as follows:

$$f(p_i) = \begin{cases} \begin{pmatrix} \frac{1}{k+d_i} \\ \\ \frac{k+d_i}{k+d_i+1} \end{pmatrix} & \text{when } i \text{ is odd} \\ \\ \begin{pmatrix} \frac{k+d_i}{k+d_i+1} \\ \\ \frac{1}{k+d_i+1} \end{pmatrix} & \text{when } i \text{ is even} \end{cases}$$

Consider vertices p_j and p_k, and without loss of generality let $j < k$. If they are adjacent then by construction $f(p_j) \odot f(p_k) \geq 1$. If they are not adjacent and both j and k are odd then the sum of the first components will be less than one, if they're both even the sum of the second components will be less than 1. If they are nonadjacent and j is odd and k is even then $k - j \geq 3$ so $d_j < d_k$ and the sum of the second components of the vectors will give $\frac{k+d_j}{k+d_j+1} + \frac{1}{k+d_k+1} < 1$ since $\frac{1}{k+d_k+1} < \frac{1}{k+d_j+1}$. Similarly if j is even and k is odd then the sum of the first components is less than 1. For all these cases $f(p_j) \odot f(p_k) < 1$ as desired.

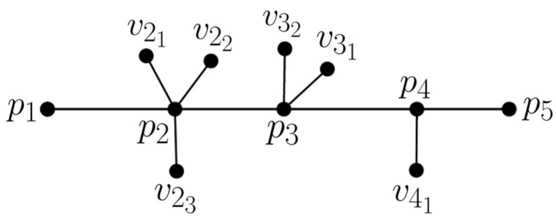

FIGURE 3.1: B: A caterpillar with its vertices labeled

Now for v_{i_j},

$$f(v_{i_j}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - f(p_i)$$

Figure 3.2 shows a 2-dot product representation of B using this construction with $k = 2$.

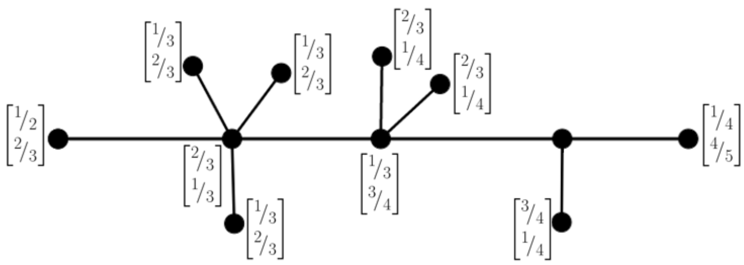

FIGURE 3.2: A min-plus 2-dot product representation of B

By construction these leaves are adjacent to p_i. There are 10 other types of vertices it should not be adjacent to. In each case the progressively increasing denominator and the alternating nature of the vector components cause the tropical dot product to fail to reach 1. In all these cases either the sum of one set of components is of two values less than $\frac{1}{2}$ or of a value of more than $\frac{1}{2}$ and a value smaller than needed to add to 1. The following tables detail each of these cases and tells which of the sets of components fails to add to 1. First consider $f(v_{i_j}) \odot f(v_{k_k})$

i	k	components whose sum is < 1
even	*even*	1^{st}
odd	*even*	1^{st}
odd	*odd*	2^{nd}
even	*odd*	2^{nd}

Now consider $f(v_{i_j}) \odot f(p_k)$

i	k	index constraints	components whose sum is < 1
even	*odd*	$i \neq k$	1^{st}
odd	*even*	$i \neq k$	2^{nd}
even	*even*	$i < k$	2^{nd}
		$i > k$	1^{st}
odd	*odd*	$i < k$	1^{st}
		$i > k$	2^{nd}

Thus $f : V \to \mathbb{R}^2$ as described above produces a tropical dot product representation of dimension 2, and $\rho_{_T}(G) \leq 2$ for any caterpillar. $\rho_{_T}(G) = 2$ when G is a caterpillar that is not a threshold graph. ■

This also gives a value for non threshold subgraphs of caterpillars, i.e. paths.

Corollary 3.10.1. $\rho_{_T}(P_n) = 2$ *for* $n > 3$.

Proof. Since P_n is a caterpillar, by Theorem 3.10 and Theorem 3.4 $\rho_{_T}(P_n) = 2$ when $n > 3$. ■

Theorem 3.10 also reduces the min-plus dot product dimension upper bound for any graph that has an induced subgraph that is a caterpillar which is at least 2 vertices larger than it's largest induced subgraph that is a threshold graph.

Corollary 3.10.2. $\rho_{_T}(C_n) \leq 3$.

Proof. If $n \leq 4$ the tropical dot product dimension has been shown. Let $n \geq 5$. For $v \in V$, $C_n - v$ is a path, which has dimension 2. Then by Lemma 3.1.1 $\rho_{_T}(C_n) \leq \rho_{_T}(C_n - v) + 1$ hence $\rho_{_T}(C_n) \leq 3$. ■

Because of the linear nature of caterpillars we are able to use the labeling system in Theorem 3.10 on disconnected graphs where each component is a caterpillar.

Corollary 3.10.3. *Let G be a graph with $k > 1$ components that are caterpillars. Then $\rho_{_T}(G) = 2$.*

Proof. Find a path of longest length for each caterpillar. Beginning with one caterpillar, label the vertices following the pattern from Theorem 3.10. When a caterpillar is completely labeled continue the labeling on another caterpillar beginning the path labels with 2 more than the highest path label already used. By design the construction in Theorem 3.10 connects path vertex p_i only to p_{i-1} and p_{i+1}. Thus the last vertex of one caterpillar and the first of another will not be adjacent, and the rest of the labeling provides only the desired adjacencies as shown in the proof of the theorem above. ■

We have already seen that ρ does not necessarily relate to $\rho_{_T}$ and is a lower bound for $\rho_{\widehat{T}}$. It is yet to be determined if there is a bounding relationship between the min-plus and max-plus dot product dimensions of a graph.

Theorem 3.11. *There exist graphs such that $\rho_{_T}(G) \neq \rho_{\widehat{T}}(G)$*

Proof. $\Theta(P_6) = 3 \Rightarrow \rho_{\widehat{T}}(P_6) = 3$ but $\rho_{_T}(P_6) = 2$ by Corollary 3.10.1. ■

Consequently, if there is to be a bounding of one by the other, we have only one option.

Conjecture 3.11.1. $\rho_{_T}(G) \leq \rho_{\widehat{T}}(G)$ *for all G.*

CHAPTER 4
APPLICATION AND FUTURE WORK

4.1 Application

The main course of our study has been working with graphs to find tropical dot product representations, and find values and bounds for tropical dot product dimensions. The situation can be viewed from the other direction. Every set of vectors yields a variety of tropical dot product graphs depending on our choice of threshold and whether we work in $\mathbb{T}$ or $\widehat{\mathbb{T}}$. This leads to the possibility of many applications for these kinds of graph representations. A natural association with tropical dot product representations is with qualification questions. Min-plus representations can evaluate situations where all of the qualifications must be met. Max-plus representations can evaluate situations where only one of many qualifications need be met. We give a few basic examples to illustrate the applications. We set our examples in a classroom with six students: Andy, Bryn, Chuck, Darby, Elian, and Fran. These students will be represented in graphs by a vertex labeled with the first letter of their name.

Example 4.0.1. A math project

The teacher wants to pair the students up to work on a research project. The student pairs will research a mathematical question, using computer programs to aid in their study, and then will write up a report on their findings. In order for the student pairs to be successful it is important that each pair has sufficient skill between them to complete all the tasks required. The teacher decided to have the students rate their skills in three areas — computer use, mathematics, and english — as an integer in the range $[-2, 2]$, with -2 meaning no skill and 2 very skilled. Then, shown in Table 4.1, a vector for each student is created representing their answers: $[computer, math, english]$.

TABLE 4.1: Student's response vectors

Name	Computer	Math	English
Andy	[1,	1,	1]
Bryn	[-1,	1,	2]
Chuck	[2,	2,	-1]
Darby	[0,	0,	1]
Elian	[1,	-1,	2]
Fran	[2,	0,	-1]

The teacher can then decide on an appropriate threshold and create a tropical dot product graph. A min-plus dot product graph is constructed since the teacher wants the student pairs to have adequate skill in all 3 areas, not just one. The teacher decides on a threshold of 0 and produces the graph in Figure 4.1.

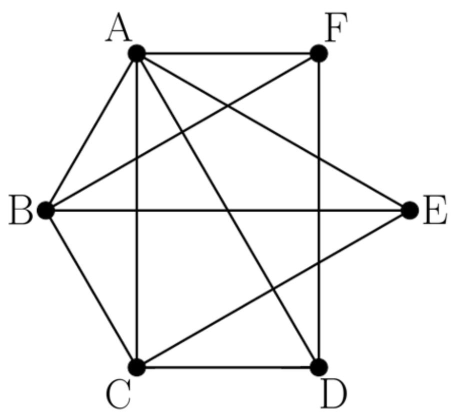

FIGURE 4.1: The min-plus dot product graph with $t = 0$

Each edge represents a pair of students whose scores add to 0 or greater and hence could be paired. Using only Figure 4.1 it is unclear how the pairs should be selected. Any edge indicates an acceptable pair, but are there pairs that would match better than others? A higher threshold would show pairs that have even higher competence levels in each subject, potentially proving for a better experience and better results on the project. If the teacher uses $t = 1$ instead the graph in Figure 4.2 is obtained. If the teacher wants each pair to have this higher skill level then Andy and Darby can only be paired with each other. Though Bryn and Chuck could be a pair Fran and Elian cannot so we use the other pairs to maintain the higher skill level in each pairing. The teacher then makes the pairs Andy with Darby, Fran with Bryn, and Elian with Chuck.

This suggests another problem to investigate.

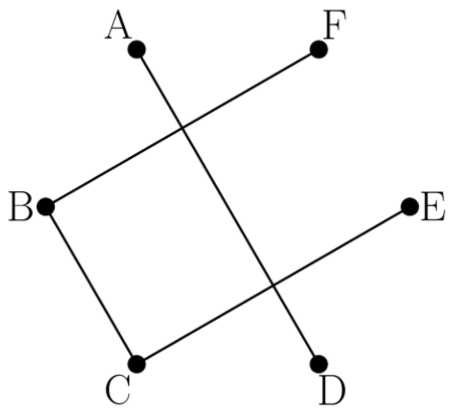

FIGURE 4.2: The min-plus dot product graph with $t = 1$

Example 4.0.2. A general project

Suppose in Example 4.0.1 the teacher wanted to allow students to select a difficult research project from one of three subject areas instead of giving the same project to each group. The students would be paired up and then complete a project that would involve extensive research or problem solving. The teacher wants each student pair to have a subject that both students excel at so they can work as equals on the project. The teacher can use the same ranking system and data from Example 4.0.1 but since each pair only needs to have high marks in one subject area, instead of all 3, the teacher used the max-plus dot product representation. Ideally each student pair would have a subject that both students ranked as a 2 so the teacher begins with a threshold of 4. This produces the graph in Figure 4.3. However, with $t = 4$ Darby is left as an independent vertex. Since Fran and Bryn are only adjacent to Chuck and Elian, respectively, the teacher could pair Andy and Darby as the only unmatched ones. This would provide the highest possible skill ranking sum for two of the three pairs, but it is uncertain if Darby

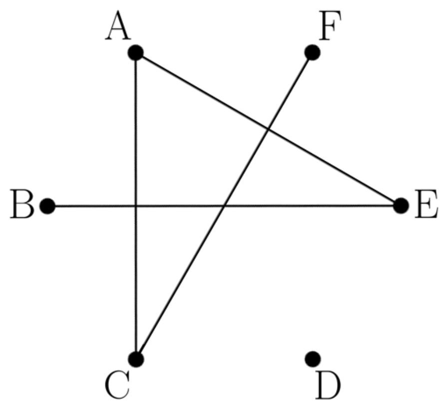

FIGURE 4.3: The max-plus dot product graph with $t = 4$

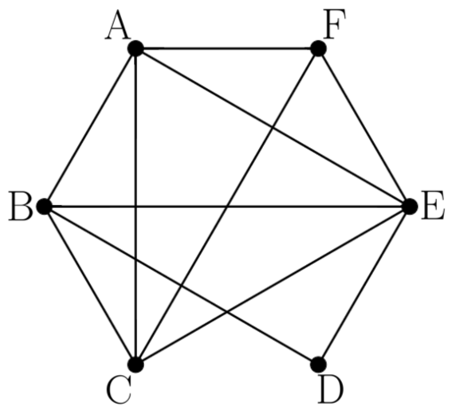

FIGURE 4.4: The max-plus dot product graph with $t = 3$

and Andy's skills would be high enough in an area to complete the project. If $t = 3$ the graph, in Figure 4.4, is connected and there are multiple ways the students can be paired such that each group has one skill that both students excel at. It is interesting to note that in the graph when $t = 3$ Andy and Darby are not adjacent. The teacher can still pair Fran and Chuck as well as Andy and Elian, which pairs were adjacent with $t = 4$, and then pair Bryn with Darby for the highest possible competency level for each group.

Example 4.0.3. Career day

The six students are to be assigned one of six jobs to experience for a day. Using the data collected for Example 4.0.1 the teacher decides to make a min-plus tropical dot product bipartite graph to show which students would be qualified to perform each job. To assign vectors for the jobs the teacher inverts the scale used by the students. In the vector for each job a value of -2 means much skill is required in that area and 2 means no particular skill in that area is required to successfully perform the job. To make the graph bipartite two additional components are added to both the student and job vectors. The first additional component is a 2 if the vector represents a student, 0 otherwise. The second additional component is a 2 if the vector represents a job, 0 otherwise. The vectors for the jobs and the student vectors with the added components are Tables 4.2 and 4.3, respectively.

All of the students need to be assigned to a job, so a perfect matching, or a set of edges such that each vertex belongs to exactly one of them, is required in our graph. An edge in the tropical dot product graph with $t = 1$ indicates that the student exceeds the skill level requirements for the job. Figure 4.5 shows that no perfect matching exists using

TABLE 4.2: Job requirement vectors

Job	Computer	Math	English	Student	Job
Cashier	[1,	1,	2,	0,	2]
Buisness Manager	[0,	0,	-1,	0,	2]
I.T.	[-2,	0,	, 2,	0,	2]
Writer	[-1,	2,	-2,	0,	2]
Secretary	[0,	1,	0,	0,	2]
Program Developer	[-2,	-1,	2,	0,	2]

TABLE 4.3: Student's response vectors with added components

Name	Computer	Math	English	Student	Job
Andy	[1,	1,	1,	2,	0]
Bryn	[-1,	1,	2,	2,	0]
Chuck	[2,	2,	-1,	2,	0]
Darby	[0,	0,	1,	2,	0]
Elian	[1,	-1,	2,	2,	0]
Fran	[2,	0,	-1,	2,	0]

this threshold. Students are still qualified for a job if the dot product of the vectors is only 0 so the teacher considers the graph with $t = 0$. This graph looks much more likely to contain a perfect matching. Keeping in mind that an edge connected Andy and Secretary when $t = 1$, if a perfect matching can be found using that edge it would

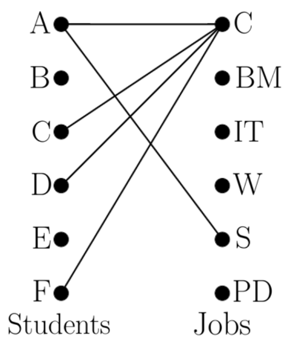

FIGURE 4.5: A min-plus dot product graph with $t = 1$

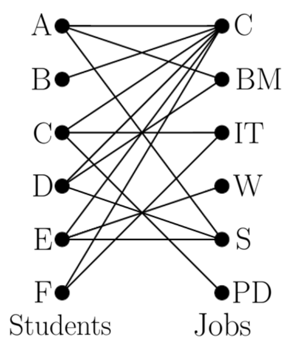

FIGURE 4.6: A min-plus dot product graph with $t = 0$

be a better choice than one that does not. In fact (fortunately for the teacher) such a matching exists as shown in Figure 4.7.

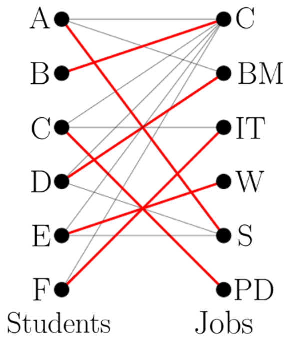

FIGURE 4.7: A perfect matching is in thick red lines

4.2 Future Work

Much remains to be determined about tropical dot product graphs and their dimensions. We conclude this work with a list of open questions.

1. Is $\rho_{_T}(G) \leq \rho_{\widehat{T}}(G)$ for all graphs?
2. Can we determine $\rho_{\widehat{T}}(G)$ of graphs for which $\Theta(G)$ is unknown?
3. Characterize k-dot product graphs under $\mathbb{T}$.
4. Does $\rho_{_T}(C_n) = 3$ for all $n \geq 5$?
5. Does $\rho_{_T}(T)$, where T is any tree, have an upper bound?
6. Threshold graphs are a subclass of interval graphs. Is there an upperbound for $\rho_{_T}(G)$ where G is an interval graph?
7. What, if any, connection is there between other graph properties and $\rho_{_T}$?

Finally, we have shown that consideration of dot product representations of graphs in algebraic systems other than the standard reals with classical addition and multiplication is worthwhile. In which other fields, rings, or semirings can we create dot product representations?

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