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MODELING OVERLAPPING AND HETEROGENEOUS PERCEPTION VARIANCE  
IN STOCHASTIC USER EQUILIBRIUM PROBLEM  
WITH WEIBIT ROUTE CHOICE MODEL

by

Songyot Kitthamkesorn

A dissertation submitted in partial fulfillment  
of the requirements for the degree

of

DOCTOR OF PHILOSOPHY

in

Civil and Environmental Engineering

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Logan, Utah

2013

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## ABSTRACT

Modeling Overlapping and Heterogeneous Perception Variance in Stochastic User  
Equilibrium Problem with Weibit Route Choice Model

by

Songyot Kitthamkesorn, Doctor of Philosophy

Utah State University, 2013

Major Professor: Dr. Anthony Chen  
Department: Civil and Environmental Engineering

In this study, a new SUE model using the *Weibull* random error terms is proposed as an alternative to overcome the drawbacks of the multinomial logit (MNL) SUE model. A path-size *weibit* (*PSW*) model is developed to relax *both* independently and identically distributed assumptions, while retaining an analytical closed-form solution. Specifically, this route choice model handles route overlapping through the path-size factor and captures the route-specific perception variance through the Weibull distributed random error terms. Both constrained entropy-type and unconstrained equivalent MP formulations for the PSW-SUE are provided. In addition, model extensions to consider the demand elasticity and combined travel choice of the PSW-SUE model are also provided. Unlike the logit-based model, these model extensions incorporate the *logarithmic* expected perceived travel cost as the network level of service to determine the demand elasticity and travel choice. Qualitative properties of these minimization programs are given to establish equivalency and uniqueness conditions. Both path-based

and link-based algorithms are developed for solving the proposed MP formulations. Numerical examples show that the proposed models can produce a compatible traffic flow pattern compared to the multinomial probit (MNP) SUE model, and these models can be implemented in a real-world transportation network.

(197 pages)

## PUBLIC ABSTRACT

Modeling Overlapping and Heterogeneous Perception Variance in Stochastic User  
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Traffic assignment problem is an important component of the transportation planning model. State-of-the-practice traffic assignment models adopt the *equilibrium* principle to equilibrate the travel demand with the travel supply (e.g., highway and transit networks) under *congestion*. These models give the transportation network performance measures to compare among transportation alternatives for supporting the decision-making processes. The deterministic user equilibrium (*DUE*) principle is perhaps the most widely used in the traffic assignment problem. In this principle, all travelers are assumed to minimize their individual travel cost, such that only the *lowest-cost* route is used at equilibrium. However, this perfect knowledge assumption is unrealistic. Travelers do not know the exact travel costs of all possible routes in the transportation network, and some travelers do not always use the minimum travel cost criterion for their route selection.

To relax this restrictive perfect knowledge assumption, the stochastic user equilibrium (*SUE*) principle was suggested. A random error term is incorporated in the

route cost function to simulate travelers' imperfect perceptions of network travel costs, such that they do not always end up selecting only the minimum cost route. In the literature, two widely used random error terms are *Gumbel* and *Normal* distributions, corresponding to the multinomial logit (*MNL*) and multinomial probit (*MNP*) route choice models, respectively. The MNL model has a closed-form probability expression, and the MNL-SUE model can be formulated as an equivalent mathematical programming (*MP*) formulation under congestion. Several efficient algorithms can be applied to solve this MNL-SUE model in a real-size network. The two major drawbacks of this model are: (1) inability to handle *route overlapping* (or correlation) among routes, and (2) inability to account for *heterogeneous perception variance with respect to different trip lengths*. These two drawbacks stem from the underlying assumption of an *independently and identically distributed (IID) Gumbel* variate. The multinomial probit (*MNP*) model, on the other hand, does not have such drawbacks. This route choice model uses the Normal distribution to allow the covariance between random error terms for pairs of routes; however, due to the lack of a closed-form solution, the MNP-SUE model is computationally burdensome when the choice set contains more than a handful of routes.

In this study, we provide a new SUE model using the *Weibull* random error terms as an alternative to overcome the drawbacks of these two classical SUE models. A path-size *weibit (PSW)* model is developed to handle *both* route overlapping among routes and heterogeneous perception variance with respect to different trip lengths, while retaining an analytical closed-form solution. Specifically, the PSW route choice model handles the route overlapping through the *path-size factor* and handles the route-specific perception variance through the Weibull distributed random error terms. Both constrained and unconstrained

equivalent MP formulations for the PSW-SUE model are provided. In addition, model extensions to consider the demand elasticity and combined travel choice of the PSW-SUE model are also provided. Unlike the logit-based model, these model extensions incorporate the *logarithmic* expected perceived travel cost as the network level of service to determine the demand elasticity and travel choice. Qualitative properties of these minimization programs are given to establish equivalency and uniqueness conditions. Both path-based and link-based algorithms are developed for solving the proposed MP formulations. Numerical examples show that the proposed models can produce a compatible traffic flow pattern compared to the MNP-SUE model, and these models can be implemented in a real-world transportation network.



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-Songyot Kitthamkesorn

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# CHAPTER 1

## INTRODUCTION

### 1.1 Transportation planning

Transportation systems have a direct impact on economics and the quality of life. These systems provide mobility for people and goods, deliver accessibility to various locations (e.g., workplaces, schools, and recreational areas), and influence the economic activities and growth patterns of a region. With this crucial component of modern society, transportation planning is critical for efficient financing, managing, operating, and maintaining of the transportation system to achieve development goals.

The most common paradigm for the transportation planning model in the United States, as used by the majority of Metropolitan Planning Organizations (MPOs), is known as the “*four-step*” travel demand forecasting model, as shown in Fig. 1. This four-step model includes four modules—trip generation, trip distribution, modal split, and traffic assignment—as a mathematical representation of the demand and supply for travel in an area. The trip generation module takes socioeconomic information to estimate the travel demand within each Traffic Analysis Zone (TAZ). The trip distribution module connects the travel demand of each TAZ to determine the travel demand between a pair of TAZs as an origin-destination (O-D) pair. The modal split module predicts how a trip between an O-D pair will be taken on a given mode of transportation. The final step is the traffic assignment module. This module is used to simulate the routes travelers choose to reach their destination on a specific mode of transportation. State-of-the-practice traffic assignment models adopt the *equilibrium* principle to equilibrate the travel demand with the travel supply (e.g., highway and transit networks) under *congestion*. These models

give the transportation network performance measures, for example, vehicle miles traveled (VMT), vehicle hours traveled (VHT), trip length, and volume/capacity (V/C) ratio. These resultant model estimations would be used to compare among transportation alternatives for supporting the decision-making processes. This study focuses on the final step of the four-step travel demand forecasting model. New mathematical programming formulations are developed to relax the shortcomings of existing models and formulations. Algorithms for solving the proposed models and formulations are also provided for real-world implementation.

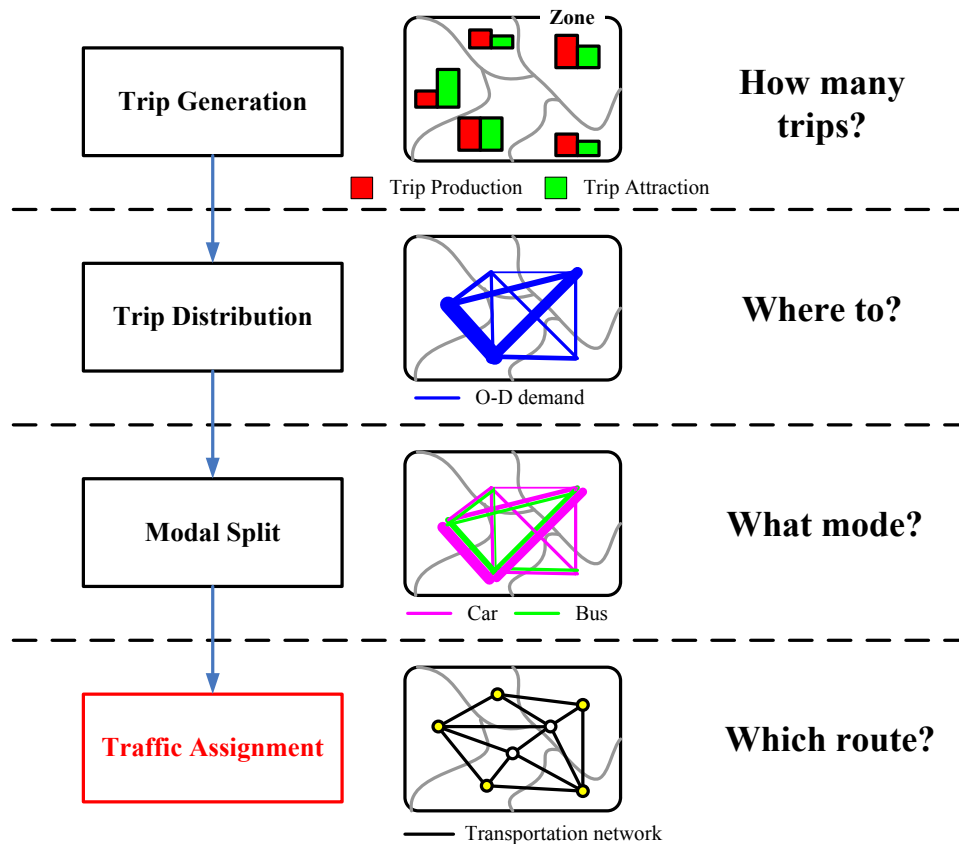


Fig. 1. Four-step transportation planning model

## 1.2 Deterministic user equilibrium model

The deterministic user equilibrium (*DUE*) is perhaps the most widely used principle for the traffic assignment problem. It is defined as follows:

The journey costs on all the routes actually used are equal and less than those which would be experienced by a traveler on any unused route. (Wardrop, 1952)

Travelers in this principle are assumed to minimize their individual travel cost, such that only the *lowest-cost* route is used at equilibrium. In 1956, Beckmann et al. (1956) developed this DUE principle into a mathematical programming (*MP*) formulation. Several efficient solution algorithms (e.g., Frank and Wolfe, 1956; Dial, 2006) can be used to solve this DUE model in a *real-size* network. However, the assumption of perfect knowledge of network conditions is unrealistic. Travelers do not know the exact travel costs of all possible routes in the transportation network, and some travelers do not always use the minimum travel cost criterion for their route selection.

## 1.3 Stochastic user equilibrium model

To relax the restrictive perfect knowledge assumption, Daganzo and Sheffi (1977) suggested the stochastic user equilibrium (*SUE*) principle. It is defined as:

No travelers can improve his or her perceived travel cost by unilaterally changing routes at SUE. (Daganzo and Sheffi, 1977)

A random error term is incorporated in the route cost function to simulate travelers' *imperfect perception* of network travel costs, such that they do not end up selecting only the minimum cost route. Therefore, the route choice behavior is *probabilistic*, as shown in Fig. 2.

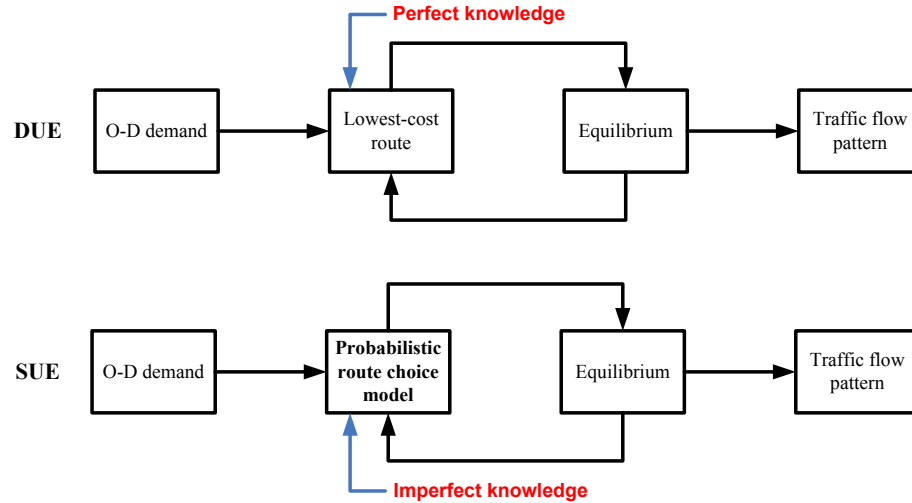


Fig. 2. DUE and SUE models

Two commonly used random error terms in the literature are *Gumbel* (Dial, 1971) and *Normal* (Daganzo and Sheffi, 1977) distributions, corresponding to the multinomial logit (MNL) and multinomial probit (MNP) (probabilistic) route choice models, respectively. The MNL model has a closed-form probability expression, and the MNL-SUE model can be formulated as an equivalent MP formulation (Fisk, 1980; Sheffi and Powell, 1982). Several efficient algorithms can be applied to solve the MNL-SUE MP formulation in a real-size network (e.g., Sheffi, 1985; Larsson and Patriksson, 1992; Damberg et al., 1996; Bell et al., 1997; Leurent, 1997; Maher, 1998; Chen et al., 2005; Chootinan et al., 2005; Zhou et al., 2012). The two main drawbacks of the MNL-SUE model are: (1) inability to account for *route overlapping* (or correlation), and (2) inability to account for *perception variance with respect to different trip lengths*. These two drawbacks stem from the underlying assumption of *independently and identically distributed (IID) Gumbel* variate (Sheffi, 1985). The MNP model, on the other hand, does not have such a drawback. It uses the Normal distribution to allow the covariance between random error terms for pairs of routes. However, due to the lack of a closed-form probability expression, solving the MNP-SUE model requires Monte

Carlo simulation (Sheffi and Powell, 1982), Clark's approximation method (Maher, 1992), or numerical method (Rosa and Maher, 2002).

To address the shortcomings of the MNL model, several closed-form route-choice models have been developed. These models can be classified into two categories: the *extended logit* models and *weibit* model, as shown in Fig. 3. The extended logit models relax the *independently distributed assumption* while retaining the Gumbel distributed random error terms. These models modify either the deterministic term or the random error term of the MNL random utility maximization (*RUM*) model. The models modifying the deterministic term include the C-logit (Cascetta et al., 1996), path-size logit (PSL) (Ben-Akiva and Bierlaire, 1999), and implicit availability/perception (IAP) (Cascetta et al., 2002) models. All three models add a correction term to the deterministic term of the disutility function to adjust the choice probability. However, the interpretation of each model is different. The C-logit model uses a commonality factor to penalize the coupling routes, while both the IAP and PSL models use a logarithmic correction term to modify the disutility (hence, the choice probability). The IAP model aims at capturing travelers' imperfect knowledge of available routes. Equivalent MP formulations for these models were recently provided by Zhou et al. (2012) and Chen et al. (2012). The models modifying the random error term include the paired combinatorial logit (PCL) (Bekhor and Prashker, 1999), cross-nested logit (CNL) (Bekhor and Prashker, 1999), and generalized nested logit (GNL) (Bekhor and Prashker, 2001) models. These models use the Generalized Extreme Value (GEV) theory (McFadden, 1978) to incorporate route correlation, hence route overlapping. Equivalent MP formulations for these extended logit models were provided by Bekhor and Prashker (1999, 2001).



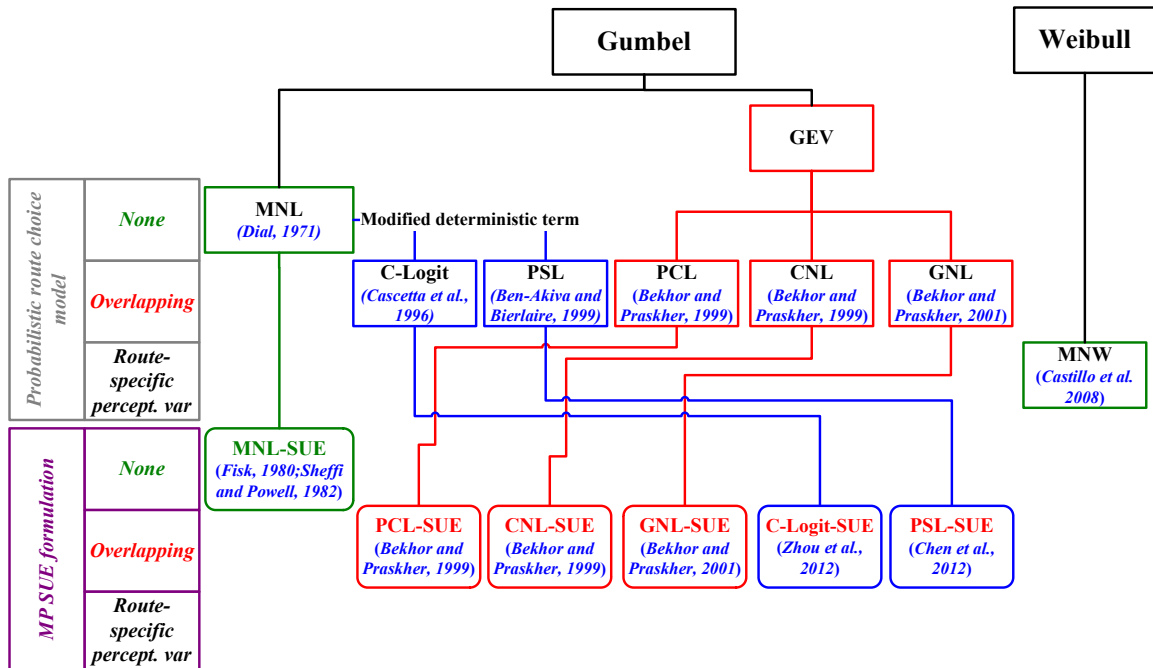


Fig. 3. Existing closed-form (probabilistic) route choice and MP SUE models

Even though all the extended models discussed above can successfully capture route overlapping, the *identically distributed* assumption (i.e. homogeneous perception variance) is still inherited; all routes are assumed to have the *same and fixed* perception variance according to the logit assumption of the *Gumbel* distribution. Hence, [Chen et al. \(2012\)](#) suggested a practical approach to partially relax the assumption by scaling the perception variance of an individual O-D pair. The *individual O-D specific scaling factors* allow the perception variance to increase or decrease according to the travel distance of each O-D pair. Specifically, the systematic disutility in the logit-based SUE models can be scaled appropriately to reflect different O-D trip lengths in a network by replacing a single dispersion parameter for all O-D pairs with *individual O-D dispersion parameters* for each O-D pair. However, it should be noted that it is not possible to scale individual routes of the same O-D pair since it would violate the logit-based SUE

models' assumption of an equal variance across the routes within the same O-D pair. For a more comprehensive review of the extended logit models used in the SUE problem, readers are directed to the reviews given by [Prashker and Bekhor \(2004\)](#) and [Chen et al. \(2012\)](#).

Recently, [Castillo et al. \(2008\)](#) proposed the multinomial weibit (*MNW*) model to relax the *identically distributed* assumption. Instead of the conventional Gumbel distribution, this route choice model adopts the *Weibull* distributed random error terms to handle the heterogeneous perception variance. Under the *independence* assumption, the MNW model has a simple analytical form with *route-specific* perception variance (i.e. non-identical perception variances with respect to trips of different lengths). However, no equivalent MP formulation has been proposed for the MNW-SUE model in the technical literature.

In this dissertation, an analytical closed-form route choice model and its MP SUE formulations are proposed to relax *both* independently and identically distributed assumptions. A *path-size* factor ([Ben-Akiva and Bierlaire, 1999](#)) is adopted to modify the MNW RUM model to create the path-size weibit (*PSW*) model, as shown in Fig. 4. Specifically, the route overlapping is captured through the path-size factor, and the route-specific perception variance is handled through the Weibull distributed random error terms. Then, both *constrained entropy-type* and *unconstrained* MP formulations for the PSW-SUE model are developed. In addition, model extensions to consider the demand elasticity and combined travel choice of the PSW-SUE model are provided. Unlike the logit-based model, these model extensions incorporate the *logarithmic* expected perceived travel cost (EPC) as the network level of service to determine the demand

elasticity and travel choice. Qualitative properties of these minimization programs are given to establish equivalency and uniqueness conditions, and algorithms to solve the proposed models are presented. Numerical examples show that the proposed models can produce a compatible traffic flow pattern compared to the MNP-SUE model under congestion, and these models can be implemented in a real-size network.

#### 1.4 Objectives

The objectives of this study were to provide:

- OBJ1. the PSW route choice model,
- OBJ2. an entropy-type MP formulation for the PSW-SUE model,
- OBJ3. a closed-form PSW EPC,
- OBJ4. an unconstrained MP formulation for the PSW-SUE model,
- OBJ5. an unconstrained MP formulation for the PSW-SUE model with elastic demand,
- OBJ6. an entropy-type MP formulation for the PSW-SUE model with elastic demand, and
- OBJ7. an entropy-type MP formulation for the combined travel choice of the PSW-SUE model with elastic demand.

The first goal was to provide the PSW route choice model to handle both route overlapping and route-specific-perception variance problems. Then, an entropy-type MP formulation for the PSW-SUE model with *route* flows as the decision variables was developed in OBJ2. Next, a closed-form PSW EPC was derived in OBJ3, which can be used to develop an unconstrained MP formulation for the PSW-SUE model with *link* flows as the decision variables in OBJ4.

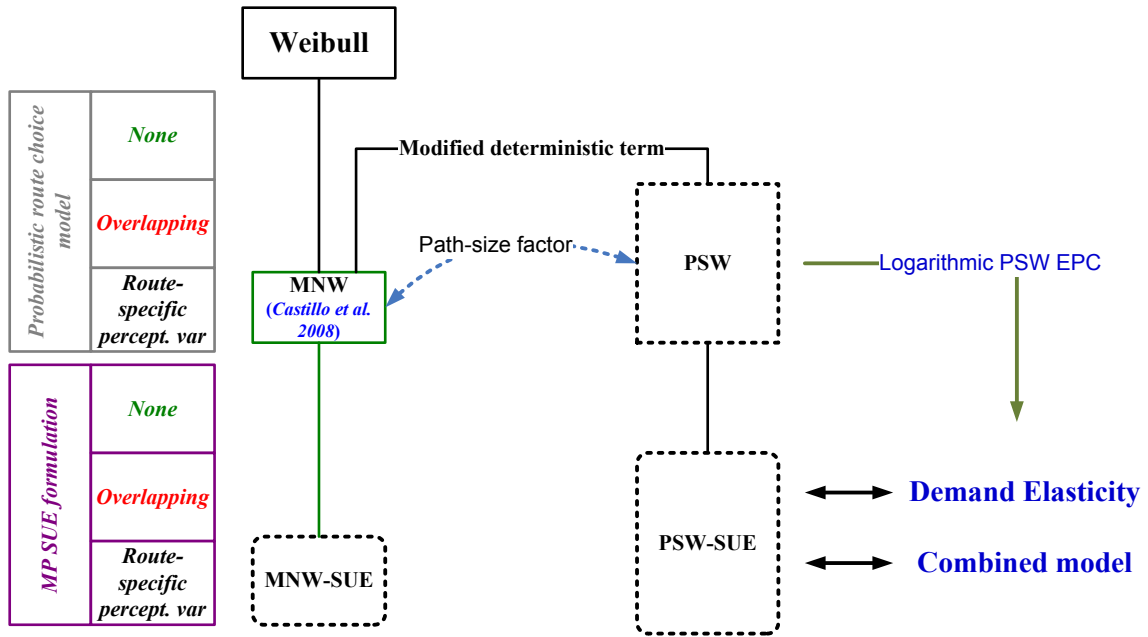


Fig. 4. Contributions of this study

The different decision variables play a significant role in the development of the solution algorithm. In addition, the PSW EPC would also be used to develop both unconstrained and entropy-type MP formulations for the PSW-SUE model with elastic demand and combined travel choice of the PSW-SUE model, where the network level of service is explicitly incorporated through the PSW EPC, in OBJ5, OBJ6, and OBJ7.

### 1.5 Organization

The organization of this dissertation is shown in Fig. 5. The next Chapter provides some background of the existing route choice models, MP SUE formulations, and solution algorithms. Chapters 3 through 5 are the main contributions of this dissertation, which consists of three technical papers. Chapter 3 provides the PSW route choice model, constrained entropy-type MP formulation for the PSW-SUE model, and a path-based solution algorithm for solving the entropy-type MP formulation. Chapter 4 derives the PSW EPC, develops an unconstrained MP formulation for the PSW-SUE model,

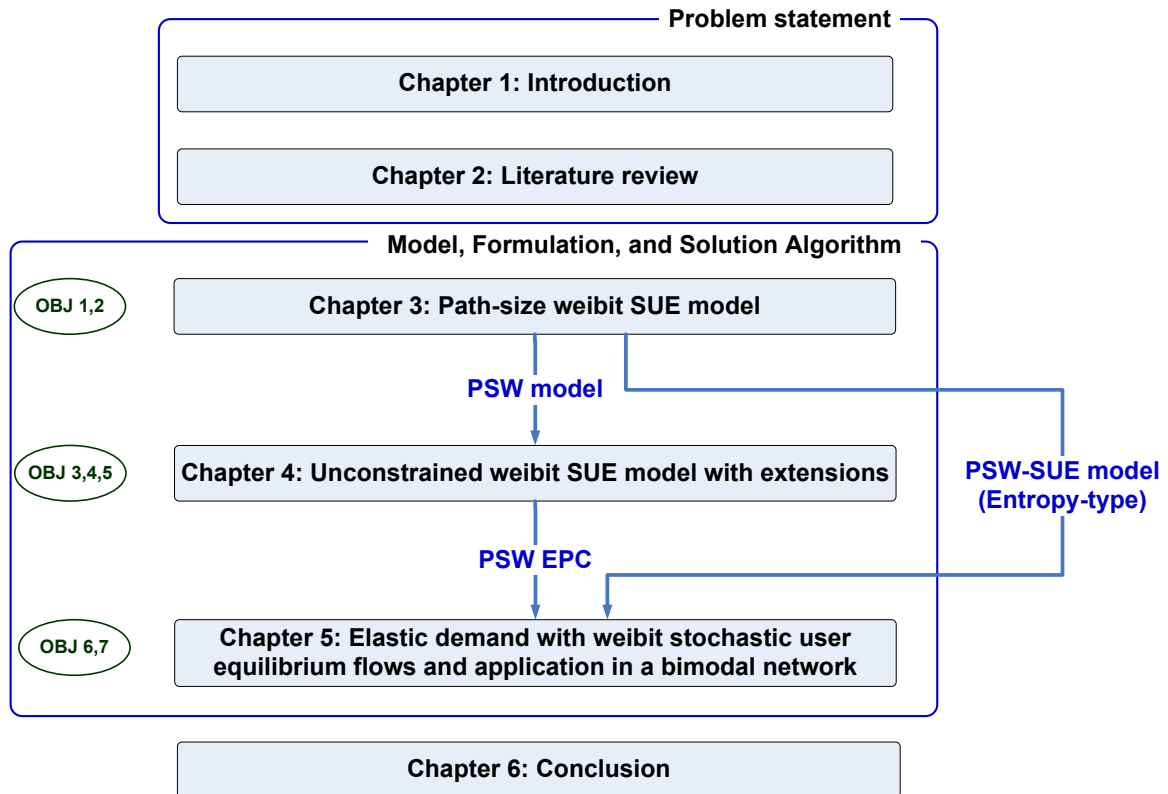


Fig. 5. Dissertation organization

provides a link-based solution algorithm for solving the unconstrained MP formulation, and delivers the model extension to consider the demand elasticity. In Chapter 5, an entropy-type MP formulation for the PSW-SUE model with an elastic demand and the combined mode choice (or modal split) of the PSW-SUE model is presented, and it provides a path-based solution algorithm for solving the proposed model in a real-size network. Conclusions and remarks for future study are provided in Chapter 6.

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## CHAPTER 2

### LITERATURE REVIEW

A strongly connected network  $[N, A]$  is considered, where  $N$  and  $A$  denote the sets of nodes and links. Let  $IJ$  denote the subsets of  $N$  to represent a set of origin-destination (O-D) pairs  $ij$ . Let  $R_{ij}$  be a set of routes (or paths) between O-D pair  $ij$ , which may consist of several links  $a \in A$ . In this section, we review the route choice models with a closed-form probability expression, their corresponding mathematical programming (MP) stochastic user equilibrium (SUE) formulations, and solution algorithms. The section begins with the route choice models, including the well-known multinomial logit (MNL) model and five extended logit models. Then, the MP SUE formulations for these logit models under congestion are provided, followed by the solution algorithms to solve these MP SUE formulations in a real-case study.

#### 2.1 Route choice models

##### 2.1.1 Multinomial logit model

The MNL model (Dial, 1971) assumes that the perceived route travel cost  $G_r^{ij}$  follows the extreme value type I distribution or the *Gumbel* distribution. With this assumption, the cumulative distribution function (CDF) of  $G_r^{ij}$ , mean route travel cost  $g_r^{ij}$ , and route perception variance  $(\sigma_r^{ij})^2$  can be expressed in Table 1. The mean travel cost  $g_r^{ij}$  is a function of the location parameter  $\lambda_r^{ij}$ , the scale parameter  $\theta_r^{ij}$ , and the Euler constant  $\gamma$ . Note that the perception variance  $(\sigma_r^{ij})^2$  is a function of  $\theta_r^{ij}$  alone.



Table 1: Gumbel distribution

CDF $F_{G_r^{ij}}(t)$	$1 - \exp\left\{-e^{\theta_r^{ij}(t-\lambda_r^{ij})}\right\}, t \in (-\infty, \infty)$	(1)
Mean route travel cost $g_r^{ij}$	$\lambda_r^{ij} - \frac{\gamma}{\theta_r^{ij}}$	(2)
Route perception variance $(\sigma_r^{ij})^2$	$\frac{\pi^2}{6\theta_r^{ij^2}}$	(3)

The MNL probability expression can be derived as follows:

$$P_r^{ij} = \Pr\left(G_r^{ij} \leq G_l^{ij}; \forall l \neq r\right), \forall r \in R_{ij}, ij \in IJ, \quad (4)$$

which corresponds to

$$P_r^{ij} = - \int_{-\infty}^{+\infty} \bar{H}_r^{ij}(\dots, t_r^{ij}, \dots) dt_r^{ij}, \quad (5)$$

where  $\bar{H}_r^{ij}$  is the partial derivative of the joint survival function w.r.t.  $t_r^{ij}$ . Under the *independently distributed* assumption, the joint survival function for the Gumbel distribution is:

$$\begin{aligned} \bar{H} &= \prod_{r \in R_{ij}} \exp\left\{-e^{\theta_r^{ij}(t_r^{ij} - \lambda_r^{ij})}\right\} \\ &= \exp\left\{-\sum_{r \in R_{ij}} e^{\theta_r^{ij}(t_r^{ij} - \lambda_r^{ij})}\right\}. \end{aligned} \quad (6)$$

Then, we have

$$\bar{H}_r^{ij} = \frac{\partial \bar{H}}{\partial t_r^{ij}} = -\theta_r^{ij} e^{\theta_r^{ij}(t_r^{ij} - \lambda_r^{ij})} \exp\left\{-\sum_{k \in R_{ij}} e^{\theta_k^{ij}(t_k^{ij} - \lambda_k^{ij})}\right\}. \quad (7)$$

Substituting Eq. (7) into Eq. (5) gives

$$P_r^{ij} = \int_{-\infty}^{+\infty} \theta_r^{ij} e^{\theta_r^{ij}(t_r^{ij} - \lambda_r^{ij})} \exp\left\{-\sum_{k \in R_{ij}} e^{\theta_k^{ij}(t_k^{ij} - \lambda_k^{ij})}\right\} dt_r^{ij}. \quad (8)$$

To obtain a closed-form probability expression,  $\theta_r^{ij}$  needs to be fixed for all routes as  $\theta$ .

With this, we have

$$\begin{aligned} P_r^{ij} &= \int_{-\infty}^{+\infty} \theta e^{\theta(t_r^{ij} - \lambda_r^{ij})} \exp\left\{-\sum_{k \in R_{ij}} e^{\theta(t_r^{ij} - \lambda_k^{ij})}\right\} dt_r^{ij} \\ &= \int_{-\infty}^{+\infty} \theta e^{\theta(t_r^{ij} - \lambda_r^{ij})} \exp\left\{-e^{\theta t_r^{ij}} \sum_{k \in R_{ij}} e^{-\theta \lambda_k^{ij}}\right\} dt_r^{ij}. \end{aligned} \quad (9)$$

By integrating Eq. (9), we have the MNL probability, i.e.,

$$P_r^{ij} = \frac{\exp(-\theta \lambda_r^{ij})}{\sum_{k \in R_{ij}} \exp(-\theta \lambda_k^{ij})}, \quad \forall r \in R_{ij}, ij \in IJ. \quad (10)$$

According to Eq. (2),  $\lambda_r^{ij}$  is related to  $g_r^{ij}$  as follows:

$$\lambda_r^{ij} = g_r^{ij} + \frac{\gamma}{\theta}. \quad (11)$$

Substituting Eq. (11) into Eq. (10) gives the MNL model, i.e.,

$$P_r^{ij} = \frac{\exp(-\theta g_r^{ij})}{\sum_{k \in R_{ij}} \exp(-\theta g_k^{ij})}, \quad \forall r \in R_{ij}, ij \in IJ. \quad (12)$$

Note that  $\theta$  is also known as the dispersion parameter (Dial, 1971).

The joint Gumbel distribution in Eq. (6) with the fixed  $\theta$  further satisfies the *stability w.r.t. the minimum operation* (Castillo et al., 2008). This important property states that joint survival extreme value function at minimum is the same family as the marginal survival extreme value function (Castillo et al., 2005). From the joint Gumbel distribution with the fixed  $\theta$ , the Gumbel survival function at minimum is also the Gumbel distribution, i.e.,

$$\bar{H}_{\min_{k \in R_{ij}}(G_k^{ij})}(t) = \exp\left\{-e^{\theta(t-\lambda^{ij,0})}\right\}, \quad (13)$$

where

$$\lambda^{ij,0} = -\frac{1}{\theta} \ln \sum_{k \in R_{ij}} \exp(-\theta \lambda_k^{ij}). \quad (14)$$

With the stability property, travelers' choice decisions are assumed to be based on their *minimum perceived* route travel cost, and the probabilistic route choice patterns can be determined by the multivariate extreme value distribution (Kotz and Nadarajah, 2000) with the Gumbel marginal. Further, we can use the Gumbel distribution to determine the EPC. From the stability property, substituting  $\lambda^{ij,0}$  in Eq. (2) gives the MNL EPC:

$$\mu_{ij} = -\frac{1}{\theta} \ln \sum_{k \in R_{ij}} \exp(-\theta \lambda_k^{ij}) - \frac{\gamma}{\theta}. \quad (15)$$

Since the constant  $-\gamma/\theta$  will not have an impact on the mathematical programming (MP) formulation considered later in this review, we can omit the term  $-\gamma/\theta$ . From Eq. (11), the MNL EPC up to a constant can be expressed as a log-sum term as follows:

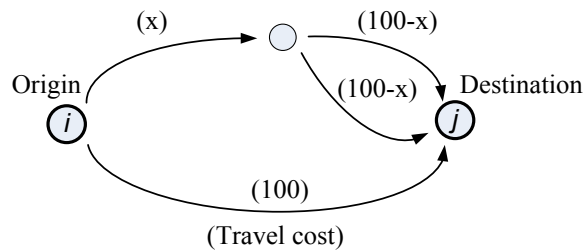
$$\mu_{ij} = -\frac{1}{\theta} \ln \sum_{k \in R_{ij}} \exp(-\theta g_k^{ij}), \quad \forall ij \in IJ. \quad (16)$$

An important property of this EPC is that the partial derivative of the MNL EPC w.r.t. the route cost gives back the MNL probability (Daganzo, 1979; Sheffi, 1985), i.e.,

$$\begin{aligned} \frac{\partial \mu_{ij}}{\partial g_r^{ij}} &= -\frac{1}{\theta} \frac{\partial \ln \sum_{k \in R_{ij}} \exp(-\theta g_k^{ij})}{\partial g_r^{ij}} \\ &= \frac{\exp(-\theta g_r^{ij})}{\sum_{k \in R_{ij}} \exp(-\theta g_k^{ij})}. \end{aligned} \quad (17)$$

Moreover, the MNL model can also be interpreted as the *Markovian* process (Akamatsu, 1996). Travelers are assumed to make a decision at each node (or *state*) until they reach the destination (or final state) according to the MNL choice probability. With this, we can use the link-based loading mechanism for the MNL loading (e.g., Dial, 1971; Sheffi, 1985; Bell, 1995; Akamatsu, 1996).

The drawback of the MNL model stems from its underlying assumption of the *independently and identically* distributed (*IID*) with Gumbel variate. The independently distributed assumption comes from the joint Gumbel distribution with independent variate in Eq. (6). The identically distributed assumption comes from the fixed  $\theta$  to obtain a closed-form probability expression in Eq. (9), since the Gumbel  $(\sigma_r^{ij})^2$  is a function of  $\theta$  alone (see Eq. (3)). As a result, the MNL model has difficulty in handling the route overlapping problem (i.e., independence assumption) and the homogeneous perception variance w.r.t. different trip lengths (i.e., identical variance assumption). Consider the loop-hole network shown in Fig. 6. In this network, all three routes have equal travel cost of 100 units. The two upper routes overlap by a portion  $x$ , while the lower route is distinct from the two upper routes. According to the *independently distributed* assumption, the MNL model gives the same probability of  $1/3$  for each route, regardless of the overlapping portion.



$$P_1^{ij} = P_2^{ij} = P_3^{ij} = \frac{\exp(-100\theta)}{\exp(-100\theta) + \exp(-100\theta) + \exp(-100\theta)} = \frac{1}{3}$$

Fig. 6. MNL probability on the loop-hole network

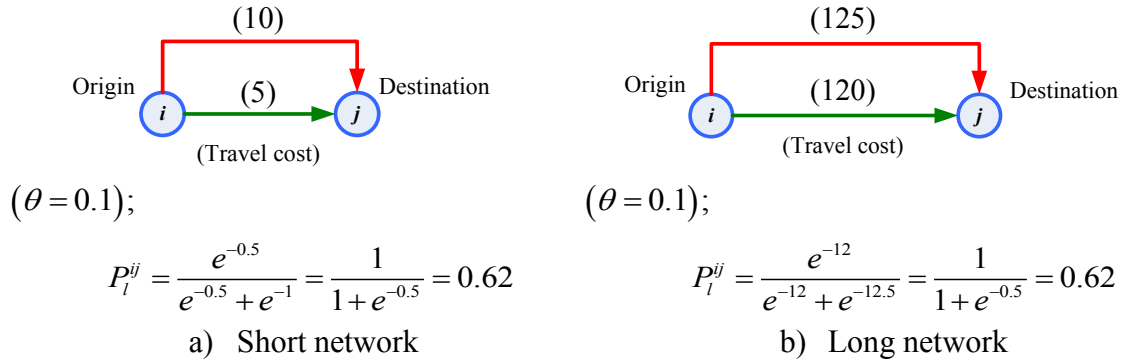


Fig. 7. MNL probabilities on the two-route networks

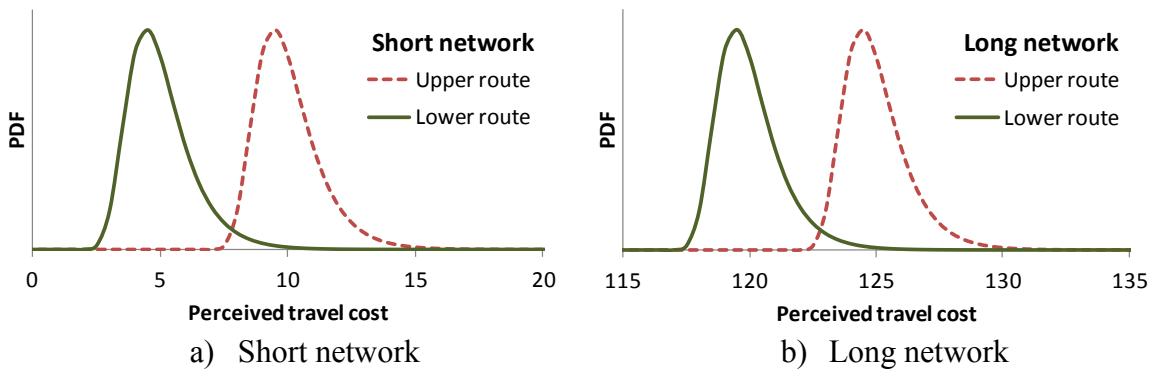


Fig. 8. MNL perception variances of the two-route networks

On the other hand, consider a two-route network configuration as shown in Fig. 7. For both networks, the upper route travel cost is larger than the lower route travel cost by 5 units. However, the upper route travel cost is two times larger than the lower route travel cost in the short network, while it is only less than 5% larger in the long network. The MNL model produces the same flow patterns for both short and long networks. This is because each route has the same perception variance of  $\pi^2/6\theta^2$  (see Eq. (3)) as shown in Fig. 8. As such, the MNL probability is solely based on the *absolute* cost difference irrespective of the overall trip lengths (Sheffi, 1985).

### 2.1.2 Extended logit models

To relax the independently distributed assumption embedded in the MNL model, several closed-form extended logit models have been developed. These models can be classified in to two categories: 1) the models modifying the deterministic term of the MNL model, and 2) the models modifying the random error term of the MNL model. Recall that the MNL model can be written in the random utility maximization (RUM) model as (Sheffi, 1985)

$$U_r^{ij} = -\theta g_r^{ij} + \xi_r^{ij}, \quad \forall r \in R_{ij}, ij \in IJ, \quad (18)$$

where  $\xi_r^{ij}$  is the *IID* Gumbel distributed random error term on route  $r$  between O-D pair  $ij$  whose CDF is

$$F_{\xi_r^{ij}}(t) = 1 - \exp(-e^t), \quad \forall r \in R_{ij}, ij \in IJ. \quad (19)$$

The models modifying the deterministic term introduce a correction factor to  $-\theta g_r^{ij}$  to adjust the probability of choosing the routes coupling with other routes, and hence the route overlapping. These models include the C-logit model (Cascetta et al., 1996) and path-size logit (PSL) model (Ben-Akiva and Bierlaire, 1999). The models modifying the random error term adopt the Generalized Extreme Value (GEV) theory (McFadden, 1978) to modify the random error term to allow the correlation, and hence the route overlapping. These models includes the cross nested logit (CNL) model (Bekhor and Prashker, 1999), generalized nested logit (GNL) model (Bekhor and Prashker, 2001), and paired combinatorial logit (PCL) model (Bekhor and Prashker, 1999). This subsection begins with the models modifying the deterministic term, followed by the models modifying the random error term (or GEV-based model).

### 2.1.2.1 Models modifying the deterministic term

We start with the C-logit model followed by the PSL model. The C-logit model uses a commonality factor  $CF_r^{ij}$  to modify the deterministic term, i.e.,

$$U_r^{ij} = -\theta(g_r^{ij} + CF_r^{ij}) + \xi_r^{ij}, \quad \forall r \in R_{ij}, ij \in IJ, \quad (20)$$

where  $CF_r^{ij}$  can be expressed as

$$CF_r^{ij} = \Omega \ln \sum_{\substack{l \neq r \\ l \in R_{ij}}} \left( \frac{L_{rl}^{ij}}{\sqrt{L_r^{ij} L_l^{ij}}} \right)^\kappa, \quad \forall r \in R_{ij}, ij \in IJ, \quad (21)$$

$L_{rl}^{ij}$  is the length of overlapping section between routes  $r$  and  $l$  between O-D pair  $ij$ ,  $L_r^{ij}$  is the length of route  $r$  between O-D pair  $ij$ , and  $\Omega$  and  $\kappa$  are the calibrated parameters. This  $CF_r^{ij}$  increases as the overlapping increases. As such, the routes coupling with other routes have a higher disutility. Note that there are several forms of  $CF_r^{ij}$  (see [Cascetta et al., 1996](#), for more information). From Eq. (20), the C-logit probability can be expressed as

$$P_r^{ij} = \frac{\exp(-\theta(g_r^{ij} + CF_r^{ij}))}{\sum_{k \in R_{ij}} \exp(-\theta(g_k^{ij} + CF_k^{ij}))}, \quad \forall r \in R_{ij}, ij \in IJ. \quad (22)$$

Since the C-logit model modifies the deterministic term, its EPC can be expressed as

$$\mu_{ij} = -\frac{1}{\theta} \ln \sum_{k \in R_{ij}} \exp(-\theta(g_k^{ij} + CF_k^{ij})), \quad \forall ij \in IJ. \quad (23)$$

Similar to the C-logit model, the PSL model uses the path-size factor  $\varpi_r^{ij}$  to modify the deterministic term as follows:

$$U_r^{ij} = \ln \varpi_r^{ij} - \theta g_r^{ij} + \xi_r^{ij}, \quad \forall r \in R_{ij}, ij \in IJ, \quad (24)$$

where  $\varpi_r^{ij}$  can be expressed as

$$\varpi_r^{ij} = \sum_{a \in \Upsilon_r} \frac{l_a}{L_r^{ij}} \frac{1}{\sum_{k \in R_{ij}} \delta_{ak}^{ij}}, \quad \forall r \in R_{ij}, ij \in IJ, \quad (25)$$

$l_a$  is the length of link  $a$ ,  $L_r^{ij}$  is the length of route  $r$  connecting O-D pair  $ij$ , and  $\Upsilon_r$  is the set of all links in route  $r$  between O-D pair  $ij$ . This path-size factor  $\varpi_r^{ij} \in (0,1]$  accounts for different route sizes determined by the length of links within a route and the relative lengths of routes that share a link. Note that several studies have provided alternative formulations for  $\varpi_r^{ij}$  (e.g., [Ramming, 2001](#); [Bovy et al., 2008](#); [Prato, 2009](#)). From Eq. (24), the PSL probability can be expressed as

$$P_r^{ij} = \frac{\varpi_r^{ij} \exp(-\theta g_r^{ij})}{\sum_{k \in R_{ij}} \varpi_k^{ij} \exp(-\theta g_k^{ij})}, \quad \forall r \in R_{ij}, ij \in IJ. \quad (26)$$

Its EPC can be expressed as

$$\mu_{ij} = -\frac{1}{\theta} \ln \sum_{k \in R_{ij}} \varpi_k^{ij} \exp(-\theta g_k^{ij}), \quad \forall ij \in IJ. \quad (27)$$

### 2.1.2.2 Models modifying the random error term

Let  $G(y_1, \dots, y_N)$  (or  $G(\cdot)$  for short) be the GEV generating function, where  $y_n \geq 0$ .

This GEV generating function satisfies the following properties ([McFadden, 1978](#)):

- 1)  $G(\cdot)$  is non-negative.
- 2)  $G(\cdot)$  is homogeneous of degree  $\mu > 0$ ; that is  $G(\alpha y_1, \dots, \alpha y_N) = \alpha^\mu G(y_1, \dots, y_N)$
- 3)  $\lim_{y_n \rightarrow \infty} G(y_1, \dots, y_N) = \infty$  for all  $n$ .
- 4) The  $l^{\text{th}}$  partial derivative of  $G(\cdot)$  w.r.t. any combination of  $l$  distinct  $y_n$ 's,  $n = 1, \dots, N$ , is non-negative if  $l$  is odd and non-positive if  $l$  is even.

With this generating function, the GEV-based probability can be derived by



$$P_r^{ij} = \frac{e^{-\theta g_r^{ij}} \tilde{G}_r^{ij} \left( e^{-\theta g_1^{ij}}, \dots, e^{-\theta g_{|R_{ij}|}^{ij}} \right)}{G \left( e^{-\theta g_1^{ij}}, \dots, e^{-\theta g_{|R_{ij}|}^{ij}} \right)}, \quad \forall r \in R_{ij}, ij \in IJ, \quad (28)$$

where  $\tilde{G}_r^{ij} \left( e^{-\theta g_1^{ij}}, \dots, e^{-\theta g_{|R_{ij}|}^{ij}} \right)$  is the partial derivative of  $G \left( e^{-\theta g_1^{ij}}, \dots, e^{-\theta g_{|R_{ij}|}^{ij}} \right)$  w.r.t. to  $e^{-\theta g_r^{ij}}$ .

Further, the EPC of the GEV-based model can be derived by

$$\mu_{ij} = -\frac{1}{\theta} \ln G \left( e^{-\theta g_1^{ij}}, \dots, e^{-\theta g_{|R_{ij}|}^{ij}} \right), \quad \forall ij \in IJ. \quad (29)$$

We start this subsection with the CNL model followed by the GNL model and the PCL model.

The CNL GEV generating function can be expressed as

$$G(\cdot) = \sum_{a \in A} \left( \sum_{r \in R_{ij}} (v_{ijar} y_r^{ij})^{\frac{1}{\phi_{ij}}} \right)^{\phi_{ij}}, \quad \forall ij \in IJ, \quad (30)$$

where  $\phi_{ij} \in (0, 1]$ , and  $v_{ijar} \in [0, 1]$  could be defined as (Bekhor and Prashker, 1999)

$$v_{ijar} = \frac{l_a}{L_r^{ij}} \delta_{ar}^{ij}, \quad \forall a \in A, r \in R_{ij}, ij \in IJ. \quad (31)$$

Both parameters represent the overlapping degree, where a larger (smaller)  $v_{ijar}$  ( $\phi_{ij}$ ) indicates a higher overlapping degree. Using the principle in Eq. (28), the CNL probability can be expressed as

$$P_r^{ij} = \frac{\sum_{a \in A} (v_{ijar})^{\frac{1}{\phi_{ij}}} \exp \left( -\frac{\theta}{\phi_{ij}} g_r^{ij} \right) \left[ \sum_{k \in R_{ij}} (v_{ijak})^{\frac{1}{\phi_{ij}}} \exp \left( -\frac{\theta}{\phi_{ij}} g_k^{ij} \right) \right]^{\phi_{ij}-1}}{\sum_{b \in A} \left[ \sum_{s \in R_{ij}} (v_{ijbs})^{\frac{1}{\phi_{ij}}} \exp \left( -\frac{\theta}{\phi_{ij}} g_s^{ij} \right) \right]^{\phi_{ij}}}, \quad \forall r \in R_{ij}, ij \in IJ. \quad (32)$$

This CNL probability can be decomposed into two levels according to the two-level tree structure, i.e.,

$$P_r^{ij} = \sum_{a \in A} P_a^{ij} P_{r|a}^{ij}, \quad \forall r \in R_{ij}, ij \in IJ, \quad (33)$$

$$P_a^{ij} = \frac{\left[ \sum_{k \in R_{ij}} (v_{ijak})^{\frac{1}{\phi_{ij}}} \exp\left(-\frac{\theta}{\phi_{ij}} g_k^{ij}\right) \right]^{\phi_{ij}}}{\sum_{b \in A} \left[ \sum_{s \in R_{ij}} (v_{ijbs})^{\frac{1}{\phi_{ij}}} \exp\left(-\frac{\theta}{\phi_{ij}} g_s^{ij}\right) \right]^{\phi_{ij}}}, \quad \forall r \in R_{ij}, ij \in IJ, \quad (34)$$

$$P_{r|a}^{ij} = \frac{(v_{ijar})^{\frac{1}{\phi_{ij}}} \exp\left(-\frac{\theta}{\phi_{ij}} g_r^{ij}\right)}{\sum_{k \in R_{ij}} (v_{ijak})^{\frac{1}{\phi_{ij}}} \exp\left(-\frac{\theta}{\phi_{ij}} g_k^{ij}\right)}, \quad \forall r \in R_{ij}, ij \in IJ. \quad (35)$$

The upper level is represented by the marginal probability ( $P_a^{ij}$ ) of selecting link  $a$  between O-D pair  $ij$ , and the lower level is represented by the conditional probability ( $P_{r|a}^{ij}$ ) of selecting route  $r$  between O-D pair  $ij$  passing through link  $a$ . Then, using Eq. (29), the CNL EPC can be expressed as

$$\mu_{ij} = -\frac{1}{\theta} \ln \sum_{b \in A} \left[ \sum_{s \in R_{ij}} (v_{ijbs})^{\frac{1}{\phi_{ij}}} \exp\left(-\frac{\theta}{\phi_{ij}} g_s^{ij}\right) \right]^{\phi_{ij}}, \quad \forall ij \in IJ. \quad (36)$$

The GNL model is a generalized version of the CNL model. Its GEV generating function can be expressed as

$$G(\cdot) = \sum_{a \in A} \left( \sum_{r \in R_{ij}} (v_{ijar} y_r^{ij})^{\frac{1}{\phi_{ja}}} \right)^{\phi_{ja}}, \quad \forall ij \in IJ, \quad (37)$$

where  $\phi_{ja}$  is specific to a link level. It could be defined as (Bekhor and Prashker, 2001)

$$\phi_{ja} = 1 - \frac{1}{\sum_{r \in R_{ij}} \delta_{ar}^{ij}} \sum_{k \in R_{ij}} v_{ijak}, \quad \forall a \in A, ij \in IJ. \quad (38)$$

Using Eq. (28), the GNL probability can be expressed as

$$P_r^{ij} = \frac{\sum_{a \in A} (v_{ijar})^{\frac{1}{\phi_{ija}}} \exp\left(-\frac{\theta}{\phi_{ija}} g_r^{ij}\right) \left[ \sum_{k \in R_{ij}} (v_{ijk})^{\frac{1}{\phi_{ija}}} \exp\left(-\frac{\theta}{\phi_{ija}} g_k^{ij}\right) \right]^{\phi_{ija}^{-1}}}{\sum_{b \in A} \left[ \sum_{s \in R_{ij}} (v_{ijbs})^{\frac{1}{\phi_{ijb}}} \exp\left(-\frac{\theta}{\phi_{ijb}} g_s^{ij}\right) \right]^{\phi_{ijb}}}, \quad \forall r \in R_{ij}, ij \in IJ. \quad (39)$$

Similar to the CNL model, the GNL probability can be decomposed into the marginal and conditional probabilities as in Eq. (33). The GNL marginal probability (upper level) can be expressed as

$$P_a^{ij} = \frac{\left[ \sum_{k \in R_{ij}} (v_{ijk})^{\frac{1}{\phi_{ija}}} \exp\left(-\frac{\theta}{\phi_{ija}} g_k^{ij}\right) \right]^{\phi_{ija}}}{\sum_{b \in A} \left[ \sum_{s \in R_{ij}} (v_{ijbs})^{\frac{1}{\phi_{ija}}} \exp\left(-\frac{\theta}{\phi_{ija}} g_s^{ij}\right) \right]^{\phi_{ija}}}, \quad \forall r \in R_{ij}, ij \in IJ, \quad (40)$$

and the GNL conditional probability (lower level) can be expressed as

$$P_{r|a}^{ij} = \frac{(v_{ijar})^{\frac{1}{\phi_{ija}}} \exp\left(-\frac{\theta}{\phi_{ija}} g_r^{ij}\right)}{\sum_{k \in R_{ij}} (v_{ijk})^{\frac{1}{\phi_{ija}}} \exp\left(-\frac{\theta}{\phi_{ija}} g_k^{ij}\right)}, \quad \forall r \in R_{ij}, ij \in IJ. \quad (41)$$

By using Eq. (29), the GNL EPC can be expressed as

$$\mu_{ij} = -\frac{1}{\theta} \ln \sum_{b \in A} \left[ \sum_{s \in R_{ij}} (v_{ijbs})^{\frac{1}{\phi_{ijb}}} \exp\left(-\frac{\theta}{\phi_{ijb}} g_s^{ij}\right) \right]^{\phi_{ijb}}, \quad \forall ij \in IJ. \quad (42)$$

Unlike the CNL and GNL models, the PCL model uses the nest between route pairs to handle the route overlapping problem. Its GEV generating function can be expressed as

$$G(\cdot) = \sum_{r=1}^{|R_{ij}|-1} \sum_{l=r+1}^{|R_{ij}|} (1 - \nu_{ijrl}) \left( y_r^{ij \frac{1}{1-\nu_{ijrl}}} + y_l^{ij \frac{1}{1-\nu_{ijrl}}} \right)^{1-\nu_{ijrl}}, \quad \forall ij \in IJ, \quad (43)$$

where  $\nu_{ijrl} \in [0,1]$  represents the degree of overlapping between routes  $r$  and  $l$ , which could be defined as (Bekhor and Prashker, 1999)

$$\nu_{ijrl} = \left( \frac{L_{rl}^{ij}}{\sqrt{L_r^{ij} L_k^{ij}}} \right)^\chi, \quad \forall r \in R_{ij}, ij \in IJ, \quad (44)$$

and  $\chi$  is a calibrated parameter. Following the same derivation as the CNL model, the PCL probability can be expressed as

$$P_r^{ij} = \frac{\sum_{l \neq r} \exp\left(\frac{-\theta g_r^{ij}}{1-\nu_{ijrl}}\right) (1-\nu_{ijrl}) \left( \exp\left(\frac{-\theta g_r^{ij}}{1-\nu_{ijrl}}\right) + \exp\left(\frac{-\theta g_l^{ij}}{1-\nu_{ijrl}}\right) \right)^{-\nu_{ijrl}}}{\sum_{k=1}^{|R_{ij}|-1} \sum_{m=k+1}^{|R_{ij}|} (1-\nu_{ijkm}) \left( \exp\left(\frac{-\theta g_k^{ij}}{1-\nu_{ijkm}}\right) + \exp\left(\frac{-\theta g_m^{ij}}{1-\nu_{ijkm}}\right) \right)^{1-\nu_{ijkm}}}, \quad \forall r \in R_{ij}, ij \in IJ. \quad (45)$$

It can be decomposed into marginal and conditional probabilities as follows:

$$P_r^{ij} = \sum_{\substack{l \neq r \\ r \in R_{ij}}} P_{rl}^{ij} P_{r|rl}^{ij}, \quad \forall r \in R_{ij}, ij \in IJ, \quad (46)$$

where

$$P_{rl}^{ij} = \frac{(1-\nu_{ijrl}) \left( \exp\left(\frac{-\theta g_r^{ij}}{1-\nu_{ijrl}}\right) + \exp\left(\frac{-\theta g_l^{ij}}{1-\nu_{ijrl}}\right) \right)^{1-\nu_{ijrl}}}{\sum_{k=1}^{|R_{ij}|-1} \sum_{m=k+1}^{|R_{ij}|} (1-\nu_{ijkm}) \left( \exp\left(\frac{-\theta g_k^{ij}}{1-\nu_{ijkm}}\right) + \exp\left(\frac{-\theta g_m^{ij}}{1-\nu_{ijkm}}\right) \right)^{1-\nu_{ijkm}}}, \quad \forall l \neq r \in R_{ij}, ij \in IJ, \quad (47)$$

$$P_{r|rl}^{ij} = \frac{\exp\left(\frac{-\theta g_r^{ij}}{1-\nu_{ijrl}}\right)}{\exp\left(\frac{-\theta g_r^{ij}}{1-\nu_{ijrl}}\right) + \exp\left(\frac{-\theta g_l^{ij}}{1-\nu_{ijrl}}\right)}, \quad \forall l \neq r \in R_{ij}, ij \in IJ. \quad (48)$$

The PCL marginal probability (upper level) is a multinomial logit probability of selecting a route pair  $rl$  among the  $R_{ij} \times R_{ij} - 1$  route pairs, and the PCL conditional probability (lower level) is simply a binary logit probability of selecting a route from the route pair.

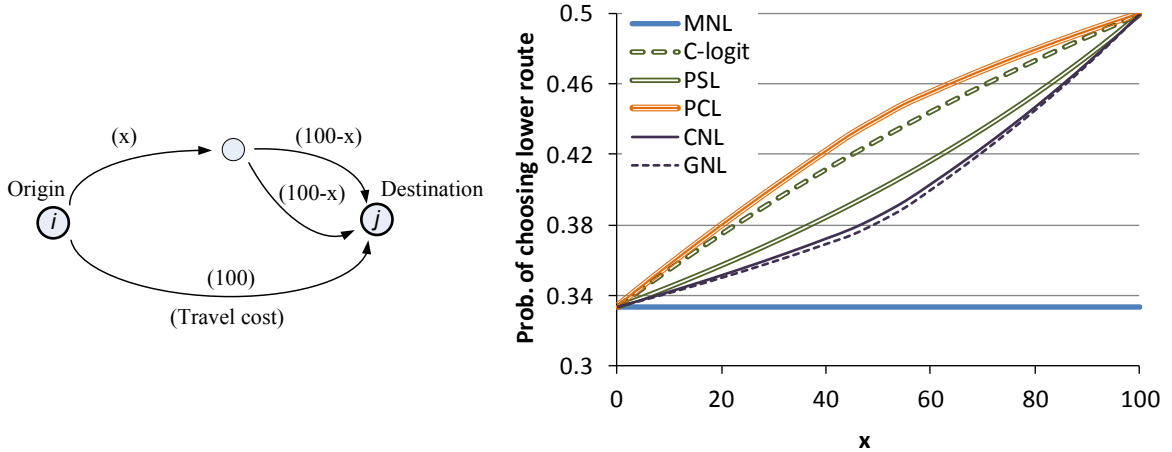


Fig. 9. Extended logit probabilities on the loop-hole network

When considering the route overlapping problem in Fig. 9, the extended logit models produce different route choice probabilities w.r.t. the overlapping portion  $x$ . Note that all models use  $\theta=0.1$ .  $\Omega$  and  $\kappa$  are equal to one for the C-logit model, and  $\chi$  is equal to one for the PCL model. Each model gives a higher probability of choosing the lower route as  $x$  increases. When  $x=100$  (i.e., only two routes with equal trip length exist), all the extended logit models produce the same probability of 0.5 in choosing two routes.

## 2.2 Mathematical programming stochastic user equilibrium formulation

In this subsection, we review a corresponding mathematical programming (MP) stochastic user equilibrium (SUE) formulation of the logit route choice models discussed in the previous subsection. The MP formulation can be classified into two categories: 1) the constrained entropy-type MP formulation and 2) the unconstrained MP formulation. The constrained formulation adopts an entropy term to handle the stochastic effect of route choice selection, while the unconstrained formulation incorporates the expected perceived cost (EPC) to develop a MP formulation. The subsection begins with the

constrained entropy-type MP formulation, followed by the unconstrained MP formulation.

### 2.2.1 Constrained entropy-type MP formulation

The constrained entropy-type MP formulation for the MNL-SUE model can be written as (Fisk, 1980)

$$\begin{aligned} \min Z &= Z_1 + Z_2 \\ &= \sum_{a \in A} \int_0^{v_a} h_a(\omega) d\omega + \frac{1}{\theta} \sum_{ij \in IJ} \sum_{r \in R_{ij}} f_r^{ij} (\ln f_r^{ij} - 1) \end{aligned} \quad (49)$$

$$\text{s.t. } \sum_{r \in R_{ij}} f_r^{ij} = q_{ij}, \quad \forall ij \in IJ, \quad (50)$$

$$f_r^{ij} \geq 0, \quad \forall r \in R_{ij}, ij \in IJ. \quad (51)$$

where  $f_r^{ij}$  denotes the flow on route  $r$  between O-D pair  $ij$ ,  $q_{ij}$  is a given demand between O-D pair  $ij$ , and  $v_a$  is the flow on link  $a$ . In Eq. (49),  $Z_1$  is the well-known “*Beckmann’s transformation*”.  $Z_2$  is the entropy term used to capture the probability flow pattern. It gives the *exponential proportion* in the equivalency conditions that is needed in the logit probability solution. Eq. (50) is the flow conservation constraint, and Eq. (51) is the non-negativity constraint.

To incorporate the route overlapping, both C-logit-SUE and PSL-SUE models add another entropy term (Chen et al., 2012; Zhou et al., 2012). The C-logit-SUE model incorporates the commonality factor through  $Z_3$ , i.e.,

$$\begin{aligned} \min Z &= Z_1 + Z_2 + Z_3 \\ &= \sum_{a \in A} \int_0^{v_a} h_a(\omega) d\omega + \frac{1}{\theta} \sum_{ij \in IJ} \sum_{r \in R_{ij}} f_r^{ij} (\ln f_r^{ij} - 1) + \sum_{ij \in IJ} \sum_{r \in R_{ij}} f_r^{ij} CF_r^{ij}, \end{aligned} \quad (52)$$

subject to the flow conservation constraint in Eq. (50) and the non-negativity constraint in Eq. (51). Similarly, the PSL-SUE model also incorporates the commonality factor through  $Z_3$ , i.e.,

$$\begin{aligned} \min Z &= Z_1 + Z_2 + Z_3 \\ &= \sum_{a \in A} \int_0^{v_a = \sum_{ij \in IJ} \sum_{r \in R_{ij}} f_r^{ij} \delta_{ar}^{ij}} h_a(\omega) d\omega + \frac{1}{\theta} \sum_{ij \in IJ} \sum_{r \in R_{ij}} f_r^{ij} (\ln f_r^{ij} - 1) - \frac{1}{\theta} \sum_{ij \in IJ} \sum_{r \in R_{ij}} f_r^{ij} \ln \omega_r^{ij} \end{aligned} \quad (53)$$

subject to the flow conservation constraint in Eq. (50) and the non-negativity constraint in Eq. (51).

On the other hand, the GEV-based models require a modified entropy term. This is because these models have a two-level tree structure (i.e., marginal and conditional probabilities). In addition, the decision variables are not the same as the MNL-SUE, C-logit-SUE, and PSL-SUE models where the decision variables are the ordinary  $f_r^{ij}$ . The decision variables for the GEV-based models also need to correspond the two-level tree structure of each model. The CNL-SUE model can be written as (Bekhor and Prashker, 1999):

$$\begin{aligned} \min Z &= Z_1 + Z_2 + Z_3 \\ &= \sum_{a \in A} \int_0^{v_a = \sum_{ij \in IJ} \sum_{r \in R_{ij}} f_r^{ij} \phi_{ar}^{ij}} h_a(\omega) d\omega + \frac{1}{\theta} \sum_{ij \in IJ} \sum_{a \in A} \sum_{r \in R_{ij}} \phi_{ij} f_{ar}^{ij} \left( \ln \frac{f_r^{ij}}{(v_{ijar})^{\frac{1}{\phi_{ij}}}} - 1 \right) \end{aligned} \quad (54)$$

$$- \frac{1}{\theta} \sum_{ij \in IJ} \sum_{a \in A} (1 - \phi_{ij}) \left( \sum_{r \in R_{ij}} f_{ar}^{ij} \right) \left( \ln \left( \sum_{r \in R_{ij}} f_{ar}^{ij} \right) - 1 \right)$$

$$\text{s.t. } \sum_{a \in Y_r} \sum_{r \in R_{ij}} f_{ar}^{ij} = q_{ij}, \quad \forall ij \in IJ, \quad (55)$$

$$f_{ar}^{ij} \geq 0, \quad \forall a \in Y_r, r \in R_{ij}, ij \in IJ, \quad (56)$$

where  $f_{ar}^{ij}$  is the flow on route  $r$  (from link  $a$ ) between O-D pair  $ij$  as the decision variable corresponding to the CNL nested structure.  $Z_2$  is adopted to incorporate the CNL conditional probability (lower level), and  $Z_3$  is adopted to incorporate the CNL marginal probability (upper level). Similar to the MNL-SUE model, Eq. (55) and Eq. (56) are the flow conservation constraint and the non-negativity constraint, respectively. Since the GNL model is a generalized version of the CNL model, the GNL-SUE model can be expressed as (Bekhor and Prashker, 2001)

$$\begin{aligned}
\min Z &= Z_1 + Z_2 + Z_3 \\
&= \sum_{a \in A} \int_0^{v_a = \sum_{ij \in IJ} \sum_{r \in R_{ij}} f_r^{ij} \delta_{ar}^{ij}} h_a(\omega) d\omega + \frac{1}{\theta} \sum_{ij \in IJ} \sum_{a \in A} \sum_{r \in R_{ij}} \phi_{ija} f_{ar}^{ij} \left( \ln \frac{f_r^{ij}}{(v_{ijar})^{\frac{1}{\phi_{ija}}}} - 1 \right) \\
&\quad - \frac{1}{\theta} \sum_{ij \in IJ} \sum_{a \in A} (1 - \phi_{ija}) \left( \sum_{r \in R_{ij}} f_{ar}^{ij} \right) \left( \ln \left( \sum_{r \in R_{ij}} f_{ar}^{ij} \right) - 1 \right)
\end{aligned} \tag{57}$$

subject to Eq. (55) and Eq. (56). Unlike the CNL and GNL models, the PCL model has a two-level tree structure according a route pair. With this, the PCL-SUE model can be written as (Bekhor and Prashker, 1999)

$$\begin{aligned}
\min Z &= Z_1 + Z_2 + Z_3 \\
&= \sum_{a \in A} \int_0^{v_a = \sum_{ij \in IJ} \sum_{r \in R_{ij}} f_r^{ij} \delta_{ar}^{ij}} h_a(\omega) d\omega + \frac{1}{\theta} \sum_{ij \in IJ} \sum_{r \in R_{ij}} \sum_{\substack{k \neq r \\ r \in R_{ij}}} (1 - v_{ijrk}) f_{r(rk)}^{ij} \left( \ln \frac{f_{r(rk)}^{ij}}{(1 - v_{ijrk})} - 1 \right) \\
&\quad - \frac{1}{\theta} \sum_{ij \in IJ} \sum_{r=1}^{|R_{ij}|-1} \sum_{k=r+1}^{|R_{ij}|} v_{ijrk} (f_{r(rk)}^{ij} + f_{k(rk)}^{ij}) \left( \ln \frac{f_{r(rk)}^{ij} + f_{k(rk)}^{ij}}{(1 - v_{ijrk})} - 1 \right)
\end{aligned} \tag{58}$$

$$\text{s.t. } \sum_{\substack{r \in R_{ij} \\ k \neq r \\ r \in R_{ij}}} f_{r(rk)}^{ij} = q_{ij}, \quad \forall ij \in IJ, \tag{59}$$

$$f_{r(rk)}^{ij} \geq 0, \quad \forall r \neq k \in R_{ij}, ij \in IJ, \tag{60}$$



where  $f_{r(rk)}^{ij}$  is the flow on route  $r$  (of route pair  $r$  and  $k$ ) between O-D pair  $ij$  as the decision variable corresponding to the PCL nested structure.  $Z_2$  is adopted to incorporate the PCL conditional probability (lower level), and  $Z_3$  is adopted to incorporate the PCL marginal probability (upper level).

### 2.2.2 Unconstrained MP formulation

Unlike the constrained MP formulation where the corresponding optimization program is different for each individual route choice model, the unconstrained MP formulation for all logit models (discussed in the previously) can be written as

$$\begin{aligned} \min Z &= Z_1 + Z_2 + Z_3 \\ &= -\sum_{a \in A} \int_0^{v_a} h_a(\omega) d\omega - \sum_{ij \in IJ} q_{ij} \mu_{ij} + \sum_{a \in A} v_a h_a(v_a). \end{aligned} \quad (61)$$

$Z_1$  is still the Beckmann's transformation.  $Z_2$  is used to incorporate the EPC to obtain a particular logit flow solution.  $Z_3$  is the network performance. For example, if we use the EPC in (16), the above MP formulation is corresponded to the MNL-SUE model.

Moreover, unlike the constrained entropy-type formulation, the unconstrained MP formulation has the *link* flow as the decision variables, regardless of the probability tree structure. with the link flow decision variables, the unconstrained MP formulation obviates the route storage in the entropy-type formulation when implementing some link-based loading techniques (e.g., Dial, 1971; Sheffi, 1985; Bell, 1995; Akamatsu, 1996). A link-based algorithm can be applied to solve the *unconstrained* SUE problem (e.g., Sheffi, 1985; Maher, 1998).

## 2.3 Solution algorithms

This subsection reviews the solution algorithm for solving the SUE MP problem. Generally, this algorithm has two main steps: search direction and line search. A search direction is obtained by solving a convex auxiliary problem through the equivalency conditions (i.e., the first-order approximation of the objective function). This can be done by performing a stochastic loading scheme that produces the flow pattern corresponding to the route choice model under consideration. A line search is computed in the search direction w.r.t. the original objective function, and then the resulting stepsize defines a new solution with a reduced objective value. Detail procedures are depended on the decision variables. The subsection begins with the path-based solution algorithm for solving the constrained entropy-type SUE MP formulation, followed by the link-based solution algorithm for solving the unconstrained SUE MP formulation.

### 2.3.1 Path-based solution algorithm

Several *path-based* solution algorithms has been developed to solve the entropy-type logit-based SUE model (e.g., [Damberg et al., 1996](#); [Bell et al., 1997](#); [Leurent, 1997](#); [Chen et al., 2005](#); [Chootinan et al., 2005](#); [Zhou et al., 2012](#)). A fundamental framework of this path-based solution algorithm is presented as follows.

---

#### Step 0: Initialization

- Set  $n = 0$ ; perform the stochastic loading according to the free flow travel costs to obtain a feasible route flow solution  $\mathbf{f}^{(1)}$ ;

#### Step 1: Direction finding

- Increment  $n = n+1$ ; update link/route travel costs; perform the MNL loading to
-

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obtain auxiliary route flows  $\tilde{f}$ , and search direction  $\begin{pmatrix} \tilde{f} \\ - \end{pmatrix}$ ;

Step 2: Line search

- Solve  $\min_{0 \leq \alpha \leq 1} Z \left[ \mathbf{f}^{(n-1)} + \alpha^{(n)} \begin{pmatrix} \tilde{f} \\ - \end{pmatrix} \right]$  to obtain  $\alpha^{(n)}$ ;

Step 3: Move

- $\mathbf{f}^{(n)} = \mathbf{f}^{(n-1)} + \alpha^{(n)} \begin{pmatrix} \tilde{f} \\ - \end{pmatrix}$ ;

Step 4: Convergence test

- If the stopping criterion is reached, terminate; otherwise go to Step 1.
- 

The stochastic loading would be according to the route choice model under consideration. For example, if we solve the MNL-SUE model, the stochastic loading will be the MNL probability from the route travel cost at each iteration.

There are some remarks for this fundamental algorithmic framework. Since the constrained entropy-type formulation has the route flow variables as its decision variable, this model need to *store* the route (or path) set. This route set can be obtained from the survey and/or route set generation (e.g., [Prato, 2009](#)). To avoid this route set enumeration, a column generation procedure ([Dantzig, 1963](#)) can be adopted which is determined by a lowest-cost route at each iteration.

For the line search procedure, we may adopt the golden section or bisection methods to determine the exact stepsize. However, both methods are computationally expensive for this complicated objective function. To overcome this problem, one may select the predetermined stepsize strategy such as the method of successive average (MSA) (e.g.,  $\alpha^{(n)} = 1/n$ , that satisfies  $\alpha(n) \rightarrow 0$  and  $\sum_{n=0}^{\infty} \alpha(n) = \infty$  to guarantee convergence). Even

though this method is easy to implement, it suffers from a sublinear convergence rate (Nagurney, 1999). In practice, inexact line search schemes are usually more practical and efficient. Among others, the Armijo-type methods (Armijo, 1966) are perhaps the most well-known inexact line search scheme. This line search scheme method has been used to solve some SUE problems and showed a better performance compared to the exact line search (e.g., Bekhor et al., 2008). It can be determined by (Bertsekas, 1976)

$$\alpha^{(n)} = \rho^{m(n)} s, \quad (62)$$

where  $m(n)$  is the first non-negative integer  $m$  such that

$$Z(\mathbf{f}^{(n)}) - Z(\mathbf{f}^{(n)}(\rho^m s)) \geq \kappa \nabla Z(\mathbf{f}^{(n)})^T (\mathbf{f}^{(n)} - \mathbf{f}^{(n)}(\rho^m s)), \quad (63)$$

where  $Z(\cdot)$  and  $\nabla Z(\cdot)$  are the objective function and its gradient w.r.t. to the solution;  $\kappa \in (0, 1)$ ,  $\rho \in (0, 1)$ , and  $s > 0$  are fixed scalars. From Eq. (63), we can obtain an appropriate stepsize (unnecessarily the optimal stepsize) with some evaluations, and therefore we can avoid solving the computationally expensive exact line search.

In the literature, there are another two promising line search schemes, recently proposed to enhance the computational performance of determining a suitable stepsize: the self regulated averaging (SRA) scheme by Liu et al. (2009) and the quadratic interpolation scheme by Maher (1998). The SRA scheme is a modified version of the MSA. It has an adjustable stepsize according to the residual error. Its stepsize can be

$$\text{expressed as } \alpha^{(n)} = \frac{1}{\eta^{(n)}}, \text{ where } \eta^{(n)} = \begin{cases} \eta^{(n-1)} + \lambda_1, & \text{if } \|\tilde{\mathbf{f}}^{(n)} - \mathbf{f}^{(n)}\| \geq \epsilon \\ \eta^{(n-1)} + \lambda_2, & \text{otherwise} \end{cases}$$

,  $\lambda_1 > 1$ , and  $0 < \lambda_2 < 1$ . With this scheme,  $\alpha^{(n)}$  is adjusted according to the residual error (i.e., the deviation between the current solution and its auxiliary solution) relationship of two

consecutive iterations. When the current residual error is larger than the previous iteration,  $\lambda_1 > 1$  makes a more aggressive reduction in the stepsize. On the other hand, when the residual error is smaller than the previous iteration,  $0 < \lambda_2 < 1$  makes the stepsize reduction more conservative.

For the quadratic interpolation scheme, the stepsize can be approximated from

$$\alpha^{(n)} = \frac{-\nabla_{\alpha} Z(\alpha)|_{\alpha=0}}{-\nabla_{\alpha} Z(\alpha)|_{\alpha=0} + \nabla_{\alpha} Z(\alpha)|_{\alpha=1}}, \quad (64)$$

where  $\nabla_{\alpha} Z(\alpha)$  is the derivative of objective function w.r.t. stepsize. In this scheme,  $\nabla_{\alpha} Z(\alpha)$  are evaluated twice per iteration (or two ends), one at  $\alpha=0$  and another at  $\alpha=1$ , to determine an approximate stepsize.

### 2.3.2 Link-based solution algorithm

A fundamental framework for this link-based algorithm can be expressed as follows (e.g., [Sheffi, 1985](#)).

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#### Step 0: Initialization

- Set  $n = 0$ ; perform the link-based stochastic loading according to the free flow travel costs to obtain a feasible link flow solution  $\mathbf{v}^{(1)}$ ;

#### Step 1: Direction finding

- Increment  $n = n+1$ ; update link travel costs; perform the link-based stochastic loading to obtain auxiliary link flows  $\tilde{\mathbf{v}}^{(n)}$ , and search direction  $(\tilde{\mathbf{v}}^{(n)} - \mathbf{v}^{(n-1)})$ ;

#### Step 2: Line search

- Solve  $\min_{0 \leq \alpha \leq 1} Z[\mathbf{v}^{(n-1)} + \alpha^{(n)} (\tilde{\mathbf{v}}^{(n)} - \mathbf{v}^{(n-1)})]$  to obtain  $\alpha^{(n)}$ ;

#### Step 3: Move

- $\mathbf{v}^{(n)} = \mathbf{v}^{(n-1)} + \alpha^{(n)} (\tilde{\mathbf{v}}^{(n)} - \mathbf{v}^{(n-1)})$ ;
-

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Step 4: Convergence test

- If the stopping criterion is reached, terminate; otherwise go to Step 1.
- 

Some link-based stochastic loading schemes have been developed in the literature. The MNL link-based stochastic loading scheme can be found, for example, in [Dial \(1971\)](#), [Sheffi \(1985\)](#), [Bell \(1995\)](#), and [Akamatsu \(1996\)](#) (also see subsection 2.2.1). Later, [Russo and Vitetta \(2003\)](#) developed a link-based loading scheme for the C-logit model, where the correction factor was modified to capture the overlapping at the link level. With this, we can obviate the path (or route) storage needed in the path-based solution algorithms.

For the line search procedure, one may adopt the MSA and SRA without the need to evaluate the (complicated) objective function. However, we may suffer from a sublinear convergence rate. Recently, [Maher \(1998\)](#) adopted the interpolation scheme to determine the stepsize of this unconstrained formulation.  $\nabla_{\alpha}Z(\alpha)$  can be determined from the first derivative of the objective function w.r.t. the stepsize, which gives

$$\nabla_{\alpha}Z(\alpha) = \frac{dZ}{d\alpha} = \sum_{a \in A} \left( v_a^{(n)} \right)^2 \frac{dh_a}{dv_a}. \quad (65)$$

Then, we can use this simple objective function gradient to approximate the stepsize in Eq. (64). Note that, to evaluate  $\nabla_{\alpha}Z(\alpha)$  at  $\alpha=1$ , we need to obtain another auxiliary link flows (see [Maher, 1998](#)). In other words, we need to perform the stochastic loading twice per iteration to approximate the stepsize.

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## CHAPTER 3

## A PATH-SIZE WEIBIT STOCHASTIC USER EQUILIBRIUM MODEL

## Abstract

The aim of this paper is to develop a path-size weibit (*PSW*) route choice model with an equivalent mathematical programming (*MP*) formulation under the stochastic user equilibrium (*SUE*) principle that can account for both route overlapping and route-specific perception variance problems. Specifically, the Weibull distributed random error term handles the identically distributed assumption such that the perception variance with respect to different trip lengths can be distinguished, and a path-size factor term is introduced to resolve the route overlapping issue by adjusting the choice probabilities for routes with strong couplings with other routes. A multiplicative Beckmann's transformation (*MBec*) combined with an entropy term are used to develop the MP formulation for the PSW-SUE model. A path-based algorithm based on the partial linearization method is adopted for solving the PSW-SUE model. Numerical examples are also provided to illustrate features of the PSW-SUE model and its differences compared to some existing SUE models as well as its applicability on a real-size network.

## 3.1 Introduction

The stochastic user equilibrium (SUE) model is well-known in the literature. It relaxes the perfect information assumption of the deterministic user equilibrium model by incorporating a random error term in the route travel cost function to simulate travelers' imperfect perceptions of travel costs. Route choice models under this approach can have different specifications according to the modeling assumptions on the random error term.

The two commonly used random error terms are *Gumbel* (Dial, 1971) and *Normal* (Daganzo and Sheffi, 1977) distributions, corresponding to the multinomial logit (MNL) and multinomial probit (MNP) route choice models, respectively. MNL model has a closed-form probability expression and can be formulated as an equivalent mathematical programming formulation (MP) by using an entropy-type model for the logit-based SUE problem (Fisk, 1980). The drawbacks of the MNL model are: (1) inability to account for overlapping (or correlation) among routes and (2) inability to account for perception variance with respect to (w.r.t.) trips of different lengths. These two drawbacks stem from the underlying assumptions that the random error terms are *independently and identically distributed* (IID) with the same and fixed perception variance (Sheffi, 1985). MNP route choice model, on the other hand, does not have such drawbacks, because it handles the route overlapping and identical perception variance problems between routes by allowing the covariance between random error terms for pairs of routes. However, the MNP model does not have a closed-form solution and it is computationally burdensome when the choice set contains more than a handful of routes. Due to the lack of a closed-form probability expression, solving the MNP model will require either Monte Carlo simulation (Sheffi and Powell, 1982), Clark's approximation method (Maher, 1992), or numerical method (Rosa and Maher, 2002).

To overcome the deficiencies of the MNL model, some analytical closed-form extensions have been proposed in the literature. These models can be broadly classified into two groups: extended logit models and Weibull-based model as shown in Fig. 10. The extended logit models were developed mainly to handle the route overlapping problem. These models modified either the deterministic term or the random error term in

the additive disutility function of the MNL model while maintaining the Gumbel distributed random error term assumption. The models modifying the deterministic term of the disutility function include the C-logit model (Cascetta et al., 1996), the implicit availability/perception (IAP) model (Cascetta et al., 2002), and the path-size logit (PSL) model (Ben-Akiva and Bierlaire, 1999). All three models add a correction term to the deterministic term of the disutility function to adjust the choice probability; however, the interpretation of each model is different. The C-logit model uses the commonality factor to penalize the coupling routes, while both the IAP and PSL models use a logarithmic correction term to modify the disutility (hence, the choice probability). Equivalent MP formulations for the C-logit model and PSL model were recently provided by Zhou et al. (2012) and Chen et al. (2012), respectively. The models modifying the random error term of the disutility function include the cross-nested logit (CNL) model (Vovsha, 1997; Bekhor and Prashker, 1999), the paired combinatorial logit (PCL) model (Chu, 1989; Bekhor and Prashker, 1999; Gliebe et al., 1999; Pravinvongvuth and Chen, 2005), and the generalized nested logit (GNL) model (Bekhor and Prashker, 2001; Wen and Koppelman, 2001). These models are based on the generalized extreme value (GEV) theory (McFadden, 1978) using a two-level tree structure to capture the similarity among routes through the random error component of the disutility function. Equivalent MP formulations for all three models were given by Bekhor and Prashker (1999, 2001). Recall that the extended logit models with closed-form solution discussed above were developed to mainly address the *independence* assumption (i.e., route overlapping problem) of the MNL-SUE model. The *identically distributed* assumption (i.e., homogeneous perception variance problem) is still inherited in these extended logit-based

SUE models. In other words, the perception variance is fixed (or constant) with respect to trips of different lengths over all routes and all origin-destination (O-D) pairs. In view of network equilibrium assignment, the *identically distributed* assumption seems unrealistic since it does not distinguish trip lengths of different O-D pairs. Hence, [Chen et al. \(2012\)](#) suggested a practical approach to partially relax the assumption by scaling the perception variance of an individual O-D pair. The *individual O-D specific scaling factors* allow the perception variance to increase or decrease according to the travel distance of each O-D pair. Specifically, the systematic disutility in the logit-based SUE models can be scaled appropriately to reflect different O-D trip lengths in a network by replacing a single dispersion parameter for all O-D pairs with *individual O-D dispersion parameters* for each O-D pair. However, it should be noted that it is not possible to scale individual routes of the same O-D pair since it would violate the logit-based SUE models' assumption of an equal variance across the routes within the same O-D pair. For a more comprehensive review of the extended logit models used in the SUE problem, readers are directed to the reviews given by [Prashker and Bekhor \(2004\)](#) and [Chen et al. \(2012\)](#).

On the other hand, [Castillo et al. \(2008\)](#) proposed the multinomial weibit (*MNW*) model to address the *identically distributed* assumption. This model assumes that the perceived route travel time follows the *Weibull* distribution, instead of the conventional Gumbel distribution. Under the *independence* assumption, the MNW model has a simple analytical form with route-specific perception variance (i.e., non-identical perception variances with respect to trips of different lengths). However, no equivalent MP formulation has been provided for the MNW-SUE model.

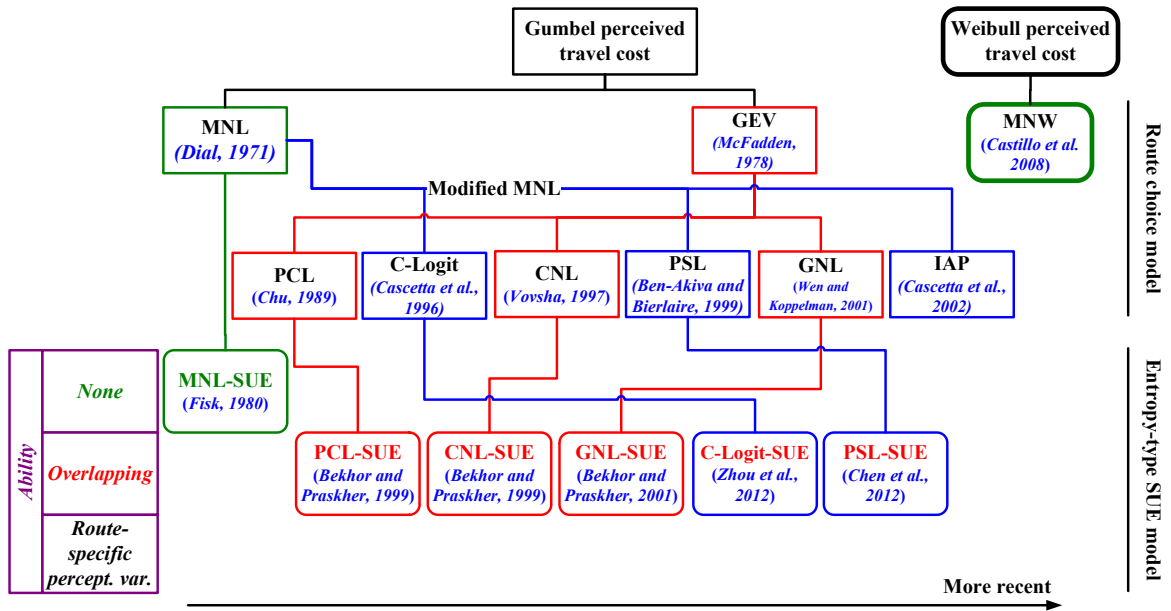


Fig. 10. Chronicle development of some closed-form route choice models and their MP formulations

To our best knowledge, no closed-form probability expression with an equivalent MP formulation has been provided to simultaneously address both route overlapping and route-specific perception variance problems in the literature. The purpose of this paper is to provide a path-size weibit (*PSW*) route choice model with an equivalent MP SUE formulation that can account for both route overlapping and route-specific perception variance problems. Specifically, the Weibull distributed random error term handles the *identically distributed assumption* such that the perception variance w.r.t. different trip lengths can be distinguished, and a path-size factor term is introduced to resolve the *route overlapping issue* by adjusting the choice probabilities for routes with strong couplings with other routes. A *multiplicative* Beckmann's transformation (*MBec*) combined with an entropy term are used to develop the MP formulation for the *PSW-SUE* model. Some qualitative properties of the *PSW-SUE* formulation are rigorously proved. A path-based algorithm based on the partial linearization method is adopted for solving the *PSW-SUE*

model. Numerical examples are also provided to illustrate features of the PSW-SUE model and its differences compared to some existing SUE models as well as its applicability on a real-size network.

The remaining of this paper is organized as follows. Section 2 provides some background of the MNW route choice model and develops the PSW model. In section 3, equivalent MP formulations for the MNW-SUE and PSW-SUE models are provided along with some qualitative properties. Section 4 presents a path-based algorithm for solving the SUE formulations. Numerical results are presented in Section 5, and some concluding remarks are provided in Section 6.

### 3.2 Weibit route choice models

In this section, we provide some background of the multinomial weibit\* (MNW) route choice model and the development of the path-size weibit (PSW) model. Specifically, we show how the MNW model resolves the identical perception variance issue inherited in the classical multinomial logit (MNL) model. Then, a path-size factor is introduced to the MNW random utility maximization (RUM) model to develop the PSW model that can address both route overlapping and non-identical route perception variance problems.

#### 3.2.1 MNW model

[Castillo et al. \(2008\)](#) developed the MNW model to relax the *identically distributed* assumption embedded in the MNL model. Instead of the Gumbel distribution, this closed-form route choice model is derived from the *Weibull* distribution presented in Table 2.

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\* The term “weibit” stands for “*Weibull* probability unit”. Note that this term has also been used in other disciplines, for example, the bioassay ([Looney, 1983](#)), contingent valuation model in economics ([Genius and Strazzer, 2002](#)), and reliability engineering ([Strong et al., 2009](#)).

Let  $G_r^{ij}$  denote travelers' perceived travel cost on route  $r \in R_{ij}$  between O-D pair  $ij \in IJ$ , and  $g_r^{ij}$  be the mean of  $G_r^{ij}$  (or the mean travel cost). Then, the cumulative distribution function (CDF) of  $G_r^{ij}$  can be expressed as the negative exponential function. The mean travel cost  $g_r^{ij}$  is a function of the location parameter  $\zeta_r^{ij} \in [0, G_r^{ij})$ , the shape parameter  $\beta_r^{ij} \in (0, \infty)$ , and the scale parameter  $\varphi_r^{ij} \in (0, \infty)$ , where  $\Gamma(\cdot)$  is the gamma function. Unlike the Gumble distribution, the perception variance of the Weibull distribution  $(\sigma_r^{ij})^2$  is also a function of  $g_r^{ij}$ .

Further, unlike the MNL model which used the conventional *additive* RUM (ARUM), the MNW model adopts the *multiplicative* RUM (MRUM) with the Weibull distribution as the random error term (Fosgerau and Bierlaire, 2009). By relating  $\varphi_r^{ij}$  with  $g_r^{ij}$  in Eq. (67), the MNW disutility function can be written as

Table 2: Weibull distribution

CDF $F_{G_r^{ij}}(t)$	$1 - \exp \left\{ - \left[ \frac{(t - \zeta_r^{ij})}{\varphi_r^{ij}} \right]^{\beta_r^{ij}} \right\}, t \in (0, \infty)$ (66)
Mean travel cost $g_r^{ij}$	$\zeta_r^{ij} + \varphi_r^{ij} \Gamma \left( 1 + \frac{1}{\beta_r^{ij}} \right)$ (67)
Route perception variance $(\sigma_r^{ij})^2$	$(\varphi_r^{ij})^2 \Gamma \left( 1 + \frac{2}{\beta_r^{ij}} \right) - (g_r^{ij} - \zeta_r^{ij})^2$ (68)



$$U_r^{ij} = \left( g_r^{ij} - \zeta^{ij} \right)^{\beta^{ij}} \varepsilon_r^{ij}, \quad \forall r \in R_{ij}, ij \in IJ, \quad (69)$$

where  $\varepsilon_r^{ij}$  is the *independently* Weibull distributed random error term on route  $r$  between O-D pair  $ij$  whose CDF is  $F_{\varepsilon_r^{ij}}(t) = 1 - \exp(-t)$ . Then, the MNW probability can be determined by

$$\begin{aligned} P_r^{ij} &= \Pr \left( \left( g_r^{ij} - \zeta^{ij} \right)^{\beta^{ij}} \varepsilon_r^{ij} \leq \left( g_l^{ij} - \zeta^{ij} \right)^{\beta^{ij}} \varepsilon_l^{ij}, \forall l \neq r \right), \quad \forall r \in R_{ij}, ij \in IJ \\ &= \Pr \left( \frac{\left( g_r^{ij} - \zeta^{ij} \right)^{\beta^{ij}}}{\left( g_l^{ij} - \zeta^{ij} \right)^{\beta^{ij}}} \varepsilon_r^{ij} \leq \varepsilon_l^{ij}, \forall l \neq r \right), \quad \forall r \in R_{ij}, ij \in IJ. \end{aligned} \quad (70)$$

Then, we can compute the choice probability from

$$P_r^{ij} = - \int_{-\infty}^{+\infty} \bar{H}_r^{ij}(\dots, \varepsilon_r^{ij}, \dots) d\varepsilon_r^{ij}, \quad \forall r \in R_{ij}, ij \in IJ, \quad (71)$$

where  $\bar{H}_r^{ij}$  is the partial derivative of the joint survival function w.r.t.  $\varepsilon_r^{ij}$ . For the weibit RUM model, the CDF of each random error term is

$$F(\varepsilon_r^{ij}) = 1 - \exp(-\varepsilon_r^{ij}), \quad (72)$$

and hence the survival function of each random error term is

$$\bar{F}(\varepsilon_r^{ij}) = 1 - F(\varepsilon_r^{ij}) = \exp(-\varepsilon_r^{ij}). \quad (73)$$

Under the *independence* assumption, the joint survival function can be expressed as

$$\bar{H} = \prod_{r \in R_{ij}} \bar{F}(\varepsilon_r^{ij}) = \exp \left( - \sum_{r \in R_{ij}} \varepsilon_r^{ij} \right), \quad (74)$$

which gives

$$\bar{H}_r^{ij} = - \exp \left( - \sum_{r \in R_{ij}} \varepsilon_r^{ij} \right). \quad (75)$$

From Eq. (70), Eq. (75) can be restated as

$$\bar{H}_r^{ij} = -\exp\left(-\left(g_r^{ij} - \zeta^{ij}\right)^{\beta^{ij}} \varepsilon_r^{ij} \sum_{k \in R_{ij}} \left(g_k^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}\right). \quad (76)$$

Substituting Eq. (76) in Eq. (71) gives

$$\begin{aligned} P_r^{ij} &= \int_0^\infty \exp\left(-\left(g_r^{ij} - \zeta^{ij}\right)^{\beta^{ij}} \varepsilon_r^{ij} \sum_{k \in R_{ij}} \left(g_k^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}\right) d\varepsilon_r^{ij} \\ &= -\frac{\left(g_r^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}}{\sum_{k \in R_{ij}} \left(g_k^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}} \left[ \exp\left(-\left(g_r^{ij} - \zeta^{ij}\right)^{\beta^{ij}} \varepsilon_r^{ij} \sum_{k \in R_{ij}} \left(g_k^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}\right) \right]_0^\infty. \end{aligned} \quad (77)$$

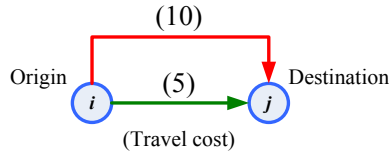
From Eq. (77), the MNW route choice probability can be expressed as

$$P_r^{ij} = \frac{\left(g_r^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}}{\sum_{k \in R_{ij}} \left(g_k^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}}, \quad \forall r \in R_{ij}, ij \in IJ. \quad (78)$$

Since  $\zeta^{ij} < G_r^{ij}$ , it naturally implies that  $\zeta^{ij} < g_r^{ij}$ . Also note that the MNW model computes the probability *directly* using proportion of the route travel costs, while the MNL model uses the exponential proportion of the route travel costs to compute the probability.

To illustrate how the MNW resolves the identical perception variance issue inherited in the MNL model, a two-route network configuration shown in Fig. 7 is adopted. For both short and long networks, the upper route travel time is larger than the lower route travel time by 5 units. However, the upper route travel time is two times larger than the lower route travel time in the short network, while it is only less than 5% larger in the long network. As expected, the MNL model produces the same flow patterns for both short and long networks. This is because the MNL model cannot handle the

perception variance with respect to (w.r.t.) different trip lengths. Each route is assumed to have the same (or identical) perception variance of  $\pi^2/6\theta^2$ , where  $\theta$  is the logit dispersion parameter, as shown in the upper two panels of Fig. 12. Hence, the MNL probability is solely based on the cost difference irrespective of the overall trip lengths (Sheffi, 1985).



MNL model

$$(\theta = 0.5);$$

$$P_l^{ij} = \frac{e^{-2.5}}{e^{-2.5} + e^{-5}} = \frac{1}{1 + e^{-2.5}} = 0.924$$

MNW model

$$(\beta^{ij} = 2.1, \zeta^{ij} = 0);$$

$$P_l^{ij} = \frac{5^{-2.1}}{5^{-2.1} + 10^{-2.1}} = \frac{1}{1 + (10/5)^{-2.1}} = 0.811$$

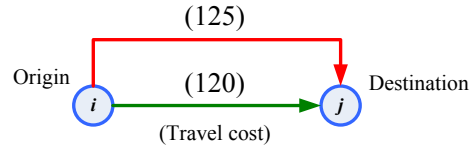
$$(\beta^{ij} = 3.7, \zeta^{ij} = 0);$$

$$P_l^{ij} = \frac{5^{-3.7}}{5^{-3.7} + 10^{-3.7}} = \frac{1}{1 + (10/5)^{-3.7}} = 0.929$$

$$(\beta^{ij} = 2.1, \zeta^{ij} = 2.5);$$

$$P_l^{ij} = \frac{2.5^{-2.1}}{2.5^{-2.1} + 7.5^{-2.1}} = \frac{1}{1 + (7.5/2.5)^{-2.1}} = 0.909$$

a) Short network



MNL model

$$(\theta = 0.5);$$

$$P_l^{ij} = \frac{e^{-60}}{e^{-60} + e^{-62.5}} = \frac{1}{1 + e^{-2.5}} = 0.924$$

MNW model

$$(\beta^{ij} = 2.1, \zeta^{ij} = 0);$$

$$P_l^{ij} = \frac{120^{-2.1}}{120^{-2.1} + 125^{-2.1}} = \frac{1}{1 + (125/120)^{-2.1}} = 0.521$$

$$(\beta^{ij} = 3.7, \zeta^{ij} = 0);$$

$$P_l^{ij} = \frac{120^{-3.7}}{120^{-3.7} + 125^{-3.7}} = \frac{1}{1 + (125/120)^{-3.7}} = 0.538$$

$$(\beta^{ij} = 2.1, \zeta^{ij} = 2.5);$$

$$P_l^{ij} = \frac{117.5^{-2.1}}{117.5^{-2.1} + 122.5^{-2.1}} = \frac{1}{1 + (122.5/117.5)^{-2.1}} = 0.523$$

b) Long network

Fig. 11. Two-route networks

The MNW model, in contrast, produces different route choice probabilities for the two networks. It uses the *relative* cost difference to differentiate the overall trip length. When considering the perception variance, the MNW model handles the *route-specific* perception variance as a function of  $\beta^{ij}$ ,  $\zeta^{ij}$ , and route travel cost, i.e.,

$$\left(\sigma_r^{ij}\right)^2 = \left[ \frac{\left(g_r^{ij} - \zeta^{ij}\right)}{\Gamma\left(1 + 1/\beta^{ij}\right)} \right]^2 \left[ \Gamma\left(1 + \frac{2}{\beta^{ij}}\right) - \Gamma^2\left(1 + \frac{1}{\beta^{ij}}\right) \right], \quad \forall r \in R_{ij}, ij \in IJ, \quad (79)$$

where  $\Gamma(\cdot)$  is the gamma function. From Eq. (79), a longer route will have a higher perception variance as shown by the probability density functions (PDFs) with different combinations of  $\beta^{ij}$  and  $\zeta^{ij}$  in Fig. 12. A larger  $\beta^{ij}$  and/or  $\zeta^{ij}$  will decrease the route perception variance. This will lead to a smaller perception variance among travelers and more flows loaded on the lower-cost route, especially on the network with a shorter trip length. For the extreme cases, when  $\beta^{ij} \rightarrow \infty$  or  $\zeta^{ij} \rightarrow g_r^{ij}; \forall r \in R_{ij}, ij \in IJ$ , the MNW model collapses to the deterministic shortest path problem, where only the lowest-cost route is selected, i.e.,

$$\lim_{\beta^{ij} \rightarrow \infty} \frac{\left(g_r^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}}{\sum_{k \in R_{ij}} \left(g_k^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}} = 1; \quad \lim_{\zeta^{ij} \rightarrow g_r^{ij}} \frac{\left(g_r^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}}{\sum_{k \in R_{ij}} \left(g_k^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}} = 1, \quad (80)$$

$$g_r^{ij} < g_l^{ij}, \forall l \neq r \in R_{ij}, ij \in IJ.$$

Meanwhile, as  $\beta^{ij} \rightarrow 0$ , the MNW model becomes the uniform traffic loading, i.e.,

$$\lim_{\beta^{ij} \rightarrow 0} \frac{\left(g_r^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}}{\sum_{k \in R_{ij}} \left(g_k^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}} = \frac{1}{|R_{ij}|}, \quad \forall r \in R_{ij}, ij \in IJ. \quad (81)$$

where  $|R_{ij}|$  is the number of routes connecting O-D pair  $ij$ .

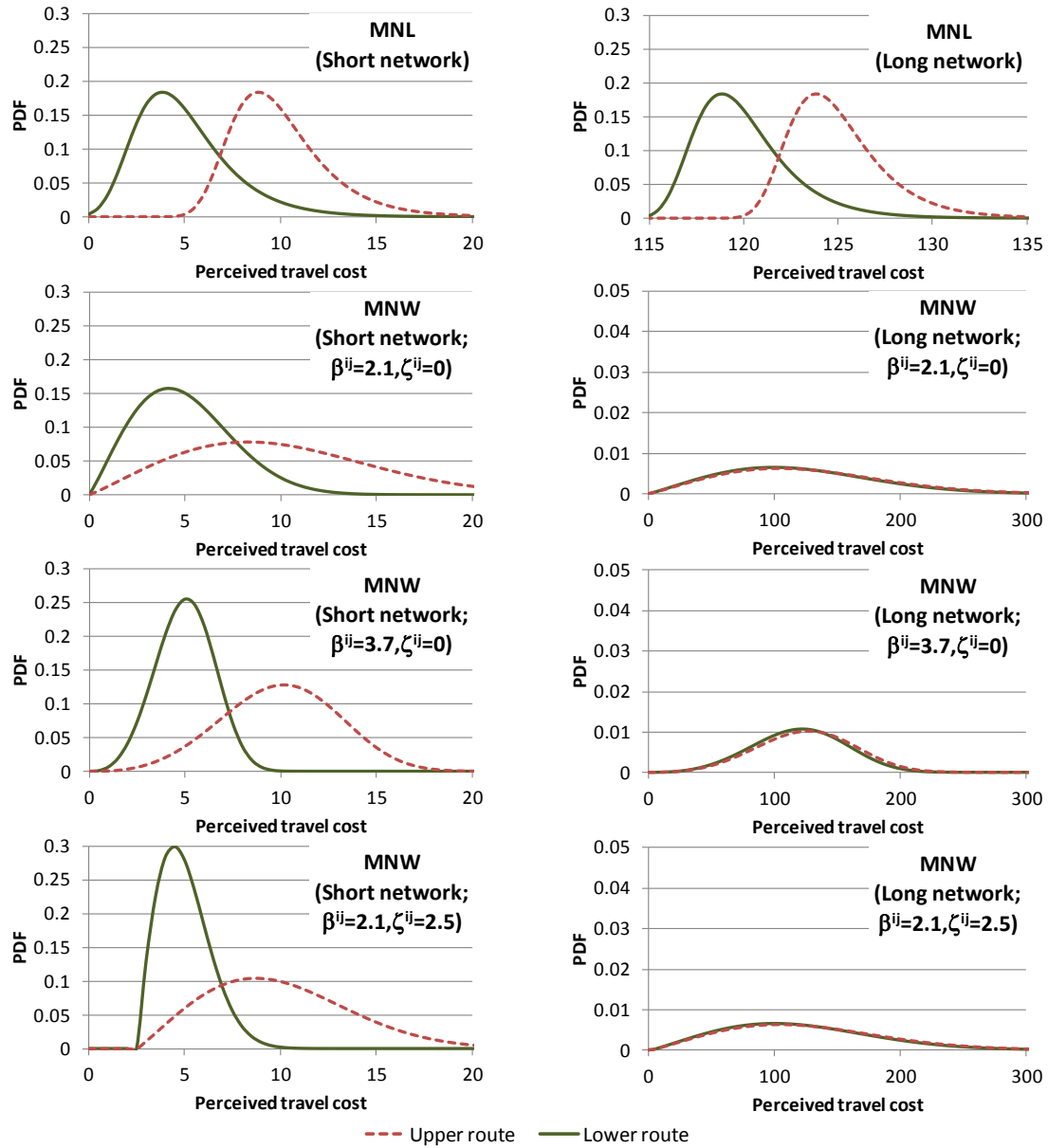


Fig. 12. Perceived travel time distributions for the two-route networks

### 3.2.2 PSW model

To relax the *independently distributed* assumption imposed on the MNW model, a path-size factor  $\varpi_r^{ij}$  is introduced to the MNW RUM in Eq. (18) to alleviate the route overlapping problem. This path-size factor  $\varpi_r^{ij} \in (0,1]$  accounts for different route sizes

determined by the length of links within a route and the relative lengths of routes that share a link, i.e., (Ben-Akiva and Bierlaire, 1999)

$$\varpi_r^{ij} = \sum_{a \in \Upsilon_r} \frac{l_a}{L_r^{ij}} \frac{1}{\sum_{k \in R_{ij}} \delta_{ak}^{ij}}, \quad \forall r \in R_{ij}, ij \in IJ, \quad (82)$$

where  $l_a$  is the length of link  $a$ ,  $L_r^{ij}$  is the length of route  $r$  connecting O-D pair  $ij$ ,  $\Upsilon_r$  is the set of all links in route  $r$  between O-D pair  $ij$ , and  $\delta_{ar}^{ij}$  is equal to 1 for link  $a$  on route  $r$  between O-D pair  $ij$  and 0 otherwise. The lengths in the common part and the route ratio (i.e.,  $l_a/L_r^{ij}$ ) is a plausible approximation of the route correlation, and  $\sum_{k \in R_{ij}} \delta_{ak}^{ij}$  measures the contribution of link  $a$  in the route correlation (Frejinger and Bierlaire, 2007). Routes with a heavy overlap with other routes will have a smaller value of  $\varpi_r^{ij}$ . The path-size factor can be used to modify the deterministic term of the MNW RUM model in Eq. (18) as follows:

$$U_r^{ij} = \frac{(g_r^{ij} - \zeta^{ij})^{\beta^{ij}}}{\varpi_r^{ij}} \varepsilon_r^{ij}, \quad \forall r \in R_{ij}, ij \in IJ, \quad (83)$$

which gives the following route choice probability:

$$P_r^{ij} = \frac{\varpi_r^{ij} (g_r^{ij} - \zeta^{ij})^{-\beta^{ij}}}{\sum_{k \in R_{ij}} \varpi_k^{ij} (g_k^{ij} - \zeta^{ij})^{-\beta^{ij}}}, \quad \forall r \in R_{ij}, ij \in IJ. \quad (84)$$

To illustrate how the path-size factors handle the route overlapping problem, we use the loop-hole network shown in Fig. 13. In this network, all three routes have equal travel cost. The two upper routes overlap by a portion  $x$ , while the lower route is distinct from the two upper routes. According to the *independently distributed* assumption, both MNL and MNW models give the same route choice probability for all  $x$  values as shown in Fig. 13.

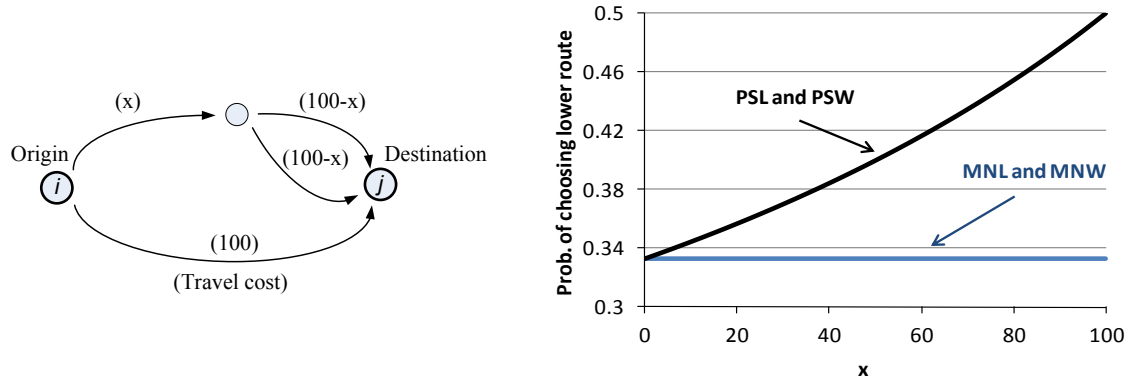


Fig. 13. Loop-hole network

In contrast, the PSW model as well as the path-size logit (PSL) model can handle the route overlapping problem via the path-size factor. As  $x$  increases, the probability of choosing the lower route increases. When  $x=100$  (i.e., only two routes with equal trip length exist), both upper and lower routes receive the same probability of being selected.

### 3.3 Equivalent mathematical programming formulations

This section presents equivalent mathematical programming (MP) formulations for the weibit route choice models under congested networks. A *multiplicative* Beckmann's transformation (MBec) combined with an entropy term are used to develop the MP formulation for the weibit SUE models. Specifically, we present the MP formulations with some qualitative properties. Before presenting the formulations, we describe the necessary assumptions.

#### 3.3.1 Assumptions

To begin with, a general assumption of link travel cost function is made, i.e.,

Assumption 3.1. The link travel cost  $\tau_a$ , which could be a function of travel time, is a strictly increasing function w.r.t. its own flow.

Since  $\zeta^{ij}$  cannot easily be decomposed into the link level, we make another assumption:

Assumption 3.2.  $\zeta^{ij}$  is equal to zero.

This assumption indicates that each route is assumed to have the *same* coefficient of variation. From Eq. (79), the route-specific coefficient of variation can be expressed as

$$g_r^{ij} = \frac{\sigma_r^{ij}}{g_r^{ij}} = \frac{(g_r^{ij} - \zeta^{ij})}{g_r^{ij}} \sqrt{\frac{\Gamma(1+2/\beta^{ij})}{(\Gamma(1+1/\beta^{ij}))^2} - 1}, \quad \forall r \in R_{ij}, ij \in IJ. \quad (85)$$

With  $\zeta^{ij} = 0$ ,  $g_r^{ij}$  of each route is equal. Note that we can adopt the variational inequality (VI) formulation (e.g., Zhou et al., 2008) to incorporate  $\zeta^{ij}$  in the MNW-SUE model.

Since the weibit model falls within the category of *multiplicative* random utility maximization model (MRUM), the deterministic part of the disutility function is simply a set of *multiplicative* explanatory variables (e.g., Cooper and Nakanishi, 1988). Then, we make an assumption of the route travel cost:

Assumption 3.3. The route travel cost is a function of multiplicative link travel costs, i.e.,

$$g_r^{ij} = \prod_{a \in \Upsilon_r} \tau_a, \quad \forall r \in R_{ij}, ij \in IJ. \quad (86)$$

This assumption not only maintains the weibit relative cost criterion (Fosgerau and Bierlaire, 2009), but it also corresponds to the *Markov process* in transportation network analysis (see Akamatsu, 1996). With a *suitable* multiplicative link cost function, travelers are assumed to make a decision at each node (or *state*) until they reach the destination (or final state) according to the weibit choice probability. In other words, travelers are assumed to follow the weibit choice probability in selecting their routes (see Chapter 4 for details).



Following the path-size logit (PSL) SUE formulation provided by [Chen et al. \(2012\)](#), the lengths used in the path-size factor for the MP formulation are assumed to be flow independent as follows.

Assumption 3.4. The lengths  $l_a$  and  $L_r^{ij}$  used in  $\varpi_r^{ij}$  are flow independent.

Note that we can also adopt the congestion-based C-logit VI formulation ([Zhou et al., 2012](#)) to incorporate the flow dependent path-size factors.

### 3.3.2 MNW SUE model

Consider the following MP formulation:

$$\begin{aligned} \min Z &= Z_1 + Z_2 \\ &= \sum_{a \in A} v_a \int_0^{v_a} \ln \tau_a(\omega) d\omega + \sum_{ij \in IJ} \frac{1}{\beta^{ij}} \sum_{r \in R_{ij}} f_r^{ij} (\ln f_r^{ij} - 1) \end{aligned} \quad (87)$$

s.t.

$$\sum_{r \in R_{ij}} f_r^{ij} = q_{ij}, \quad \forall ij \in IJ, \quad (88)$$

$$f_r^{ij} \geq 0, \quad \forall r \in R_{ij}, ij \in IJ, \quad (89)$$

where  $f_r^{ij}$  is the flow on route  $r$  between O-D pair  $ij$ ,  $q_{ij}$  is the demand between O-D pair  $ij$ , and  $v_a$  is the flow on link  $a$ . Eq. (88) and Eq. (89) are respectively the flow conservation constraint and the non-negativity constraint. The main differences between this MNW-SUE model and [Fisk's \(1980\)](#) MNL-SUE model are the *multiplicative* Beckmann's transformation (MBec) in  $Z_1$  and  $\beta^{ij}$  (the perception variance of O-D pair  $ij$ ) in  $Z_2$ . The MBec can be converted to an additive form via a log transformation to facilitate the direct route cost computations, while the O-D specific dispersion parameters

$\beta^{ij}$  is related to the route-specific perception variance (see Eq. (79)). These two differences are the key to the development of the MNW-SUE model. Note that if  $\beta^{ij}$  approaches infinity for all O-D pair  $ij$ ,  $Z_2$  approaches zero. From Assumption 3.1, the log transformation would not alter the results of the MP formulation. Minimizing  $Z_I$  would result in the deterministic user equilibrium (DUE) model where only the lowest-cost routes are used.

Proposition 3.1. *The MP formulation given in Eqs. (87) through (89) has the solution of the MNW model.*

Proof. Note that the logarithmic term in Eq. (87) implicitly requires  $\tau_a$  and  $f_r^{ij}$  to be positive. By constructing the Lagrangian function of the MNW SUE model and then setting its partial derivative to zero, we obtain

$$\sum_{a \in A} \ln \tau_a \delta_{ra}^{ij} + \frac{1}{\beta^{ij}} \ln f_r^{ij} - \lambda_{ij} = 0, \quad (90)$$

$$\sum_{r \in R_{ij}} f_r^{ij} = q_{ij}, \quad (91)$$

where  $\lambda_{ij}$  is the dual variable associated with the flow conservation constraint in Eq. (88).

Eq. (90) can be rearranged as

$$\beta^{ij} \ln \prod_{a \in Y_r} \tau_a + \ln f_r^{ij} = \beta^{ij} \lambda_{ij}. \quad (92)$$

From Assumption 3.3, Eq. (92) can be expressed as

$$\beta^{ij} \ln g_r^{ij} + \ln f_r^{ij} = \beta^{ij} \lambda_{ij}, \quad (93)$$

which indicates that the MBec provides the *logarithmic* route cost in the equivalency conditions. With this logarithmic route travel cost structure, we have the route flow as a function of  $(g_r^{ij})^{-\beta^{ij}}$ , i.e.,

$$f_r^{ij} = \exp(\beta^{ij} \lambda_{ij}) (g_r^{ij})^{-\beta^{ij}}. \quad (94)$$

From Eq. (91) and Eq. (94), the O-D demand can be written as

$$q_{ij} = \sum_{r \in R_{ij}} f_r^{ij} = \exp(\beta^{ij} \lambda_{ij}) \sum_{r \in R_{ij}} (g_r^{ij})^{-\beta^{ij}}. \quad (95)$$

Dividing Eq. (94) by Eq. (95) leads to the MNW route probability expressed as the proportion of route travel costs:

$$P_r^{ij} = \frac{(g_r^{ij})^{-\beta^{ij}}}{\sum_{k \in R_{ij}} (g_k^{ij})^{-\beta^{ij}}}, \quad \forall r \in R_{ij}, ij \in IJ. \quad (96)$$

Thus, the MP formulation given in Eqs. (87) through (89) corresponds to the SUE model for which the route-flow solution is obtained according to the MNW model. This completes the proof.  $\square$

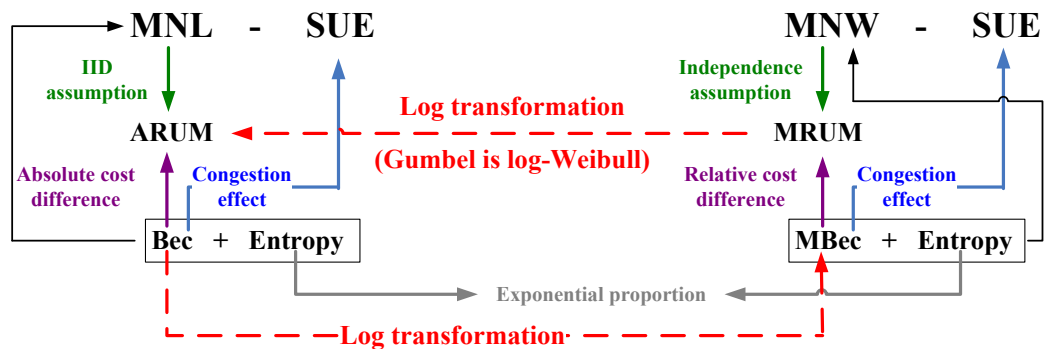


Fig. 14. Relation between MNL-SUE and MNW-SUE models

From Proposition 3.1, we can see that the first term (i.e., multiplicative Beckmann's transformation: MBec) in Eq. (87) uses the logarithm to handle the *relative cost difference* in the MNW model. This mechanism can be viewed from the relation between the MNW-SUE and MNL-SUE models shown in Fig. 14. The MBec is rooted from the relation between the Gumbel and Weibull distributions by applying a log transformation to the Beckmann's transformation (Bec) (Beckmann et al., 1956) and incorporating the exponential proportions given by the entropy term in the equivalent conditions to obtain the MNW probability. In other words, by applying a *log transformation* to the MNL travel time, we obtain the MNW model (Castillo et al., 2008; Fosgerau and Bierlaire, 2009). This is because the Gumbel distribution can be considered as the log-Weibull distribution (White, 1969).

To further illustrate the role of MBec in developing the objective function for the MNW-SUE model, we adopt a visual approach used in Bell and Iida (1997) to graphically illustrate the relation between the MNL-SUE and MNW-SUE models in Fig. 15. The supply curve gives the relationship between route flows of the upper and lower routes and their route costs. In the case of monotonically increasing link costs from Assumption 3.1, the supply curve is smooth and exhibits a logistic shape. The demand curve for this two-route network, in relation to the route cost difference between the two routes, is also smooth and logistic, but in opposite direction to the supply curve. The logarithmic route cost produced by the MBec and the exponential proportions given by the entropy term in the equivalent conditions give the MNW solution. With this, the demand curve changes according to the overall trip length. A longer trip length has a steeper demand curve, while a shorter trip length has a flatter

demand curve. Thus, the probabilities of the upper and lower routes become more similar as the overall trip length increases at the equilibrium point where the demand curve intersects the supply curve (shown by the dotted red line of the MNW SUE model in Fig. 15).

*Proposition 3.2. The solution of MNW-SUE model is unique.*

*Proof.* It is sufficient to prove that the objective function in Eq. (87) is strictly convex in the vicinity of route flow and that the feasible region is convex. The convexity of the feasible region is assured by the linear equality constraints in Eq. (88). The nonnegative constraint in Eq. (89) does not alter this characteristic.

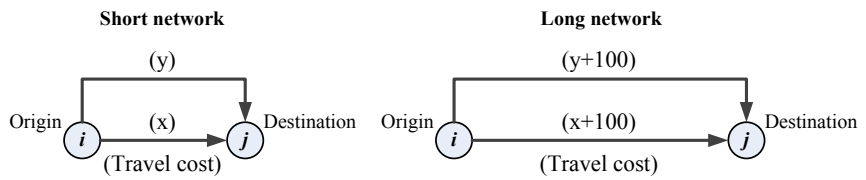
Hence, the focus is on the properties of the objective function. This is done by proving that the Hessian matrix is positive definite. According to Assumption 3.1, the Hessian matrix of the multiplicative Beckmann's transformation  $Z_1$  is positive semi-definite w.r.t. the route flow variables. This is similar to the Beckmann's transformation case. The Hessian matrix of  $Z_2$  can be shown as

$$\frac{\partial^2 (Z_2)}{\partial f_r^{ij} \partial f_k^{ij}} = \begin{cases} \frac{1}{\beta^{ij} f_r^{ij}} > 0 ; r = k \\ 0 & ; \textit{otherwise} \end{cases} . \quad (97)$$

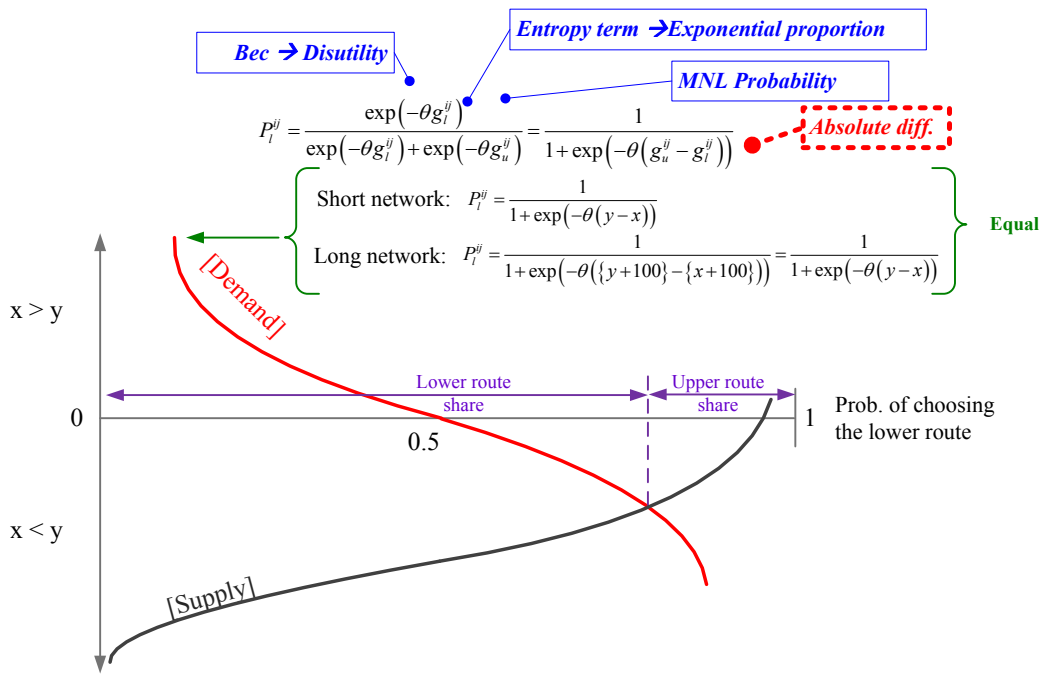
As such, the Hessian matrix of  $Z_2$  is positive definite. Hence,  $Z_1 + Z_2$  is strictly convex. The solution of MNW-SUE model is unique w.r.t. route flows. This completes the proof.  $\square$

### 3.3.3 PSW SUE model

In this section, we provide an equivalent MP formulation for the PSW-SUE model to consider both route overlapping and heterogeneous perception variance under congested



**MNL-SUE model**



**MNW-SUE model**

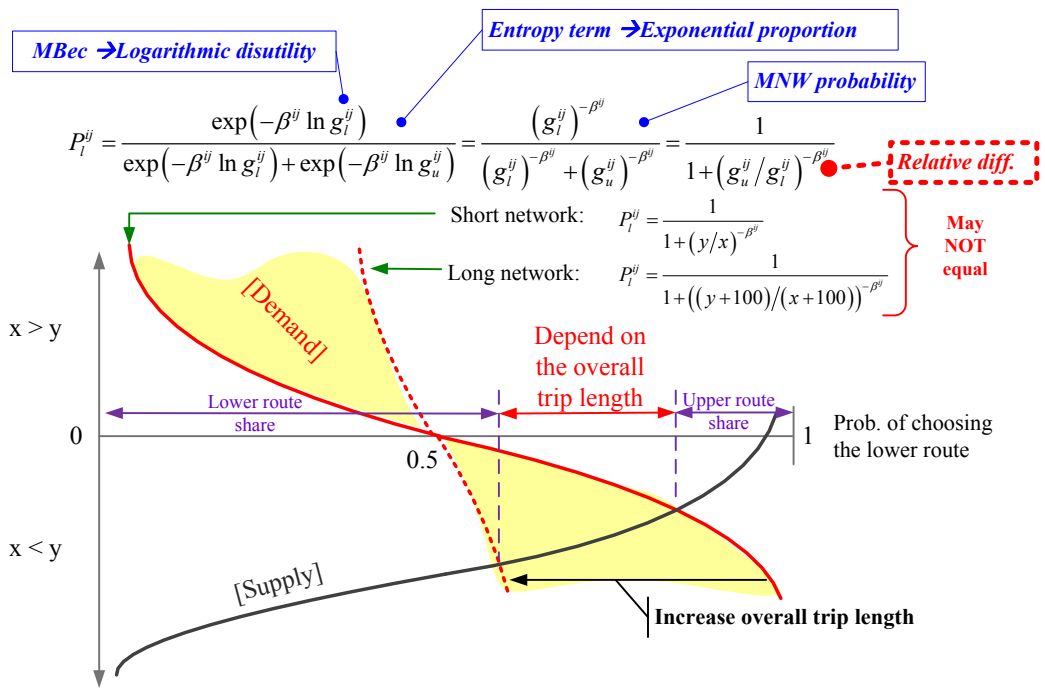


Fig. 15. Visual illustration of the MNL-SUE and MNW-SUE models

networks. Following the path-size logit (PSL) SUE formulation provided by [Chen et al. \(2012\)](#), the PSW-SUE model can be formulated as follows:

$$\begin{aligned} \min Z &= Z_1 + Z_2 + Z_3 \\ &= \sum_{a \in A} \int_0^{v_a = \sum_{ij \in IJ} \sum_{r \in R_{ij}} f_r^{ij} \delta_{ra}^{ij}} \ln \tau_a(\omega) d\omega + \sum_{ij \in IJ} \frac{1}{\beta^{ij}} \sum_{r \in R_{ij}} f_r^{ij} (\ln f_r^{ij} - 1) - \sum_{ij \in IJ} \frac{1}{\beta^{ij}} \sum_{r \in R_{ij}} f_r^{ij} \ln \varpi_r^{ij} \end{aligned} \quad (98)$$

subject to the flow conservation and non-negativity constraints in Eq. (88) and Eq. (89). The term  $Z_3$  is introduced to the MNW-SUE formulation in Eq. (87) to capture the length/size of the routes in order to correct the MNW choice probability. Note that when there is no route overlap (i.e.,  $\varpi_r^{ij} = 1$ ), the PSW-SUE model collapses to the MNW-SUE model.

*Proposition 3.3. The MP formulation given in Eqs. (98), (88), and (89) has the solution of the PSW model.*

*Proof.* Following the same principle as Proposition 3.1, we have

$$\beta^{ij} \ln g_r^{ij} + \ln f_r^{ij} - \ln \varpi_r^{ij} = \beta^{ij} \lambda_{ij}, \quad (99)$$

which gives

$$f_r^{ij} = \exp(\beta^{ij} \lambda_{ij}) \varpi_r^{ij} (g_r^{ij})^{-\beta^{ij}}, \quad (100)$$

$$q_{ij} = \sum_{r \in R_{ij}} f_r^{ij} = \exp(\beta^{ij} \lambda_{ij}) \sum_{r \in R_{ij}} \varpi_r^{ij} (g_r^{ij})^{-\beta^{ij}}. \quad (101)$$

Then, dividing Eq. (100) by Eq. (101) leads to the PSW probability expression

$$P_r^{ij} = \frac{\varpi_r^{ij} (g_r^{ij})^{-\beta^{ij}}}{\sum_{k \in R_{ij}} \varpi_k^{ij} (g_k^{ij})^{-\beta^{ij}}}. \quad (102)$$

Thus, the MP formulation given in Eqs. (98), (88), and (89) corresponds to the SUE model for which the route-flow solution is obtained according to the PSW model. This completes the proof.  $\square$

*Proposition 3.4. The solution of PSW-SUE model is unique.*

*Proof.* Following the same principle as Proposition 3.2, the Hessian matrices of  $Z_1$  and  $Z_2$  are positive semi-definite and positive definite, respectively. Since  $\varpi_r^{ij}$  in  $Z_3$  is flow independent from Assumption 3.4, we have

$$\frac{\partial^2 (Z_3)}{\partial f_r^{ij} \partial f_k^{ij}} = 0. \quad (103)$$

Thus,  $Z_1 + Z_2 + Z_3$  is strictly convex. The solution of PSW SUE model is unique w.r.t. route flows. This completes the proof.  $\square$

### 3.4 Solution algorithm

In this study, a path-based algorithm based on the partial linearization method is adopted to solve the PSW-SUE model as shown in Fig. 16. This descent algorithm iterates between the search direction and line search until the stopping criterion of a convex optimization problem is reached (Patriksson, 1994). In the PSW-SUE formulation, the search direction is obtained by solving the first-order approximation of the MBec. For the line search, we consider the classical generalized Armijo rule (Bertsekas, 1976) to find an approximate stepsize, which has been found to be effective in solving the CNL-SUE model (Bekhor et al., 2008). A column generation procedure (Dantzig, 1963) is adopted to resolve the route enumeration issue. Alternatively, a pre-generated working route set based on a behavioral route choice generation method



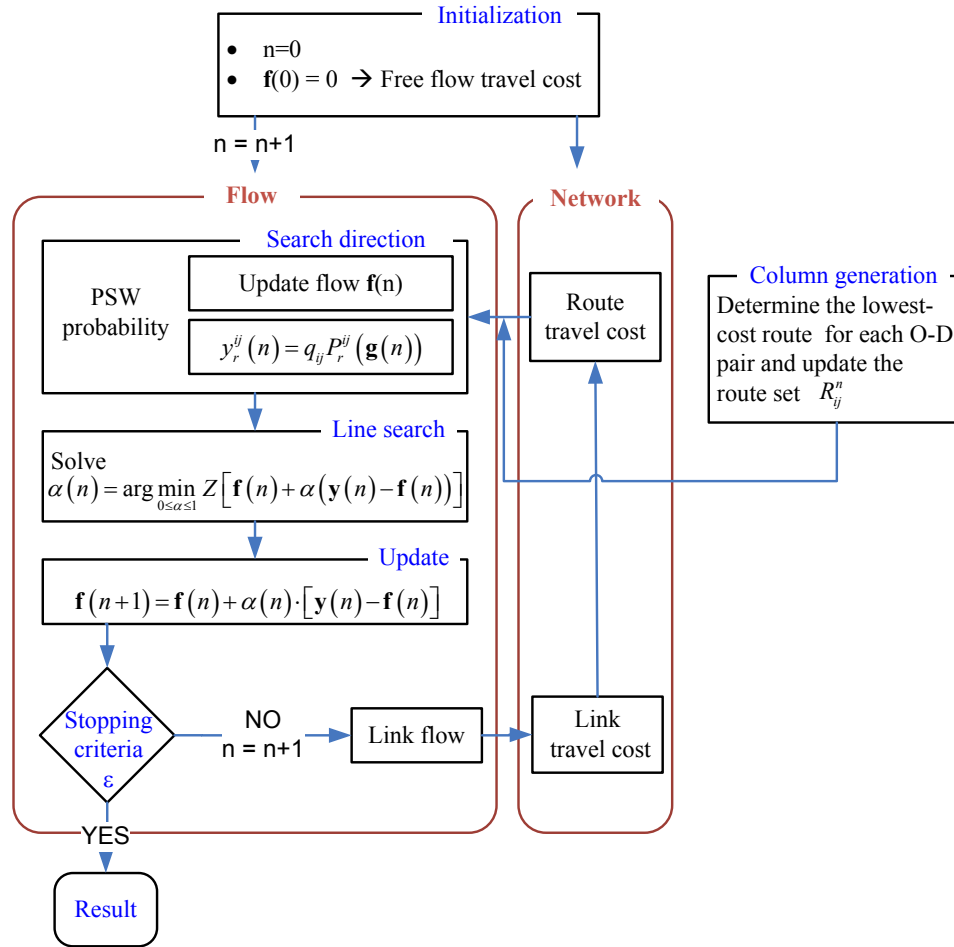


Fig. 16. Partial linearization method for solving the MNW-SUE and PSW-SUE models

(Prashker and Bekhor, 2004; Prato, 2009) could also be used instead of the column generation procedure.

Since the link cost is in a multiplicative form, we use the logarithm operator to transform it to an additive form, i.e.,

$$\bar{\tau}_a = \ln \tau_a, \forall a \in A. \quad (104)$$

This process facilitates the use of an ordinary shortest path algorithm. The route cost can then be determined from

$$g_r^{ij} = \exp\left(\sum_{a \in A} \bar{\tau}_a \delta_{ar}^{ij}\right), \quad \forall r \in R_{ij}, ij \in IJ. \quad (105)$$

It should be noted that  $\bar{\tau}_a$  could be negative if  $\tau_a$  is less than one. In such cases, a shortest path algorithm that can handle negative cycles is necessary to avoid an infinite loop. We consider [Pape's \(1974\)](#) algorithm, which is a label correcting method that can work with negative link costs. When the network contains a negative cost loop, all negative  $\bar{\tau}_a$  would be set to a very small positive number, and the algorithm is then repeated. A more appropriate modification to Pape's algorithm could be implemented to generate the shortest "simple routes" with the presence of negative cycles (e.g., the labeling and scanning method described in [Tarjan, 1983](#)). The basic idea is to include a scan operation to the shortest path algorithm to eliminate negative cost cycles. For more information, see [Tarjan \(1983\)](#).

### 3.5 Numerical results

In this section, we present three numerical examples. Example 1 uses the two-route networks in Fig. 11 with flow dependent travel cost. This example is adopted to investigate the solution from the MNW-SUE model and compare it with the MNL-SUE models (with and without scaling technique). Example 2 is the modified loop-hole network used to consider both route overlapping and route-specific perception variance problems simultaneously. Example 3 is the Winnipeg network used as a case study to demonstrate its applicability in real networks. Without loss of generality, all routes are assumed to have the same coefficient of variation  $\mathcal{G}_r^{ij}$  of 0.3 (i.e.,  $\beta^{ij} = 3.7$  for all O-D pairs, see Eq. (85)) unless specified otherwise.  $l_a$  and  $L_r^{ij}$  used in the path-

size factor  $\varpi_r^{ij}$  are set to the link free-flow travel cost and route free-flow travel cost, respectively. The dispersion parameter  $\theta$  of the MNL-SUE model is set to 0.1, and  $\theta$  of the MNL-SUE model with scaling technique (or MNLs-SUE model) is set corresponding to  $\mathcal{G}_r^{ij} = 0.3$  using the uncongested lowest-cost route (Chen et al., 2012). For the MNP-SUE model, the solution is computed by the Monte Carlo stochastic loading technique with 2000 draws to obtain stable results.

### 3.5.1 Example 1: Two-route network

The two-route networks in Fig. 11 are modified to incorporate the congestion effect as shown in Table 3. The previous route travel cost configuration is used as the free-flow travel cost, and  $f_r^{ij}/10$  is introduced to create the flow-dependent travel cost. The O-D demand is 100 vehicles per unit time. We first investigate the MNW solution from the MNW-SUE MP formulation, followed by the effect of different trip lengths.

#### 3.5.1.1 MNW solution

The MNW-SUE objective value for the short network can be expressed as (see Eq. (87))

$$\min \int_0^{f_u^{ij}} \ln\left(10 + \frac{\omega}{10}\right) d\omega + \int_0^{f_l^{ij}} \ln\left(5 + \frac{\omega}{10}\right) d\omega + \frac{1}{3.7} \left[ f_u^{ij} (\ln f_u^{ij} - 1) + f_l^{ij} (\ln f_l^{ij} - 1) \right] \quad (106)$$

s.t.

$$f_u^{ij} + f_l^{ij} = 100. \quad (107)$$

Table 3: Flow-dependent route travel cost for the two-route networks

Network	Upper route	Lower route
Short	$10 + f_u^{ij}/10$	$5 + f_l^{ij}/10$
Long	$125 + f_u^{ij}/10$	$120 + f_l^{ij}/10$

From Eq. (107), derivative of Eq. (106) w.r.t. the flow on the lower route  $f_l$  gives

$$-\ln\left(10 + \frac{100 - f_l^{ij}}{10}\right) + \ln\left(5 + \frac{f_l^{ij}}{10}\right) + \frac{1}{3.7} \left[ -\ln(100 - f_l^{ij}) + \ln f_l^{ij} \right] = 0. \quad (108)$$

Rearranging Eq. (108) gives

$$\frac{f_l^{ij}}{100 - f_l^{ij}} = \frac{(5 + f_l^{ij}/10)^{-3.7}}{(10 + (100 - f_l^{ij})/10)^{-3.7}}. \quad (109)$$

Since  $10 + (100 - f_l^{ij})/10$  and  $5 + f_l^{ij}/10$  are respectively the costs of upper and lower routes, Eq. (109) gives the MNW choice probability, i.e.,

$$\frac{f_l^{ij}}{f_u^{ij}} = \frac{(g_l^{ij})^{-3.7}}{(g_u^{ij})^{-3.7}} \text{ or } P_l^{ij} = \frac{f_l^{ij}}{f_l^{ij} + f_u^{ij}} = \frac{(g_l^{ij})^{-3.7}}{(g_l^{ij})^{-3.7} + (g_u^{ij})^{-3.7}}. \quad (110)$$

This result indicates that the multiplicative Beckmann's transformation preserves the relative cost difference criterion of the weibit model. By solving Eq. (109), we obtain the route flow solution of the MNW-SUE problem, i.e.,

$$f_u^{ij} = 35.25; f_l^{ij} = 64.75. \quad (111)$$

### 3.5.1.2 Effect of different trip lengths

Next, we consider the effect of different trip lengths under the SUE framework. The results are shown in Table 4. As expected, the MNL-SUE model produces the same flow pattern for both short and long networks, regardless of the overall trip

length. Meanwhile, the MNLs-SUE and MNW-SUE models assign different flow patterns to reflect the overall trip lengths. Specifically, both models assign a smaller amount of traffic flows on the lower route as the trip length increases. These assignment results are consistent with that there is higher opportunity for “*wrong*” perception (in the sense of choosing a larger-cost route) on the longer overall trip length network (Sheffi, 1985). Note that the MNLs-SUE model has a higher amount of flows on the lower route than the MNW-SUE model since the MNLs-SUE model still retains the identically distributed assumption; each route has the same and fixed perception variance from the classical logit assumption, despite different scaling factors (i.e.,  $\theta = 0.86$  for the short network and  $\theta = 0.04$  for the long network) are used in the two networks.

The objective values ( $Z = Z_1 + Z_2$ ) of all three models are presented in Table 5. All three models have a higher total objective value ( $Z$ ) as the overall trip length increases. The Bec value ( $Z_1$ ) of the MNL-SUE and MNLs-SUE models are higher than the MBec value ( $Z_1$ ) of the MNW-SUE model. This is because the MBec has the logarithm transformation. Note that the  $Z_1/Z_2$  ratio of the MNLs-SUE model is different from the  $Z_1/Z_2$  ratio of the MNW-SUE model. While the  $Z_1/Z_2$  ratio of the MNLs-SUE model is smaller as the overall trip length increases, the  $Z_1/Z_2$  ratio of the MNW-SUE model is larger as the overall trip length increases. These results appear to indicate that the MNW-SUE model uses the  $Z_1/Z_2$  ratio differently to capture the effect of different trip lengths compared to that of the MNLs-SUE model.

Table 4: Results of the two-route networks

Model		MNL-SUE	MNLs-SUE	MNW-SUE
Short network	Flow on the upper route	<i>41.72</i>	29.96	35.25
	Flow on the lower route	<i>58.28</i>	70.04	64.75
Long network	Flow on the upper route	<i>41.72</i>	46.23	46.84
	Flow on the lower route	<i>58.28</i>	53.77	53.16

Table 5: Objective values of MNL-SUE, MNLs-SUE and MNW-SUE models

Network	OBJ	MNL-SUE	MNLs-SUE	MNW-SUE
Short	Bec or Mbec ( $Z_1$ )	965.45	939.96	382.27
	Entropy term ( $Z_2$ )	3925.80	467.19	106.92
	Total OBJ value ( $Z=Z_1+Z_2$ )	4891.25	1407.15	489.20
	Bec or Mbec and Entropy term ( $Z_1/Z_2$ )	0.25	<i>2.01</i>	<i>3.58</i>
Long	Bec or Mbec ( $Z_1$ )	12465.45	12482.55	9813.07
	Entropy term ( $Z_2$ )	3925.80	10988.69	105.78
	Total OBJ value ( $Z=Z_1+Z_2$ )	16391.25	23471.25	9918.86
	Bec or Mbec and Entropy term ( $Z_1/Z_2$ )	3.18	<i>1.14</i>	<i>92.76</i>

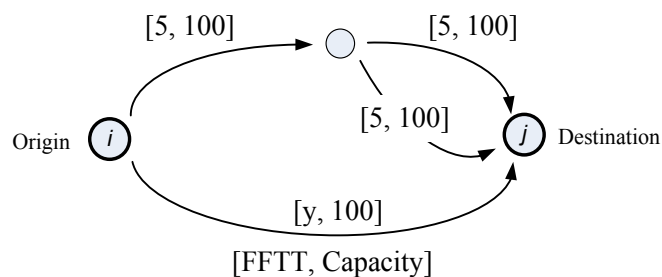


Fig. 17. Modified loop-hole network

### 3.5.2 Example 2: Modified loop-hole network

In this example, we use the modified loop-hole network given in Fig. 17 to consider both route overlapping and route-specific perception variance problems and the effects of demand level and coefficient of variation. This network has three routes. The upper two routes have a fixed overlapping section by half of the route free-flow travel time (FFTT). The lower route is truly independent, and its FFTT can be varied according to  $y \in [0,10]$ . All links have the same capacity of 100 vehicle per hour (vph), and the O-D demand is 100 vph.

The flow dependent link travel time is represented by the standard Bureau of Public Road (BPR) function

$$t_a = t_a^0 \left[ 1 + 0.15 \left( \frac{v_a}{c_a} \right)^4 \right], \quad (112)$$

where  $t_a^0$  is the FFTT of link  $a$ , and  $c_a$  is the capacity (in vph) of link  $a$ . Without loss of generality, we assume that the link travel cost (or disutility) is an exponential function (Hensher and Truong, 1985; Polak, 1987; Mirchandani and Soroush, 1987), i.e.,

$$\tau_a = e^{0.075 t_a}, \quad \forall a \in A. \quad (113)$$

#### 3.5.2.1 Effects of overlapping and heterogeneous perception variance

We first consider the route overlapping and route-specific perception variance problems. The results in Fig. 18 show that all SUE models assign a smaller amount of flows on the lower route when  $y$  increases. While the MNP-SUE and PSW-SUE models seem to give similar traffic flow patterns, the MNW-SUE model assigns a smaller flow on the lower route. This is because the MNW-SUE model does not handle the route overlapping problem. With the independently distributed assumption, the MNW-SUE

model considers each route as an independent alternative. As such, it assigns more flow on the routes with overlapping, hence a smaller amount of flow on the lower route.

### 3.5.2.2 Effects of demand level and coefficient of variation

We continue to use the modified loop-hole network with  $y = 5$  to investigate the effects of demand level and coefficient of variation. The O-D demand is varied from 25 to 300 vph, and  $\mathcal{G}_r^{ij}$  is varied from 0.1 to 1. The root mean square error (RMSE) is used as a statistical measure to compare the link-flow difference between the PSW-SUE model relative to the user equilibrium (UE) model, i.e.,

$$RMSE = \sqrt{\sum_{a \in A} \frac{(v_a^{\{UE\}} - v_a^{\{PSW\}})^2}{|A|}}, \quad (114)$$

where  $|A|$  is the number of links in the network (i.e., 4 links). A low value of RMSE means that both assignment models perform similarly.

It can be seen from Fig. 19 that as the demand level increases, the RMSE decreases. This means that the PSW-SUE model approaches the UE model when the congestion level is increased (i.e., the congestion effect due to high demand levels of 400 to 500 vph dominates the solution). Also, the RSME decreases when  $\mathcal{G}_r^{ij}$  decreases (i.e.,  $\beta^{ij}$  increases or lower perception variance). The PSW-SUE flow patterns also tend to the UE flow pattern. This implies the demand is more concentrated on the minimum cost routes (i.e., travelers are able to select the lower-cost routes more often since they have better knowledge of the network traffic conditions). Otherwise, the two models will produce different flow patterns for low demand levels and larger  $\mathcal{G}_r^{ij}$  values.



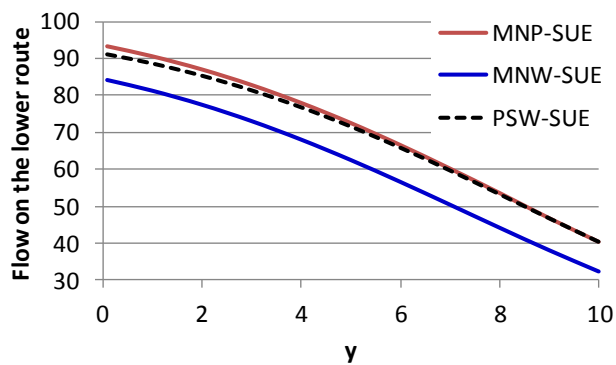


Fig. 18. Traffic flow patterns of the modified loop-hole network

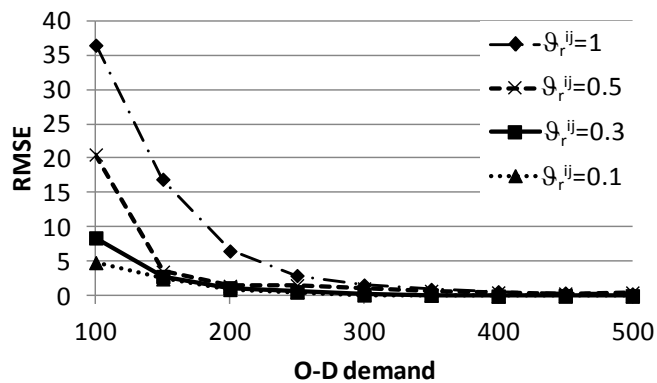


Fig. 19. Effects of demand level and coefficients of variation  $g_r^{ij}$

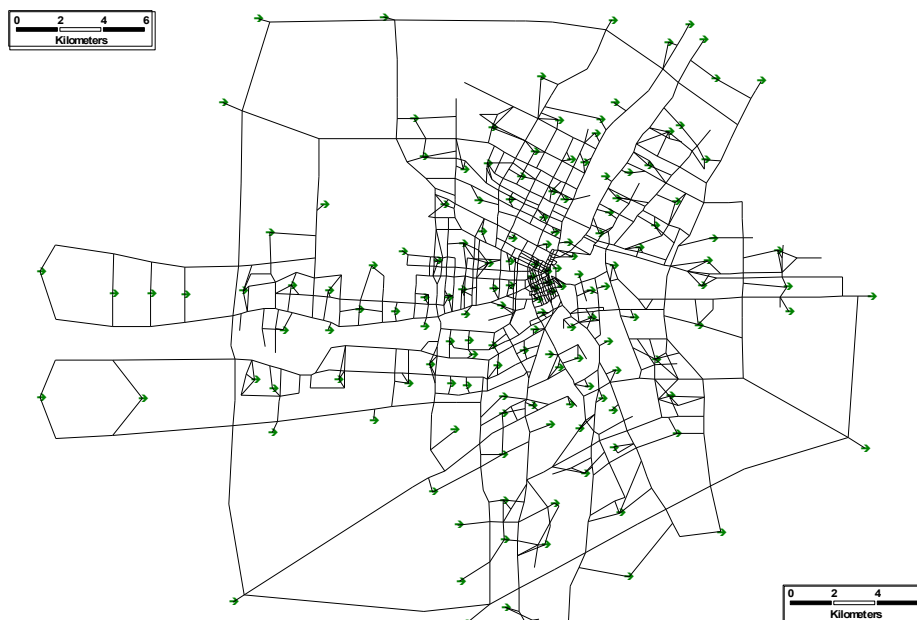


Fig. 20. Winnipeg network

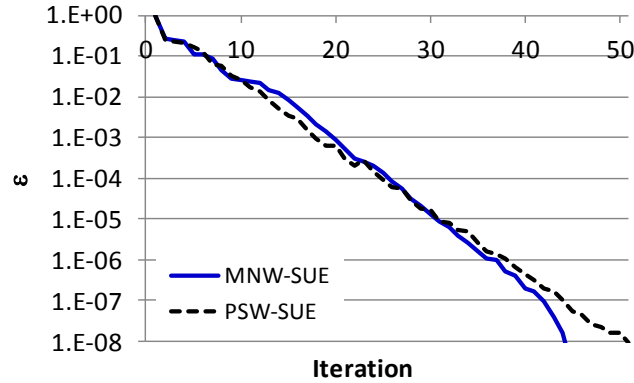


Fig. 21. Convergence characteristics of the path-based partial linearization algorithm

Table 6: Computational efforts of the MNW-SUE and PSW-SUE models

Model	# of iterations	CPU time (sec)*	CPU time per iteration	# of routes
MNW-SUE	45	10.04	0.22	15,791
PSW-SUE	51	13.41	0.26	15,442

\* All the algorithms are coded in Compaq Visual FORTRAN 6.6 and run on a personal computer with 3.8 G Pentium-IV processor and 2G RAM

### 3.5.3 Example 3: Winnipeg network

This example adopts the Winnipeg network (shown in Fig. 20) as a case study to demonstrate its applicability in a real network. This network consists of 154 zones, 2,535 links, and 4,345 O-D pairs. The network topology, link characteristics, and O-D demands can be found in Emme/2 software (INRO Consultants, 1999). We continue to use the link travel cost configuration in Eq. (113) for this real network.

#### 3.5.3.1 Computational results

For the stopping criterion of the path-based partial linearization algorithm, we adopt the residual error defined as follows.

$$\varepsilon = \frac{\sum_{ij \in IJ} \sum_{r \in R_{ij}} (gc_r^{ij} - \min(gc_r^{ij})) f_r^{ij}}{\sum_{ij \in IJ} \sum_{r \in R_{ij}} gc_r^{ij} f_r^{ij}}, \quad (115)$$

where  $gc_r^{ij} = \beta^{ij} \ln g_r^{ij} + \ln f_r^{ij} - \ln \varpi_r^{ij}$ , which should be identical for all routes at equilibrium. Without loss of generality,  $\varepsilon$  is set at  $10^{-8}$ . The convergence characteristics of the path-based partial linearization algorithm are shown in Fig. 21 and Table 6. From Fig. 21, it appears that the path-based algorithm can solve both SUE models (MNW and PSW) in a linear convergence rate, with the PSW-SUE model requiring a few more iterations to reach the desired level of accuracy. The computational efforts required by each SUE model are provided in Table 6. As expected, the PSW-SUE model does require slightly more computational efforts than the MNW-SUE model in terms of number of iterations, CPU time, and CPU time per iteration.

### 3.5.3.2 *Flow allocation comparison*

At the disaggregate level, we examine the route choice probabilities produced by the PSLs-SUE, MNW-SUE and PSW-SUE models. Note that the PSLs-SUE model use the same link travel cost configuration as the weibit SUE model in Eq. (113). For demonstration purposes, we use O-D pairs (50, 52), (14, 100), and (92, 30) to respectively represent a short, medium, and long O-D pair. The route choice probabilities shown in Fig. 22 are under the respective equilibrium route flow patterns. Recall that each SUE model handles the IID assumption (i.e., route overlapping and non-identical perception variance problems) differently as follows: (1) The PSLs-SUE model can handle the route overlapping problem, but it cannot capture the heterogeneous perception

variance among different routes, (2) the MNW-SUE model can handle the heterogeneous perception variance among different routes, but it cannot resolve the route overlapping problem, while (3) the PSW-SUE model considers both route overlapping and heterogeneous perception variance problems simultaneously. Thus, different route flows (or probabilities) can be expected. Even though O-D pair (50, 52) has only three routes with a heavy overlap between route 2 and route 3, the three SUE models produce significantly different results. The PSLs-SUE model assigns a higher probability to the independent route (i.e., route 1); the MNW SUE model, on the other hand, assigns a higher probability to the two overlapping routes (i.e., route 2 and route 3) compared to the PSL-SUE model; and the PSW-SUE model seems to allocate a flow pattern in between these two models by accounting for both overlap and heterogeneous perception variance among different routes. For the two longer O-D pairs [(14, 100) and (92, 30)], more routes are generated as a result of a longer trip length. When both number of routes and trip length are increasing, the differences among the three models also decrease.

At the aggregate level, we examine the effects of route overlapping and heterogeneous perception variance problems on the link flow patterns. The link flow pattern difference between the PSLs-SUE and PSW-SUE models can be found mostly in the central business district (CBD) area as shown in Fig. 23. The absolute maximum flow difference in the CBD area is 482 vph compared to 304 vph in the outer area (or non-CBD area). This is because there are many short O-D pairs in the CBD area with different trip lengths, and the PSLs-SUE model has difficulty in handling the heterogeneous perception variance among different routes. When comparing between the MNW-SUE and PSW-SUE models, the link flow difference can also be found mostly in

the CBD area (see Fig. 24). This is because more than 60% of the routes (or more than 9,000 routes) pass through the CBD area. As a result, route overlapping is a significant problem in the CBD area compared to the outer area.

### 3.6 Concluding remarks

In this paper, we presented a path-size weibit (PSW) route choice model with an equivalent mathematical programming stochastic user equilibrium (SUE) formulation to relax the *independently and identically distributed* (IID) assumption imposed on the MNL-SUE model. The proposed route choice model adopts the Weibull distributed random error term to handle the route-specific perception variance as a function of route travel cost and a path-size factor to resolve the route overlapping problem by adjusting the choice probabilities for routes with strong couplings with other routes. A multiplicative Beckmann's transformation (MBec) was developed to handle the multiplicative nature of this new route choice model. Incorporating this MBec with an entropy term gives the PSW traffic flow solution under congested conditions. The PSW-SUE model was tested on three networks to examine its features in comparison with some existing SUE models (MNL, PSL, MNW, and MNP) and its applicability on a real network. Through the numerical results, we observed the followings:

- The MNW-SUE model (without the *identically distributed* assumption) can account for the overall trip length by using the relative cost difference to determine the flow pattern much better than the MNL-SUE model does.
- The PSW-SUE model can produce a compatible traffic flow pattern compared to the MNP-SUE model in a congested network.

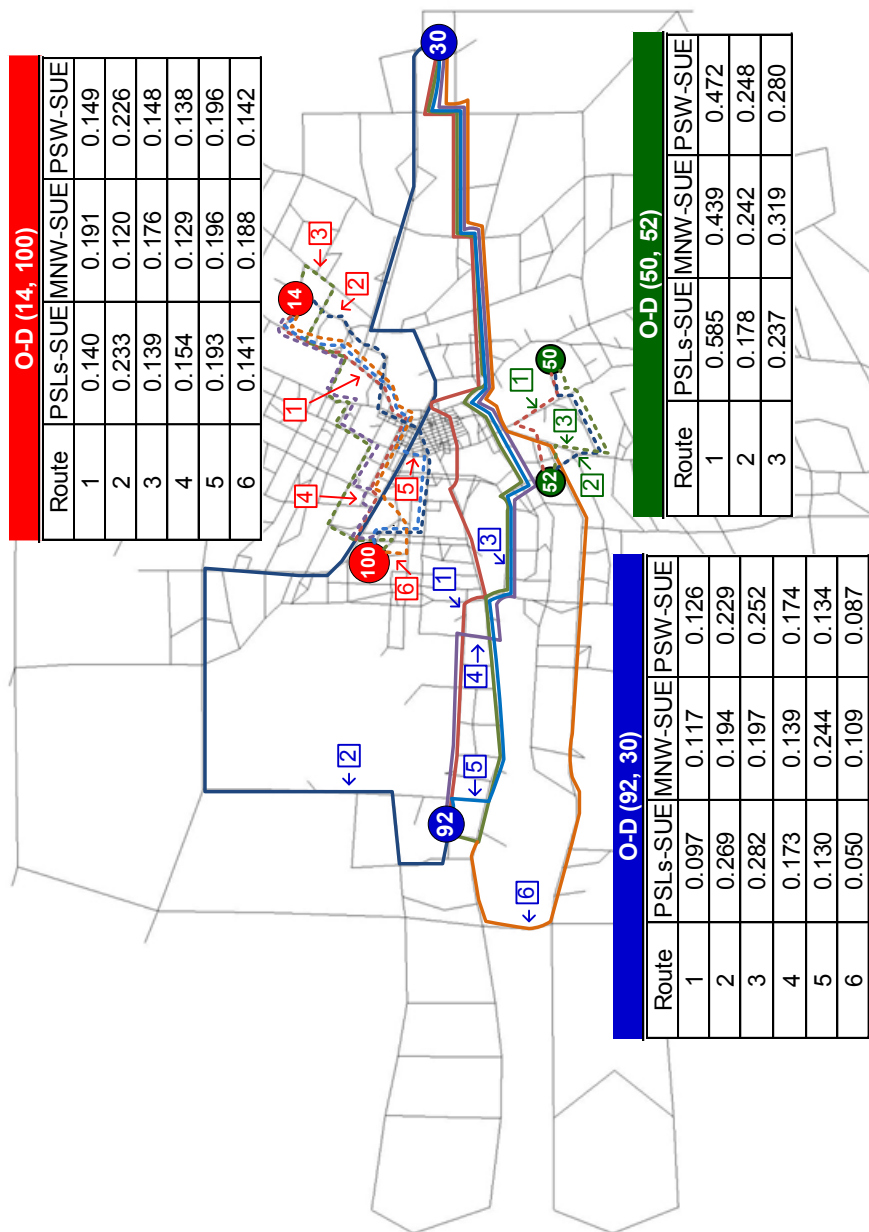


Fig. 22. Comparison of route choice probabilities of three O-D pairs

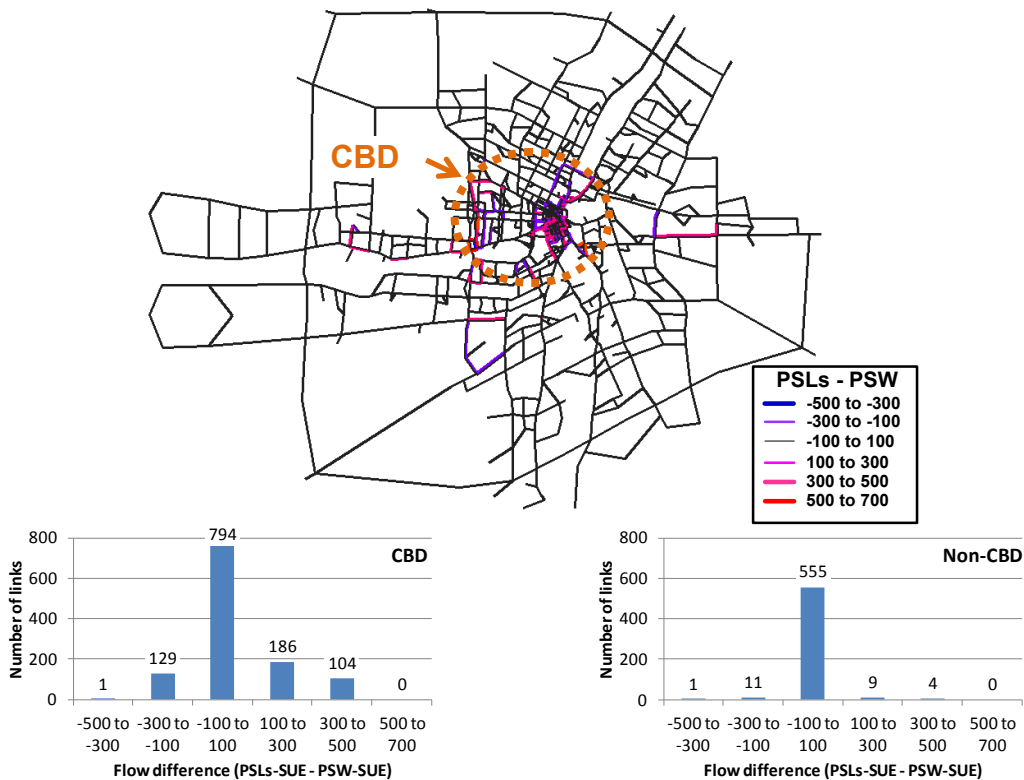


Fig. 23. Link flow difference between PSLs-SUE and PSW-SUE models

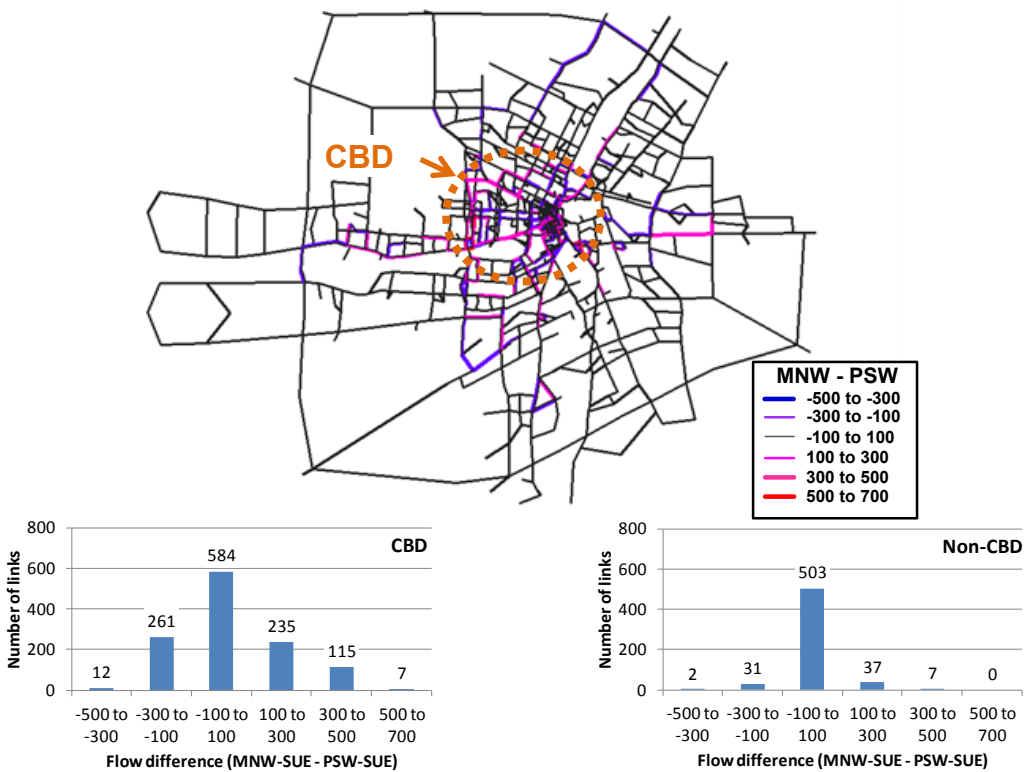


Fig. 24. Link flow difference between MNW-SUE and PSW-SUE models

- The PSW-SUE model can be applied in a real network as shown by the Winnipeg network.

For future research, parameter calibration should be conducted for the PSW-SUE model, and more tests should be conducted to validate the usefulness of the PSW-SUE model. The PSW-SUE model should be extended to consider non-zero location parameter, flow-dependent path-size factors, multiple user classes, and other travel choice dimensions (e.g., elastic demand for travel choice, modal split for mode choice, and trip distribution for destination choice). Moreover, there are several distributions (other than the Weibull distribution) that also give a closed-form probability expression (e.g., [Li, 2011](#)). A route choice model derived from these distributions would provide alternative properties (e.g., route-specific perception variance), and hence different choice probabilities.

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## CHAPTER 4

UNCONSTRAINED WEIBIT STOCHASTIC USER EQUILIBRIUM MODEL WITH  
EXTENSIONS

## Abstract

This study proposes a weibit stochastic user equilibrium (SUE) model to relax the *identically distributed* assumption of the multinomial logit (MNL) SUE model such that the *heterogeneous* perception variances with respect to different trip lengths under congested conditions are explicitly considered. Specifically, we derive an analytical closed-form expected perceived travel cost (EPC) of the multinomial weibit (MNW) model and combine it with the multiplicative Beckmann's transformation to formulate an unconstrained MNW-SUE minimization program. Qualitative properties of the unconstrained minimization program are given to establish equivalency and uniqueness of the MNW-SUE solution. A link-based algorithm combined with recent advances in line search strategies is developed for solving the unconstrained MNW-SUE minimization program. Numerical examples are also provided to illustrate the features of the MNW-SUE model along with several extensions for future research.

## 4.1 Introduction

The multinomial logit (*MNL*) model (Dial, 1971) has been widely used as a route choice model in the transportation literature. Two main advantages are its closed-form choice probability solution, and its equivalent mathematical programming (*MP*) formulation under the stochastic user equilibrium (*SUE*) framework (Fisk, 1980; Daganzo, 1982; Sheffi and Powell, 1982). However, the MNL model has two major drawbacks: (1)

inability to account for overlapping (or correlation) among routes and (2) inability to account for perception variance with respect to (w.r.t.) trips of different lengths (Sheffi, 1985). These two drawbacks stem from the MNL's underlying assumption that the perceived travel costs are *independently and identically distributed* (IID) *Gumbel* with the same and fixed perception variance. To overcome these drawbacks, the multinomial probit (MNP) model (Daganzo and Sheffi, 1977) was adopted. It uses the Normal distribution to handle the route overlapping and identical perception variance problems between routes by allowing the covariance between random error terms for pairs of routes. However, the MNP model does not have a closed-form solution and it is computationally burdensome when the choice set contains more than a handful of routes. Due to the lack of a closed-form probability expression, solving the MNP model will require either Monte Carlo simulation (Sheffi and Powell, 1982), Clark's approximation method (Maher, 1992), or numerical method (Rosa and Maher, 2002).

To overcome the drawbacks of the MNL model while retaining an analytical solution, several closed-form route choice models have been proposed in the transportation literature. These route choice models can be classified into two categories: extended logit models and weibit<sup>†</sup> model. The extended logit models with analytical closed-form probability solution were mainly developed to handle the route overlapping problem, while the weibit model was developed to address the identical perception variance problem. The extended logit models modified either the deterministic term or the random error term in the additive disutility function of the MNL model while retaining the Gumbel distributed random error term assumption. The models modifying the deterministic term of the disutility function include the C-logit model (Cascetta et al.,

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<sup>†</sup> Weibit stands for Weibull probability unit.

1996), the implicit availability/perception (*IAP*) model (Cascetta et al., 2002), and the path-size logit (*PSL*) model (Ben-Akiva and Bierlaire, 1999). All three models add a correction term to the deterministic term of the disutility function to adjust the choice probability; however, the interpretation of each model is different. The C-logit model uses the commonality factor to penalize the coupling routes, while both *IAP* and *PSL* models use a logarithmic correction term to modify the disutility, hence, the choice probability. The models modifying the random error term of the disutility function include the cross-nested logit (*CNL*) model (Bekhor and Prashker, 1999), the paired combinatorial logit (*PCL*) model (Bekhor and Prashker, 1999; Pravinvongvuth and Chen, 2005), and the generalized nested logit (*GNL*) model (Bekhor and Prashker, 2001). These models are based on the generalized extreme value (*GEV*) theory (McFadden, 1978) using a two-level tree structure to capture the similarity among routes through the random error component of the disutility function. This allows a route to belong to more than one nest (i.e., a nest is a link in the *CNL* and *GNL* models or a route pair in the *PCL* model). The route choice probability is calculated according to the two-level tree structure using the marginal and conditional probabilities. Bekhor and Prashker (1999, 2001), Zhou et al. (2012), and Chen et al. (2012) provided equivalent MP formulations for these extended logit models, while Chen et al. (2003), Bekhor et al. (2008a,b), Xu et al. (2012), and Zhou et al. (2012) developed path-based algorithms for solving these MP formulations. For a more comprehensive review of the extended logit models used in the SUE problem, readers are directed to the reviews given by Prashker and Bekhor (2004) and Chen et al. (2012).

As mentioned above, the closed-form extended logit models were developed mainly to address the independence assumption (i.e., route overlapping problem) of the MNL model. The identically distributed assumption is still inherited in the route choice problem, where *all* routes between *all* origin-destination (O-D) pairs still have the same and fixed perception variance. In view of network equilibrium assignment, the identically distributed assumption seems unrealistic since it does not distinguish trip lengths of different O-D pairs. Therefore, [Chen et al. \(2012\)](#) suggested a practical approach by scaling the perception variance of each individual O-D pair in the network. The individual O-D specific scaling dispersion parameter allows the perception variance to increase or decrease according to the travel distance of each O-D pair. Nevertheless, it should be noted that it is *not* possible to scale individual routes of the same O-D pair since it would violate the logit model's assumption of an identical variance across the routes within the same O-D pair (i.e., homogeneous perception variance) to maintain an analytical closed-form solution.

[Castillo et al. \(2008\)](#), on the other hand, provided a multinomial weibit (MNW) model to relax the *identically distributed* assumption of the MNL model. This route choice model adopts the *Weibull* distribution, instead of the conventional Gumbel distribution, to handle the *heterogeneous* perception variance at the route level. Under the independently distributed assumption, the MNW model has an analytical closed-form probability expression with *route-specific* perception variance as a function of the route travel cost. Recently, we provided an equivalent MP formulation as a *constrained* (or entropy-type) optimization problem and a path-based partial linearization algorithm for solving the MNW-SUE model (see Chapter 3).

The aim of this study is to provide an *unconstrained* minimization program as an alternative MP formulation for the MNW-SUE model. Specifically, we derive an *analytical closed-form* expected perceived travel cost (EPC) of the MNW model and combine it with the multiplicative Beckmann's transformation to formulate an *unconstrained* MNW-SUE minimization program that explicitly considers the route-specific (or *heterogeneous*) perception variances w.r.t. different trip lengths under congested networks. Some quantitative properties of the proposed *unconstrained* minimization program are provided to establish equivalency and uniqueness of the MNW-SUE solution. In addition, a link-based algorithm combined with recent advances in line search strategies is developed for solving the *unconstrained* MNW-SUE minimization program. Numerical examples using a toy network and a real network are also provided to illustrate the features of the *unconstrained* MNW-SUE model along with three extensions to consider both route overlapping and route-specific perception variance problems, demand elasticity, and multiple user classes.

The remainder of this paper is organized as follows. Section 2 provides some background of the MNW model and presents the MNW EPC. In section 3, an equivalent *unconstrained* MP formulation is provided for the MNW-SUE problem. Section 4 provides numerical results to illustrate the features of the proposed *unconstrained* MNW-SUE problem. Concluding remarks and several extensions are addressed in Section 5.

#### 4.2 Multinomial weibit model

In this section, we review the multinomial weibit (MNW) model developed by [Castillo et al. \(2008\)](#). Specifically, we present a closed-form expected perceived travel cost (EPC) of the MNW model. The section begins with the Weibull distribution,



followed by the MNW closed-form probability expression, the stability property w.r.t. the minimum operation, its choice probability, and finally the closed-form MNW EPC.

#### 4.2.1 Weibull distribution

Let  $G_r^{ij}$  denote travelers' perceived travel cost on route  $r \in R_{ij}$  between O-D pair  $ij \in IJ$ . The MNW model assumes that  $G_r^{ij}$  follows the 3-parameter Weibull distribution with the mean equals to  $g_r^{ij}$  the travel cost on route  $r$  between O-D pair  $ij$ . Similar to the Gumbel distribution, this extreme value distribution has a closed-form cumulative distribution function (CDF):

$$F_{G_r^{ij}}(t) = 1 - \exp \left\{ - \left[ \frac{(t - \zeta_r^{ij})^{\beta_r^{ij}}}{\varphi_r^{ij}} \right] \right\}, \quad \forall r \in R_{ij}, ij \in IJ, \quad (116)$$

where  $t$  is the random perceived travel cost,  $\varphi_r^{ij} > 0$  is the scale parameter,  $\beta_r^{ij} > 0$  is the shape parameter, and  $0 \leq \zeta_r^{ij} < t$  is the location parameter.  $g_r^{ij}$  can be expressed as

$$g_r^{ij} = \zeta_r^{ij} + \varphi_r^{ij} \Gamma \left( 1 + \frac{1}{\beta_r^{ij}} \right), \quad \forall r \in R_{ij}, ij \in IJ, \quad (117)$$

where  $\Gamma(\cdot)$  is the Gamma function. Unlike the Gumbel distribution, the variance of the Weibull distribution is a function of  $g_r^{ij}$ , i.e.,

$$\left( \sigma_r^{ij} \right)^2 = \left( \varphi_r^{ij} \right)^2 \Gamma \left( 1 + \frac{2}{\beta_r^{ij}} \right) - \left( g_r^{ij} - \zeta_r^{ij} \right)^2, \quad \forall r \in R_{ij}, ij \in IJ. \quad (118)$$

This feature plays an important role in handling the perception variance w.r.t. different trip lengths (to be shown in subsequent sections). For a comparison between the Gumbel and Weibull distributions, see Appendix A.

#### 4.2.2 Closed-form probability expression

The MNW choice probability can be derived as follows. According to [Ben-Akiva and Lerman \(1985\)](#) and [Castillo et al. \(2008\)](#), the route choice probability can be determined by

$$P_r^{ij} = - \int_{-\infty}^{+\infty} \bar{H}_r^{ij}(\dots, t_r^{ij}, \dots) dt_r^{ij}, \forall r \in R_{ij}, ij \in IJ, \quad (119)$$

where  $\bar{H}_r^{ij}$  is the partial derivative of the joint survival distribution  $\bar{H}$  w.r.t.  $t_r^{ij}$ . Under the *independently distributed* assumption, we have the joint Weibull survival function:

$$\bar{H} = \prod_{r \in R_{ij}} \exp \left\{ - \left[ \frac{(t_r^{ij} - \zeta_r^{ij})}{\varphi_r^{ij}} \right]^{\beta_r^{ij}} \right\}. \quad (120)$$

Partial derivative of  $\bar{H}$  w.r.t.  $t_r^{ij}$  gives

$$\bar{H}_r^{ij} = - \frac{\beta_r^{ij}}{\varphi_r^{ij}} \left[ \frac{(t_r^{ij} - \zeta_r^{ij})}{\varphi_r^{ij}} \right]^{\beta_r^{ij} - 1} \exp \left\{ - \left[ \frac{(t_r^{ij} - \zeta_r^{ij})}{\varphi_r^{ij}} \right]^{\beta_r^{ij}} \right\} \prod_{\substack{l \neq r \\ l \in R_{ij}}} \exp \left\{ - \left[ \frac{(t_l^{ij} - \zeta_l^{ij})}{\varphi_l^{ij}} \right]^{\beta_l^{ij}} \right\}. \quad (121)$$

From Eq. (119) and Eq. (121), we have

$$P_r^{ij} = - \int_{\zeta_r^{ij}}^{+\infty} - \frac{\beta_r^{ij}}{\varphi_r^{ij}} \left[ \frac{(t_r^{ij} - \zeta_r^{ij})}{\varphi_r^{ij}} \right]^{\beta_r^{ij} - 1} \exp \left\{ - \left[ \frac{(t_r^{ij} - \zeta_r^{ij})}{\varphi_r^{ij}} \right]^{\beta_r^{ij}} \right\} \prod_{\substack{l \neq r \\ l \in R_{ij}}} \exp \left\{ - \left[ \frac{(t_l^{ij} - \zeta_l^{ij})}{\varphi_l^{ij}} \right]^{\beta_l^{ij}} \right\} dt_r^{ij}. \quad (122)$$

To obtain a *closed-form* expression, both  $\beta_r^{ij}$  and  $\zeta_r^{ij}$  are fixed for *all* routes connecting O-D pair  $ij$ , i.e.,

$$P_r^{ij} = \int_{\zeta_r^{ij}}^{+\infty} \frac{\beta_r^{ij}}{\varphi_r^{ij}} \left[ \frac{(t_r^{ij} - \zeta_r^{ij})}{\varphi_r^{ij}} \right]^{\beta_r^{ij} - 1} \exp \left\{ - \sum_{k \in R_{ij}} \left[ \frac{(t_r^{ij} - \zeta_r^{ij})}{\varphi_k^{ij}} \right]^{\beta_k^{ij}} \right\} dt_r^{ij}. \quad (123)$$

By integrating the above equation, we have the MNW probability as a function of the scale and shape parameters:

$$P_r^{ij} = \frac{(\varphi_r^{ij})^{-\beta^{ij}}}{\sum_{k \in R_{ij}} (\varphi_k^{ij})^{-\beta^{ij}}}, \quad \forall r \in R_{ij}, ij \in IJ. \quad (124)$$

From Eq. (117),  $\varphi_r^{ij}$  is related to  $g_r^{ij}$  as follows:

$$\varphi_r^{ij} = \frac{(g_r^{ij} - \zeta^{ij})}{\Gamma\left(1 + \frac{1}{\beta^{ij}}\right)}, \quad \forall r \in R_{ij}, ij \in IJ. \quad (125)$$

Substituting Eq. (125) into Eq. (124) gives the MNW model:

$$P_r^{ij} = \frac{(g_r^{ij} - \zeta^{ij})^{-\beta^{ij}}}{\sum_{k \in R_{ij}} (g_k^{ij} - \zeta^{ij})^{-\beta^{ij}}}, \quad \forall r \in R_{ij}, ij \in IJ. \quad (126)$$

Note that since  $\zeta^{ij}$  is the lower bound of the travel cost (Castillo et al., 2008), it naturally implies that  $\zeta^{ij} < g_r^{ij}$ ,  $\forall r \in R_{ij}, ij \in IJ$ .

#### 4.2.3 Stability property w.r.t. the minimum operation

Stability is an important property of the extreme value distributions. It states that the joint survival function at the minimum is the same function as the marginal survival function. Castillo et al. (2008) showed that the joint Weibull distribution with fixed  $\beta^{ij}$  and  $\zeta^{ij}$  satisfies the stability property w.r.t. the minimum operation as follows:

$$\bar{H}_{\min_{k \in R_{ij}} G_k^{ij}}(t) = \exp \left\{ - \left[ \frac{(t - \zeta^{ij})}{\varphi^{ij,0}} \right]^{\beta^{ij}} \right\}, \quad (127)$$

where

$$\varphi^{ij,0} = \left( \sum_{k \in R_{ij}} (\varphi_k^{ij})^{-\beta^{ij}} \right)^{-1/\beta^{ij}}. \quad (128)$$

With this property, travelers' choice decision is assumed to be based on their *perceived minimum* travel cost of each route, and the probabilistic route choice patterns can be determined by the multivariate extreme value distribution (e.g., [Kotz and Nadarajah, 2000](#)) with the Weibull marginal. In other words, the joint distribution can be written as (see [Li, 2011](#))

$$H(t) = 1 - [1 - F(t)]^\alpha, \quad (129)$$

where  $H(t)$  is the joint distribution, and  $F(t)$  is the marginal distribution. The Weibull distribution with fixed  $\beta^{ij}$  and  $\zeta^{ij}$  satisfies this condition, i.e.,

$$H(t) = 1 - [1 - F(t)]^{(\varphi^{ij,0})^{-\beta^{ij}}}, \quad (130)$$

where  $F(t)$  is the Weibull distribution, i.e.,

$$F(t) = 1 - \exp\left\{-\left(t - \zeta^{ij}\right)^{\beta^{ij}}\right\}. \quad (131)$$

Thus, we can use the Weibull variance to *represent* the route perception variance for the MNW model. From Eq. (118) and Eq. (125), the perception variance of each route for the MNW model can be expressed as

$$\left(\sigma_r^{ij}\right)^2 = \left[ \frac{\left(g_r^{ij} - \zeta^{ij}\right)}{\Gamma\left(1 + 1/\beta^{ij}\right)} \right]^2 \left[ \Gamma\left(1 + \frac{2}{\beta^{ij}}\right) - \Gamma^2\left(1 + \frac{1}{\beta^{ij}}\right) \right], \quad \forall r \in R_{ij}, ij \in IJ. \quad (132)$$

Note that this MNW perception variance is *route-specific*. The perception variance of each route can vary according to the travel cost  $g_r^{ij}$  with fixed  $\beta^{ij}$  and  $\zeta^{ij}$  of O-D pair  $ij$ .

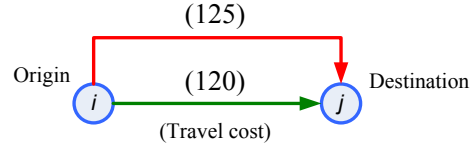
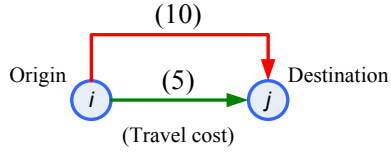
Unlike the MNL model, *all* routes between *all* O-D pairs are fixed with *identical*

perception variance in the Gumbel distribution (or a fixed dispersion parameter  $\theta$  in the MNL-SUE problem). Although it can be partially relaxed by scaling the perception variance (or dispersion parameter) of each O-D pair as demonstrated by [Chen et al. \(2012\)](#) in the context of traffic assignment, each route within the same O-D pair still requires to have identical perception variance.

#### 4.2.4 Route choice probability

We use the two-route networks in Fig. 25 to compare the choice probability produced by the MNL and MNW models. The upper route cost is larger than the lower route cost by 5 units for both networks. In the short network, the upper route cost is twice larger than the lower route cost, while it is only less than 5% larger in the long network. We assume that the MNL dispersion parameter  $\theta$  is equal to 0.1, and the MNW parameters  $\beta^{ij}$  and  $\zeta^{ij}$  are equal to 2.1 and 0, respectively.

As expected, the MNL model gives the same results for both short and long networks. The MNL choice probability is determined solely based on the *absolute* route cost difference, while the MNW model uses the *relative* route cost difference to handle different trip lengths. When considering the perception variance, the MNW model has the route-specific perception variance from Eq. (132). Unlike the same and fixed perception variance of  $\pi^2/6\theta^2$  in the MNL model, the MNW model has a lower perception variance for the shorter route as shown in Fig. 12. As the overall trip length increases, the perception variance of the upper and lower routes becomes more similar; and therefore, the probability of choosing each route is increasingly similar.



MNL

$$\theta = 0.1;$$

$$P_l^{ij} = \frac{e^{-0.1(5)}}{e^{-0.1(5)} + e^{-0.1(10)}} = \frac{1}{1 + e^{-0.1(5)}} = 0.62$$

MNL

$$\theta = 0.1;$$

$$P_l^{ij} = \frac{e^{-0.1(120)}}{e^{-0.1(120)} + e^{-0.1(125)}} = \frac{1}{1 + e^{-0.1(5)}} = 0.62$$

MNW

$$\beta^{ij} = 2.1, \zeta^{ij} = 0;$$

$$P_l^{ij} = \frac{5^{-2.1}}{5^{-2.1} + 10^{-2.1}} = \frac{1}{1 + (10/5)^{-2.1}} = 0.81$$

MNW

$$\beta^{ij} = 2.1, \zeta^{ij} = 0;$$

$$P_l^{ij} = \frac{120^{-2.1}}{120^{-2.1} + 125^{-2.1}} = \frac{1}{1 + (125/120)^{-2.1}} = 0.52$$

a) Short network

b) Long network

Fig. 25. Probability of choosing the lower route for the two-route networks

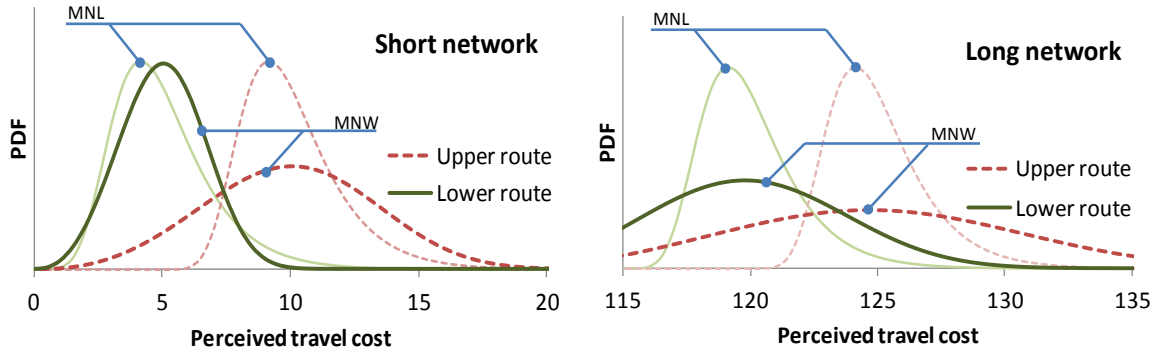


Fig. 26. Perceived travel cost distributions for the two-route networks

We further generalize the two-route networks to consider the probabilistic curve for different combinations of cost difference between the two routes and the effect of  $\beta^{ij}$  and  $\zeta^{ij}$  in Fig. 27. The MNW model produces a series of flatter probabilistic curves for

the long network compared to the series of sharper probability curves for the short network, while the MNL model just gives one probabilistic curve for both short and long networks. As  $\beta^{ij}$  and/or  $\zeta^{ij}$  increases, the probabilistic curve is getting steeper. This is due to the decrease in perception variance, which increases the chance of selecting the shorter route. Between the two networks, the impact of  $\beta^{ij}$  and  $\zeta^{ij}$  is more pronounced on the short network due to its shorter trip length.

#### 4.2.5 Expected Perceived Travel Cost

According to the stability property, we can determine the MNW EPC by substituting the Weibull scale parameter in Eq. (128) into Eq. (117) as follows:

$$\mu_{ij} = E[t] = \zeta^{ij} + \left( \sum_{k \in R_{ij}} (\varphi_k^{ij})^{-\beta^{ij}} \right)^{-1/\beta^{ij}} \Gamma \left( 1 + \frac{1}{\beta^{ij}} \right), \quad (133)$$

where  $E[ ]$  is the expected value operator. From Eq. (125), the MNW EPC (up to a constant) can be restated as

$$\mu_{ij} = \left( \sum_{r \in R_{ij}} (g_r^{ij} - \zeta^{ij})^{-\beta^{ij}} \right)^{-\frac{1}{\beta^{ij}}}, \quad \forall ij \in IJ. \quad (134)$$

Note that we can alternatively derive this MNW EPC from

$$\begin{aligned} \mu_{ij} = E[t] &= \int_{-\infty}^{\infty} t d\bar{H}_{\min_{k \in R_{ij}} G_k^{ij}}(t) \\ &= \int_{\zeta^{ij}}^{\infty} t d \exp \left\{ - \left[ (t - \zeta^{ij})^{\beta^{ij}} / (\varphi^{ij,0})^{\beta^{ij}} \right] \right\}, \end{aligned} \quad (135)$$

which would give the same result as in Eq. (133). However, the partial derivative of this MNW EPC w.r.t. the route travel cost does not give back the MNW choice probability

(Fosgerau and Bierlaire, 2009). Therefore, it could *not* be used directly to develop the unconstrained MP formulation for the MNW-SUE model.

To overcome this drawback, we consider the *logarithmic* MNW EPC, i.e.,  $E[\ln(t - \zeta^{ij})]$ . From the joint Weibull survival function in Eq. (127), the logarithmic MNW EPC can be determined by

$$\begin{aligned} \bar{\mu}_{ij} &= E[\ln(t - \zeta^{ij})] \\ &= \int_{\zeta^{ij}}^{\infty} \ln(t - \zeta^{ij}) d\bar{H}_{\min_{k \in R_{ij}} G_k^{ij}}(t) \\ &= \int_{\zeta^{ij}}^{\infty} \ln(t - \zeta^{ij}) d \exp\left\{-\left[(t - \zeta^{ij})^{\beta^{ij}} / (\varphi^{ij,0})^{\beta^{ij}}\right]\right\}. \end{aligned} \tag{136}$$

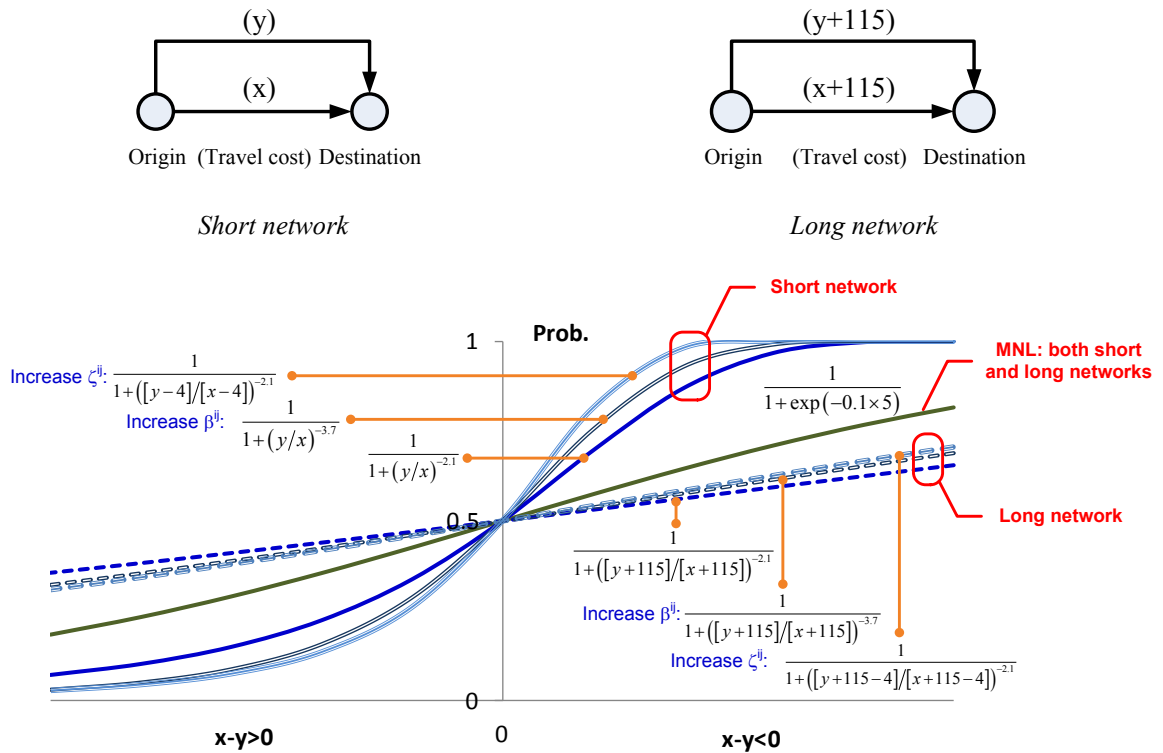


Fig. 27. Effect of  $\beta^{ij}$  and  $\zeta^{ij}$  on the probability of choosing the lower route



Let  $x$  denote  $(t - \zeta^{ij})^{\beta^{ij}} / (\varphi^{ij,0})^{\beta^{ij}}$ . Eq. (136) can be restated as

$$\bar{\mu}_{ij} = \int_0^{\infty} \ln(x^{1/\beta^{ij}} \varphi^{ij,0}) d \exp\{-x\}. \quad (137)$$

Since  $\int_0^{\infty} \ln(y) e^{-y} dy$  is a constant (see [Abramowitz and Stegun, 1972](#)), we have the logarithmic MNW EPC up to a constant:

$$\begin{aligned} \bar{\mu}_{ij} &= \int_0^{\infty} \ln(\varphi^{ij,0}) d \exp\{-x\} \\ &= -\ln \varphi^{ij,0} \exp(-x) \Big|_0^{\infty} \\ &= \ln \varphi^{ij,0} \end{aligned} \quad (138)$$

From Eq. (125) and Eq. (128), the logarithmic MNW EPC can be restated as

$$\bar{\mu}_{ij} = -\frac{1}{\beta^{ij}} \ln \sum_{r \in R_{ij}} (g_r^{ij} - \zeta^{ij})^{-\beta^{ij}}, \quad \forall ij \in IJ. \quad (139)$$

Since the constant has *no* impact on the solution, we do not consider it in the formulation development of the *unconstrained* minimization program.

Three important properties of this logarithmic MNW EPC are as follows. First, the partial derivative of this logarithmic MNW EPC w.r.t. the *logarithmic* route travel cost gives back the MNW choice probability, i.e.,

$$\begin{aligned} \frac{\partial \bar{\mu}_{ij}}{\partial \ln(g_r^{ij} - \zeta^{ij})} &= -\frac{1}{\beta^{ij}} (g_r^{ij} - \zeta^{ij}) \frac{\partial \left\{ \ln \sum_{k \in R_{ij}} (g_k^{ij} - \zeta^{ij})^{-\beta^{ij}} \right\}}{\partial (g_r^{ij} - \zeta^{ij})} \\ &= \frac{(g_r^{ij} - \zeta^{ij})^{-\beta^{ij}}}{\sum_{k \in R_{ij}} (g_k^{ij} - \zeta^{ij})^{-\beta^{ij}}}. \end{aligned} \quad (140)$$

Second, it is monotonic decreasing w.r.t. the number of routes; and third, it is concave w.r.t.  $\ln(g_r^{ij} - \zeta^{ij})$ . Details of the second and third properties are provided in Appendix B.

### 4.3 Unconstrained minimization program

In this section, we adopt the logarithmic MNW EPC to develop an *unconstrained* MP formulation for the MNW-SUE model. This section begins with some necessary assumptions, followed by the *unconstrained* MP formulation, and comparison between the proposed *unconstrained* MNW-SUE model and the *constrained* entropy-type MNW-SUE model.

#### 4.3.1 Assumptions

Before formulating the *unconstrained* MNW-SUE model, some necessary assumptions are made. To begin with, a general assumption of link travel cost function is made, i.e.,

Assumption 4.1. The travel cost on link  $a \in A$  ( $\tau_a$ ), which could be a function of travel time, is a strictly increasing function w.r.t. its own flow.

Since  $\zeta^{ij}$  cannot be easily decomposed into the link level, we make another assumption:

Assumption 4.2.  $\zeta^{ij}$  is equal to zero.

Hence, each route is assumed to have the same coefficient of variation (CoV). According to Eq. (132), the CoV can be expressed as

$$g_r^{ij} = \frac{\sigma_r^{ij}}{g_r^{ij}} = \frac{(g_r^{ij} - \zeta^{ij})}{g_r^{ij}} \sqrt{\frac{\Gamma(1+2/\beta^{ij})}{(\Gamma(1+1/\beta^{ij}))^2} - 1}, \quad \forall r \in R_{ij}, ij \in IJ. \quad (141)$$

With  $\zeta^{ij} = 0$ , all routes have the same  $\mathcal{G}_r^{ij}$ . The higher  $\beta^{ij}$ , the smaller  $\mathcal{G}_r^{ij}$ . Note that we can incorporate  $\zeta^{ij}$  into the MNW-SUE model by adopting the variational inequality formulation of the congestion-based C-logit SUE model (Zhou et al., 2012).

To handle the logarithmic MNW EPC, we make another assumption on the route travel cost:

Assumption 4.3. The route travel cost is a function of *multiplicative* link travel costs  $\tau_a$ , i.e.,

$$\mathbf{g}_r^{ij} = \prod_{a \in \Upsilon_r} \tau_a, \quad \forall r \in R_{ij}, ij \in IJ, \quad (142)$$

where  $\Upsilon_r$  is the set of links in route  $r$  between O-D pair  $ij$ .

This assumption corresponds to the *Markov process* in transportation network analysis (see Akamatsu, 1996). Under a *suitable* link travel cost function, travelers are assumed to make a decision at each node (or *state*) until they reach the destination (or *final state*) according to the weibit choice probability using the MNW EPC (see Appendix C for more details).

#### 4.3.2 Formulation

Consider the following unconstrained MP formulation:

$$\begin{aligned} \min Z &= Z_1 + Z_2 + Z_3 \\ &= - \sum_{a \in A} \int_0^{v_a} \ln \tau_a(\omega) d\omega - \sum_{ij \in IJ} q_{ij} \bar{\mu}_{ij} + \sum_{a \in A} v_a \ln \tau_a(v_a), \end{aligned} \quad (143)$$

where  $v_a$  is the flow on link  $a$ . In this MNW-SUE problem, all  $Z_1$ ,  $Z_2$ , and  $Z_3$  differ from the ordinary unconstrained SUE formulation (Sheffi and Powell, 1982; Daganzo, 1982).

Instead of the Beckmann's transformation,  $Z_1$  is the multiplicative Beckmann's

transformation (*MBec*) used to handle the multiplicative link travel cost in Assumption 4.3.  $Z_2$  includes the logarithmic EPC derived in section 2.5.  $Z_3$  can be considered as a system performance; however, it is a summation of the product of link flow and logarithmic link cost. Combining the *MBec* and  $Z_3$ , the *link* flow solution according to the weibit choice probability can be obtained from logarithmic weibit EPC (see Proposition 4.1).

Proposition 4.1. *The unconstrained SUE problem in Eq. (61) is equivalent to the MNW model.*

Proof. See Appendix D.

Proposition 4.2. *The unconstrained MNW SUE problem in Eq. (61) has the unique link flow solution.*

Proof. See Appendix E.

#### 4.3.3 Unconstrained and constrained MP formulations comparison

In this section, we compare the proposed *unconstrained* MNW-SUE formulation with the *constrained* entropy-type MNW-SUE model in Chapter 3 as shown in Table 7. The main differences between these two formulations are the decision variables. The *constrained* entropy-type formulation adopts the *route* flows as the decision variables. Hence, it has to be solved using a path-based algorithm (e.g., [Chen et al., 2002](#)). On the other hand, the *unconstrained* formulation uses the *link* flows as the decision variables. It obviates the route storage in the entropy-type formulation by implementing a link-based loading technique (e.g., [Dial, 1971](#); [Sheffi, 1985](#); [Bell, 1995](#); [Akamatsu, 1996](#)). Therefore, a link-based algorithm can be applied to solve the *unconstrained* SUE problem (e.g., [Sheffi, 1985](#); [Maher, 1998](#)).

Table 7: *Constrained* entropy-type and *unconstrained* MNW-SUE formulations

<b>Model</b>	<b>Entropy-type</b>	<b>Unconstrained</b>
<b>Formulation</b>	$\min Z = \sum_{a \in A} \int_0^{v_a} \ln \tau_a(\omega) d\omega$ $+ \sum_{ij \in IJ} \frac{1}{\beta^{ij}} \sum_{r \in R_{ij}} f_r^{ij} (\ln f_r^{ij} - 1)$ <p>s.t.</p> $\sum_{r \in R_{ij}} f_r^{ij} = q_{ij}, \quad \forall ij \in IJ$ $f_r^{ij} \geq 0, \quad \forall r \in R_{ij}, ij \in IJ$	$\min Z = - \sum_{a \in A} \int_0^{v_a} \ln \tau_a(\omega) d\omega$ $- \sum_{ij \in IJ} q_{ij} \bar{\mu}_{ij} + \sum_{a \in A} v_a \ln \tau_a(v_a)$
<b>Decision variable</b>	Route flows	Link flows
<b>Unique solution</b>	Route flows and link flows	Link flows (at the vicinity of the minimum)

#### 4.4 Solution algorithm

In this section, we provide a link-based solution algorithm for solving the *unconstrained* MNW-SUE model. Generally, this algorithm has two main steps: search direction and line search. A search direction is obtained by solving a convex auxiliary problem through the equivalency conditions (i.e., the first-order approximation of the objective function). This can be done by performing a stochastic loading scheme that produces the MNW link-flow pattern. A line search is computed in the search direction w.r.t. the original objective function, and then the resulting stepsize defines a new

solution with a reduced objective value. The major steps of the link-based solution algorithm are as follows.

---

Step 0: Initialization.

- Set iteration counter  $n=0$ ; update link travel costs; and perform the MNW loading

to obtain initial link flows: 
$$\mathbf{v}_a^{(0)} = \sum_{ij \in IJ} \sum_{r \in R_{ij}} q_{ij} P_r^{ij(0)} \delta_{ra};$$

Step 1: Direction finding

- Increment iteration counter  $n=n+1$ ; update link travel costs; perform the MNW

loading to obtain *auxiliary* link flows: 
$$\tilde{\mathbf{v}}_a^{(n)} = \sum_{ij \in IJ} \sum_{r \in R_{ij}} q_{ij} \tilde{P}_r^{ij(n)} \delta_{ra},$$
 and determine the

search direction  $(\tilde{\mathbf{v}}^{(n)} - \mathbf{v}^{(n-1)})$ ; (See Section 4.1 for details)

Step 2: Line search

- Determine the stepsize  $\alpha^{(n)}$  via some line search schemes (See Section 4.2 for details)

Step 3: Move

- $$\mathbf{v}^{(n)} = \mathbf{v}^{(n-1)} + \alpha^{(n)} (\tilde{\mathbf{v}}^{(n)} - \mathbf{v}^{(n-1)});$$

Step 4: Convergence test

- If  $RMSE = \sqrt{\|\tilde{\mathbf{v}}^{(n)} - \mathbf{v}^{(n-1)}\|_2 / |A|} \leq \varepsilon$  holds, terminate; otherwise, go to Step 1.
- 

#### 4.4.1 Link-based stochastic loading mechanism

According to Assumption 4.3, the MNW model flows the *Markov process* (see Appendix C), a link-based stochastic loading can be adopted to determine the *auxiliary*

link flows. Without loss of generality, we consider the STOCH algorithm (Dial, 1971) to perform the MNW stochastic loading.

Let  $i$  be origin,  $j$  be destination,  $h(a)$  be the head node of link  $a$ ,  $t(a)$  be the tail node of link  $a$ ,  $r(n)$  be the lowest travel cost from origin  $i$  to node  $n$ , and  $s(n)$  be the lowest travel cost from destination  $j$  to node  $n$ . Unlike the MNL model whose choice probability incorporates the exponential transformation, the MNW model makes direct use of the route costs to compute the choice probabilities. Therefore, the *link likelihood* for the MNW model does not need the exponential transformation, i.e.,

$$LL_a = \begin{cases} \tau_a^{-\beta^{ij}} \left[ \frac{r(t(a))}{r(h(a))} \right]^{\beta^{ij}}, & \text{if } r(h(a)) < r(t(a)) \text{ and } s(h(a)) > s(t(a)) \\ 0 & \text{, otherwise} \end{cases} \quad (144)$$

The *link weight* can be written as

$$w_a = \begin{cases} LL_a & \text{, if } h(a) = i \\ LL_a \sum_{\substack{t(b)=h(a) \\ \forall b \in A}} w_b & \text{, otherwise} \end{cases} \quad (145)$$

and the *link flow* can be determined by

$$v_a = \begin{cases} q_{ij} \frac{w_a}{\sum_{\substack{t(b)=j \\ \forall b \in A}} w_b} & \text{, if } t(a) = j \\ \left[ \sum_{\substack{h(c)=t(a) \\ \forall c \in A}} v_c \right] \frac{w_a}{\sum_{\substack{t(b)=t(a) \\ \forall b \in A}} w_b} & \text{, otherwise} \end{cases} \quad (146)$$

Assuming that the link travel cost is an exponential function,  $q_{ij} = 1$  and  $\beta^{ij} = 1$ .

Using the Braess network, we illustrate how to implement the STOCH loading scheme for the MNW model in Fig. 28. Fig. 28a presents the results of  $r(n)$  and  $s(n)$ . Note that  $r(n)$  ( $s(n)$ ) is not equal to zero for the origin (destination) node due to the exponential

travel cost function. Fig. 28b presents the link likelihood results, and Fig. 28c shows the link flow pattern. To verify the link flow pattern obtained from the STOCH loading scheme indeed satisfies the MNW model, we compute the route flows directly based on the MNW probability expressions in Eq. (11) and compare the results in Table 8. The results are the same, and this verifies the validity of the MNW link-based solution obtained from the STOCH loading scheme.

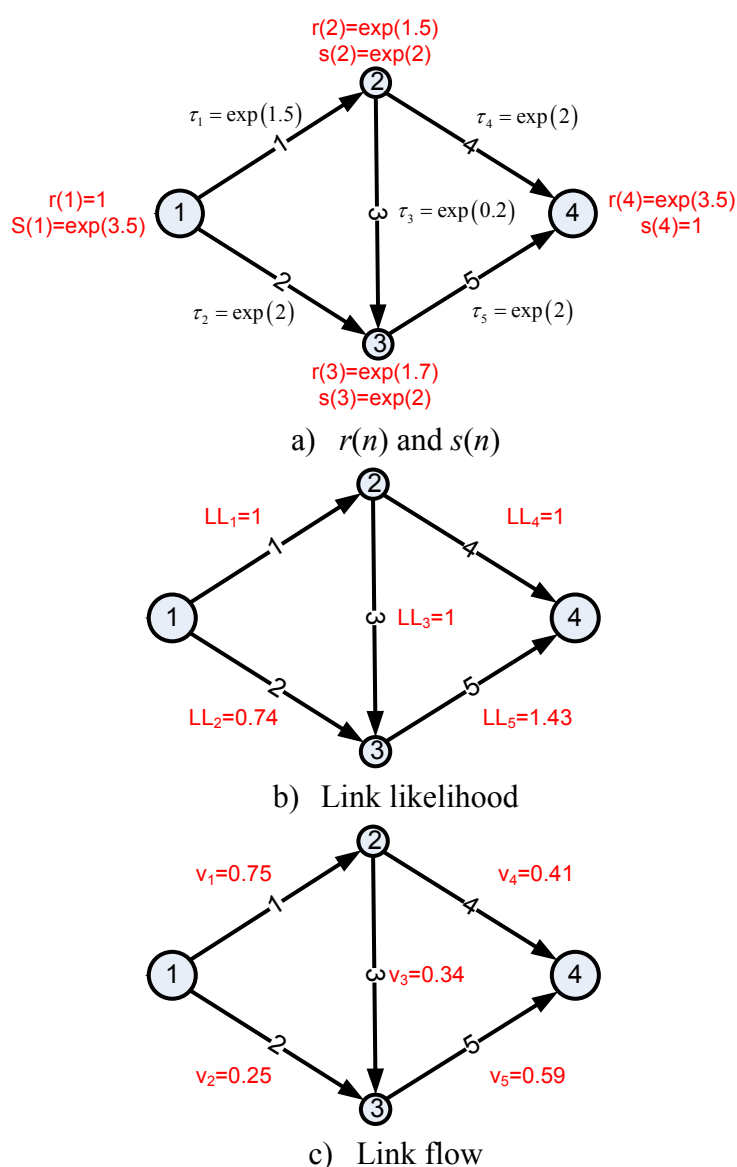


Fig. 28. STOCH stochastic loading mechanism for the MNW model



Table 8: MNW route and link flow solutions

Route	Links	Flow	Link	Flow
1	1 – 4	0.41	1	0.75
2	2 – 5	0.25	2	0.25
3	1 – 3 – 5	0.34	3	0.34
			4	0.41
			5	0.59

#### 4.4.2 Line search schemes

In this section, we consider three stepsize schemes: method of successive averages (MSA) (Sheffi, 1985), self-regulated averaging (SRA) scheme (Liu et al., 2009), and quadratic interpolation (Quad) (Maher, 1998). A brief description of each line search scheme is provided as follows.

##### 4.4.2.1 MSA scheme

The MSA scheme is perhaps the most widely used line search method to determine the stepsize for complex objective functions including those that cannot be evaluated analytically (e.g., the probit-based satisfaction function). It uses a predetermined diminishing stepsize sequence, such as  $\alpha^{(n)} = 1/n$ , that satisfies  $\alpha(n) \rightarrow 0$  and

$\sum_{n=0}^{\infty} \alpha(n) = \infty$  to guarantee convergence. This scheme is easy to implement since it does

not need to evaluate the complex objective function and/or its derivatives. However, it suffers from a sublinear convergence rate (Sheffi, 1985).

#### 4.4.2.2 SRA scheme

To overcome some drawbacks of the MSA scheme, Liu et al. (2009) developed the SRA scheme to relax the predetermined stepsize sequence. This SRA scheme determines a suitable stepsize as follows:

$$\alpha^{(n)} = \frac{1}{\eta^{(n)}}, \quad (147)$$

$$\eta^{(n)} = \begin{cases} \eta^{(n-1)} + \lambda_1, & \text{if } \|\tilde{x}^{(n)} - x^{(n-1)}\| > 1 \\ \eta^{(n-1)} + \lambda_2, & \text{otherwise} \end{cases}, \quad (148)$$

where  $\lambda_1 > 1$  and  $0 < \lambda_2 < 1$ . With this scheme,  $\alpha^{(n)}$  is adjusted according to the residual error (i.e., the deviation between the current solution and its auxiliary solution) relationship of two consecutive iterations. When the current residual error is larger than the previous iteration,  $\lambda_1 > 1$  makes a more aggressive reduction in the stepsize. On the other hand, when the residual error is smaller than the previous iteration,  $0 < \lambda_2 < 1$  makes the stepsize reduction more conservative. Note that the stepsize sequences from the SRA scheme still satisfy the diminishing stepsize sequence conditions required to guarantee convergence.

#### 4.4.2.3 Quadratic interpolation (Quad) scheme

The Quad scheme was first suggested by Maher (1998) for solving the MNL SUE problem in the link domain. This Quad scheme determines the stepsize using the objective function, unlike the MSA and SRA schemes, whose stepsizes are respectively predetermined and dependent on the residual error. The stepsize for the Quad scheme can be expressed as

$$\alpha^{(n)} = \frac{-\nabla_{\alpha} Z(\alpha)|_{\alpha=0}}{-\nabla_{\alpha} Z(\alpha)|_{\alpha=0} + \nabla_{\alpha} Z(\alpha)|_{\alpha=1}}, \quad (149)$$

where  $\nabla_{\alpha}Z(\alpha)$  is the derivative of objective function w.r.t. the stepsize. In this scheme,  $\nabla_{\alpha}Z(\alpha)$  are evaluated twice per iteration, one at  $\alpha = 0$  and another at  $\alpha = 1$ , to determine an approximate stepsize. For the *unconstrained* MNN-SUE model,  $\nabla_{\alpha}Z(\alpha)$  can be determined from the first derivative of the objective function (see Appendix D), which gives

$$\nabla_{\alpha}Z(\alpha) = \frac{dZ}{d\alpha} = \sum_{a \in A} \left( v_a^{(n)} \right)^2 \frac{d \ln \tau_a}{dx_a}. \quad (150)$$

#### 4.5 Numerical results

To demonstrate the features of the *unconstrained* MNW-SUE model, two numerical examples are conducted in this section. Example 1 uses the two-route networks to investigate the effect of different trip lengths under congestion. Example 2 adopts the Winnipeg network as a real-case study to examine the efficiency of the link-based algorithm combined with recent advances in line search strategies and the flow allocation comparison between the MNL-SUE and MNW-SUE models in a real-size network. The coefficient of variation  $\mathcal{G}_r^{ij}$  is assumed to be 0.3, which corresponds to  $\beta^{ij} = 3.7$  unless specified otherwise.

Table 9: Flow-dependent route travel cost for the two-route networks

<b>Network</b>	<b>Upper route</b>	<b>Lower route</b>
<b>Short</b>	$10 + v_u/10$	$5 + v_l/10$
<b>Long</b>	$125 + v_u/10$	$120 + v_l/10$

#### 4.5.1 Two-route Networks

The two-route networks in Fig. 25 are modified to incorporate congestion. Each route/link incorporates a flow-dependent cost component of  $v_a/10$  as shown in Table 9.

The O-D demand is assumed to be 100 vehicles per unit of time.

##### 4.5.1.1 MNW-SUE solution

We first investigate the MNW-SUE solution of the short network from its MP formulation in Eq. (61), which can be expressed as

$$\begin{aligned} \min Z = & -\int_0^{v_u} \ln\left(10 + \frac{\omega}{10}\right) d\omega - \int_0^{v_l} \ln\left(5 + \frac{\omega}{10}\right) d\omega - 100 \left( -\frac{1}{3.7} \ln\left(10 + \frac{v_u}{10} + 5 + \frac{v_l}{10}\right)^{-3.7} \right) \\ & + v_u \ln\left(10 + \frac{v_u}{10}\right) + v_l \ln\left(5 + \frac{v_l}{10}\right). \end{aligned} \quad (151)$$

Taking the derivative of Eq. (151) w.r.t. the upper link flow gives

$$\frac{\partial Z}{\partial v_u} = -\ln g_u^{ij} - 100 \frac{\partial \left( -\frac{1}{3.7} \ln \left( \left(10 + \frac{v_u}{10}\right)^{-3.7} + \left(5 + \frac{v_l}{10}\right)^{-3.7} \right) \right)}{\partial v_u} + \frac{\partial v_u \ln\left(10 + \frac{v_u}{10}\right)}{\partial v_u} = 0. \quad (152)$$

Rearranging Eq. (152) gives

$$-\ln g_u^{ij} - 10 \frac{\left(g_u^{ij}\right)^{-3.7-1}}{\left(g_u^{ij}\right)^{-3.7} + \left(g_l^{ij}\right)^{-3.7}} + \frac{v_u}{10 g_u^{ij}} + \ln g_u^{ij} = 0. \quad (153)$$

The partial derivative of the multiplicative Beckmann's transformation  $Z_1$  (i.e.,  $-\ln g_u^{ij}$ ) cancels out a part of the partial derivative of the total multiplicative travel cost  $Z_3$  (i.e.,  $+\ln g_u^{ij}$ ), which gives the MNW solution, i.e.,

$$v_u = 100 \frac{\left(g_u^{ij}\right)^{-3.7}}{\left(g_u^{ij}\right)^{-3.7} + \left(g_l^{ij}\right)^{-3.7}}. \quad (154)$$

Similarly, the lower link flow can be determined as follows:

$$v_l = 100 \frac{(g_l^{ij})^{-3.7}}{(g_u^{ij})^{-3.7} + (g_l^{ij})^{-3.7}}. \quad (155)$$

By simultaneously solving Eq. (154) and Eq. (155), we have the amount of flows assigned on each link, i.e.,

$$v_u = 35.25, v_l = 64.25. \quad (156)$$

#### 4.5.1.2 Flow allocation comparison

This section compares the results produced by the MNW-SUE model with those of two MNL-SUE models (without and with scaling). For the MNL-SUE model without scaling, the logit dispersion parameter  $\theta$  is set equal to 0.1. For the MNL-SUE with scaling (or MNLs-SUE) model,  $\theta$  is set to correspond with  $g_r^{ij} = 0.3$  for the lowest uncongested route travel cost (Chen et al., 2012). As expected in Table 10, the MNL-SUE model gives the same flow pattern for both short and long networks according to the identically distributed assumption. Meanwhile, the MNLs-SUE and MNW-SUE models produce different results for each network. Both models assign a smaller amount of traffic flows on the long network. Note that the MNLs-SUE model uses the scaled dispersion parameter to handle the overall trip length (i.e.,  $\theta = 0.86$  for the short network, and  $\theta = 0.04$  for the long network), while the MNW-SUE model uses the same  $\beta^{ij} = 3.7$  for both networks.

Note further that the MNLs-SUE model gives a slightly higher amount of flows on the lower route even though it has a larger cost than that of the MNW-SUE model. This is because the MNLs-SUE model still assumes the *same and fixed* perception variance of

Table 10: Flow allocation

SUE Model Network		MNL		MNLs		MNW	
		Cost	Flow	Cost	Flow	Cost	Flow
Short	Upper route	14.17	41.72	12.99	29.96	13.53	35.25
	Lower route	10.83	58.28	12.00	70.04	11.48	64.75
Long	Upper route	129.17	41.72	129.62	46.23	129.68	46.84
	Lower route	125.83	58.28	125.38	53.77	125.32	53.16

Table 11: Perception variance and coefficient of variation

Network		$(\sigma_r^{ij})^2$			$g_r^{ij}$		
		MNL	MNLs	MNW	MNL	MNLs	MNW
Short network	Upper route	164.49	2.22	16.46	0.91	0.11	0.30
	Lower route	164.49	2.22	11.85	1.18	0.12	0.30
Long network	Upper route	164.49	1028.08	1513.60	0.10	0.25	0.30
	Lower route	164.49	1028.08	1413.38	0.10	0.26	0.30

$(\sigma_r^{ij})^2 = \pi^2/6\theta^2$  for each route connecting the O-D pair. As such, the MNLs-SUE model underestimates the perception variance and coefficient of variation in the congested condition as presented in Table 11. As a result, the flow allocation on the larger-cost route is underestimated, especially in the short network.

#### 4.5.1.3 *Effect of demand levels*

We continue to use the short network to investigate the effect of demand levels. The O-D demand is varied from 50 to 300 vehicles per unit of time, and  $\mathcal{G}_r^{ij}$  is varied from 0.1 to 0.5. The root mean square error (RMSE) is used as a statistical measure to compare the difference between the MNW-SUE model relative to the user equilibrium (UE) model, i.e.,

$$RMSE = \sqrt{\sum_{a \in A} \frac{\left( v_a^{\{MNW-SUE\}} - v_a^{\{UE\}} \right)^2}{|A|}}, \quad (157)$$

where  $|A|$  is the number of links in the network. A low value of RMSE means that the assignment model performs similarly to the UE model.

It can be seen from Fig. 29 that as the demand level increases, the RMSE decreases. This result means that the MNW-SUE model approaches the UE model when the congestion level is increased (i.e., congestion effect due to high demand levels of 200 to 300 vehicles per unit of time dominates the solution). Also, the RSME decreases when  $\mathcal{G}_r^{ij}$  decreases (or  $\beta^{ij}$  increases with a lower perception variance). The MNW-SUE flow patterns also tend to the UE flow pattern. This result implies that the flow allocation is more concentrated on the minimum cost routes (i.e., travelers are able to select the lower-cost routes more often since they have better knowledge of the network traffic conditions). Otherwise, the two models will produce different flow patterns for low demand levels and larger  $\mathcal{G}_r^{ij}$  values.

#### 4.5.2 *Winnipeg Network*

Example 2 adopts the Winnipeg network (shown in Fig. 30) as a real-case study to examine the efficiency of the link-based solution algorithm with recent advances in line

search strategies, and the flow allocation comparison between the MNL-SUE and MNW-SUE models in a real-size network. This network consists of 154 zones, 1,067 nodes, 2,535 links, and 4,345 O-D pairs. The network topology, link characteristics, and O-D demands can be found in Emme/2 software (INRO Consultants, 1999). Without loss of generality, we assume that the link travel cost is an exponential function (Hensher and Truong, 1985; Polak, 1987; Mirchandani and Soroush, 1987), i.e.,

$$\tau_a = e^{0.075t_a}, \quad \forall a \in A, \quad (158)$$

where  $t_a$  is the travel time on link  $a$ .

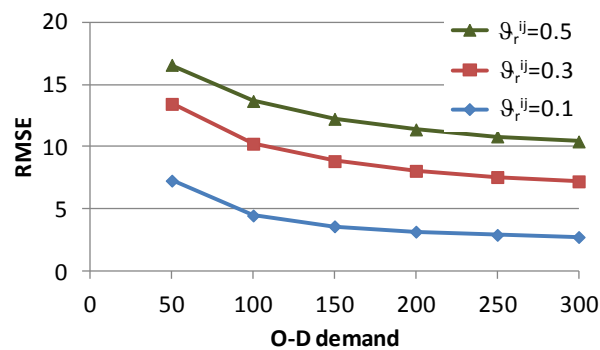


Fig. 29: Effect of demand levels

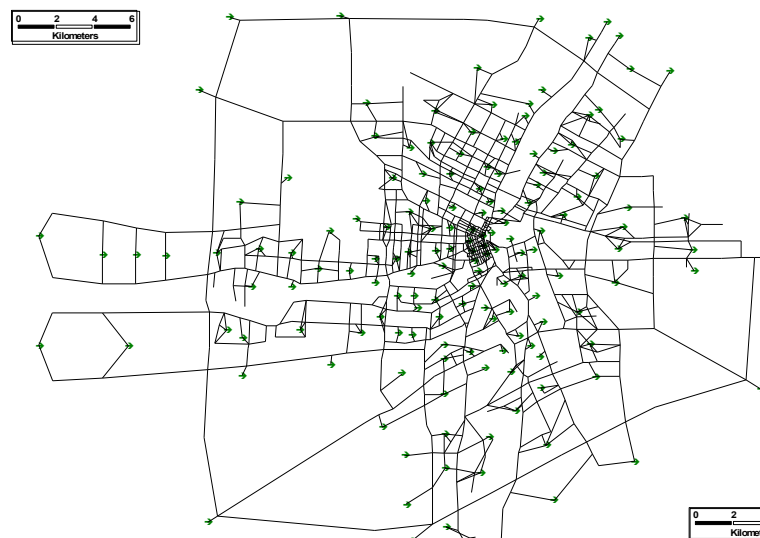


Fig. 30. Winnipeg network



#### 4.5.2.1 Computational results

The STOCH algorithm (Dial, 1971) is used to perform the link-based MNW stochastic loading in the search direction step. Three line search schemes are considered: MSA, SRA, and Quad. The MSA scheme uses a predetermined stepsize of  $1/n$ . The SRA parameters are assumed to be  $\lambda_1 = 1.5$  and  $\lambda_2 = 0.1$ . The stopping threshold  $\varepsilon$  is set at  $10^{-8}$ . The convergence characteristics of the link-based solution algorithm are shown in Fig. 31. We can see that both the SRA and Quad schemes converge linearly, while the MSA scheme cannot converge to the desired accuracy level within the maximum number of iterations (500) allowed. Between SRA and Quad schemes, the Quad scheme is faster than the SRA scheme even though the Quad scheme needs about two times longer computational cost for each iteration as presented in Table 12. Note that the Quad scheme requires more computational time per iteration than the SRA scheme since it needs to perform the link-based MNW stochastic loading twice per iteration. The first stochastic loading is to evaluate  $\nabla_{\alpha} Z(\alpha)$  at  $\alpha = 0$ , and the second stochastic loading is to evaluate  $\nabla_{\alpha} Z(\alpha)$  at  $\alpha = 1$  (see Maher, 1998).

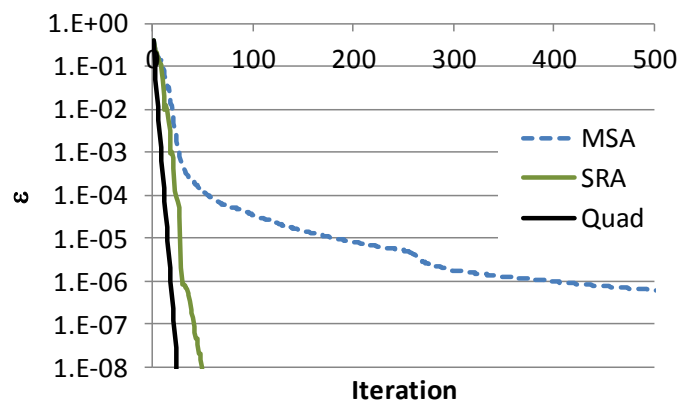


Fig. 31. Convergence characteristics of three line search schemes

Table 12: Computational efforts for solving the MNW-SUE model

Algorithm	# of iterations	CPU time (sec)	CPU time/Iteration (sec)
MSA	500 (still not converge)	5505	11.01
SRA	49	540	11.04
Quad	23	482	20.92

\*All algorithms are coded in Compaq Visual Fortran 6.6 and run on a personal computer with 3.8 G Pentium-IV processor

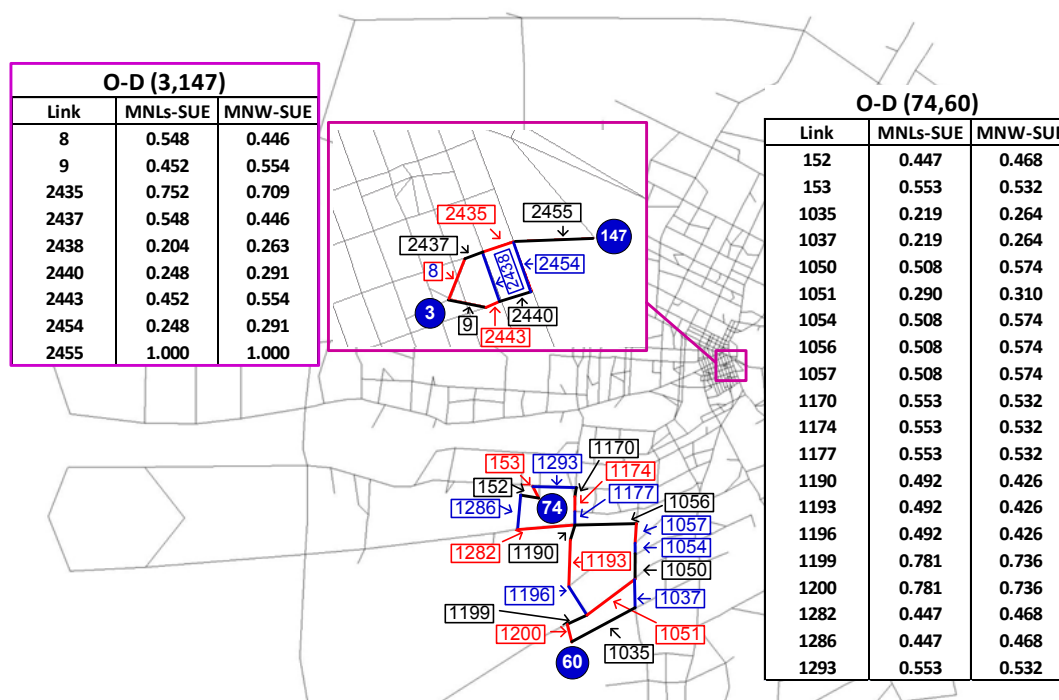


Fig. 32. Comparison of link choice probabilities of two O-D pairs

#### 4.5.2.2 *Flow allocation comparison*

At the disaggregate level, we examine the link choice probabilities produced by the MNLs-SUE and MNW-SUE models under the same cost configuration in Eq. (158). For demonstration purposes, we use O-D pairs (3,147) and (60,74) to respectively represent a short and long O-D pair. The link choice probabilities shown in Fig. 32 are under the respective equilibrium link flow pattern. Recall that the MNLs-SUE model adopts the

O-D-specific perception variance (i.e., each route connecting an O-D pair has the same and fixed perception variance) while the MNW-SUE model uses the route-specific perception variance (i.e., each route has its own perception variance as a function of route travel cost). Thus, different link choice probabilities (hence link flows) can be expected. Even though O-D pair (3,147) includes only 9 links, the two SUE models produce significantly different results. The MNLs-SUE model assigns a higher probability to link 8; the MNW-SUE model, on the other hand, assigns a higher probability to link 9. With the differences in these two beginning links, the subsequent link choice probabilities produced by each SUE model are also different. For the long O-D pair (60,74), more links are involved as a result of a longer trip length. When both number of links and trip length are increasing, the differences in the link choice probabilities between the two models also decrease.

At the aggregate level, we examine the effect of heterogeneous perception variance problem on the link flow patterns. The link flow difference between the MNLs-SUE and MNW-SUE models can be found mostly in the central business district (CBD) area as shown in Fig. 33. The complete link flow difference distributions for both CBD and non-CBD (or outer) areas are also shown in Fig. 33. The absolute maximum flow difference in the CBD area is 374 vehicle per hour (vph) compared to 268 vph in the outer area. This is because there are many short O-D pairs in the CBD area with different trip lengths, which make it difficult for the MNLs-SUE model to handle the heterogeneous perception variance among different routes. Nevertheless, these results demonstrate that the proposed unconstrained MNW-SUE formulation can handle the route-specific perception variance under congested conditions and can be implemented in a real-size network.

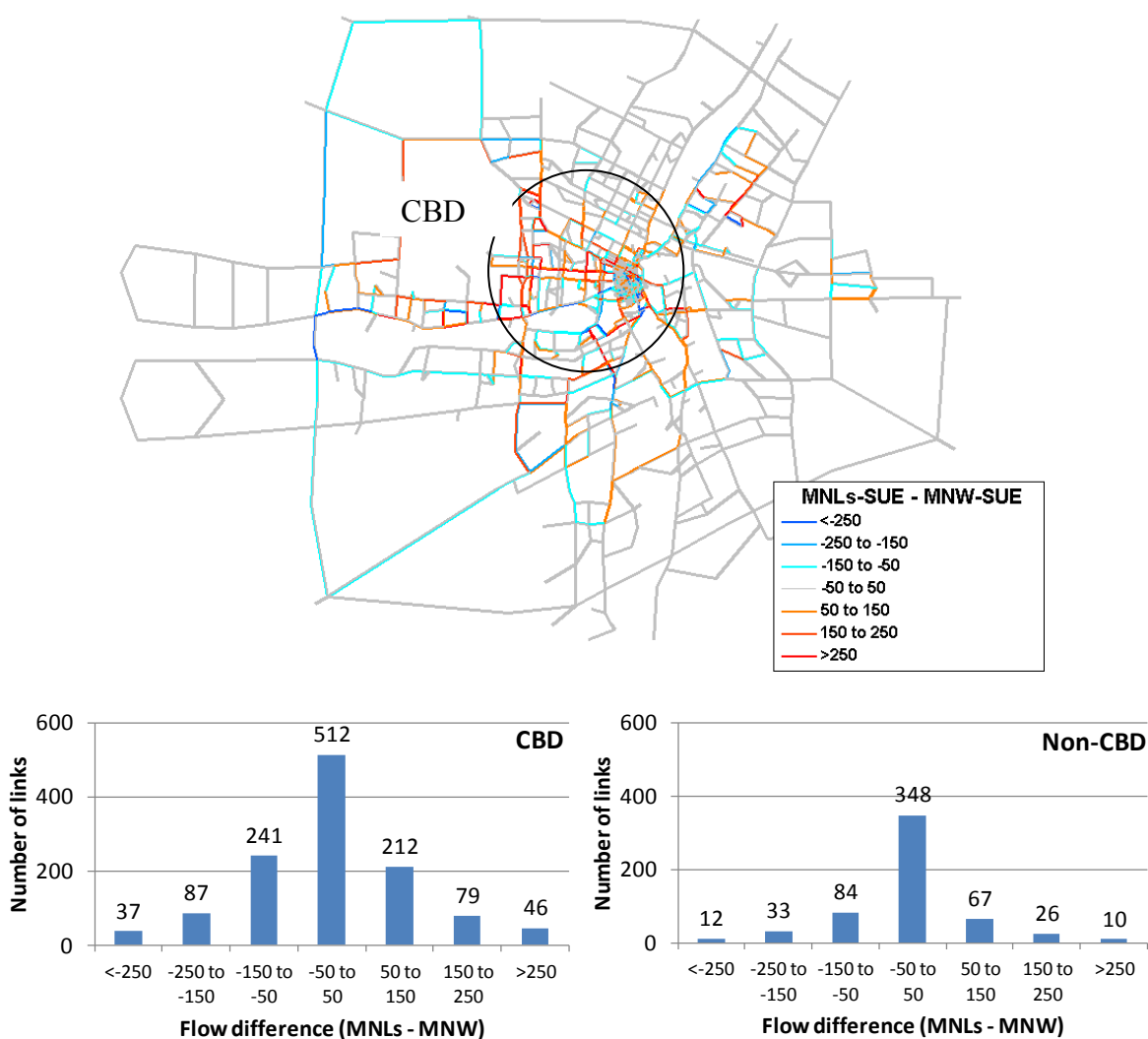


Fig. 33. Link flow comparison between the MNLs-SUE and MNW-SUE models

## 4.6 Concluding remarks and extensions

### 4.6.1 Summary

This study proposed a MNW-SUE model to relax the identically distributed assumption of the MNL-SUE model by explicitly considering heterogeneous perception variances with respect to different trip lengths under congested conditions. Specifically, a closed-form *logarithmic* expected perceived travel cost (EPC) of the MNW model

(Castillo et al., 2008) was derived and combined with the multiplicative Beckmann's transformation to develop an *unconstrained* minimization program for the MNW-SUE problem. Numerical examples revealed that

- the proposed MNW-SUE model can capture the route-specific perception variance as a function of route travel cost under congested conditions,
- it can handle the route-specific perception variance better than the MNL-SUE with scaling technique (Chen et al., 2012), which still assumes the same and fixed perception variance for all routes connecting an O-D pair, and
- it can be implemented in a real-size network as shown by the Winnipeg network.

#### 4.6.2 Extensions

Even though the MNW model can successfully relax the *identically distributed* assumption of the MNL model, it still inherits some limitations. Three extensions are suggested for further study: (1) route overlapping problem, (2) demand elasticity, and (3) multiple user classes.

##### 4.6.2.1 Handling route overlapping

The first extension relaxes the independently distributed assumption of the Weibull distribution by considering a correction factor to handle the route overlapping problem. According to Fosgerau and Bierlaire (2009), the MNW model can be posed as a random utility model with *multiplicative* error terms:

$$U_r^{ij} = \left( g_r^{ij} - \zeta^{ij} \right)^{\beta^{ij}} \varepsilon_r^{ij}, \quad \forall r \in R_{ij}, ij \in IJ, \quad (159)$$

where  $\varepsilon_r^{ij}$  is the weibull distributed random error term. A correction factor  $\rho_r^{ij}$  could be adopted to adjust the probability of routes coupling with other routes through the route travel cost (or the deterministic term), i.e.,

$$U_r^{ij} = \frac{\left(g_r^{ij} - \zeta^{ij}\right)^{\beta^{ij}}}{\rho_r^{ij}} \varepsilon_r^{ij}, \quad \forall r \in R_{ij}, ij \in IJ, \quad (160)$$

which gives the choice probability expression as

$$P_r^{ij} = \frac{\rho_r^{ij} \left(g_r^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}}{\sum_{k \in R_{ij}} \rho_k^{ij} \left(g_k^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}}, \quad \forall r \in R_{ij}, ij \in IJ. \quad (161)$$

Therefore, the route overlapping is captured through the correction factor, and the route-specific perception variance is handled through the Weibull random error term. Assuming that  $\rho_r^{ij}$  is flow independent, a logarithmic EPC for this enhanced route choice model can be expressed in closed form as follows:

$$\bar{\mu}_{ij} = -\frac{1}{\beta^{ij}} \ln \sum_{r \in R_{ij}} \rho_r^{ij} \left(g_r^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}, \quad \forall ij \in IJ. \quad (162)$$

Incorporating this logarithmic EPC (with  $\zeta^{ij} = 0$ ) in the unconstrained minimization programming formulation in Eq. (61), we have an unconstrained MNW-SUE problem that can simultaneously capture *both* route overlapping and route-specific perception variance problems under congested networks. Note that if  $\rho_r^{ij}$  is specified as the path-size factor (Ben-Akiva and Bierlaire 1999) independent of flows, we have the path-size weibit SUE model similar to the constrained convex program formulated in Chapter 3. If  $\rho_r^{ij}$  is specified as a flow-dependent commonality factor, we have a similar model as the

congestion-based C-logit SUE model formulated as a variational inequality (VI) problem by [Zhou et al. \(2012\)](#).

#### 4.6.2.2 *Considering demand elasticity*

The second extension relaxes the fixed demand assumption by explicitly considering the elasticity of travel demand. By incorporating the inverse demand function with the multiplicative Beckmann's transformation, we can formulate the MNW-SUE problem with elastic demand as a function of the logarithmic MNW EPC. Following [Maher et al. \(1999\)](#), the MNW-SUE problem with elastic demand can be written as

$$\begin{aligned}
 \min Z &= Z_1 + Z_2 + Z_3 + Z_4 + Z_5 + Z_6 \\
 &= -\sum_{a \in A} \int_0^{v_a} \ln \tau_a(\omega) d\omega - \sum_{ij \in IJ} D_{ij}(\bar{\mu}_{ij}) \bar{\mu}_{ij} + \sum_{a \in A} v_a \ln \tau_a(v_a) \\
 &\quad + \sum_{ij \in IJ} D_{ij}^{-1}(q_{ij}) D_{ij}(\bar{\mu}_{ij}) + \sum_{ij \in IJ} \int_0^{q_{ij}} D_{ij}^{-1}(\omega) d\omega - \sum_{ij \in IJ} D_{ij}^{-1}(q_{ij}) q_{ij},
 \end{aligned} \tag{163}$$

where  $D_{ij}(\cdot)$  is the (invertible) elastic demand function. In the above formulation, both travel choice and route choice are simultaneously considered. To consider both route overlapping and route-specific perception variance problems in the elastic demand, we can simply incorporate the logarithmic EPC in Eq. (162) into the objective function in Eq. (163). With this, the elastic demand can also account for the route overlapping problem through the path-size factor  $\rho_r^{ij}$  to enhance the realism of modeling demand elasticity.

#### 4.6.2.3 *Extending to multiple user classes*

The third extension relaxes the single class assumption to consider multiple user classes (e.g., multiple vehicle classes such as passenger cars, light trucks, medium trucks,

and heavy trucks; vehicles with and without equipped traveler information systems; travelers with different values of time and reliability; etc.). Modeling the multi-user class problem requires specific customization. Here we provide a general multi-class MNW-SUE model as an extended unconstrained MP formulation as follows:

$$\begin{aligned} \min Z &= Z_1 + Z_2 + Z_3 \\ &= -\sum_{a \in A} \int_0^{v_a} \ln \tau_a(\omega) d\omega - \sum_{m \in M} \sum_{ij \in IJ} q_{mij} \bar{\mu}_{mij} + \sum_{a \in A} v_a \ln \tau_a(v_a), \end{aligned} \quad (164)$$

where  $m \in M$  represents the user class. The link travel time is assumed to be a function of link flows from different user classes (i.e.,  $v_a = \sum_{m \in M} v_a^m$ ), and the logarithmic MNW-EPC could be different for each user class depending on the perception variances and other relevant attributes used to model the specific multi-class MNW-SUE problem. Again, if we want to consider both route overlapping and route-specific perception variance problems, the logarithmic EPC in Eq. (162) can be incorporated into the objective function in Eq. (164). With this, each user class can account for the route overlapping in addition to the route-specific perception variance.

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## CHAPTER 5

ELASTIC DEMAND WITH WEIBIT STOCHASTIC USER EQUILIBRIUM FLOWS  
AND APPLICATION IN A MOTORIZED AND NON-MOTORIZED NETWORK

## Abstract

In this paper, we propose a new elastic demand stochastic user equilibrium (SUE) model with application to the combined modal split and traffic assignment (CMSTA) problem. This new model, called the path-size weibit (PSW) SUE model with elastic demand (ED), is derived based on the Weibull distribution, which does not require the identically distributed assumption typically imposed in the multinomial logit (MNL) model with the Gumbel distribution. In addition, a path-size factor is included to correct the choice probabilities of routes that are not truly independent (i.e., another assumption typically required in the MNL model). Equivalent mathematical programming (MP) formulation of the PSW-SUE-ED model is developed to simultaneously consider both travel choice and route choice. The travel choice is determined based on the elastic demand function that explicitly considers the network level of service based on the logarithmic expected perceived cost of the Weibull distribution to determine the travel demand, while the route choice accounts for the route overlapping problem and the non-identical perception variance with respect to different trip lengths. Qualitative properties of the proposed MP formulation are rigorously proved. A path-based partial linearization algorithm combined with a self-regulated averaging (SRA) line search strategy is developed for solving the PSW-SUE-ED model and its application to the CMSTA problem. Numerical examples are also provided to demonstrate the features of the

proposed PSW-SUE-ED model as well as a real-case study in a bi-modal network with motorized and non-motorized mode choices.

### 5.1 Introduction

Modeling the elasticity of travel demand in network equilibrium analysis was introduced by [Beckmann et al. \(1956\)](#) to explicitly consider the equilibrium between supply and demand. The supply functions are determined by the link travel costs under congestion, and the travel demand functions are determined by the user benefits ([Florian and Nguyen, 1974](#)), generally derived based on the level of service (LOS) of the network ([Sheffi, 1985](#)). For example, as congestion increases, the network LOS decreases. Travelers may exercise their available choices by considering a different mode of travel (mode choice), going to a different destination (destination choice), foregoing some trips altogether (travel choice), in addition to choosing a different route (route choice). These choices will have an effect of the traffic flow patterns. In addition to the multi-dimensional travel choice applications (e.g., combined travel and route choice problem, combined mode and route choice problem, combined destination and route choice problem, and combined travel-destination-mode-route choice problem), modeling demand elasticity has important transportation applications in predicting future travel demand patterns and assessing network improvement. Using the network analysis with fixed demand to assess such behavior may cause a bias future travel-demand-pattern prediction, misvaluation of network performance, and result in an inefficiency budget allocation.

[Beckmann et al. \(1956\)](#) provided the pioneer work of formulating the deterministic user equilibrium (DUE) model with elastic demand (ED) (or DUE-ED) as a mathematical

programming (MP) formulation, where the ED is a function of the equilibrium route travel cost between each origin-destination (O-D) pair. For a historical review of Beckmann's DUE-ED model, readers are referred to [Boyce \(2013\)](#). Since the seminal work by [Beckmann et al. \(1956\)](#), many researchers have further developed the idea in different directions as shown in Table 13 to enhance the modeling realism and applications of the DUE-ED model.

In terms of the methodology used in formulation and analysis, [Beckmann et al. \(1956\)](#) provided the first MP formulation, or more specifically a convex programming (CP) formulation, for the DUE-ED model. [Carey \(1985\)](#) provided two dual formulations of the DUE-ED problem using node-link and link-path variables, and explored the relationship between the primal and dual formulations. [Aashtiani \(1979\)](#) gave the first nonlinear complementarity problem (NCP) formulation for modeling the interactions in a multimodal network, while [Gabriel and Bernstein \(1997\)](#) introduced the nonadditive user equilibrium (NaUE) problem as an NCP formulation in which the cost incurred on each path is not simply the sum of the link costs that constitute that path. [Dafermos \(1982\)](#) offered a variational inequality (VI) formulation for the multimodal traffic equilibrium model with elastic demand, where the link travel costs depend on the entire link flow vector and the travel demands depend on the entire mode-specific O-D cost vector. [Fisk and Boyce \(1983\)](#) provided alternative VI formulations for the network equilibrium travel choice problem, which does not require invertibility of the travel demand function. [Cantarella \(1997\)](#) provided a fixed point (FP) formulation for the multi-mode multi-user equilibrium assignment with elastic demand, where users have different behavioral characteristics as well as different choice sets.

The major drawback of the above models assumes all travelers have perfect knowledge of network conditions (i.e., know all available routes and have perfect perception of all route costs). In reality, travelers rarely know all available routes, and certainly do not always select the minimum cost route. To overcome such drawback of the DUE-ED model, several researchers extended the stochastic user equilibrium (SUE) principle suggested by [Daganzo and Sheffi \(1977\)](#) from a fixed demand (FD) to an elastic demand (ED) version, or the SUE-ED model for short. In the SUE principle, a random error term is introduced into the route cost function to mimic the perception error of network travel times due to the travelers' imperfect knowledge of network conditions. At the SUE state, *no travelers can improve his or her perceived travel time by unilaterally changing routes* ([Sheffi, 1985](#)). In the transportation literature, Gumbel and Normal distributions are the two commonly used random error terms to develop the probabilistic route choice models, which result in the multinomial logit (MNL) and multinomial probit (MNP) route choice models ([Dial, 1971](#); [Daganzo and Sheffi, 1977](#)), respectively.

[Yang and Bell \(1998\)](#) extended [Fisk's \(1980\)](#) MP formulation for the MNL-SUE model with fixed demand to the elastic demand case (or MNL-SUE-ED), while [Maher \(2001\)](#) and [Meng and Liu \(2012\)](#) provided MP and VI formulations for the MNP-SUE-ED model without and with link interactions, respectively. As is well known, the MNL model needs the independently and identically distributed (*IID*) assumption with the *Gumbel* variates in order to derive a closed-form probability expression. The IID assumption comes with two known limitations: (1) inability to handle route overlapping and (2) inability to handle perception variance with respect to (w.r.t.) different trip lengths. These drawbacks may cause a bias result in estimating the expected perceived cost (EPC) (i.e., the well-

known MNL's log-sum term) used in the elastic demand function to determine the travel demands. On the other hand, the MNP-SUE-ED model does not have the two known limitations of the MNL model since it does not require the IID Gumbel variate assumption. However, the MNP-SUE-ED model does not have a closed-form probability expression, and hence would require significant computational efforts using either Monte Carlo simulation (Sheffi and Powell, 1982), Clark's approximation method (Maher, 1992; Maher and Hughes, 1997), or numerical method (Rosa and Maher, 2002).

In terms of applications, Wu and Lam (2003a,b) adopted the VI formulation to develop a combined modal split and traffic assignment (CMSTA) model with elastic demand based on the MNL-SUE flows for modeling mode choice and route choice decisions in a bimodal (i.e., motorized and non-motorized) network. Since the CMSTA model adopts the MNL-SUE-ED model, it also inherits the same two drawbacks identified above with the same bias in estimating the travel demands for the two modes.

To partially address the drawbacks, Kitthamkesorn et al. (2013) developed an equivalent MP for the CMSTA problem that explicitly considers mode and route similarities under congested networks. The mode choice is modeled using the nested logit (NL) model (Ben-Akiva and Lerman, 1985) and the route choice is modeled through the cross nested logit (CNL) model (Bekhor and Prashker, 1999). Although the model captures the similarities of both mode and route choices in the CMSTA problem, the *identically distributed* assumption still remains (i.e., inability to account for the route-specific perception variance) due to the classical logit assumption of homogeneous perception variance.



Table 13: Summary of traffic equilibrium models with elastic demand

Approach	Reference	Model	Remark
Mathematical Programming (MP)	Beckmann et al., 1956	DUE-ED	Provide the first DUE model with elastic demand as a convex programming formulation for modeling travelers' trip-making behavior
	Carey, 1985	DUE-ED	Provide two dual formulations of the DUE-ED problem using node-link and path-link variables, and explore the relationship between the primal and dual formulations
	Yang and Bell, 1998	MNL-SUE-ED	Provide an equivalent mathematical programming formulation for the logit-based SUE problem with elastic demand
	Maher, 2001	General SUE-ED	Provide a new objective function for the SUE assignment with elastic demand, and develop a balanced demand algorithm for solving it
	Kitthamkesorn et al., 2013	CMSTA	Provide a combined modal split and traffic assignment (CMSTA) problem based on the excess demand formulation with nested logit model for mode choice and cross nested logit for route choice
Nonlinear complementarity problem (NCP)	Aashtiani, 1979	Multimodal traffic assignment problem	Provide the first NCP formulation for the multimodal traffic assignment problem, and develop one of the early path-based linearization algorithms for solving the traffic assignment problem
	Gabriel and Bernstein, 1997	Non-additive user equilibrium (NaUE)	Introduce the nonadditive traffic equilibrium problem in which the cost incurred on each path is not simply the sum of the link costs that constitute that path, and propose the nonsmooth equations/sequential quadratic programming (NE/SQP) method for solving the nonadditive traffic equilibrium problem
Variational Inequality (VI)	Dafermos, 1982	Multimodal traffic assignment problem	Provide a VI formulation for modeling the general multimodal traffic equilibrium model with elastic demand, where the link travel costs depend on the entire link flow vector and the travel demands depend on the entire mode-specific O-D cost vector, and develop an iterative relaxation algorithm for computing the equilibrium flow pattern
	Fisk and Boyce, 1983	DUE-ED	Provide alternative VI formulations for the network equilibrium travel choice problem, which does not require invertibility of the travel demand function
	Wu and Lam, 2003a,b	CMSSA-ED	Provide a combined modal split and traffic assignment model (CMSTA) with elastic demand based on the MNL-SUE flows for modeling combined mode and route choice decisions on a bimodal network, and develop a method of successive averages (MSA) with cost approximation (CA) algorithm and block Gauss-Seidel decomposition method
	Meng and Liu, 2012	MNP-SUE-ED-LI	Develop two VI models and two hybrid prediction-correction cost averaging algorithms combined with a two-stage Monte Carlo simulation based stochastic network loading method for the multinomial probit (MNP) SUE problem with elastic demand (ED) and link interactions (LI)
Fixed Point	Cantarella, 1997	Fixed point (FP)	Develop a fixed point formulation for the multi-mode multi-user equilibrium assignment with elastic demand, where users have different behavioral characteristics as well as different choice sets

In this paper, we provide an alternative to relax the IID assumption embedded in the MNL-SUE-ED model by using the *Weibull* distribution. The path-size weibit (PSW) model in Chapter 3 is used to develop an equivalent MP formulation for modeling demand elasticity in the SUE framework. The proposed PSW-SUE model with ED (or PSW-SUE-ED) has the following two significant features that are distinct from the literature shown in Table 13.

- The network level of service (LOS) is captured through the logarithm expected perceived cost (EPC) of the Weibull distribution, which is used to develop the PSW-SUE-ED model, to determine the travel demands and SUE flows. The advantage is that the log EPC explicitly considers the route overlapping and non-identical perception variance problems in the SUE assignment, and avoids the bias caused by the two known limitations of the MNL-SUE model in estimating the travel demands.
- An application of the PSW-SUE-ED model is developed to consider both mode choice and route choice as a combined modal split and traffic assignment (CSMTA) problem. It is demonstrated with a case study in a bimodal network with motorized and non-motorized modes using the Winnipeg network.

This paper not only develops a new PSW-SUE-ED model, but also provides qualitative properties, as well as a solution algorithm, accompanied by convergence results, numerical examples, and application to a bimodal network with motorized and non-motorized modes.

The remainder of this paper is organized as follows. The next section gives some background of the weibit route choice models. In section 3, the equivalent MP

formulations for the PSW-SUE-ED model and its application as a CMSTA problem are provided along with some qualitative properties. Section 4 describes a path-based partial linearization method combined with a self-regulated averaging (SRA) stepsize scheme for solving the PSW-SUE-ED model. Numerical results are presented in Section 5, and some concluding remarks are provided in Section 6.

## 5.2 Weibit route choice models

In this section, we provide some background of the weibit route choice models and their expected perceived cost (EPC). The section begins with the multinomial weibit (MNW) model and follows by the path-size weibit (PSW) model.

### 5.2.1 Multinomial weibit (MNW) model

#### 5.2.1.1 Model formulation

Castillo et al. (2008) developed the MNW model to resolve the *identical variance* issue. Unlike the MNL model which uses the conventional *additive* RUM, the MNW model adopts the *multiplicative* RUM with the Weibull random error (Fosgerau and Bierlaire, 2009), i.e.,

$$U_r^{ij} = \left( g_r^{ij} - \zeta^{ij} \right)^{\beta^{ij}} \varepsilon_r^{ij}, \quad \forall r \in R_{ij}, ij \in IJ, \quad (165)$$

where  $IJ$  is the set of O-D pairs,  $R_{ij}$  is the set of routes between O-D pair  $ij$ ,  $g_r^{ij}$  is the travel cost on route  $r$  between O-D pair  $ij$ ,  $\varepsilon_r^{ij}$  is the *independently* Weibull distributed random error term on route  $r$  between O-D pair  $ij$ . According to the MNW disutility in Eq. (18), we have the MNW route choice probability:

$$P_r^{ij} = \frac{(g_r^{ij} - \zeta^{ij})^{-\beta^{ij}}}{\sum_{k \in R_{ij}} (g_k^{ij} - \zeta^{ij})^{-\beta^{ij}}}, \quad \forall r \in R_{ij}, ij \in IJ. \quad (166)$$

To show how this MNW model handles the different trip lengths, consider a two-route network configuration as shown in Fig. 34. For both networks, the upper route cost is larger than the lower route cost by 5 units. However, the upper route cost is two times larger than the lower route cost in the short network, while it is only less than 5% larger in the long network. As expected, the MNL model produces the same route choice probability for both short and long networks. This is because the MNL model assumes that each route has the same and fixed perception variance (i.e.,  $\pi^2/6\theta^2$ ) as shown in Fig. 35a. In other words, the solution is solely based on the *absolute* cost difference irrespective of the overall trip length (Sheffi, 1985). The MNW model, in contrast, produces different route choice probabilities for the two networks. It uses a *relative* cost difference to handle different trip lengths. When considering the perception variance, the MNW model has a *route-specific* perception variance as a function of route travel cost (see Chapter 4), i.e.,

$$(\sigma_r^{ij})^2 = \left[ \frac{(g_r^{ij} - \zeta^{ij})}{\Gamma(1 + 1/\beta^{ij})} \right]^2 \left[ \Gamma\left(1 + \frac{2}{\beta^{ij}}\right) - \Gamma^2\left(1 + \frac{1}{\beta^{ij}}\right) \right], \quad \forall r \in R_{ij}, ij \in IJ, \quad (167)$$

where  $\Gamma(\cdot)$  is the gamma function. With this, a larger-cost route will have a higher perception variance as shown in Fig. 35b, such that the probability of choosing each route becomes more similar for a longer network.

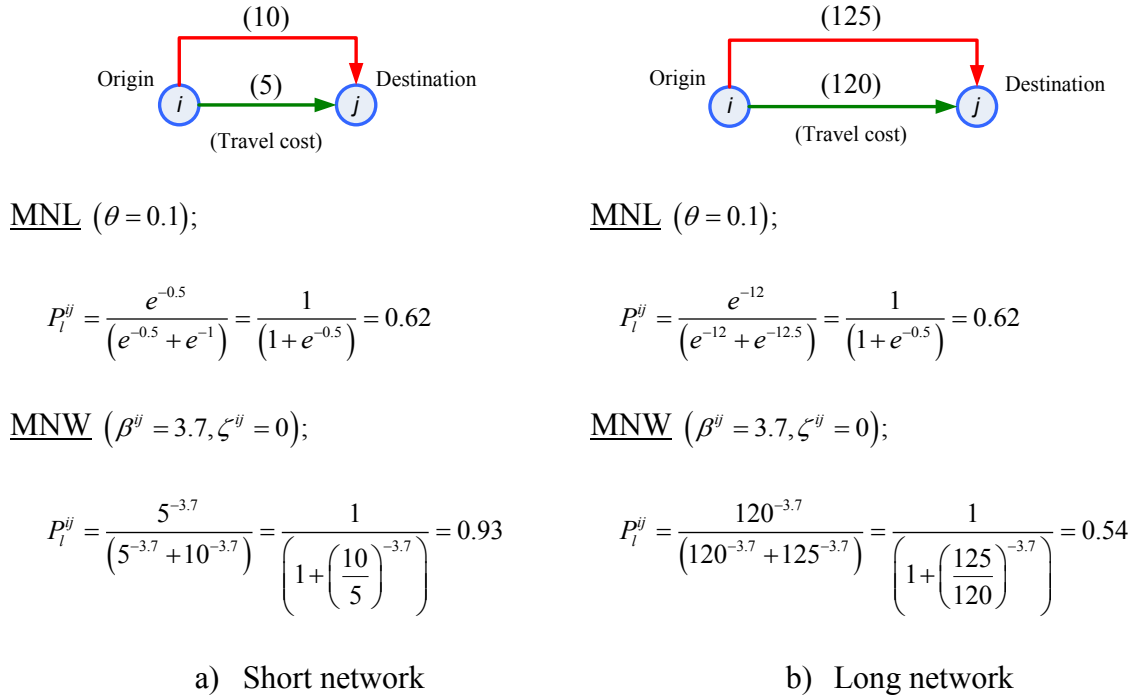


Fig. 34. Two-route networks

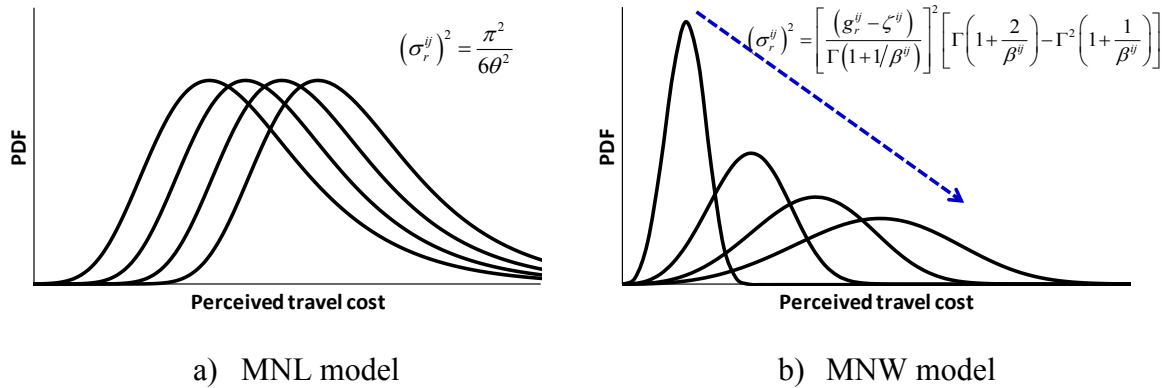


Fig. 35. Perception variance of the MNL and MNW models

Note that scaling the dispersion parameter can partially relax the identical variance issue in the MNL model in the traffic assignment context (Chen et al., 2012). By scaling the dispersion parameter according to the O-D trip length (i.e., a larger  $\theta$  for the short network or a smaller  $\theta$  for the long network), we can obtain similar results as that of the MNW model, where the probability of choosing the lower route is higher for the short

network according to a smaller perception variance. Nonetheless, this scaling technique *cannot* obtain the route-specific perception variance like the MNW model in Eq. (79). All routes between an O-D pair must still have the same and fixed perception variance of  $\pi^2/6\theta^2$  in Fig. 35a due to the classical logit assumption of homogeneous perception variance.

### 5.2.1.2 Expected perceived cost of the MNW model

The expected perceived cost (EPC) can be used to represent the level of service (LOS) of a transportation network. [Yang and Bell's \(1998\)](#) mathematical programming (MP) formulation for the MNL stochastic user equilibrium (SUE) model with elastic demand has the elastic demand as a function of the MNL EPC—the log-sum term (e.g., [Sheffi, 1985](#)). Further, [Oppenheim's \(1995\)](#) MP formulation for the MNL-based combined travel demand model also has the multi-dimensional travel choices as a function of the MNL EPCs. However, the MNW model does not have a closed-form EPC in general ([Fosgerau and Bierlaire, 2009](#)). To overcome this drawback, we consider the *logarithmic* EPC of the MNW model (see Chapter 4) to represent the LOS of a transportation network, i.e.,

$$\bar{\mu}_{ij} = -\frac{1}{\beta^{ij}} \ln \sum_{r \in R_{ij}} (g_r^{ij} - \zeta^{ij})^{-\beta^{ij}}, \quad \forall ij \in IJ. \quad (168)$$

This logarithmic MNW EPC satisfies some important properties as follows: (1) Its partial derivative w.r.t. the *logarithmic* route travel cost gives back the MNW choice probability; (2) it is monotonically decreasing w.r.t. the number of routes; and (3) it is concave w.r.t. the vector of logarithmic route travel costs (see Chapter 4).

## 5.2.2 Path-size weibit (PSW) model

### 5.2.2.1 Model formulation

Even though the MNW model can successfully address the identical perception variance problem, it still inherits the *independently distributed* assumption. To overcome this shortcoming, we adopted the path-size factor  $\varpi_r^{ij}$  (Ben-Akiva and Bierlaire, 1999) to handle the route overlapping problem (see Chapter 3). This path-size factor accounts for different route sizes determined by the length of links within a route and the relative lengths of routes that share a link, i.e.,

$$\varpi_r^{ij} = \sum_{a \in Y_r} \frac{l_a}{L_r^{ij}} \frac{1}{\sum_{k \in R_{ij}} \delta_{ak}^{ij}}, \quad \forall r \in R_{ij}, ij \in IJ, \quad (169)$$

where  $l_a$  is the length of link  $a \in A$ ,  $L_r^{ij}$  is the length of route  $r$  connecting O-D pair  $ij$ ,  $Y_r$  is the set of all links in route  $r$  between O-D pair  $ij$ , and  $\delta_{ar}^{ij}$  is equal to 1 for link  $a$  on route  $r$  between O-D pair  $ij$  and 0 otherwise. Routes with a heavy overlapping with other routes have a smaller value of  $\varpi_r^{ij}$ . Note that other functional forms of  $\varpi_r^{ij}$  can be found in Bovy et al. (2008) and Prato (2009). The path-size factor is used to modify the MNW RUM model:

$$U_r^{ij} = \frac{(g_r^{ij} - \zeta^{ij})^{\beta^{ij}}}{\varpi_r^{ij}} \varepsilon_r^{ij}, \quad \forall r \in R_{ij}, ij \in IJ, \quad (170)$$

which gives the PSW probability:

$$P_r^{ij} = \frac{\varpi_r^{ij} (g_r^{ij} - \zeta^{ij})^{-\beta^{ij}}}{\sum_{k \in R_{ij}} \varpi_k^{ij} (g_k^{ij} - \zeta^{ij})^{-\beta^{ij}}}, \quad \forall r \in R_{ij}, ij \in IJ. \quad (171)$$

### 5.2.2.2 *Expected perceived cost of the PSW model*

Since the PSW model modifies the deterministic term of the MNW model, its logarithmic EPC can be expressed as (see Chapter 4)

$$\bar{\mu}_{ij} = -\frac{1}{\beta^{ij}} \ln \sum_{r \in R_{ij}} \omega_r^{ij} (g_r^{ij} - \zeta^{ij})^{-\beta^{ij}}, \quad \forall ij \in IJ. \quad (172)$$

Note that this logarithmic PSW EPC also has the same property as the logarithm MNW EPC. The partial derivative of this logarithmic PSW EPC w.r.t. the *logarithmic* route travel cost gives back the PSW choice probability. It is monotonically decreasing w.r.t. the number of routes, and it is concave w.r.t. the vector of logarithmic route travel costs.

## 5.3 Mathematical programming formulation

This section presents equivalent MP formulations for the PSW-SUE-Ed model and its application to the combined modal split and traffic assignment (CMSTA) problem. Specifically, we present these MP formulations with some qualitative properties. The section starts with some necessary assumptions, followed by the MP formulations for the PSW-SUE-ED model and the CMSTA problem.

### 5.3.1 *Assumptions*

We start with some necessary assumptions. First, we make a general assumption on the link travel cost:

Assumption 5.1. The link travel cost  $\tau_a$ , which can be a function of link travel time, is a monotonically increasing function of its own flow.

Since the weibit model falls within the category of *multiplicative* random utility maximization model, the deterministic part of the disutility function is simply a set of



*multiplicative* explanatory variables (e.g., Cooper and Nakanishi, 1988). Then, we make an assumption of the route travel cost:

Assumption 5.2. The route travel cost is a function of multiplicative link travel costs, i.e.,

$$g_r^{ij} = \prod_{a \in Y_r} \tau_a, \quad \forall r \in R_{ij}, \forall ij \in IJ. \quad (173)$$

This assumption not only maintains the weibit relative cost criterion, but also corresponds to the weibit choice behavior similar to the *Markov process* (see Chapter 4). In addition, it allows the MP formulation to incorporate the elastic demand (travel choice) and mode choice as a function of the logarithmic PSW EPC.

Following the path-size logit (PSL) SUE formulation provided by Chen et al. (2012), the lengths used in the path-size factor for the MP formulation is flow independent. We make another assumption on the path-size factor attributes:

Assumption 5.3. The lengths  $l_a$  and  $L_r^{ij}$  used in  $\varpi_r^{ij}$  are flow independent.

Note that we can also adopt the variational inequality (VI) formulation to incorporate the flow-dependent path-size attributes similar to the congestion-based C-logit-SUE model developed by Zhou et al., 2012).

Since  $\zeta^{ij}$  cannot be decomposed into the link level easily, we make another assumption:

Assumption 5.4.  $\zeta^{ij}$  is equal to zero.

This assumption indicates that each route is assumed to have the *same* coefficient of variation (COV). From Eq. (79), the *route-specific coefficient of variation* can be expressed as

$$g_r^{ij} = \frac{\sigma_r^{ij}}{g_r^{ij}} = \frac{(g_r^{ij} - \zeta^{ij})}{g_r^{ij}} \sqrt{\frac{\Gamma(1+2/\beta^{ij})}{(\Gamma(1+1/\beta^{ij}))^2} - 1}, \quad \forall r \in R_{ij}, ij \in IJ. \quad (174)$$

With  $\zeta^{ij} = 0$ ,  $g_r^{ij}$  of each route is equal. Note that we can adopt the VI formulation of the congestion-based C-logit-SUE model developed by Zhou et al. (2012) to incorporate a non-zero  $\zeta^{ij}$  into the PSW-SUE-ED model.

Finally, we make an assumption on the elastic demand function  $D_{ij}(\cdot)$ :

Assumption 5.5. The elastic demand function is a monotonically decreasing function of the logarithmic PSW EPC.

### 5.3.2 MP formulation for the PSW-SUE-ED

In this subsection, we provide the PSW-SUE-ED model where ED is a function of the logarithmic PSW EPC as shown in Fig. 36. Consider the following MP formulation

$$\begin{aligned} \min Z(\mathbf{f}, \mathbf{q}) &= Z_1 + Z_2 + Z_3 + Z_4 + Z_5 \\ &= \sum_{a \in A} \int_0^{v_a} \sum_{ij \in IJ} \sum_{r \in R_{ij}} f_r^{ij} \delta_{ar}^{ij} \ln \tau_a(\omega) d\omega + \sum_{ij \in IJ} \sum_{r \in R_{ij}} \frac{1}{\beta^{ij}} f_r^{ij} (\ln f_r^{ij} - 1) \\ &\quad - \sum_{ij \in IJ} \int_0^{q_{ij}} D_{ij}^{-1}(\omega) d\omega - \sum_{ij \in IJ} \frac{1}{\beta^{ij}} q_{ij} (\ln q_{ij} - 1) - \sum_{ij \in IJ} \sum_{r \in R_{ij}} \frac{1}{\beta^{ij}} f_r^{ij} \ln \varpi_r^{ij} \end{aligned} \quad (175)$$

$$\text{s.t. } \sum_{r \in R_{ij}} f_r^{ij} = q_{ij}, \quad \forall ij \in IJ, \quad (176)$$

$$q_{ij} \geq 0, f_r^{ij} \geq 0, \quad \forall r \in R_{ij}, ij \in IJ, \quad (177)$$

where  $f_r^{ij}$  is the flow on route  $r$  between O-D pair  $ij$ ,  $q_{ij}$  is the travel demand between O-D pair  $ij$  from the elastic demand function  $D_{ij}(\cdot)$ , and  $v_a$  is the flow on link  $a$ . Eq. (176) is the flow conservation constraint, and Eq. (177) is the non-negativity condition. The main differences between this PSW-SUE-ED and Yang and Bell's (1998) MNL-SUE-ED

are  $Z_1$  and  $Z_5$ .  $Z_1$  is the multiplicative Beckmann's transformation. It uses the log transformation to facilitate the route cost computations. With this, the PSW-SUE-ED has the elastic demand  $D_{ij}(\cdot)$  as a function of the *logarithmic* PSW EPC at the equilibrium. On the other hand,  $Z_5$  incorporates  $\varpi_r^{ij}$  to handle the route overlapping problem. When there is no route overlapping,  $\varpi_r^{ij} = 1$  and the PSW-SUE-ED model collapses to the MNW-SUE-ED model.

*Proposition 5.1. The solution of the MP formulation given in Eqs. (175) through (177) satisfies the PSW route choice probability and the elastic demand function.*

*Proof.* Note that the logarithm terms in Eq. (175) implicitly require both  $f_r^{ij}$  and  $q_{ij}$  to be positive. By constructing the Lagrangian and then setting its partial derivative to zero, we obtain:

$$\ln g_r^{ij} + \frac{1}{\beta^{ij}} \ln f_r^{ij} - \frac{1}{\beta^{ij}} \ln \varpi_r^{ij} - \lambda_{ij} = 0, \quad (178)$$

$$\lambda_{ij} - D_{ij}^{-1}(q_{ij}) - \frac{1}{\beta^{ij}} \ln q_{ij} = 0, \quad (179)$$

$$\sum_{r \in R_{ij}} f_r^{ij} = q_{ij}, \quad (180)$$

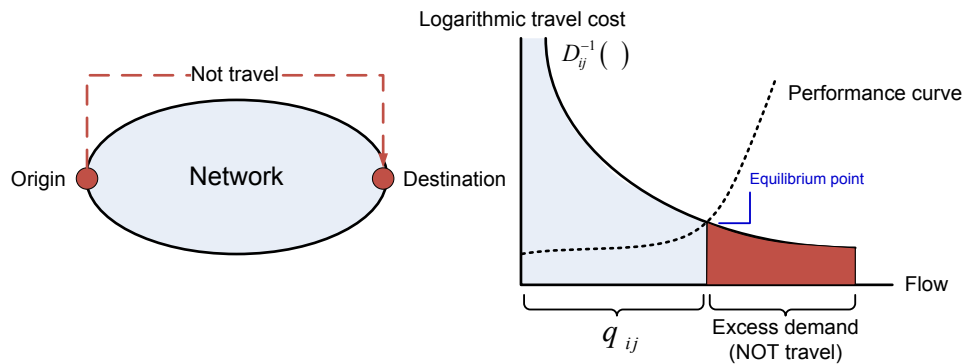


Fig. 36: Network representation and equilibrium conditions with ED

where  $\lambda_{ij}$  is the Lagrangian multiplier for the flow conservation constraint in Eq. (176).

Rearranging Eq. (90) gives:

$$f_r^{ij} = \exp(\beta^{ij} \lambda_{ij}) \varpi_r^{ij} (g_r^{ij})^{-\beta^{ij}}. \quad (181)$$

From Eq. (91), we have:

$$\sum_{r \in R_{ij}} f_r^{ij} = q_{ij} = \exp(\beta^{ij} \lambda_{ij}) \sum_{r \in R_{ij}} \varpi_r^{ij} (g_r^{ij})^{-\beta^{ij}}. \quad (182)$$

Dividing Eq. (181) by Eq. (182) gives the PSW route choice probability:

$$P_r^{ij} = \frac{f_r^{ij}}{q_{ij}} = \frac{\varpi_r^{ij} (g_r^{ij})^{-\beta^{ij}}}{\sum_{k \in R_{ij}} \varpi_k^{ij} (g_k^{ij})^{-\beta^{ij}}}, \quad \forall r \in R_{ij}, ij \in IJ. \quad (183)$$

On the other hand, by rearranging Eq. (182), we obtain:

$$\lambda_{ij} = \frac{1}{\beta^{ij}} \ln q_{ij} - \frac{1}{\beta^{ij}} \ln \sum_{r \in R_{ij}} \varpi_r^{ij} (g_r^{ij})^{-\beta^{ij}}. \quad (184)$$

Substituting Eq. (184) into Eq. (179) gives

$$D_{ij}^{-1}(q_{ij}) = -\frac{1}{\beta^{ij}} \ln \sum_{r \in R_{ij}} \varpi_r^{ij} (g_r^{ij})^{-\beta^{ij}}. \quad (185)$$

By rearranging Eq. (185), we have the following elastic demand as a function of the logarithmic PSW EPC, i.e.,

$$q_{ij} = D_{ij}(\bar{\mu}_{ij}) = D_{ij} \left( -\frac{1}{\beta^{ij}} \ln \sum_{r \in R_{ij}} \varpi_r^{ij} (g_r^{ij})^{-\beta^{ij}} \right), \quad \forall ij \in IJ. \quad (186)$$

This completes the proof.  $\square$

*Proposition 5.2. The solution of the PSW-SUE-ED model is unique.*

*Proof.* It is sufficient to prove that the objective function in Eq. (175) is strictly convex in the vicinity of route flow solution and that the feasible region is convex. The convexity of

the feasible region is assured for linear equality constraint in Eq. (176). The nonnegative constraint in Eq. (177) does not alter this characteristic. From Proposition 5.1 and Assumption 5.3, the second derivative w.r.t. route flow is

$$\frac{\partial^2 Z(\mathbf{f})}{\partial f_r^{ij} \partial f_k^{rs}} = \frac{\partial \ln g_r^{ij}}{\partial f_k^{rs}} - \frac{\partial}{\partial f_k^{rs}} D_{ij}^{-1} \left( \sum_{r \in R_{ij}} f_r^{ij} \right) + \frac{1}{\beta^{ij}} \frac{\partial \ln f_r^{ij}}{\partial f_k^{rs}} - \frac{1}{\beta^{ij}} \frac{\partial \ln \sum_{r \in R_{ij}} f_r^{ij}}{\partial f_k^{rs}}. \quad (187)$$

From Assumptions 5.1 and 5.5, both  $\frac{\partial \ln g_r^{ij}}{\partial f_k^{rs}}$  and  $\frac{\partial}{\partial f_k^{rs}} D_{ij}^{-1} \left( \sum_{r \in R_{ij}} f_r^{ij} \right)$  are positive semi-

definite. Clearly  $\frac{1}{\beta^{ij}} \frac{\partial \ln f_r^{ij}}{\partial f_k^{rs}}$  is positive definite since

$$\frac{1}{\beta^{ij}} \frac{\partial \ln f_r^{ij}}{\partial f_k^{rs}} = \begin{cases} \frac{1}{\beta^{ij} f_r^{ij}} > 0, & \text{if } (ij, r) = (rs, k) \\ 0 & , \text{ otherwise} \end{cases}. \quad (188)$$

Finally, we can observe that  $-\frac{1}{\beta^{ij}} \frac{\partial \ln \sum_{r \in R_{ij}} f_r^{ij}}{\partial f_k^{rs}}$  is positive semi-definite since all elements

of the block matrix w.r.t. O-D pair  $ij$  are equal to  $-\frac{1}{\beta^{ij} \sum_{r \in R_{ij}} f_r^{ij}}$ . Thus, the objective

function in Eq. (175) is strictly convex with respect to route flows. Therefore, the equilibrium route flow is unique. According to the flow conservation condition, the equilibrium travel demand is also unique. This completes the proof.  $\square$

### 5.3.3 Application of the PSW-SUE-ED

In this section, we provide an application of the PSW-SUE-ED model to consider both mode choice and route choice as a combined modal split and traffic assignment (CMSTA) problem. It will be demonstrated in the numerical result section with a case

study in a bimodal network with motorized and non-motorized modes using the Winnipeg network.

To begin with, a model with a *binary* logit mode choice is presented. The elastic demand term  $Z_3$  in Eq. (175) is modified using the argument-complementing function (Sheffi, 1985), i.e., (see also Fig. 37)

$$\sum_{ij \in IJ} \int_0^{q_{ij}^{ij}} D_{ij}^{-1}(\omega) d\omega = - \sum_{ij \in IJ} \frac{1}{\gamma_{ij}} \int_0^{q_{m=2}^{ij}} \left( \ln \frac{\omega}{q_{m=1}^{ij} - \omega} - \Psi_{ij} \right) d\omega, \quad (189)$$

where  $q_m^{ij}$  is the demand of mode  $m=1,2$  between O-D pair  $ij$ ,  $\gamma_{ij}$  is the binary logit model parameter between O-D pair  $ij$ , and  $\Psi_{ij}$  is the exogenous modal attractiveness difference between the two modes connecting O-D pair  $ij$ . By incorporating Eq. (189) into Eq. (175), we have a combined binary logit mode choice and PSW-SUE route choice model.

To extend the binary logit mode choice to the MNL mode choice, we restate Eq. (189) as

$$\sum_{ij \in IJ} \frac{1}{\gamma_{ij}} \int_0^{q_{m=2}^{ij}} \left( \ln \frac{\omega}{q_{m=1}^{ij} - \omega} - \Psi_{ij} \right) d\omega = \sum_{ij \in IJ} \sum_{m=1}^2 \frac{1}{\gamma_{ij}} q_m^{ij} (\ln q_m^{ij} - 1) - \sum_{ij \in IJ} \sum_{m=1}^2 \frac{1}{\gamma_{ij}} q_m^{ij} \Psi_{ijm}, \quad (190)$$

and then change the mode choice index from binary (2) to multinomial ( $M_{ij}$ ) as follows

$$\begin{aligned} \sum_{ij \in IJ} \sum_{m=1}^2 \frac{1}{\gamma_{ij}} q_m^{ij} (\ln q_m^{ij} - 1) - \sum_{ij \in IJ} \sum_{m=1}^2 \frac{1}{\gamma_{ij}} q_m^{ij} \Psi_{ijm} \\ \rightarrow \sum_{ij \in IJ} \sum_{m \in M_{ij}} \frac{1}{\gamma_{ij}} q_m^{ij} (\ln q_m^{ij} - 1) - \sum_{ij \in IJ} \sum_{m \in M_{ij}} \frac{1}{\gamma_{ij}} q_m^{ij} \Psi_{ijm}, \end{aligned} \quad (191)$$

where  $q_m^{ij}$  is now the travel demand of mode  $m \in M_{ij}$  between O-D pair  $ij$ , and  $\Psi_{ijm}$  is the exogenous modal attractiveness of mode  $m$  between O-D pair  $ij$ .

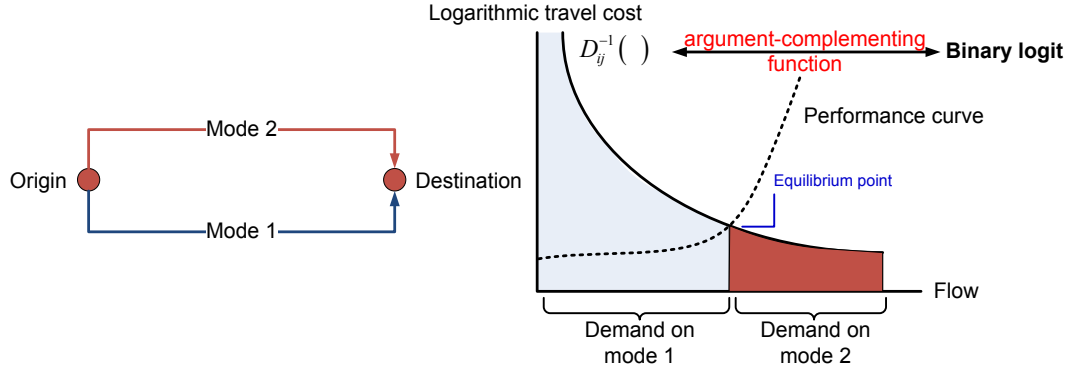


Fig. 37. Network representation and excess demand as a binary logit mode choice

By incorporating Eq. (191) into Eq. (175), we have a combined MNL mode choice and PSW-SUE route choice (or MNL-PSW-SUE) model, i.e.,

$$\begin{aligned}
 \min Z &= Z_1 + Z_2 + Z_3 + Z_4 + Z_5 \\
 &= \sum_{a \in A} \int_0^{v_a} \sum_{ij \in IJ} \sum_{m \in M_{ij}} \sum_{r \in R_{ijm}} f_{mr}^{ij} \delta_{amr}^{ij} \ln \tau_a(\omega) d\omega + \sum_{ij \in IJ} \sum_{m \in M_{ij}} \sum_{r \in R_{ijm}} \frac{1}{\beta^{ijm}} f_{mr}^{ij} (\ln f_{mr}^{ij} - 1) \\
 &\quad - \sum_{ij \in IJ} \sum_{m \in M_{ij}} q_m^{ij} \Psi_{ijm} + \sum_{ij \in IJ} \sum_{m \in M_{ij}} \left\{ \frac{1}{\gamma_{ij}} - \frac{1}{\beta^{ijm}} \right\} q_m^{ij} (\ln q_m^{ij} - 1) \\
 &\quad - \sum_{ij \in IJ} \sum_{m \in M_{ij}} \sum_{r \in R_{ijm}} \frac{1}{\beta^{ijm}} f_{mr}^{ij} \ln \varpi_{mr}^{ij}
 \end{aligned} \tag{192}$$

s.t.

$$\sum_{r \in R_{ijm}} f_{mr}^{ij} = q_m^{ij}, \quad \forall m \in M_{ij}, ij \in IJ, \tag{193}$$

$$\sum_{m \in M_{ij}} q_m^{ij} = q_{ij}, \quad \forall ij \in IJ, \tag{194}$$

$$f_{mr}^{ij} \geq 0, \quad q_m^{ij} \geq 0, \quad \forall r \in R_{ijm}, m \in M_{ij}, ij \in IJ, \tag{195}$$

where  $f_{mr}^{ij}$  is the flow on route  $r$  of mode  $m$  between O-D pair  $ij$ ,  $\beta^{ijm}$  is the weibit parameter of mode  $m$  between O-D pair  $ij$ ,  $\varpi_{mr}^{ij}$  is the path-size factor on route  $r$  of mode  $m$  between O-D pair  $ij$ , and  $\delta_{amr}^{ij}$  is equal to 1 if link  $a$  on route  $r$  of mode  $m$  between O-D

pair  $ij$  and 0 otherwise. Eq. (193) and Eq. (194) are the flow/travel demand conservation constraints, and Eq. (195) is the non-negativity constraint on the decision variables (i.e., route flows and mode-specific O-D flows).

Proposition 5.3. *The MP formulation in Eqs. (192) through (195) has the MNL mode choice solution and the PSW route choice solution.*

Proof. The Lagrangian of this problem can be expressed as

$$L = Z + \sum_{ij \in IJ} \sum_{m \in M_{ij}} \lambda_{ijm} \left( q_m^{ij} - \sum_{r \in R_{ijm}} f_{mr}^{ij} \right) + \sum_{ij \in IJ} \phi_{ij} \left( q_{ij} - \sum_{m \in M_{ij}} q_m^{ij} \right), \quad (196)$$

where  $\lambda_{ijm}$  and  $\phi_{ij}$  are Lagrangian multipliers corresponding to the constraints in Eq. (193) and Eq. (194), respectively. Following the same principle of Proposition 5.1, we have the PSW route choice solution, and  $\lambda_{ijm}$  can be expressed as a function of the mode-specific O-D demand travel demand of mode  $m$  and the logarithmic PSW EPC of mode  $m$  ( $\bar{\mu}_{ijm}$ ), i.e.,

$$\begin{aligned} \lambda_{ijm} &= \frac{1}{\beta^{ijm}} \ln q_m^{ij} - \frac{1}{\beta^{ijm}} \ln \sum_{r \in R_{ijm}} \varpi_{mr}^{ij} \left( g_{mr}^{ij} \right)^{-\beta^{ijm}} \\ &= \frac{1}{\beta^{ijm}} \ln q_m^{ij} + \bar{\mu}_{ijm}. \end{aligned} \quad (197)$$

By setting the partial derivative of the Lagrangian w.r.t.  $q_m^{ij}$  to zero, we have

$$-\Psi_{ijm} + \left\{ \frac{1}{\gamma_{ij}} - \frac{1}{\beta^{ijm}} \right\} \ln q_m^{ij} + \lambda_{ijm} - \phi_{ij} = 0, \quad (198)$$

According to Eq. (197), Eq. (198) can be restated as

$$q_m^{ij} = \exp \left( \gamma_{ij} \phi_{ij} + \gamma_{ij} \left( \Psi_{ijm} - \bar{\mu}_{ijm} \right) \right). \quad (199)$$



Then,  $q_{ij}$  can be expressed as

$$q_{ij} = \sum_{m \in M_{ij}} q_m^{ij} = \exp(\gamma_{ij} \phi_{ij}) \sum_{m \in M_{ij}} \exp(\gamma_{ij} (\Psi_{ijm} - \bar{\mu}_{ijm})). \quad (200)$$

Incorporating Eq. (199) and Eq. (200) give the MNL mode choice probability, i.e.,

$$P_m^{ij} = \frac{q_m^{ij}}{q_{ij}} = \frac{\exp(\gamma_{ij} (\Psi_{ijm} - \bar{\mu}_{ijm}))}{\sum_{n \in M_{ij}} \exp(\gamma_{ij} (\Psi_{ijn} - \bar{\mu}_{ijn}))}. \quad (201)$$

Therefore, the MP formulation in Eq. (192) through Eq. (195) has the MNL mode choice solution and PSW route choice solution. This completes the proof.  $\square$

*Proposition 5.4. The solution of MNL-PSW-SUE model is unique.*

*Proof.* Following the same principle of Proposition 5.2,  $Z_1$ ,  $Z_2$ , and  $Z_5$  are positive semi-definite, positive definite, and positive semi-definite, respectively. Since  $\Psi_{ijm}$  is flow independent, the second derivative of  $Z_3$  is zero. Assuming that  $\beta^{ijm} > \gamma_{ij}$ , the second derivative of  $Z_4$  can be expressed as

$$\frac{\partial^2 Z_4}{\partial q_m^{ij} \partial q_n^{rs}} = \begin{cases} \left\{ \frac{1}{\gamma_{ij}} - \frac{1}{\beta^{ijm}} \right\} \frac{1}{q_m^{ij}} > 0, & \text{if } (ij, m) = (rs, n) \\ 0 & , \text{ otherwise} \end{cases}. \quad (202)$$

Thus,  $Z_4$  is positive definite, and the objective function in Eq. (192) through Eq. (195) is strictly convex w.r.t. the route flows and mode-specific O-D demands. Therefore, the equilibrium route flows and mode-specific O-D demands are unique. This completes the proof.  $\square$

#### 5.4 Solution algorithm

In this section, we provide a path-based partial linearization algorithm combined with a self-regulated averaging (SRA) line search strategy for solving the PSW-SUE-ED and MNL-PSW-SUE models. The partial linearization method belongs to the descent direction algorithm (Patriksson, 1994). A search direction and a stepsize determination are iteratively performed to obtain a new iterative solution until some convergence criterion is satisfied. The search direction is obtained by solving a partial linearized subproblem. In this study, we adopt the SRA scheme recently proposed by Liu et al. (2009) to determine a stepsize. This stepsize scheme is based on the residual error and the stepsize in the current iteration to smartly determine a stepsize for the next iteration, without the computationally expensive evaluations of the complex objective function or its derivatives. The SRA scheme satisfies the convergence condition (see Robbins and Monro, 1951; Blum, 1954; Liu et al., 2009 for details). Some major detailed steps of the path-based partial linearization algorithm are provided as follows:

---

Step 0: Initialization.

- Set iteration count  $n=0$ ;
- Calculate the path-size factor;
- Update link travel costs and route travel costs;
- Calculate initial O-D demands ( $D_{ij}(\cdot)$  for the PSW-SUE-ED model and MNL mode choice for the MNL-PSW-SUE model);
- Perform the PSW loading to obtain initial route flows and link flows;

Step 1: Direction Finding.

- Increment iteration count  $n:= n +1$ ;
-

- 
- Update link and route travel costs;
  - Calculate *auxiliary* O-D demands;
  - Perform the PSW loading to obtain *auxiliary* route flows;
  - Search direction  $\left(\tilde{c} \quad - \quad \right)$ , and  $\left(\tilde{c}' \quad - \quad \right)$ ;

Step 2: Line Search.

- Use the SRA scheme to obtain  $\alpha^{(n)}$ ;

$$\alpha^{(n)} = 1/\eta^{(n)}$$

$$\eta^{(n)} = \begin{cases} \eta^{(n-1)} + \lambda_1, & \text{if } \left\| \tilde{c} \quad - \quad \right\| - \left\| \tilde{c}' \quad - \quad \right\|, \\ \eta^{(n-1)} + \lambda_2, & \text{otherwise} \end{cases}$$

where  $\lambda_1 > 1$  and  $\lambda_2 < 1$ .

Step 3: Move.

- $\mathbf{f}^{(n)} = \mathbf{f}^{(n-1)} + \alpha^{(n)} \left(\tilde{c} \quad - \quad \right)$ ;
- $\mathbf{q}^{(n)} = \mathbf{q}^{(n-1)} + \alpha^{(n)} \left(\tilde{c}' \quad - \quad \right)$ ;

Step 4: Convergence Test.

- If  $RMSE = \sqrt{\left\| \tilde{c} \quad - \quad \right\|_2 / |R|} \leq \varepsilon$  holds, terminate, where  $|R|$  is the number of routes; otherwise, go to Step 1.
- 

## 5.5 Numerical example

In this section, we provide three examples to present features of the proposed models. Example 1 is the two-route network used to investigate effect of different trip lengths. Example 2 is a loop-hole network adopted to investigate the effect of route

overlapping and route-specific perception variance problems. Lastly, Example 3 uses the Winnipeg network to show the algorithmic performance along with an application of bi-modal network considering both motorized mode (i.e., auto) and non-motorized mode (i.e., bicycle) as a case study. Without loss of generality, all routes are assumed to have the same coefficient of variation  $\mathcal{G}_r^{ij} = 0.3$  [i.e.,  $\beta^{ij} = 3.7$  for all O-D pairs, see Eq. (85)] unless specified otherwise.  $l_a$  and  $L_r^{ij}$  used in the path-size factor  $\varpi_r^{ij}$  are set to the link free-flow travel cost and route free-flow travel cost, respectively. The elastic demand function (in vehicle per hour: vph) is assumed to be a function of the *logarithmic* expected perceived cost (EPC), i.e.,

$$q_{ij} = 100 \exp(-0.05 \times \bar{\mu}_{ij}), \quad \forall ij \in IJ. \quad (203)$$

### 5.5.1 Example 1: Two-route networks

This example modifies the two-route networks in Fig. 34 to incorporate the congestion effect. The travel cost of each route includes a flow dependent cost  $f_r^{ij}/100$  as shown in Table 14. The equilibrium solutions produced by the PSW-SUE-ED model are compared with the MNL-SUE-ED and MNL-SUE-ED with scaling (or MNLs-SUE-ED) models as presented in Table 15. The MNL-SUE-ED model has the dispersion parameter  $\theta$  set equal to 0.1, and the MNLs-SUE-ED model has a scaling  $\theta$  set corresponding to  $\mathcal{G}_r^{ij} = 0.3$  using the lowest uncongested travel cost route (see [Chen et al., 2012](#)). The results show that all models give a smaller travel demand as the overall trip length increases. The MNL-SUE-ED model produces the same route choice probabilities for both short and long networks. This is because it cannot handle the heterogeneous perception variance w.r.t. different trip lengths. Meanwhile, the MNLs-SUE-ED and PSW-SUE-ED models

assign a smaller amount of flows on the lower route as the overall trip length increases. The MNLs-SUE-ED model gives a larger O-D demand. It further gives a higher probability of choosing the lower route, especially on the short network. This is because the scaling technique still assumes the same and fixed perception variance of each route within the same O-D pair.

### 5.5.2 Example 2: Loop-hole network

In this example, a loop-hole network in Fig. 38 is adopted to consider both route overlapping and route-specific perception variance problems. This loop-hole network has 3 routes, and each route has the same capacity of 100 vph. The two upper routes have the free flow travel time (FFTT) of 30 minutes with an overlapping section of  $x$ . The lower route is truly independent with the FFTT of  $y$ . The standard Bureau of Public Road (BPR) function is adopted to represent the flow dependent link travel time, i.e.,

$$t_a = t_a^0 \left[ 1 + 0.15 \left( \frac{v_a}{c_a} \right)^4 \right], \quad (204)$$

where  $t_a^0$  is the FFTT on link  $a$ , and  $c_a$  is the capacity on link  $a$ . Without loss of generality, the travel cost is assumed to be an exponential function (Hensher and Truong, 1985; Polak, 1987; Mirchandani and Soroush, 1987), i.e.,

$$\tau_a = \exp(0.05t_a), \quad (205)$$

Table 14: Flow-dependent route travel cost for the two-route networks

Network	Upper route	Lower route
Short	$10 + f_u^{ij}/100$	$5 + f_l^{ij}/100$
Long	$125 + f_u^{ij}/100$	$120 + f_l^{ij}/100$

Table 15: Travel demand and flow allocation for the two-route networks

Model	Network	Demand	Probability	
			Upper route	Lower route
MNL-SUE-ED	Short ( $\theta=1$ )	99.79	0.38	0.62
	Long ( $\theta=1$ )	78.86	0.38	0.62
MNLs-SUE-ED	Short ( $\theta=0.86$ )	91.94	0.03	0.97
	Long ( $\theta=0.04$ )	79.40	0.46	0.54
PSW-SUE-ED	Short	91.72	0.11	0.88
	Long	79.36	0.47	0.53

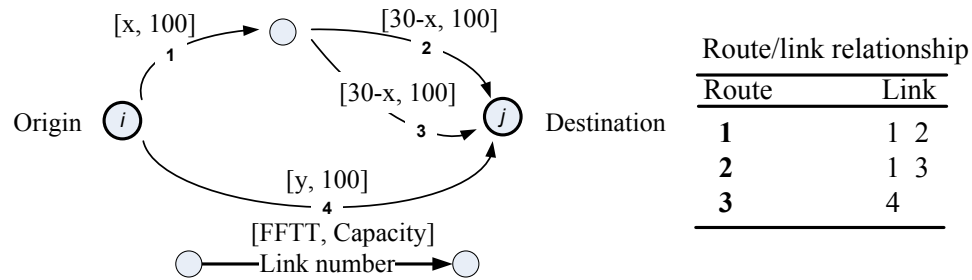


Fig. 38. Loop-hole network and its characteristics

### 5.5.2.1 Effect of route overlapping

The effect of route overlapping is investigated first where all routes have the same trip length (i.e.,  $y = 30$ ). The results produced by the PSW-SUE-ED model would be compared with the MNLs-SUE-ED model under the same cost configuration in Eq. (205).

As expected, the MNLs-SUE-ED model has difficulty in handling the overlapping problem as shown in Fig. 39. As  $x$  increases, it assigns more flows on the two upper routes (hence, a smaller amount of flows on the lower route), while the PSW-SUE-ED model assigns a relatively larger amount of flows on the lower route as  $x$  increases. At  $x = 30$ , only two routes with equal trip length exist. The PSW-SUE-ED model assigns an equal amount of flows on both routes, while the MNLs-SUE-ED model only assigns 36%

of the total flows to the lower route as it still considers all three routes as feasible routes in the loop-hole network.

However, the logarithmic PSW EPC is larger as shown in Fig. 40a. This is because the logarithmic PSW EPC incorporate the route overlapping problem through the path-size factor  $\varpi_r^{ij}$ . As the overlapping section increases,  $\varpi_r^{ij}$  of the upper routes decreases. As a result, the PSW-SUE-ED model produces a lower O-D demand level, especially for a longer overlapping section  $x$ , as shown in Fig. 40b.

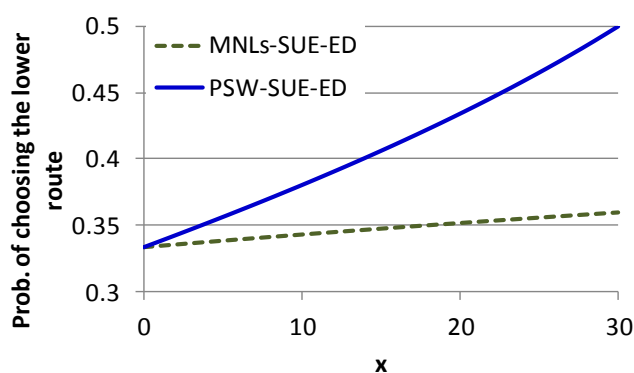
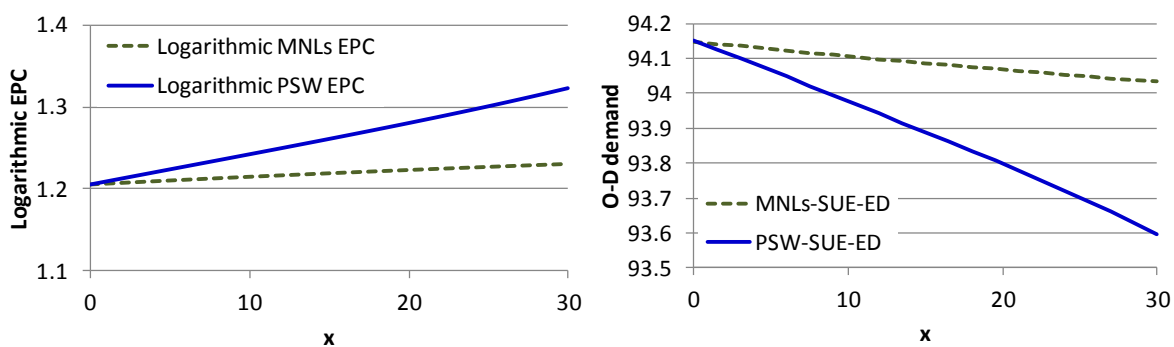


Fig. 39. Probability of choosing the lower route for the loop-hole network ( $y = 30$ )



a) Logarithmic EPC

b) O-D demand

Fig. 40. EPC and travel demand patterns for the loop-hole network ( $y = 30$ )

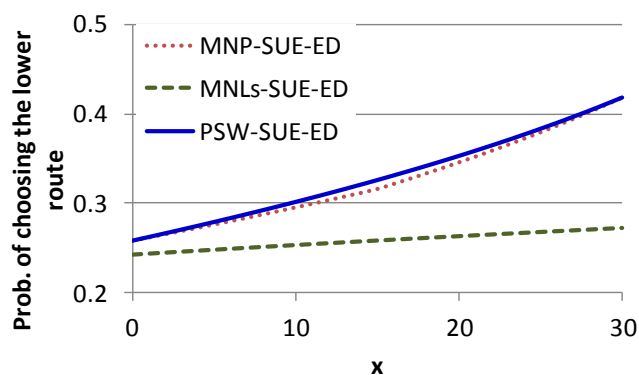


Fig. 41. Traffic flow patterns for the loop-hole network ( $y = 32$ )

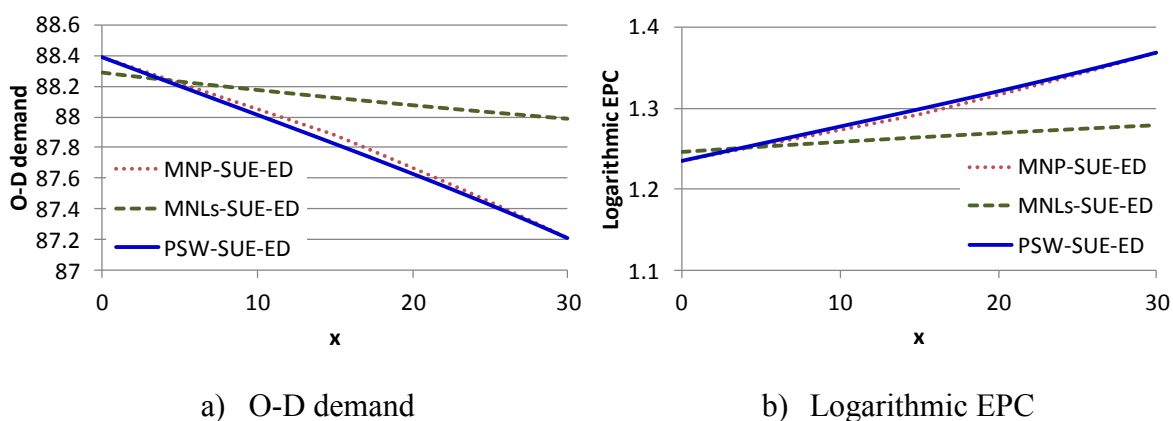


Fig. 42. EPC and travel demand patterns for the loop-hole network ( $y = 32$ )

### 5.5.2.2 Effect of both route overlapping and identical perception variance problems

To consider both route overlapping and identical perception variance problems,  $y$  is set equal 32 minutes. As such, the FFTT of the lower route is longer than that of the upper routes by 2 minutes. The results show that the PSW-SUE-ED model can produce a comparable flow pattern to the multinomial probit SUE model with elastic demand (or MNP-SUE-ED) as shown in Fig. 41. Both models give a smaller O-D demand for a larger  $x$  since the logarithmic EPC is increased as shown in Fig. 42. Note that at  $\beta^{ij} = 3.7$ , the Weibull distribution is a symmetric distribution like the Normal distribution (see [White, 1969](#)).



### 5.5.3 Example 3: Winnipeg network

This example uses the Winnipeg network shown in Fig. 43 as a real-case study. This network consists of 154 zones, 2,535 links, and 4,345 O-D pairs. The network topology, link characteristics, and O-D demands can be found in Emme/2 software (INRO Consultants, 1999). For comparison purposes, a behavioral working route set generated by Bekhor et al. (2008) is adopted. This example begins with the performance of the SRA scheme, followed by an application of the MNL-PSW-SUE model in a bi-modal network with motorized and non-motorized modes.

#### 5.5.3.1 Algorithmic performance

The algorithm is coded in Intel Visual FORTRAN 6.6 and run on a 3.80 GHz processor with 2.00 GB of RAM. The stopping criterion  $\varepsilon$  is set equal  $10^{-8}$ . We first analyze the sensitivity of the parameters  $\lambda_1$  and  $\lambda_2$  in the algorithmic performance in Table 16. It can be observed that the effect of  $\lambda_2$  seems to be higher than  $\lambda_1$  in this case. Without loss of generality, we select the value of  $\lambda_1$  and  $\lambda_2$  (1.55, 0.10) for further analyses.

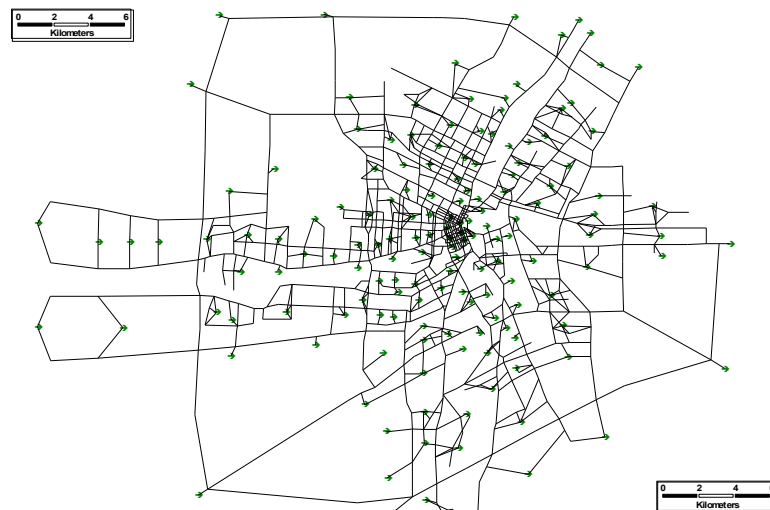


Fig. 43. Winnipeg network

The convergence characteristics of the SRA scheme are shown in Fig. 44. It appears that the SRA scheme outperforms the MSA scheme with a stepsize of  $1/n$ . The SRA scheme requires 514 iterations to reach the desired level of accuracy, while the MSA scheme requires more than 3,000 iterations as presented in Table 17. In terms of average computational effort required in each iteration, the SRA scheme needs only 0.02 second per iteration more than the MSA scheme.

Table 16: Sensitivity analysis of  $\lambda_1$  and  $\lambda_2$  in solving the PSW-SUE-ED model

No. of Iterations		$\lambda_1$										
		1.50	1.55	1.60	1.65	1.70	1.75	1.80	1.85	1.90	1.95	2.00
$\lambda_2$	0.01	562	574	572	572	548	551	553	583	594	536	531
	0.05	538	547	562	594	623	627	633	574	587	590	592
	0.10	529	514	548	565	579	598	606	597	623	675	677
	0.15	1146	1147	1251	1273	1362	1421	1455	1427	1414	1578	1596
	0.20	2015	2121	2129	2245	2387	2451	2477	2564	2679	2796	2948

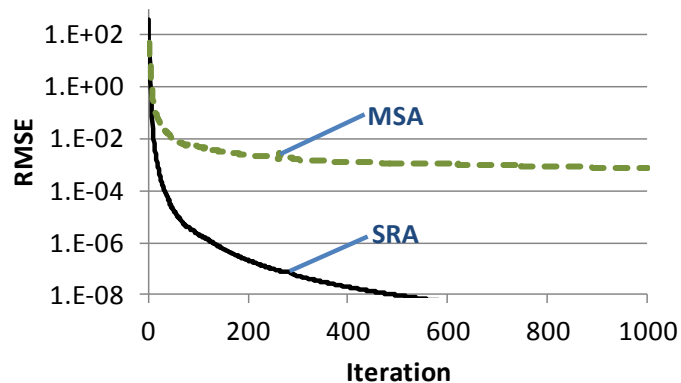


Fig. 44. Convergence characteristics for solving the MNL-PSW-SUE model

Table 17: Computational efforts for solving the PSW-SUE-ED model

Algorithm	# of iterations	CPU time (sec.)	CPU time/Iteration
MSA	3217	569	0.18
SRA	514	102	0.20

### 5.5.3.2 *Application in a bi-modal network with motorized and non-motorized modes*

To construct the bi-modal network, we incorporate the cycling routes obtained from the Winnipeg city's website to the Winnipeg network. The dedicated bike lanes are assumed to be located along some streets as shown in Fig. 45. The results produced by the MNL-PSW-SUE model will be compared with the MNL-MNLs-SUE model. For a fair comparison, we continue to use the same cost configuration in Eq. (205). Both models use the logarithmic EPC for the mode choice level with  $\gamma_{ij} = 0.5$  and  $\Psi_{ijm} = 0$  for the auto and bike modes. For the bike routes, we use Bekhor et al. (2008)'s route set with the travel distance less than 8 miles, which is the longest distance for the majority of cyclists in the United States (US Census Bureau, 2000). The bike travel time is assumed to be flow-independent determined by its trip length and average speed of 10 mph on a dedicated bike lane (Jensen et al., 2010). With this setting, we have 421 O-D pairs with both auto and bike modes available.

The computational efforts for solving the MNL-PSW-SUE model are shown in Table 18. We can see that the SRA scheme outperforms the MSA scheme. The SRA scheme requires a significant smaller computational effort than the MSA scheme.



Fig. 45. Winnipeg network with bike lanes

Table 18: Computational efforts for solving the MNL-PSW-SUE model

<b>Algorithm</b>	<b># of iterations</b>	<b>CPU time (sec)</b>	<b>CPU time/Iteration</b>
MSA	3374	650	0.19
SRA	644	138	0.21

Then, we compare the flow allocations at the disaggregate and aggregate levels. At the disaggregate level, we examine the route choice and mode choice probabilities produced by the MNL-MNLs-SUE and MNL-PSW-SUE models. For demonstration purposes, we use O-D pairs (3, 147) and (74, 60) to respectively represent a short O-D pair and a long O-D pair that include both auto and bike modes. The route choice and mode choice probabilities shown in Fig. 46 are under the equilibrium conditions. Five routes with larger auto flow proportions in each O-D pair are selected, where the first three routes (i.e., routes 1, 2, and 3) also include the bike routes. Recall that the MNLs-

SUE model can partially handle the different trip lengths through the scaling dispersion parameter whereas the MNL-PSW-SUE model can simultaneously handle both route overlapping and non-identical perception variance of different trip lengths. Therefore, the different route choice probabilities can be expected. The MNL-MNLs-SUE model seems to assign more flows to the shortest route. This is according to the same and fixed perception variance for all routes between an O-D pair. Meanwhile, the MNL-PSW-SUE model seems to produce a more dispersed assignment. It gives a lower route choice probability to routes that have couplings with other routes. At the same time, it assigns a larger amount of flows to the longer routes compared to the MNL-MNLs-SUE model. As a result, combining the effect of route overlapping and non-identical perception variance of different trip lengths, the MNL-PSW-SUE model produces a higher share for the bike mode, especially for the shorter O-D pair (i.e., O-D pair (3,147)).

At the aggregate level, we examine the effect of route overlapping and heterogeneous perception variance problems on the link flow patterns and mode shares. As expected, the link flow difference between the MNL-MNLs-SUE and MNL-PSW-SUE models can be found mostly in the central business district (CBD) area as shown in Fig. 47. This is because this area consists of much denser roads with more overlapping between routes. As such, the MNL-PSW-SUE model produces a larger probability of choosing the bike mode as presented in Table 19. In the CBD area, the bike mode share of the MNL-PSW-SUE model is more than two times higher than that of the MNL-MNLs-SUE model.

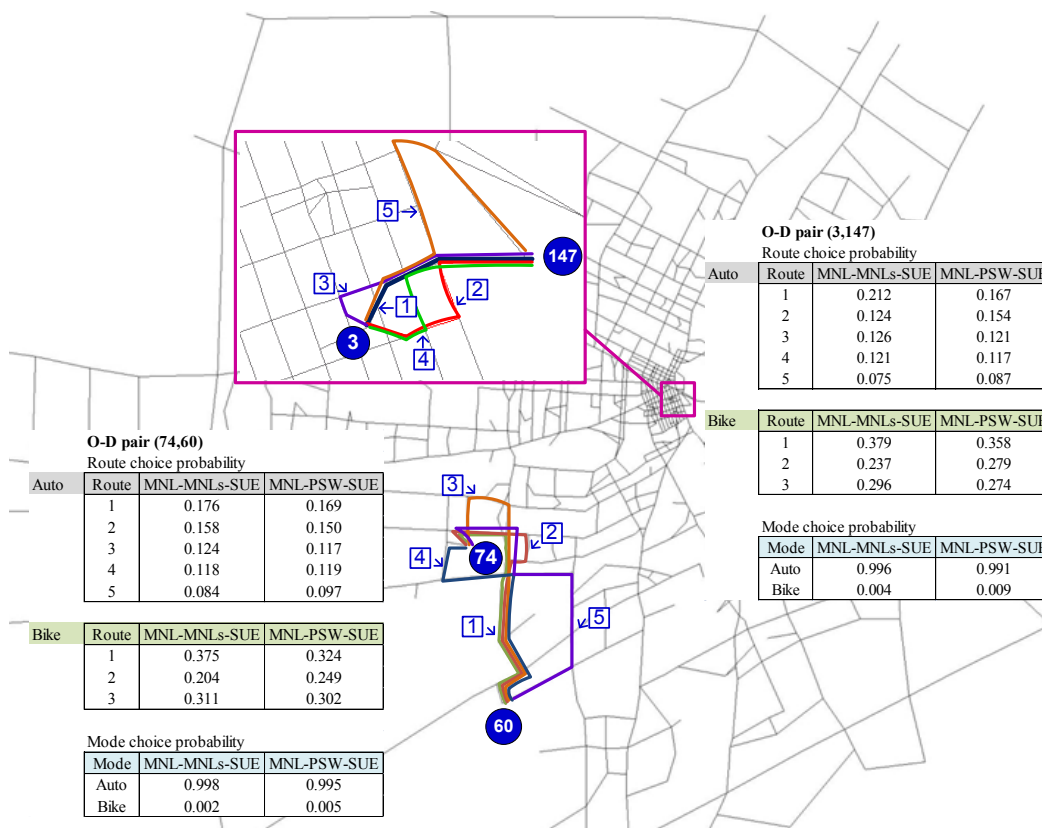


Fig. 46. Comparison of route choice and mode choice probabilities of O-D pairs (3,147) and (74,60)

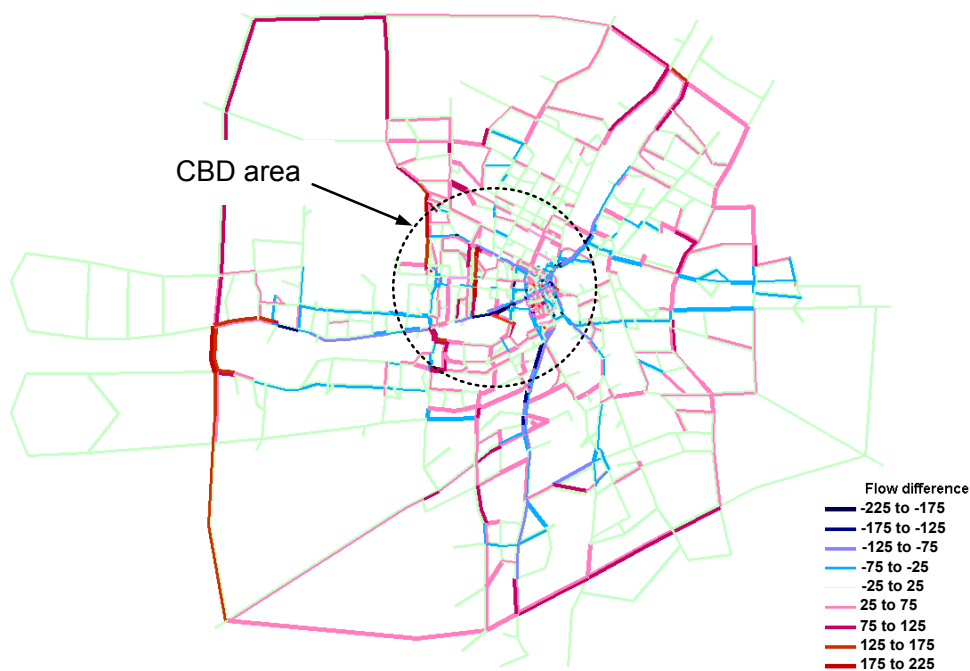


Fig. 47. Link flow difference between MNL-MNLs-SUE and MNL-PSW-SUE models

Table 19: Mode share comparison between MNL-MNLs-SUE and MNL-PSW-SUE models

Mode	MNL-MNLs-SUE			MNL-PSW-SUE		
	All areas	CBD	Non-CBD	All areas	CBD	Non-CBD
<b>Auto</b>	99.63% (54258)	99.48% (28756)	99.80% (25501)	99.26% (54056)	98.82% (28568)	99.75% (25488)
<b>Bike</b>	0.37% (201)	0.52% (151)	0.20% (50)	0.74% (403)	1.18% (340)	0.25% (63)

\*(): number of trips

## 5.6 Conclusions

This paper proposed a new model, called the path-size weibit (PSW) SUE model with elastic demand (ED), with application to the combined modal split and traffic assignment (CMSTA) problem. This new model was derived based on the Weibull distribution to handle both route overlapping and heterogeneous perception variance problems, which is lack in the widely used multinomial logit (MNL) model with Gumbel distribution. Specifically, the route overlapping is handled through the path-size factor, and the route-specific perception variance is handled through the Weibull distribution. In addition, the travel choice is determined based on the elastic demand function that explicitly considers the network level of service based on the logarithmic expected perceived cost of the Weibull distribution to determine the travel demand.

Through the numerical result and Winnipeg network studies, we observed that the PSW-SUE model with ED is capable of handling the overlapping and route-specific perception variance problems, and it could produce a compatible result to the multinomial probit model. When considering the motorized and non-motorized modes

(i.e., auto and bike), the proposed model gave a larger non-motorized travel demand especially in the downtown area, compared to the MNL-based model. For future research, we will consider other random error distributions (which also have a closed-form probability expression), and perform parameter calibration. Further, equivalent mathematical programming formulation for these models under congested networks should be developed for large-scale applications.

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## CHAPTER 6

### CONCLUDING REMARKS

#### 6.1 Summary

In this study, we provided a *closed-form* (probabilistic) route choice model and its mathematical programming (MP) formulations as an alternative to relax the *independently and identically* distributed (IID) assumption embedded in the classical multinomial logit (MNL) model (Dial, 1971) under the stochastic user equilibrium (SUE) framework (Fisk, 1980; Sheffi and Powell, 1982). A *path-size* factor (Ben-Akiva and Bierlaire, 1999) was adopted to modify the multinomial *weibit* (MNW) model (Castillo et al., 2008) to create the path-size weibit (PSW) model (see Chapter 3). Specifically, the *route overlapping* is handled through the path-size factor, and the *route-specific perception variance* is handled through the *Weibull* distribution. A *multiplicative* Beckmann's transformation (MBec) was used to develop a constrained entropy-type MP formulation and an unconstrained MP formulation for the PSW-SUE model. The MBec is used to handle the *relative* cost difference criterion of the weibit model for the constrained entropy-type PSW-SUE model. Meanwhile, it is used to handle the PSW expected perceived travel cost (EPC) for the unconstrained PSW-SUE model. Qualitative properties of these minimization programs were given to establish equivalency and uniqueness conditions. Path-based and link-based algorithms were provided for solving the constrained entropy-type and unconstrained MP formulations, respectively. Through the numerical examples, we observed the results as follows.

- The PSW-SUE model can account for both route overlapping and route-specific perception variance problems under congestion, and it can produce a compatible

traffic flow pattern compared to the MNP-SUE model.

- The elastic demand and mode choice patterns of the PSW-SUE flow can be significantly different from the logit-based model. This is because the logarithmic PSW EPC explicitly incorporates both route overlapping and route-specific perception variance problems in determining the network level of service. Therefore, the PSW-SUE model results a smaller demand for motorized vehicles.
- The proposed MP formulations can be implemented in a real-size transportation network as shown by the application of the Winnipeg network in Canada.

## 6.2 Possible drawbacks of the weibit models

The MNW model satisfies the IIA property. The choice probability ratio of any two routes is entirely unaffected by the travel cost of other routes:

$$\frac{P_r^{ij}}{P_l^{ij}} = \frac{(g_r^{ij} - \zeta^{ij})^{-\beta^{ij}} / \sum_{k \in R_{ij}} (g_k^{ij} - \zeta^{ij})^{-\beta^{ij}}}{(g_l^{ij} - \zeta^{ij})^{-\beta^{ij}} / \sum_{k \in R_{ij}} (g_k^{ij} - \zeta^{ij})^{-\beta^{ij}}} = \frac{(g_r^{ij} - \zeta^{ij})^{-\beta^{ij}}}{(g_l^{ij} - \zeta^{ij})^{-\beta^{ij}}}, \forall l \neq r \in R_{ij}, ij \in IJ. \quad (206)$$

This is because the MNW model is derived from the independent extreme value distribution (Dagsvik, 1983).

Further, since the weibit models use the relative cost difference criterion to determine the choice probability, these models may be insensitive to an arbitrary multiplier on the route cost. We use another two-route network in Fig. 48 to illustrate this drawback. The upper route of these two networks is twice longer than the lower route. For the short network, the upper route is longer than the lower route by 5 units; however, for the long network, the upper route is 50 units longer.

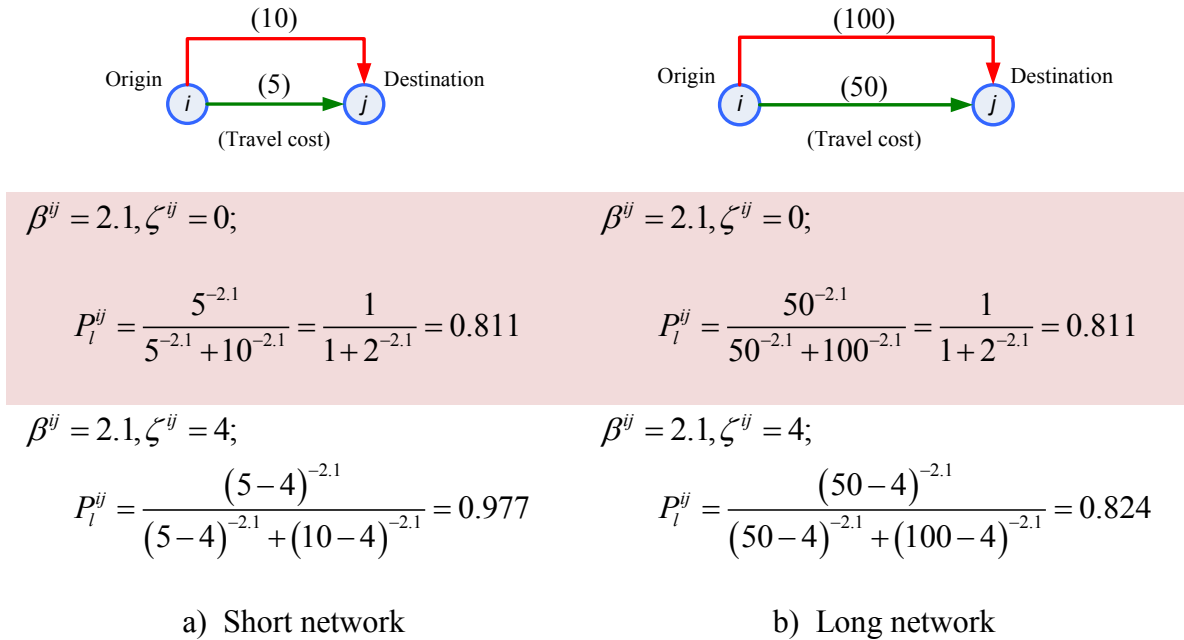


Fig. 48. Insensitive to an arbitrary multiplier on the route cost of the MNW model

With  $\zeta^{ij} = 0$ , the MNW model gives the same result for both short and long networks. Note that the PSW model also has this drawback when  $\zeta^{ij} = 0$  (in fact, it is depended on the specification of the path-size factor). Nonetheless, with a positive  $\zeta^{ij}$ , the drawback can be alleviated to a certain extent as shown in Fig. 48.

### 6.3 Future study

#### 6.3.1 Model calibration

This study focused on the theoretical properties of the weibit model and its MP formulations. Empirical studies, however, were not conducted. To implement the weibit model in a real-case study, a field traffic survey is necessary to observe travelers' route choice decision. Since the weibit model has a closed-form solution, a widely used maximum likelihood method can be adopted for estimating the model parameters.

### 6.3.2 *Incorporating the location parameter*

To develop the MP formulations for the weibit SUE model, we need to assume that the Weibull location parameter  $\zeta^{ij} = 0$  to obtain a decomposable travel cost at the link level. However, incorporating  $\zeta^{ij}$  can enhance modeling flexibility in the route choice problem. It allows modelers to incorporate the route-specific coefficient of variation (CV). With non-zero  $\zeta^{ij}$ , we have not only a higher perception variance for a longer route, but also a larger route CV for a longer route. Incorporating the non-zero  $\zeta^{ij}$  in the SUE framework can be achieved using the variational inequality (VI) formulation such as the one used in [Zhou et al. \(2012\)](#) to formulate the congestion-based C-logit SUE model.

### 6.3.3 *Incorporating both route overlapping and route-specific perception variance in the random error term*

Since this study create the PSW model by modifying the *deterministic* term of the MNW random utility maximization (RUM) model to resolve the overlapping issue, we may create another weibit route choice model by modifying the *random* error term of the MNW RUM model. Developing a joint Weibull distribution that allows the covariance between route pairs is necessary to obtain a closed-form weibit route choice model to handle both route overlapping and route-specific perception variance. One may adopt the *copula* (e.g., [Nelsen, 2006](#)), which provides a general framework to construct a joint distribution with unrestrictive marginal distributions, to develop such a joint weibull distribution.

Since the GEV theory is related to the copula (e.g., [Bhat, 2009](#); [Fosgerau et al., 2013](#)), using the inversion method we can obtain an extreme value copula ([Nelsen, 2006](#)). For example, from the paired combinatorial logit (PCL) generating function, we have the

weibit model with the route pair nest at the upper level and the individual route nest at the lower level, where the route-specific perception variance is handled through the Weibull distribution. This weibit model can be expressed as

$$P_r^{ij} = \frac{\sum_{l \neq r} (\mathbf{g}_r^{ij} - \zeta^{ij})^{-\frac{\beta^{ij}}{1-\nu_{ijrl}} (1-\nu_{ijrl})} \left( (\mathbf{g}_r^{ij} - \zeta^{ij})^{-\frac{\beta^{ij}}{1-\nu_{ijrl}} + (\mathbf{g}_l^{ij} - \zeta^{ij})^{-\frac{\beta^{ij}}{1-\nu_{ijrl}}} \right)^{-\nu_{ijrl}}}{\sum_{k=1}^{|R_{ij}|-1} \sum_{m=k+1}^{|R_{ij}|} (1-\nu_{ijkm}) \left( (\mathbf{g}_k^{ij} - \zeta^{ij})^{-\frac{\beta^{ij}}{1-\nu_{ijkm}} + (\mathbf{g}_m^{ij} - \zeta^{ij})^{-\frac{\beta^{ij}}{1-\nu_{ijkm}}} \right)^{1-\nu_{ijkm}}}, \quad (207)$$

$$\forall r \in R_{ij}, ij \in IJ.$$

Its logarithmic EPC can be written as

$$\bar{\mu}_{ij} = -\frac{1}{\beta^{ij}} \ln \sum_{k=1}^{|R_{ij}|-1} \sum_{m=k+1}^{|R_{ij}|} (1-\nu_{ijkm}) \left( (\mathbf{g}_k^{ij} - \zeta^{ij})^{-\frac{\beta^{ij}}{1-\nu_{ijkm}} + (\mathbf{g}_m^{ij} - \zeta^{ij})^{-\frac{\beta^{ij}}{1-\nu_{ijkm}}} \right)^{1-\nu_{ijkm}}, \quad \forall ij \in IJ. \quad (208)$$

With the probability expression in Eq. (207), an entropy-type MP formulation for this route choice model can be written as

$$\begin{aligned} \min Z &= Z_1 + Z_2 + Z_3 \\ &= \sum_{a \in A} \int_0^{v_a = \sum_{ij \in IJ} \sum_{r \in R_{ij}} f_r^{ij} \delta_{ar}^{ij}} \ln \tau_a(\omega) d\omega + \frac{1}{\theta} \sum_{ij \in IJ} \sum_{r \in R_{ij}} \sum_{\substack{k \neq r \\ r \in R_{ij}}} (1-\nu_{ijrk}) f_{r(rk)}^{ij} \left( \ln \frac{f_{r(rk)}^{ij}}{(1-\nu_{ijrk})} - 1 \right) \\ &\quad - \frac{1}{\theta} \sum_{ij \in IJ} \sum_{r=1}^{|R_{ij}|-1} \sum_{k=r+1}^{|R_{ij}|} \nu_{ijrk} (f_{r(rk)}^{ij} + f_{k(rk)}^{ij}) \left( \ln \frac{f_{r(rk)}^{ij} + f_{k(rk)}^{ij}}{(1-\nu_{ijrk})} - 1 \right) \end{aligned} \quad (209)$$

$$\text{s.t. } \sum_{r \in R_{ij}} \sum_{\substack{k \neq r \\ r \in R_{ij}}} f_{r(rk)}^{ij} = q_{ij}, \quad \forall ij \in IJ, \quad (210)$$

$$f_{r(rk)}^{ij} \geq 0, \quad \forall r \neq k \in R_{ij}, ij \in IJ. \quad (211)$$

In addition, we can adopt the logarithmic EPC in Eq. (208) to construct an unconstrained MP formulation through:



$$\begin{aligned}
\min Z &= Z_1 + Z_2 + Z_3 \\
&= -\sum_{a \in A} \int_0^{v_a} \ln \tau_a(\omega) d\omega - \sum_{ij \in IJ} q_{ij} \bar{\mu}_{ij} + \sum_{a \in A} v_a \ln \tau_a(v_a).
\end{aligned} \tag{212}$$

Following the same principle, we can develop other weibit models and their MP formulations by using the CNL and GNL generating functions.

#### 6.3.4 Model extension

In Chapter 4 and Chapter 5, we provided several extensions to consider the demand elasticity, modal split and assignment, and multi-user classes. One may further extend the PSW-SUE formulation to consider other choice dimensions with multi-user classes. For example, the combined trip generation, trip distribution, modal split, and traffic assignment according to the PSW-SUE framework (with multi-user classes). On the other hand, to formulate the MP formulation, the path-size factor is assumed to be flow independent. In other words, a length-based path-size factor is adopted to consider the route overlapping. To relax this assumption, we can adopt the VI problem to consider a flow dependent path-size factor to reflect a congestion-based route similarity (e.g., [Zhou et al., 2012](#)).

#### 6.3.5 Algorithmic enhancements

Since this study adopted the principle of partial linearization algorithm for solving the proposed models, alternative algorithms to improve the computational cost should be provided in the future. For example, one may adopt the gradient projection (GP) algorithm ([Bertsekas, 1976](#)). The GP algorithm has been shown as a successful path-based algorithm for solving the DUE problem ([Chen et al., 2002](#)). It adopts the diagonal inverse Hessian approximation as a scaling matrix and uses the “one-at-a-time” flow

update strategy to obtain a modest computational effort. These features make the GP algorithm computationally more efficient than the partial linearization algorithm for both additive cost (i.e., the route cost is separable into the link level) and non-additive cost (i.e., the route cost is not separable into the link level (see [Chen et al., 2002](#); [Zhou et al., 2012](#)).

### 6.3.6 *Modeling uncertainty*

This study considered only the subjective uncertainty from travelers' perception of travel costs. However, there are several uncertainties surrounding the transportation network from both demand and supply sides. We may incorporate the objective uncertainty from demand side such as daily O-D demand fluctuations and supply side such as adverse weather conditions in future research. By using the travel time budget (TTB) ([Lo et al., 2006](#)) or the mean-excess travel time framework ([Chen and Zhou, 2010](#)), one may develop the PSW traffic equilibrium model under uncertainty.

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APPENDICES

## Appendix A

## Gumbel and Weibull distributions

Distribution	Gumbel	Weibull
<b>CDF</b> $F_{G_r^{ij}}(t)$	$1 - \exp\left\{-e^{\frac{\theta_r^{ij}(t-\lambda_r^{ij})}{\theta_r^{ij}}}\right\}$ $t \in (-\infty, \infty)$	$1 - \exp\left\{-\left[\frac{(t-\zeta^{ij})}{\varphi_r^{ij}}\right]^{\beta^{ij}}\right\}$ $t - \zeta^{ij} > 0$
<b>Mean</b> $g_r^{ij}$	$\lambda_r^{ij} - \frac{\gamma}{\theta_r^{ij}}$	$\zeta^{ij} + \varphi_r^{ij} \Gamma\left(1 + \frac{1}{\beta^{ij}}\right)$
<b>Variance</b> $(\sigma_r^{ij})^2$	$\frac{\pi^2}{6\theta_r^{ij^2}}$	$(\varphi_r^{ij})^2 \Gamma\left(1 + \frac{2}{\beta^{ij}}\right) - (g_r^{ij} - \zeta^{ij})^2$

Remark:  $\lambda_r^{ij}$  is the Gumbel location parameter;  $\theta_r^{ij}$  is the Gumbel scale parameter;  $\gamma$  is the Euler constant.

## Appendix B

$\bar{\mu}_{ij}$  is monotonically decreasing w.r.t. the number of routes:

$$\bar{\mu}_{ij}\left(g_1^{ij}, \dots, g_{|R_{ij}|+1}^{ij}\right) \leq \bar{\mu}_{ij}\left(g_1^{ij}, \dots, g_{|R_{ij}|}^{ij}\right), \quad \forall ij \in IJ. \quad (213)$$

This is because  $(g_r^{ij} - \zeta^{ij})^{-\beta^{ij}} > 0, \forall r \in R_{ij}, ij \in IJ$ . Further,  $\bar{\mu}_{ij}$  is concave w.r.t. the vector of  $\ln(g_r^{ij} - \zeta^{ij})$ . The Hessian matrix of  $\bar{\mu}_{ij}$  w.r.t. the vector  $\ln(g_r^{ij} - \zeta^{ij})$  can be expressed as

$$\nabla^2 \bar{\mu}_{ij} = -\frac{1}{\beta^{ij}} \left( \frac{1}{\mathbf{1}^T \mathbf{z}} \text{diag}(\mathbf{z}) - \frac{1}{(\mathbf{1}^T \mathbf{z})^2} \mathbf{z} \mathbf{z}^T \right), \forall ij \in IJ, \quad (214)$$

where  $\mathbf{z}$  is the vector of  $z_r^{ij} = \exp[-\beta^{ij} (g_r^{ij} - \zeta^{ij})]$ , and  $T$  is the transpose operator. Let  $\mathbf{v}$  be a vector of non-zero elements  $v_r^{ij}$ . Then,  $\mathbf{v}^T \nabla^2 \bar{\mu}_{ij} \mathbf{v}$  can be expanded to

$$\mathbf{v}^T \nabla^2 \bar{\mu}_{ij} \mathbf{v} = -\frac{1}{\beta^{ij}} \frac{\left( \sum_{r \in R_{ij}} z_r^{ij} v_r^{ij^2} \right) \left( \sum_{r \in R_{ij}} z_r^{ij} \right) - \left( \sum_{r \in R_{ij}} z_r^{ij} v_r^{ij} \right)^2}{\left( \sum_{r \in R_{ij}} z_r^{ij} \right)^2}, \forall ij \in IJ. \quad (215)$$

From the Cauchy-Schwarz inequality,  $\left( \sum_{r \in R_{ij}} z_r^{ij} v_r^{ij^2} \right) \left( \sum_{r \in R_{ij}} z_r^{ij} \right) - \left( \sum_{r \in R_{ij}} z_r^{ij} v_r^{ij} \right)^2 \geq 0$ . Since

$\left( \sum_{r \in R_{ij}} z_r^{ij} \right)^2 \geq 0$ , the above Hessian matrix is negative semi-definite; hence, this logarithmic

MNW EPC is concave w.r.t. the vector  $\ln(g_r^{ij} - \zeta^{ij})$ .

## Appendix C

This appendix is to demonstrate how Assumption 4.3 relates to the *Markov process* (or *Markov chain*) under the weibit route choice decision. According to the MNW model, the probability of choosing a succeeding node can be expressed as

$$P_{p \rightarrow q}^{ij} = \frac{\left( \tau_{a(p \rightarrow q)} \mu^{qj} \right)^{-\beta^{ij}}}{\left( \mu^{pj} \right)^{-\beta^{ij}}}, \forall r \in \bar{R}_{qj}, k \in \bar{R}_{pj}, ij \in IJ, \quad (216)$$

where  $i$  is the origin node,  $j$  is the destination node,  $p$  denotes the current node in which travelers are about to leave,  $q$  denotes the succeeding node in which travelers are about to

go,  $\tau_{a(p \rightarrow q)}$  is the travel time on link  $a$  whose head node is  $p$  and tail node is  $q$ ,  $\mu^{qj}$  is the MNW EPC in Eq. (134) between node  $q$  to destination  $j$ , and  $\bar{R}_{qj}$  is the set of routes between node  $q$  to destination  $j$ . Eq. (216) indicates that travelers are assumed to make a decision at each node from available routes following node  $q$  and available routes following the current node  $p$  using the MNW EPC as shown in Fig. 49.

We use the Braess network in Fig. 50 to show that Eq. (216) leads to the MNW route choice probability. From Assumption 4.3, the probability of choosing from node 1 to node 2 can be expressed as

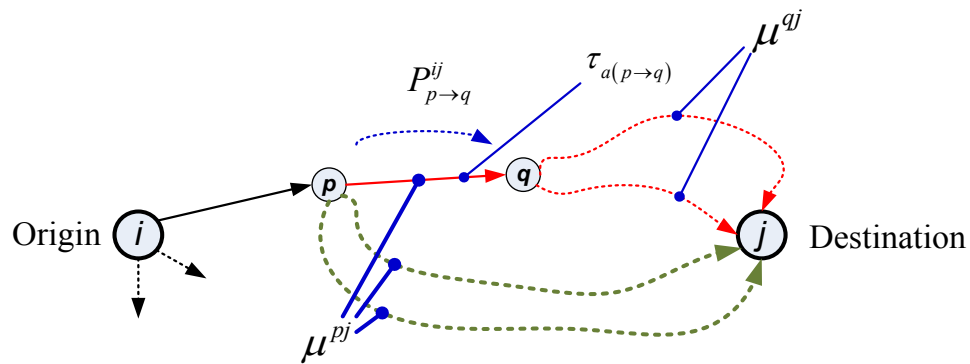


Fig. 49. Conceptual framework for the node-to-node weibit choice behavior

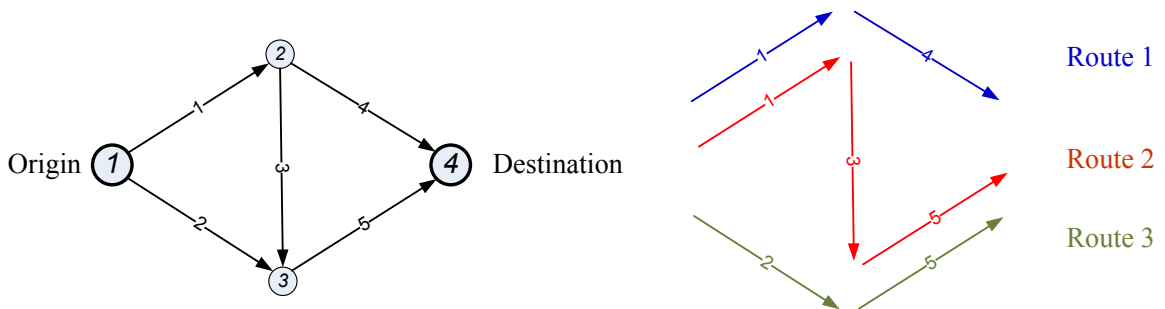


Fig. 50. Braess network and its available routes

$$\begin{aligned}
P_{1 \rightarrow 2}^{14} &= \frac{\left( \tau_1 \left( \tau_4^{-\beta^{14}} + (\tau_3 \tau_5)^{-\beta^{14}} \right)^{-\frac{1}{\beta^{14}}} \right)^{-\beta^{14}}}{\left( \left( (\tau_1 \tau_4)^{-\beta^{14}} + (\tau_1 \tau_3 \tau_5)^{-\beta^{14}} + (\tau_2 \tau_5)^{-\beta^{14}} \right)^{-\frac{1}{\beta^{14}}} \right)^{-\beta^{14}}}, \\
&= \frac{(\tau_1 \tau_4)^{-\beta^{14}} + (\tau_1 \tau_3 \tau_5)^{-\beta^{14}}}{(\tau_1 \tau_4)^{-\beta^{14}} + (\tau_1 \tau_3 \tau_5)^{-\beta^{14}} + (\tau_2 \tau_5)^{-\beta^{14}}}.
\end{aligned} \tag{217}$$

Following the same principle, we have

$$P_{1 \rightarrow 3}^{14} = \frac{(\tau_2 \tau_5)^{-\beta^{14}}}{(\tau_1 \tau_4)^{-\beta^{14}} + (\tau_1 \tau_3 \tau_5)^{-\beta^{14}} + (\tau_2 \tau_5)^{-\beta^{14}}}, \tag{218}$$

$$P_{2 \rightarrow 3}^{14} = \frac{(\tau_3 \tau_5)^{-\beta^{14}}}{(\tau_4)^{-\beta^{14}} + (\tau_3 \tau_5)^{-\beta^{14}}}, \tag{219}$$

$$P_{2 \rightarrow 4}^{14} = \frac{(\tau_4)^{-\beta^{14}}}{(\tau_4)^{-\beta^{14}} + (\tau_3 \tau_5)^{-\beta^{14}}}, \tag{220}$$

$$P_{3 \rightarrow 4}^{14} = 1. \tag{221}$$

Assuming that the decision is independent of the preceding decision (on the previous node), the choice probability of each route can be determined from

$$P_r^{ij} = \prod_{pq} (P_{p \rightarrow q}^{ij})^{\delta_{r,pq}^{ij}}, \tag{222}$$

where  $\delta_{r,pq}^{ij}$  equals to 1 if nodes  $p$  and  $q$  are on route  $r$  between O-D pair  $ij$  and 0 otherwise. From Eq. (222), we have the choice probability for route 1 as



$$\begin{aligned}
P_1^{14} &= P_{1 \rightarrow 2}^{14} \square \dots \\
&= \frac{(\tau_1 \tau_4)^{-\beta^{14}} + (\tau_1 \tau_3 \tau_5)^{-\beta^{14}}}{(\tau_1 \tau_4)^{-\beta^{14}} + (\tau_1 \tau_3 \tau_5)^{-\beta^{14}} + (\tau_2 \tau_5)^{-\beta^{14}}} \frac{(\tau_4)^{-\beta^{14}}}{(\tau_4)^{-\beta^{14}} + (\tau_3 \tau_5)^{-\beta^{14}}} \\
&= \frac{(\tau_1 \tau_4)^{-\beta^{14}}}{(\tau_1 \tau_4)^{-\beta^{14}} + (\tau_1 \tau_3 \tau_5)^{-\beta^{14}} + (\tau_2 \tau_5)^{-\beta^{14}}} \\
&= \frac{(g_1^{14})^{-\beta^{14}}}{\sum_{k \in R_{14}} (g_k^{14})^{-\beta^{14}}},
\end{aligned} \tag{223}$$

which corresponds to the MNW probability in Eq. (126). Using the same principle, we have the MNW choice probability of each route. With this, if we assume that each node is a “state”, and travelers’ movement between two adjacent nodes corresponds to the transition of the state according to  $P_{p \rightarrow q}^{ij}$ , we have Akamatsu (1996)’s Markov chain from node to node weibit choice behavior as shown in Fig. 51.

## Appendix D

Proof of Proposition 4.1. The partial derivative of  $Z_1$  can be expressed as:

$$\begin{aligned}
\frac{\partial Z_1}{\partial v_a} &= \frac{\partial \sum_{a \in A} \int_0^{v_a} \ln \tau_a(\omega) d\omega}{\partial v_a} \\
&= \ln \tau_a.
\end{aligned} \tag{224}$$

The partial derivative of  $Z_2$  w.r.t.  $v_a$  is as follows:

$$\begin{aligned}
\frac{\partial Z_2}{\partial v_a} &= \sum_{ij \in IJ} q_{ij} \sum_{r \in R_{ij}} \frac{\partial \left\{ -\frac{1}{\beta} \ln \sum_{k \in R_{ij}} (g_k^{ij})^{-\beta^{ij}} \right\}}{\partial v_a} \\
&= \sum_{ij \in IJ} q_{ij} \sum_{r \in R_{ij}} \frac{\partial \left\{ -\frac{1}{\beta} \ln \sum_{k \in R_{ij}} (g_k^{ij})^{-\beta^{ij}} \right\}}{\partial \ln g_r^{ij}} \frac{\partial \ln g_r^{ij}}{\partial v_a}.
\end{aligned} \tag{225}$$

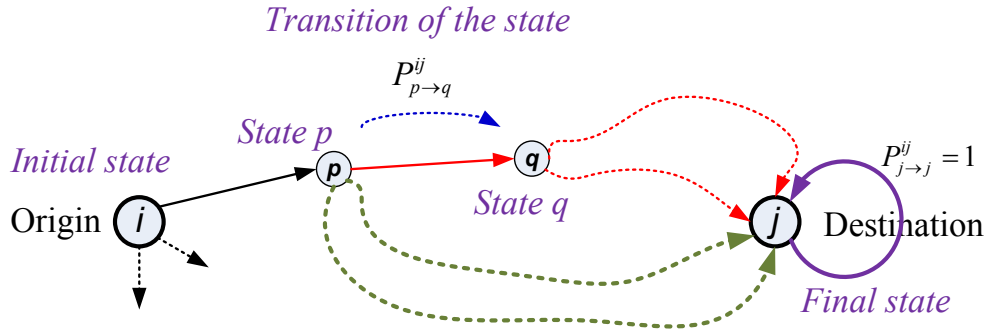


Fig. 51. Markov chain and the node-to-node weibit choice behavior

From Assumption 4.3, Eq. (225) can be restated as:

$$\begin{aligned} \frac{\partial Z_2}{\partial v_a} &= \sum_{ij \in IJ} q_{ij} \sum_{r \in R_{ij}} \frac{\partial \left\{ -\frac{1}{\beta} \ln \sum_{k \in R_{ij}} (g_k^{ij})^{-\beta^{ij}} \right\}}{\partial \ln g_r^{ij}} \frac{\partial \left\{ \ln \prod_{a \in Y_r} \tau_a \right\}}{\partial v_a} \\ &= \sum_{ij \in IJ} q_{ij} \sum_{r \in R_{ij}} \frac{(g_r^{ij})^{-\beta^{ij}}}{\sum_{k \in R_{ij}} (g_k^{ij})^{-\beta^{ij}}} \frac{\partial \left\{ \sum_{a \in A} \ln \tau_a \delta_{ar}^{ij} \right\}}{\partial v_a}, \end{aligned} \quad (226)$$

where  $\delta_{ar}^{ij}$  equals to 1 for link  $a$  on route  $r$  between O-D pair  $ij$  and 0 otherwise. From Assumption 4.1, Eq. (226) reduces to

$$\frac{\partial Z_2}{\partial v_a} = \sum_{ij \in IJ} \sum_{r \in R_{ij}} q_{ij} P_r^{ij} \delta_{ar}^{ij} \frac{\partial \ln \tau_a}{\partial v_a}. \quad (227)$$

The partial derivative of  $Z_3$  w.r.t.  $v_a$  can be expressed as

$$\begin{aligned} \frac{\partial Z_3}{\partial v_a} &= \frac{\partial \left\{ \sum_{a \in A} v_a \ln \tau_a \right\}}{\partial v_a} \\ &= v_a \frac{\partial \ln \tau_a}{\partial v_a} + \ln \tau_a. \end{aligned} \quad (228)$$

Then, the KKT condition is (the logarithmic link cost from  $Z_1$  cancels out the logarithmic link cost from  $Z_3$ )

$$\begin{aligned}
& -\frac{\partial Z_1}{\partial v_a} - \frac{\partial Z_2}{\partial v_a} + \frac{\partial Z_3}{\partial v_a} = 0 \\
& -\ln \tau_a - \sum_{ij \in IJ} \sum_{r \in R_{ij}} q_{ij} P_r^{ij} \delta_{ar}^{ij} \frac{\partial \ln \tau_a}{\partial v_a} + v_a \frac{\partial \ln \tau_a}{\partial v_a} + \ln \tau_a = 0 \\
& \left[ -\sum_{ij \in IJ} \sum_{r \in R_{ij}} q_{ij} P_r^{ij} \delta_{ar}^{ij} + v_a \right] \frac{\partial \ln \tau_a}{\partial v_a} = 0
\end{aligned} \tag{229}$$

From Assumption 4.1,  $\partial \ln \tau_a / \partial v_a$  is greater than zero. Then, we have

$$v_a = \sum_{ij \in IJ} \sum_{r \in R_{ij}} q_{ij} P_r^{ij} \delta_{ar}^{ij}, \tag{230}$$

which expresses the equilibrium link flows corresponding to the MNW model. This completes the proof.  $\square$

## Appendix E

Proof of Proposition 4.2. Following Sheffi (1985), the Hessian of the unconstrained program Eq. (61) can be expressed as

$$\frac{\partial^2 Z}{\partial v_a \partial v_b} = \begin{cases} -\sum_{ij \in IJ} q_{ij} \sum_{r \in R_{ij}} \sum_{k \in R_{ij}} \frac{\partial P_r^{ij}}{\partial \ln g_k^{ij}} \left( \frac{d \ln \tau_b}{dv_b} \delta_{br}^{ij} \right) \left( \frac{d \ln \tau_b}{dv_b} \delta_{bk}^{ij} \right) \\ + \frac{d \ln \tau_b}{dv_b} + \left[ -\sum_{ij \in IJ} \sum_{r \in R_{ij}} q_{ij} P_r^{ij} \delta_{ar}^{ij} + v_b \right] \frac{d^2 \ln \tau_b}{dv_b^2} & ; a = b \\ -\sum_{ij \in IJ} q_{ij} \sum_{r \in R_{ij}} \sum_{k \in R_{ij}} \frac{\partial P_r^{ij}}{\partial \ln g_k^{ij}} \left( \frac{d \ln \tau_a}{dv_a} \delta_{ak}^{ij} \right) \left( \frac{d \ln \tau_b}{dv_b} \delta_{br}^{ij} \right) & ; a \neq b \end{cases} \tag{231}$$

The Hessian matrix Eq. (231) can be expressed as the sum of three separate matrices:

$$\begin{aligned}
\nabla^2 Z(\mathbf{v}) &= \sum_{ij} q_{ij} \left[ (\nabla_{\mathbf{v}} \ln \boldsymbol{\tau} \cdot \Delta^{ij}) \cdot (-\nabla_{\ln \mathbf{g}} \mathbf{P}^{ij}) \cdot (\nabla_{\mathbf{v}} \ln \boldsymbol{\tau} \cdot \Delta^{ij})^T \right] \\
&+ \nabla_{\mathbf{v}} \ln \boldsymbol{\tau} + \nabla_{\mathbf{v}}^2 \ln \boldsymbol{\tau} \cdot \mathbf{B}.
\end{aligned} \tag{232}$$

This Hessian is an  $|A| \times |A|$  matrix. The matrix  $\nabla_{\mathbf{v}} \ln \boldsymbol{\tau}$  is the  $|A| \times |A|$  Jacobian of the log-multiplicative-link-travel-cost vector with the elements  $d \ln \tau_a / dv_a$ .  $\Delta^{ij}$  represents the

incidence matrix  $[\delta_{ar}^{ij}]$ . The Jacobian  $\nabla_{\ln \mathbf{g}} \mathbf{P}^{ij}$  is a  $|R_{ij}| \times |R_{ij}|$  matrix with the elements  $\partial P_r^{ij} / \partial \ln g_t^{ij}$ . According to Assumption 1,  $\nabla_v^2 \ln \boldsymbol{\tau}$  is a diagonal  $|A| \times |A|$  matrix which includes elements  $d^2 \ln \tau_a / dv_a^2$ .  $\mathbf{B}$  is a diagonal  $|A| \times |A|$  matrix, the  $a^{\text{th}}$  element of which is

$$-\sum_{ij \in IJ} \sum_{r \in R_{ij}} q_{ij} P_r^{ij} \delta_{ar}^{ij} + v_a. \quad (233)$$

To begin with, we consider the first matrix of Eq. (232). The matrix  $\nabla_{\ln \mathbf{g}} \mathbf{P}^{ij}$  is negative semi-definite since it is the Hessian of  $\bar{\mu}_{ij}$ , which is concave in  $\ln g_r^{ij}$ . Consequently,  $(-\nabla_{\ln \mathbf{g}} \mathbf{P}^{ij})$  is a positive-semidefinite matrix. The first matrix of Eq. (232) includes a quadratic form (of nonzero terms) applied to a positive-semidefinite matrix. It is therefore positive semidefinite.

The second matrix  $\nabla_v \ln \boldsymbol{\tau}$  is a diagonal matrix with positive entries. From Assumption 4.1, this matrix is positive definite. The third matrix is the product of two separate matrices. From Assumption 4.1,  $\nabla_v^2 \ln \boldsymbol{\tau}$  is a diagonal matrix with positive terms. Meanwhile matrix  $\mathbf{B}$  includes Eq. (233) along its diagonal. These terms can be either positive or negative. Thus,  $\nabla_v^2 \ln \boldsymbol{\tau} \cdot \mathbf{B}$  is not positive definite. However, when

approaching the equilibrium point,  $\left( -\sum_{ij \in IJ} \sum_{r \in R_{ij}} q_{ij} P_r^{ij} \delta_{ar}^{ij} + v_a \right) \rightarrow 0$ , and the last matrix

$\nabla_v^2 \ln \boldsymbol{\tau} \cdot \mathbf{B}$  vanishes. This implies that at the point that satisfies the SUE conditions, this unconstrained program is strictly convex (i.e., the Hessian at this point is positive definite). Note that the Hessian matrix of  $Z$  is indefinite at all other points. The only

conclusion is that the equilibrium point is a local minimum. It cannot conclude that other local minima do not exist.

To prove that the SUE solution is the only minimum of this unconstrained SUE problem, the dual variable is adopted. Let  $v_a(\ln \tau_a)$  be the inverse of  $\ln \tau_a(v_a)$ ; this inverse exists because of Assumption 1.  $v_a(\ln \tau_a)$  is increasing for all positive  $\tau_a$  and positive for all  $\tau_a > \tau_a^0$ , where  $\tau_a^0$  is the free flow travel cost on link  $a$ . Then, the objective function becomes

$$Z(\ln \boldsymbol{\tau}) = -\sum_{a \in A} \int_{\ln \tau_a^0}^{\ln \tau_a} \omega \frac{dv_a(\omega)}{d\omega} d\omega - \sum_{ij \in IJ} q_{ij} \bar{\mu}_{ij} + \sum_{a \in A} v_a(\ln \tau_a) \ln \tau_a. \quad (234)$$

After integrating by parts, we have

$$Z(\ln \boldsymbol{\tau}) = \sum_{a \in A} \int_{\ln \tau_a^0}^{\ln \tau_a} v_a(\omega) d\omega - \sum_{ij \in IJ} q_{ij} \bar{\mu}_{ij}. \quad (235)$$

The gradient of  $Z(\ln \boldsymbol{\tau})$  is given by

$$\frac{\partial Z(\ln \boldsymbol{\tau})}{\partial \ln \tau_a} = v_a - \sum_{ij \in IJ} \sum_{r \in R_{ij}} q_{ij} P_r^{ij} \delta_{ar}^{ij}. \quad (236)$$

This gradient of  $Z(\ln \boldsymbol{\tau})$  would always have the same sign and vanish at the same point as the gradient of  $Z(\mathbf{v})$ . Then, the Hessian of  $Z(\ln \boldsymbol{\tau})$  is given by

$$\nabla^2 Z(\ln \boldsymbol{\tau}) = \nabla_{\ln \boldsymbol{\tau}} \mathbf{v} + \sum_{ij} q_{ij} \left[ (\Delta^{ij}) \cdot (-\nabla_{\ln \mathbf{g}} \mathbf{P}^{ij}) \cdot (\Delta^{ij})^T \right]. \quad (237)$$

Similar to Eq. (232), the first matrix on the right hand side is semi-definite matrix. The second matrix is diagonal with nonzero terms of  $\frac{dv_a(\ln \tau_a)}{d \ln \tau_a}$ . Since  $v_a(\ln \tau_a)$  is an

increasing function,  $\nabla^2 Z(\ln \boldsymbol{\tau})$  is therefore strictly convex, having a single stationary point which is its minimum.

According to  $Z(\ln \boldsymbol{\tau})$  and  $Z(\mathbf{v})$  are related by a monotonic transformation, it is a one-to-one mapping. The gradient of  $Z(\ln \boldsymbol{\tau})$  vanishes only once at its minimum. Thus,  $Z(\mathbf{v})$  also has a unique minimum with a unique solution. Note that this unconstrained SUE problem is strictly convex at the vicinity of the minimum but not necessarily convex elsewhere. Nonetheless, its local minimum is also the global minimum. This completes the proof.  $\square$