TRANSVERSE GROUP ACTIONS ON BUNDLES

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Abstract. An action of a Lie group G on a bundle $\pi\colon E \to M$ is said to be transverse if it is projectable and if the orbits of G on E are diffeomorphic under π to the orbits of G on M. Transverse group actions on bundles are completely classified in terms of the pullback bundle construction for G-invariant maps. This classification result is used to give a full characterization of the G invariant sections of E for projectable group actions.

Keywords. transverse group actions, regular group actions, kinematic bundle, invariant sections.

March 16, 2001
Research supported by NSF grant DMS–9804833

1. Introduction. Let G be a finite dimensional Lie group which acts on a bundle E. A general problem, with a diverse range of applications in differential geometry, differential equations and mathematical physics, is that of explicitly characterizing the space of all smooth, G invariant sections of E. This problem includes the characterization of invariant metrics and connections in relativity theory and gauge theories and plays a central role in our recent work [*3*] extending the classical method of Lie for finding the group invariant solutions of differential equations with symmetry. However, in general, without certain regularity assumptions concerning the action of G on E one cannot hope for a simple, practical characterization of the G invariant sections of E.

In this note we address the problem of identifying precisely these requisite regularity assumptions. We first observe that the general problem of classifying the G invariant sections of E naturally reduces to the case where G acts transversely on E. Accordingly, we make a careful study of transverse group actions on bundles. Our main result gives a complete classification of such group actions in terms of the pullback bundle construction by G invariant maps. This, in turn, leads to a general classification theorem for invariant sections based on a minimal set of regularity conditions which are readily verified for many of the kinds of group actions which arise in applications in mathematical physics and differential equations.

To describe our results in greater detail, let $\pi\colon E \to M$ be a smooth (C^∞) submersion. For the time being we need not suppose that E is a fiber bundle over M so that, in particular, the fibers $E_x = \pi^{-1}(x)$ need not all be diffeomorphic. The Lie group G (which is not assumed connected or compact) acts ***projectably*** on E if it acts by fiber-preserving transformations — for any $p, q \in E$ and $g \in G$,

$$\pi(g \cdot p) = \pi(g \cdot q) \quad \text{whenever} \quad \pi(p) = \pi(q).$$

Since the action of G on E preserves the fibers of π, there is a smooth action of G on M for which π is G equivariant, that is, $\pi(g\cdot p) = g\cdot\pi(p)$ for all $p \in E$ and $g \in G$. We write $G_p = \{\, g \in G \,|\, g\cdot p = p \,\}$ for the isotropy subgroup of G at p. It is easily seen that for any $p \in E$, $G_p \subset G_{\pi(p)}$.

We say that G acts ***transversely*** on E if G acts projectably on E and if $G_p = G_{\pi(p)}$ for all $p \in E$. Thus for each fixed $p \in E$ and each fixed $g \in G$, the equation

$$\pi(g \cdot p) = \pi(p) \quad \text{implies that} \quad g \cdot p = p. \tag{1.1}$$

Equivalently, G acts transversely on E if the orbits of G in E project diffeomorphically under π to the orbits of G in M. We have the following examples and constructions of transverse group actions.

[i] If a group acts freely on M, then the induced action on any associated natural bundle of M, such as the tangent bundle of M, is always transverse.

[ii] Let $J^k(E) \to M$ be the k-th order jet bundle of E over M and let $\mathrm{Inv}^k(E) \subset J^k(E)$ be the bundle of k-jets of G invariant, locally defined sections of E. Then the natural action of G on $J^k(E)$ restricts to a transverse action on $\mathrm{Inv}^k(E)$. See [*10*](p. 244)

[iii] Any projectable group action on E, transverse or otherwise, naturally restricts to a tranverse action on the ***kinematic bundle*** $\kappa(E)$ ***for the action of*** G ***on*** E, the fibers of which are

$$\kappa_x(E) = \big\{\, p \in E \,|\, g \cdot p = p \quad \text{for all} \quad g \in G_x \,\big\}. \tag{1.2}$$

The kinematic bundle is the maximal subset of E over M on which G acts transversely. See [*3*].

[iv] Bundles with transverse group actions are also easily constructed as pullback bundles under G invariant maps on M. Specifically, let G be a Lie group acting on a manifold M, let $\mathfrak{q} \colon M \to X$ be a smooth, G invariant map and let $\pi \colon Y \to X$ be a bundle over X. Then the action of G on $Y \times M$ given by $g \cdot (y, x) = (y, g \cdot x)$ restricts to an action on the pullback bundle $\mathfrak{q}^*(Y) \to M$ which is both projectable and transverse.

The principle result which we wish to establish in this article states that when the quotient space $\mathfrak{q}_M \colon M \to M/G$ of M by the orbits of G is a smooth manifold, then every transverse group action on any smooth bundle $\pi \colon E \to M$ can be constructed in accordance with example **[iv]**.

Theorem 1.1 THE CLASSIFICATION THEOREM FOR TRANSVERSE GROUP ACTIONS. *Let G be a Lie group which acts transversely on a smooth bundle $\pi \colon E \to M$. Assume that the quotient space $\mathfrak{q}_M \colon M \to M/G$ of M by the orbits of G is a smooth manifold. Then*

[i] *the quotient space E/G is also Hausdorff;*

[ii] *the quotient space $\mathfrak{q}_E \colon E \to E/G$ of E by the orbits of G is a smooth manifold, $\tilde{\pi} \colon E/G \to M/G$ is a smooth bundle, and the diagram*

$$\begin{array}{ccc} E & \xrightarrow{\ \mathfrak{q}_E\ } & E/G \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{\pi}} \\ M & \xrightarrow{\ \mathfrak{q}_M\ } & M/G \end{array} \tag{1.3}$$

commutes;

[iii] *if $\pi \colon E \to M$ is a fiber bundle with fiber F, then $\tilde{\pi} \colon E/G \to M/G$ is also a fiber bundle with fiber F; and*

[iv] *the bundle $\pi \colon E \to M$ is strongly G-equivalent to the pullback bundle $\pi \colon \mathfrak{q}_M^*(E/G) \to M$ with its canonical G action.*

It should be emphasized that Theorem 1.1 establishes that for transverse group actions the regularity of the action of G on the total space E is implied by the regularity of the action of G on M. This result is useful in many applications (see, for example, [*3*]) since the regularity of G on E is often difficult to check directly. We shall give examples which show that the converse to Theorem 1.1 fails in the sense that, even for transverse actions, the regularity of G on E need not imply regularity on M.

To describe the application of Theorem 1.1 to the problem of classifying invariant sections of a bundle $\pi\colon E \to M$, let U be an open subset of M. A section $s\colon U \to E$ of E over U is said to be G invariant if for every $x \in U$ and $g \in G$ such that $g \cdot x \in U$,

$$s(g \cdot x) = g \cdot s(x).$$

It is easily seen that every G invariant section necessarily factors though the kinematic bundle $\kappa(E)$ on which G acts transversely. Thus for non-transverse group actions we are lead to the commutative diagram

$$\begin{array}{ccccc} \kappa(E)/G & \xleftarrow{\mathfrak{q}_{\kappa(E)}} & \kappa(E) & \xrightarrow{\iota} & E \\ \tilde{\pi}\big\downarrow & & \pi\big\downarrow & & \big\downarrow\pi \\ M/G & \xleftarrow{\mathfrak{q}_M} & M & \xrightarrow{\text{id}} & M. \end{array} \tag{1.4}$$

called the ***kinematic reduction diagram for the action of*** G ***on*** E. By Theorem 1.1, $\kappa(E)$ can be identified with the pullback bundle $\mathfrak{q}_M^*(\kappa(E)/G)$ and the G invariant sections of $\kappa(E)$ are precisely the pullbacks of the sections of $\mathfrak{q}_{\kappa(E)}\colon \kappa(E)/G \to M/G$. We therefore obtain, as a direct consequence of Theorem 1.1, the following result.

Theorem 1.2 The Classification Theorem for Invariant Sections. *Let $\pi\colon E \to M$ be a smooth bundle and let G be a Lie group which acts projectably on E. Assume that*

[i] *$\pi\colon \kappa(E) \to M$ is a smooth embedded subbundle of $\pi\colon E \to M$; and*

[ii] *the quotient space $\mathfrak{q}_M\colon M \to M/G$ of M by the orbits of G is a smooth manifold.*

Then for any open set $\widetilde{U} \subset M/G$, there is a one-to-one correspondence between the smooth sections $\tilde{s}\colon \widetilde{U} \to \kappa(E)/G$ and the G invariant sections of E over $U = \mathfrak{q}_M^{-1}(\widetilde{U})$ determined by

$$\tilde{s}(\mathfrak{q}_M(x)) = \mathfrak{q}_{\kappa(E)}(s(x)) \qquad \text{all } x \in U. \tag{1.5}$$

In [*3*] a wide variety of examples and applications of the kinematic reduction diagram and Theorem 1.2 are given. In particular, an explicit local coordinate description of the G invariant sections of E is given which generalizes the classical formula due to Lie (Bluman and Kumei [*5*], Olver [*10*]) . Theorem 1.2 also generalizes the well-known result that if G acts transitively on M, then the space of G invariant sections of E is parameterized by the fixed point set of the isotropy group acting on a single fiber. Therefore Theorem 1.2 includes, as a very special case, Wang's theorem ([*9*], p.106) classifying the invariant connections on a principle bundle over a homogeneous space. When the action of G on M is simple ([*6*], [*8*]), Theorem 1.2 shows that the dimension of the G invariant tensor fields of a given type on M, as a module over the ring of G invariant functions on M, is the same as the dimension of the vector space of tensors of the given type at any point $x \in M$ which are invariant under the linear isotropy representation of G_x. This theorem also provides a global generalization of the local classification of invariant sections of vector bundles given in [*7*]. In addition, the current work furnishes a general setting for the description of Kaluza-Klein reductions of general relativity and gauge theories (see, for example, [*6*]) where the reduced bundle $\tilde{\pi} : E/G \to M/G$ (or, more precisely, the reduction of the kinematic bundle $\kappa(E)$) carries the field theoretic interpretation of the Kaluza-Klein reduction. Finally, Theorem 1.2 provides the basis for extending to non-transverse group actions the geometric approach to the principle of symmetry criticality [*13*] taken in [*2*].

It is a pleasure to thank Charles Torre and Jim Stasheff for their assistance with this work.

2. Preliminaries. In this section we discuss the basic properties of projectable, transverse group actions on bundles and present an existence theorem for orbit manifolds that will be used in the proof of Theorem 1.1.

Our first proposition shows that transverse group actions satisfy the infinitesimal transversality condition given in Olver [*10*](p. 228-229). Let $\mathcal{O}_p(G)$ denote the orbit of G though the point $p \in M$.

Proposition 2.1. *If G acts transversely on E, then for any $p \in E$ the infinitesimal transversality condition*

$$\mathrm{Vert}_p(E) \cap T_p(\mathcal{O}_p(G)) = \{\, 0 \,\} \tag{2.1}$$

is satisfied, where $\mathrm{Vert}_p(E) = \{\, X_p \in T_p(E) \,|\, \pi_*(X_p) = 0 \,\}$.

Proof. Let X_p be a vector in $\mathrm{Vert}_p(E) \bigcap T_p(\mathcal{O}_p(G))$. Since $T_p(\mathcal{O}_p(G)) = \Gamma|_p$, where Γ is the Lie algebra of infinitesimal generators for the action of G on E, there is a vector field $Z \in \Gamma$ such that $Z_p = X_p$. Since X_p is π-vertical, $\pi_*(Z_p) = 0$. But since G acts projectably on E, Z is a projectable vector field on E and hence $\pi_*(Z_q) = \pi_*(Z_p)$ whenever q is in the same fiber as p. Consequently $Z_q \in \mathrm{Vert}_q(E)$ for all $q \in E_x$.

Since $\pi\colon E \to M$ is a submersion the fiber E_x, where $x = \pi(p)$, is an imbedded submanifold in E and $T_p(E_x) = \mathrm{Vert}_p(E)$. Thus Z restricts to a vector field $\tilde{Z}$ on the manifold E_x. The integral curve $g(t) \cdot p$ of Z though $p \in E_x$ coincides with the integral curve of $\tilde{Z}$ and therefore $g(t) \cdot p$ lies in E_x for all t. Transversality now implies that $g(t) \cdot p = p$ for all t. We differentiate this equation with respect to t and set $t = 0$ to deduce that $X_p = Z_p = 0$. ∎

See Examples 4.3 and 4.4 for examples of actions which are infinitesimally transverse but not transverse. If the infinitesimal transversality condition (2.1) holds for each $p \in E$ and if the isotropy group G_x is connected for each $x \in M$, then G acts transversely of E. Further properties of infinitesimal transverse group actions are given in [*4*].

Proposition 2.2. *Let G act projectably on $\pi\colon E \to M$. Then G acts transversely on E if and only if for any $p \in E$ and $x = \pi(p)$, the map $\pi\colon \mathcal{O}_p(G) \to \mathcal{O}_{\pi(p)}(G)$ is a diffeomorphism of orbits.*

Proof. For any projectable action, $\pi : \mathcal{O}_p(G) \to \mathcal{O}_x(G)$ is a submersion so that it remains to check that π and π_* are injective. Suppose that $p_1, p_2 \in \mathcal{O}_p(G)$ and $\pi(p_1) = \pi(p_2)$. Then there are $g_1, g_2 \in G$ such that $p_1 = g_1 \cdot p$ and $p_2 = g_2 \cdot p$. The condition $\pi(g_1 \cdot p) = \pi(g_2 \cdot p)$ implies that

$\pi(g_2^{-1}g_1 \cdot p) = \pi(p)$ and hence, by transversality, $g_2^{-1}g_1 \cdot p = p$ and $p_1 = p_2$. This shows that π is injective on each orbit.

If $X_q \in T_q(\mathcal{O}_p(G))$ and $\pi_*(X_q) = 0$ then, by Proposition 2.1, $X_q = 0$ and π_* is injective. This proves that π is a diffeomorphism. Conversely, let $p \in E$ and $g \in G$ and suppose that $\pi(g \cdot p) = \pi(p)$. Then $g \cdot p \in \mathcal{O}_p(G)$ and therefore, given that $\pi\colon \mathcal{O}_p(G) \to \mathcal{O}_x(G)$ is a diffeomorphism, it follows that $g \cdot p = p$. ∎

Given a projectable group action on E, let $\mathfrak{q}_E\colon E \to E/G$ and $\mathfrak{q}_M\colon M \to M/G$ be the projection maps to the quotient spaces of E and M by the orbits of G and let $\tilde{\pi}\colon E/G \to M/G$ be the induced projection map between these quotient spaces. Independent of the assumption of transversality, the diagram (1.3) commutes and all the maps in this diagram are open and continuous.

The following simple lemma unlocks one of the essential properties of transversality.

Lemma 2.3. *Let G act transversely on E. If points $\tilde{p} \in E/G$ and $x \in M$ satisfy*

$$\tilde{\pi}(\tilde{p}) = \mathfrak{q}_M(x),$$

then there is a unique $p \in E$ such that

$$\tilde{p} = \mathfrak{q}_E(p) \quad \text{and} \quad x = \pi(p). \tag{2.2}$$

Proof. To establish the existence of p we first note that since $\mathfrak{q}_E$ is surjective, there is a $p_0 \in E$ such that $\mathfrak{q}_E(p_0) = \tilde{p}$. Then

$$\mathfrak{q}_M(\pi(p_0)) = \tilde{\pi}(\mathfrak{q}_E(p_0)) = \tilde{\pi}(\tilde{p}) = \mathfrak{q}_M(x)$$

and hence there is a $g \in G$ such that $g \cdot \pi(p_0) = x$. The point $p = g \cdot p_0$ satisfies (2.2).

To prove uniqueness, suppose p and p' satisfy (2.2). Since $\mathfrak{q}_E(p) = \mathfrak{q}_E(p')$, there is a $g \in G$ such that $p' = g \cdot p$. But since $\pi(p) = \pi(p')$, we find that $\pi(g \cdot p) = \pi(p') = \pi(p)$. Transversality gives $g \cdot p = p$ and hence $p = p'$. ∎

To prove part **[ii]** of Theorem 1.1 it is necessary to briefly discuss the problem of determining when the quotient manifold M/G (or E/G) may be endowed with a manifold structure such that $\mathfrak{q}_M\colon M \to M/G$ is a bundle. We begin with an axiomatic characterization of the quotient manifold.

Definition 2.4. *Let G be a Lie group acting smoothly on a manifold M. A smooth (Hausdorff) manifold $\widetilde{M}$ together with a projection map $\mathfrak{q}_M\colon M \to \widetilde{M}$ is called a manifold of orbits or **orbit manifold** for the action of G on the manifold M if*

[i] *$\mathfrak{q}_M(x) = \mathfrak{q}_M(y)$ if and only if there is a $g \in G$ such that $y = g \cdot x$; and*

[ii] *the map $\mathfrak{q}_M$ is a smooth submersion, that is $\mathfrak{q}$ is smooth, onto, and $\mathfrak{q}_*$ is onto;*

The following properties of the orbit manifold are immediate consequences of the definition.

Proposition 2.5. *Let G be a Lie group acting smoothly on a manifold M and $\mathfrak{q}_M\colon M \to \widetilde{M}$ an orbit manifold.*

[i] *Through any given point $x \in M$, there exist local smooth sections $\varphi\colon \widetilde{U} \to M$, that is, $\mathfrak{q}_M \circ \varphi = id$ and $\varphi(\tilde{x}) = x$, where $\tilde{x} = \mathfrak{q}(x)$ and $\widetilde{U}$ is an open neighborhood of $\tilde{x}$ in $\widetilde{M}$; and*

[ii] *for each $x \in M$,*

$$\ker \mathfrak{q}_*\colon T_xM \to T_x\widetilde{M} = T_x(\mathcal{O}_x(G)).$$

[iii] *If $\mathfrak{q}_1\colon M \to \widetilde{M}_1$ and $\mathfrak{q}_2\colon M \to \widetilde{M}_2$ are two manifolds which satisfy the properties* **[i]** *and* **[ii]** *of Definition 2.4, then the bundles $\mathfrak{q}_1\colon M \to \widetilde{M}_1$ and $\mathfrak{q}_2\colon M \to \widetilde{M}_2$ are equivalent.*

Definition 2.6. *Let G be a Lie group acting smoothly on M. Then G acts* ***semi-regularly*** *on M if the dimension of each orbit $\mathcal{O}_x(G)$ is the same for all $x \in M$.*

Definition 2.7. *Let G be a Lie group acting semi-regularly on a manifold M. Then G acts* ***regularly*** *on M if, for every point $x_0 \in M$, there exist continuous local sections*

$$\varphi\colon \widetilde{U} \to M \quad \text{and} \quad \zeta\colon \widehat{U} \to G,$$

where $\widetilde{U} \subset M/G$ and $\widehat{U} \subset G/G_{x_0}$, and an open neighborhood U of x_0 such that φ and ζ pass through x_0 and e respectively and the map

$$\Phi\colon \widetilde{U} \times \widehat{U} \to U \quad \text{given by} \quad \Phi(\tilde{x}, \hat{\theta}) = \zeta(\hat{\theta}) \cdot \varphi(\tilde{x}) \tag{2.3}$$

is a homeomorphism.

Remark 2.8. In Olver [*10*] (p. 22) the action of G on M is said to be regular if it is semi-regular and if, for every point $x_0 \in M$ and every neighborhood V of x_0, there is an open set $x_0 \in U \subset V$ such that for every $x \in U$ the set $U \cap \mathcal{O}_x(G)$ is connected. Since the set $\widehat{U}$ in Definition 2.6 can be taken to be connected, the definition of regularity given here immediately implies the definition in Olver. ∎

Remark 2.9. If there are points $x_i \in M$, $i = 1, 2, \ldots$ such that

[i] $\lim_{i\to\infty} x_i = x_0$;

[ii] $x_i \in \mathcal{O}_{x_1}(G)$; and

[iii] $\mathcal{O}_{x_1}(G) \neq \mathcal{O}_{x_0}(G)$.

then a simple proof by contradiction shows that the action of G on M is not regular. ∎

Remark 2.10. With the natural local action of G on $\widehat{U}_0 \subset G/G_{x_0}$ one may ask if the diffeomorphism (2.3) is a global G equivariant map, that is, if

$$\Phi(\tilde{x}, g \cdot \hat{\theta}) = g \cdot \Phi(\tilde{x}, \hat{\theta}) \tag{2.4}$$

whenever $g \cdot \hat{\theta} \in \widehat{U}_0$. It is clear that if $g \in G_{x_0}$ and (2.4) holds , then for any $\tilde{x} \in \widetilde{U}_0$ we have that

$$g \cdot \varphi(\tilde{x}) = g \cdot \Phi(\tilde{x}, \hat{\theta}_0) = \Phi(\tilde{x}, g \cdot \hat{\theta}_0) = \Phi(\tilde{x}, g \cdot \hat{\theta}_0) = g \cdot \Phi(\tilde{x}, \hat{\theta}_0) = \varphi(\tilde{x}).$$

Therefore a necessary condition for the G equivariance of Φ is that

$$G_{x_0} \subset G_{\varphi(\tilde{x})}.$$

If M is a G manifold and the local section $\varphi\colon \widetilde{M} \to M$ can be chosen so that the local diffeomorphism (2.4) is G equivariant, then we say that M is a ***local simple*** G space and the image of the section φ is called a ***slice*** for the group action at x_0. See Palais [*12*]. ∎

Theorem 2.11. *The quotient space M/G admits a differentiable structure such that $\widetilde{M} = M/G$ satisfies the properties of the orbit manifold given in Definition 2.4 if and only if G is a regular group action on M and M/G is Hausdorff.*

Proof. If M admits a G orbit manifold, property **[ii]** of Proposition 2.5 implies that G acts semi-regularly. If M admits an orbit manifold, then the local section φ defining the map (2.3) may be taken to be smooth and so Φ itself is smooth. It is not difficult to check that the differential Φ_* is an isomorphism and hence Φ is a local diffeomorphism. The converse follows from Remark 2.8 and Theorem 3.18 in Olver [*10*]. For a direct proof of this theorem , see [*4*] ∎

We close with the following test for regularity [*4*].

Definition 2.12. *An imbedded submanifold $\psi\colon S \to M$ is called an **imbedded cross-section** for the action of G on M if the orbits of G intersect S transversely (as submanifolds) and if, for any $x \in \psi(S)$, the G orbit through x intersects $\psi(S)$ only at x, that is,*

$$\mathcal{O}_x(G) \cap \psi(S) = \{x\}. \tag{2.5}$$

Theorem 2.13. *A Lie group G acting on M acts regularly on M if and only if G acts semi-regularly on M and through each point $x_0 \in M$ there is an imbedded cross-section.*

Remark 2.14. In Abraham and Marsden [*1*] it is proved that M admits an orbit manifold if the image $\mathcal{M}$ of the map $\varphi\colon G \times M \to M \times M$ defined by $\varphi(g, x) = (x, g \cdot x)$ is a closed imbedded submanifold. It is not to difficult to show directly that though each point $x_0 \in M$ there is an imbedded cross-section if and only if $\mathcal{M}$ is an imbedded submanifold. ∎

3. The Classification of Transverse Group Actions. We shall need the following technical lemma for the proof of Theorem 1.1.

Lemma 3.1. *Let G act projectably on $\pi\colon E \to M$ and suppose that M admits a G orbit manifold $\mathfrak{q}_M : M \to \widetilde{M}$. Let $p_0 \in E$ and let V be any open neighborhood of p_0. Then there is an open neighborhood $V' \subset V$ of p_0 with the following property. For any $p_1 \in V'$ and $p_2 \in E$ such that*

[i] *$\pi(p_2) \in \pi(V')$, and*

[ii] *$\pi(p_1)$ and $\pi(p_2)$ lie on the same G orbit in M,*

there is a $g \in G$ with $g \cdot p_1 \in V$ and $\pi(g \cdot p_1) = \pi(p_2)$.

Proof. Since $\pi(p_1)$ and $\pi(p_2)$ lie on the same G orbit, there is a $g \in G$ such that $g \cdot \pi(p_1) = \pi(p_2)$ and hence $\pi(g \cdot p_1) = \pi(p_2)$. However, it may not be the case that $g \cdot p_1 \in V$. The point of the lemma is to prove that there is a g sufficiently close to the identity $e \in G$ which moves p_1 to a point which is in both the fiber of p_2 and in the set V.

Since the action of G on E is smooth, there is a open set A_0 of e in G and an open set $V_0 \subset V$ containing p_0 such that $g \cdot p \in V$ for all $p \in V_0$ and $g \in A_0$. Choose an open neighborhood $A \subset A_0$ of e such that

$$A^{-1} = A \quad \text{and} \quad A^2 \subset A_0. \tag{3.1}$$

Let $x_0 = \pi(p_0)$, $\tilde{x}_0 = \mathfrak{q}_M(x_0) \in \widetilde{M}$ and let $\hat{\theta}_0 = G_{x_0} \in G/G_{x_0}$. By Theorem 2.11 and hence there are open neighborhoods $U_0 \subset M$, $\widetilde{U}_0 \subset \widetilde{M}$ and $\widehat{U}_0 \subset G/G_{x_0}$ of the points x_0, $\tilde{x}_0$ and $\hat{\theta}_0$ respectively and smooth sections

$$\varphi^M\colon \widetilde{U}_0 \to M \quad \text{and} \quad \zeta\colon \widehat{U}_0 \to G,$$

with $\varphi^M(\tilde{x}_0) = x_0$ and $\zeta(\hat{\theta}_0) = e$, such that the map $\Phi^M\colon \widetilde{U}_0 \times \widehat{U}_0 \to U_0$ given by

$$\Phi^M(\tilde{x}, \hat{\theta}) = \zeta(\hat{\theta}) \cdot \varphi^M(\tilde{x}) \tag{3.2}$$

is a diffeomorphism. Since $\pi\colon E \to M$ is an open map, we can shrink the sets $\widetilde{U}_0$, $\widehat{U}_0$ and U_0 , if need be, so that

$$U_0 \subset \pi(V_0) \quad \text{and} \quad \zeta(\widehat{U}_0) \subset A. \tag{3.3}$$

We now claim that the open set

$$V' = \pi^{-1}(U_0) \cap V_0$$

satisfies the requirements of the proposition. Let $p_1 \in V'$ and $p_2 \in E$. Suppose that p_1 and p_2 project to points x_1 and x_2 in the same G orbit in M and that $x_2 \in \pi(V')$. Since x_1 and x_2 belong to U_0, we may write

$$x_1 = \Phi^M(\tilde{x}_1, \hat{\theta}_1) = \zeta(\hat{\theta}_1) \cdot \varphi^M(\tilde{x}_1) \quad \text{and} \quad x_2 = \Phi^M(\tilde{x}_2, \hat{\theta}_2) = \zeta(\hat{\theta}_2) \cdot \varphi^M(\tilde{x}_2),$$

where $\tilde{x}_i \in \widetilde{U}_0$ and $\hat{\theta}_i \in \widehat{U}_0$. The points x_1 and x_2 are on the same orbit and therefore $\tilde{x}_1 = \tilde{x}_2$. Because $\hat{\theta}_1$ and $\hat{\theta}_2 \in \widehat{U}_0$, (3.3) implies that $\zeta(\hat{\theta}_1), \zeta(\hat{\theta}_2) \in A$. Equation (3.1) now implies that

$g = \zeta(\hat{\theta}_2)\zeta(\hat{\theta}_1)^{-1} \in A_0$ and therefore, since $p_1 \in V' \subset V_0$, we deduce that $g \cdot p_1 \in V$. Finally, we note that

$$\pi(g \cdot p_1) = g\,\zeta(\hat{\theta}_1) \cdot \varphi^{M}(\tilde{x}_1) = \zeta(\hat{\theta}_2) \cdot \varphi^{M}(\tilde{x}_2) = \pi(p_2),$$

as required. ∎

We now prove part **[i]** of Theorem 1.1.

Proposition 3.2. *Let G act transversely on $\pi : E \to M$ with orbit manifolds $\mathfrak{q}_E : E \to \widetilde{E}$ and $\mathfrak{q}_M : M \to \widetilde{M}$. If $\widetilde{M}$ is Hausdorff then so is $\widetilde{E}$.*

Proof. Let p_1 and p_2 be two points in E such that

$$\mathcal{O}_{p_1}(G) \cap \mathcal{O}_{p_2}(G) = \phi. \tag{3.4}$$

We must prove that there exist disjoint G invariant open sets Q_i such that $\mathcal{O}_{p_i}(G) \subset Q_i$. Let $x_1 = \pi(p_1)$ and $x_2 = \pi(p_2)$.

Case [i] $\mathcal{O}_{x_1}(G) \cap \mathcal{O}_{x_2}(G) = \phi$; and

Case[ii] $\mathcal{O}_{x_1}(G) \cap \mathcal{O}_{x_2}(G) \neq \phi$.

In case **[i]** the fact that $\widetilde{M}$ is Hausdorff implies that there are disjoint open sets $P_i \subset M$ containing $\mathcal{O}_{x_i}(G)$ which are G invariant (that is, $G \cdot P_i \subset P_i$) and such that $\mathcal{O}_{x_i}(G) \subset P_i$. The sets $Q_i = \pi^{-1}(P_i)$ are then disjoint, G invariant open sets such that $\mathcal{O}_{p_i}(G) \subset Q_i$.

In case **[ii]** we first note that, without loss in generality, we can assume that $\pi(p_1) = \pi(p_2)$. Since E itself is Hausdorff, we can choose disjoint open neighborhoods V_1 around p_1 and V_2 around p_2. Choose an open set $V_1' \subset V_1$ containing p_1 in accordance with Lemma 3.1, and an open set $V_2' \subset V_2$ containing p_2 with $\pi(V_2') \subset \pi(V_1')$.

Let $Q_i = G \cdot V_i'$. Then the sets Q_i are open, G invariant sets containing $\mathcal{O}_{p_i}(G)$ and accordingly it remains to check that the Q_i are disjoint. Suppose, to the contrary, that $Q_1 \cap Q_2 \neq \phi$. Then there is a point $p' \in V_1'$ and a point $q' \in V_2'$ and a $g \in G$ such that $q' = g \cdot p'$. The points $\pi(p')$ and $\pi(q')$ lie on the same G orbit and $\pi(q') \in \pi(V_1')$. Hence, by Lemma 3.1, there is a $g \in G$ such that $p'' = g \cdot p' \in V_1$ and $\pi(p'') = \pi(q')$. Thus p'' and q' are in the same G orbit and in the same fiber of E and therefore, by transversality, $p'' = q'$. This contradicts the fact that the sets V_1 and V_2 are disjoint. The supposition that $Q_1 \cap Q_2 \neq \phi$ is false and therefore Q_1 and Q_2 are disjoint G invariant open sets which separate the G orbits though p_1 and p_2. ∎

To prove part **[ii]** of Theorem 1.1 we need to show that the action of G on E is regular and in order to do so we first need to construct continuous sections from E/G to E.

Proposition 3.3. *Let G act transversely on E and suppose M admits a G orbit manifold $\mathfrak{q}_M : M \to \widetilde{M}$. Let $\varphi^{M} : \widetilde{U} \to M$ be a section of $\mathfrak{q}_M : M \to \widetilde{M}$ and let $\widetilde{V} = \tilde{\pi}^{-1}(\widetilde{U})$. Then there is a unique* *section* $\varphi\colon \widetilde{V} \to E$ *such that the diagram*

$$\begin{array}{ccc} \widetilde{V} & \xrightarrow{\varphi} & E \\ {\scriptstyle\tilde{\pi}}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ \widetilde{U} & \xrightarrow{\varphi^M} & M \end{array} \tag{3.5}$$

commutes and this section is continuous.

Proof. Let $\tilde{p} \in \widetilde{V}$. By Lemma 2.3 there is a unique point $p \in E$ such that

$$\mathfrak{q}_E(p) = \tilde{p} \quad \text{and} \quad \pi(p) = \varphi^M \circ \tilde{\pi}(\tilde{p}) \tag{3.6}$$

and we define $\varphi(\tilde{p}) = p$. The diagram (3.5) clearly commutes and so it remains to prove that φ is continuous. Let V be any open set in E. We shall show that for every point $\tilde{p}_0 \in \varphi^{-1}(V)$ there is an open set $\widetilde{W} \subset \widetilde{E}$ such that $\tilde{p}_0 \subset \widetilde{W}$ and

$$\widetilde{W} \subset \varphi^{-1}(V). \tag{3.7}$$

Let $p_0 = \varphi(\tilde{p}_0) \in V$. Choose an open neighborhood $V' \subset V$ of p_0 with the properties established in Lemma 3.1 and let

$$\widetilde{W} = \mathfrak{q}_E(V') \cap (\tilde{\pi}^{-1} \circ (\varphi^M)^{-1} \circ \pi)(V').$$

It is clear that $\widetilde{W}$ is open in $\widetilde{E}$ and that $\tilde{p}_0 \in \widetilde{W}$. To verify the inclusion (3.7), let $\tilde{p} \in \widetilde{W}$. Then there are points p_1 and p_2 in V' such that

$$\mathfrak{q}_E(p_1) = \tilde{p} \quad \text{and} \quad \pi(p_2) = \varphi^M(\tilde{\pi}(\tilde{p})).$$

We apply the maps $\tilde{\pi}$ and $\mathfrak{q}_M$ to these two equations involving p_1 and p_2 respectively to deduce, by (3.5), that

$$\mathfrak{q}_M(\pi(p_1)) = \mathfrak{q}_M(\pi(p_2)).$$

This implies that $\pi(p_1)$ and $\pi(p_2)$ lie in the same G orbit in M and therefore, on account of the prescribed properties of the set V' established in Lemma 3.1, there is a $g \in G$ such that $p_3 = g \cdot p_1$ lies in V and satisfies $\pi(p_3) = \pi(p_2)$. We now compute

$$\mathfrak{q}_E(p_3) = \mathfrak{q}_E(g \cdot p_1) = \mathfrak{q}_E(p_1) = \tilde{p} \quad \text{and} \quad \pi(p_3) = \pi(p_2) = \varphi^M(\tilde{\pi}(\tilde{p})).$$

By comparing these equations to (3.6) we deduce that $\varphi(\tilde{p}) = p_3$ and therefore $\tilde{p} \in \varphi^{-1}(V)$. This proves the inclusion (3.7) and shows that φ is continuous. ∎

Part **[ii]** of Theorem 1.1 follows from our next proposition, Theorem 2.11 and Proposition 3.2.

Proposition 3.4. The Regularity Theorem for Transverse Group Actions. *Let $\pi\colon E \to M$ be a bundle and let G be a Lie group which acts transversely on E. If G acts regularly on M, then G acts regularly on E.*

Proof. Since the action of G on M is assumed to be semi-regular, the dimensions of the G orbits on M are fixed. By Proposition 2.2[**iii**], the orbits of G on E have constant dimension and the action of G on E is semi-regular.

Given $p_0 \in E$, let $x_0 = \pi(p_0)$. Since the action of G on M is regular there are open neighborhoods $\widetilde{U} \subset \widetilde{M}$ of $\tilde{x}_0$, and $\widehat{U} \subset G/G_{x_0}$ of G_{x_0}, and $U \subset M$ of x_0 together with continuous (in fact smooth) sections

$$\varphi^M : \widetilde{U} \to M \quad \text{and} \quad \zeta : \widehat{U} \to G,$$

such that the map $\Phi^M : \widetilde{U} \times \widehat{U} \to U$ defined by

$$\Phi^M(\tilde{x}, \hat{\theta}) = \zeta(\hat{\theta}) \cdot \varphi^M(\tilde{x})$$

is a homeomorphism (in fact, a diffeomorphism).

Let $\widetilde{V} = \tilde{\pi}^{-1}(\widetilde{U})$, $\widehat{V} = \widehat{U}$, and $V = \pi^{-1}(U)$. Let $\varphi\colon \widetilde{V} \to E$ be the continuous section defined in terms of φ^M by Proposition 3.3 and define the map $\Phi\colon \widetilde{V} \times \widehat{V} \to V$ by

$$\Phi(\tilde{p}, \hat{\theta}) = \zeta(\hat{\theta}) \cdot \varphi(\tilde{p}). \tag{3.8}$$

It is a simple matter to check that the diagram

$$\begin{array}{ccc} \widetilde{V} \times \widehat{V} & \xrightarrow{\ \Phi\ } & V \\ {\scriptstyle \tilde{\pi}}\downarrow \quad \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \pi} \\ \widetilde{U} \times \widehat{U} & \xrightarrow{\ \Phi^M\ } & U \end{array} \tag{3.9}$$

commutes.

To show that the map $\Phi\colon \widetilde{V} \times \widehat{V} \to V$ is a homeomorphism, we shall explicitly construct the inverse map $\Phi^{-1}\colon V \to \widetilde{V} \times \widehat{V}$ and prove that it is continuous. Define $\psi\colon V \to \widehat{V}$ by

$$\psi(p) = ((\Phi^M)_2^{-1} \circ \pi)(p),$$

where $(\Phi^M)_2^{-1}$ denotes the projection of $(\Phi^M)^{-1}$ onto its second factor $\widehat{U} = \widehat{V}$. The map ψ is clearly continuous and

$$\Phi^M\big(\, \tilde{\pi}(\mathfrak{q}_E(p)), \psi(p)\, \big) = \pi(p). \tag{3.10}$$

We now claim that the inverse of Φ is given by the continuous map $\Psi(p) = (\mathfrak{q}_E(p), \psi(p))$. Since $\pi \circ \Phi = \Phi^M \circ (\tilde{\pi} \times \mathrm{id})$ we have that $(\Phi^M)^{-1} \circ \pi \circ \Phi = \tilde{\pi} \times \mathrm{id}$, and therefore $\psi(\Phi(\tilde{p}, \hat{\theta})) = \hat{\theta}$ and $\Psi \circ \Phi$ is the identity on $\widetilde{V} \times \widehat{V}$.

To show that $\Phi\circ\Psi$ is the identity on V let $p\in V$, let $\Psi(p)=(\tilde p,\hat\theta)$, and let $p'=\Phi(\tilde p,\hat\theta)=\zeta(\hat\theta)\cdot\varphi(\tilde p)$. We compute

$$\mathfrak{q}_E(p')=\mathfrak{q}_E(\varphi^M(\tilde p))=\tilde p=\mathfrak{q}_E(p) \tag{3.11}$$

and, using (3.9),

$$\pi(p')=(\pi\circ\Phi)(\tilde p,\hat\theta)=\Phi^M\big(\,\tilde\pi(\tilde p),\hat\theta\,\big)=\Phi^M\big(\,\tilde\pi(\mathfrak{q}_E(p)),\psi(p))\,\big)=\pi(p). \tag{3.12}$$

By Proposition 2.3, the combination of (3.11) and (3.12) yields $\tilde p'=p$ and thus $\Phi\circ\Psi$ is the identity on V. This shows that Φ is a homeomorphism and the action of G on E is regular. ∎

In order to complete the proof of **[ii]** of Theorem 1.1, it remains simply to verify that the induced projection $\tilde\pi$ is a smooth submersion and this we leave as an exercise.

Part **[iv]** of Theorem 1.1 is established next.

Proposition 3.5. *Let G act projectably and transversely on $\pi\colon E\to M$ and suppose that (1.3) is a commutative diagram of smooth bundles. Then the bundle $\pi\colon E\to M$ is strongly G-equivalent to the pullback bundle $\pi\colon \mathfrak{q}_M^*(E/G)\to M$ with its canonical G action.*

Proof. If $\pi_1\colon E_1\to M$ and $\pi_2\colon E_2\to M$ are two bundles with projectable actions of G, then E_1 and E_2 are strongly G-equivalent if there is a G equivariant diffeomorphism from E_1 to E_2 which covers the identity on M. Define a smooth map $\psi\colon E\to M\times\widetilde E$ by $\psi(p)=(\pi(p),\mathfrak{q}_E(p))$. The commutativity of (1.3) insures that the image of ψ is in

$$\mathfrak{q}_M^*(\widetilde E)=\{\,(x,\tilde p)\in M\times\widetilde E\,|\,\mathfrak{q}_M(x)=\tilde\pi(\tilde p)\,\}.$$

Since $\mathfrak{q}_M^*(\widetilde E)$ is an imbedded submanifold of $M\times\widetilde E$, we have that ψ is actually a smooth map

$$\psi\colon E\to\mathfrak{q}_M^*(\widetilde E). \tag{3.13}$$

The map ψ covers the identity map on M and is G equivariant — for any $g\in G$

$$\psi(g\cdot p)=\big(\pi(g\cdot p),\mathfrak{q}_E(g\cdot p)\big)=(g\cdot\pi(p),\mathfrak{q}_E(p))=g\cdot\psi(p).$$

To prove that (3.13) is a diffeomorphism we first use Lemma 2.3 to deduce that ψ is one-to-one and onto. We therefore find that ψ is invertible and hence, to complete the proof, it suffices by the inverse function theorem to check that ψ_* is an isomorphism. To check that ψ_* is one-to-one, let X_p be a tangent vector to E at p. If $\Phi_*(X_p)=0$, then $\pi_*(X_p)=0$ and $(\mathfrak{q}_E)_*(X_p)=0$. Then X_p is a π-vertical vector which belongs to $T_p(\mathcal{O}_p(G))$. By Lemma 2.1, $X_p=0$.

A theorem in differential topology found in Warner [*14*](Chapter 1, exercise 6) implies that ψ_* is automatically surjective. ∎

To complete the proof of Theorem 1.1 we turn to part **[iv]** and the case where $\pi\colon E \to M$ is a fiber bundle with fiber F. Let $\{\,U_\alpha\,\}$ be a trivializing cover of M and let $\Psi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times F$ be a smooth trivialization of E. We will write the maps Ψ_α as

$$\Psi_\alpha(p) = (\pi(p), \psi_\alpha(p)).$$

There is a natural local action of G on each product $U_\alpha \times F$, namely

$$g \cdot (x, u) = (g \cdot x, u). \tag{3.14}$$

Definition 3.6. *A* ***trivialization*** $\{\,(U_\alpha, \Psi_\alpha)\,\}$ ***is said to be*** G ***invariant*** *if the maps ψ_α are all G invariant, that is for all $p \in \pi^{-1}(U_\alpha)$ and $g \in G$ such that $g \cdot p \in \pi^{-1}(U_\alpha)$*

$$\psi_\alpha(g \cdot p) = \psi_\alpha(p). \tag{3.15}$$

Equivalently, the trivialization is G invariant if the maps Ψ_α are G-equivariant, where the action of G on $U_\alpha \times F$ is given by (3.14).

Let G act on M with quotient manifold $\mathfrak{q}_M : M \to \tilde{M}$ and let $\pi\colon \widetilde{E} \to \widetilde{M}$ be a fiber bundle. Then any induced trivialization on the pullback bundle $\mathfrak{q}_M^*(\widetilde{E})$ is G invariant.

Theorem 3.7. *Let G act projectably on the fiber bundle $\pi\colon E \to M$. If E admits a G invariant trivialization, then G acts transversely on E. Conversely, if M admits a G orbit manifold $\mathfrak{q}_M\colon M \to \widetilde{M}$ and G acts transversely on E, then E admits a G invariant trivialization and $\tilde{\pi}\colon \widetilde{E} \to \widetilde{M}$ is a fiber bundle.*

Proof. Assume that E admits a G invariant trivialization $\{\,U_\alpha, \Psi_\alpha\,\}$. Let $p \in E$ and $g \in G$ satisfy $\pi(g \cdot p) = \pi(p)$ Then $p \in \pi^{-1}(U_\alpha)$ for some α and hence

$$\Psi_\alpha(g \cdot p) = (\pi(g \cdot p), \psi_\alpha(g \cdot p)) = (\pi(p), \psi_\alpha(p)) = \Psi_\alpha(p)$$

But Ψ_α is one-to-one and therefore $g \cdot p = p$. This proves that G acts transversely on E.

To prove the converse we use Theorem 2.11 to cover M with open sets U_α which gives a trivializing cover for E and for which there are diffeomorphisms

$$\Phi^M : \widetilde{U}_\alpha \times \widehat{U}_\alpha \to U_\alpha$$

defined in terms of sections

$$\varphi^M_\alpha\colon \widetilde{U}_\alpha \to M \quad \text{and} \quad \zeta_\alpha\colon \widehat{U}_\alpha \to G$$

by

$$\Phi^M_\alpha(\tilde{x}, \hat{\theta}) = \zeta_\alpha(\hat{\theta}) \cdot \varphi^M_\alpha(\tilde{x}).$$

Let $\lambda_\alpha\colon U_\alpha \to G$ and $\sigma_\alpha\colon U_\alpha \to U_\alpha$ be the maps

$$\lambda_\alpha(x) = \zeta_\alpha \circ \pi_{\widehat{U}_\alpha} \circ (\Phi^M_\alpha)^{-1}(x) \quad \text{and} \quad \sigma_\alpha(x) = \varphi_\alpha \circ \pi_{\widetilde{U}_\alpha} \circ (\Phi^M_\alpha)^{-1}(x) \tag{3.16}$$

The function σ_α is the map which takes a point $x \in U_\alpha$ to the corresponding point on the cross-section $\varphi^M_\alpha(\mathfrak{q}(x))$, while $\lambda_\alpha(x)^{-1}$ is the group element taking x to the cross-section, that is,

$$\sigma_\alpha(x) = \varphi^M_\alpha(\mathfrak{q}_M(x)) \quad \text{and} \quad \sigma(x) = \lambda_\alpha(x)^{-1} \cdot x. \tag{3.17}$$

The maps σ_α are clearly G invariant, and therefore if $g \in G$ such that $gx \in U_\alpha$ the second part of equation (3.17) gives

$$\sigma_\alpha(gx) = \lambda^{-1}(gx) \cdot g \cdot x = \lambda(x)^{-1} \cdot x = \sigma_\alpha(x)\ . \tag{3.18}$$

We now construct new maps $\psi'_\alpha\ \colon \pi^{-1}(U_\alpha) \to F$ and trivialization $\Psi'_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times F$ by letting

$$\psi'_\alpha(p) = \psi_\alpha(\lambda^{-1}_\alpha(\pi(p)) \cdot p)$$

and check that ψ'_α is G-invariant. Let $p \in \pi^{-1}(U_\alpha)$ and $g \in G$ such that $gp \in \pi^{-1}(U_\alpha)$ and let $x = \pi(p)$. Then by equation (3.18) we have

$$\pi(\lambda^{-1}_\alpha(\pi(g\cdot p)) \cdot g \cdot p) = \lambda^{-1}_\alpha(g\cdot x) \cdot g \cdot x = \pi(\lambda^{-1}(\pi(p)) \cdot p)\ .$$

Furthermore $\mathfrak{q}_E(\lambda^{-1}_\alpha(\pi(g\cdot p))\cdot g\cdot p) = \mathfrak{q}_E(p)$, and therefore, by Lemma 2.3 with $x' = \pi(\lambda^{-1}_\alpha(\pi(p))\cdot p)$, we have the equality

$$\lambda^{-1}_\alpha(\pi(g\cdot p)) \cdot g \cdot p = \lambda_\alpha(\pi(p))^{-1} p$$

which proves ${\psi'}_\alpha$ is invariant.

The inverse maps $(\Psi'_\alpha)^{-1}\colon U_\alpha \times F \to \pi^{-1}(U_\alpha)$ are given by

$$(\Psi'_\alpha)^{-1}(x,u) = \lambda_\alpha(x) \cdot \Psi^{-1}_\alpha(\sigma_\alpha(x), u). \tag{3.19}$$

This completes the proof. ∎

4. Examples. Let G act transversely on $\pi\colon E \to M$. Theorem 1.1, parts **[i]** and **[ii]** show that certain properties of the action of G on M are inherited by the action of G on E. Our first two examples show that the converse is false.

Example 4.1. Let $M = \mathbf{R}^2 - \{\,(0,0)\,\}$, let $E = M \times \mathbf{R}^+$ and let $G = \mathbf{R}$ act on E according to

$$e^\theta \cdot (x,y,u) \to (e^\theta x, e^{-\theta} y, e^\theta u).$$

This is a free action on M and hence transverse on E. Since each orbit cuts through the $u = u_0 > 0$ plane exactly once, we have that $\widetilde{E} = \mathbf{R}^2 - \{\,(0,0)\,\}$ so the action of G on E is regular and $\widetilde{E}$ is Hausdorff. Each orbit of G in M is a hyperbola (or part of a coordinate axis) and it is a simple matter to check that the action of G on M is regular but that M/G is not Hausdorff. ∎

Next we show that an action which is transverse and regular on E need not be regular on M.

Example 4.2. Let T^2 be the two-torus, $E = T^2 \times \mathbf{R}$, and π_1, π_2 be the projections of E onto its first and second factors. Let $G = \mathbf{R}$ act on $\mathbf{R}$ by translation, on T^2 by an irrational flow, and on E by the product action. This action is free on all three spaces and hence transverse on E. By applying Theorem 3.4 to the bundle $\pi_2 \colon E \to \mathbf{R}$ we deduce that the action of G on E is regular. Thus G acts transversely on $\pi_1 \colon E \to T^2$, regularly on E but not regularly on the base T^2. ∎

We now show that the conclusions of Theorem 1.1 do not hold under the weaker assumption of infinitesimal transversality.

Example 4.3. Let $M = \mathbf{R}^2 - \{\, (0,0)\,\}$ and $E = M \times \mathbf{R}$ and consider the action generated by the flow of the vector field given, in Cartesian coordinates, by

$$Z = x\partial_y - y\partial_x + (1 - \cos u)\partial_u.$$

The orbits all lie on the right cylinders $x^2 + y^2 = a^2$. If the initial value of the u coordinate is a multiple of 2π, the orbits are circles on these cylinders; otherwise, the orbits are upward moving spirals between these circles. Analytically we see that the orbit through the point (x_0, y_0, u_0), where $2(m-1)\pi < u_0 < 2m\pi$, is

$$x = x_0 \cos(t) - y_0 \sin(t), \qquad y = x_0 \sin(t) + y_0 \cos(t), \qquad u = 2\mathrm{Arccot}\big(\big(\cot(u_0/2) - t\big)\big) + 2(m-1)\pi$$

This action is infinitesimally transverse, the projected action to M is regular, the quotient M/G is Hausdorff but the action on the total space E is not regular — by Remark 2.9 it suffices to note that the points $\mathbf{x}_n = (1, 0, 2\mathrm{Arccot}(-2\pi n))$ converge to the point $(1, 0, 2\pi)$, lie on the orbit through $(1, 0, \pi)$ but the orbits through $(1, 0, \pi)$ and $(1, 0, 2\pi)$ are distinct. ∎

Finally, we construct an infinitesimally transverse group action which is regular on E and M and for which the $\widetilde{M} = M/G$ is Hausdorff but $\widetilde{E} = E/G$ is not.

Example 4.4. Let $M = \mathbf{R}^2 - \{\, (-1,0), (1,0)\,\}$, $E = M \times \mathbf{R}$ and consider the one parameter transformation group on E generated by the vector field

$$\overline{Z} = Z + 2xl\,\partial_u \quad \text{where} \quad Z = l\,\big(2xy\,\partial_x + \,(1 + y^2 - x^2)\,\partial_y\big).$$

and $l((x,y) = 1/\sqrt{4x^2y^2 + (1 + y^2 - x^2)^2}$. The explicit determination of the flow of Z shows that this vector field defines a global action of $\mathbf{R}$ on M. The orbit through the point (x_0, y_0), $x_0 \neq 0$ lies on the circle with center $(d, 0)$ and radius $\sqrt{d_0^2 - 1}$, where

$$d = \frac{1 + x^2 + y^2}{2x} \quad \text{and} \quad d_0 = \frac{1 + x_0^2 + y_0^2}{2x_0}. \tag{4.1}$$

while for $x_0 = 0$ the orbits are straight lines. We can identify the orbit manifold $\widetilde{M}$ with the imbedded cross-section $C : x^2 + y^2 = 1$, $y > 0$ so that $\widetilde{M}$ is Hausdorff.

To check that E admits an orbit manifold $\widetilde{E}$, consider the half-cylinder S in E over the half-circle C. It is easily checked that the orbits are transverse to this surface and that every orbit cuts through S. If $\mathbf{p}_0 = (x_0, y_0, u_0)$ is a point on this half-cylinder, with $x_0 \neq 0$, then the orbit through this point passes through all the points $(x_0, y_0, u_0 + 2\pi n)$. For $x_0 = 0$,$y_0 = 1$ the orbits are straight lines which intersect the half-cylinder S exactly once. Given a point $\mathbf{p}_0$ on S, every orbit crosses the set

$$S_{\mathbf{p}_0} = \{\, (x, y, u) \,|\, x^2 + y^2 = 1, |u - u_0| < 2\pi \,\} \subset S$$

exactly once and hence $S_{\mathbf{p}_0}$ is an imbedded cross-section. By Theorem 2.13, the action of G on E is regular.

The orbit manifold is

$$\widetilde{E} = \{\, C^- \times S^1 \,\} \;\cup\; \mathbf{R} \;\cup\; \{\, C^+ \times S^1 \,\},$$

where $C^- = \{\, (x, y) \in C \,|\, x < 0 \,\}$ and $C^+ = \{\, (x, y) \in C \,|\, x > 0 \,\}$. An open neighborhood of the point $(0, 1, u_0)$ in $\widetilde{E}$ consists of a small interval in $\mathbf{R}$ together with open half discs in each quarter-torus around $(0, \cos u, \sin u)$. These union together to give an open disc in $\widetilde{E}$. For fixed u_0, the points $(0, 1, u_0 + 2n\pi)$ cannot be separated in $\widetilde{E}$ and thus $\widetilde{E}$ is not Hausdorff. ∎

References

1. R. Abraham and J. Marsden, *Foundations of Mechanics*, 2nd ed., Benjamin-Cummings, Reading, Mass, 1978.
2. I. M. Anderson and M. E. Fels, *Symmetry reduction of variational bicomplexes and the principle of symmetry criticality*, Amer. J. Math. **112** (1997), 609–670.
3. I. M. Anderson, Mark E. Fels, Charles G. Torre, *Group Invariant Solutions without Transversality*, Comm. Math. Physics **212** (2000), 653–686.
4. I. M. Anderson, Mark E. Fels, Charles G. Torre, *Symmetry Reduction of Differential Equations*, In preparation.
5. G. W. Bluman and S. Kumei, *Symmetries and Differential Equations*, Applied Mathematical Sciences, 81, Springer-Verlag, New York-Derlin, 1989.
6. R. Coquereaux and A. Jadczyk, *Riemannian Geometry, Fiber Bundles, Kaluza-Klein Theories and all that*, Lecture Notes in Physics, vol. 16, World Scientific, Singapore, 1988.
7. M. E. Fels and P. J. Olver, *On relative invariants*, Math. Ann. **308** (1997), 701–732.
8. J. Harnad, S. Schnider and L. Vinet, *Group actions on principal bundles and invariance conditions for gauge fields*, J. Mathematical Physics **21** (1980), no. 12, 2719–2724.
9. S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Wiley, New York, 1963.
10. P. J. Olver, *Applications of Lie Groups to Differential Equations*, (Second Ed.), Springer, New York, 1986.
11. R. S. Palais, *A Global Formulation of the Lie theory of Transformation Groups*, Memoirs of the Amer. Math Soc., vol. 22, Amer. Math. Soc., Providence, R.I., 1957.
12. R. Palais, *Slices and equivariant imbeddings*, Seminar on Transformation Groups (A. Borel, eds.), Annals of Mathematics Studies, vol. 46, Princeton University Press, Princeton, New Jersey, 1960, pp. 101–116.
13. R. S. Palais, *The principle of symmetric criticality*, Comm. Math. Phys. **69** (1979), 19–30.
14. F. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, First published by Scott, Foresman 1971, Springer-Verlag, New York, 1983.