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## Linear Programming with Random Requirements

Nak Je Kim

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LINEAR PROGRAMMING WITH RANDOM REQUIREMENTS

by

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1968

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## INTRODUCTION

Linear programming was first developed by George B. Dantzig, Marshall Wood, and associates of the U. S. Air Force, in 1947. At that time, the Air Force organized a research group under the title of project SCOOP (Scientific Computation of Optimum Programs). This project contributed to the developing of a general interindustry model based on the Leontief input-output model, the Air Force programming and budgeting problem, and the problems which involved the relationship between two-person zero-sum games and linear programming. The result was the formal development and application of the linear programming model. This project also developed the simplex computational method for finding the optimum feasible program. Early applications of linear programming were made in the military, in economics, and in the theory of games. During the last decade, however, linear programming applications have been extended to such other fields as management, engineering, and agriculture.

As the application of linear programming has extended to many other fields, Dantzig (1955), Tintner (1955), Beale (1955), Madansky (1960), and others have been responsible for the formulation and development of stochastic linear programming. The stochastic linear programming problem occurs when some of the coefficients, in the objective function and/or in the constraint system of the linear

programming model, are subject to random variation.

In the literature, several methods are indicated for formulating the linear programming problem with random requirements to arrive at a solution. The intention of this study is to review some of these methods, and to compare one with another in terms of the optimum value of the objective function which results from each method. There are three methods that will be considered. The first method is to replace the random element with its expected value and solve the resulting linear programming problem (Hadley, 1964).

The second method is Dantzig's two-stage linear programming problem with a random requirement (Dantzig, 1955). Suppose the following linear programming problem is considered:

$$\begin{aligned} \min. \text{ (or max.) } C'X & \quad X \geq 0 \\ \text{subject to: } AX & \leq b, \end{aligned}$$

where  $C$  and  $X$  are  $n$  by  $1$  vectors,  $b$  an  $m$  by  $1$  vector, and  $A$  an  $m$  by  $n$  matrix, and  $C'$  is  $C$  transpose. If vector  $b$  is random and matrix  $A$  is known, then in the first stage, a decision is made on  $X$ , the random vector  $b$  is observed, and  $AX$  is compared with  $b$ . In the second stage, inaccuracies in the first decision are compensated for by a new decision variable  $Y$  with some penalty cost  $F$ . The problem then becomes,

$$\begin{aligned} \text{E min. (or max.) } C'X + F'Y, & \quad X \geq 0, \quad Y \geq 0, \\ \text{subject to: } AX + BY & = b, \end{aligned}$$

where  $B$  is an  $m$  by  $2n$  matrix with elements ones, minus ones, and zeroes, and  $Y$  is a  $2n$  by  $1$  vector with elements  $y_i$  and  $y_{-i}$ .  $E$  denotes an expectation.

In the third method, the constraints with random requirements are set to satisfy a given probability level. The problem then is to find values of the decision variables which optimize the expected objective function without violating the given probability measure (Charnes and Cooper, 1962).

This report surveys the literature on basic linear programming and the simplex method of solution, describes random requirements, and illustrates three methods of solution. Finally, the optimal value of the objective function of each method is compared with the others.

## LINEAR PROGRAMMING AND ITS SOLUTION METHODS

### Description

Linear programming is concerned with the problem of finding that mixture of decision variables which optimizes a linear objective function subject to a given system of linear restrictions.

To solve a linear programming problem, the solution to a set of simultaneous linear equations is required. There are various criteria which can be applied to determine whether one or more solutions exist to the problem. Suppose there are three linear equations with three decision variables. If a solution exists, it will be unique. But if there are two linear equations and three decision variables, then, in general, there exists either no solutions or an infinite number of solutions. One method of determining a set of solutions is to reduce the problem to two equations in two decision variables by letting one decision variable equal zero. This would then result in three solutions which satisfy the two equations. Decision variables are restricted to be nonnegative. Therefore, in choosing the optimum solution, only solutions with nonnegative variables will be considered.

A mathematical expression of the linear programming problem is as follows:

$$\text{min. (or max.) } Z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$





### Formulation of the problem

In practical applications of linear programming, some problems are complex and difficult to express in functional forms. However, it is possible to define certain principal parts of the system which define the separate steps for formulating the linear programming problem (Dantzig, 1963). The procedure is as follows.

The first step is to define the decision variables and write the constraint system in terms of the decision variables.

The second step is to write the objective function in terms of the decision variables. This function may either be maximized or minimized.

The third step is to determine the coefficients of the constraint system, i. e., the  $a_{ij}$ . Suppose a production model is considered in which the decision variables  $x_j$  represent different types of products, and  $b_i$  the total amount of  $i$ th resource available. Then  $a_{ij}$  is the amount of  $i$ th resource required to produce one unit of the  $j$ th product.

The fourth step is to write the constraint system so that the sum of every resource which is used for the product plus the resources that are left over equal the total available resources. After these steps are followed and the nonnegativity restriction is satisfied, the problem can be expressed in mathematical form.

### Solutions and the simplex method

Linear programming solutions consist of sets of values for the decision variables which satisfy  $m$  simultaneous linear equations in  $n$  variables where  $m < n$ . Using the matrix form given in Equations 2 and 3, the following three types of solutions can be identified (Gass, 1964).

Feasible solution--an  $n$  component column vector  $X$  that satisfies Equations 2 and 3 or a set of specific values of the  $n$  variables, which are nonnegative, and which simultaneously satisfy the  $m$  linear equations.

Basic feasible solution--a feasible solution that has no more than  $m$  positive components.

Optimal solution--either a feasible solution or a basic feasible solution that also optimizes the value of the linear objective function.

In the linear programming problem a feasible solution is sought which optimizes the objective function. But the most significant property of the set of feasible solutions is that it forms a convex set (Gass, 1964). Since the convex nature of this set is important the following description is included.

Assume that  $X_1$  and  $X_2$  are two arbitrary feasible solutions such that  $AX_1 = b$ , and  $AX_2 = b$ . Let  $X$  be any convex combination of  $X_1$  and  $X_2$ , i. e.,  $X = \alpha_1 X_1 + \alpha_2 X_2$ , where  $\alpha_1, \alpha_2 \geq 0$ , and  $\alpha_1 + \alpha_2 = 1$ . Then  $AX = A(\alpha_1 X_1 + \alpha_2 X_2) = A\alpha_1 X_1 + A\alpha_2 X_2$ .

Upon the substitution of  $\alpha_2 = 1 - \alpha_1$  into the above equation,

$$AX = \alpha_1 AX_1 + (1 - \alpha_1) AX_2 = \alpha_1 b + (1 - \alpha_1) b = b.$$

Thus  $X$  is also a point in the set of feasible solutions. Since  $X_1$  and  $X_2$  are any two points in the set of feasible solutions, the set is a convex set. This convex set is formed by the intersection of the linear constraints given in Equations 2 and 3. The boundaries of this set will be sections of the corresponding hyperplanes, and the convex set will be a region in  $n$ -dimensional Euclidian space. It can either be void, a convex polygon, or a convex region which is unbounded in some direction. If the set is void, there is no solution to the problem; if the set is a convex polygon, there is a finite number of solutions. If the convex set is unbounded, the optimum value of the objective function may also be unbounded. If the convex set is a convex polygon, the feasible solution set is a convex hull (Gass, 1964). That is, every feasible solution in the convex set can be represented as a convex combination of the extreme feasible solutions in the set. Any extreme point of the set of feasible solutions is also a basic solution. Further, a consideration of these basic feasible solutions will yield the optimum solution.

A computational scheme which evaluates the extreme point solutions in terms of the objective function until a final stage is reached is called the simplex method. While this method searches only the extreme point solutions, the final stage shows: (a) that a

finite optimal solution is found, (b) that an unbounded solution is possibly identified, or (c) that the solution of the problem is not feasible. The first step of the simplex method is to start with a basis (a set of vectors which span the vector space and are independent) that yields a basic feasible solution. The next iteration moves on to a new basis associated with a better basic feasible solution by way of replacing one of the present basis vectors with a nonbasis vector. The reason for selecting this nonbasis vector is its potential contribution to the improvement of the objective function, which can be determined beforehand. On the other hand, the present basis vector selected to be replaced is determined according to a rule which insures the continued feasibility of the new solution. Once the new basis is constructed, it forms the new starting point for repeating the same process to determine a better basic feasible solution. This continues until the final stage is reached.

The rule for selecting a vector to be replaced and the method for evaluating the vectors not currently in the basis is discussed below (Gass, 1964). Consider the linear programming problem of Equations 2 and 3. Suppose there is an extreme point solution  $X = (x_1, x_2, \dots, x_m, 0, \dots, 0)$  in terms of  $m$  column vectors out of  $n$  original column vectors of matrix  $A$ . Then there are  $m$  linearly independent vectors,

$$x_1 p_1 + x_2 p_2 + x_3 p_3 + \dots + x_m p_m = b_0 \dots \dots \dots (4)$$



This means that one of the basis vectors  $p_1, p_2, \dots, p_m$  must equal zero. Which  $p_j$  will be set to zero depends upon the nonnegative value of  $x_{i, m+1}$ . If all  $x_{i, m+1} = 0$ , then all the  $p_j$  will be positive and the same solution as Equation 4 is obtained. As  $x_{i, m+1}$  increases, then  $(x_i - x_{i, m+1})$  increases or decreases depending on whether  $\theta$  is negative or positive. In the case where  $\theta$  is positive, the minimal  $x_{i, m+1}$  which will make some  $(x_i - \theta x_{i, m+1})$  zero must be determined. If it is assumed that this occurs for  $i = k$ , then  $\theta_0 = \frac{x_k}{x_{k, m+1}}$ . By substituting  $\theta_0$  in Equation 7, there is a new basic feasible solution,  $x_i = \theta_0 x_{i, m+1}$ , and  $\theta_0$ , in terms of a new basis vector,  $p_1, p_2, \dots, p_{k-1}, p_{k+1}, \dots, p_m, p_{m+1}$ , where  $i = 1, 2, \dots, k-1, k+1, \dots, m+1$ . This, then, shows the rule for obtaining a new basic feasible solution by the inclusion of the vector  $p_{m+1}$  in place of a current vector  $p_k$  selected to maintain feasibility of the solution.

Now consider the method for evaluating nonbasis vectors to determine whether their inclusion in the basis would improve the objective function, and if so, by how much. In the above discussion, an extreme point solution  $X = (x_1, x_2, \dots, x_m, 0, \dots, 0)$  was assumed. If the objective function (Equation 1) is evaluated at this point, then

$$Z = c_1 x_1 + c_2 x_2 + \dots + c_m x_m \quad \dots \quad (8)$$

Further, assume that  $p_j$  is the new basis vector. The new value of  $\theta_0$ , denoted  $\theta'_0$ , can be determined by the rule discussed above. The

values of the new basis variables become  $x_i - \theta'_o x_{ij}$  and  $\theta'_o$ . The new objective function becomes

$$\begin{aligned} Z' &= c_1(x_1 - \theta'_o x_{1j}) + c_2(x_2 - \theta'_o x_{2j}) + \dots + c_m(x_m - \theta'_o x_{mj}) + c_j \theta'_o \\ &= c_1 x_1 + c_2 x_2 + \dots + c_m x_m - c_1 \theta'_o x_{1j} - c_2 \theta'_o x_{2j} - \dots - c_m \theta'_o x_{mj} + c_j \theta'_o \\ &= Z - \theta'_o (c_1 x_{1j} + c_2 x_{2j} + \dots + c_m x_{mj} - c_j) \\ &= Z - \theta'_o (Z_j - c_j), \text{ where } Z_j = c_1 x_{1j} + c_2 x_{2j} + \dots + c_m x_{mj}. \end{aligned}$$

If the objective function is to be maximized, improvement of the objective function occurs only when  $Z' > Z$ . This implies that  $\theta'_o (Z_j - c_j)$  must be negative. But  $\theta'_o$  cannot be negative if feasibility is maintained. Therefore,  $(Z_j - c_j)$  has to be negative to make  $\theta'_o (Z_j - c_j)$  negative. This means that if  $Z$  is to be maximized, the incoming vector must be selected from those  $p_j$  whose  $(Z_j - c_j)$  is negative. On the other hand, if the objective function is to be minimized, the indication of improvement is  $Z' < Z$ . This then implies that  $\theta'_o (Z_j - c_j)$  must be positive. In order for  $\theta'_o (Z_j - c_j)$  to be positive,  $(Z_j - c_j)$  must be positive. If  $Z$  is to be minimized, the incoming vector must be selected from those  $p_j$  whose  $(Z_j - c_j)$  is positive. With these procedures, one continues to change the basis so long as  $(Z_j - c_j) \leq 0$  for maximizing the objective function, and  $(Z_j - c_j) \geq 0$  for minimizing the objective function.



The dual problem

Suppose there is a linear programming problem,

$$\text{min. } Z = C'X$$

$$\text{subject to: } AX \leq b, \text{ and } X \geq 0.$$

One wishes to find a minimum feasible solution  $X_0$ . The solution so obtained is called the primal solution. The dual problem is to find a solution  $W$  which maximizes the linear function.

$$\text{max. } g = Wb$$

$$\text{subject to: } WA \leq c,$$

where  $W$  is a  $1$  by  $m$  vector without the nonnegativity restriction on the  $W_i$ . The following describes the relationship between the primal and dual solutions.

If either the primal or the dual problem has a finite optimum solution, then the other problem has a finite optimum solution and the extremes of the linear functions are equal, i. e.,  $\text{min. } Z = \text{max. } g$ . If either problem has an unbounded optimum solution, then the other problem has no feasible solutions. (Gass, 1964, p. 84.)

Thus the linear programming problem is to find a paired vector  $(X, W)$  which satisfies  $AX \leq b$ ,  $WA \leq c$  at the same time.



problem with random requirements is that the problem is capable of several formulations with different results for each formulation.

### Formulation methods and solutions

When random elements are introduced into the constraints of the linear programming problem, the nature of the problem changes. However the basic linear programming restrictions must still be maintained in the problem with random requirements.

Three basic methods of formulating and solving the problems will be discussed below.

The first method replaces the random coefficients with their expected value. This method provides only an approximate solution to the problem (Dantzig, 1955). Since the elements of the right-hand side of the constraints are replaced with their expected values, the formulation is still an ordinary linear programming problem.

$$\min. \text{ (or max.) } E(Z) = E(c'x)$$

$$\text{subject to: } Ax = E(b), \quad x \geq 0 \quad . \quad . \quad . \quad . \quad (10)$$

Therefore, the solution can be obtained directly from the simplex method. The values of the decision variables so obtained will be feasible.

The second method is Dantzig's two-stage linear programming problem with random requirements (Dantzig, 1955). The essential nature of this method is that the decision variables are determined in stages. Formulation of this method is as follows.



$$\text{where } B = \begin{bmatrix} 1 & -1 & 0 & 0 & \dots & \dots & \dots \\ 0 & 0 & 1 & -1 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \dots & -1 \end{bmatrix} \quad \text{and } Y = \begin{bmatrix} y_1 \\ y_{-1} \\ \vdots \\ y_m \\ y_{-m} \end{bmatrix}$$

Now consider an objective function which corresponds to the above random constraints. In the above discussion if  $b_i \geq u_i$ ,  $u_i$  falls short of the random observation  $b_i$  and the decision maker must make up the shortage with shortage cost, denoted  $f_{-i}$ ; if  $b_i \leq u_i$ ,  $u_i$  exceeds the random observation  $b_i$  and the decision maker must accept the excess cost, denoted  $f_i$ . Since  $y_i$  and  $y_{-i}$  represent a shortage and an excess respectively, these variables with their cost coefficients must be added to the objective function. Therefore the objective function can be written as,

$$\text{min. (or max.) } E(C'X + F'Y), \text{ where } F = \begin{bmatrix} f_1 \\ f_{-1} \\ \vdots \\ f_m \\ f_{-m} \end{bmatrix} \quad . \quad . \quad (13)$$

In this problem it is assumed that for every possible  $b$  and  $x$ , there exists a feasible solution which will optimize Equation 13. It is further assumed that  $X \in K > 0$ , where  $K$  is a convex set (Dantzig, Madansky, 1961). The problem, then, is to find  $X \in K$  which minimizes Equation 13 subject to Equation 12. Furthermore, assume that the random  $b_i$  are independently and normally distributed with the mean  $\mu_i$  and variance  $\sigma_i^2$ . From Equation 11,  $i$ th second

stage decision variable can be expressed as follows:

$$y_i = b_i - u_i, y_{-i} = 0 \text{ for } b_i \geq u_i \quad (14)$$

and

$$y_{-i} = u_i - b_i, y_i = 0 \text{ for } b_i \leq u_i \quad (15)$$

Expressed in summation notation, Equation 13 becomes

$$\text{min. (or max.) } E(\sum_j c_j x_j + \sum_i f_i y_i + \sum_i f_{-i} y_{-i}) \quad (16)$$

$$\text{for } j = 1, 2, \dots, n$$

$$i = 1, 2, \dots, m$$

Here  $f_i$  and  $f_{-i}$  are cost coefficients corresponding to  $y_i$  and  $y_{-i}$  respectively. If  $b_i - u_i$  is substituted for  $y_i$  and  $u_i - b_i$  is substituted for  $y_{-i}$  and if the expected value is taken, then the following expression results:

$$\text{min. (or max.)} [\sum_j c_j x_j + E \sum_i f_i (b_i - u_i) + E \sum_i f_{-i} (u_i - b_i)] \quad (17)$$

Since  $b_i$  was assumed to be normally distributed, the expectation for the  $i$ th expression of Equation 17 is the evaluation of the following integral:

$$\begin{aligned} & E f_i (b_i - u_i) + E f_{-i} (u_i - b_i), \text{ denoted } g(u_i). \text{ Then} \\ g(u_i) &= f_i \int_{u_i}^{\infty} (b_i - u_i) p(b_i) db_i + f_{-i} \int_{-\infty}^{u_i} (u_i - b_i) p(b_i) db_i \\ &= f_i \int_{u_i}^{\infty} (b_i - u_i) p(b_i) db_i + f_{-i} (u_i - \mu_i) + f_{-i} \int_{u_i}^{\infty} (b_i - u_i) db_i \end{aligned}$$

$$= (f_i + f_{-i}) \int_{u_i}^{\infty} (b_i - u_i) p(b_i) db_i + f_{-i} (u_i - \mu_i) \quad (18)$$

where  $p(b_i) = \frac{1}{\sqrt{2\pi}} \exp. \left[ -1/2 \left( \frac{b_i - \mu_i}{\sigma_i} \right)^2 \right]$ .

But  $g(u_i)$  is a convex function of  $u_i$  (Carr and Howe, 1964).

Differentiate the above expression with respect to  $u_i$  to obtain

$$\frac{dg(u_i)}{du_i} = - (f_i + f_{-i}) \int_{u_i}^{\infty} p(b_i) db_i + f_{-i} \text{ and } \frac{d^2 g(u_i)}{du_i^2} = (f_i + f_{-i}) p(u_i) \quad (19)$$

Therefore  $\frac{dg(u_i)}{du_i}$  is a nondecreasing function of  $u_i$  with  $\frac{d^2 g(u_i)}{du_i^2} \geq 0$ ,

and hence  $g(u_i)$  is a convex function of  $u_i$ . Since  $g(u_i)$  involves nonlinear terms, a polygonal approximation technique (Hadley, 1964) must be used to linearize the objective function.

The following approximation procedure is applicable only in the case where the functions are continuous and separable:

If  $0 < u_i < \beta$ , then select  $k + 1$  points of  $u_i$ , where  $u_{i0} = 0$ ,  $u_{i1} < u_{i2} < u_{i3} \dots < u_{ik}$ , and  $u_{ik}$  is the upper limit of  $u_i$ . For each  $u_{ik}$  compute  $g_i(u_{ik})$ . If  $u_i$  lies in the interval  $u_{ik-1} \leq u_i \leq u_{ik}$ , then  $u_i$  can be written as,  $u_i = u_{ik-1} + \Delta u_{ik} d_{ik}$ , where  $d_{ik}$

$$= \frac{u_i - u_{ik-1}}{\Delta u_{ik}}, \quad \Delta u_{ik} = u_{ik} - u_{ik-1}, \text{ and } 0 \leq d_{ik} \leq 1. \text{ Hence the polygonal}$$

approximation to  $g_i(u_i)$  can be written as  $\hat{g}_i(u_i) = g_{ik-1} + \Delta g_{ik} d_{ik}$ , where  $\Delta g_{ik} = g(u_{ik}) - g(u_{ik-1})$ . Next impose the condition that if  $d_{ik} > 0$  then  $d_{i\ell} = 1$ , for  $\ell = 1, 2, \dots, k-1$ . Then  $u_i$  can be written as,  $u_i = \sum \Delta u_{ik} d_{ik}$ , and hence  $\hat{g}_i(u_i)$  becomes,  $\hat{g}_i(u_i) = \sum \Delta g_{ik} + g_{io}$ .

Let  $t_{ik} = \Delta u_{ik} d_{ik}$ , so that the upper bound of  $t_{ik}$  becomes,  $0 \leq t_{ik} \leq \Delta u_{ik}$ , then  $u_i = \sum t_{ik}$ , where, if  $t_{ik} > 0$ ,  $t_{i\ell} = \Delta u_{i\ell}$ , for  $\ell = 1, 2, \dots, k-1$ . In terms of  $t_{ik}$ ,  $\hat{g}_i(u_i)$  becomes,  $\hat{g}_i(u_i) = \sum \alpha_{ik} t_{ik} + g_{io}$ , where  $\alpha_{ik} = \frac{\Delta g_{ik}}{\Delta u_{ik}}$ . Since the set  $t_{ik}$  which satisfies  $u_i = \sum t_{ik}$  is a convex set with a finite number of extreme points, any solution can be expressed as a convex combination of the extreme points (Hadley, 1964). Hence  $\sum t_{ik} = \sum \lambda_{ik} t_{ik}$  and  $\hat{g}_i(u_i) = \sum \lambda_{ik} \alpha_{ik} t_{ik} + g_{io}$ , where  $\sum \lambda_{ik} = 1$  for every  $i$  and  $\lambda_{ik} \geq 0$  for every  $i$  and  $k$ , now  $\hat{g}_i(u_i)$  replaces  $g_i(u_i)$  and  $u_i = \sum \lambda_{ik} t_{ik}$  in the original problem. Finally, Equations 12 and 17 become,

$$\text{min. (or max.) } \left[ \sum_j c_j x_j + \sum_i \sum_k \lambda_{ik} \alpha_{ik} t_{ik} + \sum_i g_{oi} \right] \quad (20)$$

$$\text{subject to: } u_i - \sum_k \lambda_{ik} t_{ik} = 0 \quad (21)$$

$$\sum_k \lambda_{ik} = 1 \text{ and } \lambda_{ik} \geq 0 \quad (22)$$

for every  $i$  and  $k$ ,

$$x_j \geq 0, \text{ for } j = 1, 2, \dots, n.$$

This method increases the number of constraints and variables.



In the third method, the constraint system with a random right-hand side must satisfy some given probability level. This method was first introduced by Charnes and Cooper (1959) for dealing with probabilistic constraints. Assume the following linear programming problem:

$$\text{min. (or max.) } C'X$$

$$\text{subject to: } AX \leq b, \quad X \geq 0,$$

where vector  $b$  is random, and each element of  $b$  is independently and normally distributed with mean  $\mu_i$  and  $\sigma_i^2$ . The linear programming problem with random  $b$  can then be formulated as follows:

$$\text{min. (or max.) } E(C'X) \quad \dots \quad (23)$$

$$\text{subject to: prob. } (AX \leq b) \geq q, \quad X \geq 0 \quad \dots \quad (24)$$

Here  $q$  is a vector of given probability levels which corresponds to the constraints. Consider now just the  $i$ th constraint:

$$\text{prob. } \left( \sum_{j=1}^n a_{ij} x_j \leq b_i \right) \geq q_i \quad \dots \quad (24a)$$

This means that the  $i$ th constraint would not be satisfied with probability  $(1 - q_i)$  in any admissible choice of the  $x_j$  values. The probability in Equation 24a can be transformed by simple subtraction and division as follows:

$$\text{prob. } \left( \sum_j a_{ij} x_j \leq b_i \right) = \text{prob. } \left( \frac{b_i - \mu_i}{\sigma_i} \geq \frac{\sum a_{ij} x_j - \mu_i}{\sigma_i} \right) \quad \dots \quad (25)$$

Since it is assumed that  $b_i$  follows a normal distribution  $\frac{b_i - \mu_i}{\sigma_i}$  is the standardized normal variable with mean zero and unit variance.

Hence Equation 24a can be written as,

$$-\Phi \left( \frac{\sum a_{ij} x_j - \mu_i}{\sigma_i} \right) \geq q_i \quad \dots \quad (26)$$

or

$$\left( \frac{\sum a_{ij} x_j - \mu_i}{\sigma_i} \right) \leq -\Phi^{-1}(q_i) \quad \dots \quad (27)$$

where  $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z \exp. (-1/2 y^2) dy$ . In this equation  $y$  is a dummy variable. Since  $q_i$  is predetermined,  $\Phi^{-1}(q_i)$  can be easily evaluated from a standard normal table. Let  $\Phi^{-1}(q_i) = h_i$ , then Equation 27 can be transformed as,

$$\sum a_{ij} x_j \leq \mu_i - h_i \sigma_i \quad \dots \quad (28)$$

Thus the problem becomes a typical linear programming problem,

min. (or max.)  $C'X$

subject to:  $\sum a_{ij} x_j \leq \mu_i - h_i \sigma_i$ ,

$$x_j \geq 0, \quad \text{for } i = 1, 2, \dots, m$$

$$j = 1, 2, \dots, n \quad \dots \quad (29)$$

## NUMERICAL EXAMPLES

In the previous chapter the discussions about the three basic methods of formulating and solving the linear programming problem with random requirements were restricted to theory. The purpose of numerical examples is to illustrate the methods with simple hypothetical problems. Assume the following linear programming problem:

$$\begin{aligned}
 \min. \quad & x_1 - 2x_2 + 3x_3 \\
 \text{subject to:} \quad & -2x_1 + x_2 + 3x_3 = b_1 \\
 & 2x_1 + 3x_2 + 4x_3 = b_2 \\
 & x_1, x_2, x_3 \geq 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)
 \end{aligned}$$

where the  $x_j$  are the decision variables,  $b_1$  and  $b_2$  are random.

Further assume that  $b_1$  and  $b_2$  are each independently and normally distributed with means,  $\mu_1 = 2$ ,  $\mu_2 = 4$ , and variances,  $\sigma_1^2 = 0.25$ ,  $\sigma_2^2 = 2.25$ .

### Method I

Since this method replaces the random variable with its expected value the problem can be formulated as follows:

$$\begin{aligned}
 \min. \quad & x_1 - 2x_2 + 3x_3 \\
 \text{subject to:} \quad & -2x_1 + x_2 + 3x_3 = E(b_1) = 2 \\
 & 2x_1 + 3x_2 + 4x_3 = E(b_2) = 4 \\
 & x_1, x_2, x_3 \geq 0
 \end{aligned}$$

Since the random elements of  $b_1$  and  $b_2$  are replaced by the constants, it is no different than an ordinary linear programming problem. The solution can be obtained as follows:

$$\begin{aligned} \text{(a)} \quad & -2x_1 + x_2 = 2 \\ & 2x_1 + 3x_2 = 4 \\ & x_1 = -0.25, \quad x_2 = 1.5. \end{aligned}$$

This solution is not feasible, since  $x_1 \leq 0$ .

$$\begin{aligned} \text{(b)} \quad & -2x_1 + 3x_3 = 2 \\ & 2x_1 + 4x_3 = 4 \\ & x_1 = 2/7, \quad x_3 = 6/7. \end{aligned}$$

This solution is feasible with the value of the objective function equal to 2.857, but it is not optimal.

$$\begin{aligned} \text{(c)} \quad & x_2 + 3x_3 = 2 \\ & 3x_2 + 4x_3 = 4 \\ & x_2 = 0.8, \quad x_3 = 0.4. \end{aligned}$$

This solution is feasible with the value of the objective function equal to -0.4.

Since the value of the objective function for (c) is less than the value of the objective function for (b), the optimum objective value is -0.4, with  $x_1 = 0$ ,  $x_2 = 0.8$  and  $x_3 = 0.4$ .

### Method II

Since in this method the penalty costs are to be known,  $f_1$ ,  $f_{-1}$ ,  $f_2$ , and  $f_{-2}$  must be assumed. If  $f_1 = 1$ ,  $f_{-1} = 0.5$ ,  $f_2 = 0.5$ ,

and  $f_{-2} = 0.5$ , the problem can be formulated as follows:

$$\min. E[x_1 - 2x_2 + 3x_3 + y_1 + 0.5y_{-1} + 0.5y_2 + 0.5y_{-2}]$$

$$\text{subject to: } -2x_1 + x_2 + 3x_3 + y_1 - y_{-1} = b_1$$

$$2x_1 + 3x_2 + 4x_3 + y_2 - y_{-2} = b_2.$$

Let  $u_1 = -2x_1 + x_2 + 3x_3$ , and  $u_2 = 2x_1 + 3x_2 + 4x_3$ .

If  $u_1 \leq b_1$ , then  $y_1 = b_1 - u_1$ , and  $y_{-1} = 0$

If  $u_1 \geq b_1$ , then  $y_{-1} = u_1 - b_1$ , and  $y_1 = 0$

If  $u_2 \leq b_2$ , then  $y_2 = b_2 - u_2$ , and  $y_{-2} = 0$

If  $u_2 \geq b_2$ , then  $y_{-2} = u_2 - b_2$ , and  $y_2 = 0$

Since  $y_1, y_{-1}, y_2, y_{-2}$  are functions of the random variables  $b_1$

and  $b_2$  respectively, the optimum expected value of the objective

function can be written as,

$$\begin{aligned} \min. x_1 - 2x_2 + 3x_3 + \int_{u_1}^{\infty} (b_1 - u_1) p(b_1) db_1 + 0.5 \int_{-\infty}^{u_1} (u_1 - b_1) p(b_1) db_1 \\ + 0.5 \int_{u_2}^{\infty} (b_2 - u_2) p(b_2) db_2 + 0.5 \int_{-\infty}^{u_2} (u_2 - b_2) p(b_2) db_2, \end{aligned}$$

$$\text{where } p(b_1) = \frac{1}{\sqrt{2\pi} \cdot 0.5} \exp. \left[ -1/2 \left( \frac{b_1 - 2}{0.5} \right)^2 \right] \text{ and}$$

$$p(b_2) = \frac{1}{\sqrt{2\pi} \cdot 1.5} \exp. \left[ -1/2 \left( \frac{b_2 - 4}{1.5} \right)^2 \right].$$

But since,

$$\int_{-\infty}^{u_1} (u_1 - b_1) p(b_1) db_1 = (u_1 - \mu_1) - \int_{u_1}^{\infty} (u_1 - b_1) p(b_1) db_1$$

. . . . . (31)

$$\int_{-\infty}^{u_2} (u_2 - b_2) p(b_2) db_2 = (u_2 - \mu_2) - \int_{u_2}^{\infty} (u_2 - b_2) p(b_2) db_2 \quad (32)$$

the problem becomes

$$\begin{aligned} \min. \quad & x_1 - 2x_2 + 3x_3 + (u_1 - 2) + (u_2 - 4) + 1.5 \int_{u_1}^{\infty} (b_1 - u_1) p(b_1) db_1 \\ & + 1.0 \int_{u_2}^{\infty} (b_2 - u_2) p(b_2) db_2 \end{aligned}$$

$$\text{subject to: } -2x_1 + x_2 + 3x_3 - u_1 = 0$$

$$2x_1 + 3x_2 + 4x_3 - u_2 = 0$$

$$x_1, x_2, x_3, u_1, u_2 \geq 0.$$

The problem is now formulated with nonlinear terms.

The next task is to linearize the nonlinear terms by polygonal approximation (Hadley, 1964). Since Equations 31 and 32 are functions of  $u_1$  and  $u_2$  respectively, Equation 31 is denoted as  $g(u_1)$ , and Equation 32 as  $g(u_2)$ . But  $g(u_1)$  and  $g(u_2)$  can be simplified as follows:

$$g(u_1) = (u_1 - 2) 1.5 \int_{u_1}^{\infty} (b_1 - u_1) p(b_1) db_1.$$

By letting  $K_1 = \frac{b_1 - 2}{0.5}$  and hence  $dK_1 = \frac{1}{0.5} db_1$ ,

$$\begin{aligned}
g(u_1) &= (u_1 - 2) + 1.5 \int_{\frac{u_1 - 2}{0.5}}^{\infty} (0.5K_1 + 2 - u_1) \frac{1}{\sqrt{2\pi} \cdot 0.5} \exp. (-1/2 K_1^2) 0.5 dK_1 \\
&= (u_1 - 2) + 1.5 \int_{\frac{u_1 - 2}{0.5}}^{\infty} (2 - u_1) \frac{1}{\sqrt{2\pi}} \exp. (-1/2 K_1^2) dK_1 \\
&\quad + 0.75 \int_{\frac{u_1 - 2}{0.5}}^{\infty} K_1 \frac{1}{\sqrt{2\pi}} \exp. (-1/2 K_1^2) dK_1.
\end{aligned}$$

Since  $K_1$  is a standard normal variable with mean zero and unit variance, if  $u_1$  is known, the first integral in the above can be evaluated from a standard normal table. The second integral in the above equation must be evaluated.

$$\text{Let } f(K_1) = 0.75 \int_{\frac{u_1 - 2}{0.5}}^{\infty} K_1 \frac{1}{\sqrt{2\pi}} \exp. (-1/2 K_1^2) dK_1, \text{ and}$$

$$Z = 1/2 K_1^2, \text{ then } dZ = K_1 dK_1.$$

If  $f(K_1)$  is transformed into  $f(Z)$ ,

$$\begin{aligned}
f(Z) &= 0.75 \int_{\frac{u_1 - 2}{0.5}}^{\infty} K_1 \frac{1}{\sqrt{2\pi}} \exp. (-1/2 K_1^2) dK_1 \\
&= 0.75 \int_{\frac{1}{2} \left( \frac{u_1 - 2}{0.5} \right)^2}^{\infty} \frac{1}{\sqrt{2\pi}} \exp. (-Z) dZ.
\end{aligned}$$

$$\text{Hence, } f(Z) = \frac{0.75}{\sqrt{2\pi}} \exp. \left[ -1/2 \left( \frac{u_1 - 2}{0.5} \right)^2 \right].$$

Therefore,

$$g(u_1) = (u_1 - 2) + \frac{0.75}{\sqrt{2\pi}} \exp. \left[ -1/2 \left( \frac{u_1 - 2}{0.5} \right)^2 \right] \\ + 1.5 \int_{\frac{u_1 - 2}{0.5}}^{\infty} (2 - u_1) \frac{1}{\sqrt{2\pi}} \exp. (-1/2 K_1^2) dK_1.$$

Since the procedure for  $g(u_2)$  is the same as for  $g(u_1)$ , the final expression for  $g(u_2)$  is given as,

$$g(u_2) = (u_2 - 4) + \frac{1.5}{\sqrt{2\pi}} \exp. \left[ -1/2 \left( \frac{u_2 - 4}{1.5} \right)^2 \right] + \int_{\frac{u_2 - 4}{1.5}}^{\infty} (4 - u_2) \frac{1}{\sqrt{2\pi}} \exp. (-\frac{1}{2} K_2^2) dK_2.$$

Next select six points in the domain of  $u_1$  and  $u_2$  and then  $g(u_1)$ ,  $g(u_2)$ ,  $\alpha_{k1}$ , and  $\alpha_{k2}$  can be computed as follows:

$u_{k1}$	$u_{k2}$	$g_{k1}$	$g_{k2}$	$\alpha_{k1}$	$\alpha_{k2}$
0.0	0.0	-1.999	-3.938		
0.5	1.0	-1.487	-2.865	0.513	1.073
1.0	2.0	-0.879	-1.613	1.217	1.252
1.5	3.0	+0.044	-0.192	1.846	1.421
2.0	4.0	+0.897	+0.997	1.706	1.187
2.5	5.0	+1.044	+1.808	0.294	0.811

where  $\alpha_{ki} = \frac{\Delta g_{ki}}{\Delta u_{ki}}$ . Let  $u_1 = \sum t_{k1} v_{k1}$  and  $u_2 = \sum t_{k2} v_{k2}$ ,

where  $0 \leq t_{k1} \leq \Delta u_{k1}$  and  $0 \leq t_{k2} \leq \Delta u_{k2}$ , then the polygonal approximate to  $g(u_1)$  and  $g(u_2)$  can be written as,



$$\hat{g}(u_1) = \sum_{k1} \alpha_{k1} t_{k1} v_{k1} + g_1(0), \text{ and}$$

$$\hat{g}(u_2) = \sum_{k2} \alpha_{k2} t_{k2} v_{k2} + g_2(0), \text{ where } \sum v_{ki} = 1.$$

The approximate function achieves its minimum when  $t_{ki}$  is the upper bound. Therefore  $\Delta u_{ki}$  replaces  $t_{ki}$ .

Finally the problem becomes

$$\begin{aligned} \min. [ & x_1 - 2x_2 + 3x_3 + 0.513v_{11} + 1.217v_{21} + 2.769v_{31} \\ & + 3.401v_{41} + 0.735v_{51} + 1.073v_{12} + 2.504v_{22} \\ & + 4.264v_{32} + 4.756v_{42} + 4.056v_{52} - 5.937] \end{aligned}$$

subject to:

$$-2x_1 + x_2 + 3x_3 - 0.5v_{11} - 1.0v_{21} - 1.5v_{31} - 2.0v_{41} - 2.5v_{51} = 0$$

$$2x_1 + 3x_2 + 4x_3 - 1.0v_{12} - 2.0v_{22} - 3.0v_{32} - 4.0v_{42} - 4.0v_{52} = 0$$

$$v_{11} + v_{21} + v_{31} + v_{41} + v_{51} = 1$$

$$v_{12} + v_{22} + v_{32} + v_{42} + v_{52} = 1,$$

$$x_1, x_2, x_3 \geq 0, \text{ and}$$

$$v_{ki} \geq 0, \quad i = 1, 2, \text{ and}$$

$$k = 1, \dots, 5.$$

The minimum value of the objective function is 4.9788 with  $x_1 = 0$ ,  $x_2 = 0.5$ ,  $x_3 = 0$ ,  $v_{11} = 1.0$ ,  $v_{22} = 0.875$ , and  $v_{52} = 0.125$ ,

Method III

This method requires preassigned probability levels on the random constraints. In this problem assume that  $q_1$  and  $q_2$  are both 0.95. Then the problem can be formulated as follows:

$$\begin{aligned} \text{min. } & E(x_1 - 2x_2 + 3x_3) \\ \text{subject to: } & \text{prob. } (-2x_1 + x_2 + 3x_3 \leq b_1) \geq 0.95 \\ & \text{prob. } (2x_1 + 3x_2 + 4x_3 \leq b_2) \geq 0.95, \text{ and} \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

Now subtract the mean values from both the right and left hand sides of the random constraints and divide each constraint by its standard deviation, then the constraints become

$$\begin{aligned} \text{prob. } & \left( \frac{-2x_1 + x_2 + 3x_3 - 2}{0.5} \leq \frac{b_1 - 2}{0.5} \right) \geq 0.95 \\ \text{prob. } & \left( \frac{2x_1 + 3x_2 + 4x_3 - 4}{1.5} \leq \frac{b_2 - 4}{1.5} \right) \geq 0.95 \end{aligned}$$

Since  $\frac{b_1 - 2}{0.5}$  and  $\frac{b_2 - 4}{1.5}$  are each standard normal variables, the probability constraints can be written as:

$$\begin{aligned} \frac{-2x_1 + x_2 + 3x_3 - 2}{0.5} & \leq -1.65 \\ \frac{2x_1 + 3x_2 + 4x_3 - 4}{1.5} & \leq -1.65. \end{aligned}$$

Therefore the problem is reduced to the following linear programming problem:

$$\text{min. } x_1 - 2x_2 + 3x_3$$

subject to:

$$-2x_1 + x_2 + 3x_3 \leq 1.175$$

$$2x_1 + 3x_2 + 4x_3 \leq 1.525, \text{ and}$$

$$x_1, x_2, x_3 \geq 0.$$

In solving this problem, an ordinary linear programming optimizing method, the simplex method, can be used.

$$(a) \quad -2x_1 + x_4 = 1.175$$

$$2x_1 + 0x_4 = 1.525$$

$$x_1 = 0.7625, \quad x_4 = 2.7,$$

This and the next three solutions are feasible.

$$(b) \quad x_2 + x_4 = 1.175$$

$$3x_2 + 0x_4 = 1.525$$

$$x_2 = 0.5083, \quad x_4 = 0.6667$$

$$(c) \quad 3x_3 + x_4 = 1.175$$

$$4x_3 + 0x_4 = 1.525$$

$$x_3 = 0.3812, \quad x_4 = 0.0314$$

$$(d) \quad x_4 + 0x_5 = 1.175$$

$$0x_4 + x_5 = 1.525$$

However the remaining six solutions are not feasible.

$$(e) \quad -2x_1 + x_2 = 1.175$$

$$2x_1 + 3x_2 = 1.525$$

$$x_1 = -0.25, \quad x_2 = 0.675$$

$$(f) \quad -2x_1 + 3x_3 = 1.175$$

$$2x_1 + 4x_3 = 1.525$$

$$x_1 = -0.041, \quad x_3 = 0.386$$

$$(g) \quad x_2 + 3x_3 = 1.175$$

$$3x_2 + 4x_3 = 1.525$$

$$x_2 = -0.025, \quad x_3 = 0.4$$

$$(h) \quad -2x_1 + x_5 = 1.175$$

$$2x_1 + x_5 = 1.525$$

$$x_1 = -0.588, \quad x_5 = 2.7$$

$$(i) \quad x_2 + 0x_5 = 1.175$$

$$3x_2 + x_5 = 1.525$$

$$x_2 = 1.175, \quad x_5 = -2$$

$$(j) \quad 3x_3 + 0x_5 = 1.175$$

$$4x_3 + x_5 = 1.525$$

$$x_3 = 0.3917, \quad x_5 = -0.0418$$

Hence the optimum solution is the set  $x_1 = 0$ ,  $x_2 = 0.5083$ ,  $x_3 = 0$ ,  $x_4 = 0.667$ , and  $x_5 = 0$ , with the optimum value of the objective function equal to  $-1.0166$ .

COMPARISON OF OPTIMUM VALUES OF THE OBJECTIVE  
FUNCTION UNDER DIFFERENT METHODS OF LINEAR  
PROGRAMMING WITH RANDOM REQUIREMENTS

So far three different methods of formulating and solving the linear programming problem with random requirements have been discussed. Since each method describes randomness differently, one would expect some differences in the optimum value of the objective function. In this chapter the methods will be compared to show how each method effects the optimum value of the objective function.

First compare the optimum value of the objective function for the first and second methods. Madansky (1960) shows that, in general, an unequal relationship exists between the two values, but under certain conditions, equality may hold. The following is a review and summary of Madansky's discussion. Only the minimizing problem will be considered, however. Consider the linear programming problem:

$$\text{min. } (C'X + F'Y) \text{ with respect to } X \text{ and } Y$$

$$\text{subject to: } AX + BY = b$$

$$X \geq 0, \text{ and } Y \geq 0.$$

Suppose vector  $X$  is given, then the objective function  $(C'X + F'Y)$  to be minimized with respect to  $Y$  for a given  $X$ , is denoted  $Z(b, X)$ .

If  $b$  is assumed random, the above problem becomes a linear programming problem with random requirements. In the two-stage

problem, the decision maker must determine the vector  $X$  in the first stage. After the vector  $X$  is determined, the expected value of the function  $(C'X + F'Y)$  is minimized with respect to the second decision variable  $Y$ , i. e.,  $E Z(b, X)$  (Dantzig, Madansky, 1961).

Consider a linear programming problem with random requirements where the decision maker first observes a random  $b$  and then solves the resulting nonlinear programming problem. That is, for each observation on  $b_1$ , one can obtain the minimum value of  $Z(b, X)$ . Therefore, the problem is to find the expected minimum value of  $Z(b, X)$  with respect to  $X$ , i. e.,  $E \min_X Z(b, X)$  (Tintner, 1955). First compare  $\min_X E Z(b, X)$  with  $E \min_X Z(b, X)$ . Since  $Z(b, X) = \min_Y (C'X + F'Y)$ , then,  $\min_X E Z(b, X) = \min_X E \min_Y (C'X + F'Y)$  and  $E \min_X Z(b, X) = E \min_X \min_Y (C'X + F'Y)$ .

Let  $X_1(b)$  be the decision variable which minimizes  $E \min_Y (C'X + F'Y)$  and let  $X_2(b)$  be the decision variable that minimizes  $\min_Y (C'X + F'Y)$ . Then  $\min_X E \min_Y (C'X + F'Y) = E (C'X_1 + F'Y)$ , and  $E \min_X \min_Y (C'X + F'Y) = E (C'X_1(b) + F'Y)$ . Since  $X_1(b)$  is the decision variable that is restricted for only the  $b$  which yields the expected value of  $\min_Y (C'X + F'Y)$ ;  $X_2(b)$  is the decision variable that minimizes for any  $b$ ,  $\min_Y (C'X_1(b) + F'Y) \geq \min_Y (C'X_2(b) + F'Y)$ . Therefore  $E \min_Y (C'X_1(b) + F'Y) \geq E \min_Y (C'X_2(b) + F'Y)$ ; hence  $\min_X E \min_Y (C'X + F'Y) \geq E \min_X \min_Y (C'X + F'Y)$ .

Next compare the  $\min_X E \min_Y (C'X + F'Y)$  with  $\min_X \min_Y (C'X (Eb) + F'Y)$  in order to compare  $E \min_X \min_Y (C'X + F'Y)$  with  $\min_X \min_Y (C'X (Eb) + F'Y)$ . Consider the dual problem of  $\min_X \min_Y (C'X + F'Y)$ , then,

$$\max. W'b,$$

$$\text{subject to: } W'A \leq C,$$

where  $W$  is dual solution vector.

Let  $W^*$  be the optimum dual solution, then  $\min_X \min_Y (C'X + F'Y) = W^*b$ , and  $W^*b \geq W'b$  for any feasible  $W$ . If the expected value of  $\min_X \min_Y (C'X + F'Y)$  is taken then  $E \min_X \min_Y (C'X + F'Y) = W^* E(b) \geq W' E(b)$ .

Let  $W^{**}$  be the vector which maximizes  $W'E(b)$  subject to  $W^{**}A = C$ , and  $W \geq 0$ , then,  $E \min_X \min_Y (C'X + F'Y) \geq W^{**}E(b) = \min_X \min_Y (C'X (Eb) + F'Y)$ . Hence  $E \min_X \min_Y (C'X + F'Y) \geq \min_X \min_Y (C'X (Eb) + F'Y)$ , and since  $\min_X E \min_Y (C'X + F'Y) \geq E \min_X (C'X + F'Y)$  as shown above,  $\min_X E \min_Y (C'X + F'Y) \geq \min_X \min_Y (C'X (Eb) + F'Y)$ . Thus the unequal relationship holds between the two values, but equality holds if and only if,  $\min_X \min_Y (C'X + F'Y)$  is a linear function of  $b$ .

Next compare the second and third methods. Thompson, Cooper, and Charnes (1963) suggest that the two-stage problem would be considered as special case of the problem with probability constraints. The probability constraints are written as,  $\text{prob. } (AX \leq b) = q$ . If

BY (second stage variable of two-stage problem) is added so that  $AX + BY = b$ , then  $q$  becomes one. This implies that when  $q = 1$  the two-stage problem is, in fact, a special case of the problem with probability constraints. In the third method (the probability constraint problem), the second stage decision variables of the second method (the two-stage problem) are included. Hence all decision variables in the objective function of the probability constraint problem depend upon a random vector whereas the first stage decision variable in the two-stage problem does not depend upon a random vector. As Thompson, Cooper, and Charnes (1963) noted, actual comparison between the optimum value of the objective function of the two-stage problem and the probability constraint problem depends upon the value of  $q_i$  in the probability constraint problem.

Finally, compare values of the objective function derived by the first and third methods. Expressions of the  $i$ th constraint for the first and second methods are,

$$\sum a_{ij} X_j \leq E(b_i) = \mu_i \quad \text{and}$$

$$\sum a_{ij} X_j \leq \mu_i - \Phi^{-1}(q_i) \sigma_i.$$

Note that the only difference between the above two constraints is  $\Phi^{-1}(q_i) \sigma_i$ . It was assumed above that random  $b_i$  is distributed normally with mean  $\mu_i$  and variance  $\sigma_i^2$ . In Equation 27 if  $q_i = 0.5$ ,  $\Phi^{-1}(q_i) = 0$ , this implies a 50 percent chance that the  $i$ th constraint



will not be satisfied for any admissible choice of the  $X_j$ .

In the case where  $q_i = 0.5$ , the  $i$ th constraint of the first method will equal the  $i$ th constraint of the third method. Now suppose  $q_i > 0.5$ ; then  $\Phi^{-1}(q_i)$  will certainly not equal zero. Therefore, the only way to make  $\Phi^{-1}(q_i)\sigma_i$  equal to zero is for  $\sigma_i$  to be zero. This means there is no variability in  $b_i$ . That is,  $b_i$  is equal to  $\mu_i$ . Hence, due to the difference of the right-hand side in the constraints between the first and third methods, a difference between the optimum value of the objective function will be expected unless the variance of  $b_i$  is as minimal as to be ignored.

### Summary

In summary, although the three methods compare with one another, each method possesses its own advantage and disadvantage in formulating or solving linear programming problems with random requirement vectors.

The first method. This method is easy and simple to apply in both formulating and solving the problem. After the random right-hand side coefficients are replaced with their expected values, the resulting problem is a regular linear programming problem. But this method can only be an approximation. However, the method will be a fairly good approximation when there is not too much variability in the random coefficients in the linear programming problem.

The second method. This method seems quite idealistic, especially in describing the random nature of the coefficients in the linear programming problem such as the penalty cost in the objective function. The penalty cost occurs as a result of the randomness of the right-hand side coefficients in the constraints. Furthermore, as was seen in chapter three, the solution method is complicated by the use of polygonal approximation.

The third method. This method seems to be more practical because there are only linear terms involved in the formulation of the problem. In some cases this method may require polygonal approximations in solving the problem, but the solution, when the random right-hand side coefficients are normally distributed, is straightforward. Furthermore, the right-hand side values of the constraints are partly determined by the parameters of the probability distribution of the random variable. The solution is less desirable when sample estimators are poor representatives of population parameters.

## LITERATURE CITED

- Beale, E. M. L. 1955. On minimizing a convex function subject to linear inequalities. *Journal of the Royal Statistical Society* 17:173-184.
- Carr, Charles R., and Charles W. Howe. 1964. *Quantitative decision procedures in management and economics*. McGraw-Hill Book Company Inc., New York, N. Y. 383 p.
- Charnes, A., and W. W. Cooper. 1959. Chance constrained programming. *Management Science* 6:73-79.
- Charnes, A., and W. W. Cooper. 1962. Chance constraints and normal deviates. *Journal of the American Statistical Association* 57:134-148.
- Dantzig, George B. 1955. Linear programming under uncertainty. *Management Science* 1:197-206.
- Dantzig, George B. 1963. *Linear programming and extensions*. Princeton University Press, Princeton, New Jersey. 632 p.
- Dantzig, George B., and Albert Madansky. 1961. On the solution of two-stage linear programs under uncertainty. *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*. University of California Press, Berkeley and Los Angeles 1:165-176.
- Gass, Saul L. 1964. *Linear programming*. McGraw-Hill Book Company Inc., New York, N. Y. 280 p.
- Hadley, G. 1964. *Nonlinear and dynamic programming*. Addison-Wesley Publication Company Inc., Reading, Massachusetts. 484 p.
- Kataoka, Shinji. 1963. A stochastic programming model. *Econometrica* 31:181-196.
- Madansky, Albert. 1960. Inequalities for stochastic linear programming problems. *Management Science* 6:197-204.
- Madansky, Albert. 1963. Linear programming under uncertainty. p. 103-110. In Robert L. Graves and Philip Walfe (ed.). *Recent advances in mathematical programming*. McGraw-Hill Book Company Inc., New York, N. Y. 347 p.

## LITERATURE CITED (Continued)

Thompson, G. L., W. W. Cooper, and A. Charnes. 1963.  
Characterizations by chance-constrained programming. p. 103-120.  
In Robert L. Graves and Philip Walfe (ed.). Recent advances in  
mathematical programming. McGraw-Hill Book Company Inc.,  
New York, N. Y. 347 p.

Tintner, G. 1955. Stochastic linear programming with applications to  
agricultural economics. Second Symposium on Linear Programming,  
National Bureau of Standards, Washington, D. C. 1:197-228.