Tutorial Series

The Chevalley Basis for a Split Semi-simple Lie Algebra

Synopsis

Let $\mathfrak{g}$ be a split, semi-simple real Lie algebra. This implies, in particular, that there exists a Cartan subalgebra for $\mathfrak{g}$ for which the associated root space decomposition of $\mathfrak{g}$ is real. For such Lie algebras there is a very special basis called the *Chevalley basis*. In this worksheet we calculate the Chevalley basis for the Lie algebra $sp(6, R)$ and illustrate the various properties of this basis.

Defining Properties of the Chevalley Basis

Let $n = \dim(\mathfrak{g})$, $r = \mathrm{rank}(\mathfrak{g})$ and $m = (n - r)/2$. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$. Let Δ be the associated set of roots, let $\Delta^+ = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ be the positive roots and let $\Delta_0 = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ be the simple roots. As usual, write the root space decomposition as

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta^+} R_\alpha + \sum_{\alpha \in \Delta^+} R_{-\alpha}.$$

The root spaces R_α are 1-dimensional subalgebras of $\mathfrak{g}$. Write $h_1, h_2, \ldots, h_r, x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m$ for the Chevalley basis. The following are the defining properties of a Chevalley basis.

Property 1. The vectors $h_1, h_2, \ldots, h_r$ are a basis for $\mathfrak{h}$, x_i spans R_{α_i} and y_i spans $R_{-\alpha_i}$. If α, β, and $\alpha + \beta$ are roots, write $[x_\alpha, x_\beta] = N_{\alpha, \beta}\, x_{\alpha+\beta}$, for some numbers $N_{\alpha, \beta}$.

Property 2. The *sl(2)* structure equations hold: $[x_i, y_i] = -h_i$, $[h_i, x_i] = 2\, x_i$, $[h_i, y_i] = -2\, y_i$, for $i = 1, 2, \ldots, r$.

Property 3. The structure equations $[h_i, x_j] = a_{ji}\, x_i$ hold, where a_{ij} is the Cartan matrix and i , j = 1, 2, ... , r .

Property 4. The mapping which sends $h_i \rightarrow -h_i$, $x_\ell \rightarrow y_\ell$ and $y_\ell \rightarrow x_\ell$ is a Lie algebra automorphism. This means that $N_{\alpha,\,\beta} = N_{-\alpha,\,-\beta}$.

Error, invalid operator parameter name

$y_\ell \rightarrow x_\ell$

Property 5. If the β string through α is $\alpha - q\beta, ..., \alpha + p\,\beta$, then $N_{\alpha,\,\beta} = \pm\,(q+1)$.

We remark that different authors choose different normalizations for the definition of the Chevalley basis. We have followed the normalizations in reference [1]. In [2], one finds that the Chevally basis satisfies $N_{\alpha,\,\beta} = -N_{-\alpha,\,-\beta}$.

1. The Chevalley basis for $sp(6, R)$

We begin by using the command SimpleLieAlgebraData to retrieve the structure equations for the 21-dimensional Lie algebra $sp(6, R)$. This is the Lie algebra of 6 *x* 6 matrices which are skew symmetric with respect to a given symplectic form. The basis used to generate the structure equations of sp(6, R) comes from the standard representation .

```
> with(DifferentialGeometry): with(LieAlgebras):
> LD := SimpleLieAlgebraData("sp(6, R)", sp6R):
```

Initialize this Lie algebra.

```
alg > DGsetup(LD);
```

Lie algebra: sp6R (2.1

For a basis of the Cartan subalgebra we take

```
sp6R > CSA := [e1, e5, e9];
```

$$CSA := [e1, e5, e9] \quad (2.2$$

The root space decomposition, the positive roots and the simple roots are:

```
sp6R > RSD := RootSpaceDecomposition(CSA);
```

$RSD := table([[1,0,-1]=e3, [-1,0,-1]=e18, [0,-2,0]=e19, [1,1,0]=e11, [0,0,-2]=e21, [0,0,2]=e15, [0,2,0]=e13, [-1,0,1]=e7, [-1,1,0]=e4, [2,0,0]=e10, [0,-1,1]=e8, [0,1,-1]=e6, [0,1,1]=e14, [-1,-1,0]=e17, [0,-1,-1]=e20, [-2,0,0]=e16, [1,-1,0]=e2, [1,0,1]=e12])$ (2.3

```
sp6R > PosRts := PositiveRoots(RSD);
```

$$PosRts := \left[\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\2\end{bmatrix}, \begin{bmatrix}0\\2\\0\end{bmatrix}, \begin{bmatrix}2\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right] \quad (2.4$$

```
chev > SimRts := SimpleRoots(PosRts);
```

$$SimRts := \left[\begin{bmatrix}0\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}\right] \quad (2.5$$

The algebra $sp(6, R)$ has rank 3 and is of root type C. With this data we can now compute the Chevalley basis.

```
sp6R > B := ChevalleyBasis(CSA, RSD, PosRts, algebratype = ["C", 3]);
```

$B := [e1 - e5, e5 - e9, e9, e2, e6, e15, e3, e14, e12, e13, e11, e10, -e4, -e8, -e21, -e7, -e20, -e18, -e19, -e17, -e16]$ (2.6

The new structure equations for $sp(6, R)$ are now calculated.

```
sp6R > newLD := LieAlgebraData(B, chev):
```

We label the basis elements in accordance with our description of the Chevalley basis given in the Synposis.

```
sp6R > DGsetup(newLD, '[h1, h2, h3, x1, x2, x3, x4, x5, x6, x7, x8, x9, y1, y2, y3, y4,
       y5, y6, y7, y8, y9]', [theta]);
```

Lie algebra: chev (2.7

2. Property 1. The Cartan Subalgebra and the Root Space Decomposition in the Chevalley Basis.

The Chevalley basis was computed in the last section. We see immediately from the the first 3 rows of the following multiplication table that the vectors h_1, h_2, h_3 define a Cartan subalgebra (the adjoint matrices are all diagonal).

```
chev > MultiplicationTable(chev, "LieTable", rows = [$1..3], columns = [$1..21]);
```

chev	$h1$	$h2$	$h3$	$x1$	$x2$	$x3$	$x4$	$x5$	$x6$	$x7$	$x8$	$x9$	$y1$	$y2$	$y3$	$y4$	$y5$	$y6$	
$h1$	0	0	0	$2\,x1$	$-x2$	0	$x4$	$-x5$	$x6$	$-2\,x7$	0	$2\,x9$	$-2\,y1$	$y2$	0	$-y4$	$y5$	$-y6$	
$h2$	0	0	0	$-x1$	$2\,x2$	$-2\,x3$	$x4$	0	$-x6$	$2\,x7$	$x8$	0	$y1$	$-2\,y2$	$2\,y3$	$-y4$	0	$y6$	$-$
$h3$	0	0	0	0	$-x2$	$2\,x3$	$-x4$	$x5$	$x6$	0	0	0	0	$y2$	$-2\,y3$	$y4$	$-y5$	$-y6$	

We can also verify that h_1, h_2, h_3 defines a Cartan subalgebra with the Query command.

```
chev > CSA1 := [h1, h2, h3]:
chev > Query(CSA1, "CartanSubalgebra");
```

$$true \tag{3.2}$$

The root space decomposition is

```
chev > RSD1 := RootSpaceDecomposition(CSA1);
```

$$RSD1 := table([[1, 0, -1] = y5, [2, -2, 0] = y7, [-2, 1, 0] = y1, [0, 1, 0] = x8, [-1, 0, 1] = x5, [2, 0, 0] = x9, [-1, 2, -1] = x2, [-1, 1, -1] = y6, [-2, 2, 0] = x7, [1, -1, 1] = x6, [0, -2, 2] = x3, [1, -2, 1] = y2, [-1, -1, 1] = y4, [0, -1, 0] = y8, [0, 2, -2] = y3, [-2, 0, 0] = y9, [1, 1, -1] = x4, [2, -1, 0] = x1]) \tag{3.3}$$

and so we see that the root spaces are spanned precisely by the vectors x_i and y_i. The simple roots are, by construction of the Chevalley basis, the roots for the vectors x_1, x_2, x_3. These are

```
sp6R > SimRts1 := [LieAlgebraRoots(x1, CSA1), LieAlgebraRoots(x2, CSA1),
       LieAlgebraRoots(x3, CSA1)];
```

$$SimRts1 := \left[\left[\begin{array}{c} 2 \\ -1 \\ 0 \end{array}\right], \left[\begin{array}{c} -1 \\ 2 \\ -1 \end{array}\right], \left[\begin{array}{c} 0 \\ -2 \\ 2 \end{array}\right]\right] \tag{3.4}$$

and the corresponding set of positive roots is

```
chev > PosRts1 := PositiveRoots(RSD1, SimRts1);
```

$$PosRts1 := \left[\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\2\\-1\end{bmatrix}, \begin{bmatrix}-2\\2\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\1\end{bmatrix}, \begin{bmatrix}0\\-2\\2\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\-1\\0\end{bmatrix}\right] \quad (3.5$$

3. Property 2. The Structure Equations for $[h_i, x_j]$

The Cartan matrix for $sp(6, R)$ is

```
chev > a := CartanMatrix("C", 3);
```

$$a := \begin{bmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -2 & 2\end{bmatrix} \quad (4.1$$

and this gives the structure constants $[h_i, x_j] = a_{ji}\, x_i$:

```
chev > MultiplicationTable(chev, "LieTable", rows = [1, 2, 3], columns = [4, 5, 6]);
```

chev	$x1$	$x2$	$x3$
$h1$	$2\,x1$	$-x2$	0
$h2$	$-x1$	$2\,x2$	$-2\,x3$
$h3$	0	$-x2$	$2\,x3$

(4.2

The structure equations for subalgebra defined by $\{h_i, x_i, y_i\}$ are

```
chev > LieAlgebraData([h1, x1, y1]);
```

$$[e1, e2] = 2\,e2,\ [e1, e3] = -2\,e3,\ [e2, e3] = -e1 \quad (4.3$$

```
chev > LieAlgebraData([h2, x2, y2]);
```

$$[e1, e2] = 2\, e2,\ [e1, e3] = -2\, e3,\ [e2, e3] = -e1 \quad (4.4$$

```
chev > LieAlgebraData([h3, x3, y3]);
```

$$[e1, e2] = 2\, e2,\ [e1, e3] = -2\, e3,\ [e2, e3] = -e1 \quad (4.5$$

These are the standard structure equations for $sl(2)$, as required.

4. Property 3. The Automorphism Property

We define the linear transformation θ which maps $h_i \to -h_i$, $x_\ell \to y_\ell$ and $y_\ell \to x_\ell$ and check that it is an automorphism.

```
chev > A := evalDG([[h1, -h1], [h2, -h2], [h3, -h3], [x1, y1], [x2, y2], [x3, y3], [x4,
       y4],[x5, y5], [x6, y6], [x7, y7], [x8, y8], [x9, y9], [y1, x1], [y2, x2], [y3,
       x3], [y4, x4],[y5, x5], [y6, x6], [y7, x7], [y8, x8], [y9, x9]]):
sp6R > Theta := LinearTransformation(A);
```

$$\Theta := h1 \to -h1,\ h2 \to -h2,\ h3 \to -h3,\ x1 \to y1,\ x2 \to y2,\ x3 \to y3,\ x4 \to y4,\ x5 \to y5,\ x6 \to y6,\ x7 \to y7,\ x8 \to y8,\ x9 \to y9,\ y1 \to x1,\ y2 \to x2,\ y3 \to x3,\ y4 \to x4,\ y5 \to x5,\ y6 \to x6,\ y7 \to x7,\ y8 \to x8,\ y9 \to x9 \quad (5.1$$

```
chev > Query(Theta, "Homomorphism");
```

$$true \quad (5.2$$

5. Property 4. The Structure Equations for $[x_i, x_j]$

To verify that the structure equations for $[x_i, x_j]$ satisfy the defining properties **4** and **5** of the Chevalley basis, we first recall that the simple roots are

```
chev > SimRts1;
```

$$\left[\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}\right] \quad (6.1$$

The root pattern for $sp(6, R)$ is given by

```
chev > AbstractRoots := PositiveRoots("C", 3);
```

$$AbstractRoots := \left[\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\2\\1\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\2\\1\end{bmatrix}\right] \quad (6.2$$

from which we get the positive roots:

```
chev > alpha := DGzip(AbstractRoots, SimRts1, "plus");
```

$$\alpha := \left[\begin{bmatrix}2\\-1\\0\end{bmatrix}, \begin{bmatrix}-1\\2\\-1\end{bmatrix}, \begin{bmatrix}0\\-2\\2\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\end{bmatrix}, \begin{bmatrix}-2\\2\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}2\\0\\0\end{bmatrix}\right] \quad (6.3$$

This ordering of the positive roots coincides with the ordering of the vectors:

```
chev > seq(RootSpace(t, RSD1), t = alpha);
```

$$x1, x2, x3, x4, x5, x6, x7, x8, x9 \quad (6.4$$

Thus, for example, since AbstractRoots[1] + AbstractRoots[2] = AbstractRoot[4], we have $\alpha_1 + \alpha_2 = \alpha_4$ and hence $[x_1, x_2] = N_{12}x_4$. Likewise , $[x_2, x_5] = N_{25}x_7$ and so on. This accounts for all the zero entries in the multiplication table for the $\{x_1, x_2, x_3, x_4 , x_5\, x_6, x_7 , x_8 , x_9\}$. (If the sum of roots is not a root, then bracket of the corresponding root spaces is 0.)

Moreover, the structure constants for the $\{y_1, y_2, y_3, y_4 , y_5\, y_6, y_7 , y_8 , y_9\}$ are identical to those for $\{x_1, x_2, x_3, x_4 , x_5\, x_6, x_7 , x_8 , x_9\}$.

```
chev > MultiplicationTable(chev, "LieTable", rows = [$4 .. 12],columns =[$4 .. 12]),
       MultiplicationTable(chev, "LieTable", rows = [$13 .. 21],columns =[$13 .. 21]);
```

chev	$x1$	$x2$	$x3$	$x4$	$x5$	$x6$	$x7$	$x8$	$x9$
$x1$	0	$x4$	0	0	$x6$	0	$x8$	$2\,x9$	0
$x2$	$-x4$	0	$x5$	0	$2\,x7$	$x8$	0	0	0
$x3$	0	$-x5$	0	$-x6$	0	0	0	0	0
$x4$	0	0	$x6$	0	$x8$	$2\,x9$	0	0	0
$x5$	$-x6$	$-2\,x7$	0	$-x8$	0	0	0	0	0
$x6$	0	$-x8$	0	$-2\,x9$	0	0	0	0	0
$x7$	$-x8$	0	0	0	0	0	0	0	0
$x8$	$-2\,x9$	0	0	0	0	0	0	0	0
$x9$	0	0	0	0	0	0	0	0	0

(6.5

6. Property 5. The Formula for $N_{\alpha,\,\beta}$

Our positive roots are:

```
chev > alpha;
```

$$\left[\begin{bmatrix}2\\-1\\0\end{bmatrix},\begin{bmatrix}-1\\2\\-1\end{bmatrix},\begin{bmatrix}0\\-2\\2\end{bmatrix},\begin{bmatrix}1\\1\\-1\end{bmatrix},\begin{bmatrix}-1\\0\\1\end{bmatrix},\begin{bmatrix}1\\-1\\1\end{bmatrix},\begin{bmatrix}-2\\2\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}2\\0\\0\end{bmatrix}\right] \tag{7.1}$$

Since the α_2 string through α_5 is

```
chev > RootString(alpha[2], alpha[5], alpha);
```

$$\left[\begin{bmatrix}0\\-2\\2\end{bmatrix},\begin{bmatrix}-1\\0\\1\end{bmatrix},\begin{bmatrix}-2\\2\\0\end{bmatrix}\right] \tag{7.2}$$

we have that q = 1 and therefore we must have that $[x_2, x_5] = \pm\ 2\ x_7$ and indeed

```
chev > LieBracket(x2, x5);
```

$$2\ x7 \tag{7.3}$$

7. A Remark on the Algorithm Used To Calculate the ChevalleyBasis

The help page ChevalleyBasisDetails provides full details on how the Chevalley basis is computed. A key step is to calculate a basis where Properties 1--3 in the Synopsis hold. This is done by the command SL2Basis which is an export of of the ChevalleyBasis.
For example:

```
sp6R > CSA2, RSD2, SR, P := ChevalleyBasis:-SL2Basis(RSD, PosRts, SimRts);
```

$$CSA2, RSD2, SR, P := [e1 - e5, e5 - e9, e9], table\left(\left[[1, 0, -1] = \sqrt{t1}\ e3, [-1, 0, -1] = -\frac{1}{\sqrt{t6}}\ e18, [0, -2, 0] = -\frac{1}{\sqrt{t3}}\ e19, [1, 1, 0] = \sqrt{t2}\ e11, [0, 0, -2] = -e21, [0, 0, 2] = e15, [0, 2, 0] = \sqrt{t3}\ e13, [-1, 0, 1] = -\frac{1}{\sqrt{t1}}\ e7, [-1, 1, 0] = -e4, [2, 0, 0] = \sqrt{t4}\ e10, [0, -1, 1] = -e8, [0, 1, -1] = e6, [0, 1, 1] = \sqrt{t5}\ e14, [-1, -1, 0] = -\frac{1}{\sqrt{t2}}\ e17, [0, -1, -1] = -\frac{1}{\sqrt{t5}}\ e20, [-2, 0, 0] = -\frac{1}{\sqrt{t4}}\ e16, [1, -1, 0] = e2, [1, 0, 1] = \sqrt{t6}\ e12\right]\right), \left[\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}\right], \{t1, t2, t3, t4, t5, t6\} \tag{8.1}$$

The output consists of:

1. A new basis for the Cartan subalgebra.
2. The root space decomposition for the new basis. Basis for the roots spaces for the simple roots are chosen so that Properties **2** and **3** hold. The remaining basis elements contain the scalar factors which are to be fixed by Properties **4** and **5**.
3. The simple roots for the new basis. The Cartan matrix calculated from these simple roots will be in standard form.

4. A set defining the scalar factors.

Highlighted Commands

- CartanSubalgebra, ChevalleyBasis, LieAlgebraRoots, LinearTransformation, Query, RootSpace, RootString, SimpleLieAlgebraData,

References

1. N. Bourbaki, *Elements of Mathematics, Lie Groups and Lie Algebras Chapters 7 -- 9,* Springer
2. J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer

Release Notes

- This worksheet was compiled with Maple 17 and DG release USU1, available by request from ian.anderson@usu.edu

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