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Network Applications and the Utah Homeless Network

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NETWORK APPLICATIONS AND THE
UTAH HOMELESS NETWORK

by

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ABSTRACT

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Major Professor: Dr. David E. Brown
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Graph theory is the foundation on which social network analysis (SNA) is built. With the flood of "big data," graph theoretic concepts and their linear algebraic counterparts are essential tools for analysis in the burgeoning field of network data analysis, in which SNA is a subset. Here we begin with an overview of SNA. We then discuss the common descriptive measures taken on network data including centrality, density, and connectivity. We also define a new data structure which we call the *location sequence matrix* (LSM). This structure has many combinatorial applications to the homeless network in Utah, including enumeration of 2-cycles, determining graph structures via intersection of chronological data, and the identification of path combinations which may lead to certain behaviors in subjects from physical data. The LSM also makes construction of dynamic graph adjacency matrices, which depict multiple location transitions, particularly easy. Finally we apply Pulse Processes, a marriage between graph theory and linear algebra, to the Utah homeless network in a new way. Of particular interest is an eigen-analysis of the the homeless network, where if $|\lambda_i| < 1$ for all eigenvalues λ_i in the network adjacency matrix, then the network is "stable." By stable, we shall mean that we do not expect the possibility of the population at any particular service to grow without bound. We believe the LSM and pulse processes, when used for analysis of the

Utah homeless network, may be useful in forming policy decisions regarding homeless services. In particular, changes in population may be tested by "removing" or "adding" vertices in the network, and then testing the resultant stability through the use of pulse processes.

(62 pages)

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Graph theory is the foundation on which social network analysis (SNA) is built. With the flood of "big data," graph theoretic concepts and their linear algebraic counterparts are essential tools for analysis in the burgeoning field of network data analysis, in which SNA is a subset. Here we begin with an overview of SNA. We then discuss the common descriptive measures taken on network data as well as proposing new measures specific to homeless networks. We also define a new data structure which we call the *location sequence matrix*. This data structure makes certain computational network analyses particularly easy. Finally we apply Pulse Processes in a new way to the homeless network in Utah. We believe the new data structure and pulse processes, when used for analysis of the Utah homeless network, may be useful in forming policy decisions regarding homeless services. In particular, pulse processes, first introduced by Brown, Roberts, and Spencer [14], to analyze energy demand, form a dynamic population model that can provide a measure of the stability in a network and the patterns of action of individuals experiencing homelessness.

(62 pages)

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Michael A. Snyder

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INTRODUCTION AND BASIC DEFINITIONS

1.1 Introduction

Social network theory (SNA) has developed as a natural evolution in the observation of how people, groups of people, and organizations are connected. While pseudo-mathematical constructions of social networks began appearing in the late 1800's, it was Moreno's sociometry that began to inspire more formal systems in the 1930's [13]. Enjoying some success prior to the 1970's, SNA was then criticized in favor of traditional linear models. These attitudes were due in large part to a misunderstanding in fundamental differences between the two methods. Linear modeling relies on the idea of independent observations, while dependence is built in to SNA as a result of the network paradigm [11]. It is the dependence in the system that every SNA researcher hopes to exploit. As a result it is considered standard practice to analyze local interactions within a network in an attempt to describe its global properties. It should be noted that traditional linear analysis should not be abandoned, but that a marriage of the two methods may help to more fully describe the attributes of a particular social dilemma.

Ironically, Frank Harary published what was at the time considered the definitive textbook on graph theory in 1969, but due to the continuing absence of networking between mathematics and the social sciences SNA still suffered a lack of support until more recently. Graph theory is the mathematical basis for SNA, and it is this theoretical basis that is at times neglected resulting in the apt adage, “Applying graph theoretic structures to sociological problems wholesale can sometimes lead to the mathematical tail wagging the sociological dog” [13]. Much research has been conducted in graph theory in the last 30 or 40 years, which, along with the ubiquity of network data, could be conjectured as the basis for the popularity that SNA has enjoyed most recently. Among the innovations currently taking place is the idea of dynamical social network theory.

That is, networks which change over time. After presenting some basic definitions, we will explore some recent research that applies both static and dynamic methods.

1.2 Chapter 1 - Basic Definitions

We now define the elements from Graph Theory that are used in the construction of social network systems. An actor in a social network is represented as a vertex or node in a graph. When there is an interaction between two actors in a network we say there is a tie between them. In a graph we say the two vertices are adjacent. In a drawing this is represented by a line segment, or edge, that connects the two vertices. These interactions are often referred to as a dyad, with an interaction between three vertices called a triad.

1.2.1 Graph

A **graph** G is an ordered triple, $(V(G), E(G), \Psi)$, with $V(G)$ the vertex set, and Ψ a relation between vertices such that if $u, v \in V(G)$ and $u\Psi v$, then $uv \in E(G)$. When $u\Psi v$, we say that u is adjacent to v. A graph may be conveniently represented by its **adjacency matrix**, which is a (0,1)-matrix, A, where a one in the ijth entry means that vertices i and j are adjacent in its graph representation, and a zero means they are not. A directed graph is a graph whose edges, which we will call arcs, have been given an orientation. Its adjacency matrix, denoted D, differs from A in that it only has a 1 in the ijth entry when there is an edge directed from i to j.

1.2.2 Graph Attributes

Graphs may contain many attributes. We name here some of the most common beginning with vertex degrees. The **degree** of a vertex $v \in V(G)$, denoted $d(v)$, is an integer equal to the number of edges incident to v. Directed graphs have both an indegree and an outdegree. The **indegree** of a vertex v in a digraph D is denoted $d^-(v)$. In D's adjacency matrix, $d^-(v) = \sum_i D_{ij}$. The **outdegree** of a vertex v in a digraph D is denoted $d^+(v)$. In D's adjacency matrix, $d^-(v) = \sum_j D_{ij}$.

Graph attributes may also apply to a graph H formed from a graph G with $V(H) \subseteq V(G)$. We call H a **subgraph** of G, provided $E(H) \subseteq E(G)$. A **clique** in a graph G is a subgraph of G, in which every pair of vertices in the subgraph are adjacent.

There are a variety of ways to describe the idea of traversing a graph. A **walk** in a graph G is a sequence of vertices and edges, $(v_0, e_1, v_1, e_2, \ldots, v_{l-1}, e_l, v_l)$. Note that

$v_i = v_j, i \neq j$ is possible, and $e_i = e_j, i \neq j$ is also possible in a walk. In G's adjacency matrix, A^r_{ij} gives the number of walks with r edges between vertices i and j, where A^r is the rth power of the matrix A. We give here a proof of this fact.

Theorem 1.1 ([10]). *If D is a digraph with adjacency matrix $A = (a_{ij}$, then the ij entry of A^r gives the number of paths of length r in D which lead from i to j.*

Proof. Let D be a digraph with $i, j, k, \in A(D)$. We argue by induction on r. If $r = 1$, the result is obvious. Assuming it for r, let us prove it for $r + 1$. Let a^{r+1}_{ij} represent the umber of paths of length $r + 1$ from i to j. Similarly, let a^r_{kj} represent the number of paths of length r from k to j. To go from i to j in $r + 1$ steps, one must go for i to some K directly and then form k to j in r steps. The number of ways to go from i to j in $r + 1$ steps with the first step going through k is $a_{ik}a^r_{kj}$. For this term is a^r_{kj} if $(i, k) \in A(D)$ and it is 0 if $(i, k) \notin A(D)$. To obtain a^r_{ij}, we simply sum the terms $a_{ik}a^r_{kj}$ fro all k. Thus

$$a^{(r+1)}_{ij} = \sum_{k=1}^{n} a_{ik}a^r_{kj}.$$

By the inductive assumption, a^r_{kj} is the kj entry of A^r. Hence, $a^{(r+1)}_{ij}$ is the ij entry of $AA^r = A^{r+1}$. □

A **trail** in a graph is a walk with no repeated edges. A path in a graph is a walk with no repeated vertices. A k**-path**, or a **path** containing k vertices, is denoted P_k. A **cycle** in a graph is a path with the "last" vertex equal to the "first" vertex. A k-cycle, or cycle containing k vertices, is denoted C_k. A **circuit** in a graph is a trail with initial and final vertex equal. If we want to describe how many vertices are **reachable** from a vertex u, then we consider the reachability of u. That is, v is reachable from u if and only if there is a walk from u to v. This definition assists in describing the idea of connectedness in a graph.

A graph G is **connected** if for $u, v \in V(G)$, u is reachable from v. A **component** is a maximally connected subgraph of G. Digraphs may have the additional attribute of being either weakly or strongly connected. A digraph D is **weakly** connected if it is connected by considering it an undirected graph, and it is **strongly connected** if for every pair of vertices $u, v \in V(D)$, v is reachable from u by a directed path.

A geodesic in a graph G is the path with the fewest number of edges between two vertices $u, v \in V(G)$. Distance in a graph G is measured by the shortest path(s), or geodesic, between u and v, for $u, v \in V(G)$. If G has two components A and B, then the distance between two vertices $a \in A$ and $b \in B$ is defined to be infinite. The diameter in a graph G is the longest distance in a graph. These ideas may be extended to graphs whose edges have weights.

If the graph has a function w that assigns a number other than 1 or 0 to each edge in G, then we consider G to be a **weighted graph**. Many graph attributes can be generalized to the case of a weighted graph. For example, the length of paths, walks, and circuits are measured by the sum of the weights on the edges transversed. Hence, distance and diameter generalize to the smallest sum and greatest sum respectively.

1.2.3 Graph Classes

Let $\mathcal{G}$ be the set of all graphs. If a subset Ξ of $\mathcal{G}$ can be completely characterized by a set of true statements, then we call Ξ a **class** of graphs. Some of the most common classes of graphs include complete graphs, regular graphs, trees, and bipartite graphs.

A **complete graph**, is a graph G such that for every $u, v \in V(G), uv \in E(G)$. A **regular graph** is a graph such that for every $u, v \in V(G)$, we have $d(u) = d(v)$.

Trees are often used to represent data structures like folders on a computer. A **tree** is a connected graph with no cycles. A **forest** is a disjoint union of trees. If we add an orientation to the vertices in a graph G, where G contains at least one cycle, and the resulting orientation contains no directed cycles, then we have a directed acyclic graph, or **DAG**.

A **bipartite graph** is a graph G consisting of the union of two sets of vertices X and Y, called **partite sets**, with $x \in X$ and $y \in Y$ adjacent if and only if $xy \in E(G)$, and no adjacencies existing within the partite sets. We may extend the definition of bipartite to include more than one partition of a a set, this results in a **multi-partite** graph. There can be many alternative characterizations for any class of graphs. For example, bipartite graphs may also be characterized as all graphs containing no odd cycles.

1.2.4 Graph Matrices

In addition to the adjacency matrix of a graph, we include three other linear algebraic graph objects. The **incidence matrix** B of a graph G is a $|V| \times |E|$ matrix with

$$B_{ij} = \begin{cases} 1 & \text{if vertex } i \text{ is incident to edge } j \\ 0 & \text{otherwise} \end{cases}$$

The **degree matrix** of a graph G is the matrix $D = diag[(d_i)_{i \in V}]$. Finally, Let G be a graph and $\tilde{B}$ be its signed incidence matrix where each 1 is given a sign indicating an arbitrarily assigned orientation. Then the **Laplacian** of G is

$$\mathcal{L} = \tilde{B}\tilde{B}^T = D - A$$

where D is the degree matrix of G and A is G's adjacency matrix.

CHAPTER 2

NETWORK ANALYSIS AND SOCIAL NETWORK ANALYSIS

2.1 Network Measures and Social Network Analysis

In applications we will be most interested in network analysis applied to homeless populations. A host of different tools and measures have been used to analyze a variety of social networks in recent literature. Data on homeless networks is difficult to obtain, and "big" homeless network data is only now available due to policies requiring service providers to collect and store a minimum amount of data using a Homeless Management Information System (HMIS). As a result, not much analysis has been done on these populations, and the most commonly studied measures are centrality and density. Throughout this chapter, we present common network measures, and where literature is available, we give examples of these measures applied to homeless networks. The development of these measures follows that found in Kolaczyk [6].

2.2 Centrality

Centrality measures come in many different forms. In general, the idea is to describe with one number, the idea of a particular vertex as being central to a graph. As one might imagine, this idea may be drastically different depending on what question is being answered. Since many notions of centrality fall closer to the statistical side of graph theory, it is important to mention the idea of normalization of a measure. Normalization is simply rescaling the range of a measure to lie within a specified interval. Most graph measures are normalized to lie within the interval [0,1] on the real line.

Before defining the most common centrality measure in SNA, we state some definitions, as in West [15], that lead to a definition of the center of a graph. Recall that distance in a graph between two vertices $u, v \in V(G)$, denoted $d(u, v)$ is the geodesic from u to v.

Definition 2.1. The diameter of a graph G is $\max_{u,v \in V(G)} d(u, v)$.

Definition 2.2. The eccentricity of a vertex u, denoted $\epsilon(u)$, is $\max_{v\in V(G)} d(u,v)$

Definition 2.3. The center of a graph G is the subgraph induced by the vertices of minimum eccentricity.

By these definitions, the center of a graph is a subgraph whose vertices have a minimum distance from every other vertex in the graph. If we consider this from a network perspective, we might consider the actors these vertices represent as the most influential vertices in the graph, since they can reach every other vertex within minimal distance. We now turn our attention to statistical centrality measures.

2.2.1 Degree Centrality

Possibly the simplest centrality measure to compute is the degree centrality. Let $G = (V, E)$ be a graph. Then the degree centrality of a vertex $v \in V$ is given by

$$C_d(v) = \frac{d(v)}{n-1}.$$

2.2.2 Closeness

Closeness is a measure of distance from a specified vertex in a graph to every other vertex in the graph. Closeness is a very basic and straightforward measure of centrality, however, the problem with the closeness centrality measure, is that it tells us nothing about disconnected graphs. Since the distance between $u, v \in V(G)$ with u and v in distinct components of G is defined to be infinite, if G is disconnected $c_{cl}(v) = 0$. Here we describe the measure due to Sabidussi [12]. Let $u, v \in V(G)$. If $dist(v, u)$ is the geodesic distance between v and u, then the closeness centrality of v is

$$c_{Cl}(v) = \frac{1}{\sum_{u\in V} dist(v,u)}.$$

This measure may be normalized for comparison by multiplying by a factor of $|V| - 1$.

2.2.3 Betweeness

Betweeness is a measure of "the extent to which a vertex is located 'between' other pairs of vertices." Applications often involve finding the importance of a vertex in terms of its access to other vertices in the graph via paths. The measure we describe is due to Freeman [3]. Let $v, s, t \in V(G)$, $\sigma(s, t|v)$ be the total number of shortest paths between

s and t that pass through v, and $\sigma(s,t) = \sum_v \sigma(s,t|v)$. Then the betweeness centrality is

$$c_B(v) = \sum_{s \neq t \neq v \in V} \frac{\sigma(s,t|v)}{\sigma(s,t)}.$$

In this case, normalization is obtained by division of the maximum possible value for $c_B(v)$, which is $\frac{(|V|-1)(|V|-2)}{2}$, which was proved by Freeman. This is the number of pairs of vertices other than v in the graph.

2.2.4 Eigenvector

Eigenvector closeness measures the closeness of the neighbors of vertex v. Then if the neighbors of v are close, it is implied that v is also close. The measure presented here is due to Bonacich [1]. Let $c_{E_i} = (c_{E_i}(1), \ldots, c_{E_i}(|V|))^T$ be the solution vector to the equation $Ac_{E_i} = \alpha^{-1}c_{E_i}$, where A is the adjacency matrix of a graph G. Then the eigenvector corresponding to the largest eigenvalue contains a measure of the centrality of each vertex in the graph. In practice, α^{-1} is the largest eigenvalue of the adjacency matrix, though any of the eigenvectors would work.

2.2.5 Centralization Index

If $c(v)$ is the centrality of a vertex $v \in G$, and $c*$ is the maximum of the $c(v)$ over G, then the centrality index is given by

$$c = \frac{\sum_{v \in V}[c^* - c(v)]}{\max \sum_{v \in V}[c^* - c(v)]}.$$

This may not be easily computed depending on the graph.

From a social network perspective, Kezar [5] notes that centrality is a measure of how central a vertex is relative to some social interaction. In social networks, we may rank vertices by their degree centrality. This may lead to an understanding of relationships within the graph based upon the ranking. The research of Fleury et al. [2], showed that members of the Montreal social services network with higher degree centrality tended to be more effective in providing their specific service. In particular, the centrality measures of this study described interactions between service members measured by volume, or degree centrality, and density. Density will be discussed later. Volume of intercommunication was then directly related to collaboration between service providers. Thus, the implication was that more collaborative service providers were more effective in assisting their users.

Centrality appears often in the SNA of business, politics, and education. An example of such analysis is that of social capital. An individual or group within a network is considered to have the greatest social capital if the degree centrality is highest. This information is useful in predicting how information or behavior will travel through a network. Predictions may also be made with respect to the minimum number of actors of highest degree, which must enact a particular behavior such that it will spread throughout the network [5].

In a study by Rice et al. [9], which analyzed how centrality and density in social networks of homeless youth may be used to disseminate information for HIV prevention, it was found that centrality played an important role in the effectiveness of online dissemination of this information. The peer leaders who had large online networks, or were more central, had better results than those who were less central. In addition, homophily, or the diverseness of a network, was also an important indicator in the acceptability of the messenger for information transfer, with less diverse networks being more effective than their more diverse counterparts. In particular, age and gender were the most important attributes with respect to diversity in the network. This was especially true for face-to-face interaction. Such relationships between networks and their diversity are not always so linear. In many cases a more diverse network results in positive correlations to a desired network behavior. Yet in other cases, too much diversity in a network may result in the inadvertent split of the network [5]. For example, suppose you have a diverse leadership team in a company. If the team is too diverse, subsets of employees may become loyal to the leader who best fits their philosophy which may cause an undesired rift in the team. Too little diversity may result in certain team members becoming alienated by having no one who shares their interests. It is important to note that while this study measured the acceptability of the HIV information, it says nothing about the effectiveness of it [9].

2.3 Other Network Analysis Tools

Beyond centrality lie a host of other analytical tools, many of which have not yet been applied to homeless networks. We state both those which have not yet been applied, as well as those which have been applied to homeless networks, giving examples where appropriate.

2.3.1 Density

A measure of the densness of a graph $G = (V, E)$ is given by

$$den(G) = \frac{2|E|}{|V|(|V|-1)} = (|V|-1)\bar{d}(G).$$

where $\bar{d}(G)$ is the average degree of G. This may be applied to graphs $H \subset G$ by considering $|V_H|$ and $|E_H|$ of the subgraph, which provides a measure of how close H is to being a clique in G. If we take $H = H_v$, that is the graph inclusive of the neighbors of $v \in V$, then we may define the **clustering coefficient** to be the average of $den(H_v)$ over all v.

Alternatively, we may define the clustering coefficient of G, to be the density of triangles among connected triples. This is given by

$$cl_T(G) = \frac{3\tau_\Delta(G)}{\tau_3(G)},$$

where $\tau_\Delta(G) = (1/3)\sum_{v\in V}\tau_\Delta(v)$ is the number of triangles in the graph, and $\tau_3(G)$ is the number of connected triples, and $\tau_\Delta(v)$ is the number of triangles to which the vertex v is connected. This clustering coefficient is known as the **transitivity** of the graph, or the "fraction of transitive triples."

According to Robins [11], density in a network is a measure of adjacency in a graph. The most commonly used measurement is the following: In a graph on n vertices with L arcs/edges, the density is often computed with the formulas $\frac{L}{n(n-1)}$ for directed graphs and $\frac{2L}{n(n-1)}$ for undirected graphs. The formula for directed graphs accounts for the out-degree of each vertex. This is why there is a factor of 2 in the formula for undirected graphs.

Again, the work of Fleury et al. [2] made use of this measure of connectedness in the homeless network of Montreal. In this case, density in the network described the global rate of adjacency between vertices. This information was then compared to local rates of density. In this way a metric was designed to discuss the effectiveness of particular members of the network with respect to the amount of collaboration in which the member was engaged. In general it was found that higher densities among members of the network related to greater collaboration, and thus an increase in effectiveness of the services provided.

2.3.2 Connectivity

Connectivity considers the level of connectedness in a graph. Of particular interest is the so called 'small world' property, which is related to the idea of six degrees of separation. Usually this analysis is done on the largest connected component of the graph by way of the the average distance. This is computed using

$$\bar{l} = \frac{2}{|V|(|V|+1)} \sum_{u \neq v \in V} dist(u, v).$$

Cuts and flows are an important characteristic of a network, but before addressing this idea, we define a few connectivity ideas which are useful in describing a network. In particular we address the idea of removing arbitrary vertices (edges) while maintaining a connected graph.

Definition 2.4. Let $G = (V, E)$ be a graph. Then G is k-vertex-connected, if $|V| > k$ and $X \subseteq V$ with cardinality $|X| < k$ is deleted, leaving $G - X$ still connected.

Definition 2.5. Let $G = (V, E)$ be a graph. Then G is k-edge-connected, if $|V| \geq 2$ and $Y \subseteq V$ with cardinality $|Y| < k$ is deleted, leaving $G - Y$ still connected.

Thus, the connectivity of a graph G is the largest k, such that G is k-vertex (edge)-connected, usually denoted $\kappa(G)$ ($\kappa'(G)$), respectively. Clearly these are both bounded by $d_{\min}(G)$ since removing all edges from a vertex of minimum degree, or all neighbors of such a vertex, will disconnect the graph, and in fact, Whitney [16] showed that $\kappa(G) \leq \kappa'(G) \leq d_{\min}(G)$.

Menger's Theorem gives us a way to find the minimum size of an x, y-separating set by considering the maximum number of pairwise internally disjoint x, y-paths, where two such paths are paths from x to y who share no edges and no vertices, save x and y. Before introducing Menger's Theorem, however, we need some technology, namely the König-Egerváry Theorem. Rather than prove this theorem, we have chosen to prove the equivalent statement of König's Theorem. These two theorems are related via a bipartite graph and its adjacency matrix. König's Theorem proves that the term rank of a bipartite graph's adjacency matrix is equal to its minimum line cover. The König-Egerváry Theorem replaces term rank with a maximum matching in the bipartite graph, and minimum line cover with a minimum vertex cover. Thus we define these two terms.

Definition 2.6. A **line cover** of an adjacency matrix, A, is a covering of the rows and columns of A such that every 1 in the matrix is covered.

Definition 2.7. The **term rank** of an adjacency matrix, denoted $t(A)$, is a set of 1's from the rows and columns of A such that no 1 shares a row or column with any other 1.

We now state and prove König's Theorem, after which we state the König-Egerváry Theorem for reference.

Theorem 2.8 (König)**.** *The term rank of a bipartite graph's adjacency matrix, A, is equal to its minimum line cover, that is $t(A) = \beta(A)$.*

Proof. To prove Könligs's Theorem we show that the term rank, $t(A)$, is less than or equal to the minimum line cover, $\beta(A)$, and that $t(A) \geq \beta(A)$ for any matrix A. This will implies that $t(A) = \beta(A)$.

$t(A) \leq \beta(A)$ Let A be any mxn matrix. Then without loss of generality, we can say that there are $r + c = \beta(A)$ rows and columns which contain all nonzero entries of A. Thus there are at most $r + c = \beta(A)$ independent entries in A. Therefore $t(A) \leq \beta(A)$.

$t(A) \geq \beta(A)$ Let A be an mxn matrix. Throw out rows or columns which contain all zero entries. Call this new pxq matrix A'. Let $S = A_1, A_2, \ldots, A_j$ with $j = min\{p, q\}$ and each A_i is a row or column of A'. By Hall's Theorem there is an SDR $X \subseteq S$ with $X = \{x_1, x_2, \ldots, x_j\}$ of size $|j|$, where each x_i can be viewed as an independent nonzero entry in A. If we take X to be of maximal size, then $|j| = t(A)$. Now since X accounts for all rows and columns in A which have nonzero entries we have that $|j| = \beta(A) = t(A)$.

Therefore $t(A) = \beta(A)$ which proves Könligs's Theorem using Hall's Theorem. □

Theorem 2.9 (König-Egerváry)**.** *If G is a bipartite graph, then the maximum size of a matching in G equals the minimum size of a vertex cover of G.*

We now state and prove Menger's Theorem. The version presented is that found in West [15].

Theorem 2.10. *Let G be a graph. If $x, y \in V(G)$ and $xy \notin E(G)$, then the minimum size of an xy-cut equals the maximum number of pairwise internally disjoint xy-paths.*

Proof. Let $\kappa(x, y)$ be the minimum size of a separating set and let $\lambda(x, y)$ be the maximum size of a set of internally disjoint paths in G. Showing that $\kappa(x, y) \geq \lambda(x, y)$ is "obvious," since each pairwise internally disjoint path must contain at least one vertex from a separating set. Each of these vertices must be distinct.

To prove equality, we use induction on n, the number of vertices in G. Base case: $n = 2$. Here $xy \notin E(G)$ yields $\kappa(x, y) = \lambda(x, y) = 0$. Induction step: $n > 2$. Let $k = \kappa_G(x, y)$. We construct k pairwise internally disjoint x, y-paths. Note that since $N(x)$ and $N(y)$ are x, y-cuts, no minimum cut properly contains $N(x)$ or $N(y)$.

Case 1: *G has a minimum x, y-cut S other than $N(x)$ or $N(y)$.* To obtain k desired paths, we combine x, S-paths and S, y-paths obtained from the induction hypothesis. Let V_1 be the set of vertices on x, S-paths, and let V_2 be the set of vertices on S, y-paths. We claim that $S = V_1 \cap V_2$. Since S is a minimal x, y-cut, every vertex of S lies on an x, y-path, and hence $S \subseteq V_1 \cap V_2$. If $v \in (V_1 \cap V_2) - S$, then following the x, v-portion of some x, S-path and then the v, y-portion of some S, y-path yields an x, y-path that avoids the x, y-cut S. This is impossible, so $S = V_1 \cap V_2$. By the same argument, V_1 omits $N(y) - S$ and V_2 omits $N(x) - S$.
Form H_1 by adding to the subgraph induced by V_1 a vertex y' with edges from S. Form H_2 by adding to the subgraph induced by V_2 a vertex x' with edges to S. Every x, y-path in G starts with an $\kappa_{H_1}(x, y') = k$, and similarly $\kappa_{H_2}(x', y) = k$. Since V_1 omits $N(y) - S$ and V_2 omits $N(x) - S$, both H_1 and H_2 are smaller than G. Hence the induction hypothesis yields $\lambda_{H_1}(x, y') = k = \lambda_{H_2}(x', y)$. Since $V_1 \cap V_2 = S$, deleting y' from the k paths in H_1 and x' from the k paths in H_2 yields the desired x, S-paths and S, y-paths in G that combine to from k pairwise internally disjoint x, y-paths in G.

Case 2: *Every minimum x, y-cut is $N(x)$ or $N(y)$.* Again we construct the k desired paths. In this case, every vertex outsied $\{x\} \cup N(x) \cup N(y) \cup \{y\}$ is in no minimum x, y-cut. If G has such a vertex v, then $\kappa_{G-v}(x, y) = k$, and applying the inductin hypothesis to $G - v$ yields the desired x, y-paths in G. Also if there exists $u \in N(x) \cap N(y)$, then u appears in every x, y-cut, and $\kappa_{G-u}(x, y) = k - 1$. Now applying the induction hypothesis to $G - u$ yields $k - 1$ paths to combine with the path x, u, y.

We may thus assume that $N(x)$ and $N(y)$ partition $V(G) - \{x, y\}$. Let G' be the bipartite graph with bipartition $N(x), N(y)$ and edge set $[N(x), N(y)]$. Every x, y-path in G uses some edge from $N(x)$ to $N(y)$, so the x, y-cuts in G are precisely the vertex covers of G'. Hence $\beta(G') = k$. By the König-Egervàry Theorem, G' has a matching of size k. These k edges yield k pairwise internally disjoint x, y-paths of length 3.

□

In light of Menger's theorem, we define the idea of cut sets. Given a graph $G = (V, E)$ and $S, \bar{S}, W \subseteq V$, a **vertex-cut (edge-cut)** is a set W of vertices (edges) which separate G into sets S and $\bar{S}$. We call this a u, v-cut if $u \in S$ and $v \in \bar{S}$. A common question related to networks and cut sets, is that of finding a minimum u, v-cut. If the graph G is weighted, then the $u.v$-cut is considered a minimum when the $\sum_i w_i$ is minimized, for $w_i \in W$. Note that for an unweighted graph (i.e. edge weights equal one), the minimum u, v-cut is $|W|$.

The theorem of Ford and Fulkerson [7], called the Max-Flow Min-Cut theorem, amounts to maximizing a measure of the 'flow' on the edges of a network to find the minimum u, v-cut.

Theorem 2.11 (Ford, Fulkerson (1962))**.** *In any network, the value of max flow equals the capacity of min cut.*

The ideas of connectivity for undirected graphs carry over easily to the case of digraphs. Recall that a digraph, D, is strongly connected if every vertex in D is reachable by every other vertex in D via a directed path. Thus connectivity can be described in terms of strongly connected subgraphs of D, and cut-sets follow by considering a set W which creates separate strongly connected subgraphs, say S and $\bar{S}$. In directed graphs, these are commonly called the source and the sink, where the flow in the graph originates at the source and terminates at the sink.

In general, directed graphs have the structure that a central strongly connected component (SCC) is flanked by an in component and an out component that are disconnected from each other. Remaining edges can flow from the in component to the out component through vertices not in the SCC, these are called tubes. Tendrils are edges which terminate after leaving the in component or the out component. Disjoint subgraphs may exist which will follow the same structure. The following figure shows the bow tie structure of directed graphs.

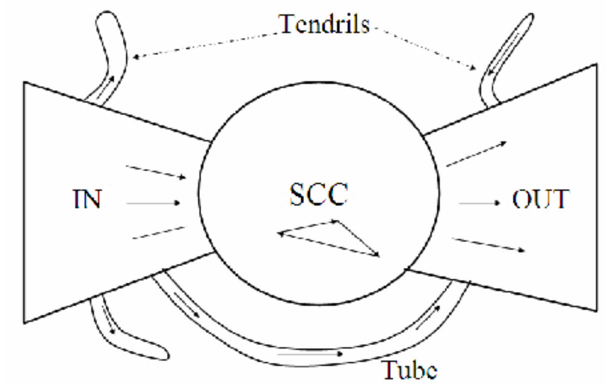

FIGURE 2.1: The bowtie structure of directed graphs.

2.3.3 Graph Partitioning

Since every graph can be described by its constituent sets, it makes sense to describe the partitioning of a graph in the parlance of sets. A **partition** $\mathcal{C} = \{C_1, C_2, \ldots, C_N\}$ of a set S is a decomposition of S into N finite subsets such that $\bigcup_{i=1}^{N} C_i = S$. Unless the graph G to be partitioned is the empty graph, we will require the C_i's to be nonempty, and $C_i \cap C_j$ to be empty. In general, partitions of graphs are sought which maximize the connectedness of the partition sets and minimize the number of edges which are adjacent to each partition set.

Hierarchical clustering, as its name suggests, refers to a partitioning of the vertices in a graph based upon a rule, or dis-similarity measure (sometimes referred to as the cost function). The **dissimilarity measure**, denoted x_{ij}, measures the dissimilarity between $v_i, v_j \in V$, and may be defined in a number of ways depending on the desired outcome. An Hierarchical clustering algorithm is usually implemented by iteratively searching all possible partitions and then either merging partitions or splitting partitions that meet the defined partition rule. The result of the algorithm may be displayed in a tree diagram, called a *dendrogram*, which shows the successive partitions due to the iterative process. There are two common algorithms for partitioning a graph based upon a hierarchical clustering.

Agglomerative: An agglomerative partitioning algorithm is one that merges partitions.

Divisive: A divisive algorithm is one that splits partitions.

The following are common methods for identifying clusters in a graph that may result in a partition.

Spectral partitioning The spectrum of a graph is the set of eigenvalues of either the adjacency matrix or the laplacian of a graph. Spectral partitioning refers to the partitioning of a graph based upon an analysis of the spectrum and the associate eigenvectors of a graph. While this can be done on any graph or partition of a graph, it has been shown that, in general, the more regular a partition is, the more accurate the spectral analysis will be when considering a physical network.

Many spectral measures exist, but the **isoperimetric number** of a graph is particularly relevant to clustering problems. This number is defined as

$$\phi(G) = \min_{S \subset V : |S| \leq |V|/2} \phi(S, \bar{S}),$$

where $\phi(S, \bar{S}) = |E(S, \bar{S})|/|S|$ is called the ratio of the cut defined by $(S, \bar{S})$. Minimizing $\phi(G)$ is an NP-Hard problem, but a bound on $\phi(G)$ is given by

$$\lambda_2/2 \leq \phi(G) \leq \sqrt{\lambda_2(2d_{max} - \lambda_2)},$$

where λ_2 is the second smallest eigenvalue of G's Laplacian, and d_{max} is the maximum vertex degree of G. If $\phi(G)$ is 'small,' meaning near zero, then it is likely that we can create a 'good' bisection of G.

Related to the idea of a bisection of G, a formal result in spectral graph theory states that G will consist of K connected components if and only if $\lambda_1(\mathcal{L}) = \cdots = \lambda_K(\mathcal{L})$ and $\lambda_{K+1}(\mathcal{L}) > 0$, where $\mathcal{L}$ is the Laplacian of G.

Assortativity and Mixing In social network literature, correlation between vertex characteristics is called **assortative mixing**. On a variation of the correlation coefficient in statistics, there is an **assortativity coefficient** which measures the correlation between vertex attributes. Let $G = (V, E)$ be a graph in which each vertex has been labeled to belong to one of $M \leq |V|$ categories. Let f_{ij} be the fraction of edges connecting vertices in category i to vertices in category j. Denote the i^{th} marginal row and column sums by f_{i+} and f_{+i}, respectively. The the assortativity coefficient is defined to be the quantity

$$r_a = \frac{\sum_i f_{ii} - \sum_i f_{i+}f_{+i}}{1 - \sum_i f_{i+}f_{+i}}.$$

This quantity is equal to one when there there is perfect assortative mixing, that is, vertices are only adjacent to other vertices belonging to the same category. Since r_a lies on the interval $(-1, 1]$ we define a minimum r_a^{min} such that when r_a is near r_a^{min}, we have strong dissasortative mixing. This refers to vertices only connecting to other vertices which lie in a different category.

The above Assortativity coefficient applies to categorical data. We may also compute the coefficient for ordinal or continuous data. For example, the Pearson correlation coefficient may be used in the case of continuous data.

Non-Shared Neighbors: Let $G = (V, E)$ be a graph. The define $\mathcal{N}_v$ to be the set of neighbors of v, Δ the symmetric difference of two sets, and d_i the i^{th} smallest element in the degree sequence of G. Then the dis-similarity measure between $v_i, v_j \in V$ is defined to be

$$x_{ij} = \frac{|\mathcal{N}_{v_i} \ \Delta \ \mathcal{N}_{v_j}|}{d_{|V|} + d_{|V|-1}},$$

which is normalized to the interval $[0, 1]$, where 0 and 1 indicate perfect similarity and dis-similarity, receptively.

Euclidean Distance: Let A be the adjacency matrix of a graph. Then the Euclidean distance dissimilarity measure is defined by

$$x_{ij} = \sqrt{\sum_{k \neq i,j} (A_{ik} - A_{jk})^2}.$$

Modularity: Let $\mathcal{C} = \{C_1, \ldots, C_n\}$ be a given partition of a graph G. Define $f_{ij} = f_{ij}(\mathcal{C})$ to be the fraction of edges in G which connect vertices in C_i to vertices in C_j. Then the modularity of $\mathcal{C}$ is the number

$$\text{mod } (\mathcal{C}) = \sum_{k=1}^{K} [f_{kk}(\mathcal{C}) - f_{kk}^*]^2,$$

where f^*_{kk} is the expected value of f_{kk} under some model of random edge assignment. Most commonly, f^*_{kk} is defined to be $f_{k+}f_{+k}$, where f_{k+} and f_{+k} are the k-th row and column sums of $\mathbf{f}$, the $K \times K$ matrix formed by the entries f_{ij}.

2.3.4 Concentration

In another study by Rice et al. [8] a high concentration of drug users in a homeless youth's social network increased the likelihood that the youth would engage in drug use. The same study attempted to find an association between the density and concentration of the individual's network. In this case, there was no statistical significance. This provides an example of the failure of a network measurement to provide an explanation of network behavior.

In addition, this study compared the networks of individuals with high concentrations of drug use to those with low concentrations. As noted, this method was successful in predicting drug use among the participants. Given a set of data describing attributes of an individual social network, this method could be readily applied to predict a variety of social behaviors.

2.3.5 Cliques

Perhaps third in the list of most commonly applied analytic techniques is that of clique analysis. Though clique analysis usually separates a graph so that centrality and density analysis may be performed on the smaller, perhaps more manageable, subgraph, it is instructive to consider cliques in their own right. Before this discussion, however, we note that a rigorous mathematical definition of a clique requires the subgraph to be complete. In the case of SNA, we have no such requirement. We may define "quasi-cliques" with a certain level of connectedness that fits the situation that the graph is modeling. For example, we look for all subgraphs in the supergraph that are one edge away from being complete. In the case of homeless service providers, it may be useful to consider how housing services interact with each other when compared to services that provide food. in this case, we apply the looser definition of a clique by requiring that the services provide food, but not require that they all communicate with each other. Such analysis was done in Fleury et al. [2] when comparing the global properties of density in the overall network with particular cliques embedded within the network.

2.3.6 Paths and Cycles

A k-path in a graph is a series of adjacent vertices of length k, while a k-Cycle in a graph is a path whose first and last vertex is the same [11]. These can be important in determining the path of an actor through a graph over time. For example, one may wish to see how a person experiencing homelessness moves through the homeless services network over a specified time period, say over a year. In this way, patterns may develop that show seasonal shifts in service usage throughout a state.

2.3.7 Dynamic versus Static Networks

Robins [11] describes dynamic systems in terms of the data that is collected to analyze them. Thus longitudinal, or panel, data is associated with dynamic systems where the network is fixed and data is collected over time. Diffusion is a measure associated with this type of analysis. This is the measure of how information or products travel through a network, and which actors are instrumental in the information transfer. For example, one may wish to know how drug use spreads through the homeless network by analyzing the proportion of drug users within the network over a given time period.

Static systems are associated with cross-sectional studies like that of the collaboration between homeless service organizations in Montreal [2]. Here we take a snapshot of a network and study attributes within the network at that particular time. Such studies are useful in describing networks that do not experience sudden changes over time. For example, organizations are far more unlikely to change over relatively short period of time when compared to individuals experiencing homelessness. It should be clear that static systems can easily be studied as dynamic systems by simply extending the data collection over time.

2.3.8 Formal versus Informal Networks

The description of formal versus informal networks seems to be closely related to the idea of organic versus artificial networks. Formal networks are usually those who have a hierarchical system with top down transfer of information, or at the very least they contain a specific set of governing properties which control their actions. These types of systems rarely happen organically, or naturally, of the participants own accord. Thus we may also refer to formal networks as artificial networks. In contrast, informal networks tend to happen organically with members being naturally connected as a result of mutual interests [5].

Kezar [5] has suggested that informal networks tend to be stronger networks, which are more easily able to transfer complex information. On the other hand, formal networks are well suited for the transfer of simple information, or to be consulted for information. This makes identifying the type of network you are analyzing important to the goals of the research being conducted.

2.3.9 Conclusion

Historically, in contrast to traditional statistical analysis, relatively little research has been done using social network theory, though this field of research is growing and expanding rapidly. Further, virtually no research has been done using social network analysis on homeless populations and the organizations that serve them. There is a rich repository of traditional analysis, which if combined with SNA may provide a more complete picture of social systems by addressing the naturally occurring interdependencies that exist. Of particular interest are graph theoretic constructs that have yet to be introduced to the social sciences, or are rarely used due to their mathematical complexity. It appears that in some cases, these may be necessary to solve problems in networks that are just now ripening with data that was heretofore nonexistent.

CHAPTER 3
PULSE PROCESSES

3.1 Pulse Processes

Pulse processes were first introduced by Roberts to analyze energy demand. The generalization of this mathematical structure, however, makes it applicable to a host of different problems involving flows in graphs or networks. In particular, pulse processes allow us to analyze how a flow in a network may result in instability of values at the vertices in the network. By instability, we will mean that values assigned to the vertices of a graph may become unbounded in the pulse process. In terms of applications, If we consider the chocolate factory in which Lucille Ball worked, in the popular series I Love Lucy, we may consider chocolate packers as vertices, and the conveyor belts to be edges in the graph. If you have seen the show, then you know that Lucy is the unbounded vertex. Since each vertex of the chocolate packing process is dependent on a previous vertex, it would be desirable to have no vertex reach a state of instability. Pulses are the means by which stability may be introduced into the system. One may add a countervailing pulse, in the form of Ethel, or another vertex to compensate for the buildup at Lucy's vertex. Here we introduce the generalized mathematical idea of a pulse process, and then we show how these processes may be applied to a network of individuals experiencing homelessness.

Signed digraphs were first used to model pulse processes, since the asymmetric relation is well suited for modeling flow in a network and the signing of edges can describe the positive or negative flow to a vertex. This idea is easily generalized to weighted digraphs, in case more specific systems must be analyzed. We begin in this order, that is, we start with signed digraphs, and then generalize to weighted digraphs, as was done in Roberts [10].

Let D be a digraph with vertex set $v_1, \ldots v_n$. Suppose the i^{th} vertex attains a value at discrete times $t = 0, 1, 2, \ldots$. We find the value $v_i(t+1)$ from $v_i(t)$, an outside pulse $p_i^o(t+1)$ applied to vertex v_i at time $t+1$, and from adjacency information in the digraph. The following rule defines this relationship:

$$v_i(t+1) = v_i(t) + p_i^o(t+1) + \sum_j sgn(v_j, v_i) p_j(t), \tag{3.1}$$

where

$$sgn(v_j, v_i) = \begin{cases} +1 & \text{if } v_j v_i \in E(D) \text{ is } + \\ -1 & \text{if } v_j v_i \in E(D) \text{ is } - \\ 0 & \text{if } v_j v_i \notin E(D) \end{cases}$$

and

$$p_j(t) = \begin{cases} v_j(t) - v_j(t-1) & \text{if } t > 0 \\ p_j^o(0) & \text{if } t = 0. \end{cases}$$

Generalization to real weighted digraphs is accomplished by replacing $sgn(v_j v_i)$ with a real number $w(v_j v_i)$ in (3.1) giving us

$$v_i(t+1) = v_i(t) + p_i^o(t+1) + \sum_j w(v_j, v_i) p_j(t), \tag{3.2}$$

with

$$w(v_j, v_i) = \begin{cases} w & \text{if } v_j v_i \in E(D) \text{ and } w \in \mathbb{R} \\ 0 & \text{if } v_j v_i \notin E(D). \end{cases}$$

We call p_j a **pulse** at vertex j at time t. With these definitions in mind, we define a **pulse process** by equation (3.1), along with two vectors, $\vec{p}^o(t) = (p_1^o(t), p_2^o(t), \ldots, p_n^o(t))$ and $\vec{v}(0) = (v_1(0), v_2(0), \ldots, v_n(0))$. The vector $\vec{p}^o(t)$ is the outside pulse to be applied to each vertex at each time step t, and $v(0)$ is the initial value of each vertex at time $t = 0$. We also define the **pulse vector** $\vec{p}(t) = (p_1(t), p_2(t), \ldots, p_n(t))$ whose i^{th} entry has been defined above as the change in value from time $t-1$ to time t at the i^{th} vertex.

Two "special" cases of pulse processes arise in the discussion of stability in weighted digraphs. A pulse process is **autonomous** when $\vec{p}(0) = 0$ for $t > 0$. A pulse process is **simple** when $\vec{v}(0) = 0$ and $\vec{p}(0)$ has a 1 in the i^{th} entry and a 0 in every other entry. Obviously a simple pulse process is also autonomous.

If we define $p^t(v_j v_i)$ as the pulse from vertex j to vertex i at time t, and $v^t(v_j v_i)$ as the value given to vertex i from vertex j at time t, then the following theorem arises from the fact that

$$
\begin{aligned}
v_i(t+1) &= v_i(t) + \sum_j w(v_j v_i) p_j(t) \\
&= v_i(t) + \sum_j w(v_j v_i)(v_j(t) - v_j(t-1)).
\end{aligned}
$$

Theorem 3.1. *The quantities $p^t(v_j v_i)$ and $v^t(v_j v_i)$ are given by the weighted number of walks from vertex v_i to v_j of length t and length less than or equal to t, respectively.*

We define the adjacency matrix of the weighted digraph by $A = [a_{ij}]$ with entry

$$
a_{ij} = \begin{cases} w(v_i v_j) & \text{if } v_j v_i \in E(D) \\ 0 & \text{if } v_j v_i \notin E(D) \end{cases}
$$

Recall that A^t_{ij} gives the number of walks of length t between vertices i and j, where A^t is the t^{th} power of the matrix A. From this fact and Theorem 3.1 we have the following theorem:

Theorem 3.2. *$p^t(x_i x_j)$ is given by the i, j entry of A^t, while $v^t(x_i x_j)$ is given by the i, j entry of $A + A^2 + \cdots + A^t$.*

Computationally we want to consider the the adjacency matrix of our weighted digraphs to apply pulse processes. As a result we now develop some results in terms of matrix theory and linear operators. In light of the explanation of Theorem 3.2, and the theorem itself, for an autonomous pulse process, we have the following theorem:

Theorem 3.3. *In an autonomous pulse process on a weighted digraph with $\vec{v}(0) = \vec{0}$,*

$$
\vec{p}(t) = \vec{p}(0) A^t
$$

As a corollary to this theorem, we have:

Corollary 3.4. *Under autonomous pulse processes, if $0 \le T \le t$, then*

$$
\vec{p}(t) = \vec{p}(T) A^{t-T}.
$$

As noted in the case of the chocolate factory, we are interested in notions of stability in pulse processes. Given the quantities of vertex value and pulse value it seems natural to define stability in terms of these values. Let B and M be finite real numbers. Then we say that vertex v_j is **pulse stable** if $|p_j(t)| < B$ and **value stable** if $|v_j(t)| < M$. We also note that a graph is pulse or value stable if each entry is pulse of value stable, respectively.

Remark 3.5. Value stability implies pulse stability.

Proof.

$$|p_j(t)| = |v_j(t) - v_j(t-1)| \leq |v_j(t)| + |v_j(t-1)| \leq M + M = 2M$$

□

For an example where pulse stability does not imply value stability, consider a simple pulse process on a directed two cycle with two vertices having $w(v_i v_j) > 1$ and $w(v_j v_i) > 1$. Since this pulse process is simple every pulse is zero, save the first, and each vertex's value will grow without bound. Thus, we have a pulse process with pulse stability, but not value stability.

We would like to know when we can expect pulse stability and value stability. Eigenvalue analysis gives us a few theorems that help begin to form a picture of stability in pulse processes.

Pulse stability in autonomous pulse processes amounts to asking when powers of the adjacency matrix of a digraph converge, since if A^t, as t grows without bound, converges to some bounded matrix L and $\vec{p}(0)$ is a finite valued vector, then $\vec{p}(t) = \vec{p}(0)A^t \leq \vec{p}(0)L$ will be bounded. We will use the following theorem to prove when this may occur.

Theorem 3.6 ([4])**.** *Let A be an $n \times n$ matrix with complex entries. Then $\lim_{m\to\infty} A^m$ exists if and only if both of the following conditions hold.*

1. *Every eigenvalue of A is contained in $S = \{\lambda \in \mathbb{C} : |\lambda| < 1 \text{ or } \lambda = 1\}$.*

2. *If 1 is an eigenvalue of A, then the dimension of the eigenspace corresponding to 1 equals the multiplicity of 1 as an eigenvalue of A.*

Proof. Let $J \in \mathbf{M}_{n\times n}(\mathbb{C})$ be a Jordan block. Then The matrix $N = (J - \lambda I)$ is an upper triangular square matrix with ones off the diagonal. By definition N is **nilpotent**, since by computation we can show that multiplying $N \cdot N$ will move the 1 in the ij^{th} entry to the $i(j+1)^{st}$ entry, and in general for $m \geq n$, $N^m = O$ the zero matrix.

Now suppose $|\lambda| < 1$. Then as $m \to \infty$ each $\lambda_i \to 0$ and by the previous argument the off diagonal entries also approach zero. Thus $\lim_{m\to\infty} A^m = O$.

Now suppose that $\lambda = 1$. Then $\lim_{m\to\infty} A^m = I_n$ since each λ is multiplied by itself m times, and the off diagonals once again approach zero.

Conversely, suppose that $\lambda > 1$. Then $\lim_{m\to\infty} A^m$ will diverge, since each λ will diverge. Again the off diagonal entries approach zero. Also, if 1 is an eigenvalue of A with multiplicity greater than 1, then A will diverge since entry a_{1n} will diverge.

Now let A be any square matrix such that its eigenvalues are members of S. Since A is square, it has a Jordan form. Hence by corollary 3 we have

$$\begin{aligned} A &= Q^{-1}JQ \\ \lim_{m\to\infty} A^m &= \lim_{m\to\infty} Q^{-1}J^mQ \\ &= Q^{-1}\lim_{m\to\infty} J^mQ \\ &= Q^{-1}LQ \end{aligned}$$

In the case that each eigenvalue is distinct, J will be a diagonal matrix. The above arguments may easily be modified. Proof is left to the reader. □

Theorem (3.6) implies that we are looking for adjacency matrices with eigenvalues that are less then one in magnitude.

Theorem 3.7. *Let D be a weighted digraph, M a real valued matrix, and $S = \{\lambda \in \mathbb{C} : |\lambda| < 1 \ \text{ or } \ \lambda = 1\}$. D is pulse stable under all autonomous pulse processes if and only if every eigenvalue of D is at most unity.*

Proof. Let A be the adjacency matrix of a weighted digraph D with eigenvalues λ_i. Suppose first that D is pulse stable under all autonomous pulse processes. Then $\lim_{t\to\infty} A^t < M$. By Theorem (3.6), every eigenvalue of A is contained in the set S.

Now suppose that every eigenvalue of A is contained in the set S. Then, again by Theorem (3.6), D is pulse stable since $\lambda_i \in S$ implies $\lim_{t\to\infty} A^t < M$. $\square$

Remark 3.8. Let D be a weighted digraph. Then D is pulse stable under all simple pulse processes if and only if every eigenvalue of D is at most 1 in magnitude. This is true since all simple pulse processes are autonomous pulse processes.

Our final stability proof gives a characterization of value stability in autonomous pulse processes.

Theorem 3.9. *Let D be a weighted digraph. Then D is value stable under all autonomous pulse processes if and only if D is pulse stable under all autonomous pulse processes and 1 is not an eigenvalue of D.*

Proof. Let A be the adjacency matrix of a digraph D. Since A is square, it has Jordan form $J = QAQ^{-1}$, whose diagonal entries are the eigenvalues, possibly repeated, of A. As in the proof of Theorem (3.6), J is an upper triangular matrix. Since $v_i(t+1) = v_i(t)+$ the i, j entry of $A + A^2 + \cdots + A^t$, we need to show that the Jordan Block corresponding to λ_i converges in $\sum J_i^t$, where J_i is the corresponding Jordan block. By Theorem (3.6), every Jordan block is nilpotent, so for t large, the matrix J_i^t will contain the eigenvalues of A on the diagonal and 0's off the diagonal. This implies that $\lim_{t\to\infty} \sum J_{ij}^t$ converges for $-1 < \lambda_i < 1$. Therefore D will be pulse stable for $-1 < \lambda_i < 1$. Since these conditions also hold for pulse stability, we conclude that D will be value stable if and only if D is pulse stable, and $-1 < \lambda_i < 1$. $\square$

As we will see in the next chapter, real data results in a right stochastic adjacency matrix. The following theorem has some implications with respect to pulse and value stability with this information in mind. Recall that the outdegree of a vertex $v_i \in V(D)$ is given by the sum of the i^{th} column of the adjacency matrix A of D, and the indegree is given by the corresponding column sum. In the following definition, $\rho(A)$ is the vertex of maximum outdegree, and $\nu(A)$ is the vertex of maximum indegree.

Definition 3.10. [4] Let $A \in M_{n\times n}(\mathbb{C})$. For $1 \leq i, j \leq n$, define $\rho_i(A)$ to the be sum of the absolute values of the entries of row i of A, and define $\nu_j(A)$ to be equal to the sum of the absolute values of the entries of column j of A. Thus

$$\rho_i(A) = \sum_{j=1}^{n} |A_{ij}| \text{ for } i = 1, 2, \ldots, n$$

and

$$\nu_j(A) = \sum_{i=1}^{n} |A_{ij}| \text{ for } j = 1, 2, \ldots, n$$

We denote the maximum row (column) sum of A by $\rho(A)(\nu(A))$. That is,

$$\rho(A) = \max\{\rho_i(A) : 1 \leq i \leq n\} \text{ and } \nu(A) = \max\{\nu_j(A) : 1 \leq j \leq n\}.$$

The following proof gives an interesting way to find the maximum value that any eigenvalue of a matrix A might achieve. The proof uses Gerschgorin disks, which are disks centered at the values of the diagonal entries a_{ii} of the matrix A. These disks have radius R_i equal to the sum of the entries along the i^{th} row, excluding the number a_{ii}. The algebraic portion of the proof contains a few nuances which will be explained after the proof to avoid cluttering its elegance.

Theorem 3.11. *[4] Let $A \in M_{nxn}(\mathbb{C})$. Then every eigenvalue of A is contained in a Gerschgorin disk.*

Proof. Let λ be an eigenvalue of A with the corresponding eigenvector

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

Then v satisfies the matrix equation $Av = \lambda v$, which can be written

$$\sum_{j=1}^{n} A_{ij} v_j = \lambda v_i \quad (i = 1, 2, \ldots, n). \tag{1}$$

Suppose that v_k is the coordinate of v having the largest absolute value; note that $v_k \neq 0$ because v is an eigenvector of A.

We show that λ lies in C_k, that is, $|\lambda - A_{kk}| \leq r_k$. For $i = k$, it follows from (1) that

$$
\begin{aligned}
|\lambda v_k - A_{kk}v_k| &= \Big|\sum_{j=1}^{n} A_{kj}v_j - A_{kk}v_k\Big| && (2)\\
&= \Big|\sum_{j\neq k} A_{kj}v_j\Big| && (3)\\
&\leq \sum_{j\neq k} |A_{kj}||v_j| && (4)\\
&\leq \sum_{j\neq k} |A_{kj}||v_k| && (5)\\
&= |v_k| \sum_{j\neq k} |A_{kj}| && (6)\\
&= |v_k| r_k. && (7)
\end{aligned}
$$

Thus

$$|v_k||\lambda - A_{kk}| \leq |v_k| r_k;$$

so

$$|\lambda| \leq r_k$$

because $|v_k| > 0$. □

As promised we now explain some of the assumptions that were made during the algebraic portion of the above proof. In step (3) to (4) we used the triangle inequality. Then from step (4) to (5) we changed the index on v by using the maximality of v_k. Finally, from step (6) to (7) we used the definition of the radius of a Gerschgorin disk to write $\sum_{j\neq k} |A_{kj}|$ as r_k.

Corollary 3.12. *[4] Let λ be any eigenvalue of $A \in M_{nxn}(\mathbb{C})$. Then $|\lambda| \leq \rho(A)$.*

Corollary 3.13. *[4] Let λ be any eigenvalue of $A \in M_{nxn}(\mathbb{C})$. Then $|\lambda| \leq min\{\rho(A), \nu(A)\}$.*

Corollary 3.13, taken with theorem 3.7 and 3.9, will be the most important for our work analyzing stability in the homeless network. We will see that analyzing the homeless network as given in empirical data results naturally in both stable and unstable systems, depending upon which aspect one is considering. Thus, some interpretation will be necessary to come up with a "new" idea of stability in these types of networks. We reserve this discussion for the next chapter, where we will develop the homeless network model.

CHAPTER 4
APPLICATIONS

4.1 Applications

As mentioned in the introduction, there are few applications of SNA to homeless populations. The most notable case being that conducted on the homeless services network in Montreal, Canada, wherein communication patterns between service providers were the primary target of analysis. While this work, and much more that has been conducted using linear and non-linear modeling, is helpful in understanding the homeless network, information describing the homeless population and the inherent interdependencies is desired. This motivates our work using the network modeling techniques outlined in the previous chapters. Below we describe, with example, some of the possibilities of using these techniques.

4.1.1 The Homeless Network

By "The Homeless Network" (THN) we shall loosely mean individuals who have used homeless services and the homeless services themselves, as well as the interaction between these two entities. We define precisely what we mean in the next section. Here we give a qualitative description of the entities which will make up the homeless network. We also describe the empirical data that was procured via the Utah Homelessness Management Information System (UHMIS) Steering Committee.

In march of 2014 we met with the UHMIS Steering Committee and proposed a study of the patterns of action of individuals experiencing homelessness in the state of Utah. By law those who receive funding for providing services to individuals experiencing homelessness must collect data and store the data in a repository called a Homeless Management Information System. The repository need not be centralized, however Utah stands among few states who have had the foresight to collect data from each service in the state, and then store it in one place. This centralized collection of data has only been occurring for the last three years, hence the timing of our request was optimal. As per our request, we obtained deidentified data on approximately $43,000$ individuals over the years 2011-2014, beginning and ending in March of those years. The data includes the unique client id's and the names of services to which each client was admitted. Entrance and exit dates for each service are included, as are individuals characteristics including age, sex, family status, etc. The UHMIS database contains further information which would be useful to later analysis on specific subsets of clients.

The types of available homeless services vary widely. A nonexhaustive list of examples in Utah include emergency shelters such as The Road Home, soup kitchens such as Saint Vincent dePaul's Soup Kitchen, rehab centers such as Catholic Community Services, refugee services such as The Asian Association of America, and services which provide all or some of these services. Individuals experiencing homelessness are just as varied, if not more varied than the list of services. In the course of this research the authors had the opportunity to experience the intake process at The Road Home in Salt Lake City, Utah. This is an urban emergency shelter for those in need of immediate housing assistance. The process for getting admitted to The Road Home is not simple and can be intimidating for someone who is newly homeless. One arrives at the area around a rather large emergency housing complex, with no clear entrance. On our first try, we ended up on the family side of the housing complex. Families and single women are separated from the single males. Upon being directed to the correct intake desk, we walked around the complex to find a large line of individuals winding down a ramp, and then around a small gated playground. On our way past the children playing at the playground we were asked to purchase illegal drugs three times. Then walking past the array of individuals milling about in the unorganized line designated for intake, we entered the shelter where we filled out the intake form and were told to wait in line outside till 10pm for intake to begin. The inside of the shelter was reminiscent of prison, as was the line outside. After waiting till 10pm, we were interviewed and found worthy of a bed for the night. We do not know if this process is common among shelters. We found the process to be inconvenient enough that we would consider sleeping outside rather than going through the intake process. Thus, any analysis of the homeless network should be considered with the perspective that data will not be available on all individuals who experience homelessness, or at the very least, the data may be incomplete. To add to the uncertainty, there is no requirement for an individual to disclose personal information to receive services. Nor is there any incentive for an individual to tell the truth. Due to the inherent instability in the lives of individuals experiencing homelessness, we might expect that one may not recall their birthday, or their last known address. Thus, again, we must be cautious in the conclusions we draw from this type of research. Nevertheless, given the large amounts of previously unavailable or unusable data, it will be useful to describe, in part, the interaction between individuals experiencing homelessness and the services they use.

4.1.2 Population Transition Model

Here we define, precisely, what we mean by the homeless network in the parlance of graph theory.

Definition 4.1. Let G be a weighted multi-digraph (WMD) with loops possible (it will turn out that loops are essential), where each vertex represents a homeless service. Then let a real valued weighted arc from location i to location j, with i possibly equal to j, represent the transition of individuals experiencing homelessness from service i to service j. With vertices and arcs defined in this way, we call G the **homeless network**.

In what follows, the weightings on the edges will represent the proportion of individuals who move from service i to service j, with $i = j$ representing the proportion of individuals who remain at service i. Note that a loop $i = j$ is necessary on each vertex, otherwise we "send" population out of a location without ever accounting for the population that remains. Some loops may have weight 0, where all of the population at a location actually does leave. This set-up leads directly to a simple population model. If we know the population at each shelter x in the network at time 0, then after individuals move according to the proportions on the weighted edges, we will know the population at each location at time 1. Continuing this process we may find the population at any shelter after t discrete time steps.

Computationally this is achieved through an application of linear algebra involving the adjacency matrix of the WMD. Let A be the adjacency matrix for a WMD. Then the ij entry of A contains the weight on the edge from location i to location j. As noted in chapter 3, the ij entry of A^t is the number of weighted walks of length t from vertex i to vertex j. In this application, the ij entry of A^t contains the proportion of people who moved from location i to location j in t discrete time steps, or by paths of length t in the WMD. Thus if we have an initial population vector $\mathcal{P}_0$ containing the initial population of each location at time $t = 0$, then we can perform a multiplication to find the population at each vertex at time $t = 1$. We make two important notes with respect to our set up so far. First, that the leftness or rightness of the multiplication makes a big difference in whether we obtain the quantities we are looking for. Second, as stated before, we will require that the diagonal entry i contains the proportions of individuals who remain at location i. If we consider the weight on the arcs leaving location i, then the diagonal entry will be

$$A_{ii} = 1 - \sum_{\{j:j \neq i\}} A_{ij}.$$

This makes A a right-stochastic matrix.

Theorem 4.2. *Let A be the right-stochastic adjacency matrix for a WMD, with $A_{ii} = 1 - \sum_{\{j:j\neq i\}} A_{ij}$. Given an initial population vector $\mathcal{P}_0$, the population $\mathcal{P}_t$ at vertex i at time t is given by the i^{th} entry in the vector resulting from the left multiplication $\mathcal{P}_0 A^t = \mathcal{P}_t$. We call this a* ***proportion process****.*

Proof. Since the j^{th} column of A gives the weighted arcs entering vertex j in the WMD including the loop from j to j, left multiplication will give $\sum_{i=1}^{n} \mathcal{P}_i A_{ij}$, the sum of the weighted population entering vertex j. □

Right multiplication results in nonsense with respect to our application, since the i^{th} entry of the resultant vector contains the sum of populations from each location scaled by the edges leaving vertex i.

The following three corollaries to Theorem (4.2) give us information about the WMD that we might like to know in terms of the homeless network. In what follows, note that $\mathcal{P}_x^t$, for $x \in \{E, L\}$, refers to a population at time t, not an exponentiation of $\mathcal{P}$. We define E to be the set of individuals entering a locations, and L to be the set of individuals leaving a location.

Corollary 4.3. *The population change at vertex i from time t_1 to t_2 is given by $\mathcal{P}_{t_2} - \mathcal{P}_{t_1}$.*

Corollary 4.4. *If we ignore loops on the WMD, then we may compute the total population entering a location at time t by*

$$\mathcal{P}_E^t = \mathcal{P}_0(A - \vec{d}I)^t,$$

where $\vec{d}$ is a vector containing the diagonal entries of A, that is, $\vec{d_i} = A_{ii} = 1 - \sum_{\{j:j\neq i\}} A_{ij}$.

Corollary 4.5. *Let $\mathcal{P}_i$ be a column vector whose entries are the population at the i^{th} location. If we ignore loops on the WMD, then we may compute the total population leaving a location at time t by*

$$\mathcal{P}_L^t = (A - \vec{d}I)_i^t \cdot \mathcal{P}_i,$$

where $\vec{d}$ is defined as in Corollary (4.4) and $(A - \vec{d}I)_i$ signifies the i^{th} row of A with 0's on the diagonal.

This population model is essentially a modified simple pulse process where the initial value vector $\vec{v}(0) = 0$ and the pulse vector $\vec{p}(0) = \mathcal{P}_0$ instead of having a 1 in entry i and 0's in all other entries. As noted above, the weight, $w(i, j)$, of the ij-entry in A^t is the proportion of individuals who move from location i to location j along paths of length t. In the context of the homeless network, we may use this information to answer questions related to the probability that after t time steps, an individual enters service j after having service i being their first admittance into the homeless network. Now j may equal i, in this case an individual may complete a cycle or circuit in the graph. In the case of loops, the individual completes a 1-cycle every time step. Since the basic structure of our adjacency matrix has not changed from that of a pulse process, all results revolving around pulse stability apply to our model. *The question is, what does pulse stability mean with respect to the homeless network?* We will answer this question momentarily.

Now we consider the idea of a simple pulse process on a WMD where the WMD has been defined to represent the homeless network. Recall that a simple pulse process is one in which $v_i(0) = 0$, and $\vec{p}(0) = [0, \cdots, 1, \cdots, 0]$ (i.e. a 1 in the i^{th} entry, and 0 else) and $\vec{p}(t) = \vec{0}$ for $t > 0$. Thus, for the homeless network, we have $v_i(0) = 0$, and $\mathcal{P}_0 = [0, \cdots, 1, \cdots, 0]$ and $\mathcal{P}_t = \vec{0}$ for $t > 0$. From this point forward we will call a simple pulse process applied to the homeless network, a **simple proportion process**, so that we avoid confusion, and because this name better represents the process we will implement. We also define an **autonomous proportion process** with the same properties as and autonomous pulse process, that is $\mathcal{P}_t = \vec{0}$ for $t > 0$. With the interpretation of the weighted entries we have given A, we gain two results for a simple proportion process which follow directly from Theorem (4.2).

Corollary 4.6. *Consider a simple proportion process applied to an adjacency matrix A representing the homeless network. Left and right multiplication by the vector $\mathcal{P}_0$ result in the following two resultant vectors:*

Left Multiplication: *In a simple proportion process the left multiplication $\mathcal{P}_0 A^t$ results in a vector $\vec{o}_i$. The j^{th} entry of this vector contains the probability that an individual transitions from i to j in t time steps.*

Right Multiplication: *In a simple proportion process, the right multiplication $A^t \mathcal{P}_0$ results in a vector $\vec{a}_i$. The j^{th} entry of this vector contains the probability that an individual transitions to i from location j in t time steps.*

We now reinterpret Theorem (3.2) in terms of the homeless network. Recall that for an adjacency matrix A, the ij-entry of $A + A^2 + \cdots + A^t$ gives us the number of walks from vertex i to vertex j of length less than or equal to t. Using the addition and multiplication property, the ij-entry of $A+A^2+\cdots+A^t$ contains the probability that an individual transitions from location i to location j in less than or equal to t time steps. Thus the i^{th}-entry in the resultant vector from the left multiplication $\mathcal{P}_0(A+A^2+\cdots+A^t)$ gives the population who moved from location i to location j in less than or equal to t time steps, with $i = j$ possible.

Theorem 4.7. *Let A be the adjacency matrix representing the homeless network. Then the population accumulation at each location in at most t time steps is given by*

$$\mathcal{P}_{\leq t} = \mathcal{P}_0(A + A^2 + \cdots + A^t).$$

This may further be interpreted as the probability that an individual is admitted to service j from service i in less than or equal to t discrete time steps. We will call this a ***cumulative proportion process****.*

Proof. The proof follows directly from Theorem (4.2) since $\mathcal{P}_0$ distributes across $(A + A^2 + \cdots + A^t)$. □

This theorem might be interpreted as the total traffic admitted to a location in the homeless network, including those who remain at each location. If we remove the diagonal entries in A, then we get the total traffic admitted to a location from every other location in the network for at most t time steps.

By rewriting Corollary (4.6) in terms of simple cumulative proportion processes, we can pluck out a vector whose entries contain the probabilities of leaving or entering a location via another in less than or equal to t times steps. Note that the modifier *simple* still refers to $\mathcal{P}_0$ being a vector of only one 1, and every other entry a 0. We now make these ideas precise.

Corollary 4.8. *Consider a simple cumulative proportion process applied to the series of adjacency matrices $\sum_{i=1}^{t} A^i$. Left and right multiplication by the vector $\mathcal{P}_0$ gives the following two resultant vectors:*

Left Multiplication: *In a simple cumulative proportion process the left multiplication $\mathcal{P}_0(A + A^2 + \cdots + A^t)$ results in a vector $\vec{o}_{i_{tot}}$. The j^{th} entry of this vector contains the probability that an individual transitions from i to j in at most t time steps.*

Right Multiplication: *In a simple cumulative proportion process, the right multiplication $(A + A^2 + \cdots + A^t)\mathcal{P}_0$ results in a vector $\vec{a}_{i_{tot}}$. The j^{th} entry of this vector contains the probability that an individual transitions to i from location j in at most t time steps.*

Finally, before considering some examples, we reinterpret stability for the homeless network. Since a proportion process is an accounting of the movements of individuals in the homeless network, we should expect that stability in such a process refers to stability in the populations in each location. This is true, with an exception for cumulative proportion processes, in the sense that stability specifically measures the likelihood that the population at a particular location is unlikely to increase without bound if the system is "stable." We will call this **proportion stability**, and make the idea precise in Theorem (4.9). Note that an autonomous proportion process is only a renaming of an autonomous pulse process, hence all properties are equivalent between the two. Recall that $\nu(A)$ is the maximum column sum in a matrix A, and $\rho(A)$ is the maximum row sum.

Theorem 4.9. *Let A be the adjacency matrix representing the homeless network. Then, by Corollary (3.13), the homeless network is proportion stable if and only if $\nu(A) < 1$.*

Proof. By Corollary (3.13) the eigenvalues of A are bound by $\min\{\nu(A), \rho(A)\}$. By Theorem (3.7) A is proportion stable under all autonomous proportion processes if and only if every eigenvalue of A is at most one. Since $\rho(A) = 1$ by the right stochasticity of A we need only guarantee that $\nu(A) \leq 1$. Therefore the homeless network is proportion stable if and only if $\nu(A) < 1$. □

We also note the exception for cumulative proportion processes and give the following reasons for the exception:

Remark 4.10. A cumulative proportion process is not guaranteed to be stable, since by our construction, each adjacency matrix A is right stochastic. Hence, by Corollary (3.13), $\sum_{i=1}^{t} A^i$ may fail the condition in Theorem (3.9), that 1 may not be an eigenvalue of A.

Theorem (4.9) gives an easy way to check if the homeless network is proportion stable. All one must do is check that there exists no column sum greater than or equal to one. This concludes our development of the homeless network.

4.1.3 Examples and Data Structures

Let us consider a small example, following which we will give an example using real data. Suppose we were interested in 4 homeless services in particular, The Road Home (TRH), Saint Ann's (SAS), Youth Crisis Center (YCC), and Volunteers of America (VOA). The first three provide emergency housing and the last provides certain services. In our example, we will let 0 represent that an individual is no longer using homeless services, or has left the services network. Obviously, a person may not be using homeless services or a shelter and still remain homeless. Thus, 0 may represent much more than simply no longer using services. We may draw a graph representing the interaction between these services, but in general data from the services is given and a graph is constructed from the data. The table below shows a typical example of the type of data from which we construct a graph. In real life, data from the Utah Homeless Network over three years contains a list approximately $90,000$ client entries long. Of those, about $40,000$ are unique clients, so the example table below is small indeed.

In order to build a graph which represents a homeless network from this data, we need a sequence of at least two locations for each client. As a result, we invented what we call the location sequence matrix (LSM). We have designed an algorithm to convert the data as given into this structure. Below is an example where entry $(2, L2) = YCC$

clientid	locationid
1	TRH
1	TRH
1	VOA
2	VOA
2	YCC
3	TRH
4	YCC
4	STA
4	STA
4	VOA
4	TRH
5	TRH
5	TRH
5	VOA
5	YCC

FIGURE 4.1: This is an example of data representing the relationship between clients and services. The data has already been cleaned and sorted in chronological order so that entries from client one begin at their first service experience and end at their last. Similarly for client 2, 3, and so on. A raw data set would contain all relevant dates and every individuals basic descriptive information (e.g. age, veteran status, family status, etc.)

means that YCC is the second location that client 2 visited. Note that this is also the last service that client 2 used, since every entry after YCC in their location sequence is a 0. In general the number of rows in an LSM is equal to the number of unique objects who have a location sequence. Then the number of columns is equal to the longest location sequence in a set of all location sequences.

Definition 4.11. Let $\mathcal{I}$ be a set of objects. Then the location sequence, l_i, for object i, or client i in the case of the homeless network, is the sequence of locations, in chronological order, with chronological overlap possible, to which the object has been admitted. The length of any location sequence is $max|l_j|$ for any $j \in \mathcal{I}$. Thus each location sequence has the same length, with 0 filling the remaining coordinates after the last location to which object i was admitted. We also note that though the locations in an LSM are in chronological order spanning the rows, there may be overlap. That is, $L1$ may occur at the same time as $L2$, but admittance to $L2$ may not occur prior to admittance to L1.

Definition 4.12. A **location sequence matrix** (LSM) is an $|\mathcal{I}| \times |l_i|$ matrix such that row i contains object i's location sequence.

clientid	L1	L2	L3	L4	L5
1	TRH	TRH	VOA	0	0
2	VOA	YCC	0	0	0
3	TRH	0	0	0	0
4	YCC	STA	STA	VOA	TRH
5	TRH	TRH	VOA	YCC	0

FIGURE 4.2: An example of a location sequence data frame. Note that the 0's represent the fact that an individual did not "check in" to a homeless service. Thus, client 1's last known location was VOA. Depending on the last know location, we might presume that the client is no longer homeless. For example, it has been conjectured that most clients who visit AAU only use the service to find permanent housing, then upon finding permanent housing, they do not experience homelessness in Utah again.

Consider columns $L1$ and $L2$ in figure 4.2. From a graph theoretic perspective, we may consider these to be ordered pairs that form an edge list for a graph G, where entry $(2, L1)$ is adjacent to entry $(2, L2)$ forming a directed edge $(L1_2, L2_2) \in E(G)$ between vertices $L1, L2 \in V(G)$. Consider row 1 and row 5 in figure 4.2. These reflect client 1 and client 5's location sequence. Note that $(L1_1, L2_1) = (L1_5, L2_5)$. This will correspond to an integer weighting of the edges in the graph G. In the case of the graph representing an $L1, L2$ transition, the weighting on the loop from TRH to TRH would have a weighting of 2. These clients also share the same $L2, L3$ transition. Thus the edge from TRH to VOA will be weighted with a 2 as well. Now we wish to build a graph representing the various transition. In general, we prefer to have real number weightings between 0 and 1, but we will first form the adjacency matrix of the graphs that will represent our data.

$$\begin{array}{c|ccccc} & TRH & VOA & YCC & STA & 0 \\ \hline TRH & 2 & 0 & 0 & 0 & 1 \\ VOA & 0 & 0 & 1 & 0 & 0 \\ YCC & 0 & 0 & 0 & 1 & 0 \\ STA & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \qquad \begin{array}{c|ccccc} & TRH & VOA & YCC & STA & 0 \\ \hline TRH & 2 & 0 & 0 & 0 & 1 \\ VOA & 0 & 0 & 1 & 0 & 0 \\ YCC & 0 & 0 & 0 & 1 & 0 \\ STA & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

FIGURE 4.3: Left we have an example of an adjacency matrix representing the transition from location 1 to location 2 in the LSM in figure 4.2 and right we have a modified version so that the weighted adjacency matrix will meet the criteria of theorem 4.2.

This adjacency matrix is rather sparse, given that we did not start with much data.

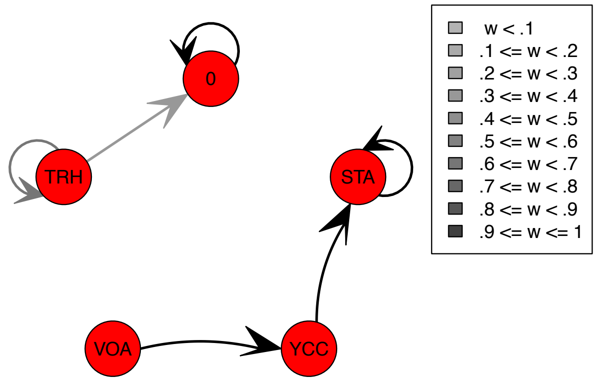

FIGURE 4.4: A graph representing the location 1 to location 2 transition as given in the LSM in figure 4.2, with the modifications of appropriate loops as noted in figure 4.3. Note that the ranges in the legend give the weightings on each edge represented by the corresponding gray scale value.

Thus we expect the graph to be sparse as well, and we see that it is, in figure 4.4. To obtain the graph in figure 4.4, we used the real number weighted adjacency matrix. This is obtained by summing the row entries and then dividing each entry along the corresponding row by its row sum.

Definition 4.13. A weighted location transition matrix, which we will call a T-matrix, is the matrix whose ij entry is

$$T_{ij} = \frac{W_{ij}}{\sum_j W_{ij}},$$

where W is the integer weighted adjacency matrix resulting from a set of data. The result of this operation makes a T-matrix right stochastic.

$$\begin{array}{c} \\ TRH \\ VOA \\ YCC \\ STA \\ 0 \end{array} \begin{array}{ccccc} TRH & VOA & YCC & STA & 0 \end{array} \qquad \begin{array}{c} \\ TRH \\ VOA \\ YCC \\ STA \\ 0 \end{array} \begin{array}{ccccc} TRH & VOA & YCC & STA & 0 \end{array}$$

$$\begin{array}{c} TRH \\ VOA \\ YCC \\ STA \\ 0 \end{array} \begin{pmatrix} \frac{2}{3} & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{array}{c} TRH \\ VOA \\ YCC \\ STA \\ 0 \end{array} \begin{pmatrix} .1333 & .0333 & .1667 & .3333 & .3333 \\ .0816 & .2041 & .3061 & .1020 & .3061 \\ .0933 & .4666 & .0800 & .0933 & .2666 \\ .5618 & .1124 & .0730 & .1404 & .1124 \\ .0000 & .0000 & .0000 & .0000 & 1.0000 \end{pmatrix}$$

FIGURE 4.5: The matrix on the left is the T-matrix for the graph in figure 4.4, while the matrix on the right is the T-matrix for the graph in figure 4.6. Notice that the sum of any row is equal to one making these right-stochastic matrices.

Figure 4.5 shows the T-matrix for the graph in figure 4.4, as well as the T-matrix for the graph in figure 4.6.

The graph in figure 4.6 contains edges and loops to and from every vertex in the graph except 0. In the applications discussed here, this will be the rule since the data only contains individuals who used a service. Any record prior to admittance to the first service use is unavailable, therefore it makes sense not to consider a transition from non-service use to service use. We might just as easily have given every individual a zero in their $L1$ column signifying their transition from non-service use to service use.

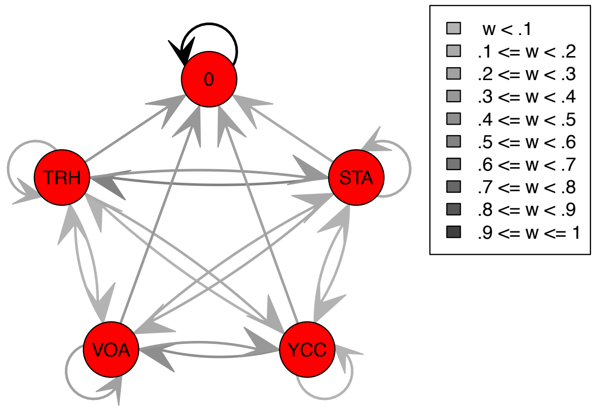

FIGURE 4.6: A graph of the adjacency matrix on the right in figure 4.5 containing a transition from every possible location. Note that since there is not much variation in the weightings, the gray scale scheme shows that there is no vertex with a probability of "transitioning" more objects than any other vertex in the graph (i.e. there is not much variation in the edge weightings. Eigen-analysis will let us formally state this result.

Before we jump into an eigen-analysis of the graphs, we want to show some of the possible analyses available via the LSM, as well as some analysis from the 3 year Utah Homeless Management Information System (UHMIS) data set. In a question answered for the state of Utah, we used the LSM to show the number of individuals in the Utah prison population who experienced homelessness after termination from prison, or during probation. This analysis could be extended to show rates of recidivism in the homeless population, that is individuals transitioning from homelessness to prison then back to homelessness. This may be represented by a two-cycle in a graph, but analysis would need to be done via the adjacency matrix in this case. Instead we consider the LSM. A person experiences recidivism, or a two-cycle, if the following conditions occur for the location subsequence, $L(i-1)_k, Li_k, L(i+1)_k$, in client k's location sequence:

$$L(i-1)_k = L(i+1)_k \neq Li_k.$$

A simple loop in the programming language of your choice will count the number of occurrences in an LSM given this condition.

By using an LSM, we might also consider the number of unique paths in a network. Given more specific information about individuals in the homeless network, unique paths may be analyzed to give some sort of measure of how successful an individual is in exiting homelessness, or an idea of what types of services an individual experiencing chronic homelessness uses. For example, one might wonder why a specific subset of individuals have longer location sequences than others. Why is it that a subset of individuals have only one non-zero entry in their location sequences? In at least one case this question has been answered by considering the LSM corresponding to the three years of homeless network data provided by UHMIS. The Asian Association of Utah (AAU) is a homeless service specializing in assisting refugees. These individuals tend to visit the center only once, and then never access any other homeless services in the state of Utah. It has been suggested that housing options are immediately offered to these refugees, and that these individuals have different motivations with respect to remaining homeless. Beyond the scope of this paper, is an analysis into the factors that contribute to such a short homeless experience.

Figure 4.8 shows a graph of the Utah homeless network using the three year data set. The edges of the graph have been colored based upon the scheme in figure 4.7.

Blue	0	$<$	w	$\leq$	10
Green	10	$<$	w	$\leq$	100
Yellow	100	$<$	w	$\leq$	1000
Red	1000	$<$	w	$\leq$	5000

FIGURE 4.7: The color scheme for the graph in figure 4.7.

There are only 49 vertices in this graph, but there are approximately 200 edges, given that for $u, v \in V(G)$, the edges uv and vu appear as one edge with arrows on both ends. The combination of these two numbers make the graph's appearance like a "bird's nest," thus we seek other analysis using the data structures that gave us this graph, as in the above case of the LSM. Nevertheless, some information may be gleaned from staring at the graph for a time. We see that the most in-degree central vertices in the graph are TRH, 0, VOA, and CCS, with some others we do not name also having a relatively high in-degree. We leave the discussion of this analysis to the more precise discussion using the in-degree and out-degree tables.

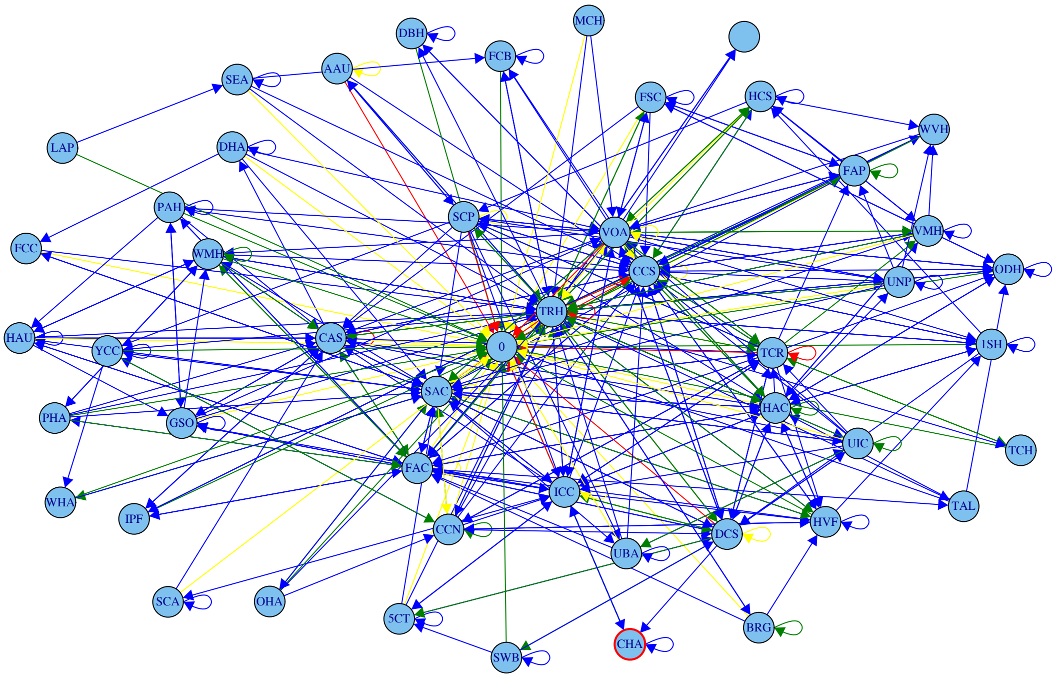

FIGURE 4.8: A graph of the service 1 to service 2 transition for all clients in the UHMIS 3 year data set. Note the large number of arrows pointing to 0, TRH, CCS, VOA, and SAC. A degree centrality analysis will reveal what is so obvious in the picture, that these services are the most degree central in the homeless network.

A global analysis may also be helpful if these types of analyses are applied to other homeless networks, thus giving us a comparison to measure. For example, we propose a simple ratio of unique paths as a global measure in a network. Let U = the number of unique location sequences in a T matrix, and P_{tot} = the total number of location sequences in a T matrix. Then the path uniqueness measure is

$$P_u = \frac{U}{P_{tot}},$$

where P_u is a number on the interval $[0,1]$, with P_u close to zero implying a network with little variation in path types and P_u close to 1 implying a network of almost all unique paths. For example, The Utah homeless network had $43,662$ client location sequences, $3,297$ of which were unique. Hence, the Utah homeless network has $P_u = .0755$, or about 8% of the paths through the homeless network were unique. This seems to be a relatively small number, but without another homeless network to compare it to, we are unsure. Without the comparison, we might ask questions such as, does P_u imply a spread out network with high mobility among homeless individuals? Does it imply a high variation in service needs among homeless individuals? These and other questions should be answered.

As mentioned above, we might also be interested in a vertex degree analysis so that we might know which locations receive the most traffic. Below are four charts, two showing the in-degree and two showing the out-degree in the Utah homeless network. Again, the three year data set was used to compute these numbers. Figures 4.9 and 4.10 represent the in-degree and out-degree of the transition from location 1 to location 2 graph, and do not account for weightings on the edges. Since the actual amount of traffic traveling to or from a location will correspond to the weighted edges, we also show the charts that include the degrees based upon the edge weightings in figures 4.11 and 4.12. Some interesting results from these figures are that, as one might expect, locations who send lots of individuals to other locations are also connected to many locations, similarly for those receiving a large number of individuals. Further, we immediately see that, since this is the location 1 to location 2 transition, a vast majority of individuals who experienced homelessness over the three year period were only admitted to one service. This seems to correlate with the fact that 10% of individuals use 90% of the services. Thus, identifying patterns of the chronically homeless is critical to our understanding of the homeless network. We also see that TRH is the highest service in both in-degree and out-degree. This makes sense since TRH, or The Road Home, is the largest and most urban shelter in the state of Utah. CCS, SAC, and CAS are also notable in the graph, and we see that this carries over in the degree tables. What is not so obvious from the graph, are the out-degree levels for each vertex. Not surprisingly, however, it turns out that the services in the network that were the most out-degree central, also happen to be the most in-degree central. It is suggested that, again, this is due to the urban location of these services.

We now consider an eigen-analysis of the adjacency matrices of the graphs we have shown so far. We want to know which graphs will be stable under an autonomous proportion process. Thus we wish to know if there are any eigenvalues of the adjacency matrices for the graphs whose magnitude is greater than 1. We used R to compute the eigenvalues of the matrices, and then find the modulus for each value to account for complex values. We note that R normalizes the eigenvectors. Using R, we found the eigenvalues of the adjacency matrices, which we label A_1, A_2, and A_3 respectively, as follows: (A_1) The three year data set, (A_2) the adjacency matrix for the graph in figure 4.4, and (A_3) the adjacency matrix for the graph in figure 4.6 all had maximum eigenvalue 1. From this information we can conclude that each of A_1, A_2, and A_3 are proportion stable under all autonomous proportion processes.

We could have begun by applying Theorem (4.9) to check the column sums directly. Recall that all column sums less than one imply proportion stability. Tables 4.1 and 4.2 show the maximum column sums and eigenvalues for each of the adjacency matrices. Note that in all three cases analysis of the column sums are inconclusive. With three examples, only one of which was computed with real data, we cannot make any conclusions about whether we would expect this to happen a majority of the time.

The above analysis of stability in the graphs allows us to conclude that the population will not grow without bound at any service. This analysis also leaves us with many questions. For example, we would like to know if the natural population limitations of each service may be affecting the data on which this analysis was conducted. This seems probable, thus further analysis might be conducted using data enumerating the number of clients a particular service must turn away. A theoretical analysis might also be conducted testing the effect of removing particular services based upon qualitative assumptions of alternative preferences of individuals experiencing homelessness. Obviously data on other homeless networks are desired for comparison.

Table 4.1: A table of the maximum column sums from each of the adjacency matrices corresponding to the graphs presented in this chapter. Matrix A_1, A_2 and A_3 correspond, respectively, to the UHMIS 3 year data set, the graph in figure 4.4, and the graph in figure 4.6. Note that each maximum column sum is greater than 1. Thus, we may not make any conclusion about stability in the graphs from this analysis.

Adjacency Matrix	Maximum Column Sum
A_1	36.48935
A_2	2
A_3	2.018482

Table 4.2: A table of the maximum eigenvalues from each of the adjacency matrices corresponding to the graphs presented in this chapter. Matrix A_1, A_2 and A_3 correspond, respectively, to the UHMIS 3 year data set, the graph in figure 4.4, and the graph in figure 4.6. Note that each eigenvalue is 1. Thus, we conclude that each of the graphs are stable under all autonomous proportion processes.

Adjacency Matrix	Maximum Eigenvalue
A_1	1
A_2	1
A_3	1

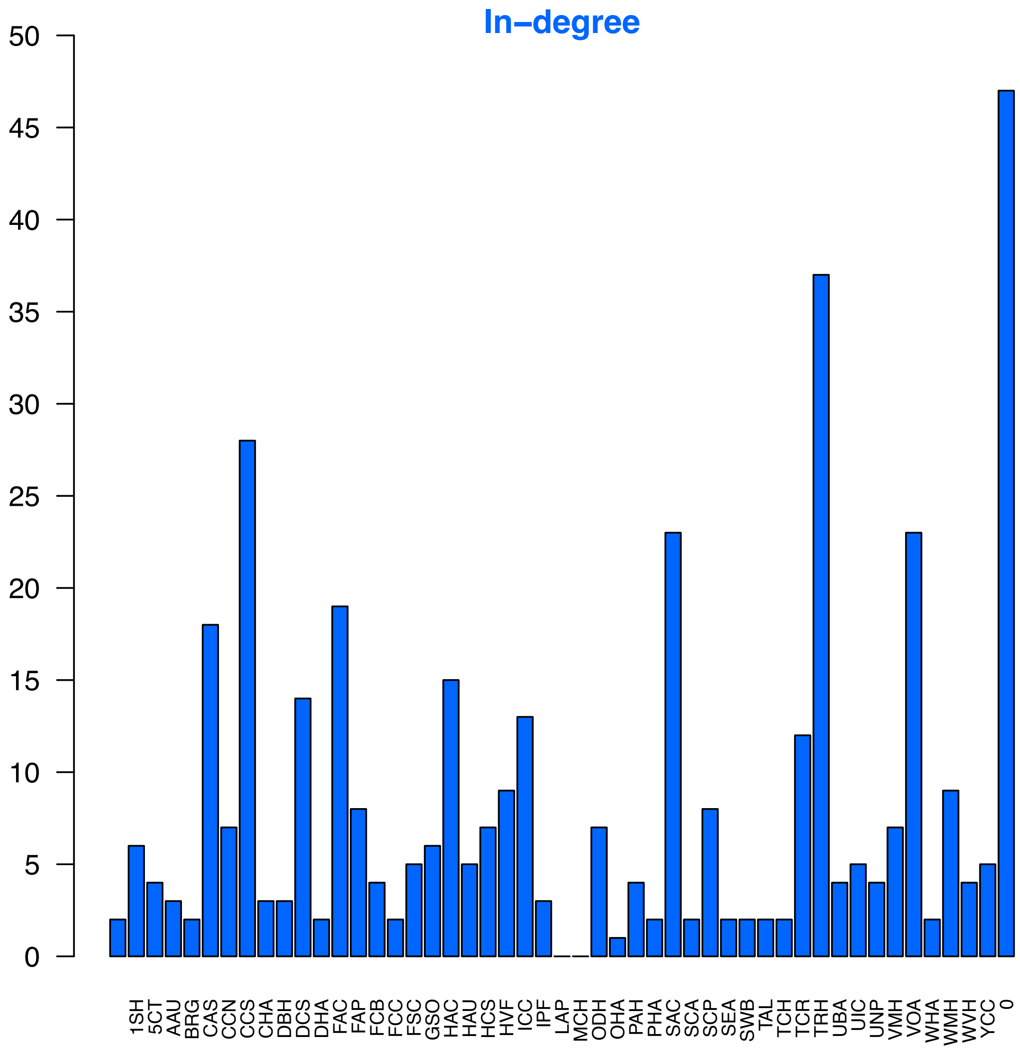

FIGURE 4.9: The in-degree for the Utah homeless network over three years. This graph does not account for edge weightings.

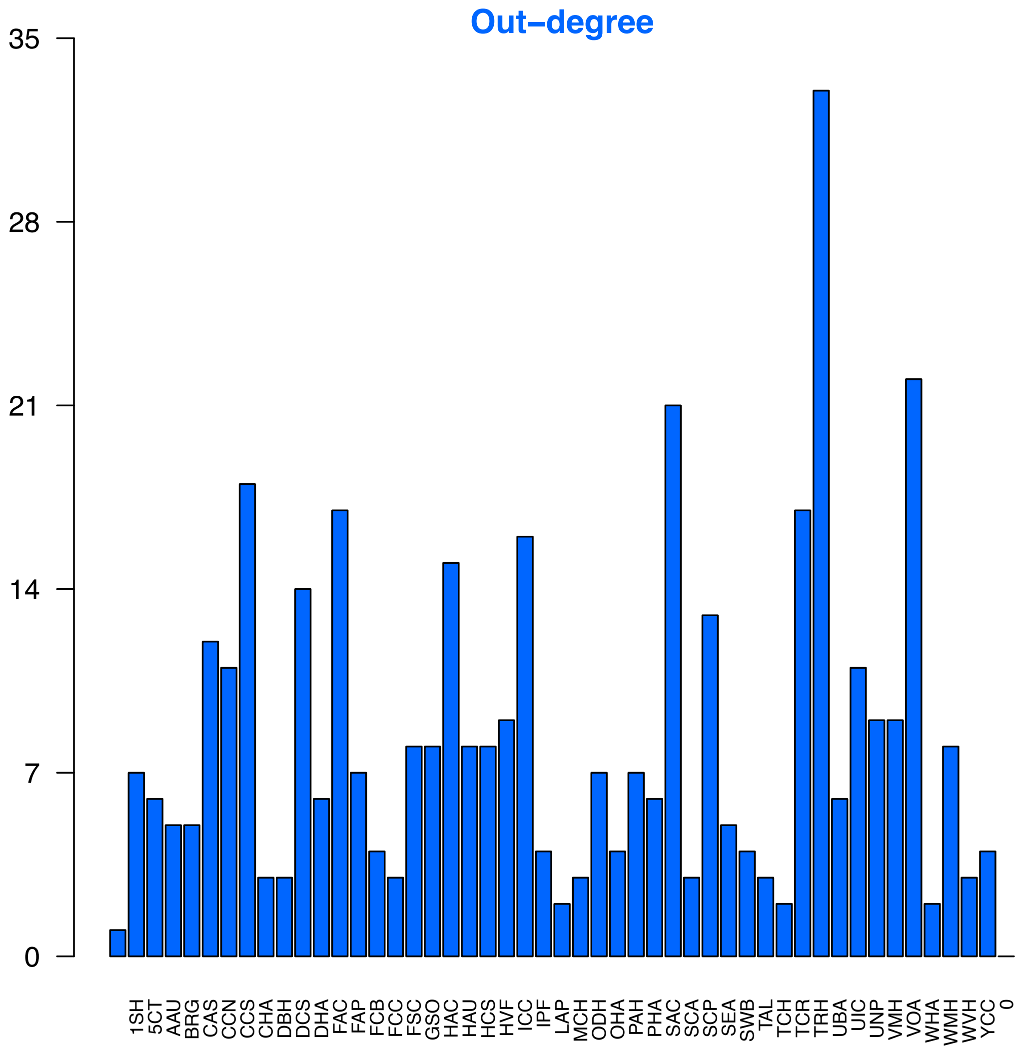

Figure 4.10: The out-degree for the Utah homeless network over three years. This graph does not account for edge weightings.

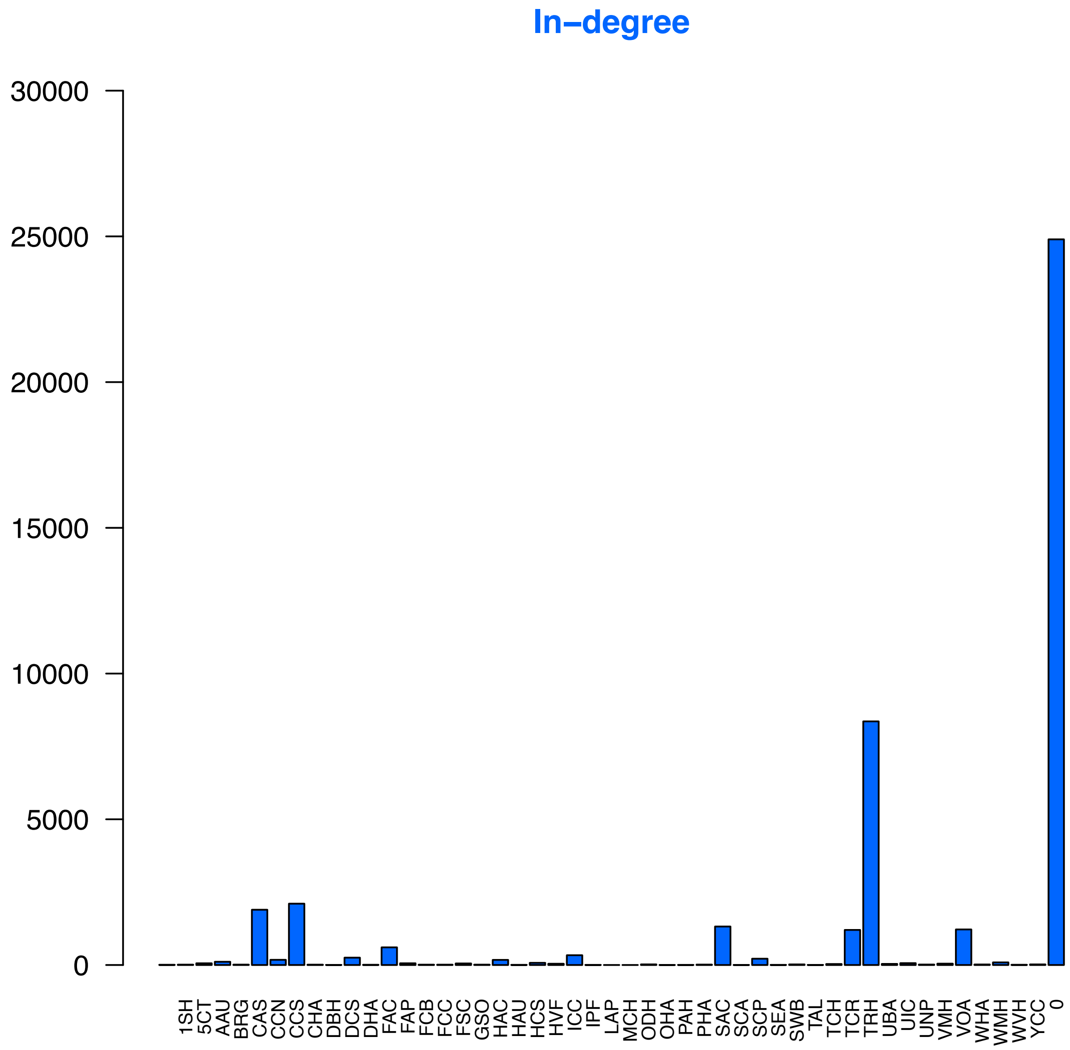

FIGURE 4.11: The in-degree for the Utah homeless network over three years. This graph does account for edge weightings.

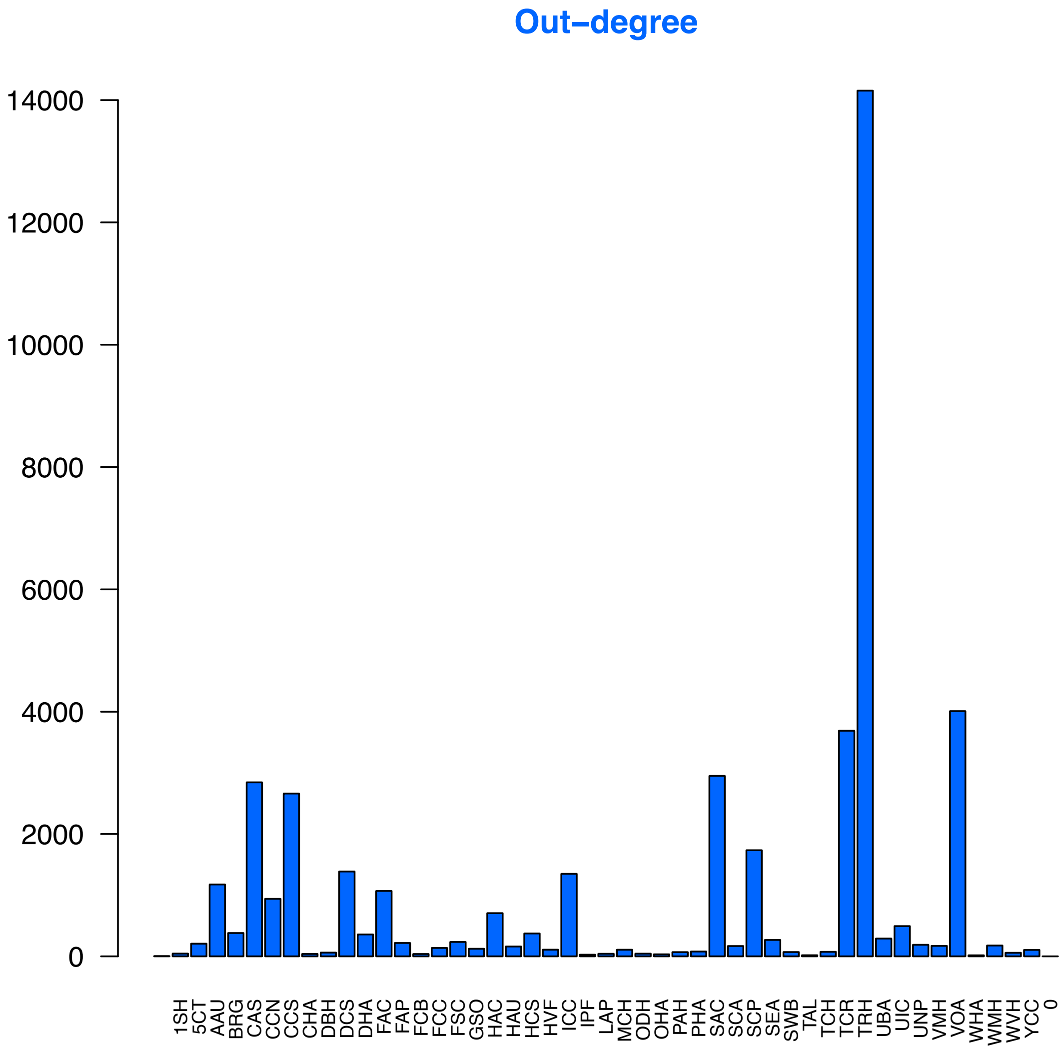

FIGURE 4.12: The out-degree for the Utah homeless network over three years. This graph does account for edge weightings.

BIBLIOGRAPHY

[1] P. Bonacich. Technique for analyzing overlapping memberships. *Sociological Methodology*, 4:176–185, 1972.

[2] M.J Fleury, G. Grenier, A. Lesage, N. Ma, and A.N. Nguii. Network collaboration of organisations for homeless individuals in the montreal region. *International Journal of Integrated Care*, 14, February 2014.

[3] L.C. Freeman. A set of measures of centrality based on betweeness. *Sociometry*, 40:35–41, 1977.

[4] S.H. Friedberg, A.J. Insel, and L.E. Spence. *Linear Algebra.* Pearson, 4th edition, 2002.

[5] A. Kezar. Higher education change and social networks: a review of research. *The Journal of Higher Education*, 85(1):91–125, January/February 2014.

[6] E.D. Kolaczyk. *Statistical Analysis of Network Data: Methods and Models.* Springer, 2009.

[7] D.R. Fulkerson L.R. Ford. *Flows in Networks.* Princeton University Press, 1962.

[8] E. Rice, N. G. Milburn, M.J. Rotheram-Borus, S. Mallett, and D. Rosenthal. The effects of peer group network properties on drug use among homeless youth. *American Behavioral Scientist*, 48(8):1102–1123, April 2005.

[9] E. Rice, E. Tulbert, J. Cederbaum, A. B. Adhikari, and N. G. Milburn. Mobilizing homeless youth for hiv pervention: a social network analysis of the acceptability of a face-to-face and online social networking intervention. *Health Education Research*, 27:226–236, January 2012.

[10] F.S. Roberts. *Discrete Mathematical Models: with applications to social biological, and environmental problems.* Prentice Hall, Inc, 1976.

[11] G. Robins. A tutorial on methods for the modeling and analysis of social network data. *Journal of Mathematical Psychology*, 57:261–274, March 2013.

[12] G. Sabidussi. The centrality index of a graph. *Psychometrika*, 31:581–603, 1966.

[13] J. Scott. Social network analysis. *Sociology*, 22(1):109–127, February 1988.

[14] J. Spencer T.A. Brown, F.S. Roberts. Pulse processes on signed digraphs: A tool for analyzing energy demand. *Rand Report*, 1972.

[15] D.B. West. *Introduction to Graph Theory*. Pearson, 2nd edition, September 2000.

[16] H. Whitney. Congruent graphs and the connectivity of graphs. *American Journal of Mathematics*, 54(1):150–168, 1932.

APPENDIX