

A Simple Targeting Procedure For Lunar Trans-Earth Injection

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A simple targeting algorithm for trans-Earth injection is developed. The techniques presented in this paper build on techniques developed for the Apollo program and other lunar and interplanetary missions. Presently, more sophisticated algorithms exist for solving this problem, but the simplicity of this particular algorithm makes it well-suited for on-board use during contingency and abort operations. In order to support a return from any lunar orbit with available fuel on the spacecraft the algorithm chooses between one-, two-, or three-burn return scenarios. The one- and two-burn cases are based on existing theory. For the three-burn case however, the existing theory is modified in order to provide a simple solution. Rather than attempting to create fuel-optimal trajectories, the algorithm presented in this paper focuses on computing a trajectory from low lunar orbit to direct atmospheric Earth entry that does not violate a fuel constraint. The algorithm attempts to use a minimal number of impulses to execute trans-earth injection. The algorithm can also be used to quickly generate good initial guesses for other more sophisticated targeting algorithms that can be used to find minimal fuel trajectories or optimize other parameters. This algorithm has three principle phases. First, an estimate of the hyperbolic excess velocity at the Lunar sphere of influence is generated. Second, a maneuver is computed that will transfer the craft from a lunar circular orbit to the hyperbolic escape asymptote. Finally, the effects of perturbations are eliminated by using linear state transition matrix targeting.

I. Introduction

The goal of this paper is to present the theory and development for a simple and robust targeting algorithm intended for on-board use during contingency and abort scenarios where a spacecraft needs to return to the Earth from an arbitrary circular lunar orbit. As, i_∞ , the angle between the plane of the lunar orbit and hyperbolic escape asymptote, becomes large, a single injection burn can no longer perform the required maneuver without violating practical fuel constraints.

There are several historically significant papers,¹⁻³ which detail the optimization of the transfer from a low parking orbit to a hyperbolic escape asymptote. However, the theories presented in the literature are built around Keplerian motion and do not account for the highly perturbed nature of the gravity field in cislunar space. This research is aimed at producing an algorithm that is based on applying the existing injection theory coupled with linear targeting techniques to the trans-earth injection problem. Since the Apollo program some advancements have been made. However, these techniques are more complex and generally require a good initial guess in order to guarantee convergence. This makes these methods potentially ill suited for on-board applications particularly during emergency situations when CPU resources may be limited or when the unforeseen initial conditions may exist.

Although only a single burn was used for trans-Earth injection during the Apollo mission, the current proposed lunar mission is much more technologically demanding than the lunar missions from the Apollo program. The Apollo lunar trajectories, and missions in general, were heavily constrained for a variety of reasons, most of these relating to the limitations of the Apollo spacecraft. The new mission demands that

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technologies be developed that will allow many of these constraints to be relaxed, or removed completely. For example, the Apollo craft had a very narrow, predetermined window for return, and relied on ground support to upload the guidance solution for trans-earth injection. Current requirements for lunar missions dictate that the space craft must be able to return from the Moon to Earth at any time and from any lunar orbit. In the event that communications with the Earth are interrupted, the space craft must be able to execute trans-earth injection autonomously. Techniques, like the one presented in this paper, are needed for calculating the trans-earth injection sequence on-board the spacecraft. The limited computing power available on-board the craft and the limited time available for performing the computations, demand that procedures for calculating the trans-earth injection sequence must be simple yet robust. The algorithm in this paper attempts to accomplish this simplicity and robustness by using fuel available as a constraint rather than trying to use the minimal amount of fuel.

As the angle between the plane of the lunar orbit and the hyperbolic escape asymptote becomes large, the limited fuel available requires the maneuver to consist of multiple burns.¹ Although only a single burn was used for the trans-earth injection during the Apollo mission, double and triple burn trans-earth procedures were studied during the Apollo mission design process.⁴⁻⁶ These studies clearly showed the utility of multi-burn maneuvers in providing global access to the moon. They also showed that multi-burn maneuvers can be used to substantially increase payload capacity. Thus, the principal advantages of these multi-burn trajectories was their ability to provide access to more of the lunar surface and substantially increase the payload that could be returned from the moon. However, these multi-burn procedures added substantial complexity to mission design and were ultimately not used during Apollo.

II. The Need to Minimize Fuel Use

During the mission planning phase it is desirable to find fuel-optimal trajectories. The cost of carrying the additional mass associated with unneeded fuel on the spacecraft is so large that every possible reduction in fuel results in substantial savings. Although this algorithm is not aimed at the mission planning stages, it may serve as a good initial guess for other more sophisticated iterative optimization algorithms which do attempt to minimize the fuel usage.^{7,8}

In abort and contingency operations there is little use in minimizing the fuel usage. Any fuel not used to execute maneuvers will be discarded or returned to Earth unused. The cost to the mission will be the same, irregardless of how much fuel is used in executing the required maneuvers. Therefore, in abort and contingency operations it is desirable to consider fuel available as a constraint rather than a parameter that should be optimized. In these situation, rather than minimizing the fuel consumed by the maneuver, it may be more appropriate to minimize parameters that could pose a more immediate danger to the mission, such as the return time, the number and the complexity of burns required to execute the maneuver, or the atmospheric interface velocity.

Each burn causes an increase in the complexity and risk of failure for the mission. Furthermore, each successive burn adds substantially to the total time of return. These two effects are generally undesirable for manned missions and are particularly unfavorable for abort situations. This algorithm attempts to avoid excessive burns and keep the time required to return to the Earth to a minimum. The algorithm also attempts to keep these factors at a minimum by simply trying to use as few burns as possible. In order to accomplish this, the one- and two- burn methods initially try to optimize fuel, meanwhile the three burn maneuver attempts to minimize the time spent in intermediate transfer orbits by consuming all of the available fuel. In all these cases, the motivation is to utilize all of the available fuel, if needed, in an effort to keep the number of burns as small as possible and return to the Earth without unnecessary delay.

III. Overview of the Algorithm

The algorithm can be summarized as follows

1. A preliminary estimate of the \vec{V}_∞ is generated using the techniques presented in Battin.⁹ This requires an estimate of the time of flight for Earth return.
2. The minimum number of burns possible given the available fuel is determined based on the angle

between the escape asymptote and the lunar orbital plane.

3. Preliminary estimates of these burns are then computed using the techniques described in sections V, VI, and VII.
4. Using the positions of the impulsive burns and their times, linear state transition matrix targeting can be used to correct for perturbations to the trajectory.

IV. Computing the \vec{V}_∞ Vector

Before the sequence of burns needed to escape the moon's gravity is computed the \vec{V}_∞ vector at the lunar sphere of influence needed for trans earth injection must be found. The direction of this vector represents the hyperbolic escape asymptote and the magnitude represents the hyperbolic excess velocity.

Battin⁹ presents a method for computing an initial estimate of the \vec{V}_∞ vector. The method presented by Battin solves the boundary value problem for Keplerian motion given a target vector (location on the lunar sphere of influence), radius of perigee (distance of closest approach to the Earth), the inclination or the return orbit, and the time of flight between perigee and the target vector. Battin is able to solve this problem by directing the \vec{V}_∞ directly away from the moon.

The radius of perigee constraint is equivalent to a constraining the flight path angle to 0° at a specific radial distance. This constraint can be loosened to allow for an arbitrary flight path angle at a specific radial distance. The ability to choose a flight path angle at a specific altitude makes the resulting algorithm more useful for problems involving atmospheric reentry.

Given an initial guess of the difference in true anomaly between the initial conditions and final conditions, θ , a parameter, A can be computed. For the lunar return problem θ is typically between 150° and 160° .

$$A = \frac{r_t [1 - \tan(\gamma) \sin(\theta)] - r_a}{r_a - r_t [\cos(\theta) - \tan(\gamma) \sin(\theta)]} \quad (1)$$

where r_t is the magnitude of the target vector and r_a is the radial distance from the center of the earth to the location of atmospheric reentry, and γ is the flight path angle at reentry. Note that

$$A = e \cos(f_a)$$

where e is the eccentricity and f_a is the true anomaly at reentry.

This parameter is used compute several of the orbital elements. The true anomaly at reentry is given by

$$\tan(f_a) = [1 + A] \tan(\gamma)$$

The eccentricity can now be found. When $f_a = 90^\circ$ there is a singularity that can be easily overcome

$$e = \begin{cases} \frac{A}{\cos(f_a)} & \text{if } A \neq 0 \\ \tan(\gamma) & \text{if } A = 0 \end{cases} \quad (2)$$

The semi-latus-rectum can now be computed

$$p = r_a (1 + A) \quad (3)$$

This makes finding the semi major axis a straightforward process.

$$a = \frac{p}{1 - e^2} \quad (4)$$

The time of flight can now be computed by using a version of Lambert's equation.

$$t_{fl} = \sqrt{\frac{a^3}{\mu_e}} [\alpha - \sin(\alpha) - \alpha' + \sin(\alpha')] \quad (5)$$

where the gravitational parameter of the Earth is given by μ_e and the values α and α' are given by

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{s}{2a}} \quad (6)$$

$$\sin\left(\frac{\alpha'}{2}\right) = \sqrt{\frac{s-c}{2a}} \quad (7)$$

Battin notes that, for this situation there is no need to be concerned with the quadrant ambiguity in α and α' . The values needed to compute α and α' are given by

$$c = \sqrt{r_t^2 + r_a^2 - 2r_t r_a \cos(\theta)} \quad (8)$$

$$s = \frac{1}{2}(r_t + r_a + c) \quad (9)$$

The value of θ needed to drive t_{fl} to the desired value can be found by using a Newtonian iteration scheme. The derivative needed to execute this scheme is given by

$$\frac{\partial t_{fl}}{\partial \theta} = \frac{3}{2} t_{fl} \left(\frac{1}{a} \frac{da}{d\theta} \right) + \sqrt{\frac{a}{\mu}} \left[\tan\left(\frac{\alpha}{2}\right) \left(\frac{ds}{d\theta} - s \left(\frac{1}{a} \frac{da}{d\theta} \right) \right) + \tan\left(\frac{\alpha'}{2}\right) \left(\frac{ds}{d\theta} - (s-c) \left(\frac{1}{a} \frac{da}{d\theta} \right) \right) \right] \quad (10)$$

where

$$\frac{ds}{d\theta} = \frac{r_t r_a}{2c} \sin(\theta) \quad (11)$$

and

$$\frac{1}{a} \frac{da}{d\theta} = \frac{1}{a} \frac{da}{dA} \frac{dA}{d\theta} \quad (12)$$

with

$$\frac{1}{a} \frac{da}{dA} = \frac{1}{1+A} + \frac{2}{A} \frac{1 - \frac{\sin(f_a)}{e} \tan(\gamma)}{\frac{1}{e^2} - 1} \quad (13)$$

and

$$\frac{dA}{d\theta} = \frac{A [\sin(\theta) + \tan(\gamma) \cos(\theta)] + \tan(\gamma) \cos(\theta)}{\cos(\theta) - \tan(\gamma) \sin(\theta) - \frac{r_a}{r_t}} \quad (14)$$

A simple Newtonian iterative scheme will rapidly converge to the desired solution. The form of equation 5 is only valid for elliptic orbit. This suits the lunar return problem well because unbounded orbits are undesirable. If the value of t_{fl} is too small the orbit will no longer be elliptic. It is easy to see if this condition is violated by simply checking the value of e or by noting that the parabolic time of flight is given by

$$t_{fl(p)} = \frac{1}{3\sqrt{\mu}} \left[\sqrt{2r_t - p} (r_t + p) - \sqrt{2r_a - p} (2r_a + p) \right] \quad (15)$$

If the true anomaly of the target is given by $f_2 = f_1 + \theta$, then orbital mechanics can now be used to find a simple expression for the velocity vector at the target.

$$\vec{v}_t = -\frac{1}{r_t} \left[\sqrt{\frac{\mu}{p}} e \sin(f_2) \vec{r}_t + \frac{\sqrt{\mu p}}{r_t} \hat{h} \times \vec{r}_t \right] \quad (16)$$

where \hat{h} is the unit vector corresponding to the angular momentum of the orbit. This vector can be computed using the inclination of the return orbit, i , and the right ascension of the return orbit, Ω .

$$\hat{h} = \begin{bmatrix} \sin(\Omega) \sin(i) \\ -\cos(\Omega) \sin(i) \\ \cos(i) \end{bmatrix} \quad (17)$$

The right ascension can be computed by^a

$$\Omega = \lambda - \sin^{-1} \left(\frac{\tan(\delta)}{\tan(i)} \right) \quad (18)$$

where λ and δ are the right ascension and the declination of \vec{r}_t .

The position vector at the point of atmospheric interface can be quickly found by using a standard elliptic solution to Kepler's equation. However, the available geometry yields a simple direct solution.

$$\vec{r}_a = \frac{r_a}{r_t} \left[\cos(\theta) \vec{r}_t + \sin(\theta) \hat{h} \times \vec{r}_t \right] \quad (19)$$

$$\vec{v}_t = -\frac{1}{r_a} \left[\sqrt{\frac{\mu}{p}} e \sin(f_1) \vec{r}_t + \frac{\sqrt{\mu p}}{r_t} \hat{h} \times \vec{r}_a \right] \quad (20)$$

A preliminary estimate of \vec{V}_∞ vector is found by placing the target vector on the sphere of influence of the moon and solving for the moon relative velocity at the target vector, \vec{v}_{tm} using the process described above. Next, the aiming point distance, r_{ap} , is computed. This computation also requires the moon relative position, \vec{r}_{tm} .

$$r_{ap} = \vec{r}_{tm} \cdot \frac{\vec{v}_{tm} \times (\vec{r}_{tm} \times \vec{v}_{tm})}{|\vec{v}_{tm} \times (\vec{r}_{tm} \times \vec{v}_{tm})|} \quad (21)$$

This distance is driven to a small value (less than a kilometer). This can easily be accomplished by noting that the moon relative velocity is nearly constant in direction and magnitude. If this is assumed an improved estimate can be computed with the following formula.

$$\vec{r}_t = \vec{r}_{em} - r_s \frac{\vec{v}_{tm}}{v_{tm}} \quad (22)$$

where \vec{r}_{em} is the position vector from the earth to the moon and r_s is the radius of the lunar sphere of influence.

V. The Single Burn Maneuver

The simplest trans-Earth injection is a single-burn maneuver like the trajectory shown in Figure 1. The Apollo missions were carefully designed in order to assure that this maneuver was possible. Fuel limits make this maneuver only realistically possible when the hyperbolic escape asymptote is located in or very near the orbital plane of the lunar parking orbit. Keeping the escape asymptote in or near the orbital plane has the effect of allowing the vast majority of the fuel to be used for boosting the orbital energy of the craft rather than performing a plane change. This single burn is intended to place the craft on a trajectory that will intersect the Earth's atmosphere at a desired flight path angle and velocity for direct atmospheric entry.

Gunther² outlines a method for computing a fuel-optimal transfer from a circular orbit to a hyperbolic escape asymptote using a single-burn. That method is presented here without derivation. The optimal single burn transfer from a circular orbit to a hyperbolic escape asymptote can be found by finding the positive root of the following quartic equation

$$w^4 - \frac{V_\infty}{V_{\text{circ}}} w - \frac{V_\infty}{V_{\text{circ}}} \sin^2 i_\infty w - \sin^2 i_\infty = 0$$

where

$$w = \frac{V_\infty}{V_{\text{circ}}} + \cos \beta \sqrt{\left(\frac{V_\infty}{V_{\text{circ}}} \right)^2 + 2}$$

This equation is easily solved for β , the angle between the position vector and the velocity directly after the burn. Note that β must satisfy the constraint $0 \leq \beta \leq 180^\circ$. The single positive root of this quartic

^a $\Omega = \lambda + \sin^{-1} \left(\frac{\tan(\delta)}{\tan(i)} \right) + \pi$, is also a valid solution. This ambiguity must be resolved manually. The solution that was used in this this document was arbitrarily chosen.

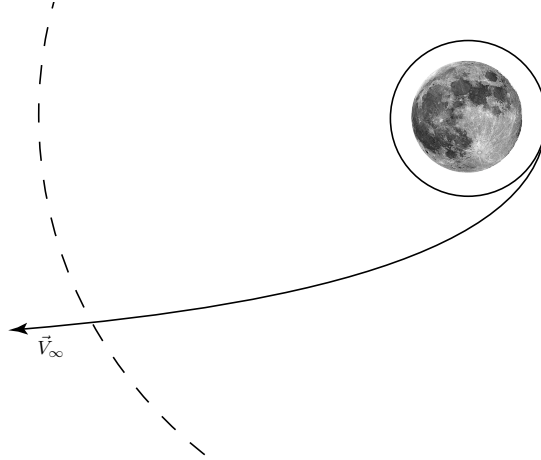


Figure 1. The geometry of the single-burn trans-earth injection maneuver. The burn is executed at perigee of the hyperbolic escape trajectory. This burn consists of a minimal plane change and a large energy boost. This single burn places the craft on a trajectory that will intercept the Earth’s atmosphere for direct entry.

equation corresponds to the optimal single impulse transfer from a low circular orbit to a hyperbolic escape asymptote. The angle between the the position vector at the maneuver and the hyperbolic escape asymptote, ψ , can now be found

$$\frac{\sin \beta}{\sin \psi} = \left(\frac{V_\infty}{V_{\text{circ}}} + \frac{1}{w} \right) \sqrt{\left(\frac{V_\infty}{V_{\text{circ}}} \right)^2 + 2}$$

The wedge angle, i , between the orbital plane of the circular orbit and the orbital plane of the escape trajectory is given by

$$\sin i = \frac{\sin i_\infty}{|\sin \psi|}$$

These two angles define the direction of the velocity vector immediately after applying the impulsive maneuver. The magnitude of the post-burn velocity vector is given by

$$V_1 = \sqrt{V_\infty^2 + 2V_{\text{circ}}^2}$$

the magnitude of the burn is given by

$$\Delta V = V_{\text{circ}} \sqrt{\left(\frac{V_\infty}{V_{\text{circ}}} \right)^2 + 3 - 2 \sqrt{\left(1 + \frac{V_\infty}{V_{\text{circ}}} w - w^2 \right) \left(2 + \frac{V_\infty}{V_{\text{circ}}} w \right)}}$$

Solutions to the single burn maneuver are shown in Figure 2 where $K = \frac{V_\infty}{V_{\text{circ}}}$, and $\Delta V' = \frac{\Delta V}{V_{\text{circ}}}$.

VI. The Double Burn Maneuver

As the angle between the plane of the lunar orbit and the hyperbolic escape asymptote becomes large, the limited fuel available requires that the maneuver consist of multiple burns. As this angle is increased, the fuel needed to execute the plane change portion of the maneuver using a single-burn increases dramatically. This dramatic increase in the fuel required can be avoided by using multiple burns to execute the maneuver rather than just one burn.

The geometry of the two-burn maneuver is shown in Figure 3. The first burn is similar to the single burn maneuver. This burn may include a minimal plane change, but the primary purpose of the first burn is to boost the orbital energy of the craft. The first burn places the craft on a low-energy hyperbolic escape trajectory. The second burn is performed at or near the sphere of influence. This is a large burn that affects the remainder of the plane change and places the craft on a trajectory that will intersect the atmosphere of the earth at a desired flight-path angle and velocity.

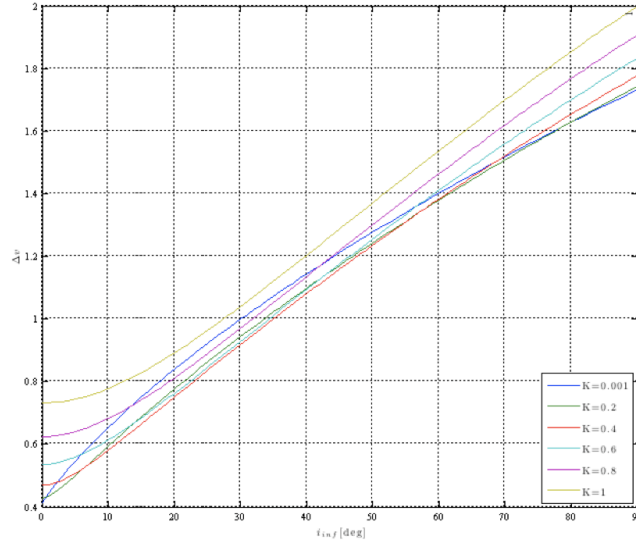


Figure 2. Nondimensional solution to the single burn maneuver.

Gunther² also presents a method for computing a fuel-optimal two-impulse transfers from a circular orbit to a hyperbolic escape asymptote. This method assumes that the second burn occurs at an infinite distance from the central body. Thus the first impulse places the craft on a hyperbolic orbit with a hyperbolic excess velocity, V_{∞_1} .

Gunther's development also assumes that the hyperbolic asymptote of the hyperbolic orbit that connects the circular orbit to the sphere of influence is in the plane of the final escape asymptote and the angular momentum vector of the first orbital plane. The angle between the plane of the circular orbit and the asymptote associated with V_{∞_1} is given by i_{∞} . While the angle between the orbit and the final escape trajectory is given by I_{∞} .

The optimal two-impulse transfer with the second burn at infinity can be found by solving the following sixth order polynomial

$$4K_1^2 w^6 + 4K_1 w^5 + (1 - 16K_1^4) w^4 + 2K_1 (1 - 20K_1^2 + 8K_1^4) w^3 + (1 - 19K_1^2 + 40K_1^4) w^2 + 2K_1 (8K_1^2 - 1) w + K_1^2 = 0$$

where $K_1 = \frac{V_{\infty_1}}{V_{\text{circ}}}$. This equation has two positive real roots, the smaller which corresponds to the optimal maneuver two impulse transfer. This polynomial requires that the magnitude of V_{∞_1} be known in order to solve for w . Thus the solution is iterative. Values of V_{∞_1} and thus K_1 , are selected until the value of I_{∞} matches the desired. Once w is found for any value of K_1 the other values necessary to compute the burn easily can be found

$$\sin^2 i_{\infty} = w^3 \frac{w - K_1}{1 + K_1 w}$$

Now the magnitude of the first burn can now be found

$$\Delta V_1 = V_{\text{circ}} \sqrt{K_1^3 + 2 - 2\sqrt{(1 + K_1 w - w^2)(2 + K_1 w)}}$$

The angle between the velocity before and after the second burn, γ , can be found

$$\cos \gamma = \frac{V_{\text{circ}}}{\Delta V_1} \left(K_1 - w \sqrt{\frac{1 + K_1 w - w^2}{2 + K_1 w}} \right)$$

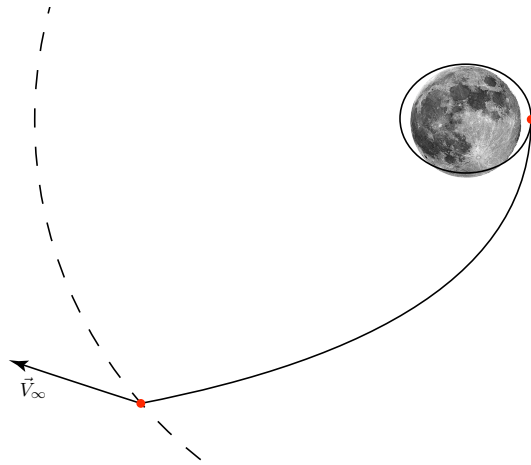


Figure 3. The geometry of the two-burn trans-Earth injection maneuver. The first burn is executed at perigee of a near low energy hyperbolic escape orbit the function of this burn is primarily to boost the orbital energy of the craft with minimal plane change. The second burn is performed at or near the sphere of influence. This is a large burn that performs the majority of plane change, boosts the orbital energy, and places the craft on target for atmospheric interface upon arrival at the Earth.

The magnitude of the second burn can be computed

$$\Delta V_2 = V_{\text{circ}} \left[\sqrt{\left(\frac{V_{\infty}}{V_{\text{circ}}}\right)^2 + \left(\frac{V_{\infty 1}}{V_{\text{circ}}} \sin \gamma\right)^2} - \frac{V_{\infty 1}}{V_{\text{circ}}} \sin \gamma \right]$$

I_{∞} can now be computed

$$I_{\infty} = i_{\infty} + \gamma - \sin^{-1} \left(\frac{V_{\infty 1}}{V_{\infty}} \sin \gamma \right)$$

Figure 4 shows the optimal values of K_1 as a function of I_{∞} and K . These solutions can rapidly be found with an iterative method such as the secant method.

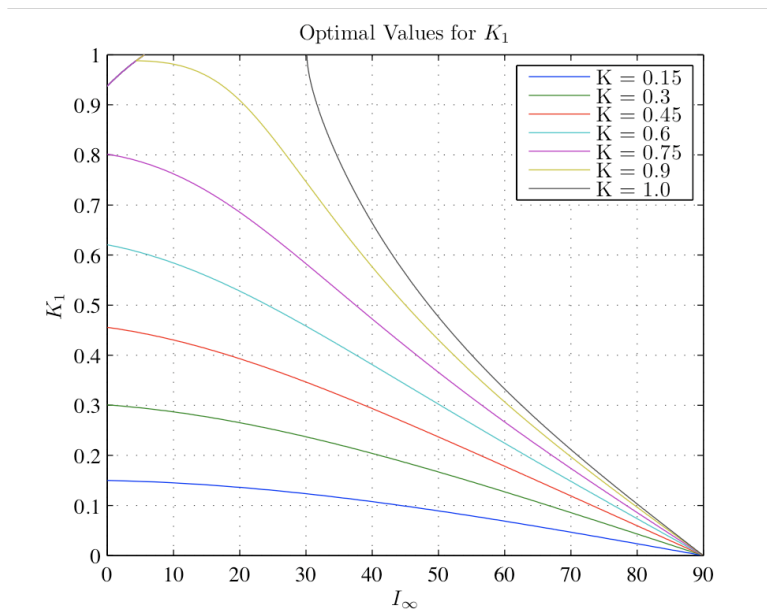


Figure 4. Nondimensional solution to the optimal two-burn escape trajectory.

VII. The Triple Burn Maneuver

Although the two-burn maneuver offers substantial fuel savings, for especially large values of i_∞ a three burn maneuver requires much less fuel than the two burn maneuver. The geometry of the three-impulse maneuver is shown in Figure 5. The first impulse of this maneuver has two purposes: to boost the orbital energy of the craft and, to place apoapsis of the orbit at the intersection of the current orbital plane and the plane containing the hyperbolic escape. The primary purpose of the second impulse is to change the orbital plane so that the resulting orbital plane includes the escape asymptote. The third impulse then serves to boost the orbital energy and place the craft on an Earth return trajectory that intersects the atmosphere of the earth at the desired flight-path angle and velocity need for direct atmospheric entry.

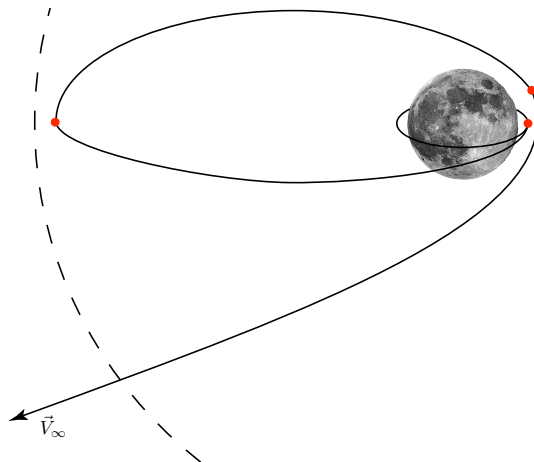


Figure 5. The geometry of the three-impulse maneuver. The first burn occurs at the periape of a high-energy elliptic orbit. The second burn occurs near the sphere of influence and primarily consists of a plane change. The third maneuver occurs periape of the final hyperbolic escape trajectory.

The triple-impulse maneuver can be viewed as a two part maneuver. The first two impulses execute a plane change while the third impulse performs the injection from a bounded elliptic orbit to a hyperbolic escape orbit. The initial plane is simply the orbital plane of the initial circular orbit. The final plane must constrain the hyperbolic escape asymptote. The intersection of these two planes is chosen so that the wedge angle between the the two planes, and thus the required plane change is minimal. Webb⁶ shows how the plane change can be optimally distributed between the first two burns. A significant amount of fuel is needed for the plane change portion of the maneuver. The fuel required to change planes is minimal when the wedge angle between the initial and final orbital planes is minimal. This occurs when the line defining the intersection of the two plane is orthogonal to the escape asymptote. The amount of fuel needed for the plane change can be minimized by solving the following equation for θ_1 , which is the of plane change executed at the first impulse, rather than the second impulse.

$$0 = \frac{V_{p1} V_{\text{circ}} \sin \theta_1}{\sqrt{V_{p1}^2 + V_{\text{circ}}^2 - 2V_{p1} V_{\text{circ}} \cos \theta_1}} - V_a \cos \frac{\theta - \theta_1}{2}$$

where V_{p1} is the velocity immediately after the first impulse, and V_a is the velocity at apoapsis. The magnitude of the first impulse is given by

$$\Delta V_1 = \sqrt{V_{p1}^2 + V_{\text{circ}}^2 - 2V_{p1} V_{\text{circ}} \cos \theta_1}$$

and the magnitude of the second impulse is given by

$$\Delta V_2 = 2V_a \sin \frac{\theta - \theta_1}{2}$$

now all that remains is to compute the third impulse.

Although several techniques could be used to compute the final burn that would place the craft on an acceptable hyperbolic trajectory. The technique that is presented in this paper places the burn in a location where the elliptic orbit and the hyperbolic orbit are tangent. One advantage of this scheme is that when a burn is performed along the velocity vector after periapse the periapse altitude must increase, this assures that no collision will occur. Also note that none of the energy from the burn is used to change direction; all of the energy imparted by the impulse increases the energy of the orbit. the location of this burn can be computed by solving the following two equations for e_h , the eccentricity of the hyperbolic orbit, and f_e , the true anomaly of the elliptic orbit. These equations are derived from equating the radial distance and its first derivative with respect to the angular rate for both the intermediate elliptic orbit and the hyperbolic orbit.

$$a_e \frac{1 - e_e^2}{1 + e_e \cos f_e} = a_h \frac{1 - e_h^2}{1 + \sin f_e + \sqrt{e_h^2 - 1} \cos f_e}$$

$$\frac{e_e \sin f_e}{1 + e_e \cos f_e} = \frac{\sqrt{e_h^2 - 1} \sin f_e - \cos f_e}{1 + \sin f_e + \sqrt{e_h^2 - 1} \cos f_e}$$

Since this final burn is directed along the velocity vector its magnitude may be determined as a function of radial distance by simply using the vis-viva integral to find the velocity of each orbit. The total impulsive maneuver is simple the sum of the three individual impulses.

A numerical search is performed which adjusts the semi major axis or the intermediate transfer ellipses until the sum of the ΔV for the three impulse sequence matches the amount of fuel that is budgeted for trans-earth injection.

VIII. Time Fixed Linear Targeting

Once an initial estimate of the trans earth injection maneuver has been generated, fixed time linear targeting techniques can be used to generate a more refined solution for each burn.

D'Souza¹⁰ outlines a technique for fixed time lunar targeting. D'Souza generally follows the a method outlined by Battin.⁹ However, D'Souza improves Battin's method by introducing linearized targets, this method is briefly outlined here.

Given a nominal trajectory, $\bar{\mathbf{x}}(t)$, the a linear system can be obtained by taking the dynamics partials about this nominal trajectory. This paper will only consider conservative systems, which have a potential function (such as gravity). For a conservative system the dynamics partials are given by

$$\left. \frac{\partial \dot{\mathbf{x}}}{\partial \mathbf{x}} \right|_{\bar{\mathbf{x}}} (t) = \begin{bmatrix} \mathbf{0}_{3 \times 3} & \mathbf{I}_{3 \times 3} \\ \mathbf{G}(t) & \mathbf{0}_{3 \times 3} \end{bmatrix} \quad (23)$$

where $\mathbf{G}(t)$ is the Hessian of the potential function with respect to the position. The differential equation for the linearized state transition matrix, Φ , can now be written as

$$\dot{\Phi}(t, t_0) = \begin{bmatrix} \mathbf{0}_{3 \times 3} & \mathbf{I}_{3 \times 3} \\ \mathbf{G}(t) & \mathbf{0}_{3 \times 3} \end{bmatrix} \Phi(t, t_0) \quad (24)$$

where $\Phi(t_0, t_0) = \mathbf{I}_{6 \times 6}$. This equation can easily be integrated to yield the linearized state transition matrix relating the initial state to the final state.

$$\Phi(t, t_0) = \left. \frac{\partial \mathbf{x}(t)}{\partial \mathbf{x}(t_0)} \right|_{\bar{\mathbf{x}}} = \begin{bmatrix} \Phi_{rr} & \Phi_{rv} \\ \Phi_{vr} & \Phi_{vv} \end{bmatrix} \quad (25)$$

The desired targets are given by^b

$$\psi = \begin{bmatrix} i \\ \gamma \\ r_a \end{bmatrix} \quad (26)$$

^bThese are the targets for the final burn. For burns prior to the final burn the target is simply the position vector where the next burn should be executed. When the target is simply a position vector, the partial derivatives that follow are greatly simplified. $\Psi = \begin{bmatrix} \mathbf{I}_{3 \times 3} & \mathbf{0}_{3 \times 3} \end{bmatrix}$

The partials of these targets are taken with respect to the state vector

$$\Psi = \frac{\partial \psi}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial \psi}{\partial \vec{r}} & \frac{\partial \psi}{\partial \vec{v}} \end{bmatrix} = \begin{bmatrix} \frac{\partial i}{\partial \vec{r}} & \frac{\partial i}{\partial \vec{v}} \\ \frac{\partial \gamma}{\partial \vec{r}} & \frac{\partial \gamma}{\partial \vec{v}} \\ \frac{\partial r_a}{\partial \vec{r}} & \frac{\partial r_a}{\partial \vec{v}} \end{bmatrix} \quad (27)$$

Linear system theory allows for the computation of an impulsive maneuver that will correct a small defect in the the final state, $\delta\psi(t_f)$, thus causing the trajectory to meet all the target constraints at the final time.

$$-\delta\psi(t_f) = \Psi \Phi(t_f, t_0) \begin{bmatrix} \mathbf{0}_{3 \times 1} \\ \Delta \vec{V} \end{bmatrix} \quad (28)$$

This equation is solved for $\Delta \vec{V}^c$

$$\Delta \vec{V} = -\mathbf{M}^{-1} \delta\psi(t_f) \quad (29)$$

where

$$\mathbf{M} = \frac{\partial \psi}{\partial \vec{r}} \Phi_{rv}(t_f, t_0) + \frac{\partial \psi}{\partial \vec{v}} \Phi_{rv}(t_f, t_0) \quad (30)$$

IX. Conclusions and Future Work

This paper presents a relatively simple and robust technique for computing a trans earth injection maneuver. This technique is particularly well suited for situations where a surplus of computational power is not available. In addition this technique can be useful for generating initial guesses for more complex targeting algorithms which attempt to optimize a trajectory with respect to some value of interest, typically fuel consumption.

It has been noted that the numeric integration of the state transition matrix can become unstable when the time of integration becomes particularly long in proximity of large bodies such as the earth or moon. The performance of the algorithm in this paper would benefit greatly from the use of symplectic or Hamiltonian integrators which do not exhibit this instability.

Although not fuel optimal this algorithm does attempt to conserve fuel to the extent possible. Studies comparing maneuvers generated by this algorithm to maneuvers found by using the same trajectory for the initial guess in an optimization scheme, such as primer vector theory, would be valuable.

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^cNote that if the rank(\mathbf{M}) < 3 then the left hand pseudo-inverse can be used to minimize the magnitude of $\Delta \vec{V}$