What is the Spacetime Geometry of an Electromagnetic Wave?

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Electrovacua: Einstein-Maxwell Equations

Einstein-Maxwell Equations:

$$R_{ab} - \frac{1}{2} R\, g_{ab} = \kappa \left(F_{ac} F_b{}^c - \frac{1}{4} g_{ab} F_{cd} F^{cd} \right)$$

$$\nabla^a F_{ab} = 0 = \nabla_{[a} F_{bc]}$$

Example: Reissner-Nordstrom

Load in required packages.

```
> with(DifferentialGeometry): with(Tensor):
```

Initialize manifold and define the metric.

```
> DGsetup([t, r, theta, phi], M);
```

frame name: M **(1.1.1)**

```
M > f := 1-2*m/r + q^2/r^2;
```

$$f := 1 - \frac{2\, m}{r} + \frac{q^2}{r^2} \quad (1.1.2)$$

```
M > g := evalDG(- f*dt &t dt + 1/f*dr &t dr + r^2*(dtheta &t
    dtheta + sin(theta)^2*dphi &t dphi));
```

$$g := \frac{2\, m\, r - q^2 - r^2}{r^2}\, dt \otimes dt - \frac{r^2}{2\, m\, r - q^2 - r^2}\, dr \otimes dr + r^2\, d\theta \otimes d\theta + r^2 \sin(\theta)^2\, d\phi \otimes d\phi \quad (1.1.3)$$

Define the electromagnetic field.

```
M > F := evalDG(-q*sqrt(2)/r^2*dt &w dr);
```

$$F := -\frac{q\sqrt{2}}{r^2}\, dt \wedge dr \tag{1.1.4}$$

Verify the Einstein equations.

```
M > G := EinsteinTensor(g);
```

$$G := -\frac{q^2}{r^2\left(2\,m\,r-q^2-r^2\right)}\, D\_t \otimes D\_t + \frac{q^2\left(2\,m\,r-q^2-r^2\right)}{r^6}\, D\_r \otimes D\_r + \frac{q^2}{r^6}\, D\_\theta \otimes D\_\theta + \frac{q^2}{r^6\sin(\theta)^2}\, D\_\phi \otimes D\_\phi \tag{1.1.5}$$

```
M > T := EnergyMomentumTensor("Electromagnetic", g, F);
```

$$T := -\frac{q^2}{r^2\left(2\,m\,r-q^2-r^2\right)}\, D\_t \otimes D\_t + \frac{q^2\left(2\,m\,r-q^2-r^2\right)}{r^6}\, D\_r \otimes D\_r + \frac{q^2}{r^6}\, D\_\theta \otimes D\_\theta + \frac{q^2}{r^6\sin(\theta)^2}\, D\_\phi \otimes D\_\phi \tag{1.1.6}$$

```
M > evalDG(G - T);
```

$$0 \tag{1.1.7}$$

Verify the Maxwell Equations.

```
M > g1 := InverseMetric(g):
M > C := Christoffel(g):
```

$\nabla_a F_{bc}$

```
M > DF := CovariantDerivative(F, C);
```

$$DF := \frac{2\,q\sqrt{2}}{r^3}\, dt \otimes dr \otimes dr + \frac{\left(2\,m\,r-q^2-r^2\right) q\sqrt{2}}{r^3}\, dt \otimes d\theta \otimes d\theta + \frac{\left(2\,m\,r-q^2-r^2\right)\sin(\theta)^2\, q\sqrt{2}}{r^3}\, dt \otimes d\phi \otimes d\phi - \frac{2\,q\sqrt{2}}{r^3}\, dr \otimes dt \otimes dr - \frac{\left(2\,m\,r-q^2-r^2\right) q\sqrt{2}}{r^3}\, d\theta \otimes dt \otimes d\theta - \frac{\left(2\,m\,r-q^2-r^2\right)\sin(\theta)^2\, q\sqrt{2}}{r^3}\, d\phi \otimes dt \otimes d\phi \tag{1.1.8}$$

$\nabla^b F_{bc}$

```
M > ContractIndices(g1, DF, [[1,2], [2,3]]);
```

$$0\,dt \tag{1.1.9}$$

$\nabla_{[a} F_{bc]}$

```
M > SymmetrizeIndices(DF, [1,2,3], "SkewSymmetric");
```

$$0\,dt \otimes dt \otimes dt \tag{1.1.10}$$

Non-null electrovacua: Rainich Geometry

The Rainich Conditions:

$$R^i_h R^h_j - \frac{1}{4}\delta^i_j R^{hk} R_{hk} = 0, \qquad R^i_i = 0, \qquad R_{ij} t^i t^j > 0$$

$$\nabla_{[j}\alpha_{i]} = 0, \quad \text{where} \quad \alpha_i = \frac{\varepsilon_{ijhk} R^j_m \nabla^h R^{mk}}{R_{cd} R^{cd}}$$

- Necessary and sufficient for the existence of a non-null solution of the Einstein-Maxwell solutions.
- Proof includes a construction of the EM field.

Rainich (1924), Misner and Wheeler (1957)

```
M > MetricSearch();
```

Example: An erroneous formula from "Exact Solutions..."

Equations (12.21) and (12.22) from *Exact Solutions of Einstein's Field Equations* for a non-inheriting electrovacuum.

$$ds^2 = a^2x^{-2}\left(dx^2 + dy^2\right) + x^2 d\phi^2 - (dt - 2\,y\,d\phi)^2$$

$$F_{tx} = \frac{F_{xy}}{2\,y} = \cos\left(\frac{2\ln(x)}{x}\right),\ F_{y\phi} = -\sin(2\ln(x)).$$

• Not a solution of the Einstein-Maxwell equations.

```
> DGsetup([t, phi, x, y], M);
```

$$\textit{frame name: M} \tag{2.1.1}$$

```
M > omega := evalDG(dt - 2*y*dphi);
```

$$\omega := dt - 2\,y\,d\phi \tag{2.1.2}$$

```
M > g := evalDG(a^2*x^(-2)*(dx &t dx + dy &t dy) + x^2*dphi &t
    dphi - omega &t omega);
```

$$g := -dt \otimes dt + 2\,y\,dt \otimes d\phi + 2\,y\,d\phi \otimes dt + \left(x^2 - 4\,y^2\right) d\phi \otimes d\phi + \frac{a^2}{x^2}\,dx \otimes dx + \frac{a^2}{x^2}\,dy \otimes dy \tag{2.1.3}$$

```
M > RainichConditions(g);
```

$$\textit{true} \tag{2.1.4}$$

```
M > F0:=RainichElectromagneticField(g);
```

$$F0 := \frac{\cos(2\ln(x) + \_C1)\sqrt{2}\sqrt{\operatorname{csgn}\left(\frac{1}{a^2}\right) + 1}\ \operatorname{csgn}\left(\frac{1}{x}\right)}{x}\,dt \wedge dx - \frac{2\cos(2\ln(x) + \_C1)\sqrt{2}\sqrt{\operatorname{csgn}\left(\frac{1}{a^2}\right) + 1}\ \operatorname{csgn}\left(\frac{1}{x}\right) y}{x}\,d\phi \wedge dx \tag{2.1.5}$$

$$-\frac{\sin(2\ln(x)+\_C1)\sqrt{\frac{x^2}{a^4}}\,a^2\sqrt{2}\,\sqrt{\operatorname{csgn}\left(\frac{1}{a^2}\right)+1}\,\operatorname{csgn}\left(\frac{1}{x}\right)}{x}\,d\phi\wedge dy$$

```
M > F := simplify(F0) assuming a > 0, x > 0;
```

$$F := \frac{2\cos(2\ln(x)+\_C1)}{x}\,dt\wedge dx - \frac{4\cos(2\ln(x)+\_C1)\,y}{x}\,d\phi\wedge dx - 2\sin(2\ln(x)+\_C1)\,d\phi\wedge dy \tag{2.1.6}$$

• The "Exact solutions..." EM field is off by some signs, factors of 2, and component labelings.

$$F_{tx} = \frac{F_{xy}}{2y} = 2\cos\left(\frac{2\ln(x)}{x}\right),\ F_{y\phi} = -2\sin(2\ln(x))$$

Null Electrovacuum Geometry

Null Maxwell Field: $F_{ab}F^{ab} = 0 = F_{ab} \star F^{ab}$

Electrovacuum: $R_{ab}R^{bc} = 0.$

Are there analogs of the Rainich conditions for solutions of the Einstein-Maxwell equations with a null electromagnetic field?

cf. Jordan and Kundt (1961), Robinson (1961), Geroch (1966), Ludwig (1970).

Main Result

Theorem: *Necessary and sufficient conditions on a metric for it to define a null electrovacuum.*

• $R_{ab}R^{bc} = 0 \Leftrightarrow R_{ab} = k_a k_b$ where $k_a k^a = 0$. ("Pure radiation" spacetime.)

• k is tangent to a shear-free, null, geodesic congruence. (cf. Robinson 1961)

• $k \wedge dk \neq 0$ (twisting/rotating) - 3 additional conditions on the congruence

(CGT, 2012)

or

- $k \wedge dk = 0$ (twist-free/irrotational) - 1 additional condition on the congruence (CGT, 2012)

Examples: Twist-free case

- One additional condition on the shear-free, null geodesic congruence.
- Introduce any null tetrad adapted to the congruence: $(k, l, m, \overline{m})$.
- In terms of the Newman-Penrose formalism:

$$\Re\left[\left(\delta + \beta - \overline{\alpha}\right)\left(\overline{\tau} - 2\overline{\beta}\right) - \frac{1}{2}\left(\epsilon - \overline{\epsilon}\right)\left(\mu - \overline{\mu}\right)\right] = 0$$

- This condition is independent of the choice of null tetrad.

Example 1: A homogeneous null electrovacuum

Originally thought to be the only homogeneous, pure radiation spacetime NOT to admit an electromagnetic source.

Define the chart and metric.

```
> DGsetup([u, v, x, y], M);
```

frame name: M **(3.2.1.1)**

```
> g := evalDG(dx &t dx + dy &t dy + 2*du &s dv - 2*exp(2*s*x)*
  du &t du);
```

$$g := -2\,e^{2sx}\,du \otimes du + du \otimes dv + dv \otimes du + dx \otimes dx + dy \otimes dy \quad \textbf{(3.2.1.2)}$$

The Ricci tensor is null, $R_{ab}R^{ac} = 0$.

```
M > Ricci := RicciTensor(g);
```

$$Ricci := 4\,s^2\,e^{2sx}\,du \otimes du \quad \textbf{(3.2.1.3)}$$

```
M > TensorInnerProduct(g, Ricci, Ricci, tensorindices = [1])
  ;
```

$$0\,du \otimes du \quad \textbf{(3.2.1.4)}$$

The Ricci tensor can be expressed as $R_{ab} = \frac{1}{4} k_a k_b$ for a null vector field k.

```
M > K := DGzip([K1, K2, K3, K4], [du, dv, dx, dy], "plus");
```
$$K := K1\,du + K2\,dv + K3\,dx + K4\,dy \quad (3.2.1.5)$$

```
M > EQTensor := evalDG(Ricci - 1/4*K &t K);
```
$$EQTensor := -\left(\frac{K1^2}{4} - 4\,s^2\,e^{2\,s\,x}\right) du \otimes du - \frac{K1\,K2}{4}\,du \otimes dv - \frac{K1\,K3}{4}\,du \otimes dx - \frac{K1\,K4}{4}\,du \otimes dy - \frac{K1\,K2}{4}\,dv \otimes du - \frac{K2^2}{4}\,dv \otimes dv - \frac{K2\,K3}{4}\,dv \otimes dx - \frac{K2\,K4}{4}\,dv \otimes dy - \frac{K1\,K3}{4}\,dx \otimes du - \frac{K2\,K3}{4}\,dx \otimes dv - \frac{K3^2}{4}\,dx \otimes dx - \frac{K3\,K4}{4}\,dx \otimes dy - \frac{K1\,K4}{4}\,dy \otimes du - \frac{K2\,K4}{4}\,dy \otimes dv - \frac{K3\,K4}{4}\,dy \otimes dx - \frac{K4^2}{4}\,dy \otimes dy \quad (3.2.1.6)$$

```
M > DGsolve(EQTensor, K, {K1, K2, K3, K4});
```
$$\left\{4\,RootOf\left(\_Z^2 - e^{2\,s\,x}\right) s\,du\right\} \quad (3.2.1.7)$$

```
M > Kdown := simplify(allvalues(DGsolve(EQTensor, K, {K1,
    K2, K3, K4}))[1], symbolic)[1];
```
$$Kdown := 4\,e^{s\,x}\,s\,du \quad (3.2.1.8)$$

```
M > Kup := RaiseLowerIndices(InverseMetric(g), Kdown, [1]);
```
$$Kup := 4\,e^{s\,x}\,s\,D\_v \quad (3.2.1.9)$$

Verify the result.

```
M > evalDG(Ricci - 1/4*Kdown &t Kdown);
```
$$0\,du \otimes du \quad (3.2.1.10)$$

```
M > TensorInnerProduct(g, Kdown, Kdown);
```
$$0 \quad (3.2.1.11)$$

It now follows that

$$f_{ab} = k_{[a} s_{b]}$$

has the correct energy-momentum tensor, where s_a is any unit vector orthogonal to k_a.

Construct a spacelike unit vector orthogonal to k^a.

```
M > S := DGzip([s1, s2, s3, s4], [du, dv, dx, dy], "plus");
```
$$S := s1\,du + s2\,dv + s3\,dx + s4\,dy \quad (3.2.1.12)$$

```
M > eq1 := TensorInnerProduct(g, Kdown, S);
```
(3.2.1.13)

```
M > 
```
$$eq1 := 4\,e^{s\,x}\,s\,s2 \quad (3.2.1.13)$$

```
M > eq2 := TensorInnerProduct(g, S, S) - 1;
```
$$eq2 := 2\,e^{2\,s\,x}\,s2^2 + 2\,s2\,s1 + s3^2 + s4^2 - 1 \quad (3.2.1.14)$$

```
M > DGsolve([eq1, eq2], S, [s1, s2, s3, s4]);
```
$$\{s1\,du + RootOf\left(\_Z^2 + s4^2 - 1\right)\,dx + s4\,dy\} \quad (3.2.1.15)$$

A simple solution is therefore:

```
M > S := Tools:-DGsimplify(eval(S, {s1=0, s2=0, s3=1, s4=0})
    );
```
$$S := dx \quad (3.2.1.16)$$

```
M > f := SymmetrizeIndices(Kdown &t S, [1,2],
    "SkewSymmetric");
```
$$f := 2\,e^{s\,x}\,s\,du \otimes dx - 2\,e^{s\,x}\,s\,dx \otimes du \quad (3.2.1.17)$$

Verify the result.

```
M > EnergyMomentumTensor("Electromagnetic", g, c*f);
```
$$4\,c^2\,e^{2\,s\,x}\,s^2\,D\_v \otimes D\_v \quad (3.2.1.18)$$

```
M > EinsteinTensor(g);
```
$$4\,s^2\,e^{2\,s\,x}\,D\_v \otimes D\_v \quad (3.2.1.19)$$

• The electromagnetic field, if one exists, must be related to f_{ab} by a "local" duality rotation,

$$\phi : M \rightarrow R,$$

$$F = \cos(\phi)\,f + \sin(\phi)\,{*f}.$$

• The Maxwell equations impose conditions on k and ϕ.

• The conditions on k give the shear-free, geodesic condition.

• The conditions on ϕ are then

$$\frac{1}{i}\delta\phi + \tau - 2\,\beta = 0, \quad \frac{1}{i}\bar{\delta}\phi - \bar{\tau} + 2\bar{\beta} = 0, \quad \frac{1}{i}D\phi - \varepsilon + \bar{\varepsilon} = 0.$$

The integrability conditions give the remaining conditions on the congruence.

• We now compute the properties of the congruence associated to the vector field k^a.

First we verify the geodesic equation $k \cdot \nabla k = 0$.

```
M > ContractIndices(Kup, CovariantDerivative(Kup,
```

```
Christoffel(g)), [[1,2]]);
```

$$0\,D\_u \qquad (3.2.1.20)$$

Next we check that the congruence is shear-free and twist-free.

```
M > CongruenceProperties(g, Kup);
```

table(["ShearNormSquared" = 0, "Expansion" = 0, "RotationNormSquared" = 0, "Raychaudhuri" = 0]) **(3.2.1.21)**

Thus k^a is tangent to a congruence which is geodesic, shear-free, non-expanding, and twist-free. (This can also be checked using the Newman Penrose spin coefficients, computed below.)

We now consider the final condition on the congruence. We begin by constructing a null tetrad whose first leg is k^a and then compute its Newman-Penrose spin coefficients and directional derivatives.

```
M > OT := DGGramSchmidt([D_v, D_u, D_x, D_y], g, signature=[
    [1,-1], 1, 1]);
```

$$OT := \left[\frac{\sqrt{2}}{2} D\_u + \frac{\sqrt{2}\left(e^{2 s x}+1\right)}{2} D\_v, -\frac{\sqrt{2}}{2} D\_u - \frac{\sqrt{2}\left(e^{2 s x}-1\right)}{2} D\_v, D\_x, D\_y\right] \qquad (3.2.1.22)$$

```
M > NT0 := NullTetrad([OT[2], OT[3], OT[4], OT[1]]);
```

$$NT0 := \left[D\_v, -D\_u - e^{2 s x} D\_v, \frac{\sqrt{2}}{2} D\_x + \frac{I}{2}\sqrt{2}\, D\_y, \frac{\sqrt{2}}{2} D\_x - \frac{I}{2}\sqrt{2}\, D\_y\right] \qquad (3.2.1.23)$$

```
M > NT := NullTetradTransformation(NT0, "boost", 4*exp(s*x)*
    s);
```

$$NT := \left[4\, e^{s x} s\, D\_v, -\frac{e^{-s x}}{4 s} D\_u - \frac{e^{s x}}{4 s} D\_v, \frac{\sqrt{2}}{2} D\_x + \frac{I}{2}\sqrt{2}\, D\_y, \frac{\sqrt{2}}{2} D\_x - \frac{I}{2}\sqrt{2}\, D\_y\right] \qquad (3.2.1.24)$$

Verify that NT is a null tetrad.

```
M > TensorInnerProduct(g, NT, NT);
```

(3.2.1.25)

$$\left[\begin{array}{cccc} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad (3.2.1.25)$$

Compute the Newman-Penrose spin coefficients and directional derivatives.

```
M > NPS:=NPSpinCoefficients(NT);
```

$$NPS := \mathrm{table}\left(\left[\text{"sigma"} = 0, \text{"gamma"} = 0, \text{"nu"} = \frac{\sqrt{2}}{16\,s}, \text{"tau"} = 0, \text{"mu"} = 0, \text{"pi"} = 0, \text{"rho"} = 0, \text{"epsilon"} = 0, \text{"alpha"} = \frac{s\sqrt{2}}{4}, \text{"lambda"} = 0, \text{"kappa"} = 0, \text{"beta"} = \frac{s\sqrt{2}}{4}\right]\right) \quad (3.2.1.26)$$

```
M > NPD := NPDirectionalDerivatives(NT):
```

Check the integrability condition:

$$\mathrm{Re}\left[\left(\delta + \beta - \overline{\alpha}\right)\left(\overline{\tau} - 2\overline{\beta}\right) - \frac{1}{2}(\epsilon - \overline{\epsilon})\left(\mu - \overline{\mu}\right)\right] = 0.$$

```
M > alpha :=NPS["alpha"]: beta:=NPS["beta"]: epsilon:=NPS
    ["epsilon"]: mu:=NPS["mu"]: tau := NPS["tau"]:
M > X := simplify(tau - 2*beta) assuming s::real:
M > X1 := simplify(conjugate(X)) assuming s::real:
M > Y := simplify(beta - conjugate(alpha)) assuming s::real:
M > Z := simplify((epsilon - conjugate(epsilonn))*(mu -
    conjugate(mu))) assuming s::real:
```

```
M > NPD["delta"](X1) + Y*X1 - Z;
```

$$0 \quad (3.2.1.27)$$

To illustrate the invariance of the integrability condition with respect to the choice of tetrad adapted to k, we consider a new tetrad obtained from NT by a combination of a spatial and null rotation.

```
M > NT1 := convert(NullTetradTransformation
    (NullTetradTransformation(NT, "null rotation", u, "L"),
    "spatial rotation", x + y), exp) assuming u::real;
```

We check that NT1 is in fact a null tetrad.

```
M > TensorInnerProduct(g, NT1, NT1);
```

We verify the integrability conditions exactly as before.

```
M > NPS1:=NPSpinCoefficients(NT1);
M > NPD1 := NPDirectionalDerivatives(NT1):
```

```
M > alpha1 :=NPS1["alpha"]: beta1:=NPS1["beta"]: epsilon1:=
    NPS1["epsilon"]: mu1:=NPS1["mu"]: tau1 := NPS1["tau"]:
M > X := simplify(tau1 - 2*beta1) assuming s::real, x::real,
    y::real, u::real:
M > X1 := simplify(conjugate(X)) assuming s::real, x::real,
    y::real, u::real:
M > Y := simplify(beta1 - conjugate(alpha1)) assuming
    s::real, y::real, x::real, u::real:
M > Z := simplify((epsilon1 - conjugate(epsilon1))*(mu1 -
    conjugate(mu1))) assuming s::real, x::real, y::real,
    u::real:
M > NPD1["delta"](X1) + Y*X1 - Z;
```

We now construct the electromagnetic field *F*. We build the solution ϕ to

$$\frac{1}{i}\delta\phi + \tau - 2\,\beta = 0, \quad \frac{1}{i}\overline{\delta}\phi - \overline{\tau} + 2\overline{\beta} = 0, \quad \frac{1}{i}D\phi - \varepsilon + \overline{\varepsilon} = 0.$$

```
N > NPSvalues := {tau = NPS["tau"], beta = NPS["tau"],
    epsilon = NPS["epsilon"]}:
M > Phi := phi(u,v,x,y);
```

$$\Phi := \phi(u, v, x, y) \tag{3.2.1.28}$$

```
M > eq1 := eval(1/I*NPD["delta"](Phi) + tau - 2*beta,
    NPSvalues);
```

$$eq1 := -\mathrm{I}\left(\frac{\sqrt{2}\left(\frac{\partial}{\partial x}\phi(u, v, x, y)\right)}{2} + \frac{\mathrm{I}\sqrt{2}\left(\frac{\partial}{\partial y}\phi(u, v, x, y)\right)}{2}\right) - \frac{s\sqrt{2}}{2} \tag{3.2.1.29}$$

```
M > eq2 := eval(-1/I*NPD["bardelta"](Phi) + conjugate(tau) -
    2*conjugate(beta), NPSvalues) assuming s::real;
```

$$eq2 := \mathrm{I}\left(\frac{\sqrt{2}\left(\frac{\partial}{\partial x}\phi(u, v, x, y)\right)}{2} - \frac{\mathrm{I}\sqrt{2}\left(\frac{\partial}{\partial y}\phi(u, v, x, y)\right)}{2}\right) - \frac{s\sqrt{2}}{2} \tag{3.2.1.30}$$

```
M > eq3 := eval(1/I*NPD["D"](Phi) - epsilon + conjugate
    (epsilon), NPSvalues);
```

$$eq3 := -4\,\mathrm{I}\,\mathrm{e}^{s\,x}\,s\left(\frac{\partial}{\partial v}\phi(u, v, x, y)\right) \tag{3.2.1.31}$$

```
M > phisol:=pdsolve({eq1, eq2, eq3}, Phi);
```

$$phisol := \{\phi(u, v, x, y) = s\,y + \_F1(u)\} \tag{3.2.1.32}$$

The electromagnetic field is obtained from the duality rotation of

$f_{ab} = k_{[a}s_{b]}$ given by ϕ:

$$F_{ab} = \cos(\phi) f_{ab} - \sin(\phi) *f_{ab}$$

```
M > F0 := evalDG(cos(Phi) * convert(f, DGform) - sin(Phi) *
    HodgeStar(g,convert(f, DGform), detmetric=-1));
```

$$F0 := 2\cos(\phi(u, v, x, y))\, e^{s x} s\, du \wedge dx - 2\sin(\phi(u, v, x, y))\, e^{s x} s\, du \wedge dy \quad \textbf{(3.2.1.33)}$$

```
M > F:=eval(F0, phisol);
```

$$F := 2\cos(s y + \_F1(u))\, e^{s x} s\, du \wedge dx - 2\sin(s y + \_F1(u))\, e^{s x} s\, du \wedge dy \quad \textbf{(3.2.1.34)}$$

We verify that (g, F) satisfy the Einstein-Maxwell equations.

The Einstein equations:

```
M > G := EinsteinTensor(g);
```

$$G := 4 s^2 e^{2 s x} D\_v \otimes D\_v \quad \textbf{(3.2.1.35)}$$

```
M > T := EnergyMomentumTensor("Electromagnetic", g, F);
```

$$T := 4 s^2 e^{2 s x} D\_v \otimes D\_v \quad \textbf{(3.2.1.36)}$$

```
M > evalDG(G - T);
```

$$0 \quad \textbf{(3.2.1.37)}$$

The Maxwell equations:

```
M > MatterFieldEquations("Electromagnetic", g, F);
```

$$0\, D\_u,\ 0\, du \wedge dv \wedge dx \quad \textbf{(3.2.1.38)}$$

• Notice that the electromagnetic field involves an arbitrary function of u, which labels the null hypersurfaces generated by k^a. This is a general feature of null electrovac spacetimes when k^a is twist-free. In the twisting case the electromagnetic field involves only one free parameter (specifying a global duality rotation).

• All homogeneous, pure radiation spacetimes are electrovacua.

• Notice also that this electromagnetic field does not inherit the symmetries of the metric.

```
M > KV := KillingVectors(g);
```

$$KV := [-y D\_v + u D\_y, D\_y, -s u D\_u + s v D\_v + D\_x, D\_u, D\_v] \quad \textbf{(3.2.1.39)}$$

```
M > LieDerivative(D_y, g);
```

(3.2.1.40)

$$0\, du \otimes du \tag{3.2.1.40}$$

```
M > LieDerivative(D_y, F);
```

$$-2\sin(s\,y + \_F1(u))\, e^{s\,x}\, s^2\, du \wedge dx - 2\cos(s\,y + \_F1(u))\, e^{s\,x}\, s^2\, du \wedge dy \tag{3.2.1.41}$$

```
M > g;
```

$$-2\, e^{2\,s\,x}\, du \otimes du + du \otimes dv + dv \otimes du + dx \otimes dx + dy \otimes dy \tag{3.2.1.42}$$

```
M > F;
```

$$2\cos(s\,y + \_F1(u))\, e^{s\,x}\, s\, du \wedge dx - 2\sin(s\,y + \_F1(u))\, e^{s\,x}\, s\, du \wedge dy \tag{3.2.1.43}$$

Example 2: A pure radiation Robinson-Trautman spacetime

```
> restart;
> with(DifferentialGeometry): with(Tensor): with(Tools):
```

Here we consider a class of pure radiation solutions of Robinson-Trautman type (See Theorem 28.6 of [2].). We determine under what conditions these solutions are in fact solutions to the Einstein-Maxwell equations.

Initialize the manifold, define the metric, and check that it is of pure radiation type.

```
> DGsetup([u, r, x, y], N);
N > g := evalDG(2/r*(-f(x)*diff(f(x), x, x)*r + diff(f(x), x)^2*r + m(u))*du &t du - 2*du &s dr + r^2/2/f(x)^2*(dx &t dx + dy &t dy));
N > Ricci := RicciTensor(g);
N > TensorInnerProduct(g, Ricci, Ricci, tensorindices = [1]);
```

This is a pure radiation spacetime only when the u-u component of Ricci is positive, $R_{uu} > 0$, which we shall assume henceforth. Define $R_{uu} = \frac{1}{4}\psi^2$.

```
N > psi := 2*sqrt(Hook([D_u, D_u], Ricci));
```

We can easily read off the covariant form of the preferred null vector field k.

```
N > Kdown := evalDG(psi*du);
N > evalDG(Ricci - 1/4*Kdown &t Kdown);
```

The contravariant form of k is as follows.

```
N > Kup := RaiseLowerIndices(InverseMetric(g), Kdown, [1]);
```

We now build a null tetrad NT adapted to k and compute its Newman-Penrose spin coefficients and directional derivatives.

```
N > OTO := DGGramSchmidt([D_u, D_r, D_x, D_y], g, signature
```

```
   = [[-1, 1], 1, 1]);
N > NT0 := simplify(NullTetrad([OT0[1], OT0[3], OT0[4], -OT0
   [2]]), symbolic);
N > NT := NullTetradTransformation(NT0, "boost", psi);
N > Spin := NPSpinCoefficients(NT):
N > NPdiff := NPDirectionalDerivatives(NT):
```

We check that the congruence is indeed geodesic and shearfree: $\kappa = 0$ and $\sigma = 0$. In addition, the spin coefficient ρ is real, indicating the congruence tangent to k is twist-free.

```
N > Spin["kappa"];
N > Spin["sigma"];
N > Spin["rho"];
```

We now compute and solve the geometric condition on the spacetime (in the twist-free case), denoted int_cond, which indicates whether the spacetime admits an electromagnetic source.

```
N > X := simplify(Spin["tau"] - 2*Spin["beta"]):
    Y := simplify(Spin["epsilon"] - DGconjugate(Spin
    ["epsilon"])):
    Z := simplify(Spin["mu"] - DGconjugate(Spin["mu"])):
    W := simplify(DGconjugate(Spin["beta"]) - Spin["alpha"])
    :
    eq:=NPdiff["bardelta"](X) + W*X - 1/2*Y*Z:
    int_cond := DGRe(eq);
```

We solve for functions f and m such that the integrability condition holds. This command may take a few minutes to complete.

```
N > PDETools:-Solve(int_cond);
```

Evidently, for generic f this pure-radiation Robinson-Trautman spacetime does not admit an electromagnetic source. There exists an electromagnetic source for the choice $f = a\, e^{bx}$, with $m(u)$ remaining arbitrary. Here is the metric and Ricci tensor in this case.

```
N > g2 := eval(g, f(x) = a*exp(b*x));
N > RicciTensor(g2);
```

We see that this is a pure radiation solution provided $\frac{dm}{du} < 0$. We now construct the corresponding electromagnetic field from a duality rotation determined by the equations

$$\frac{1}{i}\delta\phi + \tau - 2\,\beta = 0, \quad \frac{1}{i}\overline{\delta}\phi - \overline{\tau} + 2\overline{\beta} = 0, \quad \frac{1}{i}D\phi - \varepsilon + \overline{\varepsilon} = 0.$$

```
N > X := simplify(Spin["tau"] - 2*Spin["beta"]):
```

```
Y := simplify(DGconjugate(Spin["tau"]) - 2*DGconjugate
(Spin["beta"])):
Z := simplify(Spin["epsilon"] - DGconjugate(Spin
["epsilon"]) - 1/2*(Spin["rho"] - DGconjugate(Spin
["rho"]))):
```

```
N > solutions := {f(x) = a*exp(b*x)}:
    NP1:=simplify(1/I*NPdiff["delta"](phi(u,r,x,y)) + X):
    NP2:=simplify(-1/I*NPdiff["bardelta"](phi(u,r,x,y)) + Y)
    :
    NP3:=simplify(1/I*NPdiff["D"](phi(u,r,x,y)) - Z):
    simplify(eval({NP1, NP2, NP3}, solutions)):
    phisol := pdsolve(simplify(eval({NP1, NP2, NP3},
    solutions)), {phi(u,r,x,y), m(u)});
```

The only non-trivial solution is therefore $\phi = -b\,y + h(u)$, with $m(u)$ remaining arbitrary. The electromagnetic field is constructed as follows.

```
N > phi := -b*y + h(u):
    k := eval(RaiseLowerIndices(g, NT[1], [1]), solutions):
    S := eval(eval(evalDG(1/sqrt(2) * (exp(I*phi) * NT[3] +
    exp(-I*phi) * NT[4]))), solutions):
    s := eval(RaiseLowerIndices(g, S, [1]), solutions):
    F := convert(evalDG(1/2*k &w s), DGform) assuming r > 0;
```

We verify that g_2 and F do define a solution to the Einstein-Maxwell equations.

```
N > MatterFieldEquations("Electromagnetic", g2, F);
N > evalDG(EinsteinTensor(g2) - EnergyMomentumTensor
    ("Electromagnetic", g2, F));
```

This solution is (isometric to) the general form of the Petrov type D, Robinson-Trautman null electrovacuum [2].