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Hamiltonicity, Pancyclicity, and Cycle Extendability in Graphs

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HAMILTONICITY, PANCYCLICITY, AND

CYCLE EXTENDABILITY IN GRAPHS

by

Deborah C. Arangno

A dissertation submitted in partial fulfillment

of the requirements for the degree

of

DOCTOR OF PHILOSOPHY

in

Mathematical Sciences

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UTAH STATE UNIVERSITY

Logan, Utah

2014

ABSTRACT

Hamiltonicity, Pancyclicity, and

Cycle Extendability in Graphs

by

Deborah C. Arangno, Doctor of Philosophy

Utah State University, 2014

Major Professor: Dr. David E. Brown
Department: Mathematics and Statistics

The study of cycles, particularly Hamiltonian cycles, is very important in many applications.

Bondy posited his famous metaconjecture, that every condition sufficient for Hamiltonicity actually guarantees a graph is pancyclic. Pancyclicity is a stronger structural property than Hamiltonicity.

An even stronger structural property is for a graph to be cycle extendable. Hendry conjectured that any graph which is Hamiltonian and chordal is cycle extendable.

In this dissertation, cycle extendability is investigated and generalized. It is proved that chordal 2-connected $K_{1,3}$-free graphs are cycle extendable. *S-cycle extendability* was defined by Beasley and Brown, where S is any set of positive integers. A conjecture is presented that Hamiltonian chordal graphs are $\{1, 2\}$-cycle extendable.

Dirac's Theorem is an classic result establishing a minimum degree condition for a graph to be Hamiltonian. Ore's condition is another early result giving a sufficient condition for Hamiltonicity. In this dissertation, generalizations of Dirac's and Ore's Theorems are presented.

The Chvàtal-Erdős condition is a result showing that if the maximum size of an independent set in a graph G is less than or equal to the minimum number of vertices whose deletion increases the number of components of G, then G is Hamiltonian. It is

proved here that the Chvàtal-Erdős condition guarantees that a graph is cycle extendable. It is also shown that a graph having a Hamiltonian elimination ordering is cycle extendable.

The existence of Hamiltonian cycles which avoid sets of edges of a certain size and certain subgraphs is a new topic recently investigated by Harlan, et al., which clearly has applications to scheduling and communication networks among other things. The theory is extended here to bipartite graphs. Specifically, the conditions for the existence of a Hamiltonian cycle that avoids edges, or some subgraph of a certain size, are determined for the bipartite case.

Briefly, this dissertation contributes to the state of the art of Hamiltonian cycles, cycle extendability and edge and graph avoiding Hamiltonian cycles, which is an important area of graph theory.

(97 pages)

PUBLIC ABSTRACT

Hamiltonicity, Pancyclicity, and

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A significant portion of Graph Theory is devoted to determining the characteristics which guarantee the existence of long cycles.

Long cycles have roles in applications to civil engineering, chemistry, and communications, among many others, but the problem, in and of itself, of determining whether a graph has a cycle of some fixed and typically large length is one of the most important problems of both pure Mathematics and Computer Science.

A cycle containing all the vertices of the graph is called a *Hamiltonian cycle*, and a graph which possesses such a cycle is said to be *Hamiltonian*. If a graph contains cycles of every length, from three to the number of vertices of the graph it is said to be *pancyclic*. J.A. Bondy famously posited what is referred to as *Bondy's metaconjecture*: *Every condition which guarantees a graph is Hamiltonian actually guarantees it is pancyclic.* If the vertices of a cycle of length ℓ are contained in a cycle of length $\ell + 1$, the cycle is said to be *extendable*. If every non-Hamiltonian cycle in a graph is extendable, the graph is said to be *cycle-extendable*. That a graph is cycle extendable is a stronger structural property than being pancyclic which is in turn stronger than being Hamiltonian. Nevertheless George Hendry pioneered the study of cycle extendability by proving the following statement for many types of graphs possessing conditions sufficient for Hamiltonicity: *A non-Hamiltonian cycle in a graph with a property that guarantees*

it is Hamiltonian also guarantees it is cycle extendable.

A graph H is an *induced subgraph* of another graph G if H can be obtained from G by deleting vertices. A graph is said to be *chordal* if it has no cycle on four or more vertices as an induced subgraph; that is, every cycle long enough to have a chord (an edge connecting two nonconsecutive vertices on a cycle), has a chord. Hendry conjectured that any graph which is Hamiltonian and chordal is cycle extendable.

In this thesis the following questions are addressed: What are sufficient conditions for cycle extendability? And what progress can we make in resolving Hendry's Conjecture, in particular? Included among other things, results relating to Hamiltonicity, pancyclicity, and cycle extendability are developed and proved.

It is proved that a graph satisfying the Chvàtal-Erdős condition, $\kappa(G) \geq \alpha(G)$, is cycle extendable. It is proved that 2-connected claw-free chordal graphs are cycle extendable, and a forbidden subgraph pair determining cycle extendability in chordal graphs is provided. It is also proved that a graph having a Hamiltonian elimination ordering is cycle extendable.

A new minimum degree condition for Hamiltonicity is presented, and generalizations of Dirac's condition and Ore's condition are given.

Furthermore, bipartite graphs are investigated, (bipartite graphs being graphs that contain no odd cycles), and the questions are considered: what conditions guarantee there is a Hamiltonian cycle which avoids certain subsets of edges (referred to as "edge-avoiding" Hamiltonicity in a bipartite graph)? and under what conditions, given a bipartite graph G, and any graph F, can we determine whether G contains a Hamiltonian cycle that avoids some subgraph H that is isomorphic to F (such a graph G will be referred to as F-avoiding Hamiltonian bipartite)?

This dissertation presents new conditions and properties that guarantee desired cycle structure in graphs of different kinds, and ultimately contributes to a deeper understanding of the properties of graphs that affect Hamiltonicity and cycle extendability, which is important in many applications in the real world.

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Firstly, I must offer my profoundest thanks to Louis John and Marie Theresa Arangno for the monumental sacrifices and support they have given to me throughout my life. I thank them above all, for their Herculean examples as role models; indeed, I could not have come so far, nor would my labors have been so fruitful, had I not had the pattern of their own noble lives imprinted upon my own.

Secondly, it must be said that if one is lucky then perhaps once in a lifetime one might have the good fortune to meet with a wise individual of courage and integrity who stands up for justice and scholarship no matter what the challenge. My mentor, Richard Lundgren, is such a man; and I solemnly thank him for guiding me through the Straits of Charybdis and Scylla to this distinguished seat of learning where I could flourish and thrive. I owe to him an eternal debt of gratitude.

To my graduate advisor and major professor, David E. Brown, I owe a lifetime of thankfulness and fidelity. Dave Brown is a man who reflects in his own life what the great Antoíne de Saint-Exupery expressed so beautifully:

"*If you wish to build a ship, don't enlist people to collect wood, and don't assign them tasks and jobs, but rather teach them to yearn for the endless immensity of the sea.*"

Dave Brown's brilliance of scholarship, intellectual fearlessness, and profound leadership qualities, inspire in his students, graduates and undergraduates, a commitment to – and deep fascination with – the discipline of mathematics. With everyone he works beside, he shares an excitement for the open seas of mathematical research; and I thank him for letting me embark on this wonderful voyage. Dave not only had a profound influence on the direction of my study, but he knew how to challenge and inspire me. I am indebted to him for his constant patience and encouragement.

I am exceedingly grateful to Professors LeRoy Beasley, Jim Powell, and Luis Gordillo, whose provocative questions and comments at my proposal defense helped to shape my writing, and hone my arguments. I thank you LeRoy, Jim, and Luis for providing me with such generous encouragement and wonderful points of analysis during the completion of this awesome task. Dr. Beasley provided me with a thorough and

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Deborah C. Arangno

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CHAPTER 1

INTRODUCTION AND BACKGROUND

A graph is a very generic mathematical construct, representing relationships between objects, or networks of any sort – neural, social, electrical, etc.

The study of graph theory can be traced back to 1735 when Leonard Euler resolved the famous Königsberg bridge problem, which was whether someone, leaving their home, could traverse the seven bridges spanning the Pregel River, exactly once and return home. Euler interpreted the problem abstractly, with the bridges represented by edges of a graph, and formally stated the condition for which the resulting graph had a closed path which contained every edge. The condition – that all vertices of a given connected graph must have even degree – is both necessary and sufficient, and a graph having such a closed path is known as an *Eulerian graph*. To complete a cycle that uses every edge of a graph, each time you visit a vertex by one edge, you must exit by another edge, hence the necessity that each vertex be incident to an even number of edges is readily apparent. But this condition is an example of a biconditional result in graph theory in which it is usually the case, that while necessity of a condition may be obvious, its sufficiency is more difficult to prove. In proving this particular result, Euler introduced novel methods, unique to the discipline of graph theory, and laid a cornerstone for a truly new branch of mathematics. His reasoning that vertices have even degree was sufficient for an Eulerian circuit to exist, was as follows: first, he proved that a connected graph of even degree must contain a cycle (by way of contradiction, using a *longest path* argument). Next, he deleted a longest such cycle. What remains must be a connected graph of even degree. He repeated this cycle deletion, until all the edges are exhausted. Now, the graph is the union of edge-disjoint cycles, and an

Eulerian tour can be traversed by splicing together the cycles at common vertices.

In so framing the Königsberg bridge problem in mathematical terms, Euler became the Father of Graph Theory.

The field of study grew rapidly, and research in the field quickly branched into countless subfields. Cycles in particular continue to be an area of enormous interest and importance.

In 1857, William Rowan Hamilton introduced the icosian game, the object of which was to trace along the edges of a dodecahedron, passing through each vertex, and returning to the starting point. Since then, much research has been dedicated to finding sufficient conditions for the existence of a such a cycle which spans the vertex set of a graph, known as a Hamiltonian cycle. Algorithmically, this problem is known to be NP-complete. Unlike Eulerian graphs, there is no condition which is both necessary and sufficient for a graph to be Hamiltonian, other than the existence of a spanning cycle. For a graph to be Hamiltonian, it is necessary that the graph be 2-connected (which is, it has no cut vertex, hence every pair of vertices lies on a cycle), but 2-connected is not sufficient. It is also necessary that the graph be 1-tough, which means that the cardinality of the largest cut set $S \subseteq V(G)$ is greater than or equal to the number of components in the disconnected graph when S is removed. However that is not sufficient for a Hamiltonian cycle to exist, either. Graph theorists have devoted much time to the search for sufficient conditions for Hamiltonicity.

The study of cycles in graphs is a very important specialty in graph theory, not only historically in the establishment of Graph Theory as a field of Mathematics, but research in this area continues to be applied in diverse disciplines. Cycles are used in genomics and in chemistry such as in analyzing the benzene valence isomers of benzenoid rings. Cycle theory is used in studying polycyclic aromatic hydrocarbons and noncyclic aromatic systems (Trinajstik [89]). The theory of cycles is applied to bioinformatics (Jones and Pevzner [65]), DNA fragment assembly (Kaptcianos [66]), and in genome sequencing (Craven [33]).

Other applications include networks, scheduling, circuit design, software testing, optimizing transit routes and other routing problems, and job sequencing of a single

asset to multiple tasks, etc. Consider for example having to drill holes in an electrical circuit board, which requires scheduling a drilling machine to drill holes one by one. To minimize the time it takes to drill all holes, we should minimize the distance the machine (or the board) needs to move when re-positioning from hole to hole. We can model this problem as a graph with holes at the vertices, and the weight on each edge representing the geometric distance between the holes on the board. Optimal scheduling corresponds to a minimal weighted Hamiltonian cycle (see vanSteen [90]). Many problems of this type fall under the class of *Traveling Salesman Problem* (abbreviated TSP), which optimizes a route that must visit all the drop-off points on a map (see Applegate et al. [6]). The TSP problem is well-studied, and has generated a great deal of interest in the *computational complexity* of mathematical problems. In 1972, in a pioneering paper on computational complexity, Karp [67] identified the problem of finding whether a graph has a Hamiltonian cycle as one of the most fundamental NP-complete problems. On the one hand, whereas identifying whether a graph has an Eulerian cycle can be done easily by an inspection of the vertex degrees, there is no such nice characterization of a Hamiltonian graph. Algorithmically, it is an unmanageable problem.

An entire branch of mathematical study has emerged as *Hamiltonian Graph Theory.* Since Hamilton, mathematicians world-wide have labored to make progress in trying to establish sufficient conditions for the existence of cycles of special kinds – such as a spanning cycle, or the existence of cycles of all lengths. A summary of results in the area of Hamiltonian cycles was compiled in a 1991 survey by R. Gould [50], which he continuously updates [51].

One of the earliest theorems providing sufficient conditions for the existence of a Hamiltonian cycle is due to Dirac. Dirac showed that if the minimum degree of a graph G is at least $n/2$, where n is the the number of vertices in $V(G)$, referred to as the *order* of the graph, then G has a cycle that contains all the vertices – a Hamiltonian cycle. But the most pivotal result followed Dirac, and was due to Ore, who proved that for a graph to have a Hamiltonian cycle, it wasn't necessary for all the vertices to be adjacent to at least half the vertices of the graph: it was sufficient that the minimum degree sum of any pair of non-adjacent vertices be at least n (the order of G). See Theorem 1.18.

As we will see, Dirac's minimum degree condition follows as a corollary to Ore's Theorem, and indeed many other important consequences follow, as well. Ore's Theorem turns out to be significant in many ways, as we shall observe.

However, Ore's Theorem, and by consequence, Dirac's Condition, are far stronger conditions than necessary for the existence of a Hamiltonian cycle. As an example, consider a graph $G \cong C_n$, which is a cycle on n vertices, where n is large. Since G is itself a cycle, it is *ipso facto* Hamiltonian. Yet the degree of every vertex of G is 2, hence G neither satisfies Ore's Theorem nor Dirac's Condition. In fact, we will see that Ore's Theorem (and Dirac's Condition) guarantee a great deal more besides Hamiltonicity. As we will discuss later in this chapter, Bondy proved that Ore's Theorem was sufficient for a graph to be *pancyclic*, that is, to have cycles of every length from the 3-cycle to the spanning cycle (with certain exceptions). And Hendry proved that those same conditions are sufficient, with certain exceptions, for a graph to be cycle extendable (that is, any non-Hamiltonian cycle can be extended by a vertex).

So, graph theorists continue to search for sufficient conditions for Hamiltonicity that are not so strong nor so restrictive. The results of research in the area of cycles, subsequent to Ore, fall into different categories: those having to do with minimum degree or minimum edge constraints (for example Dirac [35], Bondy [18]), or with degree sum conditions (for example Ore [79], Flandrin et al. [41]). There are results identifying forbidden subgraphs (for example, Bedrossian [13], Broersma et al. [23], Duff [36], Gould et al. [52]), neighborhood union conditions (Broersma et al. [22]), conditions on toughness (Jackson et al. [59]), degree sequence type conditions (Chvàtal [30]), and the relation of connectedness relative to independence number – the so-called Chvátal-Erdős type theorems (Chvàtal [32]).

As a final point arguing that the study of cycles is a field unto itself, Nikoghosyan [77] has written a meta-mathematical survey of the development of theorems in the study of cycle structure in graphs, classifying the methods used in achieving results in this field.

Soon it was discovered that results in one area of the study of cycles, often lead

to discoveries in related areas. In proving Ore's condition for Hamiltonicity was sufficient for pancyclicity, Bondy [18] famously conjectured that any sufficient condition for a graph to have a Hamiltonian cycle must be sufficient for the graph to have cycles of all lengths. This suggestion, that almost any sufficient condition that implies Hamiltonicity also probably implies pancyclicity, is referred to as *Bondy's Meta-conjecture*. This meta-conjecture continues to challenge mathematicians, who devote years to proving whether known conditions for Hamiltonicity yield pancyclicity.

For example, Schmeichel and Hakimi [85] proved if there are two consecutive vertices on a Hamiltonian cycle with degree sum at least n, the order of the graph, then the graph is pancyclic, (unless it is bipartite or missing an $(n-1)$-cycle). As Amar points out in [3], Bauer and Schmeichel [10] then proved that a number of well-known conditions for Hamiltonicity – Dirac's minimum degree condition, Chvátal's degree sequence condition, and another condition attributed to Geng-Hua Fan, included – all imply Schmeichel and Hakimi's theorem, establishing pancyclicity except in special cases.

Bondy [19] had proven that if a graph satisfies Ore's Condition for Hamiltonicity, and C is a non-Hamiltonian cycle, then there exists a cycle containing the vertices of C and having length $|V(C)|+i$, where $i = 1$ or 2.

The classic method of proving that a graph is Hamiltonian is to assume that a longest cycle does not span the vertex set of the graph, that is, G has a longest cycle C of length $k<n$, then obtain a contradiction by finding a longer cycle C' (contradicting the maximality of the cycle C). This idea of taking a cycle, and finding a longer one which extends the given cycle, motivated another major topic of research, that of *cycle extendability*, first formally studied by George Hendry [57] in his PhD dissertation, in which he proved that Ore's sufficient condition for Hamiltonicity was also sufficient for cycle-extendability.

On the other hand, starting with a Hamiltonian cycle, if a cycle of every length can be obtained by reducing a larger cycle, we get a pancyclic ordering, and such a graph is called cycle reducible.

A *chordal* graph is a graph in which any induced cycle of length greater than three has a chord. Chordal graphs are also referred to as *triangulated*. It is easy to verify that

a graph is cycle reducible if and only if it is chordal. If a chordal graph is Hamiltonian, it is pancyclic. We will investigate this class of graphs further in Chapter 2.

The similarities between cycle extendability and cycle reducibility in graphs motivates us to look for similarities in their characterizations, and sufficient conditions. In fact, Hendry also conjectured that chordal Hamiltonian graphs are cycle extendable. That is, cycle-reducibility can be reversed, if the graph is chordal and Hamiltonian. For the last 20 years, graph theorists have been intrigued by *Hendry's conjecture*, and have made progress in identifying subclasses of chordal graphs which have the property of cycle extendability. The special kinds of chordal graphs which are now known to be cycle extendable include Hamiltonian interval graphs (Abuedia and Sritharan [1], and Chen et al. [27]), Hamiltonian planar graphs (Jiang [64]), and Hamiltonian split graphs (where a graph *splits* if its vertex set can be partitioned onto a complete graph and an independent set) [1]. And since Bender [15] proved that almost all chordal graphs split, this suggests that most Hamiltonian chordal graphs are indeed cycle extendable. However, recently LaFond and Seamone [71] have identified counterexamples to show that Hendry's conjecture is not true for a small subclass of chordal graphs.

This dissertation investigates these questions and presents new results on Hamiltonicity and cycle extendability among other things. Specifically, in Chapters 2–3, we provide answers to the important open questions: *what conditions are sufficient for the existence of a Hamiltonian cycle*, and *for which graphs is Hendry's conjecture true? What are sufficient conditions for cycle extendability?* and *What progress can we make in resolving Hendry's Conjecture, in particular?*

Among other things, we present the following results:

- A graph satisfying the Chvàtal-Erdős condition, that $\kappa(G) \geq \alpha(G)$, is cycle extendable;

- A graph having a *Hamiltonian elimination ordering* is cycle extendable;

- A *2-connected $K_{1,3}$-free chordal graph* is cycle-extendable.

In a 2-connected graph, when particular pairs of induced subgraphs are forbidden, the graph is Hamiltonian. Among those *forbidden subgraph pairs* is $\{K_{1,3}, P_6\}$. Analogous to this, we establish the existence of a forbidden subgraph pair, F_1, F_2, for cycle extendability in chordal graphs.

Although LaFond and Seamone [71] have identified counterexamples to show that Hendry's conjecture is not true, we present the conjecture that Hamiltonian chordal graphs are *{1,2}-cycle-extendable.*

Additionally, we present new minimum degree conditions for Hamiltonicity, and generalizations of Dirac's condition and Ore's condition. These results expand on the current theory to shed light on the cycle structure of graphs beyond what we now understand.

In a related topic, Kronk [70], extended results due to Posá [80], giving conditions on a graph G, such that if H is any collection of non-trivial paths in G having k edges, then there is a Hamiltonian cycle in G containing all the edges of H. This led to a new kind of problem in the study of cycle structure in graphs.

In contrast, Harlan [Harris] et al. [54] studied the conditions necessary for a Hamiltonian cycle to exist in a graph which *avoids* specified edges – referred to as edge-avoiding behavior of a graph. In this dissertation we will expand upon that subject area by providing special conditions for edge-avoiding behavior in bipartite graphs (any graph that contains no odd cycle, which is a cycle on an odd number of vertices) in Chapter 4. In that chapter, we determine conditions that give us an edge-avoiding Hamiltonian cycle in a bipartite graph, and what guarantees there is a Hamiltonian cycle which avoids certain subsets of edges. We also present conditions under which, given a a bipartite graph G, and any graph F, we can determine whether G contains a Hamiltonian cycle that avoids some subgraph H that is isomorphic to F. We will refer to such a graph G as *F-avoiding* Hamiltonian bipartite.

This chapter provides an overview of what is already known in the field of cycles, to give insight into where there are gaps in that knowledge. In the chapters that follow are results of research that fill those gaps, contributing to a better understanding of the essential problems, and expanding our knowledge in the area of cycle structure in

graphs.

Finally, Hamiltonicity and bipancyclicity in bipartite graphs is an area with much research potential. Very few graph theorists have explored the question of cycle extendability in the bipartite case – the exception including Beasley and Brown [12]. We will discuss this more in the last chapter, and consider the direction future research might take.

1.1 Notation and Terminology

In this dissertation, Bondy and Murty, *Graph Theory with Applications* [21] will be used for notation and terminology not specifically defined, and a Glossary has been furnished at the end for convenience. Only finite simple graphs are considered, with vertex set and edge set denoted $V(G)$ and $E(G)$ respectively. The degree of a vertex x will be indicated by $d(x)$. Whereas $\delta(G)$ and $\Delta(G)$ will denote the minimum and maximum degree of the graph G, respectively.

The shortest distance between vertices x and y is represented by $dist(x, y)$. If x and y lie on a cycle C, $dist_C(x, y)$ indicates the shortest distance between the vertices, traversing the cycle.

For a given vertex x of G, the neighborhood of x will be represented by $N(x)$, the set of all vertices adjacent to x. We will use $N[x]$ to represent the *closed* neighborhood of x, $N(x) \cup \{x\}$. Let $N_k[x] = \{v \in V(G) \mid d(x, v) \leq k\}$. If $H \subseteq G$, then $N_H(x)$ represents the neighbors of x in H. $G[S]$ will denote the subgraph of G induced by a set of vertices $S \subseteq V(G)$. Hence, $G[N[x]] = G[N(x) \cup \{x\}]$. We can write this subgraph as $G_1(x)$.

Definition 1.1.

- A graph G is *connected* if there exists an x, y-path between every pair of vertices, x and y.
- A *component* of a graph G is a maximal connected subgraph.
- A *cut set* is a minimal set of vertices whose deletion increases the number of components of the graph.

- A *cut vertex* is a vertex whose deletion increases the number of components of the graph.

- A graph G is *k-connected* if it requires at least k vertices to increase the number of components of G. If a graph is k-connected, the smallest cut set has cardinality k.

- A graph G is 2-connected if it has no cut set of size 1 (i.e., no *cut vertex*). This is equivalent to saying that every vertex lies on a cycle.

Definition 1.2. We will say that a vertex x is *locally connected* if the subgraph induced by its neighborhood, $G[N(x)]$, is connected, and the graph G is locally connected if every vertex of G is locally connected.

Definition 1.3. A graph G on n vertices is the *complete graph, K_n*, if all its vertices are pairwise adjacent.

Definition 1.4. A set of vertices is *independent* if no two of its elements are adjacent.

Definition 1.5. A graph is *bipartite*, if the vertex set $V(G)$ can be partitioned into two independent sets, X and Y, with all the edges of the graph, $E(G)$, having one end vertex in the partite set X and the other in the partite set Y. We denote the bipartite graph $G = (X, Y, E)$. Equivalently, G contains no odd cycles. Moreover, $G = (X, Y, E)$ is a *balanced* bipartite graph if $|X| = |Y|$.

Definition 1.6. We denote the *complete bipartite, $K_{n,m}$*, where $|X| = n$, and $|Y| = m$, and every $x \in X$ is adjacent to every $y \in Y$.

Definition 1.7. A graph is a *claw* if it is isomorphic to the complete bipartite graph $K_{1,3}$.

Definition 1.8. A graph is *Hamiltonian* if it has a spanning cycle, that is, a cycle containing all the vertices of the graph.

Definition 1.9. A graph on n vertices is *pancyclic* if it contains cycles of every length t, $3 \le t \le n$. A bipartite graph on $2n$ vertices is *bipancyclic* if it contains cycles of every even length $2t$, $2 \le t \le n$.

Definition 1.10. A cycle C in a graph is *extendable* if there exists a cycle C' such that $V(C) \subseteq V(C')$ and $|V(C')| = |V(C)| + 1$. A graph G on n vertices is said to be *cycle-extendable* if every non-Hamiltonian cycle C on $k < n$ vertices, is extendable. A graph is *fully cycle-extendable* if it is cycle extendable and every vertex lies on a 3-cycle.

Definition 1.11. A cycle C has a *chord* if there is an edge between two non-consecutive vertices on C.

Definition 1.12. A graph is *chordal* if every induced cycle of length greater than three has a chord.

Definition 1.13. A vertex v is *simplicial* if its neighborhood, $N(v)$, induces a complete subgraph in G, that is, a clique (which means, all its vertices are pairwise adjacent). A known fact is that a graph is chordal if and only if for every induced subgraph H, either H is itself a clique, or it contains two non-adjacent simplicial vertices [48].

Definition 1.14. A *perfect elimination* ordering, or a *simplicial* elimination ordering, on a graph, denoted PEO, is an ordering of the vertices $R = v_1 v_2 \dots v_n$ such that for each vertex v_i the neighbors of v_i that follow v_i in the ordering induce a clique.

For example, the chordal graph in Fig. 1.1 has a perfect elimination ordering $(v_1, v_2, v_3, v_5, v_4, v_6)$. *Note that the order is important.*
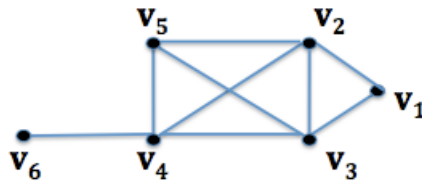

FIGURE 1.1: Perfect Elimination Ordering in a Chordal Graph

This gives us a characterization of chordal graphs.

Theorem 1.15. *The following are equivalent:*

1. *G is chordal*

2. (Dirac [35]) For every vertex pair, $x, y \in V(G)$, if S is a minimal x, y- separating set, then S induces a complete subgraph.

3. (Buneman [25]) G has a perfect elimination ordering.

Proof. $(1 \Rightarrow 2)$ If G is chordal, then any induced subgraph is chordal (a hereditary property, Theorem 1.52), then any cycle in G has a chord, which is preserved in any induced subgraph containing the cycle. So it is sufficient to show that any minimal x, y-separating set S induces a clique, for any $x, y \in V(G)$.   Let S be a minimal separating set, and let A_x, A_y be the two components of $G - S$ containing x and y, respectively. For any vertices $u, v \in S$, then since S is minimal, both u and v have edges to both components A_x and A_y. Let P_x and P_y be the shortest u, v-paths in A_x and A_y, respectively Clearly, $|P_x|, |P_x| \geq 2$, hence their union is a cycle of length ≥ 4. But G is chordal, hence this cycle has a chord. But P_x and P_y are the shortest possible u, v-paths, hence uv must be an edge in S. Since this is true for all vertex pairs, $u, v \in S$, S must be complete.

$(2 \Rightarrow 3)$ By induction on $|G|$, suppose for all $x, y \in V(G)$, a minimal separating set S is complete, show that G has a perfect elimination ordering. For $n = 1$ this is trivial. Suppose the claim holds for all graphs on fewer than n vertices. Let G have n vertices. If G is a clique, we're done, since any ordering of $V(G)$ is a perfect elimination ordering. Otherwise, there exists a vertex pair x, y such that $x, y \notin E(G)$. Then let S be a minimal x, y-separating set, and define A_x and A_y to be the components of $G - S$ containing x and y respectively. By the induction hypothesis, there exists a perfect elimination ordering in A_x. In particular, there exists a vertex $u \in A_x$ such that $N[u]$ is a clique in the induced subgraph, A_x. Since the component has no edges to the rest of the graph, and $V(S)$ forms a clique, this implies $N[u]$ forms a clique in G.   W.l.o.g., we can set $u = v_n$, and delete v_n from G, leaving us with the graph $G' = G - \{v\}$ on $n - 1$ vertices. Apply induction to get a perfect elimination ordering on the vertices of G'. Combine this ordering with v_n to get a perfect elimination ordering on G.

$(3 \Rightarrow 1)$ Suppose G has a perfect elimination ordering an let C be any cycle of length greater than or equal to 4. We start deleting vertices from the ordering until we get to a vertex $v \in V(C)$. When we delete v from the ordering, its neighbors remaining in the ordering induce a clique, so there exists an edge between two vertices on C, hence there exists a chord. $\square$

Furthermore, Gavril [46] proved that every chordal graph has a clique-tree decomposition. That is, a chordal graph can be decomposed into a clique-sum of complete graphs, as is illustrated in Fig.1.2, as an example of a graph on 8 vertices:
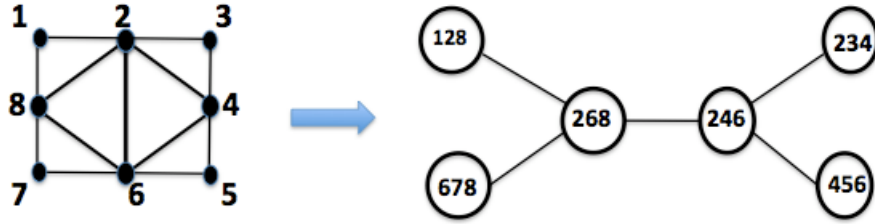
FIGURE 1.2: Clique-Tree Decomposition of a Chordal Graph

In this chapter, past results in the study of cycles are provided for background to the new results presented in Chapters 2 through 4.

These past results represent a broad survey of the literature on which the research for this dissertation was based, and upon which my new results have expanded.

1.2 Sufficient Conditions for Hamiltonicity

We must begin with Dirac's minimum degree condition for Hamiltonicity, which was one of the earliest sufficiency results in this area. His somewhat labored proof can be distilled into the simplified one provided here for contrast with a subsequent result from which Dirac's Theorem follows as a corollary.

Theorem 1.16. *(Dirac's Condition, 1952) [35] Given a graph G on $n \geq 3$ vertices, and $d(v) \geq \frac{n}{2}$ for every $v \in V(G)$, then G is Hamiltonian.*

Proof. Let C be a longest cycle in G. Suppose that C is not Hamiltonian. Then we can find a path P in G such that P is disjoint from C, and is maximal in this respect. Let u and v be the end vertices of P. Then, since $d(u) \geq \frac{n}{2}$, and $d(v) \geq \frac{n}{2}$, there must exist vertices x and y on C, which are adjacent to u and v respectively. Moreover, a path of C from x to y together with P and the edges joining x to u, and y to v, form a cycle longer than C, contradicting the maximality of C. $\qquad\square$

But the foundational result in the area of Hamiltonian Graph Theory, was provided in 1960, in Ore's Theorem [79], in which we need the following definition:

Definition 1.17. $\sigma_2(\text{G})=\min\{\text{d(x)}+\text{d(y)}: x, y \in V(G), xy \notin E(G)\}$

Ore's Theorem is pivotal to the development of cycle theory. Later degree sum restrictions that yield Hamiltonicity are referred to as *Ore-type conditions*, and Bondy and others used Ore's Theorem as a launching point to prove other significant results about graphs, having to do with pancyclicity, and cycle extendability. Moreover, the σ_2-notation is now widely used, and as we will see, similar notation has arisen to study cycles in bipartite graphs. It could be argued that the prevalence of the σ_2-notation indicates how seminal Ore's Theorem is.

The proof of Ore's theorem is furnished because it is elegant and easy to follow, and introduces a proof technique unique to graph theory, which we will use later.

Theorem 1.18. *(Ore, 1960) [79] If G is a graph on $n \geq 3$ vertices, and $\sigma_2(G) \geq n$, then G is Hamiltonian.*

Proof. Suppose, for all pairs of non-adjacent vertices u, v, $d(u) + d(v) \geq |G|$ but G is not Hamiltonian. Let G be a maximal such counter-example. That is, the addition of any edge will complete a Hamiltonian cycle. Let $P : v_1 v_2 \ldots v_n$ be a Hamiltonian path, where $x = v_1$ and y=v_n. If x is adjacent to any vertex v_i, then y cannot be adjacent to its predecessor v_{i-1}, or it would result in a Hamiltonian cycle $xv_2v_3\ldots v_{i-1}yv_{n-1}v_{n-2}v_{n-3}\ldots v_i x$. Hence, $d(x) \leq (n-1) - d(y)$, it follows that $d(x) + d(y) \leq n-1$, contradicting the hypothesis. Hence G must be Hamiltonian. $\qquad\square$

Sharpness Example: The graph G in Fig. 1.3, is not Hamiltonian. Observe that the only non-adjacent vertices lie in the independent set, $\bar{K}_{\frac{(n+1)}{2}}$, and each of these vertices has degree $\frac{(n-1)}{2}$. Hence G does not satisfy Ore's condition, because $\sigma_2(G) = n - 1$.
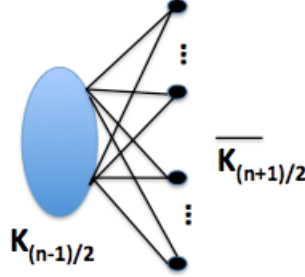
FIGURE 1.3: Sharpness Example: Ore's Condition

The earlier theorem, due to Dirac, which actually pre-dates Ore's Theorem, follows as an immediate corollary to Theorem 1.18. This was the first minimum degree restriction, which we will refer to as Dirac's Condition for Hamiltonicity.

Corollary 1. (Dirac's Condition, 1952) [35] Suppose G is a graph on $n \geq 3$ vertices. If $\delta(G) \geq n/2$, then G is Hamiltonian.

Proof. If x and y are non-adjacent vertices, then $d(x) + d(y) \geq n/2 + n/2 \geq n$, which by Ore's Theorem, implies G is Hamiltonian. $\qquad\square$

In Chapter 2, we prove a new minimum degree condition for Hamiltonicity. In Theorem 3.6, we prove that any 2-connected $K_{1,3}$-free graph of order $n>4$, with minimum degree greater than $\frac{2}{3}(n - 1)$ is Hamiltonian.

Bondy expanded upon Ore's Theorem by defining the *closure* of a graph as follows:

Definition 1.19. The *closure* of G, denoted $cl(G)$, is the graph obtained from G by recursively joining pairs of non-adjacent vertices whose degree-sum is at least n, until no such pair remains.

Then we get the following theorem:

Theorem 1.20. *(Bondy and Chvàtal, 1976) [20] A graph G is Hamiltonian if and only if $cl(G)$ is Hamiltonian.*

Additionally, if the minimum degree of the graph is at least $n/2$, then all the vertices are mutually adjacent, giving us the following corollary:

Corollary 1.21. *If G is a graph with $\delta(G) \geq \frac{n}{2}$, then $cl(G)$ is complete.*

In Chapter 4 these results are extended to bipartite graphs. The *bipartite closure* of a bipartite graph is defined, and the bipartite version of Bondy-Chvàtal's Theorem is presented (see Lemma 4.4).

Geng-Hua Fan took a slightly different approach by defining a property where if vertices having a distance 2 satisfy a degree requirement, the graph has a cycle of specified length.

Theorem 1.22. *(Geng-Hua Fan, 1984) [39] A graph G is said to have a property $P(k)$, for $3 \leq k \leq n$, if for all $x, y \in G$, if $dist(x, y) = 2$ then $max\{d(x), d(y)\} \geq k/2$. If G is 2-connected and satisfies $P(k)$, then G has a cycle of length at least k.*

Clearly, if $k = n$, Fan's theorem says the graph has a cycle of length n, and hence is Hamiltonian.

The size of the connectivity of the graph, $\kappa(G)$, relative to the size of the largest independent set, $\alpha(G)$, was discovered by Chvátal and Erdős to be another important gauge of whether a graph has a spanning cycle. Later comparable theorems, involving these measures, are referred to as Chvátal-Erdős-type conditions.

Theorem 1.23. *(Chvátal-Erdős, 1972) [32] If $G \neq K_2$ and $\kappa(G) \geq \alpha(G)$, then G is Hamiltonian.*

Proof. If $\alpha = 1$, then G is complete, hence Hamiltonian. So, take $\kappa(G) \geq 2$. Then $\delta(G) \geq \kappa(G)$ implies there exists a $(\delta + 1)$-cycle. Let $\kappa(G) = k$. Suppose C is a longest cycle. It must have at least $k + 1$ vertices. If C is not Hamiltonian, then there exists a non-empty component H in $G - C$. Since G is k-connected, there are at least k distinct

vertices in $V(C)$ with edges to H, label these as they occur clockwise, $\{u_1, u_2, \ldots uk\}$. No two can be adjacent, or we could enlarge the cycle. Let us denote their successors $\{u'_1, u'_2, \ldots u'_k\}$. None of the pairs of u'_i can be adjacent, or we could detour through H and enlarge C, contradicting the maximality of C. Hence, for some $x \in$H, this gives us an independent set of vertices $\{u'_1, u'_2, \ldots u'_k, x\}$ of size $k+1$, contradicting the hypothesis that $k \geq \alpha(G)$. Hence, if $\kappa(G) \geq \alpha(G)$, then G is Hamiltonian. $\qquad\square$

Note: this proof has been included because a similar technique will be used in later proofs of results in Chapter 2.

In particular, in Theorem 2.32, we will present a new result, that any chordal graph satisfying the Chvátal-Erdős condition, that $\kappa(G) \geq \alpha(G)$ is cycle extendable.

Sharpness Example: The Peterson Graph in Fig. 1.4 does not satisfy Chvátal-Erdős, because $\alpha(G) = 4$ whereas $\kappa(G) = 3$, and the graph is not Hamiltonian (although this is not easy to show). Another direction in Hamiltonian Graph Theory was to identify

Peterson Graph
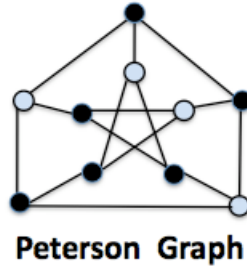
FIGURE 1.4: Sharpness Examples: Chvàtal-Erdős Condition

forbidden subgraphs. The following theorem was one of many subgraph pair restrictions (see Fig. 1.5):

Theorem 1.24. *(Gould and Jacobson, 1982) [52] If G is 2-connected, $\{K_{1,3}, Z_2\}$-free, then G is Hamiltonian. (Fig. 1.5).*

Along these lines, in Chapter 2 we will present a new forbidden subgraph pair for cycle extendability in chordal graphs (see Theorem 2.20).

Li et al. [73] gave another useful sufficient condition for Hamiltonicity, involving a
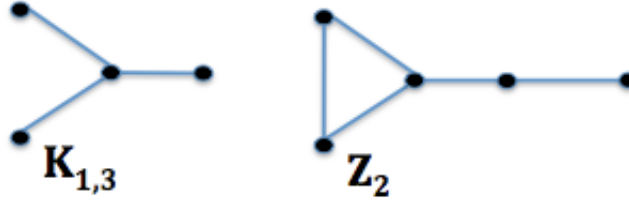
FIGURE 1.5: Forbidden Subgraph Pair – Hamiltonicity

combination of forbidden subgraph and degree-sum restrictions, and toughness, which is a parameter defined as follows:

Definition 1.25. A graph is τ-tough where $\tau > 0$, means that for every cut set $S \subseteq V(G)$, the graph $G - S$ has at most $|S|/\tau$ components, that is, $|S| \geq \tau \cdot c(G - S)$.

As we have observed, it is necessary that a graph is 1-tough for it to have a Hamiltonian cycle, but that condition is not sufficient, as illustrated in Fig. 1.6.
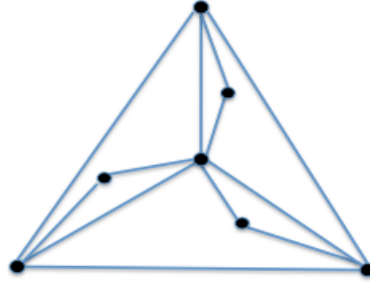
FIGURE 1.6: A Graph That Is 1-Tough But Not Hamiltonian

It was believed for a while that 2-tough graphs were Hamiltonian. But this was disproved by Bauer et al. [11]. Chvàtal made the following conjecture:

Toughness Conjecture (Chvàtal, 1973) [31] There exists some positive integer τ such that every τ-tough graph is Hamiltonian.

Whether there is a minimum value of τ remains an open question. However, Kratsch et al. proved that that $\tau = 3/2$ is the smallest toughness threshold to guarantee a

Hamiltonian cycle for split graphs [69]. Also, Chen et al. [29] proved that every 18-tough chordal graph is Hamiltonian.

Theorem 1.26. *(Li, et al., 2002) [73] A 1-tough triangle-free graph of order n such that $d(x) + d(y) + d(z) \geq n$ for every independent set of vertices $\{x, y, z\}$, is Hamiltonian.*

1.2.1 Conditions for Hamiltonicity in Partite Graphs

In 1963 Moon and Moser [76] broadened results concerning Hamiltonicity to the bipartite case. They described an Ore-type condition for which a bipartite graph would have a spanning cycle.

Definition 1.27. If $G = (X, Y, E)$ is a bipartite graph, then $\sigma_2^2(G) = min\{d(x) + d(y) : xy \notin E(G), x \in X, y \in Y\}$.

Theorem 1.28. *(Moon and Moser, 1963) [76] Suppose $G = (X, Y)$ is a bipartite graph with order $2n$, and $\sigma_2^2(G) \geq n + 1$, then G is Hamiltonian.*

Amar extended this work on bipartite graphs to find another important degree sum condition for Hamiltonicity.

Theorem 1.29. *(Amar, 1993) [3] A connected balanced bipartite graph of order $2n$ satisfying the condition that for all vertex pairs x, y where $dist(x, y) = 3$ we have $d(x) + d(y) \geq n + 1$, then G is Hamiltonian.*

Some work has been done in studying the cycle properties of tri-partite graphs, and in general, k-partite graphs. The following theorem is an example of a result concerning pancyclicity in nearly balanced k-partite graphs.

Theorem 1.30. *(Chen et al., 1995) [28] If G is a balanced k-partite graph of order kn with minimum degree $\delta(G)$ greater than $(\frac{k}{2} - \frac{1}{(k+1)})n$, if k is odd, or $(\frac{k}{2} - \frac{2}{(k+2)})n$, if k is even, then G is Hamiltonian.*

Theorem 1.31. *(Yokomura, 1998) [94] The 3-partite graph $G = (V_1 \cup V_2 \cup V_3, E)$, where $|V_i| = n$, is Hamiltonian if for any two non-adjacent vertices $u \in V_i$ and $v \in V_j$, for $1 \leq i \leq j \leq 3$, the following condition is satisfied: $|N(u) \cap V_j| + |N(v) \cap V_i| \geq n + 1$.*

In contrast, we will present new results on Hamiltonian cycles in bipartite graphs that *avoid* specified edge sets or subgraphs (see Theorems 4.5, 4.6).

1.3 Pancyclicity Results: Degree Sum Conditions

In this section, we will see that theorems giving us sufficient conditions for Hamiltonicity motivate analogous theorems relating to pancyclicity. The first is an Ore-type condition by Bondy.

Theorem 1.32. *(Bondy, 1971)* [18] *If for all non-adjacent pairs x, y, $d(x) + d(y) \geq n$, that is, if $\sigma_2(G) \geq n$, then G is pancyclic, unless n is even and $G \equiv K_{\frac{n}{2}, \frac{n}{2}}$.*

Theorem 1.33. *(Schmeichel and Hakimi, 1988)* [85] *If there exist consecutive vertices on a Hamiltonian cycle having degree sum at least $n + 1$, then G is pancyclic.*

Recall that the distance $dist_C(x, y)$ on a cycle C between two vertices x and y to be the length of the shortest x, y-path in C.

Theorem 1.34. *(She-Min Zhang, 1988)* [96] *If there exist vertices x_1, x_3 on a Hamiltonian cycle C such that $dist_C(x_1, x_3) = 2$, such that $d(x_1) + d(x_3) \geq n+1$, then G is pancyclic.*

Proof. (the proof is provided here to demonstrate the techniques used in obtaining pancyclicity.) First, it is obvious that G contains a 3-cycle, since either x_1 or x_3 has degree at least $\frac{n}{2}$. To look for cycles of all other lengths t, $4 \leq t \leq n - 1$, we must consider two cases.

Case 1) $x_1 x_3 \notin E(G)$. Then:

$$n + 1 \leq d(x_1) + d(x_3) = |N(x_1) \cup N(x_3)| + |N(x_1) \cap N(x_3)|$$

$$n + 1 \leq (n - 2) + |N(x_1) \cap N(x_3)|$$

$$\Rightarrow \quad 3 \leq |N(x_1) \cap N(x_3)|.$$

This implies there exist distinct vertices $p, q \in N(x_1) \cap N(x_3)$, giving us a 4-cycle $\{x_1, p, x_3, q, x_1\}$.

To see that G also contains cycles of every length t for $5 \leq t \leq n-1$, suppose a cycle is missing for some t. Then G cannot contain the edge pair (x_1, x_k), (x_3, x_{n+k-t}), if $4 \leq k \leq t-1$ (see Fig. 1.7a), or (x_1, x_k), (x_3, x_{k-t+4}), if $t \leq k \leq n-1$ (see Fig. 1.7b).
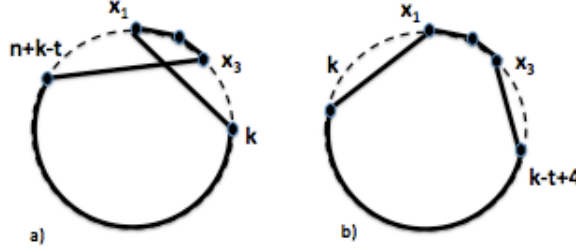
FIGURE 1.7: Restricted Edge Pairs – Proof of Zhang's Theorem

Then, $d(x_1) \leq (n-4) - (d(x_3) - 4)$, if such edge pairs are restricted, giving us: $d(x_1) + d(x_3) \leq n$, contradicting the hypothesis. $\square$

Case 2) $x_1x_3 \in E(G)$. Prove by induction on n. Base case, $n = 5$. We know there must be a 3-cycle, $C_3 : x_1x_2x_3x_1$, and by hypothesis, a 5-cycle, therefore, there exists a cycle of length 4, $C_4 : x_3x_4x_5x_1x_3$.

Suppose the result is true for all graphs of order less than n. Let $|G| = n$, and suppose G satisfies the condition that for vertices x_1, x_3 a distance 2 apart on a Hamiltonian cycle C, $d(x_1) + d(x_3) \geq n+1$.

Define $G' = G - x_2$, replacing the segment $x_1x_2x_3$ of C with the edge x_1x_3. This gives us a graph of order $n-1$, hence induction applies, and we get cycles of all lengths t, $3 \leq t \leq n-1$. Therefore, G is pancyclic. $\square$

A corollary to this theorem is a Dirac-type condition for pancyclicity:

Corollary 1.35. *(Harris, 2009) [55] If G is Hamiltonian with more than $n/3$ vertices of degree $\geq \frac{(n+1)}{2}$, then G is pancyclic.*

The next theorem has to do with a graph having an $(n-1)$-cycle, where the vertex not on the cycle is adjacent to at least half the vertices of the graph. Then pancyclicity results.

Theorem 1.36. *(Haggkvist et al., 1981) [53] If G is a graph of order $n \geq 4$ has a cycle of length $(n-1)$, and a vertex x not on the cycle has $d(x) \geq \frac{n}{2}$, then G is pancyclic. (Also due to Benhocine, et al., [16]).*

Proof. If $d(x) \geq \frac{n}{2}$, then x must be adjacent to at least two consecutive vertices x_1, x_{n-1} on the $(n-1)$-cycle. Hence G contains a n-cycle (where the edge $x_1 x_{n-1}$ can be replaced by the path $x_1 x x_{n-1}$). Also, G has a 3-cycle, $x_1 x x_{n-1} x_1$. Now suppose there is some t, $3 < t \leq n-2$, such that G has no t-cycle. Then for all i, $1 \leq i \leq n-1$, we cannot have both edges $x_i x$ and $x_{i+t-2} x$ or we would get a cycle of length t, that is, the cycle C_t: $x x_i C + x_{i+t-2} x$. But there are $(n-1)$ of these pairs, and $d(x) \leq \frac{1}{2}(n-1)$, contradicting the hypothesis. Hence G must be pancyclic. $\qquad\square$

The following important theorem examines the degree sum of the end vertices of a Hamiltonian path.

Theorem 1.37. *(Faudree et al., 1996) [40] If there exist vertices x, y at the opposite ends of a Hamiltonian path with degree sum $d(x) + d(y) \geq n+1$, then G is pancyclic.*

Note: Schelten and Scheimeyer [84] improved this bound to $\frac{(n+13)}{5}$.

1.4   Pancyclicity Results: Other Conditions

Bauer and Schmeichel considered how Fan's Condition for Hamiltonicity might apply to pancyclicity. Like Ore's Condition, Fan's Condition proves to be sufficient for more than Hamiltonicity alone, as we see in the following theorem.

Theorem 1.38. *(Bauer and Schmeichel, 1990) [10] A 2-connected graph G that satisfies Fans Condition, that for all $x, y \in G$, if $dist(x, y) = 2$ then $max\{d(x), d(y)\} \geq \frac{|G|}{2}$, is pancyclic, or $G = K_{n/2, n/2}$, or $K_{n/2, n/2} - e$ for some edge e, (or G is the the graph F_n, which is a matching between $K_{n/2}$ and the vertices of a set of $n/4$ independent edges, T).*

Harris proved a useful little theorem about pancyclicity, which can be verified with a simple inspection of the vertex degrees of the graph, and gives us some interesting insight into the local structure of some pancyclic graphs.

Theorem 1.39. *(Harris, 2009) [55] If G is Hamiltonian of order $n \geq 5$ with a vertex of degree $n - 2$, then G is pancyclic.*

The next theorem, due to Bondy, gives us a minimum edge bound to guarantee a graph is pancyclic.

Theorem 1.40. *(Bondy, 1971) [18] Every Hamiltonian non-bipartite graph of order n with at least $\frac{n^2}{4}$ edges is pancyclic.*

Note: this edge condition is satisfied if $\delta(G) \geq \frac{n}{2}$. Amar, Flandrin et al. [4], reduced this bound to $\delta(G) \geq \frac{(2n+1)}{5}$, if $n \geq 162$.

1.5 Bipancyclicity Results: Bipartite Graphs

Now, we examine some conditions for bipancyclicity in bipartite graphs. We begin with Schmeichel and Mitchem's theorem which is analogous to Theorem 1.33, due to Schmeichel and Hakimi, for ordinary graphs:

Theorem 1.41. *(Schmeichel and Mitchem, 1982) [86] A Hamiltonian bipartite graph with Hamiltonian cycle $v_1 v_2 \ldots v_{2n}$, where $d(v_1) + d(v_{2n}) \geq n + 1$, is bipancyclic.*

In Theorem 1.29, Amar proved not only does the following condition guarantee Hamiltonicity in a bipartite graph, but also pancyclicity.

Theorem 1.42. *(Amar, 1993) [3] A connected balanced bipartite graph of order $2n$ satisfying the condition that for all vertex pairs x, y where $dist(x, y) = 3$ implies $d(x) + d(y) \geq n + 1$, then G is bipancyclic.*

The next theorem gives us a minimum number of edges for which a bipartite graph is guaranteed to be bipancyclic.

Theorem 1.43. *(Entringer and Schmeichel, 1988) [37] A balanced bipartite graph of order $2n$ with $n^2 - n + 2$ edges is bipancyclic.*

Theorem 1.44. *(Harris, 2009) [55] If G is bipartite with $|X|=|Y|= n \geq 4$ and Hamiltonian cycle C, and there exists a vertex of degree $n - 1$, then G is bipancyclic.*

1.6 Hamiltonicity and Cycle Extendability of Locally Connected Graphs

Recall, a graph being locally-connected describes the connectedness of the neighborhood of any vertex in the graph (see Definition 1.2). A locally-connected graph is more likely to have other properties such as Hamiltonicity and cycle extendability. Moreover, if the neighborhoods of each vertex satisfy Ore's condition – a kind of "local Ore" behavior – the same can be said. The following theorems provide us with an understanding of how local structure affects the global properties of a graph.

In Chapter 2, we introduce new results along these lines, presenting conditions on local behavior that influences cycle extendability in a graph (see Theorem 2.23).

Theorem 1.45. *(Chartrand and Pippert, 1974) [26] If G is connected and locally-connected with maximum degree not greater than 4, then either G is Hamiltonian or $G \cong K_{1,1,3}$.*

Theorem 1.46. *(Kikust, 1975) [68] Every connected and locally-connected 5-regular graph is Hamiltonian.*

Theorem 1.47. *(Oberly and Sumner, 1979) [78] Every connected, locally-connected $K_{1,3}$-free graph is Hamiltonian.*

Theorem 1.48. *(Hendry, 1989) [56]: If G is connected and locally-connected with maximum degree $\Delta(G) \leq 5$, and $\Delta(G) - \delta(G) \leq 1$, then G is fully cycle-extendable.*

Theorem 1.49. *(Asratian and Khachatrian, 1990) [8] If G is connected and $G_1(x)$ is an Ore graph for all $x \in G$, then G is Hamiltonian.*

Theorem 1.50. *(Asratian, 1995) [7] If G is connected and $G_1(x)$ is an Ore graph for all $x \in G$, then G is fully cycle-extendable.*

1.7 Properties of Chordal Graphs

We will utilize some of the following theorems that describe important properties of chordal graphs.

Theorem 1.51. *(Buneman, 1974) [25] A graph G is chordal if and only if it admits to a perfect elimination ordering.*

In Theorem 2.34, in Chapter 2, we present a new result cycle extendability in graphs having a special kind of perfect elimination ordering, known as a *Hamiltonian elimination ordering.* This new result has interesting implications to Hamiltonicity.

Theorem 1.52. *Suppose G is a Hamiltonian chordal graph and x is a simplical vertex. Then the following properties hold:*

1. *(West, 2000) [91] Any induced subgraph of G is chordal (chordal is an* hereditary *property);*
2. *(Fulkerson and Gross, 1965) [45] G−{x} is a chordal graph;*
3. *(Abueida and Sritharan, 1980) [1] If xy is any edge incident to x, then G − xy is chordal.*

Theorem 1.53. *(Balakrishnan and Paulraja, 1986) [9] A 2-connected chordal graph G is locally connected.*

The sub-classes of Hamiltonian chordal graphs which have been proven to have the property of cycle extendability, include planar Hamiltonian chordal graphs, Hamiltonian interval graphs, and Hamiltonian split graphs, (which, as Bender et al proved, encompasses almost all chordal graphs). We can generalize that split graphs are chordal graphs whose complement is also chordal [82].

Theorem 1.54. *(Bender et al., 1985) [15] Almost all chordal graphs split.*

From this, it is easy to give a direct proof of the following conclusion:

Theorem 1.55. *(Abeuida and Sritharan, 2006) [1] Hamiltonian split graphs are cycle extendable.*

Proof. Suppose $G = (K, I, E)$ is a minimal counterexample. That is, G is a minimal graph that splits into a complete graph K and an independent set I, having some non-Hamiltonian cycle C that does not extend. Let $|G| = n$. If $x \in (K \cup I) - V(C)$ is a simplicial vertex of G, then we can define $G' = G - \{x\}$, on fewer than n vertices, which is still a Hamiltonian split graph, in which C does not extend. This contradicts the minimality of G. Hence, if x is simplicial, we can assume x lies on C, or we can easily extend the cycle. This implies that every vertex in I must lie on C, since every vertex of I is simplicial (since for all $x \in I$, $N(x) \subseteq K$, which is complete). Hence, if $x \notin V(C)$, we can suppose that x is not simplicial, and $x \notin I$, that is, $x \in K$. But x is not simpicial and K is complete, implies x is adjacent to some vertex $y \in I$. Since y lies on C, it lies on a segment yz of the cycle, which implies z, x are adjacent (since $N(x)$ is complete). Hence C can be extended by replacing the edge yz with the path yxz. $\qquad\square$

Theorem 1.56. *(Jiang, 2002) [64] Hamiltonian planar chordal graphs are cycle extendable.*

The following two theorems proved the same result, and appeared in the same issue of the same journal in 2006:

Theorem 1.57. *(Abueida and Sritharan, [1]; Chen et al.,[27]) Any Hamiltonian interval graph is cycle extendable.*

1.8   Extendability Hasse Diagram

The preceding theorems relating to cycle extendability, give us an understanding of the relationship among the properties of different graph classes, which can be summarized by the following Hasse diagram. The Hasse diagram in Figure 1.8 summarizes the state of the art in identifying the cycle extendability of structured classes of graphs. At the time this thesis is written, this figure is complete, and correct as we will presently show.

We note that if Hendry's conjecture were true the figure would collapse into a linear order. We also note, however, that there are structured classes of graphs which are not chordal and are cycle extendable, for example the *triangular grid graphs* discussed in [49]. A *triangular grid graph* is a graph which can be drawn in the Cartesian coordinate system with vertices at distinct coordinates which are linear combinations of the vectors $\vec{p} = (1,0)$ and $\vec{q} = (1/2, \sqrt{3}/2)$ and vertices adjacent if and only if their Euclidean distance is equal to 1. The graph constructed by placing a vertex at each corner of a hexagon and a vertex adjacent to each corner vertex in its center is a triangular grid graph but is not chordal since the deletion of the central vertex leaves a 6-cycle.
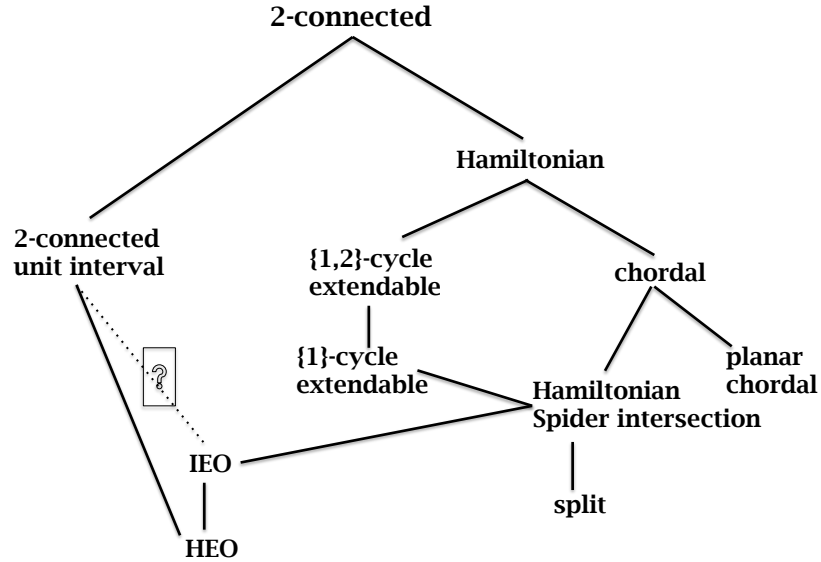
FIGURE 1.8: Hasse Diagram Indicating Containment Relationships Among Structured Classes of Graphs

Claim: *The Hasse diagram in Figure 1.8 is correct and all containment relationships are proper.*

Proof. Starting from level one, a Hamiltonian elimination ordering characterizes unit interval graphs and hence the class of graphs with an HEO is properly contained in the class of interval which are characterized via an IEO (see Jamison and Lascar [63]).

Whether there is a 2-connected unit interval graph which is not a Hamiltonian interval graph is not known and conversely. Spider intersection graphs generalize both interval graphs and split graphs and are shown to be cycle extendable in [2]. It is well-known that chordal graphs are the intersection graphs of subtrees of a tree (Gavril [46]) and a spider is a tree. Planar chordal graphs which are Hamiltonian are shown to be cycle extendable in [64], and there is no relationship between planar graphs and spider intersection graphs since a K_5 is a spider intersection graph, but there are planar graphs which require two vertices having degree greater than 2 as a host tree for their intersection representation (see the paper by Jamison et al. on using spanning trees as the basis of subtree representations of chordal graphs [61]).

In Chapter 2 we prove that 2-connected unit interval graphs are cycle extendable (see Corollary to Theorem 2.23), and we conjecture that Hamiltonian chordal graphs are $\{1, 2\}$-cycle-extendable. That $\{1\}$-cycle-extendable graphs are $\{1, 2\}$-cycle-extendable is clear and the counterexamples of Lafond and Seamone are $\{1, 2\}$-cycle-extendable but not $\{1\}$-cycle-extendable (see [71]). We observe that the 3-sun is 2-connected but is not a unit interval graph. Note that The dashed edge indicates a conjecture and the edge marked with '?' indicates the relationship is not known at this time. Finally, a sufficiently large complete bipartite graph with an unequal number of vertices in each partite set is 2-connected but not Hamiltonian. $\square$

CHAPTER 2

CYCLE EXTENDABILITY OF GRAPHS

2.1 Fundamentals

In 1980, Bondy [19] proved that Ore's Condition (Theorem 1.18), not only guarantees Hamiltonicity but also pancyclicity (except if $G \cong K_{\frac{n}{2},\frac{n}{2}}$). Furthermore, he proved that if a graph G satisfies Ore's Condition, then given any non-Hamiltonian cycle C, there is a cycle C' such that $V(C) \subseteq V(C')$ and $|V(C')|=|V(C)|+i$, where i=1 or 2. This introduced a new property, which later would be defined by Hendry.

Recall that, in 1990, Hendry [57] defined a non-Hamiltonian cycle C as being *extendable* if there exists a cycle C' such that $V(C) \subseteq V(C')$ and $|V(C')|=|V(C)|+1$, and a graph G is *cycle extendable* if every non-Hamiltonian cycle in G can be extended.

Hendry popularized the study of cycle extendability in graphs. Observing that chordal graphs are cycle reducible, Hendry conjectured in 1990 that all Hamiltonian chordal graphs are cycle-extendable. This has since sparked keen interest in whether the conjecture is true, and in their effort to prove or disprove it, a number of graph theorists made significant strides in describing behavior of chordal graphs and furthering our understanding of various graph classes. In 1999, Jiang [64] proved that Hamiltonian planar chordal graphs are cycle extendable. Not only Abueida and Sritharan [1], but Chen et al. [27], proved in 2006 that Hamiltonian interval graphs are cycle extendable, (where interval graphs are a subclass of chordal graphs, first introduced by Seymour Benzer in his study of genetics, and later characterized by Hajos and others).

Abueida and Sritharan also proved in the same paper that Hamiltonian split graphs

are cycle extendable, (recall that split graphs are graphs whose vertex set can be partitioned into a complete graph and an independent set). As mentioned earlier, Bender et al. [15], proved that "almost all" chordal graphs split. This leads us to understand that a large class of Hamiltonian chordal graphs are in fact cycle extendable, but since recently, LaFond and Seamone [71] have identified counterexamples to show that Hendrys conjecture is not true, we know that "almost all" does not include some small subclass of chordal graphs.

Beasley and Brown took steps toward describing the properties of chordal graphs that are not cycle extendable:

Proposition 2.1. *(Beasley and Brown, 2005) [12] Let G be a vertex-minimal Hamiltonian chordal graph on n vertices, with a cycle C of length $k<n$ that is not extendable. Then either $n = k + 2$, or $G - C$ contains no edges of any Hamiltonian cycle.*

From this description of what a non-cycle extendable chordal graph must look like, Beasley and Brown introduced a more general concept of S-cycle-extendability:

Definition 2.2. Given a graph G on n vertices. Let $S \subseteq \{1, 2, \ldots, n\}$. Suppose C is a cycle of length k in G. Then C is S-extendable if there is an $i \in S$ such that there exists a cycle C' in G with $|V(C')|=|V(C)|+i$, and $V(C) \subseteq V(C')$. If this is true for all non-Hamiltonian cycles in G, the graph G is said to be S-cycle-extendable.

This yields a new characterization of cycle extendable graphs:

Definition 2.3. G is cycle extendable if and only if it is $\{1\}$-cycle-extendable. That is, G is extendable if and only if every non-Hamiltonian cycle C is $\{1\}$-extendable.

In particular, in the case of chordal graphs (as we will see in the last section of this chapter), Arangno, Beasley, Brown conjecture that if G is a Hamiltonian chordal graph then G is $\{1, 2\}$-cycle-extendable.

The graph in the next figure is provided by LeMond and Seamone [71] as the base graph for their counterexample to Hendry's conjecture. This is the graph on which they

build examples of a subclass of chordal graphs which are not cycle-extendable.

The counterexample builds on the base graph H shown in Fig. 2.1, which has
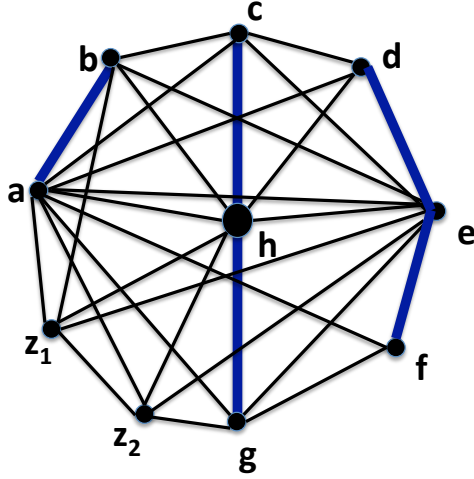
FIGURE 2.1: Counter-Example for Hendry's Conjecture: Base Graph H

heavy edges, as highlighted. Onto these heavy edges, they paste a clique of arbitrary size. So in the illustrated example, $|V(H)| = 10$, hence the minimum order of the vertex set of any graph serving as a counterexample to Hendry's conjecture, will have lower bound of 15. It has been proved that a graph obtained from a chordal graph and a disjoint set of complete graphs via "clique pasting", i.e., a *clique sum*, is still chordal. Moreover such a graph is Hamiltonian if the base graph is. Therefore, in the new graph G obtained via pasting a clique of order k onto the heavy edge $z_1 z_2$ of the base graph H shown in Fig. 2.1, we have a cycle C that spans every vertex of G except z_1 and z_2, that cannot be extended in a graph on $n + k - 2$ vertices.

Theorem 2.4. *(LaFond and Seamone, 2013) [71] For any $0 < \alpha < 1$, there exists a Hamiltonian chordal graph G with a non-extendable cycle C satisfying $|\frac{V(C)}{V(G)}| < \alpha$.*

However, consistent with Beasley and Brown's proposition, the graphs described by LeFond and Seamone are in fact $\{1, 2\}$-extendable (which results because if the cycle C spans every vertex of G except z_1 and z_2, then C can be extended by 2 vertices, z_1 and z_2. Hence, LeFond and Seamone's graphs are certainly 2-cycle-extendable.

This section examines conditions under which non-Hamiltonian cycles in such graphs can be extended. The following theorems arose from an investigation of Hendry's conjecture. Brent Thomas [88] established these theorems to lay out the base case for proof of cycle extendability in chordal graphs, inducting on the length of the cycle. These proofs give some insight into why a general proof has been elusive.

Theorem 2.5. *[88] Every 3-cycle in a chordal Hamiltonian graph is extendable.*

Proof. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Notice if $|V(G)|= 3$ the result is trivial. Let C_3 be any 3-cycle in G with $u, v, w \in V(C_3)$, $|V(G)|\geq 4$ and let C_n be a Hamiltonian cycle in G. Since $n \geq 4$, there is an edge uv of C_3 that is not an edge of C_n. Therefore there exists a cycle C_k such that $uv \in E(C_k)$, $k<n$, and $w \notin V(C_k)$. By Lemma 2.8 u and v have a common neighbor, x on C_k . Thus $uxvwu$ is a 4-cycle in G. $\square$

Theorem 2.6. *[88] Every 4-cycle in a chordal Hamiltonian graph is extendable.*

Proof. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Notice if $|V(G)|= 4$ the result is trivial. Let $C_4 = uvwxu$ be any 4-cycle in G, $|V(G)|\geq 5$, and C_n be a Hamiltonian cycle in G. Notice since G is chordal, C_4 has a chord. Without loss of generality, let $uw \in E(G)$. Since $n \geq 5$ there exist two vertices of C_4 that do not appear consecutively on C_n. If the vertices are adjacent then the cycle is extendable by Lemma 2.8. Assume that the only vertices of C_4 that do not appear consecutively are non-adjacent. Since v and x are the only two vertices of C_4 that are non-adjacent, we have either $vuwx$ or $vwux$ is a path along C_n. Without loss of generality say $vuwx$ is a path on C_n. Notice since $ux \in E(G)$, the Hamiltonian cycle can be reduced to a $(n-1)$-cycle. By Lemma 2.8, u and x have a common neighbor, y, on the $(n-1)$-cycle. Furthermore $y \neq v$ since $vx \notin E(G)$. Notice that $vuyxwv$ is a 5-cycle in G. $\square$

These proofs give a little insight into the difficulty of verfying Hendry's conjecture using an inductive approach. In both the proofs of Theorem 2.1 and Theorem 2.1

we need only consider one Hamiltonian chordal graph on 3 or 4 vertices respectively. However in a general case there are many non-isomorphic Hamiltonian chordal graphs to be considered. In fact although there are many characterizations of chordal graphs, there is no known formula for computing the number of chordal graphs on n vertices. Moreover, the proof of Theorem 2.1 shows that there may be more than one ordering of the vertices of the cycle that must be considered for each non-isomorphic graph (see [88]).

We will need to establish some properties of Hamiltonian chordal graphs for results that follow in this section.

Lemma 2.7. *Suppose G is a Hamiltonian chordal graph with simplicial vertex, v. Then $G-\{v\}$ is a Hamiltonian chordal graph.*

Proof. Since v is simplicial, then $G-\{v\}$ is chordal, by Theorem 1.52.

Suppose v is simplicial, and $M = v, v_2, v_3, \ldots v_n, v$ is a Hamiltonian cycle of G. The neighbors of v form a clique, hence $v_2 v_n \in E(G)$, therefore $M' = v_2, v_3, \ldots v_n, v$ is a Hamiltonian cycle of $G-\{v\}$. □

It is also useful to establish the following fact:

Lemma 2.8. *If G is chordal and Hamiltonian, then any two adjacent vertices lie on a 3-cycle.*

Proof. Given the edge $xy \in E(G)$, let C be the smallest cycle containing xy. If $|V(C)| \geq 4$, then C contains a chord, hence C must be a 3-cycle. □

In the same vein, we can establish the following:

Corollary 2.9. *If C is a cycle in the chordal graph G, and $e \in E(C)$, the e forms a 3-cycle with a third vertex of C.*

Proof. By induction on $|C|$: if $|C| = 3$, we're done. If $|C| > 3$, then C has a chord, f. This divides C into two paths, one of which contains the edge, e. This path, taken with

the chord, f, produces a shorter cycle, C', which contains e, such that $V(C') \subset V(C)$. Applying induction, we obtain the third vertex on C', hence on C. $\qquad\square$

Because of the properties of chordal graphs, we also get the next immediate result.

First, we will let x^-, x^+ denote the immediate predecessor and successor , respectively, of the vertex x on a cycle C, with respect to some ordering of the vertices of C.

Lemma 2.10. *If C is any cycle in a chordal graph, and x and y are consecutive vertices on that cycle, then either $x^- y$ or xy^+ are edges in $E(G)$.*

We also can characterize cycle-reducibility with the following result:

Proposition 2.11. *Suppose G is a graph on n vertices and C is a cycle of length ≥ 4. Then G is chordal if and only if there exists a cycle C' such that $|V(C')|=|V(C)|-1$.*

Proof. Let C be a cycle of length k where $3<k<n$. Let $x_1, x_2,\ldots,x_n$ be a simplicial ordering of the vertices of G. Let $x = x_i$ be the first vertex of C that is deleted, for some $1 \leq i \leq n$. The remaining vertices in the ordering that are neighbors of x form a clique, in particular those neighbors that lie on C are adjacent. Hence, deleting x yields a shorter cycle, C', where $|V(C')|=|V(C)|-1$. $\qquad\square$

Moreover, it can be observed that a Hamiltonian chordal graph must then contain cycles of all lengths, from the 3-cycle through the cycle that spans the vertex set of the graph. This was first pointed out by Hendry [56], and is stated in the following proposition:

Proposition 2.12. *A chordal Hamiltonian graph G on n vertices is pancyclic. In fact, if G is chordal, we get a pancycllc ordering of the vertices of the graph.*

Proof. Let C be a cycle of length k where $3<k<n$. Then C has a chord, x_1y_1. If $dist_C(x_1, y_1) = 2$, the chord forms a 3-cycle. Otherwise, the chord forms another cycle of length at least 4, which in turn has a chord. If this does not create a 3-cycle, the

process is repeated until a 3-cycle is obtained (since G is finite). Therefore, G contains a 3-cycle.

Let v be a simplicial vertex on the Hamiltonian cycle of G. Delete v. Since the neighbors of v are adjacent, this leaves a cycle on $(n-1)$ vertices. The chordal property is hereditary, hence this process can be repeated until we have obtained cycles of length l for all $3 \leq l \leq n$. That is, each cycle C_k of length $k \geq 4$ can be reduced to a cycle C_{l-1}, by removing a vertex v_l . This follows from the definition of chordal, and is true for all $4 \leq k \leq n$. $\qquad\square$

Note however, that the converse of these results is not true. The graph in Fig. 2.2 is pancyclic, but is neither chordal nor cycle-extendable.

FIGURE 2.2: Pancyclic Does Not Imply Chordal Nor Cycle Extendable

Since a chordal Hamiltonian graph is in fact pancyclic, as established by Proposition 2.12, we can easily reduce cycles, but it is more difficult to establish the necessary conditions to reverse this process; that is, to build larger cycles from smaller ones, equipped only with the basic property that the graph is chordal. Whence the period of time over which Hendry's conjecture has stood uncontested.

For example, Thomas [88] conjectured that if a graph has a pancyclic ordering, it must be cycle extendable. But a counterexample to this is given in the following figure (Fig. 2.3). To illustrate this, observe that the 3-cycle $v_{n+1}v_{n+2}v_{n+3}$ is not extendable.

However, if the graph is also *planar*, that is, there exists a plane representation in which the chords do not intersect, as lines in some drawing, then all the cycles may be seen as sharing consecutive vertices with the Hamiltonian circuit, and we can use the
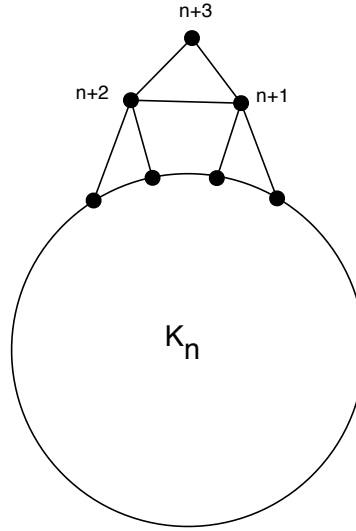
FIGURE 2.3: Pancyclic Ordering Does Not Imply Cycle Extendable

chordal property to extend any cycle to include a vertex not on it. This observation was formalized and proved by Jiang [64].

2.2   New Condition for Cycle Extendability: A Forbidden Subgraph Pair

This section explores another condition which is sufficient for a Hamiltonian chordal graph to be cycle extendable. The main theorem of this section is analogous to the previously discussed Hamiltonicity results involving *forbidden subgraph pairs.*

We will first need to establish a few facts.

Lemma 2.13. *Suppose G is a Hamiltonian chordal graph on n vertices, and suppose C is a non-Hamiltonian cycle. If v is a simplicial vertex in G not on C, then C can be extended.*

Proof. Suppose G is a minimal counter-example, that is, suppose G is a Hamiltonian chordal graph on the fewest vertices, for which a cycle not containing a given simplicial vertex cannot be extended. Let $|V(G)| = n$, let C be a non-Hamiltonian cycle, and suppose v is a simplicial vertex such that $v \notin V(C)$. Abueida and Sritharan [1] proved that any cycle of length 3 or 4 can be extended, hence we will assume $5 \leq |V(G)| \leq n-2$.

The graph G is chordal, therefore it has a simplicial elimination ordering of its vertices (or, Perfect Elimination Ordering, PEO), $x_1, x_2, \ldots, x_n$. Let $x = x_i$ be the first vertex in the ordering that lies on C. Since v is a simplicial vertex not lying on C, we know $i \geq 2$. The remaining vertices in the ordering that are neighbors of x form a clique. In particular, those neighbors of x that lie on C are adjacent. Hence deleting x yields a shorter cycle, C', where $|V(C')| = |V(G)| - 1$. The graph $G' = G - \{x\}$ is still Hamiltonian and chordal, with fewer than n vertices, such that the simplicial vertex v does not lie on C', hence the cycle C' can be extended in G', by the minimality of G. Therefore C can be extended in G. $\qquad\square$

Given a cycle C, and a vertex $v \in V(C)$, let v^+ and v^- denote the immediate successor and predecessor of a vertex v on the cycle C. By Lemma 2.13 we can assume if v is a simplicial vertex, it lies on C, otherwise, C can be extended. Moreover, since v is simplicial hence all its neighbors are adjacent, then if $w \in N(v)$, w must lie on C, otherwise we can extend C by replacing the segment $v^- v v^+$ with $v^- w v v^+$. Therefore every vertex in $N(v)$ must lie on C. This yields the following corollary to Lemma 2.13:

Corollary 2.14. *If C is a non-Hamiltonian cycle we wish to extend in a Hamiltonian chordal graph G, and v is a simplicial vertex of G, we can assume v lies on C, moreover if $w \in N(v)$ then w also lies on C.*

For our next theorem, we will need the following lemmas, (similar to a result for strongly chordal graphs proved by Abueida and Sritharan [1]).

Lemma 2.15 (Re-Routing Lemma)**.** *Suppose G is chordal and uxv is a segment of some cycle C, where w is the common neighbor of x and v on C, with $N[w] \subseteq N[v]$. Then there exists a cycle C' of G such that $V(C) = V(C')$ and uxw is a segment of C'.*

Proof. Let u, x, v occur consecutively along C, in the clockwise direction. Let z be the immediate successor of the vertex w on C. Then C contains the segment $wz \ldots ux$. Since by hypothesis, $N[w] \subseteq N[v]$ in G, v must be adjacent to z. Define A to be the segment of C from w to v, in the counterclockwise direction (see Fig. 2.4 below). Define B to

be the segment of C from z to x, in the clockwise direction. Then the desired cycle C', contains the segments B, xw, A, vz. Since both wx and ux are segments of C', it follows that uxw is a segment of C'. $\square$
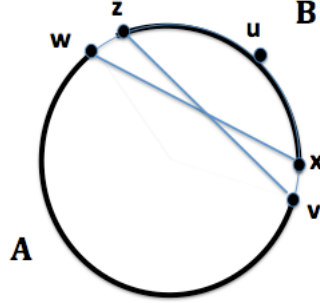

FIGURE 2.4: Segments A and B of Cycle C: Proof of Lemma 2.15

This figure shows you can re-route the cycle.

Lemma 2.16. *Let G be a chordal graph and let x be a simplicial of G. Let $p, q \in N(x)$, and suppose that $N[p] \subseteq N[q]$. If C is a cycle with $\{x, p, q\} \subseteq V(C)$, then there exists a cycle C' with $V(C) = V(C')$ such that pxq is a segment of C'.*

Proof. Suppose axb is a segment of C. If neither a nor b is p, apply the Re-Routing Lemma 2.15 to C, substituting b and p for v and w, respectively. This results in the cycle with segment axp, or pxa as applies. On the other hand, if a is not q, proceed substituting a and q in place of v and w respectively to get C'. $\square$

The next result follows immediately:

Corollary 2.17. *Let G be a chordal graph and x be a simplicial vertex of G. If there exist vertices $u, v \in N(x)$, such that $N[u] \subseteq N[v]$, then G is Hamiltonian if and only if G has a Hamiltonian cycle in which uxv is a segment.*

The next two lemmas are necessary for the main result of this section, Theorem 2.20, which is to establish a forbidden pair of subgraphs with respect to the cycle extendability of chordal Hamiltonian graphs. In the first lemma we establish that the Hamiltonian

and chordal properties are hereditary with respect to the deletion of vertices having a necessary neighborhood condition.

Lemma 2.18. *Suppose G is a Hamiltonian chordal graph with a simplicial vertex, x, and suppose there exist vertices $u, v \in N(x)$, such that $N[u] \subseteq N[v]$. Let $S = V(G) - \{x, u\}$, and $H = G[S]$. Then H is Hamiltonian and chordal.*

Proof. By Theorem 1.52, an induced subgraph of a chordal graph is chordal. Therefore, we need only show H is Hamiltonian. Take a Hamiltonian cycle in G. By Lemma 2.16, vxu is a segment of another Hamiltonian cycle, M. Let u^+ be the immediate successor of u on M. Then $vxuu_+$ is a segment of M. However, $N[u] \subseteq N[v]$, hence $vu^+ \in E(G)$, therefore we can replace the segment $vxuu^+$ with the edge vu^+, to obtain a Hamiltonian cycle M' in H. $\square$

In the second lemma we establish that the property of being "forbidden subgraph free" is hereditary with respect to the deletion of vertices having the same neighborhood condition.

Lemma 2.19. *Suppose G is a Hamiltonian chordal graph with a simplicial vertex, x, and no induced F_1 or F_2 (see Fig. 2.5). Suppose there exist vertices $u, v \in N(x)$, such that $N[u] \subseteq N[v]$. Let $S = V(G) - \{x, u\}$, and $H = G[S]$. Then H has no induced F_1 or F_2.*

Proof. If C is a cycle in G with $\{x,u,v\} \subseteq V(C)$, then by Lemma 2.16 there exists a cycle C' with $V(C) = V(C')$ such that $vxuq$ is a segment of C'. Moreover, G is Hamiltonian, hence there exists a Hamiltonian cycle M with segment $vxup$. Since $N[u] \subseteq N[v]$, it must be that vq and vp are edges in G (since $q, p \in N(u)$).

Note that if p and q are adjacent, then C can be extended. Hence we can suppose that $pq \notin E(G)$. Moreover, if x is adjacent to p, we get an extension of C either by replacing the edge vx with vpx, or the edge xu with xpu. Hence we can suppose that $xp \notin E(G)$

Delete x and u to get $S = V(G) - \{x, u\}$, and let $H = G[S]$. Then we get a Hamiltonian cycle M' in H, using the edge vp instead of the segment $vxup$. Similarly, in H, we get a cycle C', using the edge vq instead of the segment $vxuq$, such that $V(C') \subset V(C)$. Suppose F is a subgraph of G, such that possibly $F \cong F_1$ or F_2 in H.

Case 1) $u, v \notin F$: then any vertex of F that is adjacent to u, is also adjacent to v, and neighbors of the simplicial vertex x are mutually adjacent, hence deleting x and u will not result in $F \cong F_1$ or F_2.

Case 2) $u, v \in F$: then the subgraph generated by $G - \{x\}$, denoted $<G - \{x\}>$, may result in $F \cong F_1$, but $H = <G - \{x, u\}>$ will not result in either $F \cong F_1$ or $F \cong F_2$.

Case 3) $u \in F$, but $v \notin F$: then neither $<G - \{x\}>$ nor $H = <G - \{x, u\}>$ results in $F \cong F_1$ or $F \cong F_2$.

Case 4) $v \in F$, but $u \notin F$: then $<G - \{x\}>$ will not result in $F \cong F_1$ or $F \cong F_2$. Recall, p is not adjacent to either x nor q. Then suppose deletion of the vertex u results in $F \cong F_1$ or $F \cong F_2$. If there is no vertex of F, which adjacent to u, other than v itself, this means G contained a forbidden F_1 or F_2, contradicting the hypothesis. On the other hand, if a is some vertex of F, which adjacent to u, other than v, such that deleting u results in $F \cong F_2$, then G must have an induced F_2.

Hence $H = <G - \{x, u\}>$ contains no $F \cong F_1$ or $F \cong F_2$. $\qquad \square$

FIGURE 2.5: Forbidden Subgraphs

Theorem 2.20. *Let G be a chordal Hamiltonian graph with no induced F_1 or F_2. (see Fig. 2.5) If G has a simplicial vertex x with neighbors u, v such that $N[u] \subseteq N[v]$, then G is cycle-extendable.*
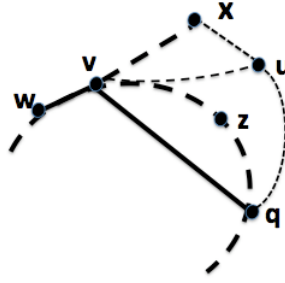
FIGURE 2.6: Inducing a Forbidden Subgraph

Proof. Let G be a minimal counter-example. That is, let G be a chordal graph on the fewest number of vertices with a Hamiltonian cycle M and no induced F_1 or F_2, having a non-Hamiltonian cycle C that cannot be extended. Let $|G| = n$. First observe that if x is a simplicial vertex of G, then by Lemma 2.13, x must lie on C. Moreover, if x is simplicial, its neighbors must also lie on C, otherwise we could extend C, (since its neighbors form a clique).

By the corollary to the preceding Lemma 2.15, w.l.o.g., vxu is a segment of the Hamiltonian cycle M, and also of C, since $N[u] \subseteq N[v]$. Let $vxup$ be a segment of M and $vxuq$ be a segment of C. It may be that q and p are different. Since $N[u] \subseteq N[v]$ in G, v must be adjacent to both p and q in G.

Let $S = V(G) - \{x, p\}$, and $H = G[S]$. If M' is the cycle obtained by replacing the segment $vxup$ with vp in M, then clearly M' is Hamiltonian in H.

Similarly, let C' be the cycle obtained by replacing the segment $vxuq$ with vq in C. It follows from Lemma 2.18 that H is a Hamiltonian chordal graph, on fewer than n vertices, containing no F_1 or F_2, by Lemma 2.19, contradicting the minimality of G.

Hence C' can be extended in $G - \{u, x\}$ to a cycle C'', containing some vertex $z \notin V(C')$, giving us a segment wvz on C''. Suppose u is not adjacent to either w or z in G. This would imply x is also not adjacent to either w or z, since x is simplicial in G. Hence $\{u, v, w, x, z\}$ induces an F_1 if w and z are adjacent; or an F_2, otherwise (see Fig. 2.6).

Therefore it must be that u is adjacent to w or to z. Without loss of generality, suppose uz is an edge in G. Starting with C''', replace the segment wvz with $wvuz$ (see Fig. 2.7a), then replace the segment $wvuz$ with $wvxuz$ to get an extension of C in G (see Fig. 2.7b).
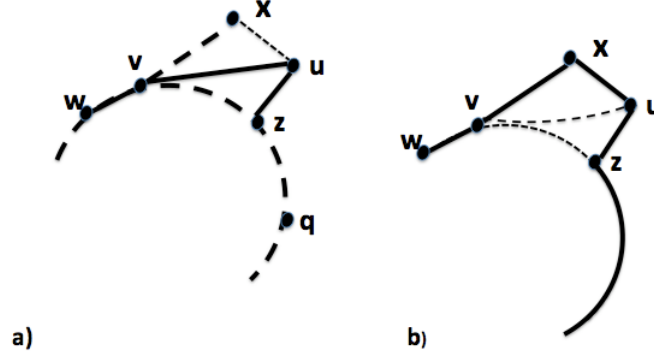

FIGURE 2.7: Obtaining an Extension of the Cycle

Specifically, if uz is an edge, then $wvxuzqCw$ is an extension of C in G, whereas, if uw is an edge, we get the cycle $wuxvzqCw$, which extends C. □

2.3   2-Connected $K_{1,3}$-Free Chordal Graphs

In this section, we will examine whether an established sufficient condition for Hamiltonicity in chordal graphs is also sufficient for cycle extendability.

2.3.1   Background

Ore proved if a graph G has n vertices, and $\sigma_2(G) \geq n$, then G is Hamiltonian. Then Bondy proved Ore's condition was sufficient for pancyclicity, moreover:

Proposition 2.21. *[18] If C is a non-Hamiltonian cycle in G, where $|G| = n$, such that $\sigma_2(G) \geq n$, then there exists a cycle C' in G such that $V(C) \subseteq V(C')$ and $|V(C')| = |V(C)| + 1$ or $|V(C)| + 2$.*

In the same vein, Oberly and Sumner proved the following [78]:

Theorem 1.47 *Any 2-connected locally-connected graph with no $K_{1,3}$ is Hamiltonian.*

Moreover, Balakrishnan and Paulraja [9] proved that any connected chordal graph is locally-connected, we get:

Theorem 1.53 *Any* 2-*connected chordal graph with no* $K_{1,3}$ *is locally-connected.*

From these two results, we conclude that any Any 2-connected chordal graph with no $K_{1,3}$ is Hamiltonian.

We will expand on this result to prove that in fact this condition for Hamiltonicity is sufficient for cycle extendability.

2.3.2 Main Results

Generalizing Ore's Condition for Hamiltonicity, Geng-Hua Fan [39] proved in 1984 that if for all vertices x and y a distance 2 apart, the maximum degree of either x or y is at least $|G|/2$, then the graph contains a Hamiltonian cycle. Bedrossian et al. [14] broadened Fan's Condition in 1993, to say that a 2-connected $K_{1,3}$-free graph is either a cycle or is pancyclic.

This suggested potential in the area of cycle extendability, specifically if applied to the well-behaved class of chordal graphs. Here we will prove any 2-connected $K_{1,3}$-free chordal graph is cycle extendable by using the following lemma from Balakrishnan and Paulraja [9]:

Lemma 2.22. *Any* 2-*connected chordal graph* G *is locally connected.*

Proof. G is locally connected if the induced subgraph $G[N(u)]$ is connected for all $u \in V(G)$. Suppose G is chordal and there exists a vertex $u \in V(G)$ such that $u_1, u_2 \in N(u)$ lie in distinct components of $G[N(u)]$, that is, $G[N(u)]$ is not connected. Then $u_1 u_2 \notin E(G)$. But G is 2-connected, implies there is a shortest u_1, u_2 path, P, of length $l \geq 2$. We can suppose not all the vertices of P lie in $N(u)$. This gives us a chordless cycle $P \cup \{u_1 u, u_2 u\}$, contradicting the fact G is chordal.

Hence G must be locally connected. $\qquad \square$

Observing this, we are able to prove the main theorem of this section:

Theorem 2.23. *Any 2-connected $K_{1,3}$-free chordal graph is cycle-extendable.*

Proof. First, observe that G is locally connected by the preceding lemma. Let C be a non-Hamiltonian cycle of length $3 \leq t \leq n-1$. G is connected. Let x be a vertex not on C which is adjacent to some $v \in V(C)$. Since G is locally connected, $N(v)$ is connected, therefore there exists a shortest v^+x path P in $N(v)$, where v^+ is the immediate successor of v on C. Suppose that $V(P) \cap V(C) = \emptyset$. Suppose $v^-v^+ \notin E(G)$: if $|V(P)| \geq 2$, then $<v^-, v, x, v^+>$ is a $K_{1,3}$ (see Fig. 2.8). Therefore the shortest path must be $P := xv^+$, that is, xv^+ must be an edge, giving us a cycle C' of length $t+1$, with $V(C) \subset V(C')$.
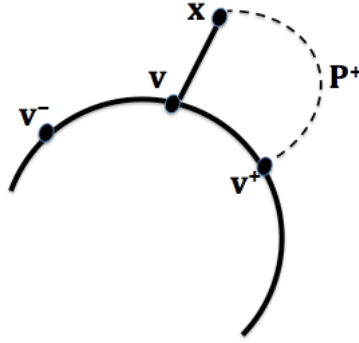

FIGURE 2.8: A $K_{1,3}$ is Induced, Provided P^+ Contains Vertices Other Than x and v^+

Otherwise, suppose $v^-v^+ \in E(G)$, and suppose $|V(P)| \geq 2$. Then there is some internal vertex $x_i \in V(P)$ adjacent to v^+. But $x_i \in N(v)$, since G is chordal, which gives us the cycle extension $C' : vx_iv^+Cv$, where C' is a cycle of length $t+1$, with $V(C) \subset V(C')$ (see Fig. 2.9).

However, suppose $V(P) \cap V(C) \neq \emptyset$. Let $\{x_1, x_2, \ldots, x_k\} = V(P) \cap V(C)$ We must consider two cases:

Case 1) none of the vertices of $V(P) \cap V(C)$ have immediate neighbors which lie in N(v). For each $x_i \in V(P) \cap V(C)$, $\{x_i^+, x_i^-, x_i, v\}$ induces a $K_{1,3}$ unless $x_i^+x_i^-$ is an edge in G. This gives us the cycle $C' : v^+x_1^-x_1^+Cx_k^-x_k^+v^-Pxvv^+$, where C' contains all the vertices of C and also v (see Fig. 2.10).
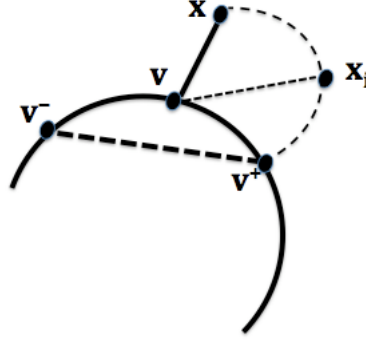
FIGURE 2.9: $V(P) \cap V(C) = \emptyset$: the Cycle Is Extendable
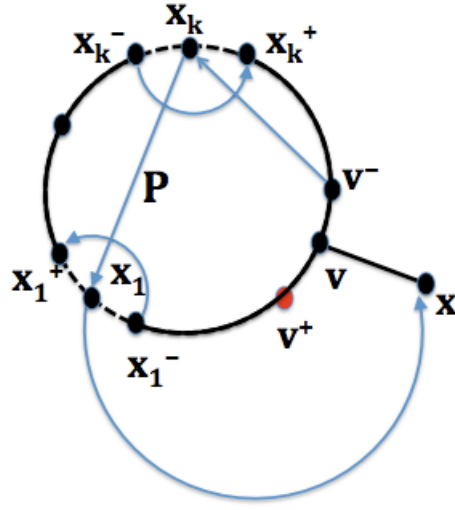
FIGURE 2.10: $V(P) \subset$ V(C): the Cycle Is Extendable

Case 2) some vertices $x_i \in V(P) \cap V(C)$, have a neighbor, either x_i^+ or x_i^- , which is in $N(v)$. Here, we follow P till we reach the first such vertex and form the new cycle C' by deleting the edges $vv^+, vv^-, x_i x_i^-$ (or $x_i x_i^+$), and using the new edges vx_i, v^+v^-, and vx_i^- (or vx_i^+), to get: $C' : v^+ C x_i^- v x_i C v^- v^+$. Doing this for all such vertices $x_i \in V(P) \cap V(C)$ with neighbors in $N(v)$, gives us a cycle C' with $x_i \in V(P) \cap V(C')$ having no neighbors in $N(v)$, which brings us back to *Case 1*, and we get a longer cycle containing v.

Therefore, any non-Hamiltonian cycle can be extended, that is, G is cycle extendable. $\quad\square$

Finally, observing that an interval graph is a chordal graph, we obtain the Balakrishnan and Paulraja's result as a corollary:

Corollary 2.24. *(Balakrishnan and Paulraja, 1986) [9] Any* 2*-connected $K_{1,3}$-free interval graph is Hamiltonian.*

From this, the next corollary follows immediately:

Corollary 2.25. *Any* 2*-connected $K_{1,3}$-free interval graph is cycle-extendable.*

The next corollary is also immediate because *unit* interval graphs – being interval graphs where all the intervals have the same length 1 – are precisely the $K_{1,3}$-free interval graphs (see Golumbic [47]).

Corollary 2.26. *Any* 2*-connected unit interval graph is cycle-extendable.*

Note: Similar results were derived earlier, as pointed out by Seamone, in a paper by Clark, who proved that Hamiltonian $K_{1,3}$-free chordal graphs are cycle-extendable, using a different proof.

2.4 The Chvátal-Erdős Condition

In 1991, Amar et al. [5] conjectured that the Chvátal-Erdős condition was sufficient for pancyclicity. However, there is a large family of K_3-free graphs for which $\kappa(G) \geq \alpha(G)$, but G is not pancyclic. These include the complete bipartite graphs, $K_{r,r}$, and the lexicographic product $G_i[\Gamma_s]$, where Γ_s is the complement of K_s.

Jackson and Ordaz [60] conjectured the following modification:

Conjecture 2.27. Let G be a k-connected graph with independence number $\alpha < k$. Then G is pancyclic.

However, Erdős proved that if the graph is sufficiently large, the Chvátal-Erdős condition is sufficient for pancyclicity:

Theorem 2.28. *(Erdős, 1966) [38] If $\kappa(G) \geq \alpha(G)$ and the order of G is greater than $4(\alpha + 1)^4$, then G is pancyclic.*

The Ramsey Theorem is needed in the statement of the next result pertaining to pancyclicity of graphs satisfying the Chvátal-Erdős condition:

Theorem 2.29. *(Ramsey, 1930 [81]) For every integer pair $k, m \geq 2$, there exists an integer $R(k, m)$ such that every graph of order $n \geq R(k, m)$ contains a clique on k vertices or an independent set of order m.*

This allows us to state the following result due to Flandrin et al., which was published in [43] and subsequently, again in [42]:

Theorem 2.30. *(Flandrin et al., 2004) [43] Let G be a k-connected graph with independence number α such that $\kappa \geq \alpha$ and $|G|>2R(4\alpha, \alpha + 1)$. Then G is pancyclic.*

Then, in 2012, Lee and Sudokov furnished yet another condition for which a graph satisfying the Chvátal-Erdős condition is pancyclic:

Theorem 2.31. *(Lee and Sudokov, 2012) [72] There exists an absolute constant c such that if G is a Hamiltonian graph with $n \geq ck^{\frac{7}{3}}$ vertices and $\kappa(G) \geq \alpha(G)$, then G is pancyclic.*

So this leads to the next question, posed by David Brown [24], extending Hendry's conjecture, whether the Chvátal-Erdős condition implies cycle-extendability in chordal graphs. We will consider that conjecture in the main result of this section.

2.4.1 Main Result

Recall: If a graph G has a non-Hamiltonian cycle C, and there exists a cycle C' such that $|V(C')| = |V(C)| + 1$, and $V(C) \subseteq V(C')$, then we say C can be extended. If every non-Hamiltonian cycle in a graph G can be extended, we say G is *cycle extendable*. We know that the following Hamiltonian graphs are cycle extendable: split graphs, spider graphs, and planar graphs. Moreover, we have proven in this chapter that 2-connected $K - 1, 3$ chordal graphs and 2-connected unit interval graphs are also cycle extendable. We have discussed a counterexample, which LeFond and Seamone introduced, proving

Hendry's conjecture that all Hamiltonian chordal graphs are cycle extendable is not true for a small subclass of chordal graphs.

At the time of this dissertation, it remains an open question. This question being, what conditions are sufficient for cycle extendability in chordal graphs? Our next result is another of our contributions to the research in this area.

Theorem 2.32. *Let G be a chordal graph on $n \geq 3$ vertices satisfying the modified Chvátal-Erdős condition, $\kappa(G) \geq \alpha(G)$. Then G is cycle extendable.*

Proof. Let C be a non-Hamiltonian cycle in G. Write $\kappa(G) = k$, then we can suppose $k \geq 2$. Clearly $|V(C)| \geq k$.

G is k-connected, hence, for any vertex $v \in V(G) - V(C)$, there is at least one edge from v to C. First suppose there is some $v \in V(G) - V(C)$, which is adjacent to at least k vertices on C. Label these vertices, as they occur clockwise along C, $\{u_1, u_2, \ldots, u_k\}$. If $|V(C)| = k$, the cycle can easily be extended. If $|V(C)| > k$, observe the following:

1. No two of these vertices u_i, u_{i+1} can be consecutive on the cycle, or we could extend C, by replacing the edge between them with the path $u_i v u_{i+1}$. Therefore, let $\{u_1^+, u_2^+, \ldots, u_k^+\}$, denote their immediate successors.
2. These vertices, $\{u_1^+, u_2^+, \ldots, u_k^+\}$, must be pairwise disjoint, or we could extend C to obtain a longer cycle $C' : u_i v u_j C^- u_i^+ u_j^+ C^+ u_i$ (see Fig. 2.11)
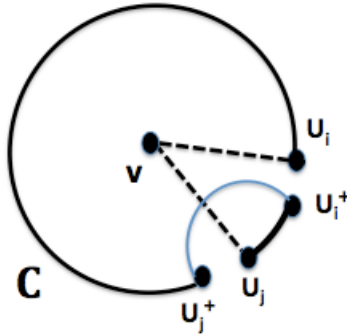

FIGURE 2.11: Extending the Cycle: There Are k Edges from v to C

This gives us an independent set, $\{u_1^+, u_2^+, \ldots u_k^+, v\}$, of size $k+1$, contradicting the hypothesis that $\alpha \leq k$. Therefore, if v has at least k edges to C, for a graph satisfying the modified Chvátal-Erdős condition, then every non-Hamiltonian cycle extends.

On the other hand, consider the case that there are k internally disjoint v, C-paths, not all of which are edges from v to a vertex on the cycle C (that is, some paths have length ≥ 2). Then we separate the vertices of $V(C)$ that are adjacent to v into two sets: Define $U = \{u_i \mid vu_i \in E(G)\}$ and for v, C-paths P_i where $|P_i| \geq 2$, define $W = \{w_i \mid w_i \in P_i \cap C\}$.

Then $U = \{u_1, u_2 \ldots, u_r\}$, and $W = \{w_1, w_2, \ldots, w_s\}$, where $r + s = k$. By assumption, the paths P_i are internally-disjoint, hence there is an independent set of vertices $\{p_1, p_2, \ldots, p_s\}$, where $p_i \in V(P_i)$, for $1 \geq i \geq s$. Also, since the successors of the vertices in U must be mutually disjoint (as in the preceding argument), this gives us an independent set of vertices $\{u_1^+, u_2^+, \ldots, u_r^+\}$, where u_i^+ is the successor of u_i on C, for $1 \leq i \leq r$. This would result in $r + s + 1 = k + 1$ independent vertices, $\{u_1^+, u_2^+, \ldots, u_r^+, \ldots, p_1, p_2, \ldots, p_s, v\}$ contradicting the hypothesis that $k \geq \alpha(G)$.

We must now only consider the possible case that any u_i^+ equals some w_i, that is, if the successor of some vertex u_i is the endpoint of another non-trivial v, C-path, P_i. If the path has length greater than 2, the choice of p_i can be made so that the vertices remain independent in our $(k+1)$-set $\{u_1^+, u_2^+, \ldots, u_r^+, p_1, p_2, \ldots, p_s, v\}$. However, if such a path contains only one internal vertex, then the vertices u_i^+ and p_i are no longer independent. However, since G is chordal, the induced cycle $u_i v p_i u_i^+ u_i$ must have a chord (see Fig. 2.12), giving us an extension of C, namely C': $C - u_i u_i^+ + \{p_i u_i^+ p_i u_i\}$.

Hence any non-Hamiltonian cycle can be extended. □

2.5 Elimination Orderings in Graphs, and $\{1, 2\}$-Cycle Extendability

In a seminal paper on the subject, Jamison and Lascar [63] defined and related different vertex elimination schemes for graphs. In particular, we mention three here: the perfect elimination ordering (PEO), which we already discussed as being intrinsic to chordal graphs; the interval elimination ordering (IEO), which is characteristic of
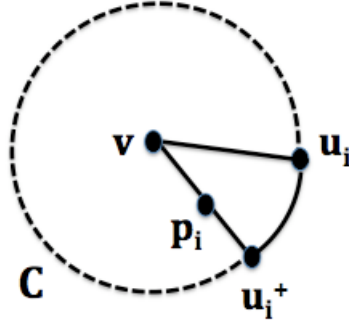
FIGURE 2.12: Successor of a v, C-edge Is the Endpoint of a v, C-Path

interval graphs (a subclass of chordal graphs); and the Hamiltonian elimination ordering (HEO), which Jamison and Lascar showed is a special sub-type of interval elimination ordering. We study these orderings because they characterize classes of chordal graphs, and we have identified the Hamiltonian elimination ordering as one that guarantees cycle extendability. This gives us another result to contribute to the open question of which classes of chordal graphs satisfy Hendry's conjecture.

Recall, a *perfect elimination ordering* (PEO) is an ordering of the vertices of a graph $R = v_1 v_2 \ldots v_n$ such that for each vertex v_i, the neighbors of v_i that follow v_i in the ordering induce a clique. (Note that we have also referred to this as a *simplicial* elimination ordering.)

Recall that in 1974, Buneman [25] established that a graph having a PEO is chordal, and vice versa.

In an ordering $v_1 v_2 \ldots v_n$, if vertex v_i precedes vertex v_j in the ordering, we write $v_i < v_j$. That is, $v_1 < v_2$, etc.

Jamison and Lascar [63] define an ordering of the vertices of a graph to be an *interval elimination ordering* (IEO) if and only if for all vertices v_i, v_j, v_k with $v_i < v_j < v_k$, if $v_i v_k \in E(G)$ then $v_j v_k \in E(G)$. The existence of such an ordering characterizes interval graph [63].

Now we define the HEO, the ordering and property we show guarantees cycle extendability.

Definition 2.33.

- A graph G has a *Hamiltonian elimination ordering* of its vertices, (denoted HEO), if $\{v_1, v_2, \ldots, v_n\}$ is a perfect elimination ordering, with v_i adjacent to v_{i+1} for all $1 \leq i \leq n-1$.

- A graph having a Hamiltonian elimination ordering, will be referred to as an *HEO graph*.

Again, since a graph is chordal if and only if it has a PEO, every HEO graph is chordal, and since interval graphs are chordal graphs, we have the following relationship among the orderings, illustrated in Fig. 2.13:

$$\text{HEO} \longrightarrow \text{IEO} \longrightarrow \text{PEO}$$
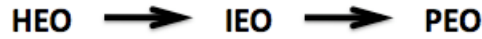
FIGURE 2.13: Elimination Orderings in Increasing Strength

As an example, for practice with the definitions, we observe the distinction among the orderings in the following graph [63]:
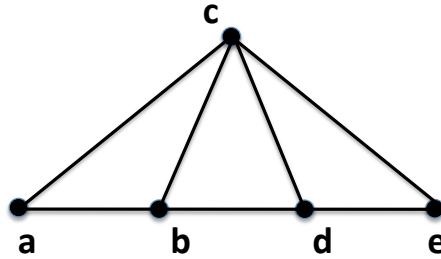

FIGURE 2.14: Distinquishing the Elimination Orderings

i) a, b, d, e, c is an HEO

ii) a, b, e, d, c is an IEO but not an HEO

iii) a, e, b, c, d is a PEO but not an IEO.

Theorem 2.34. *Any 2-connected graph G having an HEO is cycle extendable.*

Proof. Suppose G is a 2-connected graph with a Hamiltonian elimination ordering $H = \{v_1, v_2, \ldots, v_n\}$. Let C be a non-Hamiltonian cycle. Suppose v_k is the first vertex in the ordering H, that lies on C. First, let us suppose that v_k is not simplicial in G. Apply Lemma 2.13, which gives us an extension of C.

Otherwise, if v_k is simplicial, consider v_{k-1}, the vertex preceding v_k in the ordering. By definition of an HEO, $v_k v_{k-1} \in E(G)$, hence $v_{k-1} \in N(v_k)$. But v_k is simplicial, hence the neighborhood of v_k is complete, therefore $v_{k-1} v_k^+ \in E(G)$, where v_k^+ is the immediate successor of v_k on the cycle, giving us an extension of the cycle, $C' = C - v_k v_k^+ + v_k v_{k-1} v_k^+$. (See Fig. 2.15 for an example of an HEO graph.) $\qquad\square$
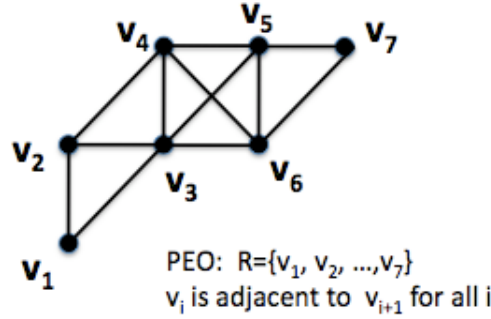
FIGURE 2.15: Example of Theorem 2.34

Applying Theorem 2.34 recursively, we obtain a cycle that spans the vertex set of the graph. This is stated in the following:

Corollary 2.35. *A 2-connected graph having an HEO is Hamiltonian.*

We present two questions, which present new or continuing lines of research. Since every HEO is an IEO, can the vertex ordering properties give us an alternate approach to proving that a Hamiltonian interval graph is cycle extendable? Since any chordal graph with an HEO is cycle extendable, but not all chordal graphs are cycle extendable, we also ask: what describes the class of chordal graphs which are cycle extendable but

have no HEO? or can Theorem 2.34 be made into a characterization of all 2-connected cycle extendable graphs?

Recall, in Proposition 5.9, Beasley and Brown described the kind of chordal graphs that would not be cycle extendable. From that description of what a non-extendable chordal graph must look like, Beasley and Brown introduced the more general concept of *S-cycle-extendability*.

From this, we obtain the following important conjecture:

Conjecture 2.36. (Arangno, Beasley, Brown, 2014) Let G be a Hamiltonian chordal graph. Then G is $\{1, 2\}$-cycle-extendable.

CHAPTER 3

NEW RESULTS: HAMILTONICITY

3.1   Generalizing Classical Theorems

3.1.1   Generalizing Dirac's Condition

Shi [87] generalized Dirac's classical theorem by proving a 2-connected graph G of order n contains a cycle which passes through all vertices of degree $\geq n/2$.

We will prove the preceding result of Shi, following a different argument, which is similar to that used by Yamashita [93] in deriving the following result: If G is a 2-connected graph and $max\{d(x) + d(y)|x, y \in S\} \geq d$ for every independent set $S \subset V(G)$ of order $k + 1$, then G has a cycle of length at least $min\{d, |V(G)|\}$.

We will use Menger's Theorem, and Bondy's Lemma, stated as follows.

Theorem 3.1. *(Menger, 1927) [75] A minimum x, y separating set, for any $x, y \in V(G)$, equals the maximum number of disjoint x, y-paths in the graph G.*

The global version of Menger's Theorem, is that a graph is k-connected if and only if it contains k edge-disjoint paths between any two distinct vertices (Whitney [92]).

We will also need :

Lemma 3.2. *(Bondy, 1971) [17] Let G be a graph on n vertices and $X \subseteq V(G)$. If C is a cycle which contains as many vertices of X as possible, and xPy is a path such that $|V(xPy) \cap X| > |V(C) \cap X|$, then $d(x) + d(y) < n$.*

Now we are ready to prove our theorem:

Theorem 3.3. *Let G be a 2-connected graph on n vertices and $X \subseteq V(G)$. If $\delta(x) \geq n/2$ for all $x \in X$, then there exists a cycle that spans X.*

Proof. Suppose C is a cycle of G that contains as many vertices of X as possible. If $V(X) \subseteq V(C)$, we're done. Otherwise, suppose there is a vertex $x_0 \in X \cap (V(G) - V(C))$. Let $k = \kappa(X)$, $k \geq 2$. Then by Menger's Theorem, there exist k internally disjoint paths from x_0 to C. Label these paths, $x_0 P_i v_i$, for distinct vertices v_i on C, $1 \leq i \leq k$. Suppose these vertices appear consecutively along C as $v_1, v_2, \ldots, v_k$ ($k \geq 2$). Let x_i be the first vertex of X following v_i on C for each i, $1 \leq i \leq k$. Then $P_{0,i}$: $x_0 P_i v_i C^- x_i$ is an x_0, x_i-path such that $|V(P_{0,i}) \cap X| > |V(C) \cap X|$ (see Fig. 3.1). Hence, by Bondy's Lemma, Lemma 3.2, $d(x_0) + d(x_i) < n$. There exist k such x_0, C-paths.

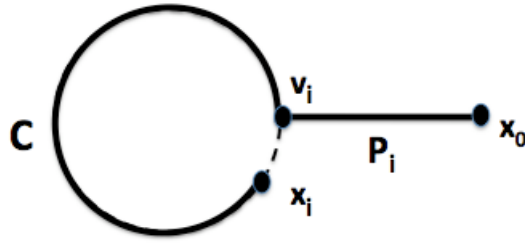FIGURE 3.1: Generalizing Dirac's Equation: Defining an x_0, x_i-Path

Also, define $P_{i,j}$: $x_i C^+ v_j P_j x_0 P_i v_i C^- x_j$, to be an $x_i x_j$-path, for any two vertices $x_i, x_j \in X$ on C, for $1 \leq i < j \leq k$, such that $|V(P_{i,j}) \cap X| > |V(C) \cap X|$ (see Fig. 3.2). Again, by Bondy's Lemma, $d(x_i) + d(x_j) < n$.
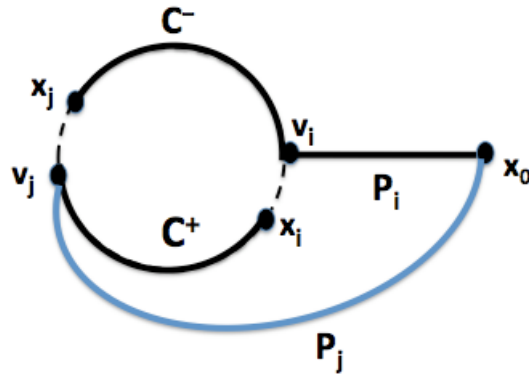

FIGURE 3.2: Generalizing Dirac's Equation: Defining an x_i, x_j-path

This gives us $d(x_i) + d(x_j) < n$ for all i, j such that $0 \leq i < j \leq k$, contradicting the hypothesis that $d(x_i) \geq n/2$ for $x_i \in X$. Hence the cycle C must include all the vertices of X. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.1.2 Generalizing Ore's Condition

We now generalize Ore's Condition, Theorem 1.18, that $d(x) + d(y) \geq n$ for each pair of non-adjacent vertices x, y in G implies that G is Hamiltonian. Our proof will follow the approach used by Shi in proving a 2-connected graph G of order n contains a cycle which passes through all vertices of degree $\geq n/2$. Our proof via contradiction will use a maximal cycle, and a minimal path.

Theorem 3.4. *Let G be a 2-connected graph on n vertices. Then G contains a cycle passing through all pair-wise non-adjacent vertices whose degree sum is at least n.*

Proof. Suppose the theorem is false. Let $S = \{x \mid \exists y \text{ s.t. } xy \notin E(G), d(x) + d(y) \geq n\}$, and let $C = a_1 a_2 \ldots a_k$ be a cycle containing as many vertices from S as possible. Denote $H = V(G) - C$. Since G is 2-connected, there exists a path connecting two vertices of C, internally disjoint from C, which contains a vertex x in S. Let P be a shortest such path. Without loss of generality, let $a_1, a_q \in V(P) \cap V(C)$. There exists a vertex $a_p \in C$ where $1 < p < q$, such that $a_p \in S$, but for all $1 < i < p$, $a_i \notin S$.

We get the following two cases:

Case 1) $p > 2$

By the minimality of P and the maximality of C, the following sets are pairwise disjoint:

$N_C(a_p)$; $N_C(x)$ - $\{a_q\}$; $N_H(a_p)$; $N_H(x)$; $\{x, a_q\}$

Therefore $\quad n \geq d_C(a_p) + (d_C(x) - 1) + d_H(a_p) + d_H(x) + 2$

$\geq d(a_p) + d(x) + 1,$ which implies

$n \geq n + 1$, contradiction.

Case 2) $p = 2$

The following sets are pairwise disjoint:

$N_C(a_p)$; $N_C(x)$; $N_H(a_p)$; $N_H(x)$; $\{x\}$, giving us a similar contradiction. □

The graph in Fig. 3.3 provides a *sharpness example*, meaning that in some sense, the result cannot be improved. As illustrated, we see that G is a 2-connected graph on n vertices, split between a clique $K = K_{\frac{(n-1)}{2}}$, and an independent set $S = \bar{K}_{\frac{(n+1)}{2}}$, and since any pair of vertices x, y in S are non-adjacent, they have degree sum $d(x) + d(y) < n$. However, G does not have a cycle that contains all the vertices of S.
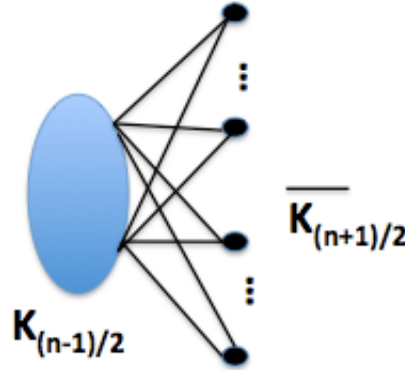
FIGURE 3.3: Sharpness Example: Generalization of Ore's Theorem

Corollary 2. Ore's condition for Hamiltonicity.

3.2 A New Minimum Degree Condition for Hamiltonicity

A graph G is locally connected if for every vertex $v \in V(G)$, the neighborhood of v is connected. Recall, a graph is Hamiltonian if it contains a spanning cycle.

In this paper, we will establish a sufficient condition for Hamiltonicity in 2-connected $K_{1,3}$-free graphs with a given minimum degree.

The principal result we will be using is Oberly and Sumner's result:

Lemma 1.47 Any 2-connected locally connected $K_{1,3}$-free graph is Hamiltonian [78].

3.2.1 Fundamentals

To obtain our next result, we need to prove the following proposition of Chartrand and Pippert [26].

Proposition 3.5. *Let G be a graph of minimum degree $\delta(G) > \frac{2}{3}(n-1)$. Then G is locally connected.*

Proof. Suppose $\delta(G) > \frac{2}{3}(n-1)$, but there exists a vertex x whose neighborhood $N(x)$ is not connected. Let y be a neighbor of x in a smallest component H_y of $N(x)$, and let $|H_y| = m_1$. Define $m_2 = |N(x)| - m_1$. Then $deg(x) = m_1 + m_2$.

Let $k = n - (m_1 + m_2 + 1)$. That is, let k be the number of the vertices in $G - (N(x) \cup \{x\})$.

Since y is not adjacent to any vertex of $N(x)$ not in H_y, it must be that $deg(y) < n - m_2 - 1$.

But by hypothesis, $deg(x) + deg(y) > \frac{2}{3}(n-1) + \frac{2}{3}(n-1)$, and so

$$deg(y) > \frac{4}{3}(n-1) - (m_1 + m_2). \text{ Hence, } n - m_2 - 1 > \frac{4}{3}(n-1) - m_1 - m_2, \text{ and so}$$

$$m_1 > \frac{1}{3}(n-1). \text{ But, by the choice of } y, \quad m_2 \geq m_1, \text{ and therefore}$$

$$m_2 > \frac{1}{3}(n-1), \text{ and}$$

$$k < \frac{1}{3}(n-1).$$

Let $z \in N(x) - H_y$, that is, a vertex in the neighborhood of x not in the component containing y. Then $deg(z) \leq |N(x)| - m_1 + k = m_2 + k$. Thus:

$deg(y) + deg(z) \leq (m_1 + k) + (m_2 + k) = (n-1) + k < \frac{4}{3}(n-1)$, contradicting the hypothesis. Therefore, it must be that G is locally connected. $\square$

3.2.2 Minimum Degree Result

Now we can prove the following theorem:

Theorem 3.6. *Any 2-connected $K_{1,3}$-free graph of order $n > 4$, with minimum degree greater than $\frac{2}{3}(n-1)$ is Hamiltonian.*

Proof. Since we proved that any graph of order n with $\delta(G) > \frac{2}{3}(n-1)$ is locally connected, then by Oberly and Sumner's lemma 1.47, the graph must be Hamiltonian. $\square$

The split graph in Fig. 3.4 illustrates this result. For example, let $n = 10$. Then G is split between the clique $K = K_6$ and the independent set $S = \bar{K}_4$. Note that G is $K_{1,3}$-free. Clearly, since $|S| < |K|$, G is Hamiltonian.
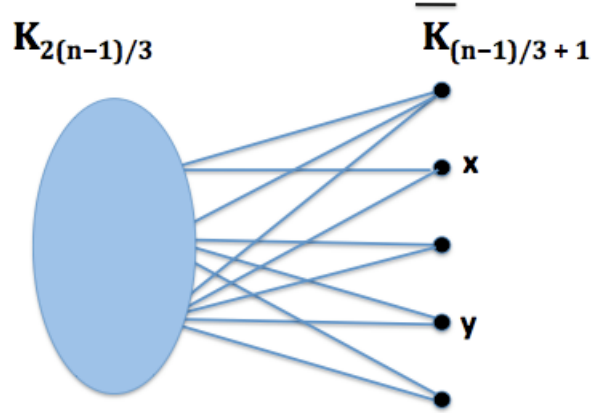

FIGURE 3.4: Example: Generalization of Ore's Theorem

CHAPTER 4

NEW RESULTS: BIPARTITE HAMILTONIAN GRAPHS

In 1962, Pósa [80] introduced a new direction in the study of Hamiltonian graphs, by examining the conditions under which specified edges are traversed. Later, Kronk [70] expanded on Pósa's results, by giving conditions for which every path of length not exceeding $k \leq n - 2$, is contained in a Hamiltonian cycle of a graph G of order n.

In contrast, Harris et al. [54] studied the the conditions under which a graph has a Hamiltonian cycle which *avoids* a specified set of edges. Such a graph is called "edge-avoiding" Hamiltonian. We expand on those results by studying bipartite graphs, and determine the conditions under which a bipartite graph has a Hamiltonian cycle which avoids a specified set of edges.

The results in [54] were obtained mainly by applying both the Bondy-Chvàtal Theorem (Theorem 1.20), and the concept of the *closure* of a graph.

Here, we define the *bipartite closure* of a bigraph, and prove a bipartite version of the Bondy-Chvàtal Theorem; namely, a graph is Hamiltonian if and only if its *bipartite closure* is Hamiltonian.

Our results on *edge-avoiding* Hamiltonicity in a bipartite graph G are obtained by removing specified edges E' to obtain a graph G', and proving the bipartite closure of G' is Hamiltonian. This yields a Hamiltonian cycle in G that doesn't use any edge of E', establishing "edge-avoiding" Hamiltonicity.

Note that for a bipartite graph to be Hamiltonian, its partite sets must have equal size. That is, given $G = (X, Y, E)$, if $|X| \neq |Y|$, then G cannot be Hamiltonian. Therefore, all bipartite graphs discussed here are *balanced*, that is, $|X| = |Y|$.

4.1   Hamiltonicity in Bipartite Graphs

In 1963, Moon and Moser [76] proved that if $G = (X, Y, E)$ is bipartite with order $2n$, and $\sigma_2^2(G) \geq n + 1$, where $\sigma_2^2(G) = min\{d(x) + d(y)\colon xy \notin E(G), x \in X, y \in Y\}$, then G is Hamiltonian.

The proof followed that used by Ore in the non-bipartite case, using a maximal counter-example, in which a contradiction resulted from the degree requirements on the end vertices of any given Hamiltonian path. We can use the same reasoning to prove the bipartite case of Bondy's famous theorem:

Lemma 4.1. *Given a bipartite graph $G = (X, Y, E)$ on $2n$ vertices, and $d(x) + d(y) \geq n + 1$ for any pair of non-adjacent vertices $x \in X$, $y \in Y$, then G is Hamiltonian if and only if $G + xy$ is Hamiltonian.*

Proof. ($\Rightarrow$)   If G is Hamiltonian, then $G + xy$ is Hamiltonian, for any edge xy, where $x \in X$, $y \in Y$.

($\Leftarrow$) suppose $G + xy$ is Hamiltonian. Delete xy, to obtain a graph G, which contains a Hamiltonian path $x = x_1, y_1, x_2, \ldots, x_n, y_n = y$. By Moon and Moser, since $d(x) + d(y) \geq n + 1$, then G is Hamiltonian.   □

If we do this recursively, for all non-adjacent vertex pairs x, y, where $x \in X$, $y \in Y$, such that $d(x) + d(y)) \geq n + 1$, we can define the bipartite closure, $bcl(G)$.

Definition 4.2. The *bipartite closure* of $G = (X, Y, E)$, denoted $bcl(G)$, is the graph with vertex set $V(G)$ obtained by iteratively adding edges between pairs of non-adjacent vertices from opposite partite sets whose degree sum is, or becomes, at least $n + 1$, until no such pair exists.

Lemma 4.3. *The bipartite closure of a bipartite graph $G = (X, Y, E)$ is well-defined.*

Proof. Let $S = (e_1, e_2, \ldots, e_r)$ be a sequence of edges added to G to form $bcl(G)$, and let $G_1 = G + \{e_1, e_2, \ldots, e_r\}$.

Also, suppose $S' = (f_1, f_2, \ldots, f_s)$ is a different sequence of edges added to G to form $bcl(G)$, which yields $G_2 = G + \{f_1, f_2, \ldots, f_s\}$. Note that if in sequence G_1, the non-adjacent vertices x and y acquire degree sum at least $n + 1$, then the edge xy must belong to the other sequence, G_2, and vice versa. Therefore, since f_1 can be added to G, it must also be in G_1. Assume f_i is the first edge of S' omitted in S; but f_i joins vertices whose degree sum is at least $n + 1$ and so these vertices must be adjacent in G_1 as well. Therefore there is no first edge of S' omitted by S. $\qquad\square$

Next, we obtain the following bipartite version of the Bondy-Chvátal Theorem, from which we further obtain a minimum degree condition for when the closure of a bipartite graph is complete.

Lemma 4.4. *If $G = (X, Y, E)$ is a bipartite graph on $2n$ vertices, then G is Hamiltonian if and only if $bcl(G)$ is.*

Proof. ($\Rightarrow$) First, suppose G is Hamiltonian. Then the addition of edges to obtain the bipartite closure does not destroy the Hamiltonian cycle. Therefore $bcl(G)$ is Hamiltonian.

($\Leftarrow$) Conversely, suppose $bcl(G)$ is Hamiltonian. Delete any edge xy. G has a Hamiltonian path, with non-adjacent vertices x and y having degree sum at least $n + 1$, hence, by Moon and Moser [76], $bcl(G) - xy$ is Hamiltonian. If $bcl(G) - xy$ is the graph G, we are done. If not, we continue to do this recursively until we obtain the graph G. Therefore $bcl(G)$ is Hamiltonian implies G is. $\qquad\square$

As advertised, the following corollary is immediate.

Corollary 3. *If G is a bipartite graph with minimum degree $\delta(G) \geq \frac{(n+1)}{2}$, then $bcl(G)$ is a biclique.*

Proof. Since $\delta(G) \geq \frac{(n+1)}{2}$, all pairs of non-adjacent vertices in different partite sets have degree sum at least $(n + 1)$, making them adjacent in the closure, hence all the vertices of G are adjacent in $bcl(G)$. $\qquad\square$

4.2 Edge-Avoiding Hamiltonicity in Bipartite Graphs

Now we can consider a Hamiltonian bipartite graph in which a specified set E' of edges, is removed. We shall establish conditions under which there remains a spanning cycle in the graph which uses none of the edges we removed. Then the graph will be referred to as *edge-avoiding Hamiltonian.*

Theorem 4.5. *Let G be a balanced bipartite graph of order $2n \geq 10$, and min degree $\delta(G) \geq \frac{3}{4}(n-1)$. If E' is any subset of $E(G)$ such that $|E'| < \frac{(n-3)}{2}$, then there exists a Hamiltonian cycle in G containing no edge from E'.*

Proof. Since $\delta(G) \geq \frac{3}{4}(n-1) \geq \frac{1}{2}(n+1)$ for all $n \geq 5$, it is clear that G has a Hamiltonian cycle. But we wish to prove there exists a Hamiltonian cycle which uses no edge of the given set E', as long as $|E'| < \frac{(n-3)}{2}$.

Define $G' = G - E'$; by showing $bcl(G')$ is Hamiltonian, we can invoke the bipartite version of Bondy-Chvàtal to establish G' is Hamiltonian, then G is E'-avoiding Hamiltonian.

Let H be the subgraph induced by the edges of E'. Let $V(H)$ denote the vertices of H. Then $|V(H)| < (n-3)$. However, we can improve this upper bound. $G-H = (X', Y')$, where $V(X') = \{x \in V(X) - V(H)\}$, and $V(Y') = \{y \in V(Y) - V(H)\}$. In the graph G', the minimum degree of $V(H)$ is at least $\frac{3}{4}(n-1) - \Delta(H)$, since the vertices of $V(H)$ are affected by the removal of the edges of E'. Whereas, the vertices in $G - V(H)$ are unaffected by the removal of the edges of E', hence their minimum degree in G' remains $\frac{3}{4}(n-1)$. Moreover, $\frac{3}{4}(n-1) \geq \frac{(n+1)}{2}$, for all $n \geq 5$. Therefore, $G - V(H)$ forms a biclique in $bcl(G')$, by Corollary 3 to Lemma 4.4. Let v be a vertex of maximum degree in H, that is, $deg(v) = \Delta(H) < (n-3)/2$.

Then:

Given $v \in V(H)$, and $u \in G - V(H)$, where v and u are in opposite partite sets, we get:

$d_{G'}(v) + d_{G'}(u) > (\frac{3}{4}(n-1) - \Delta(H)) + \frac{3}{4}(n-1)$

$\geq \frac{(3n-3)}{2} - \Delta(H)$

$> \frac{(3n-3)}{2} - \frac{(n-3)}{2}$

$\geq n+1.$

Hence, by the bipartite version of Bondy-Chvàtal (Lemma 4.4), u and v are adjacent in $bcl(G')$, which contains the biclique $G - V(H)$ joined with the independent set of vertices $V(H)$. And, since $|V(H)| \leq |G - V(H)|$, this implies $bcl(G')$ is Hamiltonian. Therefore G' is Hamiltonian, and G is E'-avoiding Hamiltonian, as desired. □

To illustrate this result, let G be a bipartite graph on $2n$ vertices, where $n = 5$, then the hypothesis is satisfied if $\delta(G) \geq \frac{3}{4}(n-1) = 3$ and the edge set removed has order $|E'| < \frac{(n-3)}{2} = 1$. But if one or more edges are removed, the graph is no longer Hamiltonian.

4.3 F-Avoiding Hamiltonicity in Bipartite Graphs

Now we consider a Hamiltonian bipartite graph in which there exists a subgraph isomorphic to a specified graph F, which is removed. We shall establish conditions under which a spanning cycle remains, which uses none of the edges from the removed subgraph, remains. Then the graph is F-avoiding Hamiltonian.

Theorem 4.6. *Let G be a balanced bipartite graph of order $2n \geq 10$, and F be a bipartite graph of order $t \leq n-1$ and max degree not greater than k. If $\sigma_2^2(G) \geq n+k+1$, then G is F-avoiding Hamiltonian.*

Proof. Let $H \cong F$. Define $G' = G - E(H)$; by showing $bcl(G')$ is Hamiltonian, we can invoke Bondy-Chvátal to establish G' is Hamiltonian, then G is F-avoiding Hamiltonian.

Define $G - H = (X', Y', E')$, where $E' = E(G) - E(H)$, and $V(X') = (V(X) - V(H))$, and $V(Y') = (V(Y) - V(H))$.

Then $|G - V(H)| = 2n - t$. Denote $\Delta(H) = k$. Let $v \in G - V(H)$. Hence $d_{G'}(v) = d_G(v)$. Let w be any vertex in G not adjacent to v, where w and v are in opposite partite sets. Then $d_{G'}(w) \geq d_G(w) - k$, since we removed at most k edges from any vertex by removing $E(H)$. We now have the following:

$$d_{G'}(w) + d_{G'}(v) \geq d_G(w) + d_G(v) - k$$

but $d_G(w) + d_G(v) \geq n + k + 1$, by hypothesis, and so

$$d_{G'}(w) + d_{G'}(v) \geq n + 1.$$

Thus, wv is an edge in $bcl(G')$.

Moreover, since this is true for every vertex in $G - V(H)$, the vertices of $G - V(H)$ are mutually adjacent in $bcl(G')$, and they are each adjacent to every vertex in $V(H)$. Hence the closure of G' contains a biclique joined to an independent set S of order t. And since $t \leq n - 1$, this implies $t < (2n - t)$, that is, $|S| < |G - V(H)|$, hence $bcl(G')$ is Hamiltonian. By the bipartite version of Bondy-Chvtal, Lemma 4.4, this implies G' is Hamiltonian, therefore G is F-avoiding Hamiltonian, as desired. □

As illustrated in Figure 4.1, the graph G has degree $2n = 10$. The maximal F has order 4 and max degree 2, which is a 4-cycle, marked in red. In the first bipartite graph below, $\sigma_2^2(G) \geq 8$, satisfying the hypothesis of Theorem 4.6, we can trace a Hamiltonian cycle, spanning the vertex set of G, while avoiding the red cycle.

In contrast, consider the second bipartite graph shown in Figure 1. Here, $\sigma_2^2(G) = 7$, which fails to satisfy the conditions of the theorem, and we discover that there is no Hamiltonian cycle that avoids the red edges.
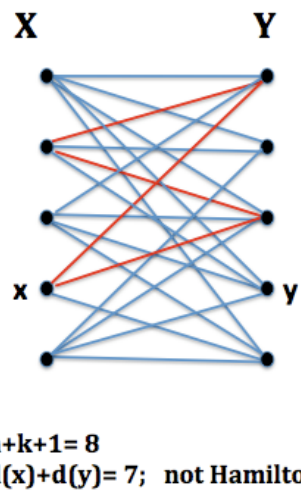

n+k+1= 8
min{d(x)+d(y) | x,y non adjacent} ≥ 8
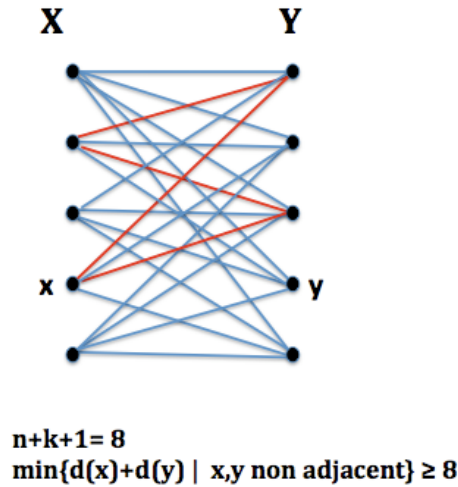
FIGURE 4.1: Sharpness Example: F-Avoiding Hamiltonicity

CHAPTER 5

DIRECTIONS FOR FURTHER STUDY

In this final chapter we will summarize some key findings of the previous chapters in context of potential for further study.

5.1 Cycle Extendability

In 1974, Fleischner [44] provided an elegant proof that the square of any 2-connected graph, G, is Hamiltonian (where G^2, the square of G, is the graph on $V(G)$ in which two vertices are adjacent if and only if they have distance at most 2 in G). This naturally leads to questions whether the square of a graph is pancyclic, or cycle extendable.

Secondly, Zamfirescu [95] defined a graph G to be Hamiltonian if and only if there exists a family of cycles $F=\{C_1, C_2, \ldots C_n\}$ such that:

- every vertex of G is in at least one cycle in F;
- the intersection-like graph F^* is a tree, where $V(F^*)=F$, and $C_1 C_2 \in E(F^*)$ if and only if the cycles C_1, C_2 share exactly one edge;
- the intersection graph $\Omega(F)$ is a tree, where each C_i is considered a subset of vertices of $V(G)$).

Using this definition of a Hamiltonian graph, what properties can be described for different graph classes? does this give us different methods for proving results about pancyclicity or cycle extendability for chordal graphs, for example?

Similarly, Scheinerman [83] defined a Hamiltonian cycle in a graph G, to be a subest $F \subset E(G)$, for which:

- $|F|= n$;

- every edge cut $[S,\bar{S}]$ contains at least 2 edges of F.

Again, we might gain new insights from this perspective, and discover conditions for cycle structure in graphs using this alternate definition.

But most significantly, the consequences of Ore's Condition have been seen to tie together Hamiltonicity, pancyclicity, and cycle extendability in graphs. This is recapitulated in the following:

Theorem 5.1. *Let G be a graph of order $n \geq 3$ such that $\sigma_2(G) \geq n$, then:*

1. *(Ore) [79]: G is Hamiltonian;*
2. *(Bondy) [18]: G is pancyclic unless p is even and $G = K_{\frac{n}{2},\frac{n}{2}}$;*
3. *(Bondy) [18]: if C is a non-Hamiltonian cycle, then there exists a cycle C' in G such that $V(C) \subset V(C')$ and $|V(C')| = |V(C)|+i$, where $i = 1$ or 2.*

In 1982, Schmeichel and Mitchem [86] proved that the Ore-type condition established by Moon and Moser for Hamiltonicity in bipartite graphs, was sufficient for bipancyclicity.

In 1991, Hendry [58] expanded this, to prove the same condition was also sufficient for bi-cycle extendability. Specifically, Hendry proved if a balanced bipartite graph G of order $2n$ satisfied the condition that for all non-adjacent vertices $x \in X$ and $y \in Y$ such that $d(x) + d(y) \geq n + 1$, then every non-Hamiltonian cycle C could be extended (unless G is one of a class of exceptional graphs, too technical to define here, see [58]).

Hendry defined a bipartite graph to be bi-cycle extendable as follows:

Definition 5.2. A bipartite graph $G = (X, Y)$ is *bi-cycle extendable* if G contains a cycle, and given any non-Hamiltonian cycle C, there exists a cycle C' in G such that $V(C) \subset V(C')$ and $|V(C')| = |V(C)|+2$.

This gives us a similar summary of consequences of the Ore-type Condition for bipartite graphs, relating Hamiltonicity, bipancyclicity, and bi-cycle extendability, as follows:

Theorem 5.3. *Let $G = (X, Y)$ be a balanced bipartite graph of order $2n \geq 4$ such that $\sigma_2^2(G) \geq n + 1$, then:*

- *G is Hamiltonian (Moon and Moser) [76]*

- *G is bipancyclic (if G is not one of exceptional graphs) (Schmeichel, Mitchem) [86]*

- *G is bi-cycle extendable. (Hendry) [58]*

In Chapter 2, we looked at new conditions for cycle extendability in graphs. This leads us to wonder if analogous conditions also result in bi-cycle extendability in bigraphs.

We suggest the following conjectures.

Conjecture 5.4. Let G be a connected bipartite graph on $2n$ vertices with a Hamiltonian cycle $x_1, y_1, \ldots x_n, y_n, x_1$. If $dist(x_i, x_j) = 2$ implies $max\{d(x_i), d(x_j)\} \geq \frac{(n+1)}{2}$, then:

- G is bipancyclic

- G is bi-cycle extendable.

Conjecture 5.5. Let G be a connected bipartite graph on $2n$ vertices with a cycle C of length $n - 2$, and a vertex $v \notin V(C)$ such that $d(v) \geq \frac{(n+1)}{2}$, then G is bipancyclic.

Conjecture 5.6. Let G be a Hamiltonian bipartite graph with vertices $x_1 \in X$ and $y_n \in Y$ on a Hamiltonian cycle $x_1, y_1, \ldots x_n, y_n, x_1$ such that $d(x_1) + d(y_n) \geq n + 1$, then G is bipancyclic.

Conjecture 5.7. Let G be a 2-connected bipartite graph on $2n$ vertices, and let S be a subset of $V(G)$. If $d(x) + d(y) \geq n + 2$ for all vertices $x \in X \cap S$ and $y \in Y \cap S$ such that $dist(x, y) = 3$, then there is a cycle that spans S.

5.2   Chordal Graphs

We looked at a number of theorems dealing specifically with chordal graphs, in responding to Hendry's question whether Hamiltonian chordal graphs were cycle extendable.

As we saw, this constituted a challenge that invigorated much research activity in the study of cycles. In particular, a challenge to see which conditions that proved sufficient for Hamiltonicity might also prove sufficient for cycle extendability, in the special case of chordal graphs. This suggests the following conjecture:

Conjecture 5.8. Let G be a 2-connected chordal graph satisfying Fan's Condition for Hamiltonicity, then G is cycle extendable.

Next, we consider the bipartite case. Golumbic and Goss [48] defined a bipartite graph to be *chordal bipartite* if each cycle of length greater than or equal to 6 has a chord.

In the same manner that Jamison [62] characterized chordal graphs as graphs in which every k-cycle is the sum of $(k-2)$ 3-cycles, (where cycles will be viewed as sets of edges with the sum of cycles meaning their symmetric difference). McKee [74] defined a graph to be chordal bipartite if and only if every k-cycle is the sum of $(\frac{k}{2} - 1)$ 4-cycles.

Dalhaus et al. [34] defined a chordal graph to be strongly chordal if and only if for every $k \geq 6$, every k-cycle has a 2-chord, (which is, there are two chords of the cycle, forming a triangle with one edge of the cycle). McKee proved this is equivalent to saying a graph is strongly chordal if and only if every k-cycle C is the sum of $(k-2)$ 3-cycles, each of which contains an edge of C, and furthermore, that a graph is strongly chordal bipartite if every k-cycle C is the sum of $(\frac{k}{2}-1)$ 4-cycles, each of which contains an edge of C (see Fig. 5.1).

Beasley and Brown observed the following [12]:

Proposition 5.9. *If G is a chordal bipartite Hamiltonian graph on n vertices with a non-extendable cycle C, and contains no 8-fans (that is, induced subgraphs consisting*
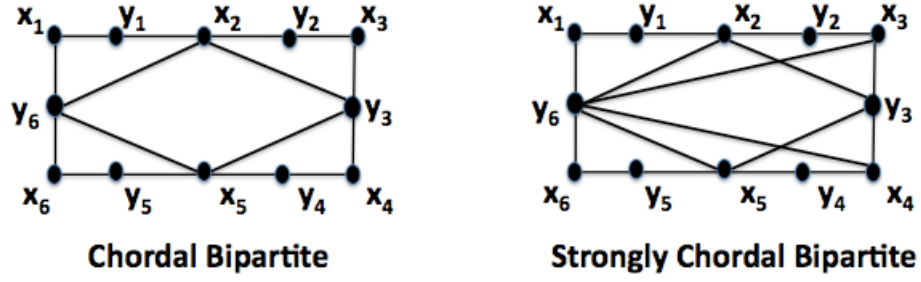
FIGURE 5.1: Chordal and Strongly Chordal Bipartite Graphs

of the cycle $<v_1, v_2,\ldots v_8, v_1>$ with chords v_1v_i for $i = 1, 2\ldots,6$), and if G is vertex-minimal with respect to these properties, then $|V(G)|=|V(C)|+4$, or $G - C$ contains no 4-cycle with a 2-path in common with any Hamiltonian cycle.

With this in mind, we pose the following questions:

1. What conditions are sufficient for a chordal Hamiltonian bipartite graph to be bi-cycle extendable?

2. What conditions guarantee a strongly chordal bipartite graph is bi-cycle extendable?

3. Can the results of this thesis be extended or applied to these questions, as we believe they can?

These questions certainly provide more opportunities for further study.

BIBLIOGRAPHY

[1] A. Abueida and R. Sritharan. Cycle extendability and Hamiltonian cycles in chordal graph classes. *Discrete Math.*, 20(3):669–681, 2006.

[2] A. Abueida, R. Sritharan, and A. Busch. Hamiltonian spider intersection graphs are cycle extendable. *SIAM J. Discrete Math.*, 27:4:1913 – 1923, 2013.

[3] D. Amar. Applying a condition for a Hamiltonian bipartite graph to be bipancyclic. *Discrete Math.*, 111:19–25, 1993.

[4] D. Amar, E. Flandrin, I. Fournier, and A. Germa. Pancyclism in Hamiltonian graphs. *Discrete Math.*, 89(2):111–131, 1983.

[5] D. Amar, I. Fournier, and A. Germa. Pancyclism in Chvátal-Erdös graphs. *Graphs Combin.*, 7:101–112, 1991.

[6] D. L. Applegate, R. E. Bixby, V. Chvàtal, and W. J. Cook. *The Traveling Salesman Problem: A Computational Study.* Princeton University Press, 2007.

[7] A.S. Asratian. Some properties of graphs with local Ore conditions. *Ars Combinatoria*, 41:97–106, 1995.

[8] A.S. Asratian and N.K. Khachatrian. Some localization theorems on Hamiltonian circuits. *J. Combin. Theory*, (Ser B 49):287–294, 1990.

[9] R. Balakrishnan and P. Paulraja. Chordal graphs and some of their derived graphs. *Congress. Numer.*, 53:71–74, 1986.

[10] B. Bauer and E. Schmeichel. Hamiltonian degree conditions which imply a graph is pancyclic. *J. Combin. Theory*, 48(Ser B):111–116, 1990.

[11] D. Bauer, H. J. Broersma, and H. J Veldman. Not every 2-tough graph is Hamiltonian. *Discrete Appl. Math.*, 99(1-3):317–321, 2000.

[12] L. Beasley and D. Brown. On cycle and bi-cycle extendability in chordal and chordal bipartite graphs. *Congressus Numerantum*, 174:41–47, 2005.

[13] P. Bedrossian. *Forbidden Subgraphs and Minimum Degree Conditions for Hamiltonicity*. PhD thesis, Memphis State University, 1991.

[14] P. Bedrossian. A generalization of Fan's condition for Hamiltonicity, pancyclicity, and Hamiltonian-connectedness. *Discrete Math.*, 115:39–50, 1993.

[15] E.A. Bender, L.B. Richmond, and N.C. Wormald. Almost all chordal graphs split. *J. Austral. Math. Soc.*, 38(Ser A):214–221, 1985.

[16] A. Benhocine and A.P. Wojda. Geng-Hua Fan conditions for pancyclic or Hamilton-connected graphs. *J. Combin. Theory*, 42(Ser B):167–180, 1987.

[17] J.A. Bondy. Large cycles in graphs. *Discrete Math.*, 1:121–132, 1971.

[18] J.A. Bondy. Pancyclic graphs. *J. Combin. Theory*, 11(Ser B):41–46, 1971.

[19] J.A. Bondy. *The Theory and Applications of Graphs*. Wiley, 1981.

[20] J.A. Bondy and V. Chvátal. A method in graph theory. *Discrete Math.*, 15(2):111–135, 1976.

[21] J.A. Bondy and U.S.R. Murty. *Graph Theory with Applications*. Elsevier, 1980.

[22] H. Broersma, J. van den Heuvel, and H. Veldman. A generalization of Ore's theorem involving neighborhood unions. *Discrete Math.*, 122:37–49, 1993.

[23] H. Broersma and H. Veldman. *Restrictions on induced Subgraphs Endsuring Hamiltonicity or Pancyclicity of $K_{1,3}$-free Graphs*. Contemporary Methods in Graph Theory, 1990.

[24] D.E. Brown. (personal communication). Utah State University, September 2013.

[25] P. Buneman. A characterization of ridgid circuit graphs. *Discrete Math.*, 9:204–212, 1974.

[26] G. Chartrand and R. Pippert. Locally connected graphs. *Casopis Pest. Mat.*, 99:158–163, 1974.

[27] G. Chen, R. Faudree, R. Gould, and M. Jacobson. Cycle extendability of Hamiltonian interval graphs. *SIAM Journal of Discrete Math*, 20(3):682–689, 2006.

[28] G. Chen, R. Faudree, R. Gould, M. Jacobson, and L. Lesniak. Hamiltonicity in balanced k-partite graphs. *Graphs and Combinatorics*, 11:221–231, 1995.

[29] G. Chen, M Jacobson, A. Kezdy, and L. Jeno. Tough enough chordal graphs are Hamiltonian. *Networks*, (31):29–38, 1998.

[30] V. Chvátal. On Hamilton's ideals. *J. Combin. Theory*, 12(Ser B):163–168, 1972.

[31] V. Chvátal. Tough graphs and Hamiltonian circuits. *Discrete Math.*, 5:215 – 228, 1973.

[32] V. Chvátal and P. Erdős. A note on Hamiltonian circuits. *Discrete Math.*, 2:111–113, 1972.

[33] M. Craven. Sequence assembly. [Lecture], 2011.

[34] E. Dahlhaus, P.D. Manuel, and M. Miller. A characterization of strongly chordal graphs. *Discrete Math.*, 187:269–27, 1998.

[35] G. A. Dirac. Some theorems on abstract graphs. *Proc. Lond. Math. Soc*, 2:69–81, 1952.

[36] D. Duffus, R. Gould, and M.S. Jacobson. The theory and applications of graphs, 1981.

[37] R. Entringer and E. Schmeichel. Edge conditions and cycle structure in bipartite graphs. *Ars Combinatoria*, 26:229–232, 1988.

[38] P. Erdös. On the construction of certain graphs. *J. Combin. Theory*, 1:149–152, 1966.

[39] Geng-Hua Fan. New sufficient conditions for cycles in graphs. *J. Combin. Theory*, 37(Ser B):221–227, 1984.

[40] G. Faudree, O. Favaron, E. Flandrin, and H. Li. Pancyclism and small cycles in graphs. *Discussiones Mathematicae Graph Theory*, 16(1):27–40, 1996.

[41] E. Flandrin, H. Jung, and H. Li. Hamiltonism, degree sum and neighborhood intersection. *Discrete Math.*, pages 41–52, 1991.

[42] E. Flandrin, H. Li, A. Marczyk, and M. Wozniak. A chvátal-Erdös type condition for pancyclability. *Discussiones Mathematicae Graph Theory*, 26:335–342, 2006.

[43] E. Flandrin, H. Li, H. Marczyk, and M. Wozniak. A note on pancyclism of highly connected graphs. *Discrete Math.*, 286:57–60, 2004.

[44] H. Fleischner. The square of every two-connected graph is Hamiltonian. *J. Combin. Theory*, 16:29–34, 1974.

[45] D. Fulkerson and O. Gross. Incidence matrices and interval graphs. *Pacific J. Math.*, 15:835–855, 1965.

[46] F. Gavril. The intersection graphs of subtrees in trees are exactly the chordal graphs. *Journal of Combinatorial Theory*, 16(Ser B):47–56, 1974.

[47] M. C. Golumbic. *Algorithmic Graph Theory and Perfect Graphs*. Academic Press, 1980.

[48] M. C. Golumbic and C. F. Goss. Perfect elimination and chordal bipartite graphs. *J. Graph Theory*, 2:155–163, 1978.

[49] V. S. Gordon, Y. L. Orlovich, and F. Werner. Hamiltonian properties of triangular grid graphs. *Discrete Math.*, 308:6166 – 6188, 2008.

[50] R. Gould. Updating the Hamiltonian problem – a survey. *J. Graph Theory*, 15:121–157, 1991.

[51] R. Gould. Recent advances on the Hamiltonian problem: Survey III. *Graphs and Combinatorics*, 30(1):1–46, 2014.

[52] R. Gould and M.S. Jacobson. Forbidden subgraphs and Hamiltonian properties of graphs. *Discrete Math.*, 42:189–196, 1982.

[53] R. Haggkvist, R.J. Faudree, and R.H. Schelp. Pancyclic graphs. *Ars Combinatoria*, 11:37–49, 1981.

[54] A. Harlan, M. Ferrara, and M. Jacobson. Hamiltonian cycles avoiding sets of edges in a graph. *Australasian J. Combinatorics*, 48:191–203, 2010.

[55] A. Harris. *Cycle Structures in Graphs*. PhD thesis, University of Colorado Denver, 2009.

[56] G. Hendry. A strengthening of Kikust's theorem. *J. Graph Theory*, 13(2):257–260, 1989.

[57] G. Hendry. Extending cycles in graphs. *Discrete Math.*, 85:59–72, 1990.

[58] G. Hendry. Extending cycles in bipartite graphs. *J. Combin. Theory*, 51(Ser B):292–313, 1991.

[59] B. Jackson, H. Li, and Y. Zhu. Dominating cycles in regular 3-connected graphs. *Discrete Math.*, 102:163–176, 1991.

[60] B. Jackson and O. Ordaz. Chvátal-Erdös conditions for paths and cycles in graphs and digraphs: A survey. *Discrete Math.*, 84:241–254, 1990.

[61] R. E. Jamison and R. C. Laskar. Intersection representation of a chordal graph by subtrees of a spanning tree. *Congr. Numer.*, 172:193 – 200, 2005.

[62] R.E. Jamison. On the null-homotopy of bridged graphs. *European J. Combin.*, 8:421–428, 1987.

[63] R.E. Jamison and R. Laskar. Elimination orderings of chordal graphs. *Combinatorics and applications (Indian Statist. Inst., Calcutta)*, pages 192–200, 1982.

[64] T. Jiang. Planar Hamiltonian chordal graphs are cycle extendable. *Discrete Math.*, 257(2-3):441–444, 2002.

[65] N. Jones and P. Pevzner. *An Introduction to Bioinformatics Algorithms.* The MIT Press, www.bioalgorithms.info, 2004.

[66] J. Kaptcianos. Graph theory aiding DNA fragment assembly. [Talk]. St. Michael's College, 2011.

[67] R. M. Karp. *Reducibility Among Combinatorial Problems.* Plenum, 1972.

[68] P. B. Kikust. The existence of a Hamiltonian circuit in a regular graph of degree 5. In *Latvian mathematical yearbook, 16 (Russian)*, pages 33–38, 271. 1975.

[69] D. Kratsch, J. Lehel, and H. Müller. Toughness, Hamiltonicity and split graphs. *Discrete Math.*, 150(1-3):231–245, 1996.

[70] H. Kronk. Generalization of a theorem of Pósa. *Proceed. of the AMS*, 21(1):77–78, 1969.

[71] M. LaFond and B. Seamone. Some Hamiltonian chordal graphs are not cycle extendable. [Manuscript], September 2013.

[72] C. Lee and B. Sudakov. Hamiltonicity, independence number, and pancyclicity. *European J. Combin.*, 33:449–457, 2012.

[73] X. Li, B. Wei, Z. Yu, and Y. Zhu. Hamiltonian cycles in 1-tough triangle-free graphs. *Discrete Math.*, 254:275–287, 2002.

[74] T. A. McKee. Chordal bipartite, strongly chordal, and strongly chordal bipartite graphs. *Discrete Math.*, 260:231–238, 2003.

[75] K. Menger. Zur allgemeinen kurventhoerie. *Fundamenta Mathematicae*, 10:96–115, 1927.

[76] J.W. Moon and L. Moser. On Hamiltonian bipartite graphs. *Israel J. Math*, 1:163–165, 1963.

[77] Z. G. Nikoghosyan. On evolution and fundamental results in large cycles theory, 2012.

[78] D.J. Oberly and D.P. Sumner. Every connected, locally connected nontrivial graph with no induced claw is Hamiltonian. *J. Graph Theory*, 3:351–356, 1979.

[79] O. Ore. A note on Hamiltonian circuits. *Amer. Math. Monthly*, 67:55, 1960.

[80] L. Pósa. On the circuits of finite graphs. *Magyar Tud. Akad. Mat. Kutató Int. Közl*, 8:355–361, 1964.

[81] F.P. Ramsey. On a problem of formal logic. *Proc. Lond. Math. Soc*, 30:264–286, 1930.

[82] E.R. Scheinerman. On the interval number of a chordal graph. *J. Graph Theory*, 12(3):311–316, 1988.

[83] E.R. Scheinerman. *Fractional Graph Theory: A Rational Approach to the Theory of Graphs*. Dover Books on Mathematics, 2013.

[84] U. Schelten and I. Schiermeyer. Small cycles in Hamiltonian graphs. *Discrete Appl. Math.*, 79(1-3):201–211, 1997.

[85] E. Schmeichel and S. Hakimi. A cycle structure theorem for Hamiltonian graphs. *J. Combin. Theory*, 45(Ser B):99–107, 1988.

[86] E. Schmeichel and J. Mitchem. Bipartite graphs with cycles of all even lengths. *J. Graph Theory*, 6:429–439, 1982.

[87] R. Shi. 2-neighborhoods and Hamiltonian conditions. *J. Graph Theory*, 16(3):267–271, 1992.

[88] B. Thomas. Cycle extendability in graphs. [manuscript], Denver, CO, 2010.

[89] N. Trinajstik. *Chemical Graph Theory*. CRC Press, 1983.

[90] Maarten van Steen. *An Introduction to Graph Theory and Complex Networks*. Maarten van Steen, April 2010.

[91] D.B. West. *Introduction to Graph Theory.* Pearson, 2nd edition, 2000.

[92] H. Whitney. Congruent graphs and the connectivity of graphs. *Am. J. Mathematics,* 54(1):150–168, 1932.

[93] T. Yamashita. On degree sum conditions for long cycles and cycles through specified vertices. *Discrete Math.,* 308:6584–6587, 2008.

[94] K. Yokomura. A degree sum condition on Hamiltonian cycles in balanced 3-partite graphs. *Discrete Math.,* 178:293–297, 1998.

[95] T. Zamfirescu. A characterization of Hamiltonian graphs. *Atti Acad. Sci. Ist. Bologna Cl. Sci. Fis. Rend,* 1(13):39–40, 1974.

[96] She-Min Zhang. Pancyclism and bipancyclism of Hamiltonian graphs. *J. Combin. Theory,* 60(Series B):159–168, 1994.

GLOSSARY

$\alpha(G)$ *independence number of the graph G*

$\kappa(G)$ *connectivity of the graph G*

$\delta(G)$ *minimum vertex degree of G*

$\Delta(G)$ *maximum vertex degree of G*

$bcl(G)$ *the bipartite closure of G*

$dist(x, y)$ *the length of the shortest xy-path in G*

$G_1(x)$ $G[N[x]] = G[N(x) \cup \{x\}]$

K_n *complete graph on n vertices*

$K_{n,n}$ *complete balanced bipartite graph on $2n$ vertices*

$K_{1,3}$ *claw*

$K_{1,1,3}$ *complete tripartite graph, with two partite sets of order 1, the other of order 3*

$|E(G)|$ *size of the graph G*

$|V(G)|$ *order of the graph G*

$N(x)$ *neighborhood of a vertex, $x \in V(G)$*

$N_H(x)$ *neighborhood of a vertex, $x \in H$, where H is a subgraph of G*

v^- *immediate predecessor of a vertex v, on a path or a cycle*

v^+ *immediate successor of a vertex v, on a path or a cycle*

$\sigma_2(G) = min\{d(x) + d(y) :\ x, y \in V(G), xy \notin E(G)\}$

$\sigma_2^2(G) = min\{d(x)+d(y) : xy \notin E(G), x \in X, y \in Y\}$, for bipartite graph $G = (X, Y, E)$

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EDUCATION

- Ph.D. – Utah State University (USU), June 2014

  Pure Mathematics, Graph Theory. Dissertation: "Hamiltonicity, Pancyclicity, and Cycle Extendability in Graphs". Chair: David E. Brown, Ph.D.

- PhD Study – University of Colorado, Boulder, 1992-1993

  University of Colorado, Denver, 2009-2012

- M.S. – Emory University, 1980

  Mathematics. Thesis: On the Duality of First Category and Measure Zero

- B.S. – Mercer University, 1977

  Magna Cum Laude, Mathematics, Physics, Latin. Mentors: Willis B. Glover, PhD, Dr. Norwood, Ph.D.

TEACHING EXPERIENCE

- 1978-1980, Emory University.

  Graduate Student Instructor Taught undergraduate math students

- 1980-1981, DeKalb College

  Instructor of Mathematics. Taught Trigonometry through Calculus

- 1982-1993, University of Colorado at Colorado Springs.

  Prof. Honoraria, Mathematics: Taught Calculus I-III, Lin Alg/Diff EQ, and Trigonometry; Taught non-traditional students, professional students returning to school, etc.; Taught almost 12 years, including full evening loads and some weekends; Consistently outstanding Evaluations.

- 1993-1996, Pikes Peak Community College.

  Adjunct Professor of Mathematics: Taught traditional and non-traditional students, Trigonometry through Calculus III.

- Spring 1999, The Colorado College.

  (temp) Instructor: Taught two blocks: Calculus I, and Calculus II; Consistently outstanding Evaluations.

- 1998-1999, Regis University

  Adjunct Professor of Mathematics.

- 1999-2002, US Air Force Academy.

  Instructor of Mathematics: Served as Associate Air Officer Commanding for Academics for Cadet Squadron 20; Assisted with the annual MORSS conference hosted at the Academy (June 2000); Presented paper on Cosmology research at World Congress in Rome, Sept. 2000; Appointed as USAFA nominee for the 2002 NEH Fellowship; Assisted with USAFA Forensics team in national multi-institution tournament; Assisted with the National Science Olympiad 2001, hosted at USAFA and UCCS; Assisted with the USAFA Fencing Team, and served on the Fencing Eligibility Committee; Assisted with the USAFA Collegiate Pistol Team (2001-2); Served as USAFA Editor of Mathematica Militaris (inter-service academic journal); Served as a judge for the Native American Science Bowl; Consistently outstanding Evaluations.

- 2002-2012, University of Maryland University College.

  Associate Professor of Mathematics Physical Science: Taught worldwide: teaching assignments included such locations as Kosovo, Kuwait, Afghanistan, Iceland,

Italy, Germany, etc. Helped develop pilot course which became a mainstay of the mathematics curriculum; Helped develop course module for an on-line mathematics course; Served on Faculty Advisory Council; Served on University Advisory Council, shared governance; Consistently outstanding Evaluations.

- Jan 2009-Dec 2012, University of Colorado Denver.
  PhD Teaching Instructor, Dept of Math Statistical Sciences: Taught undergraduate courses, Calculus I III, and Lin Alg/Differential Equations; Consistently outstanding Evaluations.

- Spring 2011, International College of Beijing/UCD, China Agricultural University.
  UCDenver Mathematics Instructor: Taught Calculus II-III.

- 2013-present, Utah State University, Dept of Mathematical Statistical Sciences.
  PhD student: Taught undergraduate courses, Calculus I-II and Business Calculus; Completed a Teaching Workshop and Sexual Harrassment training; Served as Director of Graduate Campus Affairs Utah State Graduate Senate: Duties included serving on the University Fee Board, the Calendar Committee, and assisting with Research Week. Responsible for Graduate Enhancement Awards ($80,000 grant money), and assisted with the Graduate Research and Projects Grants ($10,000 grant money); Initiated the launch of the Utah State Student Chapter of the AMS, and chaired the Student Colloquium Committee; Helping to launch the Utah State Mathematics Journal, publishing department research activities; Was given the active role in directing a graduate seminar, responsible for weekly presentation of my research and related topics; Helped organize and host a Math Study Hall for finals week, Spring 2014.

- 2001-2008 AP Calculus Reader/ Faculty Consultant, ETS.

PROFESSIONAL EXPERIENCE

- 1980-1983 Mathematician, NORAD/SpaceCommand, Directorate of Astrodynamics, Peterson AFB, CO.
  Performed Mathematical modeling of space systems and data analysis in support

of Strategic Defense operations conducted at NORAD/Cheyenne Mountain Complex.

- 1987-1988 Subject Matter Expert/ Project Specialist, United Airlines.

  Designed developed the Math curriculum and courseware for the USAF Undergraduate Space CBT program.

- 1989-1993 Senior Systems Analyst, Geodynamics Corp, NTB/Falcon AFB, CO.

  Helped end the Cold War through the development of space-based Strategic Defense systems. Designed missile defense architectures, strategies and advanced weapons systems (directed particle beams, space-based lasers et al.); Modeled defense systems and weapon elements in support of large-scale simulations for the conduct of multi-command War Games hosted at the JNTF; Performed Studies Analysis in support of policy decisions at the level of the SDIO and Joint Chiefs of Staff; Performed IVV activities to study the validity and applicability of numerous large-scale end-to-end simulations from various service branches.

- 1997 Senior Systems Analyst, National Systems Research, Colorado Springs, CO.
  Worked on the GPS program.

RESEARCH INTERESTS

- Cosmology

- Graph Theory

- Mathematics and Philosophy

SELECTED LIST OF PUBLICATIONS

Books

- Simulation Validation, Arangno and Knepell, IEEE/ ComSci Press, 1993.

- Abstract Algebra , Arangno, Schaum's/McGraw-Hill, 1998.

- Advanced Calculus , Deborah Arangno, Murray Spiegel, Schaum's/McGraw-Hill, 1999.

- Foundations of Calculus: UCCS, 1985.

- A Wise Companion, Arangno, iUniverse.com, 2001
- The Modern Heretic: Principles for a New Humanism - A Manifesto, Arangno, PubAmerica, June 2006

PhD Dissertation

- "Hamiltonicity, Pancyclity, and Cycle Extendiblity in Graphs", Utah State Univ, 2014.

Masters Thesis

- "A Study of Duality: Measure Zero and First Category", Emory University, 1980.

Articles

- "Formulation Analysis of Lunar/Solar Perturbative Forces", NORAD/Spacecommand, 1983.
- "Algorithm for Solutions to Complex Quartics", NORAD/Spacecommand, 1983.
- "Ground-Based Radar - Design, Function Definition, Algorithm", NTB/SDIO, 1989.
- "The Terminal Phase Defense - Analysis of Problem and Recommendation of Strategies Architectures", NTB/SDIO, 1989.
- "Alternative Brilliant Pebble Architecture Designs, Trade Analysis", NTB/SDIO 1989.
- "Bias Assessment Regulatory Guide", NTB/SDIO, 1989.
- "The Brilliant Pebble Battle Management Study", NTB/SDIO, 1989
- "Global Protection Against Limited Strikes - Mission Statement Policy Analysis", NTB/SDIO, 1989.
- "Handbook of Ellipsoids Geoids in Computing Earth Models", Geodynamics Corp, 1993
- "Entropy A Paradigm of Universe Dynamics", Quarterly of the Fellowship of Catholic Scholars, Notre Dame University, Spring 1999.

- Simple Mathematics in Orbital Analysis", Mathematica Militaris, Vol.11, no.4, Sp 2001

Delivered Papers/ Talks

- A Particle Theory of Cosmology", Physics for the 21st Century, World Congress of University Professors, Rome, 2000

- "Mathematics - Thinking Outside the Box", paper delivered at the John Fauvel Memorial Conference, the Colorado College, 2001

- "From Intuition to Esoterica", Joint AMS Meetings, San Diego, CA, Jan 2013

- "Edge-Avoiding Hamiltonian Graphs", MAA Intermountain Section Meeting, Idaho, 2013

- "Edge-Avoiding and F-Avoiding Hamiltonicity in Bipartite Graphs", AMS Western Regional Meeting, NM, (presented April 2014).

MEMBERSHIPS, CLEARANCES

- Top Secret clearance, SCI (Geodynamics/SDIO)

- Secret Clearance (NORAD/SpaceCommand)

- NEH nominee (USAFA, 2001)

- Golden Key Honor Society

- Sigma Tau Gamma Honor society

- Pi Mu Epsilon Mathematical Honor Society

- Association of University Professors

- American Mathematical Society

- Previously AIAA and FCS

- Deans List

- Latin Honor Society

AWARDS

- 1973 Freshman Honor Scholarship, Mercer University

- 1978-80 Emory University Graduate Teaching Assistantship:

- 2014 Utah State University Graduate Student Travel Award

- 2009-12 Univ of Colorado Graduate Teaching Assistantship:

- 2013-14 Utah State University Graduate Teaching Assistantship

- 2013-14 Utah State University Director of Graduate Campus Affairs