Biconformal supergravity and the AdS/CFT conjecture

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Abstract

Biconformal supergravity models provide a new gauging of the superconformal group relevant to the Maldacena conjecture. Using the group quotient method to biconformally gauge $SU(2,2|N)$, we generate a 16-dim superspace. We write the most general even- and odd-parity actions linear in the curvatures, the bosonic sector of which is known to descend to general relativity on a 4-dim manifold.

1 Introduction

For the past half decade, studies of M/string theory have been dominated by interest in the relationships between string on specified backgrounds and Yang-Mills gauge theories in lower dimensions. These investigations, triggered by Maldacena [1], include the conjecture that type IIB string on an $AdS_5 \times S^5$ background is dual to $N = 4$, $d = 4$ supersymmetric Yang-Mills theory. Since the isometry group of the manifold $AdS_5 \times S^5$ is the superconformal group, the corresponding IIB string theory could have a ghost-free, conformal supergravity theory as its low energy limit. Therefore, it is interesting to revisit – and extend – the set of conformal supergravity theories.

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Here we find a alternative to the classical conformal supergravity theories. The new theory has an action linear in superconformal curvatures.

Intense activity in the 70s and early 80s provided what seemed to be a complete picture of possible supergravity models. The demonstration in 1975 by Haag, Łopusański and Sohnius [2] of all possible supersymmetries of the S matrix showed clearly how supersymmetry could overcome the limitations of the Coleman-Mandula theorem [3]. It was only a short time before systematic classifications of the graded Lie algebras emerged. The simple graded Lie algebras were classified by Freund and Kaplansky [4], and Kac [5], [6] added the exceptional algebras. Some work on classification was also provided in [7], [8].

With these classifications available, Nahm [9] was able to identify those graded algebras suitable for physical models in arbitrary spacetime dimension by restricting to algebras with physical spin-statistics behavior, compact internal symmetry, and an adjoint operation. Nahm went on to determine the structure of all their flat space representations.

Simultaneously, other authors ([10]-[26]) explored Poincaré and conformal supergravity theories based on the new symmetries and developed the theory of supermanifolds. Of particular interest for our purpose is the development of the group manifold (or group quotient) method for constructing supermanifolds (see, eg., [19]-[22] and in particular, the review by Castellani, Fré and van Nieuwenhuizen [23]). The method described in [23] (who cite [20]; see also [19]) is a modified version of techniques developed by Cartan (for a complete treatment see Kobayashi and Nomizu [27]), which has been generalized to supergroups. We now turn to a discussion of conformal supergravity and examine the use of the group manifold method in its construction.

The first comments on conformal supergravity by Freund ([28]) identify some of the properties of the superconformal gauge fields. The full theory was then developed simultaneously and independently by Freund, Ferber and Crispim-Romao in one series of articles ([28], [29], [30], [31]) and by Kaku, Townsend, van Nieuwenhuizen and Ferrara ([32], [33], [34], [35], [36]) in another series of articles. One cannot doubt the sense of excitement and urgency that accompanied these developments. Because the review article, [23] uses similar methods to our own, we will refer to the construction presented there.

The group manifold method provides a systematic way to implement a given local symmetry as a gravity theory. Essentially, one begins with a Lie group or graded Lie group, G, containing the local symmetry (super)group of interest, H, as a sub-(super)group. Then the quotient G/H is a manifold with local H symmetry. The cosets of H in G provide a projection from G to G/M, so the resulting structure is a principal fiber bundle with fibers isomorphic to H.

In their implementation, Castellani, et al., independently choose the dimension, d, of the final spacetime manifold. Their manifold may be any d-dimensional submanifold of the fiber bundle as long as $d \leq \dim(G/H)$. They then introduce a connection one form, h^A, on the bundle and write its curvature, R^A. The form of the curvature is fully determined by the graded Lie algebra. Their construction is completed by implementing two assumptions:

1. The action is H-invariant integral of a d-form,

$$S = \int \left(\Lambda + R^A v_A + R^A \wedge R^B v_{AB} + \cdots\right) \tag{1}$$

2. The vacuum (which they define to be $R^A = 0$) is a solution of the field equations.

This last condition is necessary because of the arbitrary choice of the spacetime dimension. Castellani, et al., must specify some condition of this sort to fix the H-tensors $\Lambda, v_A, , v_{AB}, \cdots$. The so-called "cohomology condition" follows from the variation of the action when the curvatures vanish,

$$\frac{\delta}{\delta h^A}\Lambda + Dv_A = 0 \tag{2}$$

This equation supplements the usual variational field equations. Solutions to the combined cohomology and variational equations exist only for certain subgroups H and dimensions d.

Our construction begins with the same group quotient and fiber bundle structure, but our subsequent assumptions differ in three ways. First, we do not allow the choice of spacetime dimension in our construction. Instead, we let the group and subgroup symmetries determine the dimension of the physical spacetime by requiring $d = \dim(G/H)$. Second, we do not require the physical space to be a submanifold of the group manifold. Rather, it may be any manifold consistent with the local structure of the principal fiber bundle $(G, G/H)$. Finally, we do not require vanishing curvature to be a solution to the field equations. Indeed, we note that vanishing curvature may be inconsistent with reduction to the AdS background. Because of these last two differences, we are not required to separately impose the cohomology condition.

As in [23], the action may be any H-invariant d-form. However, we note that since the action is built in an H-invariant way from the curvatures, one cannot write an action before constructing the geometric background.

In this way, the group structure is all that is required to construct a general class of geometries with the desired symmetries, including dimension of physical space, the expressions for curvatures of the connection, and the relevant fields of the theory. From these curvatures it is straightforward to write the most general linear action functional..

In summary, Castellani, Fré, and van Nieuwenhuizen consider, in principle, any subgroup H of G and any spacetime dimension consistent with the cohomology and field equations, while we require manifold dimension $d = dim(G/H)$, drop the cohomology equations, and drop the constraint to zero curvature solutions. Our method is likely more rigid and therefore more predictive. In any case, demanding $d = dim(G/H)$ has interesting consequences for the conformal and superconformal groups, and possibly for M theory as well. We now take a brief look at these theories, starting with the bosonic case.

Gauging of the conformal group is implicit in any gauging of the superconformal group, and thus the conformal supergravity theories referenced above all contain conformal gaugings. A systematic presentation of the possible gaugings that can be used to construct Poincaré and conformal supergravity theories was provided by Ivanov and Niederle ([37],[38]). Using group manifold methods, they reproduced the gaugings present in the known conformal supergravity theories, and in [38] were the first to recognize an alternative gauging. The alternative gauging sets the local symmety to the homothetic group, comprised of Lorentz transformations and dilatations.

The use of the homothetic group as the local symmetry is not in itself a new result. Indeed, the extension of the homothetic group by an internal symmetry provides the residual bosonic symmetry of the conformal supergravities considered in [23] and ([28]-[36]). However, Ivanov and Niederle also set $d = \dim(G/H) = 8$, thereby introducing four new coordinates to the physical manifold. To deal with the additional dimensions, they then resticted the 4 new dimensions to a submanifold generated by conformal transformations, thereby essentially making these directions pure gauge degrees of freedom. The 4 new coordinates (or n new coordinates for n-dim spacetimes) were freed from this constraint in [39], using what is now called biconformal gauging of the conformal group. The enlarged space still permits general relativity on a Lorentzian submanifold. The extra dimensions participate as conjugate (momentum-like) variables in a symplectic structure. Because the volume element of the 8- or $2n$-dim space is dimensionless, the biconformal space allows actions linear in the curvature. Wehner and Wheeler [40] showed that, with minimal or vanishing torsion, the most general linear action is extremal only when there is a symplectic form and the Einstein equation holds on a 4- or n-dim submanifold. The remaining dimensions may be identified (in all known classes of solutions) with coordinates on the cotangent space.

The goal of the present work is to supersymmetrize 4-dim gravity using the alternative gauging. That is, we study 4-dim biconformal supergravity.

2 The Superconfornal graded Lie algebra

The conformal group of a four dimensional spacetime is locally isomorphic to $O(4,2)$. $Spin(4,2)$, also locally isomorphic to $O(4,2)$, gives a spinor representation for the conformal group. Using the 4×4 Dirac matrices,

$$\left\{\gamma^a,\gamma^b\right\} = 2\eta^{ab} = 2\ diag\,(-1,1,1,1) \tag{3}$$

and defining

$$\begin{aligned}\sigma^{ab} &= -\frac{1}{8}\left[\gamma^a,\gamma^b\right]\\ \gamma_5 &= i\gamma^0\gamma^1\gamma^2\gamma^3\end{aligned}$$

the full Clifford algebra has basis

$$\Gamma \in \left\{1, i1, \gamma^a, i\gamma^a, \sigma^{ab}, i\sigma^{ab}, \gamma_5\gamma^a, i\gamma_5\gamma^a, \gamma_5, i\gamma_5\right\}$$

where $a,b = 0,1,2,3$. We use this representation for the conformal sector of the superconformal group.

The structure of the superconformal group, $SU(2,2|N)$ is well known ([28], [4], [6]). To construct $SU(2,2|N)$ we demand that the generators of the graded Lie algebra preserve a complex super-metric, $\mathcal{H}$, diagonally composed of a 4-dim Hermitian matix, Q (Q is in fact a spinor metric– see [41]), and an N-dim, anti-hermitian, (symmetric) matrix P,

$$\mathcal{H} = \begin{pmatrix} Q & \\ & P \end{pmatrix} \tag{4}$$

The invariance condition is

$$\mathcal{H}T + T^{\ddagger}\mathcal{H} = 0 \tag{5}$$

where

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, T^{\ddagger} = \begin{pmatrix} A^{\dagger} & -C^{\dagger} \\ B^{\dagger} & D^{\dagger} \end{pmatrix} \tag{6}$$

where $T^{\ddagger}$ is the usual super-adjoint. We obtain,

$$\begin{aligned} QA + A^{\dagger}Q &= 0 & (7) \\ C &= -P^{-1}B^{\dagger}Q & (8) \\ PD + D^{\dagger}P &= 0 & (9) \end{aligned}$$

We use a spinor representation for the conformal Lie algebra. Note that by choosing,

$$Q = -i\gamma^5 \tag{10}$$

the invariance condition for the A-sector bosonic generators selects the subset of Clifford generators

$$\Gamma_0 \in \left\{ i1, \gamma^a, \sigma^{ab}, \gamma_5\gamma^a, \gamma_5 \right\}$$

The last 15 of these 16 generators provide a manifest basis for the Lie algebra $su(2,2)$.

It is straightforward to find a representation of the Dirac matrices for which $Q = diag(1,1,-1,-1)$, and thus demonstrate that the 15 matrices listed above generate $SU(2,2)$. Thinking of Q as a Hermitian spinor metric, we are justified in calling the Q-invariant subalgebra the isometry subalgebra of the Clifford algebra. Note that the defining relationship of the Clifford algebra, eq.(3), is invariant under $U(4)$ transformations, and we may use this freedom to select a real representation of the Dirac matrices. In a real basis, Q remains Hermitian but is necessarily antisymmetric, $Q = -Q^t$. It follows that the generators of $SU(2,2)$ are unitiarily equivalent to a set preserving a symplectic form.

Choosing generators for the Lie algebra, we identify

$$M^a{}_b = \eta_{bc}\sigma^{ac} = -\frac{1}{8}\eta_{bc}\left[\gamma^a, \gamma^c\right] \tag{11}$$

$$P_a = \frac{1}{2}\eta_{ab}\left(1+\gamma_5\right)\gamma^b = \frac{1}{2}\eta_{ab}\gamma^b\left(1-\gamma_5\right) \quad (12)$$

$$K^a = \frac{1}{2}\left(1-\gamma_5\right)\gamma^a = \frac{1}{2}\gamma^a\left(1+\gamma_5\right) \quad (13)$$

$$D = -\frac{1}{2}\gamma_5 \quad (14)$$

and compute the Lie algebra, which is listed in Appendix 1. Here, $M_{ab} = -M_{ba} = \eta_{ac}M^c_b$ are the Lorentz rotation generators, P_a the translations, K^a the special conformal transformations, and D the dilatation. With modified sign conventions, these agree with the work of [32] and [33].

In addition to giving a symplectic representation, a real representation for the Dirac matrices is a convenience in writing real-valued action functionals. With real Dirac matrices, ${M^a}_b$ is real, D is pure imaginary, and P_a , K^a are complex conjugates of one another.

Returning to the invariance conditions, we note that the D-sector generators preserve the anti-Hermitian form P, which in the real representation is symmetric, $P^t = P$. Therefore, the internal symmetry is $U(N)$. The set of generators includes $N(N-1)/2$ real, antisymmetric generators and $N(N+1)/2$ imaginary, symmetric generators to constitute the required N^2 generators.

Among the bosonic symmetries are two commuting generators: $i1_4$ in the A-sector and $i1_N$ in the D-sector, where 1_4 and 1_N denote the 4- and N-dim identies, respectively. By demanding vanishing superdeterminant for the superconformal group, we eliminate these in favor of the supertraceless combination

$$E = -\frac{i}{4}\begin{pmatrix} 1_4 & \\ & \frac{4}{N}1_N \end{pmatrix}$$

The generator E functions as a central charge in the conformal subalgebra and appears in one fermionic anticommutator.

There are therefore a total of N^2+16 bosonic generators and $8N$ fermionic generators. Explicit forms for all of the generators (and the complete $su(2,2|N)$ algebra) are presented in Appendix 1.

3 Maurer Cartan Structure Equations:

We now define the set of super differential forms dual to the generators of the super Lie algebra. In general, the dual differential forms are defined as follows.

$$\left\langle G_\Sigma, \omega^\Pi \right\rangle \equiv \delta^\Pi_\Sigma$$

Here the indices Π, Σ run over the types of indices present in the Lie algebra and differential forms are bold. G_Σ represents an arbitrary generator of the super Lie algebra and ω^Π is the corresponding Lie algebra valued one form. We utilize differential forms in order to make the expressions for the action more manageable ([23]). Note that by definition, we are assigning the differential forms, ω_Λ, to have the opposite conformal weight of their corresponding generators, G^Σ. Explicitly we define the form $\omega^\Pi \in \left\{\omega^a_b, \omega_a, \omega^a, \omega, \psi^B_\beta, \chi^B_\beta, \alpha, \pi^{\rho\sigma}_R, \pi^{\rho\sigma}_I\right\}$ by:

$$\begin{array}{cc} \langle M^a{}_b, \omega^c_d \rangle = \delta^a_d \delta^c_b - \eta^{ac}\eta_{bd} & \left\langle G^{\alpha+}_A, \chi^B_\beta \right\rangle = \delta^\alpha_\beta \delta^B_A \\ \left\langle P_a, \omega^b \right\rangle = \delta^b_a & \left\langle G^{\alpha-}_A, \psi^B_\beta \right\rangle = \delta^\alpha_\beta \delta^B_A \\ \langle K^a, \omega_b \rangle = \delta^a_b & \left\langle D^{\mu\nu}_R, \pi^R_{\alpha\beta} \right\rangle = \delta^\mu_\alpha \delta^\nu_\beta \\ \langle D, \omega \rangle = 1 & \left\langle D^{\mu\nu}_I, \pi^I_{\alpha\beta} \right\rangle = \delta^\mu_\alpha \delta^\nu_\beta \\ \langle E, \alpha \rangle = 1 & \end{array} \tag{15}$$

The first four forms listed on the left are associated with the Lorentz, translation, co-translation, and dilatation generators, respectively. We refer to ω^a_b as the spin-connection, ω^a as the solder form, ω_a as the co-solder form, and ω as the Weyl vector. ψ^B_β and χ^B_β correspond to the fermionic generators while the remaining three one-forms, $\alpha, \pi^{\rho\sigma}_R, \pi^{\rho\sigma}_I$ are associated with the internal symmetry.

The Maurer-Cartan structure equations are defined in general by

$$0 = \mathbf{d}\omega^\Sigma + \frac{1}{2} c_{\Gamma\Delta}{}^\Sigma \omega^\Gamma \omega^\Delta \tag{16}$$

and are fully equivalent to the Lie algebra relations, with $\mathbf{d}^2 = 0$ providing the Jacobi identies. Here $c_{\Gamma\Delta}{}^\Sigma$ are the structure constants of the graded Lie algebra and the standard wedge product is assumed between all differential forms. Using the group quotient method ([27], [19], [20], [23]), the connection is generalized, giving the curvature 2-forms $\mathbf{\Omega}^\Sigma$,

$$\mathbf{\Omega}^\Sigma = \mathbf{d}\omega^\Sigma + \frac{1}{2} c_{\Gamma\Delta}{}^\Sigma \omega^\Gamma \omega^\Delta \tag{17}$$

according to the Cartan structure equations. When we form the group quotient between $SU(2,2|N)$ and our chosen isotropy subgroup, the curvature

2-forms are required to be *horizontal*, that is, expandable only in the basis forms spanning the co-tangent space to the quotient manifold. For example, the usual Riemannian structure of general relativity arises from the quotient of the Poincaré group by its Lorentz subgroup. The dual forms for the Poincaré group are of two types: the spin connection ω_b^a and the solder form $\mathbf{e}^a$. The horizontal curvatures may be expanded bilinearly in the solder forms only, $\mathbf{R}_b^a = \frac{1}{2}\mathbf{R}_{bcd}^a \mathbf{e}^c \mathbf{e}^d$.

Regardless of the group quotient we choose, the general form of the curvatures is the same until they are expanded in basis forms. Thus, we may immediately write the following expressions for the curvature 2-forms using the structure constants from Appendix 1. We have

$$\begin{aligned}
\mathbf{\Omega}_b^a &= \mathbf{d}\omega_b^a - \omega_b^c\omega_c^a - 2\omega_b\omega^a + 2\eta^{ac}\eta_{bd}\omega_c\omega^d - P^{\alpha\beta}\left[\sigma^a{}_b\right]_{AB}\chi_\alpha^A\psi_\beta^B && (18)\\
\mathbf{\Omega}^a &= \mathbf{d}\omega^a - \omega^c\omega_c^a - \omega\omega^a + \frac{1}{2}P^{\alpha\beta}\left[\gamma^a\right]_{(AB)}\psi_\alpha^A\psi_\beta^B && (19)\\
\mathbf{\Omega}_a &= \mathbf{d}\omega_a - \omega_a^c\omega_c - \omega_a\omega + \frac{1}{2}P^{\alpha\beta}\left[\gamma_a\right]_{(AB)}\chi_\alpha^A\chi_\beta^B && (20)\\
\mathbf{\Omega} &= \mathbf{d}\omega - 2\omega^a\omega_a - \frac{1}{2}P^{\alpha\beta}Q_{AB}\chi_\alpha^A\,\psi_\beta^B && (21)
\end{aligned}$$

for the usual supersymmetric generalization of the bosonic curvatures, and

$$\begin{aligned}
\mathbf{\Theta}_\beta^B &= \mathbf{d}\psi_\beta^B - \left[\frac{1}{2}\omega_a^b\sigma^a{}_b\right]^B{}_A\psi_\beta^A + \left[\omega^a\gamma_a\right]^B{}_A\chi_\beta^A - \frac{1}{2}\omega\psi_\beta^B \\
&\quad -2iP^{\alpha\mu}\pi_{\mu\beta}^R\psi_\alpha^B - 2P^{\alpha\mu}\pi_{\mu\beta}^I\left[\gamma_5\right]^B{}_A\psi_\alpha^A && (22)\\
\overline{\mathbf{\Theta}}_\beta^B &= \mathbf{d}\chi_\beta^B - \left[\frac{1}{2}\omega_a^b\sigma^a{}_b\right]^B{}_A\chi_\beta^A + \left[\omega_a\gamma^a\right]^B{}_A\psi_\beta^A + \frac{1}{2}\omega\chi_\beta^B \\
&\quad -2iP^{\alpha\mu}\pi_{\mu\beta}^R\chi_\alpha^B + 2P^{\alpha\mu}\pi_{\mu\beta}^I\left[\gamma_5\right]^B{}_A\,\chi_\alpha^A && (23)\\
\mathbf{\Pi}_{\rho\sigma}^R &= \mathbf{d}\pi_{\rho\sigma}^R + 2iP^{\beta\mu}\left(\pi_{\rho\beta}^R\pi_{\mu\sigma}^R - \pi_{\rho\beta}^I\pi_{\mu\sigma}^I\right) - \frac{i}{4}Q_{AB}\left(\chi_\rho^A\psi_\sigma^B - \chi_\sigma^A\psi_\rho^B\right) && (24)\\
\mathbf{\Pi}_{\rho\sigma}^I &= \mathbf{d}\pi_{\rho\sigma}^I + 2iP^{\beta\mu}\left(\pi_{\rho\beta}^R\pi_{\mu\sigma}^I + \pi_{\sigma\beta}^R\pi_{\mu\rho}^I\right) && (25)\\
&\quad +\frac{1}{4}\left[\gamma_5\right]_{AB}\left(\chi_\rho^A\psi_\sigma^B + \chi_\sigma^A\,\psi_\rho^B\right) && (26)\\
\mathbf{A} &= \mathbf{d}\alpha + \frac{1}{2}\left(\left(\gamma_5\right)_B^C Q_{CA}\chi_\alpha^A\psi_\beta^B P^{\alpha\beta} + \left(\gamma_5\right)_A^C Q_{CB}\psi_\alpha^A\chi_\beta^B P^{\alpha\beta}\right) && (27)
\end{aligned}$$

for the fermionic and internal curvatures.

We define $\mathbf{\Theta}_\beta^B$ to be the fermionic curvature and $\mathbf{\Pi}^R_{\rho\sigma}, \mathbf{\Pi}^I_{\rho\sigma}, \mathbf{A}$ to be the internal symmetry curvatures ($\mathbf{\Pi}^R_{\rho\sigma}, \mathbf{\Pi}^I_{\rho\sigma}$ refer to the real and imaginary components, respectively, of the internal symmetry curvatures). Note that in the equations listed above ω^a, ω_a refer to two independent one forms associated with distinct generators. The position of the lower case Latin indices is used to designate the generator, it does *not* refer to any use of the metric.

It is important to recognize that eqs.(18-27) do not yet fully define the curvatures because we have not yet specified the subgroup which determines horizontality. Knowing this group determines not only the cotangent basis forms in which they are to be expanded, but also how these component curvatures mix under the residual fiber (gauge) symmetry. We now turn to these questions.

4 Gauging the Supergroup

In general, to gauge the group, an isotropy subgroup (any subgroup containing no subgroup normal in the full group) is chosen. In keeping with our comments in the introduction, this choice determines the dimension and local character of the physical superspace. The quotient of the full group by the isotropy subgroup is a manifold whose dimension is given by the difference in the dimensions between the full group and the isotropy subgroup. The curvatures of the manifold follow by generalizing the connection ω^Σ and demanding horizontality. We allow any global structure consistent with this local structure.

Much is known about the bosonic case of conformal gauging. As described in [38], [37], and [39], the demand that the local symmetry contain both Lorentz transformations and dilatations leaves only two choices for the quotient. The first of these is to take the quotient of the conformal group by the subgroup built from Lorentz transformations, dilatations and special conformal transformations (i.e., the inhomogeneous homothetic group). The quotient manifold is then 4-dimensional, and is immediately identified with spacetime. The alternative *biconformal* gauging takes the quotient of the conformal group by the homogeneous homothetic group, consisting of Lorentz transformations and dilatations only.

The first case has been treated abundantly in the literature, as discussed in the introduction. While the homothetic group is used for the isotropy subgroup in [23], the manifold is still taken to be 4-dimensional and the cohomology equations are imposed by hand.

By contrast, the biconformal gauging of the conformal group of a compactified, n-dim spacetime produces a $2n$-dim space. This space is spanned by n coordinates with units of length and another n coordinates with units of inverse length. As a result, the volume form is dimensionless and it is possible to write a gravity action which is linear in the curvature. Assuming minimal [40] or vanishing [39] torsion, the resulting field equations reduce in a particular subset of conformal gauges to the Einstein equation of general relativity on an n-dimensional submanifold. In the final symmetry of the space, the translational and special conformal symmetries become general coordinate symmetry on the base manifold, leaving the Lorentz and dilatational curvatures as local symmetries. The extra n dimensions participate in a symplectic structure that has been shown to be consistent with the Hamiltonian dynamics [43] of an n-dim configuration space.

Our central aim is now to reproduce this result for a supersymmetric extension of the conformal group and write linear actions over the resulting space.

In order to define a fiber bundle over a supermanifold we must generalize the isotropy subgroup, the homogeneous homothetic group, to a sub-supergroup H of the entire supergroup. We demand two properties of H:

1. The A-sector of the bosonic part of H must consist of Lorentz transformations and dilatations.

2. The D-sector of the bosonic part of H must be $U(N)$, thereby retaining the entire D-sector as a local internal symmetry.

We now show that there are three possible choices for H satisfying these conditions. Two of these are mathematically equivalent, so there are two distinct group quotients that give rise to the desired homothetic bosonic fiber symmetry. The proof is as follows.

First consider condition 1. The homothetic algebra may be characterized as the dilatationally invariant subalgebra of the conformal algebra. That is, the homothetic generators, W, are exactly those that satisfy,

$$\begin{aligned} [\gamma_5, W] &= 0 && (28) \\ QW + W^\dagger Q &= 0 && (29) \end{aligned}$$

Alternatively, since all elements of the graded Lie algebra already satisfy eq.(29), we may say that the subalgebra that preserves

$$\alpha Q + \beta Q \gamma_5$$

for any fixed $\alpha, \beta \neq 0$, is homothetic.

Next, we generalize this condition to an arbitrary element of $su(2,2|N)$ and impose both conditions 1 and 2. Thus, the subset of $su(2,2|N)$ generators leaving any matrix, $\mathcal{M}$, of the form

$$\mathcal{M} = \begin{pmatrix} \alpha Q + \beta Q \gamma_5 & R \\ S & J \end{pmatrix}$$

invariant generates a subgroup of the superconformal group. We demand this subgroup to be our isotropy, H. Letting T be as in eq.(6), invariance of $\mathcal{M}$, namely, $\mathcal{M}T + T^{\ddagger}\mathcal{M} = 0$ for all T leads directly to the conditions $R = S = 0$ and $J = \lambda P$. The three possible solutions depend on whether

$$\det\left(\left(\alpha - \lambda\right) Q + \beta Q \gamma_5\right)$$

vanishes or not (see Appendix II). The three possibilities are:

1. $(\alpha - \lambda) \neq \beta$. No nonzero spinor B survives in the local symmetry. All fermionic degrees of freedom are then coordinate degrees of freedom, so we have a 16-dim superspace spanned by ω^a, ω_a, χ and ψ with local homothetic and $U(N)$ symmetry. The supervolume element is dimensionless.

2. $(\alpha - \lambda) = \beta \neq 0$. Only left handed spinors B provide local symmetries. In this case, each local fermionic symmetry B must satisfy:

$$\left(1 + \gamma_5\right) B = 0$$

   Therefore, half of the fermionic degrees of freedom (those generated by G_A^+) lie on the fiber and half become coordinate degrees of freedom. The volume element, $\sim \chi^1\chi^2\chi^3\chi^4$, has scaling dimension $(length)^{2N}$ and, since ψ and χ are complex conjugates, there is no evident real-valued action linear in the curvatures.

3. $(\alpha - \lambda) = -\beta \neq 0$. Only right handed spinors B provide local symmetry The local fermionic symmetries satisfy

$$(1 - \gamma_5) B = 0$$

so they are those generated by $G_{\bar{A}}^{-}$. Again, the fermionic degrees of freedom are split between coordinate and fiber, giving a volume element $\sim \psi^1\psi^2\psi^3\psi^4$, of scaling dimension $(length)^{-2N}$. There is no real-valued action linear in the curvatures.

The two chiral solutions may ultimately prove to be of interest, since they display heteroticity, but they will not occupy us further here. The first case, in which all fermionic degrees of freedom are realized as superspace coordinates, has the dimensionless volume form characteristic of biconformal gauging. We now examine this case in detail.

First, to accomplish the group quotient, we note that both Q and $Q\gamma_5$ are independent invariants. Then the isotropy subgroup is generated by those transformations leaving both the super-metric, $\mathcal{H}$, and

$$\mathcal{M} = \begin{pmatrix} Q\gamma_5 & 0 \\ 0 & 0 \end{pmatrix} \tag{30}$$

invariant. This amounts to all Lorentz transformations, dilatations, the central charge E, and $U(N)$ transformations. These span a $8 + N^2$ dimensional submanifold of $SU(2,2|N)$, while the quotient G/H is 16-dimensional.

Next, we implement the requirement for horizontal curvatures. Each curvature is now defined to be bilinear in the set of basis forms

$$\omega^{\Pi} \in \left\{ \omega_a, \omega^a, \psi_{\beta}^{B}, \chi_{\beta}^{B}, \right\}$$

Thus, for each curvature,

$$\mathbf{\Omega}^{\Sigma} = \frac{1}{2}\Omega^{\Sigma}{}_{\Pi\Lambda}\omega^{\Pi}\omega^{\Lambda}$$

Expanding explicitly, we adopt the following notational conventions for each curvature:

$$\begin{aligned} \mathbf{\Omega}^{\Sigma} &= \frac{1}{2}R^{\Sigma}{}_{ab}\,\omega^a\omega^b + R^{\Sigma a}{}_{b}\omega_a\omega^b + \frac{1}{2}R^{\Sigma ab}\omega_a\omega_b \\ &\quad + R^{\Sigma}{}_{a\widetilde{B}}\omega^a\psi^{\widetilde{B}} + \bar{R}^{\Sigma}{}_{aA}\omega^a\chi^A \\ &\quad + R^{\Sigma a}{}_{\tilde{A}}\omega_a\psi^{\widetilde{A}} + R^{\Sigma a}{}_{A}\omega_a\chi^A \\ &\quad + \frac{1}{2}R^{\Sigma}{}_{\widetilde{A}\widetilde{B}}\psi^{\widetilde{A}}\psi^{\widetilde{B}} + R^{\Sigma}{}_{A\widetilde{B}}\chi^A\psi^{\widetilde{B}} + \frac{1}{2}\bar{R}^{\Sigma}{}_{AB}\chi^A\chi^B \end{aligned} \tag{31}$$

where indices with tildes are contracted with $\psi^{\widetilde{A}}$ as opposed to χ^{A}.

Finally, each connection form and curvature is now regarded as an H-tensor rather than a superconformal tensor. This means that eqs.(18-27) now represent nine independent H-tensors instead of a single $su(2,2|N)$ tensor. Indeed, even the ten components $R^{\Sigma}{}_{ab}, \ldots, \bar{R}^{\Sigma}{}_{AB}$ of each have no mixing under homothetic transformations, and therefore provide a total of 90 independent tensor fields.

The basis forms $\omega_a, \omega^b, \chi^A, \psi^B$ all transform tensorially, under the homothetic gauge group, as do the curvatures, $\mathbf{\Omega}_a.\mathbf{\Omega}^a, \mathbf{\Theta}^A, \overline{\mathbf{\Theta}}^A$ and $\mathbf{\Omega}^a_b$. The remaining curvatures, $\mathbf{\Omega}$ and $\mathbf{A}$, are invariant under the gauge transformations. Thus, the supersymmetry of the space is entirely in the coordinate transformations of the superspace formulation. General coordinate transformations on the base manifold can mix the components of the fermionic and bosonic basis forms.

These gauge transformations differ markedly from those of ([28]-[36]), since these theories retain local Lorentz, dilatational and special conformal symmetry, together with the fermionic transformations. In [23], the bosonic local symmetry is reduced to Lorentz and dilatational, plus supersymmetries. In the present formalism, the full local symmetry is simply Lorentz and dilatational, while the fermionic transformations become coordinate transformations of superspace. Thus, our construction results in far more invariant quantities, and makes it possible to easily write purely geometric invariant actions.

Below, we describe actions for the biconformal gauging of $SU(2,2|N)$. We present the most general, gauge invariant, linear actions of both even and odd parity. The even parity action gives rise to general relativity in the bosonic sector of the superspace, and we show that the Rarita-Schwinger equation is contained in the fermionic sector.

5 Biconformal Actions

Before addressing possible actions, we must consider the volume element for super-biconformal space. Since the bosonic portion of the base manifold is spanned by the solder and co-solder forms we will first define,

$$\phi_{bosonic} = \phi_b = \varepsilon_{acde}\varepsilon^{bfgh}\omega_{bfgh}\omega^{acde}$$

where

$$\omega_{bfgh} = \omega_b \omega_f \omega_g \omega_h$$

and ε_{acde} is the four dimensional Levi-Civita tensor. Again, the mixed index position indicates the scaling weight of the indices and not any use of the metric. A full derivation and justification of this volume element is given in [40]. It is important to note that the real-valued 8-form ϕ_b is both dilatationally and Lorentz invariant. The Levi-Civita tensor is normalized such that traces are given by

$$\varepsilon_{a_1 \ldots a_p c_{p+1} \ldots c_n} \varepsilon^{b_1 \ldots b_p c_{p+1} \ldots c_n} = p!\,(n-p)!\,\delta^{b_1 \ldots b_p}_{a_1 \ldots a_p}$$

where δ represents the following totally antisymmetric tensor,

$$\delta^{b_1 \ldots b_p}_{a_1 \ldots a_p} \equiv \delta^{[b_1 \ldots b_p]}_{a_1 \ldots a_p}$$

The fermionic portion of the base manifold is spanned by the spinor-valued one forms, ψ and χ. The volume form is therefore proportional to

$$\psi^1 \psi^2 \psi^3 \psi^4 \chi^1 \chi^2 \chi^3 \chi^4$$

However, it is desirable to write a manifestly tensorial expression. This makes subsequent calculations simpler, but there is a difficulty in constructing one. Unlike bosonic differential forms, fermionic forms commute so the wedge product does not automatically eliminate quadratic terms such as $\psi^1 \wedge \psi^1$. However, this difficulty is readily overcome by noting the following ideas.

Differential p-forms may be defined as maps from p-dimensional volumes into the reals. Thus, for example, the 1-form $\mathbf{f} = f(x)\mathbf{d}x$ maps $\mathbf{f}: C \to R$ according to $r = \int f(x)\mathbf{d}x$. This is a useful point of view for fermionic forms. By the rules of Berezin integration ([13], [15], [17]), integrals over a pair of identical fermionic forms vanish, regardless of the integrand. Specifically, although $\mathbf{d}\theta\mathbf{d}\theta$ is not manifestly zero by symmetry, it vanishes on every complete superspace integral. Ignoring the obvoius ambiguities, consider what map $\mathbf{d}\theta\mathbf{d}\theta$ must be:

$$\begin{aligned} \int\int f(\theta)\,\mathbf{d}\theta\mathbf{d}\theta &= \int\int (a + b\theta)\,\mathbf{d}\theta\mathbf{d}\theta \\ &= \int \left(\int (a + b\theta)\,\mathbf{d}\theta \right) \mathbf{d}\theta \\ &= \int b\mathbf{d}\theta \\ &= 0 \end{aligned}$$

Alternatively, we note that in order for two $\mathbf{d}\theta$ integrals to fail to vanish, we would require two factors of θ in the integrand. This also vanishes. Therefore, we are justified in defining quadratic terms such as $\psi^1 \wedge \psi^1$ to be equivalent to the zero map, and therefore zero.

For two fermionic degrees of freedom, θ_1 and θ_2, we may therefore write

$$\begin{aligned}\frac{1}{2}\left(\mathbf{d}\theta_1+\mathbf{d}\theta_2\right)^2 &= \frac{1}{2}\mathbf{d}\theta_1\mathbf{d}\theta_1+\mathbf{d}\theta_1\mathbf{d}\theta_2+\frac{1}{2}\mathbf{d}\theta_2\mathbf{d}\theta_2 \\ &\cong \mathbf{d}\theta_1\mathbf{d}\theta_2\end{aligned}$$

where we use the equivalence of quadratics to zero in the last step. With this convention in mind, any purely eighth order polynomial in ψ and χ is proportional to the $N=1$ volume element. A simple covariant expression for the volume form is therefore

$$\phi_f = \frac{1}{4!}\left(Q_{AB}\chi^A\psi^B\right)^4$$

It is straightforward to check that this reduces to $\psi^1\psi^2\psi^3\psi^4\chi^1\chi^2\chi^3\chi^4$. The generalization to arbitrary N is immediate:

$$\phi_f = \left(Q_{AB}P^{\alpha\beta}\chi^A_\alpha\ \psi^B_\beta\right)^{4N}$$

Finally, the full volume form over the superspace is

$$\mathbf{\Phi} = \phi_b\phi_f$$

To construct an action linear in the eight curvatures,

$$\left\{\mathbf{\Omega}^a_b, \mathbf{\Omega}^a, \mathbf{\Omega}_a, \mathbf{\Omega}, \mathbf{A}, \mathbf{\Pi}^R_{\alpha\beta}, \mathbf{\Pi}^I_{\alpha\beta}, \mathbf{\Theta}^B_\beta, \overline{\Theta}^B_\beta\right\}$$

we first note the additional available tensors fields. These include the Dirac matrices,

$$\{\gamma^a, \gamma_5, \sigma^a_b, \gamma_5\gamma^a\}$$

together with the set consisting of the Minkowski metric, η_{ab}, the spinor metric, $Q_{AB,}$ the $U(N)$ metric, $P^{\alpha\beta}$, and the Levi-Civita tensor, ε_{abcd}. We define the Dirac matrices to be of zero conformal weight and covariantly constant, $D_\Sigma\Gamma_0 = 0$.

We next use the following fact to construct the action. Let Φ_Σ be a general tensor-valued $8(N+1)-2$ form with index Σ of arbitrary type and let $\mathbf{\Omega}^\Sigma$ be any curvature 2-form. Then their product,

$$\mathbf{\Omega}^\Sigma\Phi_\Sigma$$

must be proportional to a complete volume form,

$$\mathbf{\Omega}^{\Sigma}\Phi_{\Sigma} = \Omega^{\Sigma\Lambda}{}_{\Pi\Delta}S_{\Sigma\Lambda}{}^{\Pi\Delta}\Phi$$

where $S_{\Sigma\Lambda}{}^{\Pi\Delta}$ is built from the available tensor fields characterized above.

The most general, even parity, homothetic gauge-invariant action linear in the curvatures for the case of a $U(N)$ internal symmetry is

$$\begin{aligned} S \; = \; & \int \Big\{ \Big(\alpha_1 \mathbf{\Omega}^a_b \left[\sigma^b_a\right]_{AB} + \left(\alpha_2 \mathbf{\Omega} + \alpha_3 \mathbf{A}\right) Q_{AB} \Big) \chi^A_\alpha \psi^B_\beta P^{\alpha\beta} \left(\chi\psi\right)^2 \phi_b \\ & + \left(\alpha_4 \mathbf{\Omega}^a_b + \left(\alpha_5 \; \mathbf{\Omega} + \alpha_6 \mathbf{A}\right) \delta^a_b\right) \varepsilon_{acde} \varepsilon^{bfgh} \omega_{fgh} \omega^{cde} \phi_f \\ & + \alpha_7 (\mathbf{\Omega}^m \left[\gamma_m\right]_{BD} \; \chi^B_\alpha \chi^D_\beta - \mathbf{\Omega}_m \left[\gamma^m\right]_{BD} \psi^B_\alpha \psi^D_\beta) P^{\alpha\beta} \left(\chi\psi\right)^2 \phi_b + \alpha_8 \Phi \\ & + \beta \left(\mathbf{\Theta}^M_\alpha \left[\gamma^a\right]^A{}_M \omega_b \psi^B_\beta - \overline{\mathbf{\Theta}}^M_\alpha \left[\gamma_b\right]^A{}_M \omega^a \chi^B_\beta \right) \\ & \times Q_{AB} P^{\alpha\beta} \left(\chi\psi\right)^3 \varepsilon_{acde} \varepsilon^{bfgh} \omega_{fgh} \omega^{cde} \\ & + \lambda_1 \left(\mathbf{\Pi}^R_{\alpha\lambda} + \mathbf{\Pi}^I_{\alpha\lambda} \right) \chi^A_\beta \psi^B_\rho P^{\lambda\rho} P^{\alpha\beta} Q_{AB} \left(\chi\psi\right)^3 \delta^a_b \varepsilon_{acde} \varepsilon^{bfgh} \omega_{fgh} \omega^{cde} \phi_f \\ & + \lambda_2 \left(\mathbf{\Pi}^R_{\alpha\lambda} + \mathbf{\Pi}^I_{\alpha\lambda} \right) \left[\gamma_5\right]_{BD} \psi^B_\rho \chi^D_\beta P^{\rho\lambda} P^{\alpha\beta} \left(\chi\psi\right)^2 \phi_b \Big\} \end{aligned} \tag{32}$$

where

$$\left(\chi\psi\right)^n = \left(\chi^A_\alpha Q_{AB} P^{\alpha\beta} \psi^B_\beta \right)^n$$

for any integer, n and where $\alpha_1, \alpha_2 ... \alpha_8, \beta, \lambda_1$ and λ_2 are arbitrary constant coefficients. The most general, linear, odd parity action is given in Appendix III.

By integrating over the fermionic degrees of freedom, S reduces to the most general linear action found in [40], together with a generic matter term of the form $g(x)\phi_b$. This bosonic action is known to produce general relativity over a 4-dim subspace [40]. Relations to other superconformal actions are discussed in the final section.

6 The N=1 Case

We investigate the $N = 1$ case in some detail. Since any fermionic 8-form is proportional to the volume form, we may define a tensor $\sigma^{ABCDEFGH}$ by

$$\chi^A \chi^B \chi^C \chi^D \psi^E \psi^F \psi^G \psi^H = \sigma^{ABCDEFGH} \phi_f$$

Therefore,

$$\sigma^{ABCDEFGH} \equiv \left| Q^{A[E} Q^{|B|F} Q^{|C|G} Q^{|D|\,H]} \right|$$

Note that the antisymmetrization removes the "diagonal" terms as required by the discussion of the previous section, while the absolute value restores symmetry under interchanges. It is also convenient to define

$$\sigma^{A_1\ldots A_k}{}_{B_1\ldots B_k} \equiv \sigma^{A_1\ldots A_k A_{k+1}\ldots A_N}{}_{B_1\ldots B_k A_{k+1}\ldots A_N} = \frac{k!\,(n-k)!}{n!}\left|\delta^{A_1\ldots A_k}_{B_1\ldots B_k}\right|$$

Again, we wish to consider gauge invariant actions over the superspace, linear in the curvature two forms. The curvatures under consideration are the seven two-forms,

$$\left\{\mathbf{\Omega}^a_b, \mathbf{\Omega}^a, \mathbf{\Omega}_a, \mathbf{\Omega}, \mathbf{A}, \mathbf{\Theta}^B, \overline{\mathbf{\Theta}}^B\right\}$$

Then replacing $P_{\alpha\beta} \to i$ in eq.(32), the most general, gauge-invariant, even parity, action linear in the curvatures is

$$\begin{aligned} S \;=\; & \int \Big\{ \Big(\alpha_1 \mathbf{\Omega}^a_b \left[\sigma^b_a\right]_{AB} + (\alpha_2 \mathbf{\Omega} + \alpha_3 \mathbf{A}) Q_{AB} \Big) \chi^A \psi^B \left(\chi\psi\right)^2 \phi_b \\ & + (\alpha_4 \mathbf{\Omega}^a_b + (\alpha_5 \mathbf{\Omega} + \alpha_6 \mathbf{A}) \delta^a_b) \varepsilon_{acde} \varepsilon^{bfgh} \omega_{fgh} \omega^{cde} \phi_f + \alpha_7 \Phi \\ & + \alpha_8 (\mathbf{\Omega}^m \left[\gamma_m\right]_{BD} \chi^B \chi^D - \mathbf{\Omega}_m \left[\gamma^m\right]_{BD} \psi^B \psi^D) \left(\chi\psi\right)^2 \phi_b \\ & + \beta_1 \mathbf{\Theta}^M \left[\gamma^a\right]^A{}_M \omega_b \psi^B Q_{AB} \left(\chi\psi\right)^3 \varepsilon_{acde} \varepsilon^{bfgh} \omega_{fgh}\, \omega^{cde} \\ & - \beta_1 \overline{\Theta}^M \left[\gamma_b\right]^A{}_M \omega^a \chi^B\, Q_{AB} \left(\chi\psi\right)^3 \varepsilon_{acde} \varepsilon^{bfgh} \omega_{fgh} \omega^{cde} \Big\} \end{aligned} \qquad (33)$$

We next examine some properties of the field equations for S.

7 The Field Equations

Twenty-eight tensor equations result from variation of the action with respect to the seven one-forms $\omega_a, \omega^a, \psi^B, \chi^B, \omega, \omega^a_b$ and α. From the discussion of the gauge transformations we recall that under local symmetry transformations the bosonic and fermionic curvatures do not mix. This observation naturally gives rise to the question: how does the supersymmetry of the model appear? The answer lies in the field equations. The field equations generated by the action relate the components of the bosonic curvatures, $\mathbf{\Omega}_a, \mathbf{\Omega}^a, \mathbf{\Omega}, \mathbf{\Omega}^a_b, \mathbf{A}$, with those of the fermionic curvatures $\mathbf{\Theta}^A, \overline{\mathbf{\Theta}}^A$, and under general coordinate transformations the components of the two types of curvatures will mix.

For example, there are four equations generated by varying the action with respect to the solder form, ω^a. One of these is:

$$\begin{aligned} 0 &= 144\alpha_1 \Omega_{l\ \ n\widetilde{N}}^{m} \left[\sigma_{m}^{l}\right]_{AB} \sigma^{AMB\widetilde{N}} - 144\alpha_2 \Omega_{n\widetilde{N}} \sigma^{M\widetilde{N}} - 144\alpha_3 A_{n\widetilde{N}} \sigma^{M\widetilde{N}} \\ &\quad -2\alpha_8 \overline{\Theta}^{A}{}_{G\widetilde{H}} \left[\gamma_n\right]{}_{BD} \sigma^{MDG\widetilde{H}} - \alpha_8 \overline{\Theta}^{E}{}_{\widetilde{G}\widetilde{H}} \left[\gamma_n\right]{}_{BD} \sigma^{BNDG\widetilde{H}}{}_{E} \\ &\quad -144\alpha_8 \Omega_{mnN} \left[\gamma^m\right]_{BD} \sigma^{MNBD} - 48\beta_1 \Theta^{L}{}_{an} \left[\gamma^a\right]^{A}{}_{L} \sigma^{M}{}_{A} \end{aligned} \tag{34}$$

We see that in this expression, one component of the spacetime curvature tensor, $\Omega_{l\ \ n\widetilde{N}}^{m}$, is related to the fermionic curvature tensors. We can therefore eliminate certain fermionic *components* of the spacetime curvature in favor of the fermionic *curvatures*. Since supercoordinate transformations mix the bosonic and fermionic parts of the spacetime curvature, the different curvatures must mix. Similar comments apply to the torsion, co-torsion and dilatation.

Note that eq.(34), a fermionic piece of the Einstein equation, is a Rarita-Schwinger type equation since the final term is proportional to

$$\begin{aligned} \left[\gamma^a\right]^{M}{}_{L} \Theta^{L}{}_{an} &\sim \gamma^a \left(\partial_a \psi_n - \partial_n \psi_a\right) \\ &\sim \not{\partial} \psi_n \end{aligned}$$

where we have supressed the spinor index on ψ and γ. Thus, the special cases with $\alpha_1 = \alpha_2 = \alpha_8 = 0$ give the massless Dirac equation for a spin-3/2 particle.

8 Conclusion

We have formulated the biconformal supergravity theory of the superconformal group, $SU(2,2|N)$, writing the most general even and odd parity action linear in the curvatures. The result is a 16-dim superspace with local Lorentz and dilatational symmetries. Finding the field equations for the $N = 1$ case illustrates how supercoordinate transformations will mix the fermionic and bosonic curvatures.

These results are important for several reasons:

1. The $N = 5$ case is a gauging of the $AdS_5 \times S^5$ background of the Maldacena conjecture, and therefore provides (at least) the linear curvature or low energy limit of the string theory of the conjecture.

2. There is a large class of actions which are linear in the supercurvatures without auxiliary fields, permitting (1) GR-type gravity theory and (2) Dirac-type and Rarita-Schwinger-type spinor equations.

3. Supersymmetrization introduces matter systematically into biconformal space.

4. Our use of Cartan's group manifold methods gives superspace automatically.

Two of these points merit further discussion. We begin by comparing the class of curvature-linear actions we have written with the actions used by previous authors. Then we comment briefly on the relationship between the Maldacena conjecture and $SU(2,2|5)$ biconformal supergravity.

Numerous papers ([23], [28]-[36]) have examined properties of conformal supergravity in four dimensions. With the exception of [30] and [23], all of these authors use actions quadratic in the curvatures, typically making use of the MacDowell-Mansouri [26] approach.

For example, the MacDowell-Mansouri approach is used by Crispim-Romao, Ferber and Freund [30], who write a curvature squared action

$$\begin{aligned} A_1 &= \int d^4x \varepsilon^{\mu\nu\rho\sigma} R^A_{\mu\nu} R^B_{\rho\sigma} M_{AB} \\ &= \int d^4x \varepsilon^{\mu\nu\rho\sigma} R^{ab}{}_{\mu\nu} R^{cd}{}_{\rho\sigma} \varepsilon_{abcd} \end{aligned}$$

where the indices $A, B = 1, \ldots, 15$ range over all generators of the conformal group, including special conformal transformations. In this approach, the curvature is expanded in terms of a reduced symmetry group and the leading quadratic term becomes topological. The action is then essentially the Einstein-Hilbert action plus a cosmological term. These authors also consider the case of only Lorentz and dilatational gauge fields, with a linear curvature action and linear torsion, coupled to an auxiliary tensor ψ^{AB}, all in superspace:

$$A_2 = \int d^{4+4N} z \det\left(e_M{}^A\right)\left((-1)^{b+bc} \psi^{CA} R_{ABC}{}^B + (-1)^a \mu D_A \psi^{BC} T_{CB}{}^A\right)$$

They modify this, replacing ψ^{AB} with two copies of a vectorial superfield χ^A to eliminate a second derivative term.

Ferrara, Kaku, Townsend and van Nieuwenhuizen ([32]-[36]) begin with the same MacDowell-Mansouri action, A_1, including torsion, co-torsion, dilatation and internal $U(1)$ quadratic terms:

$$
\begin{aligned}
I &= \int d^4x \varepsilon^{\mu\nu\rho\sigma} \Big(\alpha R^{ab}{}_{\mu\nu} R^{cd}{}_{\rho\sigma} \varepsilon_{abcd} \\
&\quad + \beta R^{\alpha}_{\mu\nu}(Q) (\gamma_5 C)_{\alpha\beta} R^{\beta}_{\rho\sigma}(S) + \gamma R_{\mu\nu}(D) R_{\rho\sigma}(A) \Big)
\end{aligned}
$$

In addition, they demand vanishing torsion and self dual gravitino field,

$$
\begin{aligned}
0 &= R^{a}_{\mu\nu}(P) \\
0 &= R^{n}_{\mu\nu}(Q) + \frac{1}{2}\gamma_5 \tilde{R}^{n}_{\mu\nu}(Q)
\end{aligned}
$$

When the constraints are substituted, I develops a term linear in the curvature and also terms built from additional fields. In [35], the 4-dim action is shown to be invariant under all of the remaining superconformal transformations.

In all of these papers except [30], the integrals are four dimensional, so the local gauge group is the co-Poincaré group. The gauge transformations therefore mix essentially all of the curvatures. This is in sharp constrast to our action, since the group quotient method requires only local Lorentz and local dilatational invariance.

Only [30] and [23] claim actions which are linear in the curvatures. This is accomplished using the auxiliary H-invariant tensors that exist by virtue of the reduced local symmetry (from co-Poincaré to homothetic) combined with the demand that $R^A = 0$ solve the field equations. It seems likely that these fields may be derived from the biconformal approach by integrating over the extra 4 coordinates.

Finally, the classes of biconformal action we present in eqs.(33) and (43) are constructed purely from the geometry without auxiliary fields. The supersymmetry is carried entirely by supercoordinate transformations in the underlying 16-dim superspace.

Next, we comment on conformal supergravity as the low energy limit of string theory on $AdS_5 \times S^5$. Since string theory is free of ghosts while most conformal gauge theories are not, such a relationship might be thought impossible. However, our linear curvature action removes this obstacle, and the issue must be examined in further detail. Here we discuss some features of such a possible correspondence.

Normally, gravitational gauge theories start with the symmetry of a highly symmetric space. Gauging then leads to a class of geometries closely related to the first. For example, the gauge theory of the Poincaré symmetry of Minkowski spacetime leads to the class of pseudo-Riemannian spacetimes. Each of these spacetimes has a copy of the original Minkowski space as the tangent space at each point. The connection is then clear: the gauge theory is a perturbation, or deformation, of the original space.

With this in mind, the standard gaugings of the superconformal group are expected to give spaces that are locally $AdS_5 \times S^5$. Indeed, the internal S^5 is not disturbed in these models – the gauged spacetimes retain a copy of S^5 at each point. The AdS_5 becomes curved in a way that depends on the field content.

Biconformal gauging is different from the standard gauging, because it doubles the dimension of the bosonic base space. Therefore, instead of a 10 -dim generalization of $AdS_5 \times S^5$, the gauging leads to a superspace with $(8+5)$-dimensional bosonic sector. The question naturally arises, what is this space? Clearly, 5 dimensions reflect the $SU(5)$ internal symmetry. To understand the meaning of the remaining 8 dimensions, consider the bosonic biconformal spaces studied in [39] and [40]. In these cases, biconformal space is found to have symplectic structure relating coordinates with opposite scaling dimension (note the similarity to the U coordinate with dimensions of mass in Maldacena [1]). The interpretation of biconformal space as a generalization of phase space has proved quite successful. Indeed, applying the technique to a conformally invariant generalization of Newton's second law [43] produces Hamiltonian dynamics as a gauge theory. Therefore, despite the increased dimension, we still expect the biconformal supergravity theory to describe curved $AdS_5 \times S^5$ on a submanifold, with the remaining dimensions providing momentum information. The exact character of this additional momentum information will be the subject of further study.

The introduction of string into the $AdS_5 \times S^5$ background poses another problem. How does string move in the biconformal superspace? The only previous study of matter in biconformal space [44] shows that scalar fields which are *a priori* dependent on all $2n$-dimensions of biconformal space reduce under the field equations to fields defined on n-dimensions satisfying the usual n-dim scalar field equations. We conjecture that string is similarly constrained by its equations of motion and interactions with the curvatures to its usual $AdS_5 \times S^5$ motions. Once again, further study is required before we have a complete answer

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Appendix I: Superconformal generators and Lie algebra

Let a generic element of $su(2,2|N)$ be written as

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where

$$\begin{aligned} QA + A^\dagger Q &= 0 & (35) \\ C &= -P^{-1}B^\dagger Q & (36) \\ PD + D^\dagger P &= 0 & (37) \end{aligned}$$

Identifying A-type conformal generators with $B = C = D = 0$, G-type fermionic generators with $A = D = 0$, and D-type generators with $A = B = C = 0$, we may choose the A-type generators as in eqs.(11-14), the D-type as

$$\begin{aligned} \left[D_R^{\alpha\beta}\right]^\mu_{\ \ \nu} &= i\left(P^{\mu\alpha}\delta^\beta_\nu - P^{\mu\beta}\delta^\alpha_\nu\right) \\ \left[D_I^{\alpha\beta}\right]^\mu_{\ \ \nu} &= P^{\mu\alpha}\delta^\beta_\nu + P^{\mu\beta}\delta^\alpha_\nu \end{aligned}$$

where we have chosen,

$$P_{\alpha\beta} = \begin{pmatrix} i1 & \\ & i1 \end{pmatrix}$$

and the fermionic generators as

$$\left[G_A^{\alpha-}\right] = \begin{pmatrix} & \frac{1}{2}\delta^\alpha_\beta \left[1+\gamma_5\right]^B_{\ \ A} \\ \frac{1}{2}P^{\alpha\beta}Q_{BC}\left[1-\gamma_5\right]^B_{\ \ A} & \end{pmatrix}$$

and

$$\left[G_A^{\alpha+}\right] = \begin{pmatrix} & \frac{1}{2}\delta^\alpha_\beta \left[1-\gamma_5\right]^B_{\ \ A} \\ \frac{1}{2}P^{\alpha\beta}Q_{BC}\left[1+\gamma_5\right]^B_{\ \ A} & \end{pmatrix}$$

where $\alpha, \beta = 1...N$ and $A, B = 1...4$.

The Lie algebra is as follows. For the symplectic (A) sector one finds the Lie algebra of the conformal group,

$$\begin{aligned} \left[M^a_{\ \ b}, M^c_{\ \ d}\right] &= \frac{1}{2}\left(\delta^c_b\delta^a_e\delta^f_d - \eta^{ac}\eta_{be}\delta^f_d - \eta_{bd}\eta^{cf}\delta^a_e + \eta_{be}\eta^{cf}\delta^a_d\right. \\ &\quad \left. -\eta^{af}\eta_{de}\delta^c_b + \eta^{ac}\eta_{de}\delta^f_b + \eta_{bd}\eta^{af}\delta^c_e - \delta^f_b\delta^c_e\delta^a_d\right) M^e_{\ \ f} \\ \left[M^a_{\ \ b}, P_c\right] &= \eta_{bc}\eta^{ad}P_d - \delta^a_c P_b = -2\Delta^{ad}_{cb}P_d \\ \left[M^a_{\ \ b}, K^c\right] &= \delta^c_b K^a - \eta_{bd}\eta^{ac}K^d = 2\Delta^{ac}_{db}K^d \end{aligned}$$

$$\left[P_a, K^b\right] = -2\Delta^{bd}_{ca} M^c{}_d - 2\delta^b_a D \qquad [D, P_a] = -P_a$$
$$[K^a, P_b] = 2\Delta^{ac}_{db} M^d{}_c + 2\delta^a_b D \qquad [D, K^a] = K^a$$

The unitary D-sector has the commutation relations

$$\begin{aligned}
\left[D^{\alpha\beta}_R, D^{\mu\nu}_R\right] &= \frac{1}{2}\Big(iP^{\beta\mu}\delta^\alpha_\rho\delta^\nu_\sigma - iP^{\beta\nu}\delta^\alpha_\rho\delta^\mu_\sigma - iP^{\alpha\mu}\delta^\beta_\rho\delta^\nu_\sigma + iP^{\alpha\nu}\delta^\beta_\rho\delta^\mu_\sigma \\
&\quad -iP^{\beta\mu}\delta^\alpha_\sigma\delta^\nu_\rho + iP^{\beta\nu}\delta^\alpha_\sigma\delta^\mu_\rho + iP^{\alpha\mu}\delta^\beta_\sigma\delta^\nu_\rho - iP^{\alpha\nu}\delta^\beta_\sigma\delta^\mu_\rho\Big) D^{\rho\sigma}_R \\
\left[D^{\alpha\beta}_R, D^{\mu\nu}_I\right] &= \frac{1}{2}\Big(iP^{\beta\mu}\delta^\alpha_\rho\delta^\nu_\sigma + iP^{\beta\nu}\delta^\alpha_\rho\delta^\mu_\sigma - iP^{\alpha\mu}\delta^\beta_\rho\delta^\nu_\sigma - iP^{\alpha\nu}\delta^\beta_\rho\delta^\mu_\sigma \\
&\quad +iP^{\beta\mu}\delta^\alpha_\sigma\delta^\nu_\rho + iP^{\beta\nu}\delta^\alpha_\sigma\delta^\mu_\rho - iP^{\alpha\mu}\delta^\beta_\sigma\delta^\nu_\rho - iP^{\alpha\nu}\delta^\beta_\sigma\delta^\mu_\rho\Big) D^{\rho\sigma}_I \\
\left[D^{\alpha\beta}_I, D^{\mu\nu}_R\right] &= \frac{1}{2}\Big(iP^{\beta\mu}\delta^\alpha_\rho\delta^\nu_\sigma - iP^{\beta\nu}\delta^\alpha_\rho\delta^\mu_\sigma + iP^{\alpha\mu}\delta^\beta_\rho\delta^\nu_\sigma - iP^{\alpha\nu}\delta^\beta_\rho\delta^\mu_\sigma \\
&\quad +iP^{\beta\mu}\delta^\alpha_\sigma\delta^\nu_\rho - iP^{\beta\nu}\delta^\alpha_\sigma\delta^\mu_\rho + iP^{\alpha\mu}\delta^\beta_\sigma\delta^\nu_\rho - iP^{\alpha\nu}\delta^\beta_\sigma\delta^\mu_\rho\Big) D^{\rho\sigma}_I \\
\left[D^{\alpha\beta}_I, D^{\mu\nu}_I\right] &= \frac{1}{2}\Big(-iP^{\beta\mu}\delta^\alpha_\rho\delta^\nu_\sigma - iP^{\beta\nu}\delta^\alpha_\rho\delta^\mu_\sigma - iP^{\alpha\mu}\delta^\beta_\rho\delta^\nu_\sigma - iP^{\alpha\nu}\delta^\beta_\rho\delta^\mu_\sigma \\
&\quad +iP^{\beta\mu}\delta^\alpha_\sigma\delta^\nu_\rho + iP^{\beta\nu}\delta^\alpha_\sigma\delta^\mu_\rho + iP^{\alpha\mu}\delta^\beta_\sigma\delta^\nu_\rho + iP^{\alpha\nu}\delta^\beta_\sigma\delta^\mu_\rho\Big) D^{\rho\sigma}_R
\end{aligned}$$

Finally, the fermionic generators satisfy

$$\left[D, G^{\alpha+}_A\right] = \tfrac{1}{2} G^{\alpha+}_A \qquad \left[P_a, G^{\alpha+}_A\right] = \delta^\alpha_\mu \left[\gamma_a\right]^C{}_A \left[G^{\mu-}_C\right] \qquad \left[P_a, G^{\alpha-}_A\right] = 0$$
$$\left[D, G^{\alpha-}_A\right] = -\tfrac{1}{2} G^{\alpha-}_A \qquad \left[K^a, G^{\alpha-}_A\right] = \delta^\alpha_\mu \left[\gamma^a\right]^C{}_A \left[G^{\mu+}_C\right] \qquad \left[K^a, G^{\alpha+}_A\right] = 0$$

$$\left[M^a{}_b, G^{\alpha+}_A\right] = \delta^\alpha_\mu \left[\sigma^a{}_b\right]^C{}_A \left[G^{\mu+}_C\right]$$
$$\left[M^a{}_b, G^{\alpha-}_A\right] = \delta^\alpha_\mu \left[\sigma^a{}_b\right]^C{}_A \left[G^{\mu-}_C\right]$$

with the conformal generators,

$$\begin{aligned}
\left[E, G^{\alpha+}_A\right] &= -\frac{i\,(N-4)}{4N} \left[\gamma_5\right]^C{}_A \delta^\alpha_\lambda \left[G^{\lambda+}_C\right] \\
\left[E, G^{\alpha-}_A\right] &= +\frac{i\,(N-4)}{4N} \left[\gamma_5\right]^C{}_A \delta^\alpha_\lambda \left[G^{\lambda-}_C\right]
\end{aligned}$$

with the E,

$$\begin{aligned}
\left[D^{\mu\nu}_R, G^{\alpha+}_A\right] &= \left[\gamma_5\right]^C{}_A \left(P^{\alpha\mu}\delta^\nu_\lambda + P^{\alpha\nu}\delta^\mu_\lambda\right) \left[G^{\lambda+}_C\right] \\
\left[D^{\mu\nu}_R, G^{\alpha-}_A\right] &= -\left[\gamma_5\right]^C{}_A \left(P^{\alpha\mu}\delta^\nu_\lambda + P^{\alpha\nu}\delta^\mu_\lambda\right) \left[G^{\lambda-}_C\right] \\
\left[D^{\mu\nu}_I, G^{\alpha+}_A\right] &= -i\delta^C_A \left(P^{\alpha\mu}\delta^\nu_\lambda - P^{\alpha\nu}\delta^\mu_\lambda\right) \left[G^{\lambda+}_C\right] \\
\left[D^{\mu\nu}_I, G^{\alpha-}_A\right] &= i\delta^C_A \left(P^{\alpha\mu}\delta^\nu_\lambda - P^{\alpha\nu}\delta^\mu_\lambda\right) \left[G^{\lambda-}_C\right]
\end{aligned}$$

with the $U(N)$ generators, and

$$\begin{aligned}\left\{G_A^{\alpha+}, G_B^{\beta-}\right\} &= \Big(\frac{1}{2}P^{\alpha\beta}\left(-Q_{AB}D + [\sigma^a{}_b]_{AB}\, M^b{}_a\right) \\ &\quad -\frac{i}{2}Q_{AB}D_R^{\alpha\beta} + \frac{1}{2}\left[\gamma_5\right]_{AB} D_I^{\alpha\beta} + \left[\gamma_5\right]^C{}_B Q_{CA}P^{\alpha\beta}E\Big)\end{aligned}$$

$$\left\{G_A^{\alpha+}, G_B^{\beta+}\right\} = P^{\alpha\beta}\left[\gamma_a\right]_{(AB)}\left[K^a\right]^C{}_D \quad \left\{G_A^{\alpha-}, G_B^{\beta-}\right\} = P^{\alpha\beta}\left[\gamma^a\right]_{(AB)}\left[P_a\right]^C{}_D$$

with one another.

Appendix II:

In this Appendix we prove that there are three possible choices of supersymmetric extension of the local symmetry H which are consistent with the following two properties:

1. The A-sector of the bosonic part of H must consist of Lorentz transformations and dilatations.
2. The D-sector of the bosonic part of H must be $U\left(N\right)$, thereby retaining the entire D-sector as a local internal symmetry.

The proof is as follows.

As noted above, any subalgebra that preserves

$$\alpha Q + \beta Q\gamma_5$$

for any fixed $\alpha, \beta \neq 0$, is homothetic. We generalize this condition to an arbitrary element of $su(2,2|N)$ and impose both conditions 1 and 2. Thus, the subset of superconformal generators leaving any matrix $\mathcal{M}$ invariant generates a subgroup of $SU(2,2|N)$. We want this subgroup to be our isotropy, H. Let the invariant matrix $\mathcal{M}$ and a generic generator T be given by

$$\begin{aligned}\mathcal{M} &= \begin{pmatrix} \alpha Q + \beta Q\gamma_5 & R \\ S & J \end{pmatrix} \\ T &= \begin{pmatrix} A & B \\ C & D \end{pmatrix}\end{aligned}$$

Then invariance,

$$\mathcal{M}T + T^{\ddagger}\mathcal{M} = 0$$

for all T gives

$$
\begin{aligned}
0 &= (\alpha Q + \beta Q \gamma_5) A + RC + A^{\dagger} (\alpha Q + \beta Q \gamma_5) - C^{\dagger} S && (38) \\
0 &= AS + JC + B^{\dagger} (\alpha Q + \beta Q \gamma_5) + D^{\dagger} S && (39) \\
0 &= (\alpha Q + \beta Q \gamma_5) B + RD + A^{\dagger} R - C^{\dagger} J && (40) \\
0 &= SB + JD + B^{\dagger} R + D^{\dagger} J && (41)
\end{aligned}
$$

Applying condition 2 with $A = B = C = 0$ we see that

$$
\begin{aligned}
0 &= D^{\dagger} S \\
0 &= RD \\
0 &= JD + D^{\dagger} J
\end{aligned}
$$

Since the D-sector generators span all N-dim matrices, the first two equations require $R = S = 0$, while the third will constrain D unless J is proportional to the unitary metric P. With $R = S = 0$, the first constraint, eq.(38), is satisfied for homothetic generators A, while (41) is satisfied in agreement with Property 2 if and only if $J = \lambda P$. Therefore, the invariance equations reduce to

$$0 = (\alpha Q + \beta Q \gamma_5) B - \lambda C^{\dagger} P \qquad (42)$$

and its adjoint. We are therefore left with conditions on B and C. Replacing $C^{\dagger} = QBP^{-1}$ in eq(42), we have

$$
\begin{aligned}
0 &= (\alpha Q + \beta Q \gamma_5) B - \lambda Q B \\
&= ((\alpha - \lambda) Q + \beta Q \gamma_5) B
\end{aligned}
$$

Now, since B is comprised of N spinors, we can treat this as a matrix equation for each B. Nonzero B requires

$$
\begin{aligned}
0 &= \det ((\alpha - \lambda) Q + \beta Q \gamma_5) \\
&= \left(\beta^2 - (\alpha - \lambda)^2\right)^2
\end{aligned}
$$

so there are three possible cases:

$$
\begin{aligned}
(\alpha - \lambda) &\neq \beta \\
(\alpha - \lambda) &= \beta \neq 0 \\
(\alpha - \lambda) &= -\beta \neq 0
\end{aligned}
$$

The consequences of each of these conditions are described in the text.

Appendix III: Odd parity action

The most general, linear, odd parity action is given by

$$\begin{aligned} S &= \int \Big\{ \left(\alpha_1 \mathbf{\Omega}^a_b \left[\sigma^b_a\right]_{AB} + (\alpha_2 \mathbf{\Omega} + \alpha_3 \mathbf{A}) \left[\gamma_5\right]_{AB} \right) \chi^A_\alpha \psi^B_\beta P^{\alpha\beta} \left(\chi\psi\right)^2 \phi_b \\ & + \left(\alpha_4 \mathbf{\Omega}^a_b + (\alpha_5 \ \mathbf{\Omega} + \alpha_6 \mathbf{A}) \, \delta^a_b\right) \varepsilon_{acde} \varepsilon^{bfgh} \omega_{fgh} \omega^{cde} \phi_f \\ & + \alpha_7 (\mathbf{\Omega}^m \left[\gamma_5 \gamma_m\right]_{BD} \ \chi^B_\alpha \chi^D_\beta - \mathbf{\Omega}_m \left[\gamma_5 \gamma^m\right]_{BD} \psi^B_\alpha \psi^D_\beta) P^{\alpha\beta} \left(\chi\psi\right)^2 \phi_b + \alpha_8 \Phi \\ & + \beta \left(\mathbf{\Theta}^M_\alpha \left[\gamma_5 \gamma^a\right]^A{}_M \omega_b \psi^B_\beta - \overline{\mathbf{\Theta}}^M_\alpha \left[\gamma_5 \gamma_b\right]^A{}_M \omega^a \chi^B_\beta \right) \\ & \times Q_{AB} P^{\alpha\beta} \left(\chi\psi\right)^3 \varepsilon_{acde} \varepsilon^{bfgh} \omega_{fgh} \omega^{cde} \\ & + \lambda_1 \left(\mathbf{\Pi}^R_{\alpha\lambda} + \mathbf{\Pi}^I_{\alpha\lambda} \right) \chi^A_\beta \psi^B_\rho P^{\lambda\rho} P^{\alpha\beta} Q_{AB} \left(\chi\psi\right)^3 \delta^a_b \varepsilon_{acde} \varepsilon^{bfgh} \ \omega_{fgh} \omega^{cde} \phi_f \\ & + \lambda_2 \left(\mathbf{\Pi}^R_{\alpha\lambda} + \mathbf{\Pi}^I_{\alpha\lambda} \right) \left[\gamma_5\right]_{BD} \psi^B_\rho \chi^D_\beta P^{\rho\lambda} P^{\alpha\beta} \left(\chi\psi\right)^2 \phi_b \Big\} \end{aligned} \tag{43}$$

As with the even parity case, there are 11 arbitrary parameters.

Appendix IV: Field equations

The full set of field equations from the $N = 1$, even parity action is as follows.

From the ω variation we have,

$$\begin{aligned} 0 &= -3\alpha_2 \overline{\Theta}^A{}_{M\widetilde{H}} \sigma^{NM\widetilde{H}}{}_A - \frac{3}{2} \alpha_2 \Theta^B{}_{\widetilde{M}\widetilde{H}} Q_{AB} \sigma^{AN\widetilde{M}\widetilde{H}} \\ & + 576 \alpha_2 \Omega_b{}^b{}_{\widetilde{M}} \sigma^{N\widetilde{M}} + 576 \alpha_2 \Omega^a{}_{a\widetilde{M}} \sigma^{N\widetilde{M}} \\ & + 144 \alpha_5 \overline{\Theta}^{Am}{}_m \sigma^N{}_A \end{aligned}$$

and

$$\begin{aligned} 0 &= 576 \alpha_2 \Omega_{nN\widetilde{M}} \sigma^{N\widetilde{M}} + 72 \alpha_5 \Omega_f{}^m{}_l \delta^{lf}_{nm} - 36 \alpha_5 \Omega^c{}_{nc} \\ & - 144 \alpha_5 \overline{\Theta}^A{}_{nN} \sigma^N{}_A - 144 \alpha_5 \Theta^B{}_{n\widetilde{M}} \sigma^{A\widetilde{M}} Q_{AB} \end{aligned}$$

together with the complex conjugates of these expressions.

The ω^a_b variation gives

$$0 = \alpha_1 \left[\sigma^n_m\right]_{AB} \Big(-\overline{\Theta}^A{}_{M\widetilde{H}} \sigma^{NMB\widetilde{H}} - \frac{1}{2} \Theta^B{}_{\widetilde{M}\widetilde{H}} \sigma^{AN\widetilde{M}\widetilde{H}}$$

$$
\begin{aligned}
&-2\overline{\Theta}^{C}{}_{M\widetilde{H}}\sigma^{ANM\,B\widetilde{H}}{}_{C}-\Theta^{D}{}_{\widetilde{M}\widetilde{H}}\sigma^{ACN\,B\widetilde{M}\widetilde{H}}Q_{CD}\\
&+576\Omega_{b}{}^{b}{}_{\widetilde{N}}\sigma^{AN\,B\widetilde{N}}+576\Omega^{a}{}_{a\widetilde{N}}\sigma^{AN\,B\widetilde{N}}\Big)\\
&-144\alpha_4\left[\sigma^{n}_{m}\right]_{AB}\overline{\Theta}^{Cn}{}_{m}\sigma^{N}{}_{C}
\end{aligned}
$$

and

$$
\begin{aligned}
0 \;=\;& 576\alpha_1\left[\sigma^{n}_{m}\right]_{AB}\Omega_{pM\widetilde{H}}\sigma^{AM\,B\widetilde{H}}+72\alpha_4\Omega_{f}{}^{q}{}_{m}\delta^{nf}_{pq}\\
&-36\alpha_4\Omega^{c}{}_{mc}\delta^{n}_{p}-144\alpha_4\overline{\Theta}^{C}{}_{mN}\sigma^{N}{}_{C}\delta^{n}_{p}\\
&-144\alpha_4\Theta^{D}{}_{m\widetilde{N}}\sigma^{C\,\widetilde{N}}Q_{CD}\delta^{n}_{p}
\end{aligned}
$$

and the complex conjugates of these expressions.

For the α variation, the field equations are

$$
\begin{aligned}
0 \;=\;& -\frac{3}{2}\alpha_3\overline{\Theta}^{A}{}_{GH}\sigma^{GH\,\widetilde{E}}{}_{A}-3\alpha_3\Theta^{B}{}_{G\widetilde{H}}\sigma^{GA\,\widetilde{E}\widetilde{H}}Q_{AB}\\
&+576\alpha_3\Omega_{b}{}^{b}{}_{G}\sigma^{G\,\widetilde{E}}+576\alpha_3\Omega^{m}{}_{mG}\sigma^{G\,\widetilde{E}}\\
&+144\alpha_6\Theta^{Bm}{}_{m}\sigma^{A\,\widetilde{E}}Q_{AB}
\end{aligned}
$$

and

$$
\begin{aligned}
0 \;=\;& 576\alpha_3\Omega_{mG\widetilde{H}}\sigma^{G\,\widetilde{H}}+72\alpha_6\Omega_{f}{}^{n}{}_{b}\delta^{bf}_{mn}\\
&-36\alpha_6\Omega^{c}{}_{mc}-144\alpha_6\overline{\Theta}^{A}{}_{mE}\sigma^{E\,B}Q_{AB}\\
&+144\alpha_6\Theta^{B}{}_{m\widetilde{E}}\sigma^{A\,\widetilde{E}}Q_{AB}
\end{aligned}
$$

together with conjugate equations.

Variation of ω^a leads to

$$
\begin{aligned}
0 \;=\;& 144\alpha_1\Omega^{m}_{l}{}_{M\widetilde{N}}\left[\sigma^{l}_{m}\right]_{AB}\sigma^{AM\,B\widetilde{N}}\delta^{p}_{n}+144\alpha_2\Omega_{M\widetilde{N}}\sigma^{M\,\widetilde{N}}\delta^{p}_{n}\\
&+144\alpha_3A_{M\widetilde{N}}\sigma^{M\,\widetilde{N}}\delta^{p}_{n}-108\alpha_4\delta^{p}_{n}+72\alpha_5\delta^{p}_{n}+24\alpha_4\Omega^{ab}_{b}{}_{l}\delta^{pl}_{an}\\
&+24\alpha_5\Omega^{a}{}_{l}\delta^{pl}_{an}+24\alpha_6A^{a}{}_{l}\delta^{pl}_{an}+144\alpha_7\delta^{p}_{n}\\
&+288\alpha_8\Omega^{p}{}_{\widetilde{M}\widetilde{H}}\left[\gamma_n\right]_{BD}\sigma^{BD\,\widetilde{M}\widetilde{H}}+72\alpha_8\Omega^{m}{}_{\widetilde{M}\widetilde{N}}\left[\gamma_m\right]_{BD}\sigma^{BD\,\widetilde{M}\widetilde{N}}\delta^{p}_{n}\\
&-72\alpha_8\Omega_{mMN}\left[\gamma^{m}\right]_{BD}\sigma^{MN\,BD}\delta^{p}_{n}-144\beta_1\left[\gamma^{p}\right]^{A}{}_{M}\left[\gamma_n\right]^{M}{}_{N}\sigma^{N}{}_{A}\\
&+96\beta_1\Theta^{M}{}_{mN}\left[\gamma^{a}\right]^{A}{}_{M}\delta^{pm}_{an}\sigma^{N}{}_{A}-36\beta_1\overline{\Theta}^{Mm}{}_{\widetilde{N}}\left[\gamma_m\right]^{A}{}_{M}\sigma^{B\,\widetilde{N}}Q_{AB}\delta^{p}_{n}
\end{aligned}
$$

and

$$\begin{aligned}
0 &= 144\alpha_1\Omega_l^m{}_{n\widetilde{N}}\left[\sigma_m^l\right]_{AB}\sigma^{AM\,B\widetilde{N}} - 144\alpha_2\Omega_{n\widetilde{N}}\sigma^{M\,\widetilde{N}} - 144\alpha_3 A_{n\widetilde{N}}\sigma^{M\,\widetilde{N}} \\
&\quad -2\alpha_8\overline{\Theta}^A{}_{G\widetilde{H}}\left[\gamma_n\right]_{BD}\sigma^{MDG\widetilde{H}} - \alpha_8\overline{\Theta}^E{}_{\widetilde{G}\widetilde{H}}\left[\gamma_n\right]_{BD}\sigma^{BNDG\widetilde{H}}{}_E \\
&\quad -144\alpha_8\Omega_{mnN}\left[\gamma^m\right]_{BD}\sigma^{MN\,BD} - 48\beta_1\Theta^L{}_{an}\left[\gamma^a\right]^A{}_L\sigma^M{}_A
\end{aligned}$$

and

$$\begin{aligned}
0 &= 144\alpha_1\Omega_l^m{}_{nN}\left[\sigma_m^l\right]_{AB}\sigma^{AN\,B\widetilde{M}} - 144\alpha_2\Omega_{nN}\sigma^{N\,\widetilde{M}}\delta_n^p - 144\alpha_3 A_{nN}\sigma^{N\,\widetilde{M}} \\
&\quad -2\alpha_8\overline{\Theta}^B{}_{L\widetilde{H}}\left[\gamma_n\right]_{BD}\sigma^{LD\,\widetilde{M}\widetilde{H}} - 2\alpha_8\overline{\Theta}^E{}_{L\widetilde{H}}\left[\gamma_n\right]_{BD}\sigma^{BDL\,\widetilde{M}\widetilde{H}}{}_E \\
&\quad -\alpha_8 Q_{EF}\Theta^F{}_{\widetilde{L}\widetilde{H}}\left[\gamma_n\right]_{BD}\sigma^{BDE\,\widetilde{H}\widetilde{L}}{}_A + 576\alpha_8\Omega_b{}^b{}_{\widetilde{L}}\left[\gamma_n\right]_{BD}\sigma^{BD\,\widetilde{M}\widetilde{L}} \\
&\quad +576\alpha_8\Omega^a{}_{a\widetilde{H}}\left[\gamma_n\right]_{BD}\sigma^{BD\,\widetilde{M}\widetilde{H}} + 144\alpha_8\Omega^m{}_{n\widetilde{N}}\left[\gamma_m\right]_{BD}\sigma^{BD\,\widetilde{M}\widetilde{H}} \\
&\quad +36\beta_1\overline{\Theta}^{Lb}{}_n\left[\gamma_b\right]_{BL}\sigma^{B\,\widetilde{M}}
\end{aligned}$$

and

$$\begin{aligned}
0 &= -12\alpha_4\Omega^a_{pan} - 12\alpha_5\Omega_{pn} - 12\alpha_6 A_{pn} + 288\alpha_8\Omega_{p\widetilde{M}\widetilde{N}}\left[\gamma_n\right]_{BD}\sigma^{BD\,\widetilde{M}\widetilde{N}} \\
&\quad +36\beta_1\overline{\Theta}^M{}_{n\widetilde{N}}\left[\gamma_p\right]^A{}_M Q_{AB}\sigma^{B\,\widetilde{N}}
\end{aligned}$$

The complex conjugates of these equations are obtained through the ω_a variation.

Finally, variation of ψ^A gives,

$$\begin{aligned}
0 &= \frac{3}{2}\alpha_1\Omega^a_{b\widetilde{M}\widetilde{N}}\left[\sigma_a^b\right]_{AN}\sigma^{AL\,\widetilde{M}\widetilde{N}} + \frac{3}{2}\alpha_2\Omega_{\widetilde{M}\widetilde{N}}Q_{AN}\sigma^{AL\,\widetilde{M}\widetilde{N}} \\
&\quad +\frac{3}{2}\alpha_3 Q_{AN}A_{\widetilde{M}\widetilde{N}}\sigma^{AL\,\widetilde{M}\widetilde{N}} - 2\alpha_8\Omega_{mP\widetilde{H}}\left[\gamma^m\right]_{ND}\sigma^{PL\,D\widetilde{H}} \\
&\quad -2\alpha_8\Omega_{mP\widetilde{H}}\left[\gamma^m\right]_{BD}\sigma^{PLEG\,BD\widetilde{H}}Q_{EN} + 144\beta_1\left[\gamma^m\right]^A{}_N\Theta^B{}_{m\widetilde{H}}\sigma^{L\,\widetilde{H}}Q_{AB} \\
&\quad +432\beta_1\left[\gamma^m\right]^A{}_N\Theta^D{}_{m\widetilde{H}}Q_{AB}Q_{CD}\sigma^{CL\,B\widetilde{H}} + 144\beta_1\left[\gamma^n\right]_{BN}\Omega_b{}^b{}_n\sigma^{L\,B} \\
&\quad +144\beta_1\left[\gamma^a\right]^A{}_N\Omega^c{}_{ac}\sigma^L{}_A + 144\beta_1\left[\gamma^m\right]_{NM}\Theta^M{}_{m\widetilde{H}}\sigma^{L\,\widetilde{H}} \\
&\quad -432\beta_1\left[\gamma^m\right]^A{}_M\Theta^M{}_{m\widetilde{H}}\sigma^{LE\,\widetilde{H}}Q_{EN} - 432\beta_1\left[\gamma^m\right]_{BN}\overline{\Theta}^C{}_{mH}\sigma^{LHBD}Q_{CD}
\end{aligned}$$

and

$$0 = -144\alpha_4\Omega_m^{n\,m}{}_{\widetilde{N}}\sigma^{A\,\widetilde{N}}Q_{AN} - 144\alpha_5\Omega^n{}_{\widetilde{N}}\sigma^{A\,\widetilde{N}}Q_{AN}$$

$$
\begin{aligned}
&-144\alpha_6 A^n{}_{\widetilde{N}}\sigma^{A\widetilde{N}}Q_{AN} - 144\beta_1\left[\gamma^n\right]_{BN}\Theta^B{}_{L\widetilde{H}}\sigma^{L\widetilde{H}} \\
&-432\beta_1\left[\gamma^n\right]_{BN}\Theta^D{}_{L\widetilde{H}}Q_{CD}\sigma^{CLB\widetilde{H}} + 144\beta_1\Omega_b{}^b{}_L\left[\gamma^n\right]^A{}_N\sigma^L{}_A \\
&+288\beta_1\left[\gamma^a\right]^A{}_N\Omega^c{}_{lL}\sigma^L{}_A\delta^{nl}_{ac} + 144\beta_1\left[\gamma^n\right]_{NM}\Theta^M{}_{L\widetilde{H}}\sigma^{L\widetilde{H}} \\
&+576\beta_1\left[\gamma^n\right]^A{}_M\Theta^M{}_{L\widetilde{H}}\sigma^{LE}{}_AQ_{EN} + 432\beta_1\left[\gamma^n\right]_{BN}\overline{\Theta}^C{}_{LH}\sigma^{LHB}{}_C
\end{aligned}
$$

and

$$
\begin{aligned}
0 \;=\;& 144\alpha_4\Omega^m_{n\;m\widetilde{N}}\sigma^{A\widetilde{N}}Q_{AN} + 144\alpha_5\Omega_{n\widetilde{N}}\sigma^{A\widetilde{N}}Q_{AN} - 144\alpha A_{n\widetilde{N}}\sigma^{A\widetilde{N}}Q_{AN} \\
&-144\beta_1\left[\gamma^a\right]^A{}_N\Omega_{naL}\sigma^L{}_A - 432\beta_1\left[\gamma_n\right]_{BM}\overline{\Theta}^M{}_{\widetilde{L}\widetilde{N}}\sigma^{BE\widetilde{L}\widetilde{N}}Q_{EN}
\end{aligned}
$$

and

$$
\begin{aligned}
0 \;=\;& \left(2i\alpha_1\left[\sigma^n_m\right]_{AB}\left[\sigma^m_n\right]_{MN} - \frac{i}{2}\alpha_2 Q_{MN}Q_{AB} + \left[\gamma_5\right]_{MN}Q_{AB}\right)\sigma^{AMB\widetilde{N}} \\
&+3\alpha_1\left[\sigma^a_b\right]_{AN}\Omega^b_{aL\widetilde{M}}\sigma^{AL\widetilde{M}\widetilde{N}} + 3\alpha_2\Omega_{L\widetilde{M}}Q_{AN}\sigma^{AL\widetilde{M}\widetilde{N}} \\
&+3\alpha_3 A_{L\widetilde{M}}Q_{AN}\sigma^{AL\widetilde{M}\widetilde{N}} - 144\alpha_4\Omega^{n\;m}_{m\;\;\;n}\sigma^{A\widetilde{N}}Q_{AN} \\
&+144\alpha_5\Omega^n{}_n\sigma^{A\widetilde{N}}Q_{AN} - 144\alpha_6 A^n{}_n\sigma^{A\widetilde{N}}Q_{AN} \\
&+4\alpha_7 Q_{AN}\sigma^{A\widetilde{N}} + \alpha_8\frac{i}{2}\left(\left[K^m\right]_{L\widetilde{N}} + \left[K^m\right]_{\widetilde{N}L}\right)\left[\gamma_m\right]_{BD}\sigma^{BD\widetilde{L}\widetilde{N}} \\
&+\alpha_8\Omega^m{}_{\widetilde{L}\widetilde{H}}\left[\gamma_m\right]_{BD}Q_{EN}\sigma^{BDE\widetilde{L}\widetilde{H}\widetilde{N}} - \alpha_8\Omega_{mLM}\left[\gamma^m\right]_{ND}\sigma^{LMD\widetilde{N}} \\
&-\alpha_8\Omega_{mLM}\left[\gamma^m\right]_{BD}\sigma^{LMEBD\widetilde{N}}Q_{EN} + 144\beta_1\left[\gamma^m\right]^A{}_N\Theta^B{}_{mL}Q_{AB}\sigma^{L\widetilde{N}} \\
&+432\beta_1\left[\gamma^m\right]^A{}_N\Theta^D{}_{mL}Q_{AB}Q_{CD}\sigma^{CLB\widetilde{N}} \\
&-144\beta_1\left[\gamma^m\right]^A{}_M\Theta^M{}_{mL}Q_{AN}\sigma^{L\widetilde{N}} - 432\beta_1\left[\gamma^m\right]^A{}_M\Theta^M{}_{mL}Q_{EN}\sigma^{LE\widetilde{N}}{}_A \\
&-432\beta_1\left[\gamma_b\right]^A{}_M\overline{\Theta}^{Mb}{}_{\widetilde{L}}\sigma^{BE\widetilde{N}\widetilde{L}}Q_{EN}
\end{aligned}
$$

The complex conjugates of these equations are obtained through the χ^A variation.

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