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MULTIPLICITY RESULTS OF PERIODIC SOLUTIONS
FOR TWO CLASSES OF NONLINEAR PROBLEMS

by

Kazuya Hata

A dissertation submitted in partial fulfillment
of the requirements for the degree

of

DOCTOR OF PHILOSOPHY

in

Mathematical Sciences

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UTAH STATE UNIVERSITY
Logan, Utah

2014

ABSTRACT

Multiplicity Results of Periodic Solutions

for Two Classes of Nonlinear Problems

by

Kazuya Hata, Doctor of Philosophy

Utah State University, 2014

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Department: Mathematics and Statistics

In this dissertation, we study the existence, multiplicity, and some qualitative properties of periodic solutions for the following two classes of nonlinear differential equations:

I) (Special) Relativistic Pendulum Equations (RPEs):

$$(\phi(u'))' = \nabla_u F(x, u) + h(x),$$

where $\phi : (-a, a) \to \mathbb{R}$ is an increasing homeomorphism satisfying $\phi(0) = 0$, and $\phi(s)s > 0$ for all $s \in (-a, a) \setminus \{0\}$. This type of equation also arises from geometric problems such as the minimum surfaces with various choices of ϕ.

II) (2-coupled) Gross-Pitaevskii Equations (GPEs):

$$\begin{cases} -u'' + \lambda_1 u = \mu_1 u^3 + \beta v^2 u, \\ -v'' + \lambda_2 v = \mu_2 v^3 + \beta u^2 v, \end{cases}$$

where λ_i, μ_i, and β are parameters.

For I), under some conditions, we establish a multiplicity result depending on the periodic condition of F by applying the Generalized Saddle Point Theorem. Our result partially answers the open problems raised by H. Brezis and J. Mawhin in 2010.

For II), we establish a multiplicity result by Variational Methods and we investigate local and global bifurcations by Bifurcation Analysis.

For the multiple existence part, we treat the case of full symmetry: $\lambda_1 = \lambda_2 = \lambda > 0$ and $\mu_1 = \mu_2 = \mu > 0$. By applying $\mathbb{Z}_2$-Index Theory, we show that there are infinitely many solutions for $\beta \leq -\mu$, and, for any integer k, there exist at least k pairs of solutions (u, v) and (v, u) depending on β for $\beta > -\mu$.

For the bifurcation part, we take advantage of the system where it has some constant solutions curves. In the general case, there are constant semi-trivial solutions:

$$(u, v) = (\omega_1, \theta), \ (u, v) = (\theta, \omega_2), \quad \omega_i := \sqrt{\tfrac{\lambda_i}{\mu_i}}.$$

In the case $\lambda_1 = \lambda_2 = 1$, $(u, v) = (A_\beta, B_\beta)$, $\quad A_\beta := \sqrt{\tfrac{\mu_2 - \beta}{\mu_1 \mu_2 - \beta^2}}, \ B_\beta := \sqrt{\tfrac{\mu_1 - \beta}{\mu_1 \mu_2 - \beta^2}}.$

We study the bifurcation structures which bifurcate from the solution curves:

$$\mathcal{T}_1^\beta := \{(\omega_1, \theta) \in H_T^1 \times H_T^1 : \beta \in \mathbb{R}\}, \quad \mathcal{T}_1^{\lambda_1} := \{(\omega_1, \theta) \in H_T^1 \times H_T^1 : \lambda_1 > 0\},$$

$$\mathcal{T} := \{(A_\beta, B_\beta) \in H_T^1 \times H_T^1 : \beta \in (-\sqrt{\mu_1 \mu_2}, \mu_1) \cup (\mu_2, \infty)\},$$

where H_T^1 is the Sobolev space $W^{1,2}$ with the inner product and the T-periodic conditions on $[-T/2, T/2]$. To show the existence of bifurcation points, we apply the Crandall-Rabinowitz Local Bifurcation Theorem which is proved by the Lyapunov-Schmidt Reduction. This allows us to convert the problem in the infinite-dimensional space of functions into a problem in a finite dimensional space due to the linearized equations which are of Fredholm operators. We also use critical groups for showing there are infinitely many bifurcation points. The nontrivial solutions in these connected bifurcation branches are not given by solving the Dirichlet problems. In addition, we show there are some global bifurcations by restricting the functional's domain into the space of even functions with the Symmetric Criticality Principle. We also show some connected branches are disjoint by the Strong Maximum Principle. For bifurcations from $\mathcal{T}$, we also show that the bifurcation branches extend to $-\infty$ for β if $\mu_1 \neq \mu_2$.

(95 pages)

PUBLIC ABSTRACT

Multiplicity Results of Periodic Solutions

for Two Classes of Nonlinear Problems

by

Kazuya Hata, Doctor of Philosophy

Utah State University, 2014

Major Professor: Dr. Zhi-Qiang Wang
Department: Mathematics and Statistics

We investigate the existences and qualitative properties of periodic solutions of the following two classes of nonlinear differential equations:

I) (Special) Relativistic Pendulum Equations (RPEs);

II) (2-coupled) Gross-Pitaevskii Equations (GPEs).

The pendulum equation describes the motion of a pendulum. According to Special Relativity, which was published by A. Einstein in 1905, causality is more fundamental than constant time-space, thus time will flow slower and space will distort to keep causality if the speed of motion is near the speed of light. In such high speed situations, the pendulum equation needs to be revised due to Special Relativity. The revised equation is called RPE. Our result answers some open questions about the existence of multiple periodic solutions for RPEs.

GPEs are sometimes called coupled nonlinear Schrödinger equations. The Schrödinger equation is the fundamental equation of Quantum Mechanics which is the "exotic" probabilistic fundamental physics law of the "micro" world – the world of atoms and molecules. A well-known physicist and Nobel laureate, R. Feynman, said "I think I can safely say that nobody understands quantum mechanics." which indicates the physical/philosophical difficulty of interpretations. It raises paradoxical problems such as

the well-known Schrödinger's Cat. Setting aside these difficulties, if we combine Special Relativity and Quantum Mechanics as a many-body system, then we have Quantum Field Theory (QFT) which is more deterministic, and governs even elementary particle physics. GPEs are also related to QFT. For example, superconductivity and Bose Einstein Condensates (BEC). These phenomena in condensed matter physics can be thought of as the emergence of the mysterious micro world physics at "macro" level.

We study these equations from the viewpoint of mathematical interest. It is generally difficult to solve nonlinear differential equations. It is also generally difficult even to prove the existence of solutions. Although we show there exist solutions, we still do not know how to solve the differential equations analytically.

Variational Methods (or Calculus of Variations) are useful tools to show there exist solutions of differential equations. The idea is to convert the problem of solving equations into the problem of finding critical points (i.e. minimum/maximum points or saddle points) of a functional, and each critical point can generally correspond to a weak solution. However, it is also generally difficult to find out such critical points because we look for critical points in an infinite-dimensional functions space. Thus many advanced mathematical theories or tools have been developed and used for decades in nonlinear analysis. We use some topological theories. From information of the functional's shape, these theories deduce if there exists a critical point, or how many critical points exist. The key of these theories is to use the symmetry of the equations.

We also investigate bifurcation structures for II), i.e. the connection structures between the solutions. By linearizations which look at the equations "locally," we reduce the problem in the infinite dimension to one in a finite dimension. Furthermore, it allows us to apply Morse Theory, which connects between local and global aspects of the functional's information. In several cases, we show that there are infinitely many bifurcation points that give rise to global bifurcation branches.

(95 pages)

ACKNOWLEDGMENTS

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CHAPTER 1

INTRODUCTION

In this dissertation, by applying Variational Methods and Topological Methods, in particular, Critical Point Theory and Bifurcation Theory, we study the existence and properties of periodic solutions for the following two classes of systems:

I) (Special) Relativistic pendulum systems;

II) (2-coupled) Gross-Pitaevskii systems.

In this chapter, we first explain the backgrounds of these equations, and some preliminaries on the methods used. Finally, a short summary of our main results is given in the last section.

1.1 Background

1.1.1 Relativistic Pendulum Equations (RPEs)

First we explain and review the (relativistic) pendulum systems. The Classical (forced) Pendulum Equation (CPE), which represents the behavior of a pendulum, is given by

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = h(t), \tag{1.1}$$

where $h(t)$ is an external force, g is the gravitational constant, l is the length of the pendulum (see Figure 1.1).

If $\int_0^T h(t) = 0$, then the Mountain Pass Theorem (Theorem 1.3) assures that the CPE has a weak solution geometrically distinct from the one which minimizes the action functional ([30] [31]).

In 1905 Einstein published ([18]) Special Relativity which shows that the speed of any physical object cannot exceed the speed of light because of causality, and the time-space will change to avoid exceeding of the speed of light. According to Special Relativity, if the speed of the pendulum is sufficiently near the speed of light, then
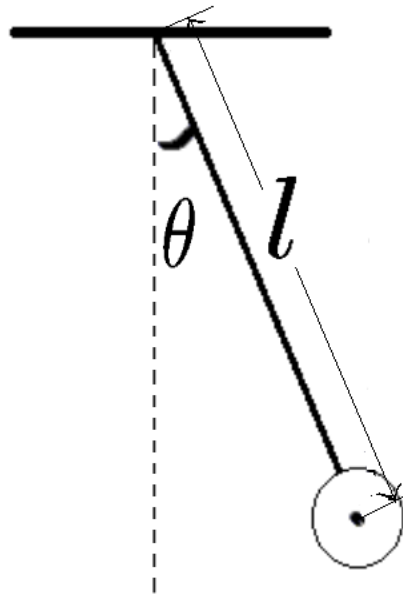
Figure 1.1: Pendulum system

the classical pendulum equation must be changed into the special relativistic pendulum equation given by

$$\frac{d}{dt}\left(\frac{u'(t)}{\sqrt{1-|u'(t)|^2}}\right) + A\sin u(t) = h(t), \tag{1.2}$$

where we set the speed of light $c = 1$, a quasilinear equation.

As the generalized form of (1.2), the (special) Relativistic Pendulum Equation (RPE) with the periodic boundary condition is written as:

$$\begin{cases} (\phi(u'))' - g(x,u) = h(x), \\ u(0) = u(T), \quad u'(0) = u'(T), \end{cases} \tag{1.3}$$

where, for some $a > 0$, $\phi : (-a, a) \to \mathbb{R}$ is an increasing homeomorphism satisfying $\phi(0) = 0$ and

$$\phi(s)s > 0,$$

for all $s \in (-a, a) \setminus \{0\}$. Equations like (1.3) also arise from geometric problems (e.g. the minimum surface) with various choices of ϕ.

In 2010, under specified conditions, Brezis and Mawhin ([8]) showed that there

exists a classical periodic solution of RPE by using Variational Methods, and by taking advantage of convex properties, and by using the Topological Index Theory (the Leray-Schauder degree), and Fixed-Point Theory of compact operators. They proved their results including the existence of one solution, and also suggested some open problems. Some research work has been done on the open problems.

More generally, we think of the case where the range of u is $\mathbb{R}^n$. If g is a potential force in (1.3), then the RPE with the periodic condition is rewritten in the following form:

$$
\begin{cases}
(\phi(u'))' = \nabla_u F(x, u) + h(x), \\
u(0) = u(T), \quad u'(0) = u'(T),
\end{cases}
\tag{1.4}
$$

which is the system we will study.

Let denote the $L^p_{\text{loc}}(\mathbb{R})$ spaces with the T-periodic condition as $L^p_T(\mathbb{R})$. We make the following assumptions:

(R1) $\phi : B_a \subset \mathbb{R}^n$ is onto $\mathbb{R}^n$ such that $\psi := \phi^{-1} : \mathbb{R}^n \to B_a$ and there is a C^1 function $\Psi : \mathbb{R}^n \to \mathbb{R}$ such that $\nabla\Psi = \psi$ and Ψ is bounded below.

(R2) $F \in C^1(S \times \mathbb{R}^n, \mathbb{R})$. For an integer $0 \leq k \leq n$, F is T-periodic in x, 2π-periodic in $u_1, \cdots, u_k$ and $\nabla_u F$ is bounded. $h \in (L^2_T(\mathbb{R}))^n$ is 2π-periodic in x and $\int_0^T h_i(x)dx = 0$ for $i = 1, \cdots, k$. Writing $u = (v, w)$ with $v \in \mathbb{R}^k$ and $w \in \mathbb{R}^{n-k}$. Assume $\int_0^T (F(x, u) + h(x)u)dx \to -\infty$ as $|w| \to \infty$ uniformly in $v \in \mathbb{R}$.

We showed ([23]) that at least $k+1$ solutions of (1.4) exist depending on the periodicity of F by a new approach with the abstract Generalized Saddle Point Theorem which is related to the cuplength of cohomology rings in algebraic topology. Hence we have partially answered some of the open problems concerning multiplicity of periodic solutions. Our main theorem will be stated in the last section of this chapter. The proof is given in Chapter 2.

1.1.2   Gross-Pitaevskii Equations (GPEs)

One of the important problems in physics is to study quantum (field) systems which describes physical behaviors in the 'micro' world. After setting theoretical assumptions and Hamiltonians, the problem becomes a mathematical problem.

The Gross-Pitaevskii Equations, which are induced by the Hartree-Fock approximation and the pseudo-potential interaction model, describe the ground state of bosons

in a quantum (field) physical system, e.g. the Bose-Einstein condensate (BEC) system. The equation is sometimes referred to as the nonlinear Schrödinger equation, too. The Schrödinger equation is the fundamental equation of quantum mechanics.

For a domain $\Omega \subset \mathbb{R}^n$, the two-component system of nonlinear Schrödinger equation (which is also called Gross-Pitaevskii equations) without the trapping potential for hyperfine states is given by

$$
\begin{cases}
-i\frac{\partial}{\partial t}\Phi_1 = \frac{\hbar^2}{2m}\Delta\Phi_1 + \mu_1|\Phi_1|^2\Phi_1 + \beta|\Phi_2|^2\Phi_1, \text{ for } y \in \Omega, \ t > 0, \\
-i\frac{\partial}{\partial t}\Phi_2 = \frac{\hbar^2}{2m}\Delta\Phi_2 + \mu_2|\Phi_2|^2\Phi_2 + \beta|\Phi_1|^2\Phi_2, \text{ for } y \in \Omega, \ t > 0, \\
\Phi_j = \Phi(y,t) \in \mathbb{C}, \ j = 1,2, \\
\Phi_j(y,t) = 0 \text{ for } y \in \partial\Omega, \ t > 0, \ j = 1,2,
\end{cases}
\tag{1.5}
$$

where m is atom mass, $\hbar = \frac{h}{2\pi}$, and h is the Planck constant, μ_i and β are real constants. Each Φ_i corresponds to the quantum field of condensed matters.

If Φ_i is the solitary wave solution of the form

$$
\Phi_1(x,t) = e^{i\lambda_1 t}u(x), \quad \Phi_2(x,t) = e^{i\lambda_2 t}v(x),
$$

then the system is reduced to the following elliptic system:

$$
\begin{cases}
-\frac{\hbar^2}{2m}\Delta u = -\lambda_1 u + \mu_1 u^3 + \beta u v^2, \\
-\frac{\hbar^2}{2m}\Delta v = -\lambda_2 v + \mu_2 v^3 + \beta v u^2.
\end{cases}
$$

By setting $\frac{\hbar^2}{2m} = 1$, we have

$$
\begin{cases}
-\Delta u + \lambda_1 u = \mu_1 u^3 + \beta v^2 u, \\
-\Delta v + \lambda_2 v = \mu_2 v^3 + \beta u^2 v.
\end{cases}
$$

When $N = 1$, the following 2-coupled Gross-Pitaevskii system with the periodic condition is the system that we will study:

$$
\begin{cases}
-u'' + \lambda_1 u = \mu_1 u^3 + \beta v^2 u, \\
-v'' + \lambda_2 v = \mu_2 v^3 + \beta u^2 v, \\
u, \ u' \text{ are } T\text{-periodic}, \\
v, \ v' \text{ are } T\text{-periodic}.
\end{cases}
\tag{1.6}
$$

Although these constants λ_i, μ_i, and β can be any real numbers, to show multiple

existence of solutions, we restrict $\lambda_i > 0$ and $\mu_i > 0$, and $\beta < 0$ for some technical reasons of the variational structure to establish multiple existence. Note that β is called attractive (repulsive) when $\beta > 0$ ($\beta < 0$ respectively) because of physical reasons. To investigate bifurcations, we treat $\beta \in \mathbb{R}$.

In the case where the domain is a smooth bounded domain $\Omega \subset \mathbb{R}^m$ for $m \leq 3$ under the condition $u = v = 0$ on $\partial\Omega$ and $u, v > 0$ on Ω, in the case $\lambda_1 = \lambda_2 = 1$ and $\mu_1 = \mu_2 = 1$, Dancer, Wei, and Weth showed ([16])

   i) the 2-coupled Gross-Pitaevskii system admits an infinite sequence of solutions of the system for $\beta \leq -1$, and

   ii) there exists $\beta_k > -1$ such that k pairs of solutions exist for $\beta < \beta_k$.

This was done by using Variational Methods with the symmetric advantage, i.e., $\mathbb{Z}_2$-Index Theory with the Krasnosel'skii genus ([26]). Sato and Wang applied $\mathbb{Z}_2$-Index Theory too ([42]). $\mathbb{Z}_p$-Index Theory has also been applied to the N-coupled system (see [45]). Nguyen proved ([34]) there exist infinitely smooth periodic traveling wave solutions for (1.5) by using the Topological Degree Theory for positive operators.

In the periodic condition case of the fully symmetric equations:

$$\begin{cases} -u'' + \lambda u = \mu u^3 + \beta v^2 u \\ -v'' + \lambda v = \mu v^3 + \beta u^2 v \\ u, \ u' \text{ are } T\text{-periodic,} \\ v, \ v' \text{ are } T\text{-periodic,} \end{cases} \tag{1.7}$$

we obtained a similar result for [16] by using the same technique and $\mathbb{Z}_2$-Index Theory for the periodic condition case too. The proofs are given in Chapter 3.

We also investigate the bifurcation structures from some semi-trivial solutions and a synchronized solution curve for GPEs. More precisely, we study bifurcations from the following solution curves:

$$\begin{aligned} \mathcal{T}_1^\beta &:= \{(\omega_1, \theta) \in H_T^1 \times H_T^1 : \beta \in \mathbb{R}\}, \\ \mathcal{T}_1^{\lambda_1} &:= \{(\omega_1, \theta) \in H_T^1 \times H_T^1 : \lambda_1 > 0\}, \\ \mathcal{T} &:= \{(A_\beta, B_\beta) \in H_T^1 \times H_T^1 : \beta \in (-\sqrt{\mu_1\mu_2}, \mu_1) \cup (\mu_2, \infty)\}, \end{aligned}$$

where H_T^1 is the Sobolev space $W^{1,2}$ with the inner product and the T-periodic conditions on $[-T/2, T/2]$, and

$$\omega_i := \sqrt{\frac{\lambda_i}{\mu_i}}, \quad A_\beta := \sqrt{\frac{\mu_2 - \beta}{\mu_1 \mu_2 - \beta^2}}, \quad B_\beta := \sqrt{\frac{\mu_1 - \beta}{\mu_1 \mu_2 - \beta^2}}.$$

We will apply the Crandall-Rabinowitz Local Bifurcation Theorem which is proved by the Lyapunov-Schmidt Reduction. This allows us to change the problem in the infinite dimensional space of functions into a problem in a finite dimensional space due to the fact that the linearized equations are of Fredholm operators. By using the Morse Theory, we show there are countably infinitely many bifurcation points. We also show the nontrivial solutions in these connected bifurcation branches are not given by solving the Dirichlet problems. In addition, we show that there are some global bifurcations by restricting the domain of the functional into the space of even functions with the Symmetric Criticality Principle. By using the Strong Maximum Principle, we also show that some of the connected branches do not connect with each other. For bifurcations from $\mathcal{T}$, we also show that these bifurcation branches extend to $-\infty$ for β if $\mu_1 \neq \mu_2$. The existence of uncountable possible bifurcations is also shown.

Our main theorems will be listed in the last section in this chapter. The proofs of theorems for bifurcations are given in Chapter 4.

1.2 Some methods in nonlinear analysis

1.2.1 Variational Methods

We give a short review of Variational Methods here. Variational Methods (or Calculus of Variations) in functional analysis ([7]) or nonlinear analysis ([17]) are frequently used to show the existence of (weak) solutions of differential equations. The strategy of Variational Methods is the following: For a given differential equation (system), define the corresponding functional E (which is called energy functional if $E = \int H dt$ for the Hamiltonian H) whose critical points are (weak) solutions of the original system, then show there exists a critical point(s) of E. Since we look for critical points in an infinite dimensional functions space (e.g. Banach space or Sobolev space), it is generally difficult to find critical points. The simplest direct way is to show that a bounded *minimizing sequence* $\{u_k\}$ *of* E (i.e. $E(u_k) \to \inf E$) converges strongly, thus compactness is important naturally. However, compactness is not generally assured in an infinite dimensional space. Coarser topology has more compact sets. For example, the unit ball in a dual

space is always compact in the weak* topology by the Banach-Alaoglu-Bourbaki Theorem (for example, see [20]). Therefore not only weak topology but also weak* topology is sometimes used.

In the case that the functional E on a reflexive Banach space is *weakly lower semi-continuous (w.l.s.c)* (i.e. if u_k converges to u weakly, then $\liminf_{k\to\infty} E(u_k) \geq E(u)$) and has a bounded minimizing sequence, E achieves a minimum. Since the *Sobolev space* $W^{1,p}$ for $1 < p < \infty$ (i.e. the space of functions $u \in L^p(\Omega)$ having a weak derivative $u' \in L^p(\Omega)$, where a *weak derivative of u* is a function v satisfying $\int_\Omega u \cdot f' = -\int_\Omega v \cdot f$ for every $f \in C_c^\infty(\Omega)$) is a reflexive Banach space and E is frequently *lower semi-continuous (l.s.c)* (i.e. if u_k converges to u strongly, then $\liminf_{k\to\infty} E(u_k) \geq E(u)$), to guarantee that E is w.l.s.c. and E has a bounded minimizing sequence are frequently difficult parts. Although if E on a normed space is l.s.c. and a convex function, then E is w.l.s.c., but E is not convex generally. On the other part, if E on a reflexive Banach space is *coercive* (i.e. $E(u) \to \infty$ if $\|u\| \to \infty$), then the existence of a bounded minimizing convergent sequence is guaranteed. However, the corresponding functional may not be coercive generally too. For example, a physical system can be represented as the Hamiltonian system:

$$-J \cdot \frac{du}{dt} = \nabla H(t, u), \quad J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}, \tag{1.8}$$

where J is the symplectic matrix.

Let us think of the system with the periodic boundary condition. Then the corresponding functional is not generally coercive. To overcome these difficulties, some varieties of advanced Variational Methods have been developed with functional/nonlinear analysis and topological theories. Especially, the following two developed theories are known:

a) **Morse Theory** (see [12] [31] [33] for the details): The theory gives the relation between the topological information of a manifold and the critical points of a smooth *Morse function* (i.e. all critical points of the function are nondegenerate) by using the *Morse index of a critical point p*, which is defined as the number of negative eigenvalues of the Hessian matrix at p, representing the number of independent directions around p in which the functional decreases.

b) **Minimax Theory** (see [31] [39] [43] [51] for the details): The theory gives the information of critical points having the critical value

$$c := \inf_{A \in S} \max_{u \in A} E(u)$$

with a suitable class of sets S. The Mountain Pass Theorems, the Saddle Point Theorems, and G-Index Theories (which are explained later) are well known results in this direction.

These theories use frequently level sets:

Sublevel set: For a functional $E : X \to \mathbb{R}$, the following set is called the *sublevel set of E*:

$$\mathcal{M}^a := \{u \in X : E(u) \leq a\}$$

which is sometimes denoted as E^a.

Applications of topological theories for Variational Methods focus on topological information of $\mathcal{M}^a$ generally. Thus we introduce a deformation of sublevel sets. A theorem which assures such deformations exist, is called the Deformation Theorem/Lemma. To construct such deformation in Banach spaces, pseudo-gradient is used (see [17] [31] [51]).

Palais-Smale condition

Palais-Smale condition: For a functional $E : X \to \mathbb{R}$, if every sequence $\{u_j\}$ in X such that $\{E(u_j)\}$ is bounded and $E'(u_j) \to 0$ has a convergent subsequence, then we say E satisfies the *Palais-Smale condition*, denoted as (PS)-condition. If there is a sequence $\{u_j\}$ in X such that

$$E(u_j) \to c, \quad E'(u_j) \to 0,$$

then we say E satisfies *the Palais-Smale condition on the level c*, denoted as $(\text{PS})_c$ condition. This implies that c is a critical value of E.

Remark 1.1. The (PS)-condition assures the compactness of the set of critical points on the level c. The (PS)-condition implies the $(\text{PS})_c$-condition but the opposite is not generally true. If a functional E on X is bounded below and satisfies the $(\text{PS})_c$ condition

with $c = \inf_X E$, then every minimizing sequence has a convergent subsequence. This is proved by Ekland's Variational Principle (see [17] [31] [51]).

To look for critical points on a restricted domain is sometimes useful. Especially the following manifold is frequently used (We also use it in Chapter 3).

Nehari manifold: The following set is a manifold (if all its points are regular and $E \in C^2$. See [17] Remark 4.3.40), which is called *Nehari manifold*:

$$\mathcal{N} := \{u \in X \setminus \{\theta\} : \langle E'(u), u \rangle = 0\}.$$

Remark 1.2. Various modified versions of Nehari manifold are also frequently used. $\langle E'(u), u \rangle$ is frequently written as just $E'(u)u$.

Minimax Theorems

Now we introduce some Minimax Theorems. The following Mountain Pass Theorem by Ambrosetti and Rabinowitz is probably the most famous one.

Theorem 1.3 (Mountain Pass Theorem). *([51] p.42) Let X be a Banach space, $\varphi \in C^1(X, \mathbb{R})$, $e \in X$ and $r > 0$ be such that $\|e\| > r$ and*

$$b := \inf_{\|u\| = r} \varphi(u) > \varphi(0) \geq \varphi(e).$$

If φ satisfies the $(PS)_c$ condition with

$$
\begin{aligned}
c &:= \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t)), \\
\Gamma &:= \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\},
\end{aligned}
$$

then c is a critical value of φ.

The following Saddle Point Theorem by Rabinowitz is a generalized version of the Mountain Pass Theorem.

Theorem 1.4 (Saddle Point Theorem). *([51] p.42) Let $X = Y \oplus Z$ be a Banach space with $\dim Y < \infty$. Define, for $\rho > 0$,*

$$
\begin{aligned}
M &:= \{u \in Y : \|u\| \leq \rho\}, \\
M_0 &:= \{u \in Y : \|u\| = \rho\}.
\end{aligned}
$$

Let $\varphi \in C^1(X, \mathbb{R})$ be such that

$$b := \inf_Z \varphi > a := \max_{M_0} \varphi.$$

If φ satisfies the (PS)$_c$ condition with

$$\begin{aligned} c &:= \inf_{\gamma \in \Gamma} \max_{u \in M} \varphi(\gamma(u)), \\ \Gamma &:= \{\gamma \in C(M, X) : \gamma|_{M_0} = \mathrm{Id}\}, \end{aligned}$$

then c is a critical value of φ.

For the proof of the main theorem in Chapter 2, we use the Generalized Saddle Point Theorem (Theorem 2.5) which is a generalized version of the Saddle Point Theorem.

1.2.2 G-Index Theories

Let G be a finite abelian group. For a differential equation having a variational structure which is G-invariant, G-Index Theory shows the existence and multiplicity of solutions by using the *(Krasnosel'skii's) genus index* and Borsuk-Ulam Type Theorems. The genus is defined as the least dimension of the range in which a special continuous function from the invariant set can exist. The concept of the genus is related to the Lusternik-Schnirelman category which can measure the 'size' of symmetric sets ([39]).

Lusternik-Schnirelman category

Lusternik-Schnirelman category: For a closed set A in a topological space X, the *Lusternik-Schnirelman category* is the least integer n such that there exists a covering of A by n closed sets which is contractible in X, denoted as $\mathrm{cat}_X(A)$.

The key idea of the Lusternik-Schnirelman Theory is the following: if a contractible local neighborhood of each critical point exists, then the number of critical points of E on A is greater than equal to $\mathrm{cat}_X(A)$. The case where X is a compact manifold is typically treated. The corresponding critical values are given by

$$c_k := \inf_{A \in \mathcal{A}_k} \sup_{u \in A} E(u), \quad \mathcal{A}_k := \{A \subset X : A \text{ closed}, \ \mathrm{cat}_X(A) \geq k\}.$$

Remark 1.5. The Krasnosel'skii's genus is a 'simpler' version of the Lusternik-Schnirelman category. The genus can be thought of as equivalent to the category in a symmetric space

which is invariant under the symmetry group. The Lusternik-Schnirelman category is related to the cuplength of cohomology rings.

We apply $\mathbb{Z}_2$-Index Theory for the fully symmetric case (hence $\lambda_1 = \lambda_2$ and $\mu_1 = \mu_2$) of the 2-coupled GPEs. Let σ be a $\mathbb{Z}_2$-group action given by $\sigma(u, v) = (v, u)$ in $\mathcal{M}$, where $\mathcal{M}$ is our 'nice' manifold having critical points. For any closed σ-invariant subset A of $\mathcal{M}$, the $\mathbb{Z}_2$-genus $\gamma(A)$ is defined as the smallest $k \in \mathbb{N} \cup \{0\}$ such that there exists a continuous function $h : A \to \mathbb{R}^k \setminus \{0\}$ with $h(\sigma(u, v)) = -h(u, v)$ for all $(u, v) \in A$.

Let S be the boundary of a bounded symmetric neighborhood of zero in a k-dimensional normed vector space. We can also construct a continuous function $\psi : S \to \mathcal{M}$ satisfying $\psi(-u) = \sigma(\psi(u))$. Then $h \circ \psi$ makes a contradiction with the Borsuk-Ulam Theorem which says every continuous odd map $f : \partial U \to \mathbb{R}^{n-1}$ has a zero, where U is an open bounded symmetric neighborhood of 0 in $\mathbb{R}^n$. This argument with properties of genus (Lemma 3.6) allows us to conclude the existence of at least k critical points.

To see this, suppose there exists a continuous function

$$\exists h : \psi(S) \to \mathbb{R}^{k-1} \setminus \{0\},$$

then we get the following diagram.

$$
\begin{array}{ccccc}
S^{k-1} \cong S \ni u & \xrightarrow{\psi} & \psi(u) & \xrightarrow{h} & h(\psi(u)) \subseteq \mathbb{R}^{k-1} \setminus \{0\} \\
\downarrow{\scriptstyle -\mathbf{1}} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle -\mathbf{1}} \\
S^{k-1} \cong S \ni -u & \xrightarrow{\psi} & \psi(-u) & \xrightarrow{h} & h(\psi(-u)) \subseteq \mathbb{R}^{k-1} \setminus \{0\}
\end{array}
$$

Figure 1.2: $\mathbb{Z}_2$-Index Theory and the Borsuk-Ulam Theorem

The diagram contradicts the Borsuk-Ulam Theorem.

Remark 1.6. For a finite abelian group G, generalized (or pseudo) Borsuk-Ulam Type Theorems are needed to obtain the dimensional property (Lemma (3.6) (vi)) necessary for establishing the G-Index Theory (for example, see [14] [40] [45] [50]). Generalizations of the Borsuk-Ulam Theorem are still studied nowadays (for example, see [48]). For a nonabelian group G, the construction of G-Index Theory seems to be an open problem although some research has already been done (for example, [15]).

1.2.3 Bifurcation Methods

Recall the definition of a bifurcation point.

Bifurcation point: Suppose X and Y are Banach spaces, $F : X \to Y$, and the preimage $F_{-1}(\theta)$ contains a curve $\mathcal{Z}$. If every neighborhood U of z contains zeros of F in $U \setminus \mathcal{Z}$, then the interior point $z \in \mathcal{Z}$ is called a *bifurcation point for F with respect to $\mathcal{Z}$*.

Bifurcation Methods are methods used to investigate the existence of bifurcation points and their properties. In the case we already know there exists a solution curve $\mathcal{Z}$, the existence of a bifurcation point on $\mathcal{Z}$ implies that another solution curve $\mathcal{Y}$ may exist locally. We call the other solution curve a *local bifurcation* or a *bifurcation branch*. To show the existence of a bifurcation point, we look at the original problem $F = \theta$ locally; hence we linearize the original problem. If the linearized problem has a nontrivial kernel, then there is the possibility that a bifurcation point exists. However, it is a necessary condition. Thus we need to show that the possible bifurcation point is actually a bifurcation point. To show the existence of bifurcation points, we use the *Morse index* which is defined as the number of negative eigenvalues of the Hessian matrix of a functional. Note that the Morse index implies the number of dimensional directions to decrease the functional, and the definition in the infinite dimensional space makes sense if it is finite.

To show that possible bifurcation points are bifurcation points, we also use the Crandall-Rabinowitz Local Bifurcation Theorem (Theorem 4.14). The Crandall-Rabinowitz Local Bifurcation Theorem is proved by the Lyapunov-Schmidt Reduction.

Lyapunov-Schmidt Reduction

Lyapunov-Schmidt Reduction is a method for changing the problem in an infinite dimensional space into a problem in a finite dimensional space by using the advantage of Fredholm operators. Let X, Y be Banach spaces, and T be a topological space. Let $U(\theta)$ be a neighborhood of θ in X. Suppose $F : U \times T \to Y$ is a continuous map satisfying $F(\theta, \lambda_0) = \theta$. Suppose $F_x(\theta, \lambda_0)$ is a Fredholm operator, i.e.,

a) im $F_x(\theta, \lambda_0)$ is closed in Y,

b) $d := \dim \ker F_x(\theta, \lambda_0) < \infty$.

c) $d^* := \operatorname{codim} \operatorname{im} F_x(\theta, \lambda_0) < \infty$.

Let us think of solving the following equation:

$$F(x, \lambda) = \theta. \tag{1.9}$$

There are direct sum decompositions of X and Y by the properties of b) and c). Thus the problem (1.9) can be changed into:

$$\begin{cases} PF(\tilde{x} + u(\tilde{x}, \lambda), \lambda) = \theta, \\ (\text{Id} - P)F(\tilde{x} + u(\tilde{x}, \lambda), \lambda) = \theta, \end{cases} \tag{1.10}$$

where $P : Y \to \text{im}F_x(\theta, \lambda_0)$ is the projection operator, and $\tilde{x} \in \ker F_x(\theta, \lambda_0)$. $u(\tilde{x}, \lambda)$ is a solution satisfying the first equation of (1.10) due to the Implicit Function Theorem. Thus the problem (1.9) is reduced to the second equation of (1.10) which is a system of d^* equations of d variables. This procedure is called the *Lyapunov-Schmidt Reduction* (for the details, see [13] [17] [31]).

We also investigate to see if bifurcations are global. Begin by Recalling the definition of global bifurcation.

Global bifurcation: Let $\mathcal{Z}$ be a solution curve and $z_0 \in \mathcal{Z}$ a bifurcation point. Let $\mathcal{Y}$ be the connected component of bifurcation solutions emanating from z_0. $\mathcal{Y}$ is called global if $\mathcal{Y}$ is unbounded or $\mathcal{Y}$ meets $\mathcal{Z}$ at a different bifurcation point (see Figure 1.3).

The Rabinowitz Global Bifurcation Theorem (Theorem 4.9), which can be proved by using the Leray-Schauder degree, is a well-known theorem to see if a bifurcation is global.
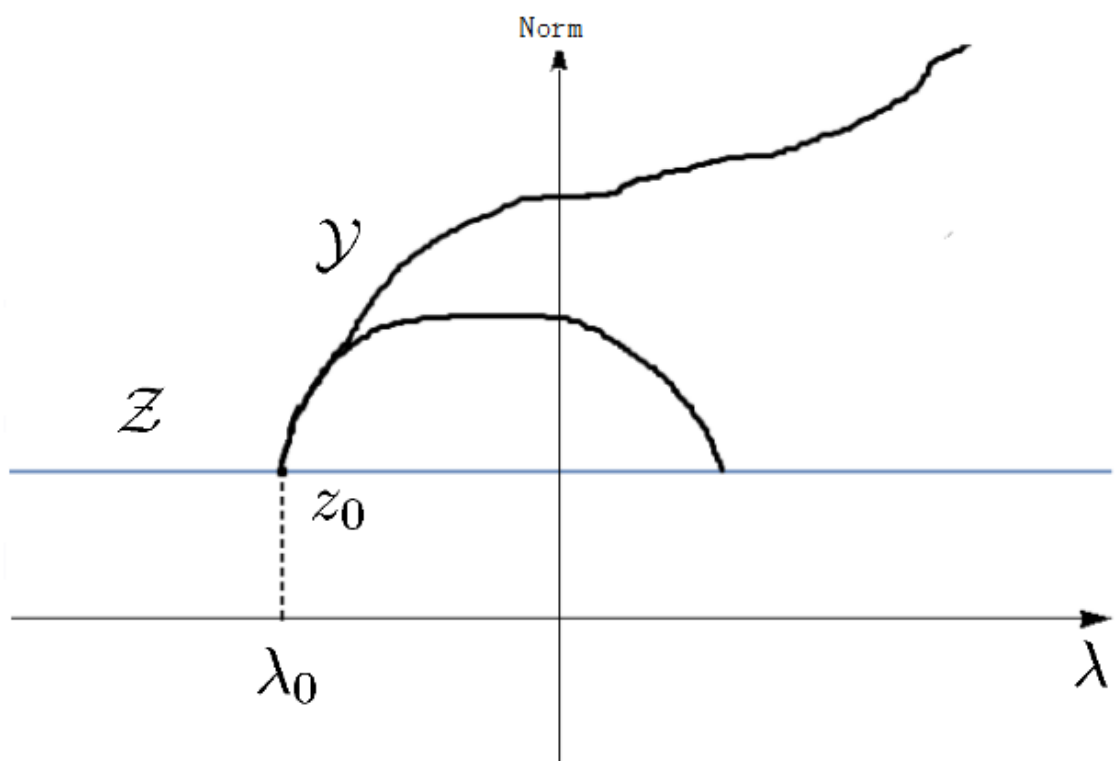
Figure 1.3: Example of global bifurcation

1.3 A brief summary of main results

Here we list our main theorems:

1.3.1 Multiple existence theorem for RPEs

Let set $S := \mathbb{R}/\{2\pi\mathbb{Z}\}$, and let denote the L^p spaces with the T-periodic conditions as L_T^p.

(R1) $\phi : B_a \subset \mathbb{R}^n$ is onto $\mathbb{R}^n$ such that $\psi := \phi^{-1} : \mathbb{R}^n \to B_a$ and there is a C^1 function $\Psi : \mathbb{R}^n \to \mathbb{R}$ such that $\nabla \Psi = \psi$ and Ψ is bounded below.

(R2) $F \in C^1(S \times \mathbb{R}^n, \mathbb{R})$. For an integer $0 \leq k \leq n$, F is T-periodic in x, 2π-periodic in $u_1, \cdots, u_k$ and $\nabla_u F$ is bounded. $h \in (L_T^2(\mathbb{R}))^n$ is 2π-periodic in x and $\int_0^T h_i(x)dx = 0$ for $i = 1, \cdots, k$. Writing $u = (v, w)$ with $v \in \mathbb{R}^k$ and $w \in \mathbb{R}^{n-k}$. Assume $\int_0^T (F(x, u) + h(x)u)dx \to -\infty$ as $|w| \to \infty$ uniformly in $v \in \mathbb{R}$.

Theorem 1.7 (Theorem A). *([23]) Assume (R1) and (R2). Then the system (1.4) has at least $k + 1$ classical T-periodic solutions.*

1.3.2 Multiple existence theorem for 2-coupled GPEs

Theorem 1.8 (Theorem B). *a) If $\beta \leq -\mu$, then (1.6) with the fully symmetric condition $\mu_1 = \mu_2 = \mu > 0$ and $\lambda_1 = \lambda_2 = \lambda > 0$ admits a sequence $(u_k, v_k)_k$ of solutions with*

$$\|u_k\|_{L_T^\infty} + \|v_k\|_{L_T^\infty} \to \infty.$$

b) For any positive integer k there exists a number $\tilde{\beta}_k > -\mu$ such that, for $\beta < \tilde{\beta}_k$, (1.6) with the fully symmetric condition $\mu_1 = \mu_2 = \mu > 0$ and $\lambda_1 = \lambda_2 = \lambda > 0$ has at least k pairs (u, v), (v, u) of solutions.

1.3.3 Bifurcation theorems for 2-coupled GPEs

Set

$$\beta_k := (\Lambda_k + \lambda_2)\frac{\mu_1}{\lambda_1}, \quad \Lambda_k := (k\frac{2\pi}{T})^2,$$

where $k = 0, 1, \cdots$.

Set

$\mathcal{S}_k^\beta :=$ The connected bifurcation component through $((\omega_1, \theta), \beta_k)$,

$$X_{\text{even}} := \{u \in H^1[-T/2, T/2] : u \text{ is an even function }\}.$$

Theorem 1.9 (Theorem C). *Along $\mathcal{T}_1^{\beta}$, there are infinitely many bifurcation points of (1.6): $\beta_0 < \beta_1 < \cdots$ in β, having the following properties:*

a) *The nontrivial solutions in the bifurcation branches are not given by solving Dirichlet problems.*

b) *These bifurcation branches are global.*

c) *In $X_{\text{even}} \times X_{\text{even}}$, $\mathcal{S}_k^{\beta} \cap \mathcal{S}_l^{\beta} = \emptyset$, for $k \neq l$.*

d) *Each $\mathcal{S}_k^{\beta}$ is unbounded.*

Set

$$\tilde{\lambda}_k := \frac{1}{2}\Lambda_k, \quad \tilde{\tilde{\lambda}}_k := \frac{\mu_1}{\beta}(\lambda_2 + \Lambda_k),$$

where $k = 0, 1, \cdots$.

With fixed β, set

$\mathcal{S}_k^{\lambda_1} :=$ The connected bifurcation component through $((\omega_1, \theta), \tilde{\lambda}_k)$, $k = 1, 2, \cdots$.

$\mathcal{R}_k^{\lambda_1} :=$ The connected bifurcation component through $((\omega_1, \theta), \tilde{\tilde{\lambda}}_k)$, $k = 0, 1, \cdots$.

Theorem 1.10 (Theorem D). *Along $\mathcal{T}_1^{\lambda_1}$, there are infinitely many bifurcation points of (1.6): For fixed β, $\tilde{\lambda}_1 < \tilde{\lambda}_2 < \cdots$, and $\tilde{\tilde{\lambda}}_0 < \tilde{\tilde{\lambda}}_1 < \cdots$, in λ_1, having the following properties:*

a) *The nontrivial solutions in the bifurcation branches are not given by solving Dirichlet problems.*

b) *These bifurcation branches are global if $\Lambda_k \neq \frac{2\lambda_2\mu_1}{\beta - 2\mu_1}$.*

c) *In $X_{\text{even}} \times X_{\text{even}}$, $\mathcal{S}_k^{\lambda_1} \cap \mathcal{S}_l^{\lambda_1} = \emptyset$ and $\mathcal{R}_k^{\lambda_1} \cap \mathcal{R}_k^{\lambda_1} = \emptyset$ for $k \neq l$.*

d) *Each $\mathcal{S}_k^{\lambda_1}$ and $\mathcal{R}_k^{\lambda_1}$ is unbounded.*

Set

$\mathcal{S}_k :=$ The connected bifurcation component though $((A_{\beta_k}, B_{\beta_k}), \beta_k)$.

Theorem 1.11 (**Theorem E**)**.** *Suppose*

$$T \neq \sqrt{2}\pi k, \quad k = 1, 2, 3, \cdots .$$

Then along $\mathcal{T}$, there are infinitely many bifurcation points of (1.6) with the condition $\lambda_1 = \lambda_2 = 1$: $\beta_1 > \beta_2 > \beta_3 > \cdots \beta_{k_0} > 0 > \beta_{k_0+1} > \beta_{k_0+2} > \cdots > \beta_k$, $\beta_k \to -\sqrt{\mu_1 \mu_2}$ as $k \to \infty$, having the following properties:

a) *The nontrivial solutions in the bifurcation branches are not given by solving Dirichlet problems, and are positive solutions.*

b) *These bifurcation branches are global.*

c) *Each $\mathcal{S}_k$ extends to $-\infty$ for β if $\mu_1 \neq \mu_2$.*

d) *In $X_{\text{even}} \times X_{\text{even}}$, $\mathcal{S}_k \cap \mathcal{S}_l$ for $k \neq l$.*

Theorem 1.12 (**Theorem F**)**.** *Suppose*

$$T = \sqrt{2}\pi j$$

for some $j = 1, 2, \cdots$. Then any point in $(-\sqrt{\mu_1 \mu_2}, \mu_1) \cup (\mu_2, \infty)$ in β is a possible bifurcation point of (1.6) with the condition $\lambda_1 = \lambda_2 = 1$.

The proofs are given in later chapters. In Chapter 2, we study RPEs. In Chapter 3, we see the similar multiple existence results for 2-coupled GPEs in [16] also holds for the periodic condition. In Chapter 4, we study about bifurcations for 2-coupled GPEs. In Chapter 5, we discuss summaries and mention some possible future research topics.

CHAPTER 2

MULTIPLE EXISTENCE RESULTS FOR RELATIVISTIC PENDULUM EQUATIONS [1]

2.1   Introduction

2.1.1   The main theorem (Theorem A)

Here we consider the following system of RPEs for $u \in (H^1([0,T]))^n$:

$$\begin{cases} (\phi(u'))' = \nabla_u F(x,u) + h(x), \\ u(0) = u(T), \quad u'(0) = u'(T). \end{cases} \tag{2.1}$$

Let set $S := \mathbb{R}/\{2\pi\mathbb{Z}\}$, and let denote the L^p spaces with the T-periodic conditions as L_T^p. Suppose the following condition:

(R1) $\phi : B_a \subset \mathbb{R}^n$ *is onto* $\mathbb{R}^n$ *such that* $\psi := \phi^{-1} : \mathbb{R}^n \to B_a$ *and there is a* C^1 *function* $\Psi : \mathbb{R}^n \to \mathbb{R}$ *such that* $\nabla\Psi = \psi$ *and* Ψ *is bounded below.*

(R2) $F \in C^1(S \times \mathbb{R}^n, \mathbb{R})$. *For an integer* $0 \leq k \leq n$, F *is* T-*periodic in* x, 2π-*periodic in* $u_1, \cdots, u_k$ *and* $\nabla_u F$ *is bounded.* $h \in (L_T^2(\mathbb{R}))^n$ *is* 2π-*periodic in* x *and* $\int_0^T h_i(x)dx = 0$ *for* $i = 1, \cdots, k$. *Writing* $u = (v,w)$ *with* $v \in \mathbb{R}^k$ *and* $w \in \mathbb{R}^{n-k}$. *Assume* $\int_0^T (F(x,u) + h(x)u)dx \to -\infty$ *as* $|w| \to \infty$ *uniformly in* $v \in \mathbb{R}$.

Remark 2.1. As an example of (R1),

$$\phi(x) = \frac{x}{\sqrt{1 - |x|^2}}, \quad \psi(x) = \frac{x}{\sqrt{1 + |x|^2}}, \quad \text{and } \Psi(x) = \sqrt{1 + |x|^2}.$$

We give the proof of the following theorem in this chapter.

Theorem 2.2 (**Theorem A**). *([23]) Assume (R1) and (R2). Then the system (2.1) has at least $k + 1$ classical T-periodic solutions.*

---

[1]Coauthored by Kazuya Hata, Jiaquan Liu, and Zhi-Qiang Wang [23].

Note that these solutions are actually *geometrically distinct solutions*, i.e. solutions whose i-th component does not differ by a multiple of T_i, $i = 1, \cdots, n$.

2.1.2 Historical background

For CPEs, the historical original paper [22] was in 1922. From the late 1980's, many studies about the existence and multiplicity of periodic solutions were done (see references in [10] [11] [27] [29] [35] [39]).

For RPEs, this class of equations has received much attention in recent years starting from papers by Torres ([46] [47]). Then in a series of interesting papers ([3] [4] [6] [8] [9] [28]) the problem of the existence and multiplicity of periodic solutions for RPEs has been studied. Fixed Point Theorem Method was mainly used in these works. Brezis and Mawhin ([8] [9]) have explored by using Minimization Methods in convex sets of Banach spaces, and raised some open questions concerning multiplicity of periodic solutions for RPEs.

The original form of RPEs is given by the following form:

$$\left(\frac{u'}{\sqrt{1 - (u')^2}}\right)' + A \sin u = h(x),$$

where A is a constant, $h \in L_T^1(\mathbb{R})$ and $\int_0^{2\pi} h(x)dx = 0$.

RPEs are generalized into the following form:

$$(\phi(u'))' - g(x, u) = h(x),$$

where $\phi(-a, a) \to \mathbb{R}$ is an increasing homeomorphism, g is T-periodic in x and 2π-periodic in u, and h is T-periodic and has mean value zero, with some smoothness conditions. For the following RPEs with continuous periodic forcing h and arbitrary dissipation f

$$(\phi(u'))' + f(u)u' + A \sin u = h(x),$$

Torres proved ([46] [47]) the existence of at least two T-periodic solutions when

$$aT < 2\sqrt{3} \quad \text{and} \quad |\bar{h}| < A(1 - \frac{aT}{2\sqrt{3}}), \tag{2.2}$$

and of at least one T-periodic solution when

$$aT = 2\sqrt{3} \quad \text{and} \quad \bar{h} = 0, \tag{2.3}$$

by using a Schauder Fixed-Point Theorem. The assumption has been improved in [6] by using a Leray-Schauder degree argument, and yielded another multiplicity result by using an Upper and Lower Solution Method.

Under the conditions that $A \in \mathbb{R}$ and $\bar{h} = 0$, Brezis and Mawhin established the existence of a periodic solution of (2.2) and its corollary with the existence of a T-periodic solution for (2.2). Thus the conditions (2.2) and (2.3) are removed, and this was done by a Minimization Argument in closed convex subsets of a Banach space.

More precisely under the following conditions:

(R3) Φ is continuous on $[-a, a]$, of class C^1 on $(-a, a)$, strictly convex, and $\phi := \Phi' : (-a, a) \to \mathbb{R}$ is a homeomorphism such that $\phi(0) = 0$.

(R4) g is a Carathéodory function, bounded on $\mathbb{R}^2$, $g(\cdot, u)$ is T-periodic for any $u \in \mathbb{R}$ and some $T > 0$, $g(x, \cdot)$ is 2π-periodic for a.e. $x \in \mathbb{R}$, $G(x, u) := \int_0^u g(x, s)ds$ is bounded on $\mathbb{R}^2$, and $G(x, \cdot)$ is 2π-periodic for a.e. $x \in \mathbb{R}$.

Brezis and Mawhin proved the following theorem:

Theorem 2.3 (Brezis-Mawhin). *([8]) Under conditions (R3) and (R4), (2.2) has a classical periodic solution.*

In [9], under the following assumptions:

(R5) ϕ is a homeomorphism from $B_a \subset \mathbb{R}^n$ onto $\mathbb{R}^n$ such that $\phi(0) = 0$, $\phi = \nabla\Phi$ with $\Phi : \overline{B}_a \to (-\infty, 0]$ of class C^1 on B_a, continuous and strictly convex on $\overline{B}_a$.

(R6) $F(\cdot, u)$ is measurable on $[0, T]$ for every $u \in \mathbb{R}^n$, $F(x, \cdot)$ is continually differentiable on $\mathbb{R}^n$ for a.e. $x \in [0, T]$, and $\nabla_u F$ satisfies the L^1-Carathéodory conditions.

It was proved:

Theorem 2.4 (Brezis-Mawhin). *([9]) Under (R5) and (R6) and if F is also periodic in each component of u and $\bar{h} = 0$, the system (2.1) has at least one periodic solution.*

Open problems are answered by Theorem A

In [9], some open problems were formulated with the following CPE:

$$\begin{cases} u'' = \nabla_u F(x, u) + h(x), \\ u(0) = u(T), \quad u'(0) = u'(T). \end{cases} \tag{2.4}$$

(Q1) ([9] Remark 9.3) *Under (R6), the periodicity of F in u, $h \in (L^1(0,T))^n$ and $\overline{h} = 0$ holds, then the CPE (2.4) has at least $n+1$ geometrically distinct solutions. Does it hold for RPEs?*

(Q2) ([9] Remark 7.4) *The CPE (2.4) has at least one solution when the assumption*

$$|\nabla_u F(x, u)| \le g(x)|u|^\alpha + k(x)$$

for $g, k \in L^1$ nonnegative with $\alpha = 0$ holds and

$$\overline{F}(u) + \langle \overline{h}, u \rangle \to -\infty \quad \text{as } |u| \text{ to } \infty.$$

Does it hold for RPEs?

Theorem 2.2 answers the question (Q1) when $k = n$, and answers the question (Q2) when $k = 0$.

2.2 Generalized Saddle Point Theorem

In this section, for the proof of our main theorem, we introduce Liu's Generalized Saddle Point Theorem which is a generalized version of Chang's Saddle Point Theorem which is related to cuplength. The cuplength is the invariant number of a cohomology ring whose operator is the cup product on the direct sum of cohomology groups of a space X. For the details, see [24] [32] etc.

Theorem 2.5 (Generalized Saddle Point Theorem). *([27]) Let X be a Banach space having a decomposition: $X = Y \oplus Z$ where Y, $Z \subset X$ with $\dim Z < \infty$. Let V be a finite-dimensional compact C^2-manifold without boundary. Let $f : X \times V \to \mathbb{R}$ be a C^1-function that satisfies the Palais-Smale condition.*

Suppose that f satisfies

(i) $\inf_{x \in Y \times V} f(x) \ge \beta$,

(ii) $\sup_{x \in S \times V} f(x) \le \alpha < \beta$,

where $S = \partial D$, $D = \{y \in Z : |y| \le r\}$ and r, α, β are constants.

Then the function f has at least cuplength$(V) + 1$ *critical points.*

The Generalized Saddle Point Theorem is proved by the following theorem which is related to the Lusternik-Schnirelman category. We do not give the proofs of these theorems here. See [27] for the proofs.

Theorem 2.6. *([27]) Let X be a real Banach space such that $X = Y \oplus Z$, where Y and Z are closed subspaces of X and $\dim Z < \infty$. Let V be a finite-dimensional compact C^2-manifold without boundary.*

Set $D = \{z \in Z : |z| \le R\}$, $S = \partial D = \{z \in Z : |z| = R\}$. To add, set $Q = S \times V \subset X \times V$, $L = Y \times V \subset X \times V$. Then

$$\mathrm{cat}^*(D \times V) \ge \mathrm{cuplength}(V) + 1.$$

Here

$$\mathrm{cat}^*(A) := \inf_{h \in H} \mathrm{cat}(h(A) \cap L),$$

where H is the family of all homeomorphisms of $X \times V$ which are homotopic to the identity mapping while keeping the subset Q fixed.

According to the Theorem 2.5, the following two corollaries (see [11], [27] for the proofs) hold:

Theorem 2.7 (**Chang's Critical Point Theorem**). *([11]) Suppose that A satisfies the following assumptions*

(H_1) $A_\pm := A|_\pm$ has a bounded inverse on $H_\pm$,

(H_2) $\gamma := \dim(H_- \oplus H_+) < \infty$.

Let V^n be a C^2 compact n-manifold without boundary, and let $g \in C^1(H \times V^n, \mathbb{R})$ be a function having a bounded and compact differential $\mathrm{d}g(x)$. Assume that

$$g(P_0 x, v) \to -\infty \quad \text{as } |P_0 x| \to \infty, \quad \text{if } \dim H_0 \ne 0,$$

where P_0 is the orthogonal projection onto H_0. Then the function

$$f(x, v) = \frac{1}{2}(Ax, x) + g(x, v)$$

possesses at least cuplength$(V^n) + 1$ *distinct critical points.*

If further, we assume that $g \in C^2(H \times V, \mathbb{R}^1)$, and that f is nondegenerate, then f has at least $\sum_{i=1}^{n} \beta_i(V^n)$ critical points, where $\beta_i(V^n)$ is the i-th Betti number of V^n.

Theorem 2.8 (**Liu's Critical Point Theorem**). *([27] Theorem 1.8) Let H be a Hilbert space and A be a bounded self-adjoint operator on H which splits the space H into $H_0 \oplus H_+ \oplus H_-$ according to its spectral decomposition. Denote by $P_\pm$ and P_0 the orthogonal projections onto positive, negative spectral space $H_\pm$ and $\ker(A)$, H_0, respectively.*

Assume that

(A1) The restriction $A|_{H_\pm}$ is invertible, i.e., $A|_{H_\pm}$ has a bounded inverse on $H_\pm$.

(A2) The spaces H_- and H_0 are finite-dimensional.

(A3) $G : H \times V \to \mathbb{R}$ is a C^1-function, where V is a finite-dimensional compact C^2-manifold. Suppose that G has a bounded compact gradient dG and

$$G(P_0 u, v) \to -\infty \ (\text{or} + \infty), \ \text{uniformly in } v \text{ as } |P_0 u| \to +\infty.$$

Then the function $f : H \times V \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{2}(Au, u) + G(u, v) \text{ for } x = (u, v)$$

has at least cuplength$(V) + 1$ *critical points.*

Remark 2.9. Because the sum of the Betti numbers is the lower bound of cuplength, Theorem 2.8 improves the result of Theorem 2.7. To add, note that the condition of the finite dimension of H_- on Theorem 2.8 can be dropped ([27] Theorem 3.3).

We apply the Theorem 2.8 for the proof of our main theorem.

2.3 Proof of Theorem A

In this section, we give the proof of Theorem A.

Proof. Let us introduce relativistic kinetic momentum $v = \phi(u')$. Due to the condition (R3), we have $u' = \psi(v)$. Then the original system is equivalent to the following first

order Hamiltonian system:

$$
\begin{cases}
u' = \psi(v), \quad v' = \nabla_u F(x, u) + h(x), \\
u(0) = u(T), \quad u'(0) = u'(T).
\end{cases}
\tag{2.5}
$$

Under the conditions (R1) and (R2), that weak solutions of the original system are classical solutions is known. By the known fact (for example, see [8]), we only need to consider the weak solutions. Without loss of generality, we assume $T = 2\pi$.

Claim 1: *The Euler-Lagrange functional associated with the above system (2.5):*

$$
I(u, v) = \int_0^T u' \cdot v \, dx - \int_0^T \Psi(v) \, dx + \int_0^T F(x, u) \, dx + \int_0^T h(x) \cdot u \, dx
$$

which is defined on the product space $H^{1/2}(S, \mathbb{R}^n) \times H^{1/2}(S, \mathbb{R}^n)$ with $S := \mathbb{R}/\{2\pi\mathbb{Z}\} \cong \mathbb{T}^k$.

Remark 2.10. The functional I corresponds to the action of the Lagrangian, and the integrand L is the Lagrangian. The Hamiltonian of the system (2.5)

$$
H(x, (u, v)) := (u', v) - L(u, v),
$$

which is given by the Legendre transform of the Lagrangian, is

$$
H(x, (u, v)) = -\Psi(v) + F(x, u) + h(x) \cdot u.
$$

On the system (2.1), multiply any vector a into $u' = \psi(v)$, and multiply any vector b into $v' = \nabla_u F(x, u) + h(x)$. Then add these equations, integrate from 0 to T, to get the functional. By taking the functional derivatives, we know Claim 1 holds.

Let $w = (u, v)$. Because of

$$
\int_0^T u' \cdot v \, dx = \frac{1}{2}(Jw', w),
$$

the quadratic part $\int_0^T u' \cdot v \, dx$ defines a linear operator A and its domain is $W^{1,2}(S, \mathbb{R}^{2n})$ whose elements are w, hence

$$
Aw = Jw',
$$

where w_i is the i-th component of w.

Claim 2: *The spectrum of the operator A is $\sigma(A) = \mathbb{Z}$ with eigenvalue being of multiplicity $2n$, and the eigenspace of A corresponding to the eigenvalue $k \in \mathbb{Z}$ is*

$$
\begin{aligned}
E_k &= \exp(ktJ)\mathbb{R}^{2n} = ((\cos kt)I + (\sin kt)J)\mathbb{R}^{2n}, \\
J &:= \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}.
\end{aligned}
$$

The eigenvalue problem

$$
Jw' = \lambda w
$$

with the periodic condition is a well-known ordinary differential equation's problem. For example, p.43 in [31] shows that Claim 2 holds.

Note that

$$
\ker A = E_0 = \mathbb{R}^{2n}.
$$

Define

$$
E = \{u \in H^{1/2}(S, \mathbb{R}^2) : \int_0^T u(x)dx = 0\}.
$$

Then we have $H^{1/2}(S, \mathbb{R}^n) = E \oplus \mathbb{R}^n$. By the conditions, we have I is a translation invariant in $u_1, \cdots, u_n$ with an integer multiple of 2π. Then I can be regarded as defined on $X = (E \oplus E \oplus \mathbb{R}^n \oplus \mathbb{R}^{n-k}) \times \mathbb{R}^k/\{2\pi\mathbb{Z}\} = (E \oplus E \oplus \mathbb{R}^{2n-k}) \times \mathbb{T}^k$.

Claim 3: *The conditions on the Theorem 2.8 are satisfied, i.e., the Theorem 2.8 can be applied to our problem.*

Let $V = \mathbb{T}^k$, $H_0 = E_0$ and $H_+ = E = H_-$ in the theorem.

(A1) *The restriction $A|_{H_\pm}$ is invertible.*

We remind that $w = (u, v)$. We show that the quadratic part

$$
b(u, v) := \int_0^T u'vdx = \frac{1}{2}(Aw, w)
$$

is a non-degenerate (or nonsingular) bilinear form on $E \oplus E$ that means A is invertible. Suppose $b(u, v) = 0$ for any v. Then

$$0 = \int_0^T u'v \, dx = -\int_0^T uv' \, dx.$$

Thus we know $u = \theta$. Next, suppose $b(u, v) = 0$ for any u. Then, from the same discussion, we know $v = \theta$.

(A2) *The spaces H_- and H_0 are finite-dimensional.*

We showed that A has kernel $\mathbb{R}^n \oplus \mathbb{R}^n$ having finite-dimension in Claim 2.

(A3) *$G : H \times V \to \mathbb{R}$ is a C^1-function, where V is a finite-dimensional compact C^2-manifold. Suppose that G has a bounded compact gradient dG and*

$$G(P_0 u, v) \to -\infty \ (or + \infty), \ \text{uniformly in } v \text{ as } |P_0 u| \to +\infty.$$

Note that $V = \mathbb{T}^k$ is a compact, and C^∞-manifold (with the standard metric g_0). Set

$$G(x, u) = -\int_0^T \Psi(u) \, dx + \int_0^T F(x, u) \, dx + \int_0^T h(t)u \, dx.$$

Then G has the bounded compact gradient because of (R1) and (R2).

Finally, we need to verify the Landersman-Lazer type condition, which is,

$$G(P_0 u, v) \to -\infty \ (or \ +\infty) \ \text{uniformly in } v \text{ as } |P_0 u| \to +\infty.$$

The condition assures the Palais-Smale condition. Note that $P_0 u \in \mathbb{R}^{2n}$. Recall that part of the condition on (R2): for $u = (v, w)$,

$$\int_0^1 (F(x, u) + h(x)u) \, dx \to -\infty \text{ as } |w| \to \infty \text{ uniformly in } v \in \mathbb{R}.$$

Thus we just need to show $\Psi(v) \to \infty$ as $|v| \to \infty$.

<u>Claim Ψ</u>: $\Psi(v) \to \infty$ *as* $|v| \to \infty$.

Assume $\psi(0) = 0$. Consider the flow generated by the negative pseudo-gradient vector field $V(\eta)$ of Ψ

$$\frac{d\eta}{dt} = -V(\eta), \quad \eta(0) = x,$$

where V is a pseudo-gradient vector field of Ψ such that

$$\langle V(x), \nabla\Psi(x)\rangle \geq \|\nabla\Psi\|^2, \quad \|V(x)\| \leq 2.$$

Since $\nabla\Psi = \psi$ is a homeomorphism from $\mathbb{R}^n$ onto B_a, there exists $\delta > 0$ such that for all $v \in \mathbb{R}^n$ with $|v| \geq 1$,

$$|\nabla\Psi(v)| = |\psi(v)| \geq \delta. \tag{2.6}$$

First for each v with $|v| = 1$ consider the flow line $\eta(t, v)$. Then as Ψ is non-increasing for t and Ψ is bounded from below there exists $t_0 \leq 0$ such that $\eta(t, v)$ is outside B_1 for $t \leq t_0$ and to integrate (2.6) gives

$$\Psi(\eta(t)) \geq -\delta t + \Psi(\eta(t_0, v)) \to \infty$$

and $|\eta(t)| \to \infty$ as $t \to -\infty$.

Now if there is a $C > 0$ and

$$w_n \in \mathbb{R}^n$$

such that $|w_n| \to \infty$ and $\Psi(w_n) \leq C$, then there are $t_n > 0$ such that $v_n = \eta(t_n, w_n) \in \partial B_1$ (in fact $\eta(t, w) \to 0$ as $t \to +\infty$ due to Ψ being bounded below). Then $w_n = \eta(-t_n, v_n)$.

Assume $v_n \to v_0$. There is $t_0 < 0$ such that $\Psi(\eta(-t_0, v_n)) \to \Psi(\eta(-t_0, v_0)) \geq C + 1$ and $\Psi(\eta(-t_n, v_n)) \geq \Psi(\eta(-t_0, v_n))$ a contradiction for large n with the statement that Ψ is non-increasing for t. Thus Claim Ψ holds.

The following fact:

$$\text{cuplength}(\mathbb{T}^k) = k,$$

is known. Therefore, applying Theorem 2.8 with this fact proves the main theorem, establishing the existence of at least $k + 1$ critical points of I. □

When $n = 1$ and $k = 1$, (R1) can be weakened as the following (H_ϕ) which does not need Ψ to be defined at $x = \pm a$ and the convexity of Φ in (R3).

(H_ϕ) $\phi : (-a, a) \to \mathbb{R}$ is a homeomorphism.

To see this, let $\psi = \phi^{-1}$ then $\psi : \mathbb{R} \to (-a, a)$ and $\psi(t) \to \pm a$ (if ϕ is increasing) as $t \to \pm \infty$. Let $\Psi(t) = \int_0^t \psi(s)ds$. Then $\Psi(t) \to +\infty$ as $t \to \infty$, hence (R1) is satisfied. Thus we have the following corollary:

Corollary 2.11. *Assume (H_ϕ). For any $T > 0$, $A \in \mathbb{R}$ and $h \in L^2_T(\mathbb{R})$ such that $\overline{h} = 0$, (2.1) has at least two classical T-periodic solutions.*

Remark 2.12. Under the condition (R5), the Legendre-Frenchel transform of Φ (denoted as Φ^*) is well-defined and is also strictly convex and of class C^2 (see [9]). Furthermore, $\phi^{-1} = \nabla \Phi^*$. As Φ is bounded from above and Φ^* is bounded from below. Thus (R5) implies (R1). We do not need Φ to be defined on the close ball $\overline{B}_a$.

2.4 Further results

In [8] [9], the convexity of Φ (and thus Ψ) was assumed as the condition (R3). Without requirement of a bounded derivative of F, our first order systems approach gives rise to some further results immediately from some classical works as in [31]. We mention a couple of examples here.

(R7) *There is a $l \in L^4$ such that for all x, u, $F(x, u) \geq (l, u)$.*

(R8) *There is $\alpha \in (0, 2\pi/T)$ and $\gamma \in L^2$ such that $F(x, u) \leq \frac{\alpha}{2}|u|^2 + \gamma(x)$.*

(R9) $\int_0^T F(x, u)dx \to +\infty$ *as $|u| \to \infty$ for $u \in \mathbb{R}^n$.*

(R10) *Uniformly in x, $F(x, u)/u^2 \to 0$ and $F(x, u) \to -\infty$ as $|u| \to \infty$.*

  a) Under (R1) and (R7)-(R9) with Ψ being convex, assume $-F(x, u) \in C^1$ is convex in u. Then (2.1) has at least one T-periodic solution by Theorem 3.5 in [31] because $\Psi(v) - F(x, u)$ is convex in (u, v) now.

  b) Under (R1) and (R10) with Ψ being convex, assume $-F(x, u) \in C^1$ is convex in u. Then for each integer $k \geq 1$, (2.1) has a kT-periodic solution u_k with minimal period T_k such that $T_k \to \infty$ as $k \to \infty$. When $F = F(u)$ for each large k the minimal period of u_k is kT. This follows from Theorem 3.2 in [31].

2.5   Summary

We established ([23]) multiplicity results of periodic solutions for RPEs which answers some open problems raised in [8] [9]. Our approach is to convert the problem into the first order Hamiltonian system and apply the Generalized Saddle Point Theorem (Theorem 2.5). Our first order systems approach allows us to apply some theorems in [31], and follow some further results immediately.

CHAPTER 3

MULTIPLE EXISTENCE RESULTS FOR 2-COUPLED

GROSS-PITAEVSKII EQUATIONS

3.1   Introduction

In 2010, Dancer, Wei, and Weth ([16]) established multiple existence results of positive solutions for 2-coupled GPEs with the Dirichlet boundary condition by using $\mathbb{Z}_2$-Index Theory. We prove that the similar multiple existence results in [16] hold for 2-coupled GPEs with a periodic condition.

In this chapter, we think of 2-coupled GPEs in the fully symmetric case: $\lambda_1 = \lambda_2 = \lambda$ and $\mu_1 = \mu_2 = \mu$. First, we set up the variational structure. Then we apply $\mathbb{Z}_2$-Index Theory which takes advantage of the symmetry.

Recall the problem to be studied, GPEs:

$$\begin{cases} -u'' + \lambda u = \mu u^3 + \beta v^2 u, \\ -v'' + \lambda v = \mu v^3 + \beta u^2 v, \\ u, \ u' \text{ are } T\text{-periodic}, \\ v, \ v' \text{ are } T\text{-periodic}. \end{cases} \tag{3.1}$$

Let denote the Sobolev space $W^{1,2}$ with a T-periodic condition as H_T^1.

Set $\mathcal{H} := H_T^1 \times H_T^1$.

For $(u, v) \in \mathcal{H}$, the corresponding functional of (3.1) is given by

$$\begin{aligned} E(u,v) &= \int_T [\frac{1}{2}(|u'|^2 + \lambda|u|^2 + |v'|^2 + \lambda|v|^2) - \frac{1}{4}(\mu|u|^4 + \mu|v|^4) \\ &\quad - \frac{1}{2}\beta u^2 v^2] dt, \end{aligned} \tag{3.2}$$

where we denoted the integral over the T-period interval as $\int_T$.

Lemma 3.1. *The corresponding energy functional E of (3.1) is of class C^2, and E satisfies the Palais-Smale condition.*

Proof. By the Sobolev Embedding Theorem and Proposition B.34 in [39], E is of class C^2.

E is compactly embedded into L_T^4, where L_T^4 is the L^4 space with the T-periodic condition. E' is a compact perturbation of $\mathrm{Id}_{\mathcal{H}}$ and E satisfies the Palais-Smale condition. $\square$

3.2 Application of $\mathbb{Z}_2$-Index Theory

3.2.1 The variational structure

For the setting up of Variational Methods, we follow the way in [16]. We assume that $\mu > 0$, $\lambda > 0$, and $\beta < 0$.

First, we refer the following lemma which is proved by the Standard Elliptic Regularity (for example, see [19] [21]):

Lemma 3.2. *Every nontrivial critical point* $(u, v) \in \mathcal{H}$ *of* E *is a classical solution of (3.1).*

Let define the norm for $u \in H_T^1$:

$$\|u\|^2 := \int_T (|u'|^2 + |u|^2) dt$$

Note that with any constant $\lambda > 0$,

$$\|u\|_\lambda^2 := \int_T |u'|^2 + \lambda |u|^2 dt$$

defines an equivalent norm of $\|\cdot\|$.

Define

$$
\begin{aligned}
\mathcal{M} \quad := \quad & \{(u, v) \in \mathcal{H}, u, v \neq 0 : \partial_u E(u, v)u = 0, \partial_v E(u, v)v = 0\} \\
= \quad & \left\{ (u, v) \in \mathcal{H}, u, v \neq 0 \; \middle| \;
\begin{array}{l}
\|u\|_\lambda^2 - \beta \int_T u^2 v^2 = \mu \int_T u^4, \\
\|v\|_\lambda^2 - \beta \int_T u^2 v^2 = \mu \int_T v^4
\end{array}
\right\}.
\end{aligned}
$$

Note that all nontrivial critical points (u, v) of E are contained in $\mathcal{M}$ because for a critical point (u, v) of E with any $(x, y) \in \mathcal{H}$,

$$0 = E'(u, v)(x, y) = \partial_u E(u, v)x + \partial_v E(u, v)y.$$

Thus if we choose $(x, y) = (u, \theta)$ or $(x, y) = (\theta, v)$, then we get $\partial_u E(u, v)u = 0$ and $\partial_v E(u, v)v = 0$, respectively.

Lemma 3.3.   (i) $\mathcal{M}$ is a C^2-submanifold of $\mathcal{H}$ of codimension two.

(ii) If (u, v) is a critical point of the restriction $E|_{\mathcal{M}}$ of E to $\mathcal{M}$, then (u, v) is a nontrivial critical point of E.

(iii) $E(u, v) = \frac{1}{4}[\|u\|_\lambda^2 + \|v\|_\lambda^2]$ for $(u, v) \in \mathcal{M}$.

(iv) $E|_{\mathcal{M}} : \mathcal{M} \to \mathbb{R}$ satisfies the Palais-Smale condition $(PS)_c$.

Proof.   (i) By the Sobolev Embedding $H_T^1 \hookrightarrow L_T^4$, we have for $(u, v) \in \mathcal{M}$,

$$C_1 \|u\|^4 \geq |u|_4^4 \geq \|u\|^2 \quad \Rightarrow \quad \|u\| \geq \frac{1}{\sqrt{C_1}},$$

$$C_1 \|v\|^4 \geq |v|_4^4 \geq \|v\|^2 \quad \Rightarrow \quad \|v\| \geq \frac{1}{\sqrt{C_1}},$$

where $C_1 > 0$ is the constant of the Sobolev Embedding. Define $F(u, v)$ given by

$$F(u, v) = \begin{pmatrix} F_1(u, v) \\ F_2(u, v) \end{pmatrix} = \begin{pmatrix} \|u\|_\lambda^2 - \beta \int_T u^2 v^2 - \mu \int_T u^4 \\ \|v\|_\lambda^2 - \beta \int_T u^2 v^2 - \mu \int_T v^4 \end{pmatrix}.$$

Then

$$T_{u,v} := \begin{pmatrix} \partial_u F_1(u, v)u & \partial_u F_2(u, v)u \\ \partial_v F_1(u, v)v & \partial_v F_2(u, v)v \end{pmatrix} = \begin{pmatrix} -2\mu \int_T u^4 & -2\beta \int_T u^2 v^2 \\ -2\beta \int_T u^2 v^2 & -2\mu \int_T v^4 \end{pmatrix},$$

by the following:

$$\partial_u F_1(u, v)u = 2\|u\|_\lambda^2 - 2\beta \int_T u^2 v^2 - 4\mu \int_T u^4 = -2\mu \int_T u^4 \neq 0,$$

$$\partial_v F_1(u, v)v = -2\beta \int_T u^2 v^2 = \partial_u F_2(u, v)u,$$

$$\partial_v F_2(u, v)v = 2\|v\|_\lambda^2 - 2\beta \int_T u^2 v^2 - 4\mu \int_T v^4 = -2\mu \int_T v^4 \neq 0.$$

Note that for $(u, v) \in \mathcal{M}$, because of $\beta < 0$,

$$\mu \int_T u^4 > -\beta \int_T u^2 v^2 \geq 0,$$

$$\mu \int_T v^4 > -\beta \int_T u^2 v^2 \geq 0,$$

which implies that $T_{u,v}$, which is a Hermitian matrix, is negative definite. Thus all non-zero eigenvalues of $T_{u,v}$ are negative, so the number of non-zero eigenvalues, which is equal to the rank of $T_{u,v}$, is two. Hence the vectors

$$
\begin{aligned}
F'(u,v)(u,\theta) &= \begin{pmatrix} \partial_u F_1(u,v)u + \partial_v F_1(u,v)\theta \\ \partial_u F_2(u,v)u + \partial_v F_2(u,v)\theta \end{pmatrix} = \begin{pmatrix} -2\mu \int_T u^4 \\ -2\beta \int_T u^2 v^2 \end{pmatrix}, \\
F'(u,v)(\theta,v) &= \begin{pmatrix} \partial_u F_1(u,v)\theta + \partial_v F_1(u,v)v \\ \partial_u F_2(u,v)\theta + \partial_v F_2(u,v)v \end{pmatrix} = \begin{pmatrix} -2\beta \int_T u^2 v^2 \\ -2\mu \int_T v^4 \end{pmatrix},
\end{aligned}
$$

are linearly independent in $\mathbb{R}^2$ by

$$
T_{u,v} = \begin{pmatrix} F'(u,v)(u,\theta) & F'(u,v)(\theta,v) \end{pmatrix}.
$$

Thus $F'(u,v) : \mathcal{H} \to \mathbb{R}^2$ is surjective. In conclusion, $\mathcal{M}$ is a C^2-submanifold of codimension two.

(ii) Suppose $(u,v) \in \mathcal{M}$ is a critical point of $E|_{\mathcal{M}}$. Then there are Lagrangian multipliers $l_1, l_2 \in \mathbb{R}$ such that

$$
l_1 F_1'(u,v) + l_2 F_2'(u,v) = E'(u,v)
$$

in $\mathcal{H}^*$.

Applying this to (u,θ) and (θ,v) gives us

$$
\begin{aligned}
E'(u,v)(u,\theta) &= 0 = l_1 F_1'(u,v)(u,\theta) + l_2 F_2'(u,v)(u,\theta) \\
&= l_1 \partial_u F_1(u,v)u + l_2 \partial_u F_2(u,v)u, \\
E'(u,v)(\theta,v) &= 0 = l_1 F_1'(u,v)(\theta,v) + l_2 F_2'(u,v)(\theta,v) \\
&= l_1 \partial_v F_1(u,v)v + l_2 \partial_v F_2(u,v)v,
\end{aligned}
$$

respectively. Thus we get

$$
T_{u,v} \cdot \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \begin{pmatrix} \partial_u F_1(u,v)u & \partial_u F_2(u,v)u \\ \partial_v F_1(u,v)v & \partial_v F_2(u,v)v \end{pmatrix} \cdot \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
$$

Then we get $l_1 = 0 = l_2$ because $T_{u,v}$ is negative definite, therefore $E'(u,v)(x,y) = 0$ for any $(x,y) \in \mathcal{H}$ (which is sometimes written as $E'(u,v) = 0$) by the above

Lagrangian multipliers equation.

(iii) For any $(u, v) \in \mathcal{M}$, we have

$$
\begin{aligned}
E(u, v) &= \frac{1}{2}[\|u\|_\lambda^2 + \|v\|_\lambda^2] - \frac{1}{4}\int_T (\mu u^4 + \mu v^4)dt - \frac{\beta}{2}\int_T u^2 v^2 dt \\
&= \frac{1}{2}[\|u\|_\lambda^2 + \|v\|_\lambda^2] \\
&\quad -\frac{1}{4}[\|u\|_\lambda^2 + \|v\|_\lambda^2 - 2\beta \int_T u^2 v^2 dt] \\
&\quad -\frac{\beta}{2}\int_T u^2 v^2 dt \\
&= \frac{1}{4}[\|u\|_\lambda^2 + \|v\|_\lambda^2].
\end{aligned}
$$

(iv) Suppose $(u_k, v_k)_k \subset \mathcal{M}$ is a Palais-Smale sequence for $E|_\mathcal{M}$ (hence, $E(u_k, v_k) \to c$ and $E'(u_k, v_k) \to 0$). Note that for any positive number λ, there exists a constant C_2 such that

$$
0 \leq \|u\|_\lambda^2 + \|v\|_\lambda^2 \leq C_2(\|u\|^2 + \|v\|^2)
$$

for any $(u, v) \in \mathcal{H}$. Thus $(u_k, v_k)_k$ is bounded in $\mathcal{H}$ by (iii).

Therefore there exists a weekly convergent subsequence of $(u_k, v_k)_k$. Passing to the subsequence, then by the Sobolev Embedding that we used before, we may assume that the subsequence converges weekly in $\mathcal{H}$ and converges strongly in L_T^4, hence $(u_k, v_k) \rightharpoonup (u, v) \in \mathcal{H}$ and $u_k \to u, v_k \to v$ in L_T^4. We think of only the case of $u \neq 0$ and $v \neq 0$.

We have to show that $(u_k, v_k) \to (u, v) \in \mathcal{H}$. Note that

$$
o(1) = E'|_\mathcal{M}(u_k, v_k) = E'(u_k, v_k) - l_1^k F_1'(u_k, v_k) - l_2^k F_2'(u_k, v_k)
$$

as $k \to \infty$ for appropriate Lagrangian multiplier's sequences $(l_1^k)_k, (l_2^k)_k \subset \mathbb{R}$.

Thus since $(u_k, v_k)_k$ is bounded in $\mathcal{H}$,

$$
\begin{aligned}
o(1) &= \begin{pmatrix} E'(u_k, v_k)(u_k, 0) - [l_1^k F_1'(u_k, v_k) + l_2^k F_2'(u_k, v_k)](u_k, 0) \\ E'(u_k, v_k)(0, v_k) - [l_1^k F_1'(u_k, v_k) + l_2^k F_2'(u_k, v_k)](0, v_k) \end{pmatrix} \\
&= - \begin{pmatrix} [l_1^k F_1'(u_k, v_k) + l_2^k F_2'(u_k, v_k)](u_k, 0) \\ [l_1^k F_1'(u_k, v_k) + l_2^k F_2'(u_k, v_k)](0, v_k) \end{pmatrix} \\
&= -T_{u_k, v_k} \cdot \begin{pmatrix} l_1^k \\ l_2^k \end{pmatrix} \\
&= [-T_{u,v} + o(1)] \cdot \begin{pmatrix} l_1^k \\ l_2^k \end{pmatrix}.
\end{aligned}
$$

Because of $(u_k, v_k) \in \mathcal{M}$ for every k and $\limsup_{k \to \infty} \beta \int_T u_k^2 v_k^2 \leq 0$, where we used $\limsup$ since the limit might not converge, the weak convergence implies that

$$
\|u\|_\lambda^2 - \beta \int_T u^2 v^2 \leq \mu \int_T u^4,
$$
$$
\|v\|_\lambda^2 - \beta \int_T u^2 v^2 \leq \mu \int_T v^4.
$$

Thus, the same as the proof in (i), $T_{u,v}$ is negative definite, and thus $l_1^k, l_2^k \to 0$ by the above $o(1)$ equation. Therefore $E'(u_k, v_k) \to 0$ strongly because $F_1'(u_k, v_k)$ and $F_2'(u_k, v_k)$ are still bounded in $\mathcal{H}^*$ as $k \to \infty$.

Therefore (u, v) is the weak solution of (3.1). Because of $(u, v) \in \mathcal{M}$, multiplying the first equation of (3.1) by u and integrating by parts, we get

$$
\|u\|_\lambda^2 = \mu |u|_4^4 + \beta \int_T v^2 u^2 = \lim_{k \to \infty} [\mu |u_k|_4^4 + \beta \int_T v_k^2 u_k^2] = \lim_{k \to \infty} [\|u_k\|_\lambda^2]
$$

where the strong convergence in L_T^4 made the strong convergence for $\beta \int_T v^2 u^2$. Since $\|\cdot\|_\lambda^2$ defines an equivalent norm of $\|\cdot\|^2$, $\|u\|_\lambda^2 = \lim_{k \to \infty} [\|u_k\|_\lambda^2]$ implies that $\|u\|^2 = \lim_{k \to \infty} \|u_k\|^2$, hence $u_k \to u$ strongly in H_T^1. Likewise, we get $v_k \to v$ strongly in H_T^1.

$\square$

3.2.2 $\mathbb{Z}_2$-Index Theory

Now we apply $\mathbb{Z}_2$-Index Theory for the problem (3.1).

Define

$$
\begin{aligned}
K_c &:= \{(u,v) \in \mathcal{M} : E(u,v) = c, E'(u,v) = 0\} \\
&= \{(u,v) \in \mathcal{M} : E|_{\mathcal{M}}(u,v) = c, (E|_{\mathcal{M}})'(u,v) = 0\}.
\end{aligned}
$$

Let us also define a group action of $\mathbb{Z}_2$ on $\mathcal{H}$:

$$
\sigma : \mathcal{H} \to \mathcal{H}, \quad (u,v) \mapsto \sigma(u,v) = (v,u),
$$

and define the least level of E having a fixed point of σ as

$$
c(\beta) := \inf\{E(u,v) : (u,v) \in \mathcal{M} \text{ is a fixed point of } \sigma\}.
$$

Lemma 3.4. $\lim_{\beta \searrow -\mu} c(\beta) = \infty.$

Proof. The proof is derived from [16]. $\mathcal{M}$ does not have fixed points of σ for $\beta \leq -\mu$ because of $\|u\|_\lambda^2 = (\mu + \beta)\int_T u^4$, thus $c(\beta) = \infty$.

When $-\mu < \beta < 0$ and $(u,u) \in \mathcal{M}$ for some $u \in H_T^1$, then

$$
\|u\|_\lambda^2 = (\mu + \beta)|u|_4^4 \leq (\mu + \beta)C_\lambda \|u\|_\lambda^4
$$

where C_λ is the constant of the Sobolev Embedding $H_T^{1,\lambda} \hookrightarrow L_T^4$ as we used on Lemma 3.3 (i) where $H_T^{1,\lambda}$ is H_T^1 with the norm $\|\cdot\|_\lambda^2$. Thus we get $\|u\|_\lambda^2 \geq \frac{1}{C_\lambda(\mu+\beta)}$, so $E(u,u) \geq \frac{1}{2C_\lambda(\mu+\beta)}$ by Lemma 3.3 (iii). Therefore $\lim_{\beta \searrow -\mu} c(\beta) \geq \lim_{\beta \searrow -\mu} \frac{1}{2C_\lambda(\mu+\beta)} = \infty$. $\square$

Proposition 3.5. *Let $c \in \mathbb{R}$, and let $N \subset \mathcal{M}$ be a relatively open σ-invariant neighborhood of K. Then there exists $\epsilon > 0$ and a C^1-deformation $\eta : [0,1] \times \mathcal{M}^{c+\epsilon} \setminus N \to \mathcal{M}^{c+\epsilon}$ such that, for all $(u,v) \in \mathcal{M}^{c+\epsilon} \setminus N$ and $s \in [0,1]$,*

$$
\begin{aligned}
\eta(0,(u,v)) &= (u,v), \\
\eta(s,(u,v)) &\in M^{c-\epsilon}, \\
\sigma[\eta(s(u,v))] &= \eta(s, \sigma(u,v)).
\end{aligned}
$$

Proof. The proof is done in the standard way by using the Palais-Smale condition and the fact that $\mathcal{M}$ is C^1-manifold. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Recall the following lemma:

Lemma 3.6. *Let $A, B \subset \mathcal{M}$ be closed and σ-invariant.*

  (i) *If $A \subset B$, then $\gamma(A) \leq \gamma(B)$,*

  (ii) *$\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$,*

  (iii) *If $h : A \to \mathcal{M}$ is continuous and σ-equivalent, then $\gamma(A) \leq \gamma(\overline{h(A)})$.*

If A does not contain fixed points of σ, then

  (iv) *if $\gamma(A) > 1$, then A is an infinite set;*

  (v) *if A is compact, then $\gamma(A) < \infty$, and there exists a relatively open σ-invariant neighborhood N of A in $\mathcal{M}$ such that $\gamma(A) = \gamma(N)$.*

Finally,

  (vi) *if S is the boundary of a bounded symmetric neighborhood of zero in a k-dimensional normed vector space and $\psi : S \to \mathcal{M}$ is a continuous map satisfying $\psi(-u) = \sigma(\psi(u))$, then $\gamma(\psi(S)) \geq k$.*

Proof. The proofs are done in the standard way of Index Theory. Here we give the proof of (vi) only to see the relation with the Borsuk-Ulam Theorem. For other proof, for example, see [16] [39].

  (vi) Suppose the contrary, $\gamma(\psi(S)) < k$, hence there exists a continuous map $h : \psi(S) \to \mathbb{R}^{k-1} \setminus \{0\}$ satisfying the property $h(\sigma(u,v)) = -h(u,v)$. Then $h \circ \psi : S \to \mathbb{R}^{k-1} \setminus \{0\}$ is an odd continuous map which contradicts the Borsuk-Ulam Theorem (see Figure 1.2). Therefore $\gamma(\psi(S)) \geq k$.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Proposition 3.7. *For every $c < c(\beta)$ we have $\gamma(K_c) < \infty$, and there exists $\epsilon > 0$ such that*

$$\gamma(\mathcal{M}^{c+\epsilon}) \leq \gamma(\mathcal{M}^{c-\epsilon}) + \gamma(K_c).$$

Proof. For example see [16] [39]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Define

$$c_k := \inf\{c \in \mathbb{R} : \gamma(\mathcal{M}^c) \geq k, \ k \in \mathbb{N}\}.$$

Proposition 3.8. *(i) For every k, $c_k < \infty$ is bounded independently of $\beta < 0$.*

(ii) $c_k \to \bar{c}$ as $k \to \infty$, where $c(\beta) \leq \bar{c} \leq \infty$.

(iii) If $c := c_k = c_{k+1} = \cdots = c_l < c(\beta)$ for some $l \geq k$, then $\gamma(K_c) \geq l - k + 1$.

(iv) If $c_k < c(\beta)$ then $K_{c_k} \neq \emptyset$, and $\mathcal{M}^{c_k}$ contains at least k pairs (u, v), (v, u) of critical points of E.

Proof. (i) The proof is derived from [16]. Let $W^k \subset H_T^1$ be a k-dimensional subspace consisting of functions $u \in H_T^1$, satisfying the condition $\int_T u = 0$.

Define

$$S^{k-1} := \{u \in W^k : \|u\| = 1\},$$

then for any $u \in S^{k-1}$,

$$u^+ := \max\{u, 0\}, \quad u^- := -\min\{u, 0\}$$

are not θ because of the condition of W^k, $\int_T u = 0$.

Define

$$\psi : S^{k-1} \to \mathcal{M}, \quad \psi(u) = \left(\sqrt{\frac{\|u^+\|_\lambda^2}{\mu|u^+|_4^4}} u^+, \sqrt{\frac{\|u^-\|_\lambda^2}{\mu|u^-|_4^4}} u^-\right),$$

which is continuous, and satisfies

$$\psi(-u) = \sigma(\psi(u))$$

for every $u \in S^{k-1}$ because of $(-u)^+ = u^-$ and $(-u)^- = u^+$. Thus $\gamma(\psi(S^{k-1})) \geq k$ by Lemma 3.6. Therefore we have

$$c_k \leq \sup_{u \in S^{k-1}} E(\psi(u)) < \infty.$$

By the definition of ψ and Lemma 3.3 (iii), $\sup_{u\in S^{k-1}} E(\psi(u))$ is independent of β.

(ii)-(iv) The proofs are done by the standard method of Index Theory. For example, see [16] [39]. $\qquad\square$

3.2.3 Proof of Theorem B

Theorem 3.9 (Theorem B). *a) If $\beta \leq -\mu$, then (3.1) with the fully symmetric condition $\mu_1 = \mu_2 = \mu > 0$ and $\lambda_1 = \lambda_2 = \lambda > 0$ admits a sequence $(u_k, v_k)_k$ of solutions with*

$$\|u_k\|_{L_T^\infty} + \|v_k\|_{L_T^\infty} \to \infty.$$

b) For any positive integer k there exists a number $\tilde{\beta}_k > -\mu$ such that, for $\beta < \tilde{\beta}_k$, (3.1) with the fully symmetric condition $\mu_1 = \mu_2 = \mu > 0$ and $\lambda_1 = \lambda_2 = \lambda > 0$ has at least k pairs (u, v), (v, u) of solutions.

Proof. a) By Lemma 3.4, we can choose a sequence $(u_k, v_k)_k \subset K_{c_k}$ for every k which is a sequence of nontrivial critical points of E such that $E(u_k, v_k) \to \infty$. Thus $\|u_k\|_\lambda^2 + \|v_k\|_\lambda^2 \to \infty$ by Lemma 3.3 (iii). Note that

$$T\mu(|u_k|_\infty^4 + |v_k|_\infty^4) \geq \mu(|u_k|_4^4 + |v_k|_4^4) \geq \|u_k\|_\lambda^2 + \|v_k\|_\lambda^2.$$

Therefore $|u_k|_\infty + |v_k|_\infty \to \infty$ as $k \to \infty$.

b) By the Lemma 3.4 and Proposition 3.8 (i), for any positive integer k there exists $\tilde{\beta}_k > -\mu$ such that for $\beta < \tilde{\beta}_k$, $c_k < c(\beta)$ holds. Thus E has at least k pairs of nontrivial critical points by Proposition 3.8 (iv).

$\qquad\square$

3.3 Summary

In this chapter, we applied $\mathbb{Z}_2$-Index Theory for 2-coupled GPEs (in the case $\lambda_1 = \lambda_2 = \lambda$ and $\mu_1 = \mu_2 = \mu$) with the periodic condition. We followed the way of [16] which uses the decomposition of u into u^+ and u^- to show that c_k is bounded independent of β, and uses the well-constructed function ψ. Finally, we concluded that the similar multiple existence results in [16] hold for the periodic condition case too.

CHAPTER 4

LOCAL AND GLOBAL BIFURCATIONS FOR 2-COUPLED GROSS-PITAEVSKII EQUATIONS

4.1   Introduction

In this chapter, we study the bifurcation structures of 2-coupled GPEs with the periodic boundary condition, and prove some theorems (Theorem 4.20, 4.31, 4.42, 4.44) of existences and qualitative properties of local and global bifurcations. For making the proof simpler, we think of the radial domain, hence we focus on the following system:

$$
\begin{cases}
-u'' + \lambda_1 u = \mu_1 u^3 + \beta v^2 u, \\
-v'' + \lambda_2 v = \mu_2 v^3 + \beta u^2 v, \\
u(-T/2) = u(T/2), \quad u'(-T/2) = u'(T/2), \\
v(-T/2) = v(T/2), \quad v'(-T/2) = v'(T/2),
\end{cases}
\tag{4.1}
$$

where $\lambda_1, \lambda_2 > 0$, $\mu_1, \mu_2 > 0$, and $\beta \in \mathbb{R}$.

Let denote the Sobolev space $W^{1,2}$ with the inner product and the T-periodic conditions on $[-T/2, T/2]$ as H_T^1. The corresponding energy functional $E : H_T^1 \times H_T^1 \to \mathbb{R}$ is given by

$$
\begin{aligned}
E(u, v) &= \int_T [\frac{1}{2}(|u'|^2 + \lambda_1 |u|^2 + |v'|^2 + \lambda_2 |v|^2) - \frac{1}{4}(\mu_1 |u|^4 + \mu_2 |v|^4) \\
&\quad - \frac{1}{2}\beta u^2 v^2] dt.
\end{aligned}
\tag{4.2}
$$

Remark 4.1. Just as with the Lemma 3.1, we know the functional E of (4.2) is of C^2, and satisfies the Palais-Smale condition.

The system (4.1) has the following constant solutions:

In the general case, $(u, v) = (\omega_1, \theta)$, $(u, v) = (\theta, \omega_2)$, $\quad \omega_i := \sqrt{\dfrac{\lambda_i}{\mu_i}}$.

In the case $\lambda_1 = \lambda_2 = 1$, $(u, v) = (A_\beta, B_\beta)$, $\quad A_\beta := \sqrt{\dfrac{\mu_2 - \beta}{\mu_1 \mu_2 - \beta^2}}$, $B_\beta := \sqrt{\dfrac{\mu_1 - \beta}{\mu_1 \mu_2 - \beta^2}}$.

We study the bifurcation structures bifurcating from the following solution curves:

$$
\begin{aligned}
\mathcal{T}_1^\beta &:= \{(\omega_1, \theta) \in H_T^1 \times H_T^1 : \beta \in \mathbb{R}\}, \\
\mathcal{T}_1^{\lambda_1} &:= \{(\omega_1, \theta) \in H_T^1 \times H_T^1 : \lambda_1 > 0\}, \\
\mathcal{T} &:= \{(A_\beta, B_\beta) \in H_T^1 \times H_T^1 : \beta \in (-\sqrt{\mu_1 \mu_2}, \mu_1) \cup (\mu_2, \infty)\},
\end{aligned}
$$

Remark 4.2. Obviously, the bifurcation structures from $\mathcal{T}_1^\beta$ and $\mathcal{T}_1^{\lambda_1}$ are similar to the bifurcation structures from

$$
\begin{aligned}
\mathcal{T}_2^\beta &:= \{(\theta, \omega_2) \in H_T^1 \times H_T^1 : \beta \in \mathbb{R}\}, \\
\mathcal{T}_2^{\lambda_1} &:= \{(\theta, \omega_2) \in H_T^1 \times H_T^1 : \lambda_1 > 0\},
\end{aligned}
$$

respectively.

For the bifurcating from $\mathcal{T}$, we set $\lambda_1 = \lambda_2 = 1$, hence the following is the system to be studied:

$$
\begin{cases}
-u'' + u = \mu_1 u^3 + \beta v^2 u, \\
-v'' + v = \mu_2 v^3 + \beta u^2 v, \\
u(-T/2) = u(T/2), \quad u'(-T/2) = u'(T/2), \\
v(-T/2) = v(T/2), \quad v'(-T/2) = v'(T/2).
\end{cases}
\tag{4.3}
$$

In [2] [44], for the following system:

$$
\begin{cases}
-u'' + u = \mu_1 u^3 + \beta v^2 u, & \text{in } \Omega, \\
-v'' + v = \mu_2 v^3 + \beta u^2 v, & \text{in } \Omega,
\end{cases}
\tag{4.4}
$$

where $\Omega \subset \mathbb{R}^m$, $m \leq 3$, the bifurcation structures of positive solutions from

$$\mathcal{T}_\omega = \{(\beta, u_\beta, v_\beta \in \mathbb{R} \times H_0^1(\Omega) \times H_0^1(\Omega)) : \beta \in (-\sqrt{\mu_1\mu_2}, \mu_1) \cup (\mu_2\infty)\},$$

$$u_\beta := \sqrt{\frac{\mu_2 - \beta}{\mu_1\mu_2 - \beta^2}}\omega, \quad v_\beta := \sqrt{\frac{\mu_1 - \beta}{\mu_1\mu_2 - \beta^2}}\omega,$$

have been studied, where $\omega \in H_0^1(\Omega)$ is a solution of

$$\begin{cases} -\omega'' + \omega = \omega^3, \\ \omega > 0 \quad \text{in } \Omega. \end{cases}$$

To show the theorem of bifurcations from $\mathcal{T}$ (Theorem 4.42), we use some techniques of [2] [44]. In our case, to show there are some local or global bifurcations, we need to restrict the functions space into the space of even functions, thus we study the bifurcation structures of the following auxiliary problem too:

$$\begin{cases} -u'' + \lambda_1 u = \mu_1 u^3 + \beta v^2 u, \\ -v'' + \lambda_2 v = \mu_2 v^3 + \beta u^2 v, \\ u(-T/2) = u(T/2), \quad u'(-T/2) = u'(T/2), \\ v(-T/2) = v(T/2), \quad v'(-T/2) = v'(T/2), \\ u(-t) = u(t), \quad v(-t) = v(t) \quad \text{for } \forall t \in [-T/2, T/2]. \end{cases} \tag{4.5}$$

4.2 Bifurcations in β from the semi-trivial solution curve $\mathcal{T}_1^\beta$ (Theorem C)

Set

$$\Lambda_k := (k\frac{2\pi}{T})^2, \quad k = 1, 2, \cdots$$

and we fix $\lambda_1 > 0$ such that

$$\lambda_1 \neq \frac{\Lambda_k}{2}, \quad \forall k.$$

We also fix $\lambda_2, \mu_i > 0$ in the following. Set

$$\omega_i := \sqrt{\frac{\lambda_i}{\mu_i}},$$

$$\mathcal{T}_1^\beta := \{(\omega_1, \theta) \in \mathcal{H} : \beta \in \mathbb{R}\}.$$

Then $\mathcal{T}_1^\beta$ is a solution curve for $\beta \in \mathbb{R}$ containing semi-trivial solutions of the form (ω_1, θ).

Lemma 4.3. *Along $\mathcal{T}_1^\beta$, all possible bifurcation points of (4.1) are: $\beta_0 < \beta_1 < \beta_2 < \cdots$ in β, where*

$$\beta_k := (\Lambda_k + \lambda_2)\frac{\mu_1}{\lambda_1}.$$

Proof. The linearization of (4.1) at (ω_1, θ) is given by

$$\begin{cases} -\phi'' = 2\lambda_1 \phi, \\ -\psi'' = (\beta\frac{\lambda_1}{\mu_1} - \lambda_2)\psi, \\ \phi(-T/2) = \phi(T/2), \quad \psi(-T/2) = \psi(T/2), \\ \phi'(-T/2) = \phi'(T/2), \quad \psi'(-T/2) = \psi'(T/2). \end{cases}$$

Recall the following eigenvalue problem

$$\begin{cases} -\phi'' = \Lambda_k \phi, \\ \phi(-T/2) = \phi(T/2), \quad \phi'(-T/2) = \phi'(T/2). \end{cases}$$

has eigenvalues

$$\Lambda_k := (k\frac{2\pi}{T})^2, \quad k = 0, 1, \cdots$$

and its eigenfunctions

$$\{\mathbf{1}\} \quad (k = 0),$$
$$\{\sin(k\frac{2\pi}{T}t), \cos(k\frac{2\pi}{T}t)\} \quad (k \neq 0).$$

Therefore when

$$\beta_k = (\Lambda_k + \lambda_2)\frac{\mu_1}{\lambda_1},$$

the linearized problem has a non-trivial kernel. On the other hand, when $\beta_k \neq (\Lambda_k + \lambda_2)\frac{\mu_1}{\lambda_1}$, θ is nondegenerate for the second equation of (4.1), hence the linearized problem has only zero solutions. Therefore we have the possibility that a bifurcation happens at only $\beta_0 < \beta_1 < \beta_2 < \cdots$ in β. $\qquad\square$

Remark 4.4. We can choose $\mu_1 \in (0, \infty)$ as the bifurcation parameter instead of λ_1.

Recall the definition of the critical groups of an isolated critical point u:

$$C_n(\varphi, u) := H_n(\varphi^{\varphi(u)} \cap U, \varphi^{\varphi(u)} \cap U \setminus \{u\}), \quad n = 0, 1, 2, \cdots$$

where $H_n(A, B)$ are the relative homology groups.

We also recall some properties here:

If $\varphi'(u) = 0$ and $\varphi''(u)$ is invertible (hence, u is a non-degenerate critical point) then

$$\dim C_n(\varphi, u) = \delta_{n,i},$$

where i is the Morse index of $\varphi''(u)$.

Theorem 4.5. *([31] Theorem 8.9 p.198) Let U be an open neighborhood of 0 in a Hilbert space V, let Λ be an open interval and let $f(\lambda, u)$ be the gradient with respect to u of $\varphi \in C^2(\Lambda \times U, \mathbb{R})$. Assume that the following conditions are satisfied:*

a) *0 is a critical point of $\varphi_\lambda = \varphi(\lambda, \cdot)$ for every $\lambda \in \Lambda$ and 0 is an isolated critical point of φ_a and φ_b for some reals $a < b$ in Λ.*

b) *φ_λ satisfies the Palais-Simale condition over a closed ball $B[0, r] \subset U$ for every $\lambda \in [a, b]$.*

c) *There exists $n \in \mathbb{N}$ such that*

$$\dim C_n(\varphi_a, 0) \neq \dim C_n(\varphi_b, 0).$$

Then there exists a bifurcation point $(\lambda_0, 0) \in [a, b] \times \{0\}$ for $f(\lambda, u) = 0$.

Lemma 4.6. *Along $\mathcal{T}_1^\beta$, there are bifurcation points $\beta_0 < \beta_1 < \beta_2 < \cdots$ in β.*

Proof. We need to verify that the bifurcations actually happen.

Recall

$$E''(\omega_1, \theta)|_{(\phi, \psi)^2} = \int_T |\phi'|^2 + \lambda_1 \phi^2 - 3\mu_1 \omega_1^2 \phi^2 + |\psi'|^2 + \lambda_2 \psi^2 - \beta \omega_1^2 \psi^2 dt.$$

Define V_k:

V_k = a subspace of $H^1[-T/2, T/2]$ spanned by eigenfunctions associated to Λ_k of the linearized problem.

The product space $\{\theta\} \times V_k$ is a subspace of $\ker[E''(\omega_1, \theta)]$, and has a finite dimension 1 when $k = 0$, and 2 when $k \neq 0$. We show that the Morse index changes at the bifurcation point on the subspace $\{\theta\} \times V_k$ which is in the kernel space of the linearized problem.

<u>Claim 1</u>: *For $\beta_{k-1} < \beta < \beta_k$ (set as $\beta_{-1} = 0$) $E''(\omega_1, \theta)|_{(\phi,\psi)^2}$ is negative definite on $\{\theta\} \times (V_0 \oplus \cdots \oplus V_{k-1})$, and positive definite on $\{\theta\} \times (V_k \oplus V_{k+1} \oplus \cdots)$.*

<u>Claim 2</u>: *For $\beta_{k+1} > \beta > \beta_k$ $E''(\omega_1, \theta)|_{(\phi,\psi)^2}$ is negative definite on $\{\theta\} \times (V_0 \oplus \cdots \oplus V_k)$, and positive definite on $\{\theta\} \times (V_{k+1} \oplus V_{k+2} \oplus \cdots)$.*

Note that

$$
\begin{aligned}
E''(\omega_1, \theta)|_{(\phi,\psi)^2} &= \int_T [-\phi'' - 2\lambda_1 \phi]\phi \\
&\quad + [-\psi'' - (\beta_l \frac{\lambda_1}{\mu_1} - \lambda_2)\psi]\psi - (\beta - \beta_l)\frac{\lambda_1}{\mu_1}\psi^2 dt.
\end{aligned}
$$

Thus in $\{\theta\} \times V_l$

$$
E''(\omega_1, \theta)|_{(\theta,\psi_l)^2} = \int_T -(\beta - \beta_l)\frac{\lambda_1}{\mu_1}\psi_l^2 dt.
$$

Under the assumption of Claim 1, if $l < k$, then because of $\beta_l < \beta$, we get $E''(\omega_1, \theta)|_{(\theta,\psi_l)^2} < 0$. If $l \geq k$ then because of $\beta_l > \beta$, we get $E''(\omega_1, \theta)|_{(\theta,\psi_l)^2} > 0$. Thus Claim 1 holds.

Under the assumption of Claim 2, if $l \leq k$, then because of $\beta_l < \beta$, we get $E''(\omega_1, \theta)|_{(\theta,\psi_l)^2} < 0$. If $l > k$, then because of $\beta_l > \beta$, we get $E''(\omega_1, \theta)|_{(\theta,\psi_l)^2} > 0$. Thus Claim 2 holds.

Therefore by Claim 1 and Claim 2, the Morse index of E changes at β_k. By Theorem 4.5, the statement holds. □

Lemma 4.7. *The bifurcation solutions from $\mathcal{T}_1^\beta$ include solutions which are not given by solving Dirichlet problems.*

Proof. Let $(u, v) \in \mathcal{H}$ be a solution in the bifurcation from $\mathcal{T}_1^\beta$, which is sufficiently close to the bifurcation point but not in the bifurcation point.

By using sufficiently small $\epsilon > 0$, u and v can be represented by $u = \omega_1 + h_1$ with $\|h_1\| < \epsilon$ and $v = h_2$ with $\|h_2\| < \epsilon$, then by taking a limit $\epsilon \to 0$, we get

$$\begin{cases} -h_1'' + \lambda_1 h_1 = 3\mu_1 \omega_1^2 h_1, \\ -h_2'' + \lambda_2 h_2 = \beta \omega_1^2 h_2. \end{cases} \tag{4.6}$$

Suppose (u, v) is given by solving Dirichlet problems, hence $u(-T/2) = u(T/2) = 0$ and $v(-T/2) = v(T/2) = 0$. Thus $h_1(-T/2) = h_1(T/2) = -\omega_1$ and $h_2(-T/2) = h_2(T/2) = 0$. However because ω_1 is nondegenerate for $-u'' + \lambda_1 u = \mu_1 u^3$ with the periodic condition on $[-T/2, T/2]$, the first equation of (4.6) has only the zero solution because it is not in the bifurcation point. Therefore we get a contradiction.

$\square$

We show a stronger conclusion as in the following:

Lemma 4.8. *The bifurcation nontrivial solutions from $\mathcal{T}_1^\beta$ are not given by solving Dirichlet problems.*

Proof. Suppose (u, v), which is in the bifurcation branch from $\mathcal{T}_1^\beta$, are given by solving Dirichlet problems, hence $u(-T/2) = u(T/2) = 0$ and $v(-T/2) = v(T/2) = 0$. Note that $u'(-T/2) = u'(T/2)$ by the periodic condition. Because the solution is in the bifurcation from $\mathcal{T}_1$, the solution can be earned to change continuously from (ω_1, θ). In such continuous change, there is a part satisfying $u \geq 0$ connected to $\mathcal{T}_1^\beta$, so assume $u \geq 0$. Note that a non-simple zero cannot be earned inside the domain at the first time due to the Strong Maximum Principle. Thus the first zero of the solutions on the branch part is realized on the boundary. If $u'(-T/2)$ is positive (or negative), then $u < 0$ around $t = T/2$ (or $t = -T/2$, respectively) because of

$$u = u'(T/2)(t - T/2) + o(t - T/2),$$

thus we get a contradiction. Therefore $u'(-T/2) = u'(T/2) = 0$. Since θ is a solution of the first equation of (4.1) satisfying the condition, then we get $u = \theta$ because of the uniqueness of solutions for differential equations.

$\square$

With fixed λ_1, set

$$\mathcal{S}_k^\beta := \text{The connected bifurcation component through } ((\omega_1, \theta), \beta_k).$$

Recall the Rabinowitz Global Bifurcation Theorem which can be proved by using the Leray-Schauder Degree Theory:

Theorem 4.9 (**Rabinowitz Global Bifurcation Theorem**). *([38]) Let X be a real Banach space and $F(x, \lambda) = x - \lambda L x - N(x, \lambda)$, where $L \in \mathcal{L}(X, X)$ and $N : X \times \mathbb{R} \to X$ are compact. Let S be the solution set of $F(x, \lambda) = \theta$, $S_+ = S \setminus (\{\theta\} \times \mathbb{R})$, and let ζ be the component of S_+, containing (θ, λ_1). Assume that $N(x, \lambda) = o(\|x\|)$ uniformly on any finite interval in λ and that $\lambda_1^{-1} \in \sigma(L)$ is an eigenvalue of odd multiplicity. Then the following alternatives hold: Either*

a) *ζ is unbounded; or*

b) *there are only a finite number of points $\{(\theta, \lambda_i) : i = 1, \cdots, l\}$ lying on ζ where $\lambda_i^{-1} \in \sigma(L)$, $i = 1, 2, \cdots, l$. Furthermore, if β_i is the algebraic multiplicity of λ_i^{-1}, then $\sum_{i=1}^l \beta_i$ is even.*

Remark 4.10. Note that the branch that satisfies the conditions of the Rabinowitz Global Bifurcation Theorem is called a global bifurcation. The part b) on Theorem 4.9 implies that each connected component:

$\mathcal{S}_{\lambda_i} := \text{The connected bifurcation component through } (\theta, \lambda_i),$

meets $\mathcal{S}_{\lambda_j}$, $i \neq j$.

Lemma 4.11. *The bifurcation branch from β_0, $\mathcal{S}_0^\beta$, is global.*

Proof. Note that for β_0, the multiplicity of the corresponding eigenvalue 0 of the linearized problem is 1. Thus Theorem 4.9 is applied, and the bifurcation branch is global. $\qquad \square$

Define

$$X_{\text{even}} := \{u \in H^1[-T/2, T/2] : u \text{ is an even function }\}.$$

Then the following lemma holds:

Lemma 4.12. *If (u, v) is a critical point of a functional corresponding to the problem (4.5) in $X_{\text{even}} \times X_{\text{even}}$, then the (u, v) is a critical point in $\mathcal{H}$.*

Proof. We use the Symmetric Criticality Principle ([37]). Define a group representation $T(g)$ of $g \in \mathbb{Z}_2$ on $\mathcal{H}$ given by

$$(u(t), v(t)) \mapsto (u(-t), v(-t)).$$

Then $\{T(g)\}_{g \in G}$ is an isometric representation of the topological group G over a Hilbert space $\mathcal{H}$. The functional (3.2) is in C^2, and is invariant under G. In addition, a critical point $(u, v) \in X_{\text{even}} \times X_{\text{even}}$ is in $\text{Fix}(G)$ which is the set of fixed points for all elements in G. Therefore by the Symmetric Criticality Principle, (u, v) is a critical point in $\mathcal{H}$.
□

Remark 4.13. Lemma 4.12 can be proved in a direct way:

Proof. Suppose (u, v) is a critical point in $X_{\text{even}} \times X_{\text{even}}$, then for any $\phi, \psi \in X_{\text{even}}$,

$$
\begin{aligned}
0 &= E'(u, v)(\phi, \psi) \\
&= \int_{-T/2}^{T/2} (-u'' + \lambda_1 u - \mu_1 u^3 - \beta u v^2)\phi + \int_{-T/2}^{T/2} (-v'' + \lambda_2 v - \mu_2 v^3 - \beta v u^2)\psi \, dt.
\end{aligned}
$$

Note that $-u'' + \lambda_1 u - \mu_1 u^3 - \beta u v^2$ and $-v'' + \lambda_2 v - \mu_2 v^3 - \beta v u^2$ are even functions if $(u, v) \in X_{\text{even}} \times X_{\text{even}}$. Let define

$$X_{\text{odd}} := \{u \in H^1[-T/2, T/2] : u \text{ is an odd function }\}.$$

Then because of $X_{\text{even}} \perp X_{\text{odd}}$, for any $\phi, \psi \in X_{\text{odd}}$, we get $E'(u, v)(\phi, \psi) = 0$ in $\mathcal{H}$.
□

Recall the Candall-Rabinowitz Local Bifurcation Theorem:

Theorem 4.14 (Crandall-Rabinowitz Local Bifurcation Theorem). *([26] [39])*
Suppose that $U \subset X$ is an open neighborhood of θ, and that $F \in C^2(U \times \mathbb{R}^1, Y)$ satisfies $F(\theta, \lambda) = \theta$. If $F'_x(\theta, \lambda_0)$ is a Fredholm operator with

$$\dim \ker F'_x(\theta, \lambda_0) = \operatorname{codim} \operatorname{im} F'_x(\theta, \lambda_0) = 1,$$

and if

$$F''_{x\lambda}(\theta, \lambda_0) u_0 \notin \operatorname{im} F_x(\theta, \lambda_0)$$

for all $u_0 \in \ker F_x(\theta, \lambda_0) \setminus \{\theta\}$, then (θ, λ_0) is a bifurcation point, and there exists a unique C^1 curve $(\lambda, \psi) : (-\delta, \delta) \to \mathbb{R} \times Z$ satisfying

$$\begin{cases} F(su_0 + \psi(s), \lambda(s)) = \theta, \\ \lambda(0) = \lambda_0, \quad \psi(0) = \psi'(0) = \theta, \end{cases}$$

where $\delta > 0$, and Z is the complement space of $\mathrm{span}\{u_0\}$ in X. Furthermore, there is a neighborhood of (θ, λ_0), in which

$$F_{-1}(\theta) = \{(\theta, \lambda) : \lambda \in \mathbb{R}\} \cup \{(su_0 + \psi(s), \lambda(s)) : |s| < \delta\}.$$

Lemma 4.15. *Along $\mathcal{T}_1^\beta$, there are infinitely many bifurcation points of (4.5): $\beta_1 < \beta_2 < \cdots$ in β, where*

$$\beta_k := (\Lambda_k + \lambda_2)\frac{\mu_2}{\lambda_1}.$$

Moreover the bifurcation branch in $X_{\text{even}} \times X_{\text{even}}$ is represented by a unique C^1 curve in the neighborhood of the bifurcation point.

Proof. To show there are bifurcations for (4.5), we could use almost the same discussion on the above lemma by applying Theorem 4.5. Here we get a stronger conclusion that we do not need later, hence the unique C^1 curve which is stated on Theorem 4.14 exists.

We need to reset the problem in a subspace $X_{\text{even}} \times X_{\text{even}}$ for applying Theorem 4.14.

Define

$$\begin{aligned} F : X_{\text{even}} \times \mathbb{R}^1 &\to C([-T/2, T/2]) \\ (u, \beta) &\mapsto u'' + \Lambda(\beta)u \end{aligned}$$

then F is a Fredholm operator. We will show that Theorem 4.14 can be applied into the linearized equation at (ω_1, θ):

$$F(\theta, \beta_k) = \theta,$$

where $k = 1, 2, \cdots$.

Note that for the operator $(\frac{d^2}{dt^2}) + \Lambda_k \mathrm{Id}$ with the domain $X_{\mathrm{even}} \subset H^1([-T/2, T/2])$,

$$\ker[(\frac{d^2}{dt^2}) + \Lambda_k \mathrm{Id}] = \{s\cos(\sqrt{\Lambda_k}t) : s \in \mathbb{R}\}.$$

Because the operator $(\frac{d^2}{dt^2})$ under the free end point condition is self-adjoint, and the cokernel of a self-adjoint linear operator is isomorphic to its kernel, thus

$$\mathrm{coker}\, F_u'(\theta, \beta_k) = \{s\cos(\sqrt{\Lambda_k}t) : s \in \mathbb{R}\}.$$

Moreover, we have

$$F_{u\beta}''(\theta, \beta_k)\cos(\sqrt{\Lambda_k}t) = \cos(\sqrt{\Lambda_k}t) \notin \mathrm{im} F_u(\theta, \beta_k),$$

because of

$$F_{u\beta}''(\theta, \beta_k) = \cos u|_{u=\theta} = \mathrm{Id}.$$

Now we can apply Theorem 4.14, and it gives the conclusion. $\qquad\square$

Lemma 4.16. *The bifurcation branches from β_k, $\mathcal{S}_k^\beta$ ($k \geq 1$) are global in $X_{\mathrm{even}} \times X_{\mathrm{even}}$.*

Proof. By applying Theorem 4.9 with Lemma 4.15, we know that the bifurcation branches from β_k, $\mathcal{S}_k^\beta$ ($k \geq 1$) of (4.5) are global in $X_{\mathrm{even}} \times X_{\mathrm{even}}$. $\qquad\square$

Remark 4.17. Lemma 4.16 implies that the bifurcation branches $\mathcal{S}_k^\beta$ ($k \geq 1$) are global in $\mathcal{H}$.

Lemma 4.18. *In $X_{\mathrm{even}} \times X_{\mathrm{even}}$, for any solution (u,v) in $\mathcal{S}_k^\beta$, v has $2k$ zeros on $[-T/2, T/2)$.*

Proof. In $X_{\mathrm{even}} \times X_{\mathrm{even}}$, for β which is sufficiently close to β_k,

$$v = (\beta - \beta_k)\psi_k + o(\beta - \beta_k),$$

where ψ_k, which is the eigenfunction of the linearized problem, has $2k$ zeros on $[-T/2, T/2)$, hence v satisfies the statement around β_k.

Next, note that v satisfies

$$-v'' + \lambda_2 v = (\mu_2 v^2 + \beta u^2)v.$$

Therefore v must have only simple zeros because if a zero which is not simple is realized inside the domain, then we get $v = \theta$ due to the Strong Maximum Principle. Next, suppose a zero is realized on the boundary of the domain the first time in the branch connected to $\mathcal{T}_1^\beta$, then if v' is not zero on the boundary, then we find a negative point around the boundary. Thus v' on the boundary must be zero, so we get $v = \theta$ again by the uniqueness of solutions for the differential equations. Therefore v cannot make the number of zeros change continuously. $\qquad\square$

Lemma 4.19. *In* $X_{\text{even}} \times X_{\text{even}}$,

$$\mathcal{S}_k^\beta \cap \mathcal{S}_l^\beta = \emptyset, \quad k \neq l$$

and each $\mathcal{S}_k^\beta$ *is unbounded.*

Proof. The previous lemma implies $\mathcal{S}_k^\beta \cap \mathcal{S}_l^\beta = \emptyset$ for $k \neq l$, which implies that each $\mathcal{S}_k^\beta$ is unbounded by Theorem 4.9.

$\qquad\square$

Now, our main theorem in Section 4.2 (Theorem C).

Theorem 4.20 (Theorem C). *Along* $\mathcal{T}_1^\beta$, *there are infinitely many bifurcation points of (4.1):* $\beta_0 < \beta_1 < \cdots$ *in* β, *having the following properties:*

a) *The nontrivial solutions in the bifurcation branches are not given by solving Dirichlet problems.*

b) *These bifurcation branches are global.*

c) *In* $X_{\text{even}} \times X_{\text{even}}$, $\mathcal{S}_k^\beta \cap \mathcal{S}_l^\beta = \emptyset$, *for* $k \neq l$.

d) *Each* $\mathcal{S}_k^\beta$ *is unbounded.*

Proof. Collecting lemmas in the above, we know the theorem holds. $\qquad\square$

4.3 Bifurcations in λ_1 from the semi-trivial solution curve $\mathcal{T}_1^{\lambda_1}$ (Theorem D)

We use the same notation in the previous section. We fix $\lambda_2, \mu_i, \beta > 0$ in the following. Recall

$$\omega_i := \sqrt{\frac{\lambda_i}{\mu_i}}.$$

Set

$$\mathcal{T}_1^{\lambda_1} = \{(\omega_1, \theta) \in \mathcal{H} : \lambda_1 > 0\}.$$

Then $\mathcal{T}_1^{\lambda_1}$ is a solution curve for $\lambda_1 > 0$ containing semi-trivial solutions of the form (ω_1, θ).

Lemma 4.21. *Along $\mathcal{T}_1^{\lambda_1}$, all possible bifurcation points of (4.1) are: $\tilde{\lambda}_1 < \tilde{\lambda}_2 < \cdots$ and $\tilde{\tilde{\lambda}}_0 < \tilde{\tilde{\lambda}}_1 < \cdots$ in λ_1, where*

$$\tilde{\lambda}_k := \frac{1}{2}\Lambda_k, \quad \tilde{\tilde{\lambda}}_k := \frac{\mu_1}{\beta}(\lambda_2 + \Lambda_k).$$

Proof. The linearization of (4.1) at (ω_1, θ) is given by

$$\begin{cases} -\phi'' = 2\lambda_1\phi, \\ -\psi'' = (\beta\frac{\lambda_1}{\mu_1} - \lambda_2)\psi, \\ \phi(-T/2) = \phi(T/2), \quad \psi(-T/2) = \psi(T/2), \\ \phi'(-T/2) = \phi'(T/2), \quad \psi'(-T/2) = \psi'(T/2). \end{cases} \tag{4.7}$$

Note that the following eigenvalue problem

$$\begin{cases} -\phi'' = \Lambda_k\phi, \\ \phi(-T/2) = \phi(T/2), \quad \phi'(-T/2) = \phi'(T/2), \end{cases} \tag{4.8}$$

has eigenvalues

$$\Lambda_k := (k\frac{2\pi}{T})^2, \quad k = 0, 1, \cdots$$

and its eigenfunctions

$$\begin{aligned} &\{\mathbf{1}\} \quad (k = 0), \\ &\{\sin(k\frac{2\pi}{T}t), \cos(k\frac{2\pi}{T}t)\} \quad (k \neq 0). \end{aligned}$$

Therefore when

$$2\lambda_1 = \Lambda_k, \quad k = 1, 2, \cdots.$$

the first equation of the linearized problem has non-trivial kernel, and when $2\lambda_1 \neq \Lambda_k$, it has only zero solutions. On the other hand, when

$$(\beta\frac{\lambda_1}{\mu_1} - \lambda_2) = \Lambda_k, \quad k = 0, 1, 2, \cdots .$$

the second equation of the linearized problem has non-trivial kernel, and when $(\beta\frac{\lambda_1}{\mu_1} - \lambda_2) = \Lambda_k$, it has only zero solutions.

Therefore we have the possibility that a bifurcation happens at only $\tilde{\lambda}_1 < \tilde{\lambda}_2 < \cdots$ and $\tilde{\tilde{\lambda}}_0 < \tilde{\tilde{\lambda}}_1 < \cdots$ in λ_1. $\qquad\square$

Remark 4.22. There is also a possible bifurcation point $\tilde{\lambda}_0$ which will bifurcate from $\{(\omega_1, \theta) \in \mathcal{H} : \lambda_1 \geq 0\}$.

We rearrange the order of $\tilde{\lambda}_1 < \tilde{\lambda}_2 < \cdots$ and $\tilde{\tilde{\lambda}}_0 < \tilde{\tilde{\lambda}}_1 < \cdots$ in λ_1 by using the natural order of real numbers, as

$$\lambda'_1 < \lambda'_2 < \cdots ,$$

where we do not distinguish between $\tilde{\lambda}_k$ and $\tilde{\tilde{\lambda}}_l$ when $\tilde{\lambda}_k = \tilde{\tilde{\lambda}}_l$.

Lemma 4.23. *Along* $\mathcal{T}_1^{\lambda_1}$, $\lambda'_0 < \lambda'_1 < \cdots$ *in* λ_1 *are bifurcations parameters.*

Proof. We need to verify if the bifurcations actually happen.

Note that

$$
\begin{aligned}
E''(\omega_1, \theta)|_{(\phi,\psi)^2} &= \int_T |\phi'|^2 + \lambda_1\phi^2 - 3\mu_1\omega_1^2\phi^2 + |\psi'|^2 + \lambda_2\psi^2 - \beta\omega_1^2\psi^2 dt \\
&= \int_T |\phi'|^2 - 2\lambda_1\phi^2 + |\psi'|^2 + (\lambda_2 - \frac{\beta\lambda_1}{\mu_1})\psi^2 dt.
\end{aligned}
$$

Recall

V_k = a subspace of $H^1[-T/2, T/2]$ spanned by eigenfunctions associated to Λ_k of the linearized problem.

Note that V_k has a finite dimension 1 when $k = 0$, and 2 when $k \neq 0$, and the product space $V_k \times V_l$ is a subspace of $\ker[E''(\omega_1, \theta)]$.

We show that the Morse index changes at the bifurcation point on the subspace $V_k \times V_l$ which is in the kernel space of the linearized problem.

Define the subspaces of the kernel space of the linearized problem:

$$
W_i := \begin{cases} V_k \times \{\theta\} & \text{if } \lambda_i' = \tilde{\lambda}_k \neq \tilde{\tilde{\lambda}}_l \text{ for all } l, \\ \{\theta\} \times V_k & \text{if } \lambda_i' = \tilde{\tilde{\lambda}}_k \neq \tilde{\lambda}_l \text{ for all } l, \\ V_k \times V_l & \text{if } \lambda_i' = \tilde{\lambda}_k = \tilde{\tilde{\lambda}}_l. \end{cases}
$$

In the kernel space of the linearized problem, we show that $E''(\omega_1, \theta)|_{(\phi_k, \psi_l)^2}$ changes the sign by going though λ_i'.

<u>Claim 1</u>: *For $\lambda_{i-1}' < \lambda_1 < \lambda_i'$ (set as $\lambda_{-1}' = 0$) $E''(\omega_1, \theta)|_{(\phi, \psi)^2}$ is negative definite on $W_0 \oplus \cdots \oplus W_{i-1}$, and positive definite on $W_i \oplus W_{i+1} \oplus \cdots$.*

<u>Claim 2</u>: *For $\lambda_{i+1}' > \lambda_1 > \lambda_i'$ $E''(\omega_1, \theta)|_{(\phi, \psi)^2}$ is negative definite on $W_0 \oplus \cdots \oplus W_i$, and positive definite on $W_{i+1} \oplus W_{i+2} \oplus \cdots$.*

Suppose $W_j = V_k \times V_l$. Note that

$$
\begin{aligned}
E''(\omega_1, \theta)|_{(\phi, \psi)^2} &= \int_T [-\phi'' - 2\tilde{\lambda}_k \phi]\phi - 2(\lambda_1 - \tilde{\lambda}_k)\phi^2 \\
&\quad + [-\psi'' - (\frac{\beta\tilde{\tilde{\lambda}}_l}{\mu_1} - \lambda_2)\psi]\psi - \beta\frac{\lambda_1 - \tilde{\tilde{\lambda}}_l}{\mu_1}\psi^2 dt.
\end{aligned}
$$

Thus in W_j

$$
E''(\omega_1, \theta)|_{(\phi_k, \psi_l)^2} = \int_T -2(\lambda_1 - \tilde{\lambda}_k)\phi_k^2 - \beta\frac{\lambda_1 - \tilde{\tilde{\lambda}}_l}{\mu_1}\psi_l^2 dt. \tag{4.9}
$$

Under the assumption of Claim 1, if $j < i$, then because of $\tilde{\lambda}_k < \lambda_1$ and $\tilde{\tilde{\lambda}}_l < \lambda_1$, we get $E''(\omega_1, \theta)|_{(\phi_k, \psi_l)^2} < 0$. If $j \geq i$, then because of $\tilde{\lambda}_k > \lambda_1$ and $\tilde{\tilde{\lambda}}_l > \lambda_1$, we get $E''(\omega_1, \theta)|_{(\phi_k, \psi_l)^2} > 0$. Next, suppose $W_j = V_k \times \{\theta\}$ or $W_j = \{\theta\} \times V_k$ instead of $W_j = V_k \times V_l$, then because one of the terms on (4.9) disappears, we still get $E''(\omega_1, \theta) < 0$ in W_j for $j < i$, and $E''(\omega_1, \theta) > 0$ in W_j for $j \geq i$. Thus Claim 1 holds.

Under the assumption of Claim 2, if $j \leq i$, then because $\tilde{\lambda}_k < \lambda_1$ and $\tilde{\tilde{\lambda}}_l < \lambda_1$, we get $E''(\omega_1, \theta)|_{(\phi_k, \psi_l)^2} < 0$. If $j > i$, then because $\tilde{\lambda}_k > \lambda_1$ and $\tilde{\tilde{\lambda}}_l > \lambda_1$, we get $E''(\omega_1, \theta)|_{(\phi_k, \psi_l)^2} > 0$. Next, suppose $W_j = V_k \times \{\theta\}$ or $W_j = \{\theta\} \times V_k$ instead of $W_j = V_k \times V_l$, then because one of the terms on (4.9) disappears, we still get $E''(\omega_1, \theta) < 0$ in W_j for $j \leq i$, and $E''(\omega_1, \theta) > 0$ in W_j for $j > i$. Thus Claim 2 holds.

Therefore by Claim 1 and Claim 2, the Morse index of E changes at λ_i'. By Theorem 4.5, the statement holds. $\square$

Lemma 4.24. *The bifurcation solutions from $\mathcal{T}_1^{\lambda_1}$ include solutions which are not given by solving Dirichlet problems.*

Proof. Let $(u,v) \in \mathcal{H}$ be a solution in the bifurcation from $\mathcal{T}_1^{\lambda_1}$, which is sufficiently close to the bifurcation point but not in the bifurcation point.

By using sufficiently small $\epsilon > 0$, u and v can be represented by $u = \omega_1 + h_1$ with $\|h_1\| < \epsilon$ and $v = h_2$ with $\|h_2\| < \epsilon$, then by taking a limit $\epsilon \to 0$, we get

$$\begin{cases} -h_1'' + \lambda_1 h_1 = 3\mu_1 \omega_1^2 h_1, \\ -h_2'' + \lambda_2 h_2 = \beta \omega_1^2 h_2. \end{cases} \tag{4.10}$$

Suppose (u,v) is given by solving Dirichlet problems, hence $u(-T/2) = u(T/2) = 0$ and $v(-T/2) = v(T/2) = 0$. Thus $h_1(-T/2) = h_1(T/2) = -\omega_1$ and $h_2(-T/2) = h_2(T/2) = 0$. However because ω_1 is nondegenerate for $-u'' + \lambda_1 u = \mu_1 u^3$ with the periodic condition on $[-T/2, T/2]$, the first equation of (4.10) has only the zero solution because it is not in the bifurcation point. Therefore we get a contradiction. $\square$

We show a stronger conclusion with the following:

Lemma 4.25. *The bifurcation nontrivial solutions from $\mathcal{T}_1^{\lambda_1}$ are not given by solving Dirichlet problems.*

Proof. Suppose (u,v), which is in the bifurcation branch from $\mathcal{T}_1^{\lambda_1}$, are given by solving Dirichlet problems, hence $u(-T/2) = u(T/2) = 0$ and $v(-T/2) = v(T/2) = 0$. Note that $u'(-T/2) = u'(T/2)$ by the periodic condition. Because the solution is in the bifurcation from $\mathcal{T}_1$, the solution can be earned to change continuously from (ω_1, θ). In such continuous change, there is a part satisfying $u \geq 0$ connected to $\mathcal{T}_1^{\lambda_1}$, so assume $u \geq 0$. If $u'(-T/2)$ is positive (or negative), then $u < 0$ around $t = T/2$ (or $t = -T/2$, respectively) because of

$$u = u'(T/2)(t - T/2) + o(t - T/2),$$

thus we get a contradiction. Therefore $u'(-T/2) = u'(T/2) = 0$. Since θ is a solution of the first equation of (4.1) satisfying the condition, then we get $u = \theta$ because of the uniqueness of solutions for differential equations. $\square$

With fixed β, define

$$\mathcal{S}_k^{\lambda_1} := \text{The connected bifurcation component through } ((\omega_1, \theta), \tilde{\lambda}_k), \ k = 1, 2, \cdots.$$

$$\mathcal{R}_k^{\lambda_1} := \text{The connected bifurcation component through } ((\omega_1, \theta), \tilde{\tilde{\lambda}}_k), \ k = 0, 1, \cdots.$$

Lemma 4.26. *Suppose*

$$\Lambda_k \neq \frac{2\lambda_2\mu_1}{\beta - 2\mu_1}.$$

The bifurcation branch from $\tilde{\tilde{\lambda}}_0$, $\mathcal{R}_0^{\lambda_1}$, is global.

Proof. Note that for $\tilde{\tilde{\lambda}}_0$, the multiplicity of the corresponding eigenvalue 0 of the linearized problem is 1. Thus Theorem 4.9 is applied, the bifurcation branch is global. $\square$

Define an ordered set in real numbers given by:

$$\{\lambda_i''\}_i := (\{\tilde{\lambda}_i\}_i \cup \{\tilde{\tilde{\lambda}}_i\}_i) \setminus (\{\tilde{\lambda}_k : \tilde{\lambda}_k = \tilde{\tilde{\lambda}}_l \text{ for an } l\}_k \cup \{\tilde{\tilde{\lambda}}_k : \tilde{\tilde{\lambda}}_k = \tilde{\lambda}_l \text{ for an } l\}_k).$$

Lemma 4.27. *Along $\mathcal{T}_1^{\lambda_1}$, there are infinitely many bifurcation points of (4.5): $\lambda_0'' < \lambda_1'' < \cdots$ in λ_1. Moreover the bifurcation branch is represented by a unique C^1 curve in the neighborhood of the bifurcation point.*

Proof. To show there are bifurcations for (4.5), we could use almost the same discussion on the above lemma by applying Theorem 4.5. Here we get a stronger conclusion that we do not need later, hence the unique C^1 curve which is stated in Theorem 4.14 exists.

Recall

$$X_{\text{even}} := \{u \in H^1[-T/2, T/2] : u \text{ is an even function }\}.$$

We need to reset the problem in a subspace $X_{\text{even}} \times X_{\text{even}}$ for applying Theorem 4.14. Define

$$\begin{aligned} F : X_{\text{even}} \times \mathbb{R}^1 &\rightarrow C([-T/2, T/2]) \\ (u, \lambda) &\mapsto u'' + \Lambda(\lambda)u \end{aligned}$$

then F is a Fredholm operator. We will show that Theorem 4.14 can be applied into the linearized equation at (ω_1, θ):

$$F(\theta, \beta_k) = \theta,$$

where $k = 1, 2, \cdots$.

Note that for the operator $(\frac{d^2}{dt^2}) + \Lambda_k \mathrm{Id}$ with the domain $X_{\text{even}} \subset H^1([-T/2, T/2])$,

$$\ker[(\frac{d^2}{dt^2}) + \Lambda_k \mathrm{Id}] = \{s \cos(\sqrt{\Lambda_k}t) : s \in \mathbb{R}\}.$$

Because the operator $(\frac{d^2}{dt^2})$ under the free end point condition is self-adjoint, and the cokernel of a self-adjoint linear operator is isomorphic to its kernel, thus

$$\mathrm{coker}\, F_u'(\theta, \beta_k) = \{s \cos(\sqrt{\Lambda_k}t) : s \in \mathbb{R}\}.$$

Moreover we have

$$F_{u\beta}''(\theta, \beta_k) \cos(\sqrt{\Lambda_k}t) = \cos(\sqrt{\Lambda_k}t) \notin \mathrm{im} F_u(\theta, \beta_k),$$

because

$$F_{u\beta}''(\theta, \beta_k) = \cos u|_{u=\theta} = \mathrm{Id}.$$

Now we can apply Theorem 4.14 and, then it gives the conclusion. $\qquad\square$

Lemma 4.28. *Suppose*

$$\Lambda_k \neq \frac{2\lambda_2 \mu_1}{\beta - 2\mu_1}.$$

The bifurcation branches from $\tilde{\lambda}_k$, $\mathcal{S}_k^{\lambda_1}$ ($k \geq 1$) and the branches from $\tilde{\tilde{\lambda}}_k$, $\mathcal{R}_k^{\lambda_1}$ ($k \geq 1$) are global.

Proof. By applying Theorem 4.9 with the previous lemma, we know that the bifurcation branches from $\tilde{\lambda}_k$, $\mathcal{S}_k^{\lambda_1}$ ($k \geq 1$) of (4.5) are global. Since the bifurcation branches must be a subset of the bifurcation branches from $\tilde{\lambda}_k^{\lambda_1}$, $\mathcal{S}_k^{\lambda_1}$ ($k \geq 1$) of (4.1), the bifurcation branches of (4.1) must also be global.

By applying the above discussion with replacing $\tilde{\lambda}_k$ and $\mathcal{S}_k^{\lambda_1}$ with $\tilde{\tilde{\lambda}}_k$ and $\mathcal{R}_k^{\lambda_1}$, we know that the branches from $\tilde{\tilde{\lambda}}_k$, $\mathcal{R}_k^{\lambda_1}$ ($k \geq 1$), are also global. $\qquad\square$

Lemma 4.29. *In $X_{\text{even}} \times X_{\text{even}}$, for any solution (u,v) in $\mathcal{S}_k^{\lambda_1}$, v has $2k$ zeros on $[-T/2, T/2)$.*

Proof. For λ_1 which is sufficiently close to $\tilde{\lambda}_k$,

$$v \;=\; (\lambda_1 - \tilde{\lambda}_k)\psi_k + o(\lambda_1 - \tilde{\lambda}_k),$$

where ψ_k, which is the eigenfunction of the linearized problem, has $2k$ zeros on $[-T/2, T/2)$, hence v satisfies the statement around $\tilde{\lambda}_k$.

Next, note that v satisfies

$$-v'' + \lambda_1 v = (\mu_1 v^2 + \beta u^2)v.$$

Therefore v must have only simple zeros because if a zero which is not simple is realized inside the domain, then we get $v = \theta$ due to the Strong Maximum Principle. Next, suppose a zero is realized on the boundary of the domain the first time in the branch connected to $\mathcal{T}^{\lambda_1}$, then if v' is not zero on the boundary, we find a negative point around the boundary. Thus v' on the boundary must be zero, so we get $v = \theta$ again by the uniqueness of solutions for the differential equations. Therefore v cannot make the number of zeros change continuously. $\qquad\square$

Lemma 4.30. *In $X_{\text{even}} \times X_{\text{even}}$,*

$$\mathcal{S}_k^{\lambda_1} \cap \mathcal{S}_l^{\lambda_1} = \emptyset, \quad k \neq l$$

and each $\mathcal{S}_k^{\lambda_1}$ is unbounded.

Proof. The previous lemma implies $\mathcal{S}_k^{\lambda_1} \cap \mathcal{S}_l^{\lambda_1} = \emptyset$ for $k \neq l$, which implies that each $\mathcal{S}_k^{\lambda_1}$ is unbounded by Theorem 4.9. $\qquad\square$

Now, our main theorem in Section 4.3 (Theorem D).

Theorem 4.31 (Theorem D). *Along $\mathcal{T}_1^{\lambda_1}$, there are infinitely many bifurcation points of (4.1): For fixed β, $\tilde{\lambda}_0 < \tilde{\lambda}_1 < \cdots$, and $\tilde{\tilde{\lambda}}_1 < \tilde{\tilde{\lambda}}_2 < \cdots$, in λ_1, having the following properties:*

a) *The nontrivial solutions in the bifurcation branches are not given by solving Dirichlet problems.*

b) *These bifurcation branches are global if $\Lambda_k \neq \frac{2\lambda_2 \mu_1}{\beta - 2\mu_1}$.*

c) *In $X_{\text{even}} \times X_{\text{even}}$, $\mathcal{S}_k^{\lambda_1} \cap \mathcal{S}_l^{\lambda_1} = \emptyset$ and $\mathcal{R}_k^{\lambda_1} \cap \mathcal{R}_k^{\lambda_1} = \emptyset$ for $k \neq l$.*

d) *Each $\mathcal{S}_k^{\lambda_1}$ and $\mathcal{R}_k^{\lambda_1}$ is unbounded.*

Proof. Collecting lemmas in the above, we know the theorem holds. $\qquad\square$

4.4 Bifurcations from the synchronized solution curve $\mathcal{T}$ (Theorem E and F)

Set $\lambda_1 = \lambda_2 = 1$. Without loss of generality, we assume $0 < \mu_1 \leq \mu_2$, and fix μ_1 and μ_2. We also set

$$A_\beta := \sqrt{\frac{\mu_2 - \beta}{\mu_1 \mu_2 - \beta^2}}, \quad B_\beta := \sqrt{\frac{\mu_1 - \beta}{\mu_1 \mu_2 - \beta^2}},$$

$$\mathcal{T} := \{(A_\beta, B_\beta) : \beta \in (-\sqrt{\mu_1 \mu_2}, \mu_1) \cup (\mu_2, \infty)\}.$$

Then $\mathcal{T}$ is a solution curve of (4.3) for $\beta \in (-\sqrt{\mu_1 \mu_2}, \mu_1) \cup (\mu_2, \infty)$ containing solutions of the form (A_β, B_β).

Define

$$a(\beta) := \frac{3\mu_1 \mu_2 - 2\mu_1 \beta - \beta^2}{\mu_1 \mu_2 - \beta^2},$$

$$b(\beta) := \frac{2\beta \sqrt{(\mu_1 - \beta)(\mu_2 - \beta)}}{\mu_1 \mu_2 - \beta^2},$$

$$c(\beta) := \frac{3\mu_1 \mu_2 - 2\mu_2 \beta - \beta^2}{\mu_1 \mu_2 - \beta^2},$$

$$\gamma_\pm := \frac{a - c \pm \sqrt{(a - c)^2 + 4b^2}}{2b},$$

which are naturally defined in the proofs of bifurcation theorems later. First, we give the following two lemmas to show there are possible bifurcation points:

Lemma 4.32. *When $\beta \neq 0$, $\gamma_+ \neq \gamma_-$, hence $(a - c)^2 + 4b^2 \neq 0$.*

Proof. Note that

$$a - c = \frac{3\mu_1\mu_2 - 2\mu_1\beta - \beta^2}{\mu_1\mu_2 - \beta^2} - \frac{3\mu_1\mu_2 - 2\mu_2\beta - \beta^2}{\mu_1\mu_2 - \beta^2} = \frac{2(\mu_2 - \mu_1)\beta}{\mu_1\mu_2 - \beta^2},$$

$$b = \frac{2\beta\sqrt{(\mu_1 - \beta)(\mu_2 - \beta)}}{\mu_1\mu_2 - \beta^2}.$$

Because of $\beta \in (-\sqrt{\mu_1\mu_2}, \mu_1) \cup (\mu_2, \infty)$, the lemma holds. $\square$

Lemma 4.33. *Suppose*

$$T \neq \sqrt{2}\pi l, \quad l = 1, 2, \cdots.$$

Then all possible parameters of the bifurcation from $\mathcal{T}$ are given by

$$\beta_k := f^{-1}(\Lambda_k), \quad k = 0, 1, \cdots$$

where $f : (-\sqrt{\mu_1\mu_2}, \mu_1) \to (0, \infty)$ is defined by

$$f(\beta) := 2\frac{\mu_1\mu_2 - \beta(\mu_1 + \mu_2) + \beta^2}{\mu_1\mu_2 - \beta^2},$$

which is a strictly decreasing diffeomorphism $(-\sqrt{\mu_1\mu_2}, \mu_1) \to (0, \infty)$. In particular, there exists no bifurcation point for $\beta > \mu_2$.
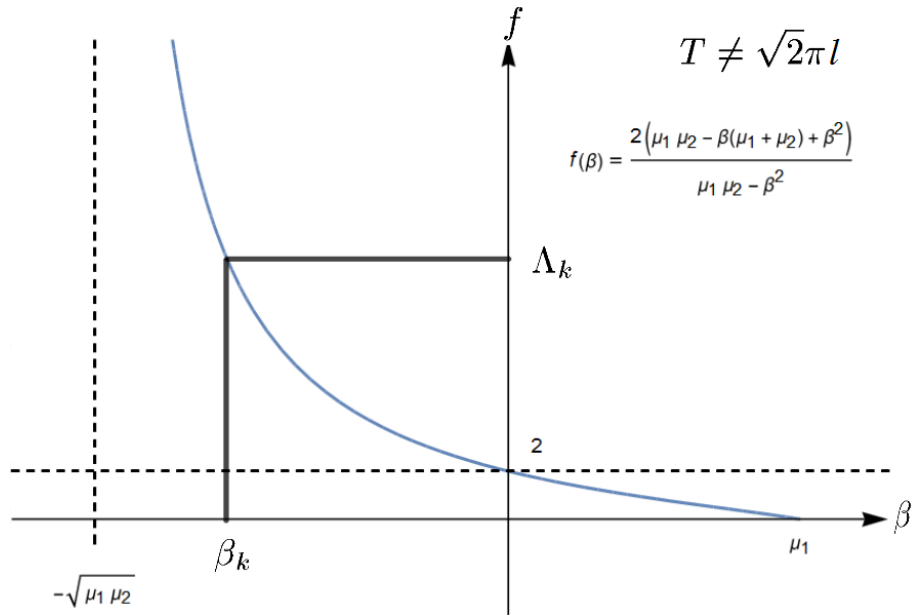

Figure 4.1: The function f and bifurcation points

Proof. The linearized problem of (4.3) at (A_β, B_β) is

$$
\begin{cases}
-\phi'' + \phi = 3\mu_1 A_\beta^2 \phi + \beta B_\beta^2 \phi + 2\beta B_\beta A_\beta \psi, \\
-\psi'' + \psi = 3\mu_2 B_\beta^2 \psi + \beta A_\beta^2 \psi + 2\beta A_\beta B_\beta \phi, \\
\phi(-T/2) = \phi(T/2), \quad \psi(-T/2) = \phi(T/2), \\
\phi'(-T/2) = \phi'(T/2), \quad \psi'(-T/2) = \phi'(T/2),
\end{cases}
$$

Hence

$$
\begin{cases}
-\phi'' + \phi = a(\beta)\phi + b(\beta)\psi, \\
-\psi'' + \psi = c(\beta)\psi + b(\beta)\phi, \\
\phi(-T/2) = \phi(T/2), \quad \psi(-T/2) = \phi(T/2), \\
\phi'(-T/2) = \phi'(T/2), \quad \psi'(-T/2) = \phi'(T/2),
\end{cases}
$$

where

$$
a(\beta) = \frac{3\mu_1\mu_2 - 2\mu_1\beta - \beta^2}{\mu_1\mu_2 - \beta^2},
$$
$$
b(\beta) = \frac{2\beta\sqrt{(\mu_1 - \beta)(\mu_2 - \beta)}}{\mu_1\mu_2 - \beta^2},
$$
$$
c(\beta) = \frac{3\mu_1\mu_2 - 2\mu_2\beta - \beta^2}{\mu_1\mu_2 - \beta^2}.
$$

<u>Claim:</u> ϕ and ψ in the solution (ϕ, ψ) of the linearized problem are dependent. Moreover the nontrivial kernel form of the linearized problem is given by

$$
(\phi, \psi) = \begin{cases}
(\gamma_+\psi, \psi) & \text{if } \beta \leq 0, \\
(\gamma_-\psi, \psi) & \text{if } \beta > 0.
\end{cases}
$$

Recall

$$
\gamma_\pm = \frac{a - c \pm \sqrt{(a-c)^2 + 4b^2}}{2b}.
$$

$\phi - \gamma_\pm\psi$ solves

$$
-(\phi - \gamma_\pm\psi)'' + (\phi - \gamma_\pm\psi) = (a - b\gamma_\pm)(\phi - \gamma_\pm\psi).
$$

Simple calculation shows that

$$a - b\gamma_+ = 3, \quad \text{if } \beta \leq 0,$$

$$a - b\gamma_- = 3, \quad \text{if } \beta > 0.$$

Thus we study the case of $\beta \leq 0$ and the case of $\beta > 0$ separately.

The Case $\beta \leq 0$

Suppose $\beta \leq 0$. Then $\phi - \gamma_+ \psi$ solves

$$-(\phi - \gamma_+ \psi)'' = 2(\phi - \gamma_+ \psi),$$

then because of $T \neq \sqrt{2}\pi k$, $k = 1, 2, \cdots$, $\frac{1}{\sqrt{\mu_1}}$ is nondegenerate for $-u'' + u = \mu_1 u^3$ on $[-T/2, T/2]$ with the periodic condition, thus the above equation has only the zero solution in $\phi - \gamma_+ \psi$. Therefore we get $\phi = \gamma_+ \psi$.

On the other hand, since $\gamma_+ \neq \gamma_-$ by Lemma 4.32, we need to check for γ_- too. $\phi - \gamma_- \psi$ solves

$$-(\phi - \gamma_- \psi)'' = (a - b\gamma_- - 1)(\phi - \gamma_- \psi).$$

Suppose that $a - b\gamma_- - 1 \neq \Lambda_j$ for some j, then the above equation must have only the zero solution in $\phi - \gamma_- \psi$. Then by substituting $\phi = \gamma_- \psi$ into the linearized problem, we get

$$\begin{aligned} -\psi'' &= (b\gamma_- + c - 1)\psi \\ &= 2\psi, \end{aligned}$$

for $\beta \leq 0$. However because of $T \neq \sqrt{2}\pi k$, $k = 1, 2, \cdots$, this equation has only the zero solution, so we get the trivial kernel. Thus the Claim holds for $\beta \leq 0$.

The Case $\beta > 0$

Suppose $\beta > 0$. Then $\phi - \gamma_- \psi$ solves

$$-(\phi - \gamma_- \psi)'' = 2(\phi - \gamma_- \psi),$$

then because of $T \neq \sqrt{2}\pi k$, $k = 1, 2, \cdots$, $\frac{1}{\sqrt{\mu_1}}$ is nondegenerate for $-u'' + u = \mu_1 u^3$ on $[-T/2, T/2]$ with the periodic condition, thus the above equation has only the zero

solution in $\phi - \gamma_- \psi$. Therefore we get $\phi = \gamma_- \psi$.

On the other hand, since $\gamma_+ \neq \gamma_-$ by Lemma 4.32, we need to check for γ_+ too. $\phi - \gamma_+ \psi$ solves

$$-(\phi - \gamma_+ \psi)'' = (a - b\gamma_+ - 1)(\phi - \gamma_+ \psi).$$

Suppose that $a - b\gamma_+ - 1 \neq \Lambda_j$ for some j, then the above equation must have only the zero solution in $\phi - \gamma_+ \psi$. Then by substituting $\phi = \gamma_+ \psi$ into the linearized problem, we get

$$\begin{aligned} -\psi'' &= (b\gamma_+ + c - 1)\psi \\ &= 2\psi \end{aligned}$$

for $\beta > 0$. Because of $T \neq \sqrt{2}\pi k$, $k = 1, 2, \cdots$, this equation has only the zero solution, so we get the trivial kernel. Thus the Claim holds for $\beta > 0$.

Finally, by substituting $\phi = \gamma_+ \psi$ for $\beta \leq 0$, and $\phi = \gamma_- \psi$ for $\beta > 0$ into the linearized problem, we know that ψ solves

$$-\psi'' = f(\beta)\psi,$$

where

$$f(\beta) = b\gamma_+ + c - 1 = \frac{3\mu_1\mu_2 - 2\beta(\mu_1 + \mu_2) + \beta^2}{\mu_1\mu_2 - \beta^2} - 1 = 2\frac{\mu_1\mu_2 - \beta(\mu_1 + \mu_2) + \beta^2}{\mu_1\mu_2 - \beta^2}.$$

It is not difficult to see that f is a strictly decreasing diffeomorphism $(-\sqrt{\mu_1\mu_2}, \mu_1)$ to $(0, \infty)$. Note that there is a nontrivial kernel if $f(\beta) = \Lambda_k$ for $k = 1, 2, \cdots$.

Therefore the linearized problem has a nontrivial kernel if $f(\beta) = \Lambda_k$, for $k = 1, 2, \cdots$. $\qquad\square$

Lemma 4.34. *Suppose*

$$T \neq \sqrt{2}\pi l, \quad l = 1, 2, \cdots.$$

Then along $\mathcal{T}$, there are infinitely many bifurcation points: $\beta_1 > \beta_2 > \beta_3 > \cdots \beta_{k_0} > 0 > \beta_{k_0+1} > \beta_{k_0+2} > \cdots > \beta_k$, $\beta_k \to -\sqrt{\mu_1\mu_2}$ as $k \to \infty$, where k_0 is the maximum

positive integer satisfying

$$\Lambda_{k_0} < 2.$$

Proof. We show that the Morse index changes at the bifurcation point.

$$
\begin{aligned}
E''(A_\beta, B_\beta)|_{(\phi,\psi)^2} &= \int_T |\phi'|^2 + (1 - 3\mu_1 A_\beta^2 - \beta B_\beta^2)\phi^2 + |\phi'|^2 + (1 - 3\mu_2 B_\beta^2 - \beta A_\beta^2)\psi^2 \\
&\quad -4\beta A_\beta B_\beta \phi\psi dt.
\end{aligned}
$$

Define V_k^β:

$V_k^\beta = \{(\gamma_\pm \psi, \psi) \in \mathcal{H}\colon \psi$ is an eigenfunction associated to Λ_k of the linearized problem, where $\gamma_\pm = \gamma_+$ if $\beta \leq 0$, and $\gamma_\pm = \gamma_-$ if $\beta > 0.$ $\}$.

V_k^β has a finite dimension 1 when $k = 0$, and 2 when $k \neq 0$, and is a subspace of $\ker[E''(\omega_1, \theta)]$ as we saw on the proof on the previous lemma. We show that the Morse index changes at the bifurcation point on the subspace V_k^β which is in the kernel space of the linearized problem.

Let denote $E''(A_\beta, B_\beta)|_{(\phi,\psi)^2}$ as H_β. Then around β_k,

$$H_\beta = H_{\beta_k} + (\beta - \beta_k)H'_{\beta_k} + o(\beta - \beta_k).$$

Due to the Claim on the proof of Lemma 4.33, we study the case of $-\sqrt{\mu_1 \mu_2} < \beta < 0$ and the case of $0 < \beta < \mu_1$ separately.

The Case $-\sqrt{\mu_1 \mu_2} < \beta < 0$:

Under the assumptions of the statement, we show the following two claims:

<u>Claim 1</u>: *For $\beta > \beta_k$ and close to β_k, H_β is positive definite on $V_0^\beta \oplus \cdots \oplus V_k^\beta$ and negative definite on $V_{k+1}^\beta \oplus V_{k+2}^\beta \oplus \cdots$.*

<u>Claim 2</u>: *For $\beta_k > \beta$ and close to β_k, H_β is positive definite on $V_0^\beta \oplus \cdots \oplus V_{k-1}^\beta$ and negative definite on $V_k^\beta \oplus V_{k+1}^\beta \oplus \cdots$.*

To prove these claims, we show that H'_β is positive definite on V_k^β. Note that

$$H'_\beta[(\phi, \psi)^2] = \frac{\partial}{\partial \beta} H_\beta|_{\beta=\beta_k} = -\int_T a'(\beta)\phi^2 + 2b'(\beta)\phi\psi + c'(\beta)\psi^2 dt.$$

Direct calculations show

$$
\begin{aligned}
a'(\beta) &= -\frac{2\mu_1(\mu_1\mu_2 - 2\beta\mu_2 + \beta^2)}{(\mu_1\mu_2 - \beta^2)^2}, \\
b'(\beta) &= \frac{2\mu_1^2\mu_2^2 - 4(\mu_1 + \mu_2)\mu_1\mu_2\beta + 4\mu_1\mu_2\beta^2 - 2(\mu_1 + \mu_2)\beta^3 + \beta^4}{(\mu_1\mu_2 - \beta^2)^2\sqrt{\mu_1 - \beta}\sqrt{\mu_2 - \beta}}, \\
c'(\beta) &= -\frac{2\mu_2(\mu_1\mu_2 - 2\beta\mu_1 + \beta^2)}{(\mu_1\mu_2 - \beta^2)^2}.
\end{aligned}
$$

Substitute $(\gamma_+(\beta)\psi, \psi)$ into (ϕ, ψ). Then

$$H'_\beta[(\gamma_+(\beta)\psi, \psi)^2] = -[a'(\beta)\gamma_+^2 + 2b'(\beta)\gamma_+(\beta) + c'(\beta)] \int_T \psi^2 dt.$$

For $-\sqrt{\mu_1\mu_2} < \beta < 0$, we get

$$\gamma_+ < 0, \quad a'(\beta) < 0, \quad b'(\beta) > 0, \quad c'(\beta) < 0.$$

Thus Claim 1 and Claim 2 hold.

The Case $0 < \beta < \mu_1$:

Let m_ω be the Morse index of the solution of (4.5). Then

$$m_\omega = n_1 + \cdots + n_{k_0},$$

where k_0 is the maximum integer satisfying $\Lambda_{k_0} < 2$.

Let $m(\beta)$ be the Morse index of (A_β, B_β). Under the assumptions of the statement, we show the following two claims:

Claim 3: $m(\beta) = m_\omega$ for $\beta < \mu_1$ and close to μ_1.

Claim 4: $m(0) = 2m_\omega$.

Denote $E''(A_\beta, B_\beta)|_{(\phi,\psi)^2}$ as H_β again. Note that

$$H_\beta = \int_T |\phi'|^2 + |\psi'|^2 dt - \int_T (a-1)\phi^2 + 2b\phi\psi + (c-1)\psi^2 dt.$$

Let W^- be the eigenspace of (4.8) associated to the eigenvalues $0 < \Lambda_1 < \Lambda_2 < \cdots < \Lambda_{k_0} < 2$ and let W^+ be the eigenspace of (4.8) associated to the eigenvalues $2 < \Lambda_{k_0+1} < \Lambda_{k_0+2} < \cdots$.

For $\phi \in W^- \setminus \{0\}$, we get

$$\int_T |\phi'|^2 dt \le \Lambda_{k_0} \int_T \phi^2 dt < 2 \int_T \phi^2 dt,$$

and for $\phi \in W^+ \setminus \{0\}$, we get

$$\int_T |\phi'|^2 dt \ge \Lambda_{k_0+1} \int_T \phi^2 dt > 2 \int_T \phi^2 dt.$$

By the above inequalities with $a(\beta) \to 3$, $b(\beta) \to 0$, and $c(\beta) \to 1$ as $\beta \to \mu_1$, we know that H_β is negative definite on $W^- \times \mathbb{R}\omega \subset \mathcal{H}$, and positive definite on $W^+ \times (\mathbb{R}\omega)^\perp \subset \mathcal{H}$. Therefore Claim 3 holds.

Next, note that H_0 is negative definite on $W^- \times W^-$, and positive definite on $W^+ \times W^+$. In the same as the above with $a(0) = 3$, $b(0) = 0$, and $c(0) = 3$, Claim 4 holds.

Let n_k be the multiplicity of Λ_k of eigenvalues of (4.8). Define i_k as

$$i_k := \lim_{\epsilon \searrow 0} [m(\beta_k - \epsilon) - m(\beta_k + \epsilon)].$$

Note that $|i_k| \le n_k$ by Lemma 4.33. By Claim 3 and Claim 4, we have

$$m_\omega = m(0) - m(\beta) = i_1 + \cdots + i_{k_0} \le n_1 + \cdots + n_{k_0} = m_\omega.$$

Thus we know that $i_k = n_k$ for $1 \le k \le k_0$.

Therefore the Morse index of E changes at each bifurcation point. By Theorem 4.5, the statement holds. □

Lemma 4.35. *The nontrivial solutions bifurcating from $\mathcal{T}$ are not given by solving the Dirichlet problems, and are positive solutions.*

Proof. Suppose (u, v), which is in the bifurcation branch from $\mathcal{T}$, is given by solving Dirichlet problems, hence $u(-T/2) = u(T/2) = 0$ and $v(-T/2) = v(T/2) = 0$. Note that $u'(-T/2) = u'(T/2)$ by the periodic condition. Because the solution is in the bifurcation from $\mathcal{T}$, the solution can be earned to change continuously from $(A_{\beta_k}, B_{\beta_k})$. In such continuous change, there is a part satisfying $u \geq 0$ connected to $\mathcal{T}$, so we assume $u \geq 0$. According to the Strong Maximum Principle, a zero cannot be earned inside the domain at the first time of the branch part, Thus the first zero of the solutions on the branch part is realized on the boundary. If $u'(-T/2)$ is positive (or negative), then $u < 0$ around $t = T/2$ (or $t = -T/2$, respectively) because of

$$u = u'(T/2)(t - T/2) + o(t - T/2),$$

thus we get a contradiction. Therefore $u'(-T/2) = u'(T/2) = 0$. Since θ is a solution of the first equation of (4.1) satisfying the condition, then we get $u = \theta$ because of the uniqueness of solutions for differential equations. The same thing holds for v.

The above discussion implies that the solutions cannot have a zero. Therefore the solutions must be positive. $\qquad \square$

Set

$\mathcal{S}_k :=$ The connected bifurcation component though $((A_{\beta_k}, B_{\beta_k}), \beta_k)$.

Lemma 4.36. *In $X_{\text{even}} \times X_{\text{even}}$, the branch $\mathcal{S}_k$ from β_k is global for all k. Hence, the branch $\mathcal{S}_k$ from β_k is global for all k in $\mathcal{H}$.*

Proof. As the same as the proofs of Lemma 4.11, 4.15, and 4.16. $\qquad \square$

Lemma 4.37. *For $\mu_1 < \beta < \mu_2$, there is no positive/negative solution of (4.3).*

Proof. Multiplying v (or u) with the first (second, respectively) equation of (4.3) then integrating with the interval T we have

$$\int_T u'v' + \int_T uv = \mu_1 \int_T u^3 v + \beta \int_T v^3 u,$$
$$\int_T v'u' + \int_T uv = \mu_2 \int_T v^3 u + \beta \int_T u^3 v.$$

Subtracting the second equation from the first equation, we get

$$0 = \int_T [(\mu_1 - \beta)u^2 + (\beta - \mu_2)v^2]uv.$$

Note that the inside bracket on the right hand side must be negative because of $\mu_1 - \beta < 0$ and $\beta - \mu_2 < 0$, so we get that $uv = \theta$ or uv must change sign which contradicts with either $u > 0$ and $v > 0$, or $u < 0$ and $v < 0$. $\qquad\square$

Lemma 4.38. In $X_{\text{even}} \times X_{\text{even}}$, for any solution $(u, v) \in X_{\text{even}} \times X_{\text{even}}$ in $\mathcal{S}_k$,

$$\sqrt{\mu_1 - \beta}u - \sqrt{\mu_2 - \beta}v$$

has precisely $2k$ zeros on $[-T/2, T/2)$.

Proof. In $X_{\text{even}} \times X_{\text{even}}$, for β which is sufficiently close to β_k,

$$
\begin{aligned}
u &= A_{\beta_k} + (\beta - \beta_k)\gamma_\pm \psi_k + o(\beta - \beta_k), \\
v &= B_{\beta_k} + (\beta - \beta_k)\psi_k + o(\beta - \beta_k),
\end{aligned}
$$

where $\gamma_\pm = \gamma_+$ if $\beta \leq 0$ and $\gamma_\pm = \gamma_-$ if $\beta > 0$, and ψ_k, which is the eigenfunction of the linearized problem, has $2k$ zeros on $[-T/2, T/2)$.

Then for β which is sufficiently close to β_k, we know

$$\sqrt{\mu_1 - \beta}u - \sqrt{\mu_2 - \beta}v = [\sqrt{\mu_1 - \beta}\gamma_+ - \sqrt{\mu_2 - \beta}](\beta - \beta_k)\psi_k + o(\beta - \beta_k).$$

Thus around β_k, the statement holds.

Next, set

$$
\begin{aligned}
\alpha &:= \sqrt{\frac{\mu_1 - \beta}{\mu_2 - \beta}}, \\
w &:= \alpha u - v,
\end{aligned}
$$

then in the radial coordinate, w satisfies

$$
\begin{aligned}
-w'' &= (\mu_1 u^2 + \sqrt{\mu_1 - \beta}\sqrt{\mu_2 - \beta}uv + \mu_2 v^2)w, \\
w(-T/2) &= w(T/2), \quad w'(-T/2) = w'(T/2).
\end{aligned}
$$

Therefore w must have only simple zeros because if a zero which is not simple is realized inside the domain, then we get $w = \theta$ due to the Strong Maximum Principle. Suppose that a zero is realized on the boundary of the domain the first time in the branch connected to $\mathcal{T}$, then if w' is not zero on the boundary, we find a negative point

around the boundary. Thus w' on the boundary must be zero, so we get $w = \theta$ again by the uniqueness of solutions for the differential equations. Therefore w cannot make the number of zeros change continuously. $\qquad\square$

We refer to the following theorem as a consequence of a Liouville Type Theorem by a Standard Blow-up Argument:

Theorem 4.39 (Bartsch-Dancer-Wang). *([2] Theorem 2.5) For $\Omega \subset \mathbb{R}^m$, $m \leq 3$, suppose $m = 1$ otherwise Ω is radial. Then given a compact set $B \subset \mathbb{R}$ and $k \in \mathbb{N}$, the set*

$$\Bigg\{ \, (\beta, u, v) \in \mathbb{R} \times H_0^1(\Omega) \times H_0^1(\Omega) : (\beta, u, v) \text{ solves the problem (4.11):}$$

$$\begin{cases} -u'' + u = \mu_1 u^3 + \beta v^2 u, \\ -v'' + v = \mu_2 v^3 + \beta u^2 v, \\ u, v > 0. \end{cases} \qquad (4.11)$$

in Ω, $\beta \in B$, and $\sqrt{\mu_1 - \beta}u - \sqrt{\mu_2 - \beta}v$ has at most k zeros. $\Bigg\}$

is bounded.

Remark 4.40. Theorem 4.39 holds for $(u, v) \in H_T^1 \times H_T^1$ too since the proof does not depend on $H_0^1(\Omega)$. See the proof of [2] Theorem 2.5.

Lemma 4.41. $\mathcal{S}_k$ *extends to* $-\infty$ *for* β *if* $\mu_1 \neq \mu_2$.

Proof. The branches are unbounded due to Theorem 4.9 with Lemma 4.38, and cannot pass through the region $(\mu_1, \mu_2) \times \mathcal{H}$ because of no solutions by Lemma 4.37. Recall that the solutions on the branches are positive by Lemma 4.35, so we can apply Theorem 4.39 with Lemma 4.38. Thus each $\mathcal{S}_k$ in $X_{\text{even}} \times X_{\text{even}}$ must be bounded above in the compact set in Theorem 4.39. It implies that the statement holds. $\qquad\square$

Now, our main theorem in Section 4.4 (Theorem E).

Theorem 4.42 (Theorem E). *Suppose*

$$T \neq \sqrt{2}\pi k, \quad k = 1, 2, \cdots .$$

Then along $\mathcal{T}$, there are infinitely many bifurcation points of (4.3): $\beta_1 > \beta_2 > \beta_3 > \cdots \beta_{k_0} > 0 > \beta_{k_0+1} > \beta_{k_0+2} > \cdots > \beta_k$, $\beta_k \to -\sqrt{\mu_1\mu_2}$ as $k \to \infty$, having the following properties:

a) The nontrivial solutions in the bifurcation branches are not given by solving Dirichlet problems, and are positive solutions.

b) These bifurcation branches are global.

c) Each $\mathcal{S}_k$ extends to $-\infty$ for β if $\mu_1 \neq \mu_2$.

d) In $X_{\text{even}} \times X_{\text{even}}$, $\mathcal{S}_k \cap \mathcal{S}_l$ for $k \neq l$.

Proof. Collecting lemmas in the above, we know the theorem holds. □

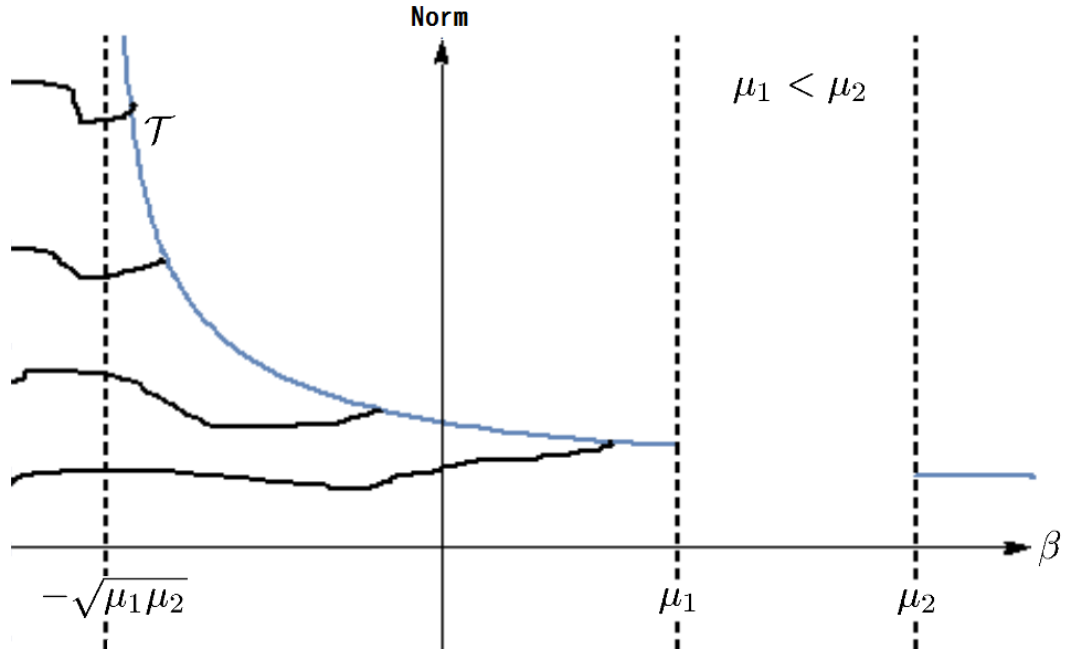

Figure 4.2: Aspects of bifurcation branches from $\mathcal{T}$

Finally, we study the case $T = \sqrt{2}\pi j$ for some $j = 1, 2, \cdots$.

Lemma 4.43. *Suppose $T = \sqrt{2}\pi j$ for some $j = 1, 2, \cdots$. Then for a fixed β, the problem*

$$-\psi'' = \Lambda\psi + b(\beta)\Theta_{A,B}, \quad \Lambda_j \neq \Lambda,$$

$$\psi(-T/2) = \psi(T/2), \quad \psi'(-T/2) = \psi'(T/2),$$

where

$$\Theta_{A,B} := A\cos\frac{2\pi}{T}jt + B\sin\frac{2\pi}{T}jt,$$

with constants A and B has a nontrivial solution which is given by

$$\tilde{\psi} := \tilde{A}\cos\frac{2\pi}{T}jt + \tilde{B}\sin\frac{2\pi}{T}jt,$$

where

$$\tilde{A} := \frac{b(\beta)A}{\Lambda_j - \Lambda}, \quad \tilde{B} := \frac{b(\beta)B}{\Lambda_j - \Lambda}.$$

Proof. Substitute $\tilde{\psi}$ into the problem, then we get

$$\Lambda_j\tilde{\psi} = \Lambda\tilde{\psi} + b(\beta)\Theta_{A,B},$$

thus

$$\begin{cases} \Lambda_j\tilde{A} = \Lambda\tilde{A} + b(\beta)A, \\ \Lambda_j\tilde{B} = \Lambda\tilde{B} + b(\beta)B. \end{cases}$$

So if we set

$$\tilde{A} := \frac{b(\beta)A}{\Lambda_k - \Lambda}, \quad \tilde{B} := \frac{b(\beta)B}{\Lambda_k - \Lambda},$$

then $\tilde{\psi}$ solves the problem. $\quad\square$

Our result for the case $T = \sqrt{2}\pi j$ (Theorem F).

Theorem 4.44 (Theorem F). *Suppose*

$$T = \sqrt{2}\pi j$$

for some $j = 1, 2, \cdots$. Then any point in $(-\sqrt{\mu_1\mu_2}, \mu_1) \cup (\mu_2, \infty)$ in β is a possible bifurcation point of (4.1) with the condition $\lambda_1 = \lambda_2 = 1$.

Proof. Lemma 4.43, setting $\Theta_{A,B} = \phi - \gamma_{\pm}\psi$ implies that there is a nontrivial kernel at any point in the interval in β. $\quad\square$

Remark 4.45. As a different way, we could show that there are the possible bifurcation points of the above lemma by

$$-\psi'' \;\; = \;\; (b\gamma_- + c - 1)\psi = 2\psi$$

for $\beta \leq 0$, and

$$-\psi'' \;\; = \;\; (b\gamma_+ + c - 1)\psi = 2\psi$$

for $\beta > 0$, in the proof of Lemma 4.33.

4.5 Summary

In this chapter, the bifurcation structures of GPEs were studied. We showed there are countably infinitely many bifurcation points which bifurcates from semi-trivial solutions exists in β and in λ_1 (thus in λ_2), by the Crandall-Rabinowitz Local Bifurcation Theorem (Theorem 4.14). We also showed that some qualitative properties including global bifurcations by the restrictions of the functional's domain into the space of even functions. Similarly, we also showed that there are countably infinitely many bifurcation points which bifurcates from a synchronized solution by using the Morse index with some techniques of [2]. In the case $T = \sqrt{2}\pi j$ for some j, any point in β is a possible bifurcation point along $\mathcal{T}$, hence there are uncountable possible bifurcation points.

CHAPTER 5

CONCLUSIONS AND FUTURE RESEARCHES

5.1   Summary of results

In this dissertation, we showed the existence, multiplicity, and some qualitative properties of periodic solutions for the following two classes of nonlinear differential equations:

I) (Special) Relativistic Pendulum Equations (RPEs):

$$(\phi(u'))' = \nabla_u F(x, u) + h(x),$$

where $\phi : (-a, a) \to \mathbb{R}$ is an increasing homeomorphism satisfying $\phi(0) = 0$, and $\phi(s)s > 0$ for all $s \in (-a, a) \setminus \{0\}$. This type of equation also arises from geometric problems such as the minimum surfaces with various choices of ϕ.

II) (2-coupled) Gross-Pitaevskii Equations (GPEs):

$$\begin{cases} -u'' + \lambda_1 u = \mu_1 u^3 + \beta v^2 u, \\ -v'' + \lambda_2 v = \mu_2 v^3 + \beta u^2 v, \end{cases}$$

where λ_i, μ_i, and β are parameters.

For RPEs, more generally, we treated the case where the range of u is $\mathbb{R}^n$. Under some conditions, we established a multiplicity result depending on the periodic condition of F by applying the Generalized Saddle Point Theorem (Theorem 2.5). Our result partially answered open problems which were raised by Brezis and Mawhin in 2010 ([8]).

For 2-coupled GPEs, in the symmetric case of $\lambda_1 = \lambda_2 = \lambda > 0$ and $\mu_1 = \mu_2 = \mu > 0$, by applying $\mathbb{Z}_2$-Index Theory, we showed that there are infinitely many solutions for $\beta \leq -\mu$, and, for any integer, there exist at least k pairs of solutions (u, v) and (v, u) depending on β for $\beta > -\mu$.

In addition, the system has the following constant solutions:

In the general case, $(u,v) = (\omega_1, \theta)$, $(u,v) = (\theta, \omega_2)$, $\quad \omega_i := \sqrt{\dfrac{\lambda_i}{\mu_i}}$.

In the case $\lambda_1 = \lambda_2 = 1$, $(u,v) = (A_\beta, B_\beta)$,

$$A_\beta := \sqrt{\frac{\mu_2 - \beta}{\mu_1 \mu_2 - \beta^2}}, \ B_\beta := \sqrt{\frac{\mu_1 - \beta}{\mu_1 \mu_2 - \beta^2}}.$$

The bifurcation structures bifurcating from the following solution curves:

$$
\begin{aligned}
\mathcal{T}_1^\beta &:= \{(\omega_1, \theta) \in H_T^1 \times H_T^1 : \beta \in \mathbb{R}\}, \\
\mathcal{T}_1^{\lambda_1} &:= \{(\omega_1, \theta) \in H_T^1 \times H_T^1 : \lambda_1 > 0\}, \\
\mathcal{T} &:= \{(A_\beta, B_\beta) \in H_T^1 \times H_T^1 : \beta \in (-\sqrt{\mu_1 \mu_2}, \mu_1) \cup (\mu_2, \infty)\},
\end{aligned}
$$

were also studied. To show the existence of bifurcation points, we applied the Crandall-Rabinowitz Local Bifurcation Theorem (Theorem 4.14) which is proved by the Lyapunov-Schmidt Reduction which changes the problem in the infinite dimensional space of functions into a problem in a finite dimensional space due to that the linearized equations are of Fredholm operators.

In the case of the bifurcations from $\mathcal{T}$, we used the Morse Theory to show there are countably infinitely many bifurcation points with some techniques of [2]. In addition, we showed that there are global bifurcations, and also showed their some properties by restricting the domain of the functional into the space of even functions with the Symmetric Criticality Principle.

5.2 Future researches

Finally, we mention some possible future research themes for RPEs and GPEs.

5.2.1 Further research theme for RPEs

Minimal period

To ask if there exists a solution with the minimal period will be an interesting question. In [49] Wang, Wang, and Shi showed that there exists a solution having minimal period pT, where p is an integer such that $p \neq 1$, was proved in the CPE

$$x'' + A \sin x = h(t), \tag{5.1}$$

under the following conditions:

(H_h) : *The function h is T-periodic with minimal period T and odd in t.*

(H_p) : *For an integer $p > 1$,*

$$0 < \frac{\omega^2}{A} < p^2 < \frac{\omega^2 s_p^2}{A},$$

where s_p is the least prime factor of p.

The proof relies on Variational Methods and Fourier Analysis. Yu also studied for the minimal period ([52] [53]). It would be natural to ask if the similar results hold for RPEs.

Maximum number of solutions

In [36] Ortega showed at most how many solutions exist in the classical pendulum equation by using the idea in [25] to change the problem into counting the zeros of a corresponding holomorphic function via the Jensen Inequality in complex analysis (which is not the Jensen Inequality known in real analysis or statistics. See [1] [41]). It would be natural to ask if it can be applied for RPEs.

Bifurcation structures from CPEs

To study bifurcation structures of RPEs bifurcating from solutions of CPEs (e.g. (1.1), (5.1)) will be naturally interesting. It is even unclear that what kind of parameters can be a bifurcation parameter. Parameters relating to Lorentz transformations would be natural candidates of bifurcation parameters.

5.2.2   Further research theme for GPEs

How to distinguish the multiple solutions

In Chapter 3, we saw that the multiple existence results of the Dirichlet condition in [16] hold for the periodic condition too. In addition, we got the bifurcation branches that might include these solutions. However, we are not sure how these solutions are identified together.

T-dependency of the number of solutions

On the Dirichlet boundary condition, a positive solution cannot be extended as an even function, due to the Strong Maximum Principle. Thus it is natural to guess that the number of solutions will depend on period T. To show such dependency of

the number of solutions, Fourier Analysis will naturally be one of the tools to use. However if we apply the similar discussion with the Fourier Analysis for T instead of β, it will be a question if the infimum level of fixed points bounded independently from T (Lemma 3.8 (i)). In addition, it will be a problem to discuss the limits of T because the coefficients of a Fourier expansion depend on T. Thus, how to get multiple existence results depending on T will be an interesting question. In [34], the existence of solutions with T-dependency were proved by using the Topological Degree Theory of (nonlinear) compact operators.

Bifurcations of uncountable possible bifurcation points

We showed there are uncountable infinitely many possible bifurcation points as in Theorem 4.44. In this case, we cannot apply Theorem 4.5 to show these points are actually bifurcation points because each critical point is not isolated, hence we cannot use the Morse Theory directly. To show bifurcations actually happen for such uncountable possible bifurcation points is an open problem.

Mixed couplings

In 3-coupled Gross-Pitaevskii system:

$$
\begin{cases}
-u_1'' + \lambda_1 u_1 = \mu_1 u_1^3 + \beta_{12} u_1 u_2^2 + \beta_{13} u_1 u_3^2, \\
-u_2'' + \lambda_2 u_2 = \mu_2 u_2^3 + \beta_{21} u_2 u_1^2 + \beta_{23} u_2 u_3^2, \\
-u_3'' + \lambda_3 u_3 = \mu_3 u_3^3 + \beta_{31} u_3 u_1^2 + \beta_{32} u_3 u_2^2,
\end{cases}
\tag{5.2}
$$

where λ_i, μ_i, and β_{ij} are constants, and $\beta_{ij} = \beta_{ji}$ $(i \neq j)$ since otherwise the variational structure does not exist. Each β_{ij} can be either positive or negative. In Variational Methods, the positive/negative beta case is generally treated in differently defined manifolds. The mixed couplings case, which is the case in which the system has both negative beta and positive beta, will be an interesting problem. If we combine these different manifolds together, then we would expect to be able to treat the mixed couplings case.

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APPENDIX

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3 messages

HATA Kazuya <hata@mail1.big.or.jp>   Fri, Dec 5, 2014 at 9:10 AM
To: wkrysz@mat.umk.pl
Cc: Kazuya HATA <kazuya.hata@aggiemail.usu.edu>

Dear Editor of Topological Methods in Nonlinear Analysis

How are you? I am Kazuya Hata who is a coauthor of the following paper:

K. Hata, J. Liu, and Z.-Q. Wang: Note on periodic solutions of relativistic pendulum type systemps, Topological Methods in Nonlinear Analysis 42, 2, (2013), 417-425.

Now I am writing my dissertation which includes the materials of the paper, so I want to get your permission to reprint as soon as possible.

Thanks.
HATA, Kazuya.

Wojciech Kryszewski <wkrysz@mat.umk.pl>   Fri, Dec 5, 2014 at 8:04 PM
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Dear Author

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W dniu 2014-12-05 o 01:10, HATA Kazuya pisze:
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Ta wiadomość została sprawdzona na obecność wirusów przez oprogramowanie antywirusowe Avast.
http://www.avast.com

HATA Kazuya <hata@mail1.big.or.jp>   Sat, Dec 6, 2014 at 12:44 AM
To: Wojciech Kryszewski <wkrysz@mat.umk.pl>
Cc: Kazuya HATA <kazuya.hata@aggiemail.usu.edu>

Dear Editor of Topological Methods in Nonlinear Analysis, Wojciech Kryszewski

Thank you very much. To show the consent, I will attach this e-mail communication with you on my dissertation at Utah State University.

Sincerely,
HATA, Kazuya.
[Quoted text hidden]

A permission-to-use

3 messages

HATA A. Kazuya <hata@mail1.big.or.jp>   Wed, Dec 10, 2014 at 2:49 AM
To: jiaquan@math.pku.edu.cn
Cc: Kazuya HATA <kazuya.hata@aggiemail.usu.edu>

Dear Dr. Jiaquan Liu.

How are you? I am Kazuya Hata who is a coauthor on the following paper with you:

K. Hata, J. Liu, and Z.-Q. Wang: Note on periodic solutions of relativistic pendulum type systemps, Topological Methods in Nonlinear Analysis 42, 2, (2013), 417-425.

I am writing my Ph.D. dissertation in mathematics at Utah State University (2014), and I want to get your permission to use the materials of the paper for the dissertation.

Sincerely,
HATA, Kazuya.

---

jiaquanliu <jiaquan@math.pku.edu.cn>   Sun, Dec 14, 2014 at 7:53 AM
To: "HATA A. Kazuya" <hata@mail1.big.or.jp>
Cc: Kazuya HATA <kazuya.hata@aggiemail.usu.edu>

No problem. Wish you success with your dissertation
J.Q.Liu

在 2014-12-10 01:49:21，"HATA A. Kazuya" <hata@mail1.big.or.jp> 写道：
[Quoted text hidden]

HATA A. Kazuya <hata@mail1.big.or.jp>   Mon, Dec 15, 2014 at 7:18 AM
To: jiaquanliu <jiaquan@math.pku.edu.cn>
Cc: Kazuya HATA <kazuya.hata@aggiemail.usu.edu>

Dear Dr. Jaquan Liu

Thank you very much! I will also attach this e-mail communication with you to my dissertation at Utah State University (2014).

Thank you very much for the collaboration, again.

Sincerely,
HATA, Kazuya.
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