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Finite Difference Solution for Drainage of Heterogeneous Sloping Lands

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FINITE DIFFERENCE SOLUTION FOR DRAINAGE OF HETEROGENEOUS SLOPING LANDS

by

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Approved:

Major Professor	Committee Member
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NOTATION

A	=	Soil parameter
a	=	Soil parameter
B	=	Soil parameter
b	=	Soil parameter
C	=	Soil parameter
C_1, C_2, C_3	=	Coefficients of the heterogeneity equation
D	=	Soil parameter
D_e	=	Effective flow depth in linearized Boussinesq equation (L)
d	=	Depth from drains to impermeable bed (L)
d_e	=	Depth from drains to impermeable bed as modified by Hooghoudt's theory (L)
g	=	Gravity acceleration (L/T^2)
$H = \frac{P}{\rho g}$	=	Soil water pressure head (L)
$h = \frac{-P}{\rho g}$	=	Soil water suction head (L)
h	=	Thickness of flow in Boussinesq's equation (L)
$h_b = \frac{P_b}{\rho g}$	=	Soil bubbling pressure head (L)
i	=	Node index in the x-direction
$i(t)$	=	Rate of unsaturated accretion to the water table (L/T)
j	=	Node index in the y-direction
K	=	Unsaturated hydraulic conductivity (L/T)

K = Newton's iteration index

K_o = Saturated hydraulic conductivity (L/T)

$K_r = \frac{K}{K_o}$ = Relative hydraulic conductivity

L = Spacing between drains (L)

m = Soil parameter

m = iteration index (SOR iteration)

m = Height of water table at mid point between the drains above datum for the steady state in sloping lands (L)

n = Time level index

n = Soil parameter

$\vec{n}$ = Normal vector

P = Soil water pressure (F/L^2)

P_b = Soil bubbling pressure (F/L^2)

$P_c = -P$ = Soil water suction (F/L^2)

P_1 = Soil parameter (F/L^2)

$q(x, t)$ = Flux rate at soil surface (L/T)

S = Specific yield in Boussinesq's equation

$S = \frac{\theta}{\eta}$ = Soil saturation = Ratio of volume of water to volume of voids in soil.

$S_e = \frac{S - S_r}{1 - S_r}$ = Effective saturation

S_r = Residual Saturation = Irreducible saturation

S_o = Percentage of specific yield that is drained at the onset of desaturation.

T	=	Depth from soil surface (L)
t	=	Time (T)
$\vec{V}$	=	Darcian velocity vector (L/T)
W	=	Over-relaxation parameter
x	=	Horizontal coordinate (L)
y	=	Vertical coordinate (L)
y	=	Position head above datum (L)
y	=	Height of water table at mid point between drains above datum (L)
y_o	=	Initial height of water table at mid point between drains above datum (L)
Z	=	Soil parameter (L)
α	=	Angle of slope of the impermeable bed
β	=	$\frac{K_o D_e}{S}$ (L^2/T)
ρ	=	Density of water (M/L^3, or FT^2/L^4)
η	=	Soil porosity
Θ	=	Angle of slope of surface of soil
Θ	=	Volumetric water content
Θ_o	=	Volumetric water content at saturation
λ	=	Pore size distribution index - a soil parameter
τ	=	Soil parameter
∇	=	The "del" operator
Δx	=	Mesh size in the x-direction
Δy	=	Mesh size in the y-direction

Δt = Magnitude of the time step

ϕ = y + H = Total hydraulic potential (FL/F, or L)

ABSTRACT

Finite Difference Solution for Drainage of

Heterogeneous Sloping Lands

by

Fahd S. Natur, Doctor of Philosophy

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Department: Civil Engineering

The two-dimensional problem of tile drainage on sloping heterogeneous lands was considered. The land surface and the impermeable boundaries of the problem were of a general shape. The flow in both the saturated and unsaturated zones was considered and the system was treated as one composite system. The problem was solved by a finite difference numerical method using the successive over-relaxation iterative (SOR) method for the steady state case with no local recharge, and a combined Newton inner iteration and successive over-relaxation outer iteration for the transient state case with local recharge. Both the rising water table and the falling water table cases were simulated. A computer program was written in Fortrain IV Language for this purpose, and a UNIVAC 1108 computer system was used. The results of two runs for a hypothetical problem and one run for a field testing problem are presented. The results were compared with some approximate mathematical solutions for the falling water table.

(180 pages)

INTRODUCTION

Background

Artificial drainage, simply defined, is the artificial removal of excess unwanted water from a locality of interest. It has a wide range of application in the field of engineering. It is very important in soil engineering, foundation works, earth works, earth fill dams, highways, railroads, airports, in stabilization of slopes and wherever it is desired to protect against excessive soil pore pressure, or against frost damage or where it is desired to increase the shearing resistance of the soil. (See Terzaghi and Peck, 1968, and Sherard et al. 1963 for more on non-agricultural drainage.)

Probably more drainage work is done for agricultural lands than for all of the above fields put together. This research is concerned with drainage of agricultural lands and for the rest of the paper the word "drainage" will mean drainage of agricultural lands unless otherwise qualified.

The purpose of agricultural land drainage is the removal of excess moisture from the plant root zone to provide an optimum environment for the plant roots for optimum production and to maintain a favorable salt balance in the root zone so that economic and good production is sustained perpetually.

The beginnings of the art of drainage are lost in pre-history, but probably man practiced drainage not very long after he practiced agriculture itself. Remnants of very old drainage works can still be found at the seats of very old civilizations. In recorded history, the Greek historian Hirodotus, as early as 400 B.C., mentioned the drainage networks in the Nile Valley (Ayres and Scoates, 1939; Framji and Mahajan, 1969).

The importance of land drainage cannot be overemphasized. A significant volume of research confirmed the detrimental effects of a high water table on crop production. Vast areas of present world deserts were once very productive lands, but they deteriorated because of excessive accumulation of salts due to lack of drainage. It is now recognized that drainage is an effective means of managing river-basin salt balance, that it is very important in maintaining a successful irrigated agriculture for long periods and that, except for some very rare localities where natural drainage is adequate, irrigation and drainage developments are complimentary.

Some statistics may add to this emphasis on the importance of drainage. Framji and Mahajan (1969) reported that there were 247 million acres of land in the world provided with artificial drainage. The United States of America (USA) Water Resources Council (1968) estimated that up to 1959 a total of 131 million acres of land had been drained for agricultural purposes in the USA, and that 189,000 miles of open ditches and 58,000 miles of tile drains had been constructed. In 1966 alone, the

United States Department of Agriculture (USDA) Soil Conservation Service assisted in design and construction of 12,720 miles of open drains and 25,553 miles of tile drains (USA Water Resources Council, 1968). Luthin (1966) estimated that 20 percent of the land in the major corn belt states was drained mostly by tile drains.

Although drainage development had been undertaken for centuries, still vast areas of land in the world are in need of improved drainage. Gulhati (1955) estimated that 150 to 200 million acres of irrigated crop land in the world needed improved drainage. Nearly 84 million acres of crop and pasture land in the USA need some drainage improvement (USA Water Resources Council, 1968). With the increasing recognition of the importance of drainage, construction of drainage works is being undertaken at an ever-increasing pace.

The costs involved in drainage developments are not insignificant. Luthin (1966) estimated that the cost of tile drains installed in the state of Iowa alone was more than the cost of the Panama Canal. The cost of the irrigation projects that depend on proper drainage for their success is many folds greater. This makes it very important to have good drainage design. No design can be any better than the theory on which it is based and our understanding of the physical processes.

Until about three decades ago the design and construction of drainage works were just an art that depended solely on the experience and judgment of the designer. In fact there were no scientific bases for the design nor any theory until about a century ago, that is, until

Henry Darcy discovered his famous linear law of flow of water through porous materials in 1856. Probably the first drainage problem to be investigated scientifically was the famous Boussinesq problem towards the turn of the century. Hooghoudt is accredited by Luthin (1966) to be the first to present a complete rational analysis of the drainage problem. Since that time, and especially in the last three decades, much research with rewarding results went into the science and engineering of drainage. Nevertheless, drainage science is still not an exact science at the present day. This is due to the complexity of the physical problem and the complexity of the factors that enter into it.

The greater part of the research was on steady state problems. The greater percentage was for homogeneous soils with or without a horizontal impermeable barrier. Most of the drain spacing formulas were developed for these cases. Most design procedures are based either on empirical information or on analysis incorporating simplifications the validity of which is at best difficult to prove (Hedstrom, Corey, and Duke, 1971). Many solutions assume no flow above the water table (Hedstrom, Corey, and Duke, 1971) which in most cases introduces gross errors. Some of the solutions that attempted to account for the flow above the water table did so only by increasing the cross sectional area of the saturated flow which is in most cases inadequate representation of the flow system (Childs, 1945). Many of the unsteady state solutions consider only the saturated flow and assume instantaneous and complete desaturation of the soil above the water table (Jensen and Hanks, 1967).

It is well demonstrated now that this is not the case. These simplified solutions, suffering as they do from restrictions and assumptions, are nevertheless great steps in increasing our knowledge and understanding of the complex problem of drainage.

The steady state condition cannot exist practically for any appreciable time. Heterogeneity of the soil is the rule rather than the exception in nature. More often than not, the land surface is not flat and the impermeable barrier is not horizontal. In humid and subhumid areas it is very important to provide drainage on hillside forested or pasture lands and to protect adjacent crop lands from seepage from the hillside. In many arid and semi-arid areas the increase in population and the resulting increase in demand for new crop land is forcing development and irrigation of sloping areas. This is enhanced further by the development and spreading use of sprinkler irrigation systems. This creates the need for drainage of sloping lands. Robinson (1959) estimated that about 90 percent of the drain installations in Colorado are of the interceptor type.

Although there has been a large volume of literature written on the drainage problem in recent years, not very much is found on drainage of sloping lands. This served as a motivation for this study. It was decided to include the effects of slope, heterogeneity and unsaturation in the solution. For this last consideration, the whole soil-water system comprising the saturated and the unsaturated regions is treated as one system.

The resulting partial differential equations of flow are non-linear and, to the knowledge of the writer, no analytical method of solution of these equations is available. Recourse is therefore made to a numerical solution.

Objectives of the Study

The following two objectives were set at the start of the study.

1. To use the finite difference numerical method to solve the general two-dimensional unsteady state drainage problem in a heterogeneous sloping soil with a general geometry of the ground surface, impermeable barrier and drain placement. The solution would be general enough to be used at many different locations. The solution would treat the saturated and the unsaturated parts of the medium as one system since it is already recognized that the same governing equations (Darcy's and the continuity) hold for both parts, and that the two parts form a physical and mathematical continuity. Both infiltration and drainage conditions will be simulated.

2. To test the solution with data already available in the literature for the actual performance of some field drainage systems.

REVIEW OF LITERATURE

Drainage of Sloping Lands

Seepage in sloping lands

Seepage and drainage on sloping lands are so intimately interrelated that a brief review of this subject is warranted. Seepage studies, whether their purpose be drainage (Kirkham, 1947), erosion control and slope stability (Whisler, 1969) or hydrology (Jeppson, 1969b) give us a better understanding of the drainage problem.

The classical view (Since Dupuit, Forchheimer and Boussinesq) was that the flow in steady state seepage on sloping lands was parallel to the sloping impermeable layer. While this may be characteristic of many areas and situations, it is not universal. Kirkham (1947) studying piezometric surfaces in a hillside sloping farm in the Iowan glacial drift area found that water moved approximately vertically downward at the top of the slope, horizontally outward in the middle slope and approximately vertically upward near the bottom of the slope. Bornstein, Bartlett, and Howard (1965), studying piezometric data in a sloping area underlain by a fragipan layer, concluded that water moved generally parallel to the ground surface although the impermeable layer had protrusions and depressions and no continuous slope.

Klute, Scott, and Whisler (1965) using analytical solutions for seepage in a saturated inclined rectangular homogeneous soil slab confirmed, in a general qualitative sense, Kirkham's (1947) findings. Whisler (1969),using an electric resistance network analog to simulate the above mentioned inclined soil slab,confirmed the analytical results and found that the minimum rate of recharge necessary to keep the slab saturated increased as the slope angle was increased. He concluded that long slabs acted like pipes with soil at both ends.

Powers, Kirkham, and Snowden (1967), and Selim and Kirkham (1972a, 1972b) presented analytical solutions for the problem of seepage through sloping saturated homogeneous soils overlying horizontal impermeable layers such as found in systems of drainage by bedding (See Luthin, 1966, pp. 232-239, for drainage by bedding). Their results confirmed those of Whisler (1969) and in addition showed that increasing the depth of the soil increased the magnitude of interflow and the percentage of the total recharge that goes into interflow. Warrick (1970), and Morin and Warrick (1973) used conformal mapping to solve a similar problem with infinite depth and obtained similar general results.

Jeppson (1969b) used a finite difference numerical method to solve the flow equation for infiltration of water on a watershed of heterogeneous soil and any prescribed shape of bed and soil surface. He formulated the problem in the inverse plane of the potential function and the stream function.

Youngs (1971) presented a mathematical solution of the flow of a free surface aquifer resting on an impermeable layer of any shape with no limitation to small slopes. This solution however needs a prior knowledge of the pressure distribution along the impermeable boundary.

Drainage of sloping lands

General

Bouwer (1955a, 1955b) using mathematical reasoning and a sand tank model found that alignment of tile drains longitudinally with the slope or transversely across the slope did not affect the drainage capacity of the drains. Kirkham (1947) reported that in some seep areas on the slope, drains were not effective even at a spacing as small as 50 feet. This was especially true in the artesian areas near the bottom of the slope. Bornstein (1964), Benoit, Fisher and Bornstein (1967), Bornstein, Thiel and Benoit (1967), Bornstein and Benoit (1967) and Benoit and Bornstein (1972) reported the results of a long term field experiment on sloping land underlain by a fragipan layer. Their results showed that shallow (20 inches) surface diversion ditches were not effective in draining the slope and that most of the drainage was done by deep subsurface drains. They also concluded that random drains or a single cutoff drain were not sufficient and recommended a system of parallel drains. Thiel and Bornstein (1965) confirmed these results with an electric resistance network analog and showed that a backfill over the drains with high conductivity was highly effective. Willardson (1968) recommended a backfill of high conductivity to cut off the flow above and below the tile drain on slopes.

Solutions to the drainage problem

Steady state with no surface recharge. The differential equation of steady state, saturated, free surface flow in homogeneous soil resting on a sloping impermeable bed with no local surface replenishment was solved by direct integration by many investigators. Bear (1972) attributed the solution to Dupuit in 1863 and to Pavlovski in 1931. Bear, Zaslavsky and Irmay (1968) presented the solution and called the problem "Pavlovski's problem." Solutions in one or another of the horizontal-vertical, the longitudinal-vertical and the longitudinal-normal sets of coordinates were given by Jaeger (1957), Polubarinova-Kochina (1962), Werner (1957), Glover according to Donnan (1959) and Todd and Bear (1959). (Here longitudinal axis means the axis along the slope and normal axis means the axis normal to the slope.) All of the above solutions used one or the other of two approximations used originally by Boussinesq. According to Wooding and Chapman (1966), Boussinesq in 1877 extended the Dupuit-Forchheimer assumptions (called D-F assumptions hereafter) to flow systems on sloping barriers so that the streamlines are taken parallel to the sloping bed, and in 1904 he used the original D-F assumptions (horizontal streamlines) for the same problem. Consequently all of these solutions are limited to small slopes. A solution by the method of functional analysis and which does not utilize the D-F assumptions (based on the hydrodynamic theory) was given by Polubarinova-Kochina (1962). This solution would not be limited to small slopes. Childs (1971) using the first of Boussinesq's formulations (stream lines parallel to slope) and a more accurate expression for the hydraulic gradient obtained a more accurate solution that was claimed to be not limited to small slopes only.

Childs (1946) used electric conductor sheet analogs to investigate the problem of drainage of foreign water by open ditches or tile drains on sloping lands.

According to Childs (1946), Hopf and Trefftz (1921) obtained solutions to restricted cases of seepage to an open ditch on sloping lands by conformal mapping. (For more on conformal mapping see Vallentine 1967 or Bieberbach, 1964.) Polubarinova-Kochina (1962) used conformal mapping to solve the case of drainage to a horizontal slit drain in sloping land. According to Maasland (1959), conformal mapping was used by Gustaffson (1946) for the solution of this problem. Conformal mapping was used in a qualitative manner by Brooks (1959) and Nelson (1960) for the same problem.

Many of the solutions cited above assumed that the source of foreign water was at an infinite distance from the drain. Keller and Robinson (1959) using dimensional analysis and the results of a sand tank model, modified Glover's equation (Donnan, 1959) to apply to cases where the source of seepage was at a finite distance from the drain. This case would be encountered when an intercepter is used to intercept seepage from a higher unlined canal. For such a case Willardson, Boles and Bouwer (1971) applied the electric resistance network analog method and investigated effects of distance and depth of the interceptor relative to the canal on seepage and interception.

The solutions mentioned above give the shape of the water table upstream of a drain and assume that the water table downstream of the drain will be parallel to the sloping bed starting from the water level in the drain. However it was pointed out by Brooks (1959), Nelson (1960) and, for some cases, by Polubarinova-Kochina (1962) that the water table downstream of the drain would rise to a level higher than the level of the water in the drain.

Steady state with surface recharge. This problem was investigated less extensively than the previous one. Werner (1957) used the Boussinesq 1877 formulation and solved the approximate linearized differential equation by Laplace transformations (operational calculas) for a steady recharge rate with and without foreign seepage water. (For more on the Laplace transforms see Spiegel,1965). Schmid and Luthin (1964) used the Boussinesq 1904 formulation and solved the linearized differential equation for the case of drainage ditches penetrating to the sloping impermeable boundary neglecting seepage surfaces. As design aids they presented curves of $\frac{q}{K}$ vs. $\frac{H}{L}$ for slopes from zero to 70 percent, where q is the rate of replenishment, K is the hydraulic conductivity of the soil, H is the maximum height of the water table above the bed between two drains and L is the spacing of the drains. Guitjens (1964) and Guitjens and Luthin (1965) checked the solution of Schmid and Luthin (1964) with a Hele-Shaw viscous flow model for slopes from zero to 80 percent. They found that the model showed the existance of significant seepage surfaces and that the horizontal stream lines of the D-F assumptions introduced an error in Schmid and Luthin's (1964) results and that this error increased with increasing slope for a fixed $\frac{q}{K}$ ratio and with increasing $\frac{q}{K}$ ratio for a fixed slope. Wooding and Chapman (1966) compared the solutions of Werner (1957) and Schmid and Luthin (1964) with an exact solution by conformal mapping. They found good agreement between Werner's (1957) solution and the conformal mapping solution, but a discrepancy that increased with increasing slope when Schmid and Luthin's (1964) solution was compared with the comformal mapping solution. They presented design curves similar to those of Schmid and Luthin (1964) but based on Werner's (1957) solution. Wooding (1966) showed excellent agreement between Werner's (1957) solution and conformal mapping for some other particular cases of flow in sloping lands. Childs (1971) extended his refined expression of the hydraulic gradient to the case of steady recharge and obtained a solution which was claimed to be not limited to small slopes. An analytical solution to this case was also obtained by Henderson and Wooding (1964).

Mein and Turner (1968) using an electric resistance network analog to study drainage on slopes of sand dunes for up to 10 percent slope recommended that for slopes up to this value either Schmid and Luthin's (1964) curves or Wooding and Chapman's (1966) curves could be used for design.

Luthin and Taylor (1966a, 1966b) used a digital computer to solve the more exact Laplace equation by the finite difference numerical method for a homogeneous, sloping soil with steady replenishment rate and open drains penetrating to the barrier.

Ziegler (1972) studied drainage on sloping land with steady recharge using a sand tank model and concluded that drainage on sloping lands had many aspects similar to drainage in flat lands. Carlson (1971) using a sand tank model compared the results of the model with results calculated by formulas developed for flat lands and concluded that spacing formulas developed for flat lands could be used for spacing of mid-slope drains.

Unsteady state. This problem has been investigated the least. Werner (1957) gave an analytical solution for the cases of a sudden change in the elevation of the tail water (drain water in drainage), a sudden change in the rate of replenishment and a uniformly increasing rate of replenishment. Henderson and Wooding (1964) gave a solution for the build-up of the water table under constant recharge.

Luthin and Guitjens (1967) used a Hele-Shaw viscous flow model to study the case of a falling water table after cessation of a steady recharge. They wrote:

> It appears that flat land drainage theory can be applied to sloping land without much error if the drainage facility reaches the impermeable layer. (Luthin and Guitjens, 1967, p. 50)

Chauhan (1967) and Chauhan, Schwab and Hamdy (1968) studied the case of the falling water table and compared the results of an analytical solution of the linearized Boussinesq equation and the results of an analog computer finite difference solution of the nonlinear equation with the results of a Hele-Shaw viscous flow model. They found that the three methods had good agreement up to 8 percent slope, but the two approximate methods (analytical and analog) deviated from the model at higher slopes. They also confirmed the results of Luthin and Guitjens (1967) that the rate of the fall of the water table at its highest point for the moderately sloping case was the same as that for flat lands.

Heterogeneous soils. All of the drainage studies cited above were for homogeneous soils. To the extent of my knowledge the only paper that dealt with drainage of heterogeneous sloping land was that of Nelson (1961) who presented theory and a graphical method for transforming a heterogeneous soil into a homogeneous one, applying the homogeneous drainage theory, then transforming the soil back to the original heterogeneous condition.

Numerical Solutions

General

There are many problems in physical sciences and engineering for which the differential equations governing the phenomena under study can be formulated, yet the analytical solutions of these equations are beyond the reach of pure mathematics as it stands at the present. The partial differential equations of flow in porous media are almost always in this class, (unless they are much simplified by sometimes valid and sometimes totally unrealistic assumptions). In such cases the numerical methods of solving these equations are among the most powerful tools to deal with the problem.

There are several numerical methods of solving partial differential equations, but the most general, the most versatile and the most widely used method is the method of finite differences. In Lawrenson's words:

> There are several numerical techniques that can be used ..., but the one which is still supreme, and which can be applied equally to linear and non-linear problems, to steady-state and transient ones without limits as to boundary shapes and conditions, is the method of finite differences. (Lawrenson, 1966, p. 102)

The calculus of finite differences is an old branch of calculus that started not very many years after the invention of the differential (infinitismal or continuous) calculus, to deal with discontinuous functions and discrete observations. (It was sometimes called the calculus of observations). According to Jordan (1960), it was started by Brook Taylor in 1717, and its theory was laid down by Jacob Sterling in 1730. Before the Twentieth Century, the main application of this calculus was for evaluation of terms and sums of series (Boole, 1860), and in mathematical statistics (Jordan, 1960).

A very close analogy between the calculus of finite differences and the differential calculus was recognized from the beginning (Boole, 1860; Spiegel 1971), but it seems that this analogy was mostly used to carry the theory and methods of solution of differential equations over to difference equations. Although Luthin and Scott (1952) stated that the use of numerical methods for the solution of differential equations was as old as Newton, it is widely accepted that Richardson (1910), was the first to solve a differential equation by the method of finite differences, (Thom and Apelt, 1961; Remson, Hornberger, and Molz, 1971). Thom and Apelt (1961) mentioned that both Boltzman in 1892 and Runge in 1908 gave the finite difference operator for the Laplace equation. Southwell (1940, 1946) used finite differences and his relaxation method in solving differential equations. Few other investigators used the method with desk-type computations. With the advent of high speed computers, numerous investigators used this method for a large number of problems in many fields of application.

The theory of the discretization of a differential equation by the method of finite differences as given by Forsythe and Wasow (1960) could be described by the following. If we have a partial differential equation defined on an open connected domain R of the independant variables (x, y, . . .) with a solution u(x, y, . . .) defined on R, we replace the domain R (infinite number of points) with a set S which has a finite (but relatively large) number N of elements P. The solution u(x, y, . . .) is replaced with a function U(P) defined on S. U(P) is then found by solving a system of simultaneous algebraic equations. Usually each element P of the set S is taken as a point in or near the set $\overline{R} = R \cup C$, where C is the boundary of R. Normally these points are taken as the grid points of a regular mesh dividing $\overline{R}$.

The mechanics of the method consist of dividing the domain of the problem by a regular mesh (most usually a square mesh), then replacing the derivatives in the differential equation with differences of the values of the dependant and the independant variables at the grid points (nodes). This done at each node, gives a system of simultaneous algebraic equations. The method of evaluating the derivatives in terms of differences by use of expansion of functions into Taylor's series is given by Remson, Hornberger and Molz (1971) and Carnahan, Luther, and Wilkes (1969). There are other methods of obtaining the difference equations from a differential equation, such as the methods of the calculus of variation discussed by Forsythe and Wasow (1960), and Remson, Hornberger, and Molz (1971). Usually there is also a choice of several schemes for representing a derivative in terms of finite differences. (See Davis and Polonsky, 1964; Richtmyer, 1957). The choice usually depends on the problem, the domain geometry, the requirements imposed on the solution and the individual solving the problem.

Once differencing is done, we need to solve the resulting system of simultaneous algebraic equations (which is usually a large system). Methods for solving such systems can be grouped into the direct methods and the iterative methods. Iterative methods are usually preferred for large systems (Forsythe and Wasow, 1960; Lawrenson, 1966; Thom and Apelt, 1961; Carnahan, Luther, and Wilkes, 1969; Remson, Hornberger, and Molz, 1971). Direct methods, however, may be practical and advantageous in certain cases (Jeppson, 1968b). Iterative methods, starting from given or assumed values of the unknown at the nodes of the mesh, seek to improve these values in successive iterations until the changes in these values with more iterations are made less than a small error prescribed by the solver.

One iterative method that offers an optimum combination of simplicity, flexibility and high speed is the method of "Successive Over Relaxation" (referred to as SOR hereafter) (Lawrenson, 1966). In this method the improved value of the unknown at a node is taken as the sum of the starting value of the unknown at the node plus the product of a relaxation parameter and the difference between the computed and the starting values at the node. Furthermore, this improved value is used directly after it is computed in all subsequent calculations in the iteration. The value of the relaxation parameter ranges from zero to two. Usually an optimum value between one and two exists for this parameter to give the quickest convergence (although in some problems a value less than unity- underrelaxation may be needed for the stability of the solution). In any case the optimum value of this parameter heavily depends on the type of the differential equation, the geometry of the domain, and the type of the boundary conditions (Forsythe, and Wasow, 1960), and on the difference scheme, (Jeppson, 1968b).

Relaxation methods were used by Gauss in 1823 and by Seidel in 1874 (Forsythe and Wasow 1960), and by Southwell (1940, 1946). According to Forsythe and Wasow (1960), over-relaxation was used by Fox in 1948 and its theory was given for limited types of systems by Frankel (1950) and Young (1954). Later investigators successfully extended the use of SOR to systems outside the limitations of Young (1954) and Frankel (1950). For more on the theory of iterative methods and the SOR see Forsythe and Wasow (1960), Varga (1962),and Wachspress (1966).

The equation of flow in porous media under unsaturated conditions is a nonlinear partial differential equation. Finite difference numerical solutions were obtained for a number of non-linear partial differential equations although the theory of such solutions is very scant (Forsythe and Wasow 1960). According to Forsythe and Wasow (1960), the only nonlinear parabolic partial differential equation for which an approximate difference has been studied systematically was of the form:

$$\frac{\partial u}{\partial t} = a_o (x, t) \frac{\partial^2 u}{\partial x^2} + a_1 (x, t) \frac{\partial u}{\partial x} + d (x, t, u)$$

where the nonlinearity is in the last term only and not associated with any of the derivatives. Jeppson (1972) stated that the schemes used for nonlinear equations were principally extensions of methods that worked with linear equations, without a developed theory for the nonlinear equations. He pointed out that this was no guarantee for convergence, stability or representation of the solution and that the scheme of differencing and the method of solution were of extreme importance in dealing with non-linear equations. He also pointed out that each of the many types of nonlinearities introduces its own pecularities and difficulties in the problem. (For more on solution of partial differential equations by the method of finite differences see Richtmyer, 1957; Forsythe and Wasow, 1960; Thom and Apelt, 1961; Carnahan, Luther and Wilkes, 1969.)

Finite difference solutions in porous media flow problems

Even before the development of high speed computers some investigators used the method of finite differences with relaxation schemes suited to desk calculators to solve problems in porous media flow (Southwell, 1946; Luthin and Gaskell, 1950; Kirkham and Gaskell, 1950; Day and Luthin, 1956). With the development and widely spread use of high speed computers, a large number of more complex porous media flow problems were solved using this method. The volume of literature on this subject has become so large that a complete review of it is beyond the scope of this study. Freeze (1969) reviewed a number of one-dimensional problems solved by this method, and Remson, Hornberger and Molz (1971) gave a large number of references on this subject. To set a background, some articles will be mentioned here with no claim that the list is exhaustive or comprehensive.

This method was used in almost all areas of the field of flow in porous media, such as underground hydrology (Hornberger, Ebert and Remson, 1970; Lin, 1972, 1973), seepage through earth dams (Jeppson 1968b; Freeze, 1971), seepage from earth canals (Jeppson, 1968a; 1968c; Burejev and Burejeva, 1966, Jeppson and Nelson, 1970), infiltration (Brutsaert, 1971; Jeppson, 1972; Hanks, Klute, and Bresler, 1969; Ibrahim and Brutsaert, 1968; Selim and Kirkham, 1973), trickle irrigation

(Brandt et al., 1971), flow towards wells (Luthin, and Scott, 1952; Taylor and Luthin, 1969; Brutsaert, Breitenbach, and Sunada, 1971; Cooley, 1971), soil column drainage (Watson, 1967, Brooks et al., 1971) and land drainage, which will be reviewed in a later section.

The problems solved represent a variety of combinations of equation form (hydrodynamic or Boussinesq), flow dimensions (one, two or three-dimensional), flow states (steady or unsteady), flow conditions (saturated, unsaturated or a composite of both) and medium characteristics (homogeneous or heterogeneous). Reference to these solutions is given by the following.

Boussinesq's equation: Moody (1966), Terzidis (1968), Hornberger, Ebert and Remson (1970), Lin (1972) and Zucker et al. (1973).

One-dimensional, unsteady flow in homogeneous, unsaturated media: Wang, Hassan and Franzini (1964), Whisler and Klute (1965), Remson et al. (1965), Remson, Resnicoff and Scott (1974), Remson,Fungaroli and Hornberger (1967), and Whisler and Watson (1969).

One-dimensional, unsteady; homogeneous, composite: Whisler and Klute (1967).

One-dimensional, unsteady; heterogeneous, unsaturated: Ashcroft et al. (1962), Hanks and Bowers (1962), Day and Luthin (1956), Rubin and Steinhardt (1963), Klute, Whisler and Scott (1965), Whisler and Klute (1966), Rubin (1966), Kobayashi (1966), Jensen and Hanks (1967), Klute and Bresler (1969) and Jeppson (1970b).

One-dimensional, unsteady, heterogeneous, composite: Freeze (1969).

Two-dimensional, steady, homogeneous, composite: Jeppson (1968a, 1968b).

Two-dimensional, steady, heterogeneous, saturated: Freeze and Witherspoon (1966) and Jeppson (1969a).

Two-dimensional, steady, heterogeneous, composite: Sewell and van Schilfgaarde (1963) and Jeppson (1968c, 1969b).

Two-dimensional, unsteady, homogeneous, saturated: Isherwood (1959), Todsen (1971) and Tseng and Ragan (1973).

Two-dimensional, unsteady, homogeneous, composite: Rubin (1968).

Two-dimensional, unsteady, heterogeneous, saturated: Burejev and Burejeva (1966), and Taylor and Luthin (1969).

Two-dimensional, unsteady, heterogeneous, unstaurated: Brandt et al. (1971) and Green, Dabiri and Weinang (1970).

Two-dimensional, unsteady, heterogeneous, composite: Brutsaert, Breitenbach and Sunada (1971), Freeze (1971a) Hornberger, Remson and Fungarolli (1969), and Brutsaert (1971).

Three-dimensional, steady, heterogeneous, composite: Nelson (1962), Reisenauer (1963) and Reisenauer, Nelson and Knudsen (1963).

Three-dimensional, unsteady, heterogeneous, composite: Freeze (1971b).

Three-dimensional, axisymmetric: (these problems collapse to two-dimensional ones). Jeppson (1968d, 1970a) and Wei (1971).

Various methods of differencing the transient equation were used. The explicit difference scheme was used by Kobayashi (1966). The implicit scheme was used by Brutsaest (1971), Brutsaert et al. (1971) and Freeze (1969, 1971a, 1971b) among others. The Crank-Nicolson Scheme was used by Ashcroft et al. (1962), Hanks and Bowers (1962), Hanks, Klute and Bresler (1969), Jensen and Hanks (1967), Klute Whisler and Scott (1965), Rubin (1966), Whisler and Klute (1965), Brandt et al. (1971) and Jeppson (1972).

Methods of solution of the resulting system of algebric equations were also numerous. The relaxation method was used by Day and Luthin (1956), Isherwood (1959), Luthin and Gaskell (1950), Kirkham and Gaskell (1950) and Luthin and Scott (1952). The SOR was used by Nelson (1962), Reisenauer (1963), Reisenauer, Nelson and Knudsen (1963), Taylor and Luthin (1963), Freeze and Witherspoon (1966), Jeppson (1968a, 1968b 1968c) and Tseng and Ragan (1973). The line successive over-relaxation (LSOR) method was used by Freeze (1971a, b). The alternating direction implicit (ADI) method was used by Rubin (1968) and Lin (1972, 1973). The Newton Iterative method was used by Jeppson (1968d, 1972) and Brutsaert (1971). The Newton-SOR (Newton inner iteration and SOR outer iteration) was used by Jeppson and Nelson (1970) and Wei (1971). The Newton-LSOR was used by Brutsaert, Breitenbach and Sunada (1971). The Newton-ADI was used by Brandt et al. 1971.

Finite difference solutions of drainage problems

Although drainage problems were among the earliest porous media flow problems solved numerically by the method of finite differences, very little is found in the literature on this subject. Luthin and Gaskell (1950) and Taylor and Luthin (1963) used the method of finite differences to study steady state drainage to tile lines in layered soils with a ponded soil surface. Kirkham and Gaskell (1950) used the method to study the transient case of the falling water table in homogeneous soils drained by tiles or open ditches. Finite differences and the method of Kirkham and Gaskel (1950) were used by Isherwood (1959) to study the effect of tile depth, tile spacing, barrier depth, hydraulic conductivity and drainable porosity on the rate of fall of the water table between tile drains. The same methods were used by Todsen (1971) to study the transient behavior of the water table between ditch drains in the presence of local accretion. Sewell and van Schilfgaarde (1963) used finite differences to study steady drainage to tile drains in a composite saturated-unsaturated system, and Rubin (1968) investigated the case of the falling water table in a composite system using finite differences. Luthin and Taylor (1966) studied steady drainage to open ditches on sloping homogeneous lands with accretion using finite differences. Remson, Hornberger and Molz (1971) gave a finite difference solution to the Boussinesq's transient equation for tile lines lying on the impermeable layer. Moody (1966) solved the Boussinesq's transient equation for the falling water table between tiles above the impermeable barrier, using finite differences.

Composite Saturated-Unsaturated Systems

General

Until very recently the saturated and the unsaturated regions of water flow in soils were treated separately; the first by groundwater hydrologists neglecting the unsaturated flow and the second by soil physicists with no consideration of the saturated flow (Freeze 1969). In most solutions of drainage problems the unsaturated flow above the water table was either neglected or an equivalent saturated fringe thickness was added to the saturated flow cross section to compensate for it. The validity of this method of compensation, especially in flows which are not predominantly horizontal, was never confirmed (Hedstrom, Corey and Duke, 1971).

Neglect of the unsaturated flow above the water table may lead, in many cases, to serious errors (Jensen and Hanks, 1967; Brutsaert, Breitenbach and Sunada, 1971; Hedstrom, Corey and Duke, 1971). The importance of including the unsaturated flow was emphasized by Bouwer (1959, 1964), Kraijenhoff Van de Leur (1962), Reisenauer (1963) and Freeze and Harlan (1969). Bird and McCorquodale (1971) studying performance of tile drains reported that the seepage in the unsaturated zone was very significant. Luthin and Day (1955) showed experimentally and by a numerical solution that the volume of unsaturated flow can exceed that of saturated flow in some problems in certain cases. Freeze (1971a) showed that the inclusion of the unsaturated flow was not a matter of trivial consequences on the results.

Many of the solutions of the transient drainage problem (especially the case of the falling water table) were obtained as a succession in time of steady state solutions, where, starting from an initial water table position, a new water table position, after a time increment, is calculated by some equation, then the steady state equation of flow is solved for the new saturated region and the procedure is repeated for new time steps. (See for example Kirkham and Gaskell, 1950; Isherwood, 1959; Burejev and Burejeva, 1966; Todsen, 1971; Tseng and Ragan, 1973). In addition to neglecting the flow in the unsaturated zone, most of these solutions assume instantaneous and complete desaturation of the medium at a point as soon as the water table falls beyond that point. They assume furthermore that the drainable porosity or specific yield is a constant quantity independent of time or position above the water table. These assumptions were criticized as inaccurate by Childs (1960), Kraijenhoff Van de Leur (1962), Jensen and Hanks (1967), Rubin (1968) and Freeze (1971b), all emphasizing the fact that drainable porosity was a dynamic quantity that depended on time as well as position above the water table. Hewlett and Hibbert (1963), in a sloping soil tank experiment, reported that they were still getting measurable amounts of drainage outflow from a 10.85 cubic meters volume of soil (38,3 cubic feet) 145 days after the soil mass was desaturated.

Treatment of the problem in a composite saturated-unsaturated system eliminates the inaccuracies mentioned above. Two contrasting theories exist in the literature concerning composite systems. These theories were reviewed by Fujioka and Kitamura (1964) and by Hornberger and Freeze (1970). The first theory, although admitting the physical continuity of flow between the saturated and the unsaturated zones, claims that there are differences between the flows in the two zones in that the water in the unsaturated zone possesses relative compressibility and that the curve of moisture content versus pressure head exhibits a first derivative discontinuity at saturation. The results of an experiment conducted by Fujioka and Kitamura (1964) to test this theory did not support it. The second theory claims that there is physical and mathematical continuity of flow in both the saturated and the unsaturated zones, and that the distinction between the two zones is only an arbitrary distinction of definition. This second theory is more widely accepted by workers in the field than the first (Rubin, 1968; Freeze and Harlan, 1969; Freeze, 1969, 1971a, 1971b; Hornberger, Remson and Fungaroli, 1971; Brutsaert, Breitenbach and Sunada, 1971).

In solving transient free-surface problems in porous media, the composite system treatment has another great advantage. It eliminates the need for calculating the position of the water table a priori, leaving this to emerge as part of the solution (Freeze, 1971b).

The advantages of treating a composite system were recognized early by many researchers, but the task of solving such a system analytically was (and still is) formidable if not impossible. This treatment was made feasible only through the availability of high speed computers and the wide spread use of numerical methods in solution. This probably explains why this trend is only very recent. Some of the works that treated composite systems were referenced previously in this review under headings with the adjective "composite."

Treatment of a composite system would only be possible through the extension of the flow equation to the unsaturated zone. This would necessitate the extension of Darcy's Law to unsaturated flow and the consideration of variable medium hydraulic conductivity and water content (or saturation) as functions of the pore pressure head; it being understood that the mass continuity equation is universally true in any medium (at least for the velocities we consider in porous media flow).

Extension of Darcy's law to unsaturated media

In 1856, Henry Darcy published his famous experimental law that stated that flow through saturated sand was directly proportional to the head loss and inversely proportional to the length of path of flow. This law which gave a linear relation between flow and hydraulic gradient (which is the loss in hydraulic head divided by the length of the flow path), was originally found for vertical downward flow through saturated homogeneous sand columns. It has been since shown to be independant of the direction of flow (van Schilfgaarde, 1970) and it has been extended to two and three dimensions by many investigators, first heuristically and then by planned experiments and success in application (Bear, 1972).

It has also been obtained theoretically by various workers by statistical averaging of the Navier-Stokes equations (neglecting inertial terms) over the flow section, and by various conceptual models of the porous medium to which the hydrodynamic theory was applied. (See Hall, 1956; DeWiest, 1969; Rumer, 1969; Bear, 1972). This showed that Darcy's law reflected the macroscopic statistical average of the hydrodynamic behavior of water flow through the multitude of the tortuous flow paths in a porous medium.

Buckingham (1907), investigating capillary flow of soil moisture and utilizing analogies of this flow to heat flow (Fourier's law) and to electric current flow (Ohm's law), suggested a law for unsaturated moisture flow in soils which was in actuality an extension of Darcy's law for saturated flow. Although Buckingham did not mention Darcy's law, the analogy between saturated and unsaturated flows would have been as close (if not closer) as that of the unsaturated flow to heat and electric current flows (Swartzendruber, 1969). Buckingham's law was accepted and recognized as an extension of Darcy's law to unsaturated media by Israelsen (1927), Richards (1928, 1931) and Gardner (1936). This extension of Darcy's Law was verified experimentally by Childs and Collis-George (1950) and analytically by Hall (1956). This is now universally accepted except maybe at very low moisture contents (Swartzendruber, 1963; Churayev, and Gorokhov, 1970). Using this law and the continuity equation, Richards (1931) derived the general equation of flow in isothermal unsaturated media.

Soil water content and hydraulic conductivity as functions of capillary pressure

Buckingham (1907) and later investigators (Richards, 1931; Gardner, 1936) recognized that the water content and the hydraulic conductivity of unsaturated media were functions of capillary pore pressure. In the last two decades much work has been done to investigate these relationships both in the field of petroleum engineering (Rose, 1949; Fatt and Dykstra, 1961; Burdine, 1953) and in the field of soil water (Gardner, 1958; King 1965).

It is well established now that the conductivity-water content relation is unique with no hysterisis, Water content-capillary pressure relation is hysteritic and (consequently) the conductivity-capillary pressure relation is also hysteritic (Childs, 1969; Bear, Zaslavsky and Irmay, 1968; van Bavel, 1969 among others). Although some authors mentioned dependence of some or all of these relationships on the water content gradient (Gardner and Gardner, 1950) or on the hydraulic gradient (Churayev and Gorokhov, 1970; Rogers and Klute, 1971), yet according to Bear (1972) no definite conclusion has been reached on this matter.

Researchers using numerical solutions for problems of unsaturated flow dealt with these relationships in three different ways. One group of researchers used tables of corresponding values of water content, conductivity and capillary pressure for their particular media (Day and Luthin, 1956; Hanks and Bowers, 1967; Jensen and Hanks, 1967; Whisler and Klute, 1967; Whisler and Watson, 1969; Hanks, Klute and

Bresler, 1969; Freeze, 1969; Green, Dabiri and Weinang, 1970; Watson and Whisler, 1972). Another group of researchers fitted their particular data with special functions (Brandt et al., 1971, used cubic spines, and Selim and Kirkham, 1973 used exponential fits). A third group used some general equations developed for these relationships. The equations given in the literature are many and their origins are various. Some were based on empirical fitting of data (Gardner, 1958; King, 1965) and some were based on conceptual idealized models of porous media (bundles of capillary tubes) coupled with empirical fitting (Burdine, 1953; Brooks and Corey, 1964). In reviewing some of these equations here, the symbols of some authors will be changed to conform to a single set of symbols and to avoid confusion especially between negative values of pressure (P, H) and positive values of pressure (P_c, h).

Water content-pressure relations. Swartzendruber (1969) used an approximate linear relation of the form:

$$\Theta = \eta - bh \qquad (1)$$

where

Θ = volumetric water content

η = soil porosity

$h = -H = -\dfrac{P}{\rho g}$ = suction head

$H = \dfrac{P}{\rho g}$ = pressure head

b = a soil parameter

This was a gross simplification. Taylor and Luthin (1969) used an equation of the form

$$\theta = \frac{\theta_o}{Ah^3 + 1} \tag{2}$$

where

θ_o = water content at saturation

A = a soil parameter

Brooks and Corey (1964) studied a large number of experimental data and suggested the equation:

$$S_e = \frac{1}{\left(\frac{P_c}{P_b}\right)^{\lambda}} \qquad \text{for } P_c \geq P_b \tag{3}$$

where

$S_e = \frac{S - S_r}{1 - S_r}$ = effective saturation

$S = \frac{\theta}{\eta}$ = saturation = ratio of volume of water to volume of voids

S_r = residual saturation (irreducible saturation)

= saturation when the water phase becomes discontinuous and conductivity becomes practically zero

P_c = $-P$ = suction

P = pressure

P_b = bubling pressure, a positive soil parameter with units as P_c

λ = pore size distribution index, a dimensionless soil parameter (For definition and methods of finding P_b, λ, S_r see Brooks and Corey, 1964). Brutsaert (1968) suggested a more general form:

$$S_e = \frac{A}{A + h^b} \tag{4}$$

Where A and b are parameters. A similar form was used by Cooley (1971). Wei (1971) following Brutsaert (1968) used the form

$$S_e = \frac{1}{\left(\frac{P_c}{P_b}\right)^b + A} \tag{5}$$

where A, b and P_b are parameters.

More complicated equations were suggested by King (1965), Rubin, Steinhardt,and Reiniger (1964), Visser (1969), Rogowski (1971) and White et al. (1970).

Conductivity-water content relation. Irmay (1954) suggested a relation of the form:

$$K = K_o S_e^3 \tag{6}$$

which could be written as

$$K_r = S_e^3$$

where

K = unsaturated hydraulic conductivity

K_o = saturated hydraulic conductivity

K_r = relative hydraulic conductivity

$= \frac{K}{K_o}$ by definition

Bruch and Street (1967) used the same form. Wang, Hassan, and Franzini (1964), Singh and Franzini (1967) and Brutsaert (1968) used the same form but with a general parameter exponent for S_e instead of three. Brooks and Corey (1964) using Burdine (1953) theory and their saturation-pressure relation (Equation 3) suggested the relation:

$$K_r = (S_e)^{\frac{2+3\lambda}{\lambda}}$$

<u>Conductivity-pressure relation.</u> Richards(1931) used an equation of the form:

$$K = aH + b \qquad (8)$$

where a and b are parameters. This linear relation was used because it was helpful in some analytical solutions. Gardner (1958) reviewed previous equations and from a study of available data at the time suggested the following equation:

$$K = \frac{a}{h^n + b} \qquad (9)$$

where a, b and n are parameters. Taylor and Luthin (1969) used the following form:

$$K = \frac{K_o}{Ah^3 + 1} \tag{10}$$

where A is a parameter. Sewell and van Schilfgaarde (1963) used the following equation:

$$K_r = \frac{m}{P_c^n + m} \tag{11}$$

Wesselling and Wit (1966) used the equation:

$$K = a h^{-b} \tag{12}$$

where a and b are parameters. It should be noted that Equations (10), (11) and (12) are actually special versions of Gardner's Equation (9). Raats (1971) used the form

$$K = b e^{aH} \tag{13}$$

where a and b are parameters. Burdine (1953), starting with a conceptual model of the porous medium (a bundle of capillary tubes) and using empirical fitting of data to evaluate tortousity suggested the equation:

$$K_r = \left(\frac{S - S_r}{1 - S_r}\right)^2 \frac{\int_o^s \frac{ds}{P_c^2}}{\int_o^1 \frac{ds}{P_c^2}} \tag{14}$$

which could be written (Brooks and Corey, 1964) as

$$K_r = (S_e)^2 \frac{\int_o^{S_e} \frac{dS_e}{P_c^2}}{\int_o^1 \frac{dS_e}{P_c^2}} \tag{15}$$

Obviously, to evaluate the integrals in Equation (14) or Equation (15) one needs to know the $S-P_c$ relationship. Brooks and Corey (1964) used their suggested relationship (Equation 3) to evaluate the Burdine integrals in Equation (15) to get

$$K_r = \frac{1}{\left(\frac{P_c}{P_b}\right)^\tau} \qquad \text{for } P_c \geq P_b \tag{16}$$

$$\tau = 2 + 3\lambda \tag{17}$$

Where τ is a positive parameter. Equation (16) gives good fits with experimental data except for values of P_c very near to P_b. Because of the nature of their derivation, Equations (15) and (16) are more representative of the drainage branch of the $\Theta - P$ relation. King (1965) noted that Gardner's (1958) equation (Equation 9) is dimensionally inconsistent, and suggested modifying it to the dimensionally consistent form

$$K_r = \frac{1}{\left(\frac{P_c}{P_1}\right)^\tau + b} \tag{18}$$

where P_1 is a positive soil parameter of dimensions similar to those of P_c, and b and τ are positive dimensionless soil parameters. This equation gave a very good fit to imbibition as well as drainage data. For K_r to have a value of unity at saturation, the parameter b in Equation (18) would be held equal to 1 at saturation (King, 1965). Wei (1971) used a generalized form of Equation 18 which was given as:

$$K_r = \frac{b}{\left(\frac{P_c}{P_b}\right)^{\tau} + b} \tag{19}$$

The relationship between the parameters τ and λ given by Brooks and Corey (1964) (Equation 17) is supported by many sets of data (King, 1973, in a verbal communication). More on other equations for K_r could be found in Smith (1966) and Raats and Gardner (1971).

THE PHYSICAL PROBLEM

Although the drainage problem in nature is a transient three dimensional flow problem, it will be reduced to a two dimensional transient one by assuming that all conditions along the horizontal axis normal to the general direction of flow are similar. This assumption was used by the majority of investigators in the drainage field and in most drainage problems it is quite reasonable and gives no serious errors. A general two dimensional geometry is envisaged as shown in Figure 1, with a general shape and slope of the land surface and the impermeable boundary. The land receives foreign water seeping from higher lands and local accretion from irrigation or rainfall. Tile drains are assumed at specified depths and spacings and of sufficient diameter to be able to carry away all the water that seeps into them with no back pressure in the tile lines.

A complete irrigation cycle will be simulated, where irrigation water is applied for some time, followed by a period of a few days of no irrigation. This will encompass the cases of water table build-up and water table recession. The effects of evapotranspiration during the cycle will be neglected. It is assumed that the effects of evapotranspiration on the position of the water table are normally small, unless the water table is very close to the land surface. This is supported

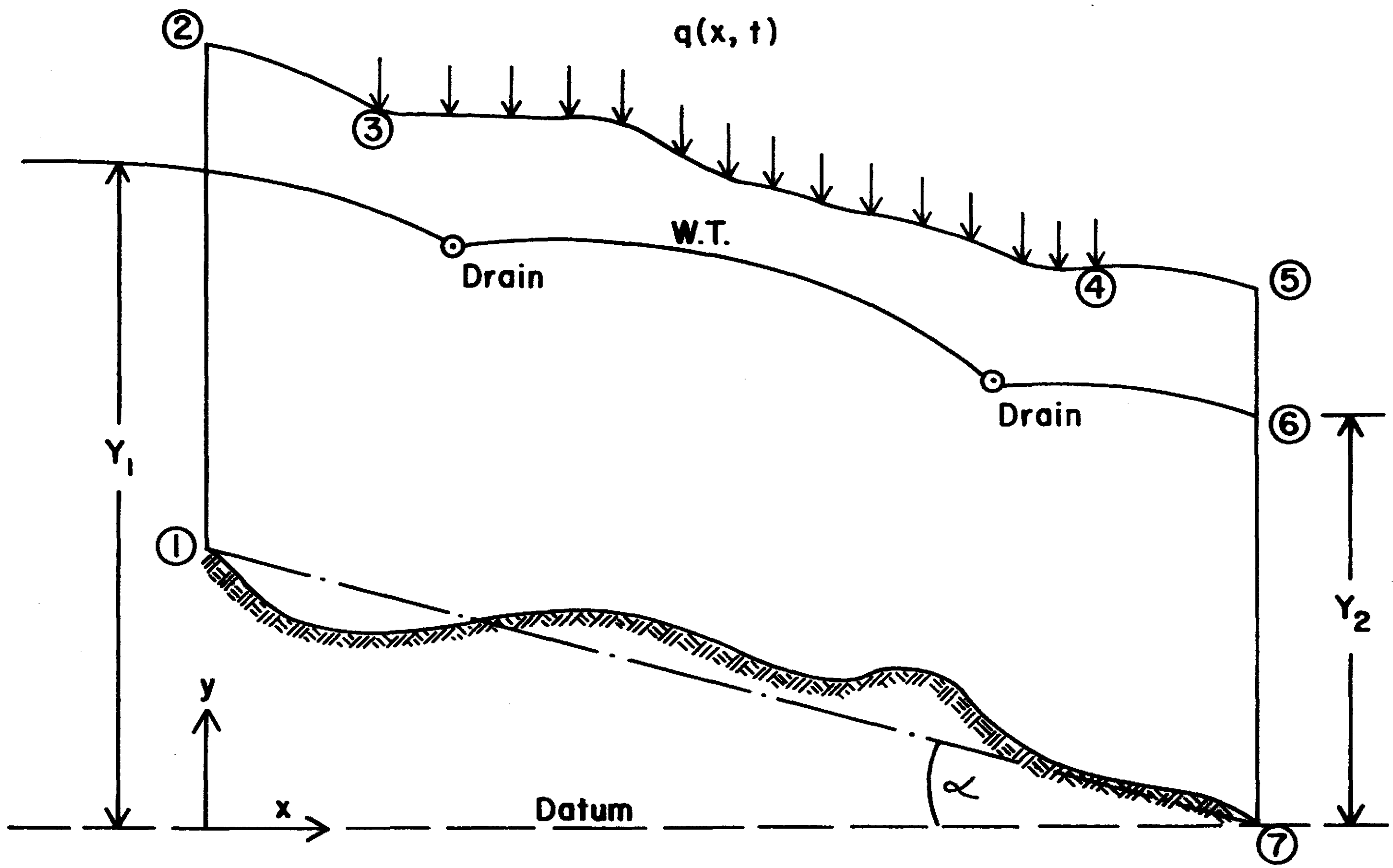


Figure 1. Domain and boundaries of the problem.

by the facts that evaporation is a soil-surface phenomenon and that its effects decrease rapidly with depth (Remson, Fungarolli, and Hornberger, 1967), and that root extraction for transpiration is limited to the zone above the water table. Evaporation (if its magnitude is known a priori) could be included in the model as a specified negative flux at the land surface, but the problem is not as simple as this statement implies and its investigation is not considered in this dissertation.

The soil is treated as a heterogeneous medium with respect to the saturated conductivity. King's (1974) definition of heterogeneity with respect to a property is adopted; namely that the value of the saturated conductivity at a point varies with the position in space of that point. The soil is assumed homogeneous with respect to the other soil parameters that enter into the soil characteristic relationships ($K-\theta-P$). While this might not be strictly true, yet the inclusion of heterogeneity in these parameters is not warranted by the scant amount of data one can find about them in the literature for field soils. Evaluation of these parameters is not yet a routine part of land drainage investigations and it is highly improbable (at the present) that a drainage engineer will find enough available data on these parameters to characterize heterogeneity with respect to them. In fact, other than a forthcoming paper by Jeppson, the writer is not aware of any work that dealt with heterogeneity of these parameters. The soil will also be assumed isotropic. It is suggested by King (1974, p. 12) that "most field materials could be described as heterogeneous rather than anisotropic, provided the scale of resolution of conductivity measurement is sufficiently small."

Saturated and unsaturated zones of the domain are treated as one composite flow system. This eliminates many of the weaknesses of treating the two zones separately. The position of the water table emerges as part of the solution as the zero pressure isobar.

No hysterisis in the soil characteristics relationships is considered, and the drainage envelop curves of these relationships are used to characterise the soil. It is true that during some parts of the irrigation-drainage cycle (for example at the start of irrigation) parts of the soil mass will be desaturating while other parts will be increasing in saturation, yet it is contemplated that hysterisis, although important in detailed studies of some fine phenomena, will not have a large effect on the gross hydrologic phenomenon of water table fluctuation. It is also important to note that during the greater part of the cycle, the soil mass will be desaturating.

THE MATHEMATICAL MODEL

The mathematical model consists of the partial differential equations of flow together with the boundary and initial conditions. The assumptions in the mathematical formulation are also included since it is believed, after Nelson (1962), that the capabilities and limitations of any formulation are best understood by a careful examination of the underlying assumptions.

The Differential Equations

The classical method of derivation given by many textbooks on porous media flow is followed here. Starting with Darcy's Law in vector form:

$$\vec{V} = -K\nabla\phi \tag{20}$$

and the differential form of the mass continuity equation:

$$\nabla \cdot (\rho \vec{V}) = -\frac{\partial}{\partial t}(\rho\Theta) \tag{21}$$

and substituting Equation (20) into Equation (21) we get:

$$\nabla \cdot [\rho K \nabla\phi] = \frac{\partial}{\partial t}(\rho\Theta) \tag{22}$$

Considering ρ to be constant gives:

$$\nabla \cdot [K \nabla \phi] = \frac{\partial \Theta}{\partial t} \tag{23}$$

where:

$\vec{V}$ = The Darcian velocity vector.

K = $K_o K_r(H)$ = The hydraulic conductivity at any pressure head (H).

K_o = The saturated hydraulic conductivity.

$K_r(H)$ = The relative hydraulic conductivity = The ratio of the hydraulic conductivity at any pressure head (H) to the saturated hydraulic conductivity.

ϕ = $y + H$ = Hydraulic head or total potential energy on a unit weight basis.

y = Position head above datum.

H = $\frac{P}{\rho g}$ = Pressure head (negative in unsaturated media).

P = Soil pore water pressure (negative in unsaturated media).

ρ = Density of water.

g = Gravity acceleration

Θ = Soil water content on a volumetric basis

t = Time

∇ = The del operator

In two-dimensional Cartesian coordinates Equation (23) can be written as:

$$\frac{\partial}{\partial x}\left(K \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(K \frac{\partial \phi}{\partial y}\right) = \frac{\partial \Theta}{\partial t} \qquad (24)$$

Setting $K = K_o\, K_r(H)$ and $\Theta = \eta\, S(H)$ by definition and expanding Equation (24) gives

$$K_o\, K_r(H) \left[\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right] + K_o \cdot \frac{\partial K_r(H)}{\partial x} \cdot \frac{\partial \phi}{\partial x} + K_r(H) \cdot \frac{\partial K_o}{\partial x} \cdot \frac{\partial \phi}{\partial x}$$

$$+ K_o \frac{\partial K_r(H)}{\partial y} \cdot \frac{\partial \phi}{\partial y} + K_r(H) \cdot \frac{\partial K_o}{\partial y} \cdot \frac{\partial \phi}{\partial y} = \eta \frac{\partial S(H)}{\partial t} \qquad (25)$$

where:

η = Soil porosity

S = Soil saturation = Ratio of volume of water to volume of voids in a soil elemental volume.

Using the chain rule of differentiation on the terms that involve space derivatives of $K_r(H)$ and time derivative of $S(H)$ in Equation (25), and dropping the functional notation of K_r and S, Equation (25) can be written as:

$$K_o K_r \left[\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right] + K_r \cdot \frac{\partial K_o}{\partial x} \cdot \frac{\partial \phi}{\partial x} - K_o \cdot \frac{\partial K_r}{\partial h} \cdot \left(\frac{\partial \phi}{\partial x}\right)^2$$

$$+ K_r \cdot \frac{\partial K_o}{\partial y} \cdot \frac{\partial \phi}{\partial y} - K_o \cdot \frac{\partial K_r}{\partial h} \cdot \left(\frac{\partial \phi}{\partial y}\right)^2 = -\eta \cdot \frac{\partial S}{\partial h} \cdot \frac{\partial \phi}{\partial t} \qquad (26)$$

Since the intermediate terms in the chain are $\frac{\partial h}{\partial H} = -1$ and $\frac{\partial H}{\partial \phi} = 1$.

where:

$$h = -H = -\frac{P}{\rho g} = \text{suction head (positive in unsaturated media).}$$

The following relationships for K_r, S and h are used:

$$K_r = \frac{A}{\left(\frac{h}{h_b}\right)^{\tau} + B} \qquad \text{for } h > Z \qquad (27)$$

$$K_r = 1 \qquad \text{for } h \leq Z \qquad (28)$$

$$S_e = \frac{C}{\left(\frac{h}{h_b}\right)^{\lambda} + D} \qquad \text{for } h > Z \qquad (29)$$

$$S_e = 1 \qquad \text{for } h \leq Z \qquad (30)$$

$$S_e = \frac{S - S_r}{1 - S_r} \qquad (31)$$

where:

A, B, C, D, τ, λ: are dimensionless soil parameters

h_b, Z: are soil parameters with units as those of h

S_e = effective saturation as defined in Equation (31)

S_r = residual saturation.

These relationships are of similar form to most of the equations cited in the section on review of literature and are general enough to allow a certain amount of freedom to the program user in fitting his data. By special choices of the parameters A, B, C, D and Z, Equation (27) can revert to King's (1965) modification of Gardner's (1958) equation (Equation (18)), to Brooks and Corey's (1964) equation (Equation (16)), or to Equation (19), used by Wei (1971). Equation (29) can revert to Brooks and Corey's (1964) relationship (Equation (3)), to Brutsaert's (1968) relationship (Equation (4)) or to Wei's (1971) equation (Equation (5)). The disadvantage of having two many unrelated parameters on the other hand is that more experimental data are needed to evaluate the parameters, (Jeppson 1973, in a verbal discussion).

From Equations (27), (29), and (31) we get:

$$\frac{\partial K_r}{\partial h} = - \frac{A\tau (h)^{\tau - 1}}{(h_b)^{\tau}\left[\left(\frac{h}{h_b}\right)^{\tau} + B\right]^2} = - G1 \cdot K_r^{\,2} \cdot (h)^{\tau - 1} \qquad (32)$$

$$\frac{\partial S}{\partial h} = (1 - S_r) \frac{\partial S_e}{\partial h} = - (1 - S_r) \cdot G2 \cdot S_e^{\,2} \cdot (h)^{\lambda - 1} \qquad (33)$$

where:

$$G1 = \frac{\tau}{A(h_b)^{\tau}} \quad \text{and} \quad G2 = \frac{\lambda}{C(h_b)^{\lambda}}$$

K_o usually varies with depth rather than with the y - coordinate, and since $T = y_o - y$, we get

$$\frac{\partial K_o}{\partial y} = - \frac{\partial K_o}{\partial T} \qquad (34)$$

where:

T = depth.

y_o = y − coordinate of the soil surface.

Substituting Equations (32), (33) and (34) into Equation (26), and dividing by $K_o K_r$ gives:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + F1 \cdot \left(\frac{\partial \phi}{\partial x}\right)^2 + F2 \cdot \frac{\partial \phi}{\partial x} + F1 \left(\frac{\partial \phi}{\partial y}\right)^2$$

$$- F3 \cdot \frac{\partial \phi}{\partial y} = G3 \cdot F13 \cdot \frac{\partial \phi}{\partial t} \qquad (35)$$

where:

$$F1 = G1 \cdot K_r \cdot (h)^{\tau - 1}$$

$$F2 = \frac{1}{K_o} \cdot \frac{\partial K_o}{\partial x}$$

$$F3 = \frac{1}{K_o} \cdot \frac{\partial K_o}{\partial T}$$

$$F13 = \frac{1}{K_o K_r} \cdot S_e^2 \cdot (h)^{\lambda - 1}$$

$$G3 = \eta \cdot G2 \cdot (1 - S_r)$$

Equation (35) is the equation of unsteady unsaturated flow. In the case of saturated flow in both the steady and the unsteady states, $\frac{\partial K_r}{\partial h} = 0$, and $\frac{\partial S}{\partial h} = 0$, giving $F1 = 0$, and $F13 = 0$, and Equation (35) becomes:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + F2 \frac{\partial \phi}{\partial x} - F3 \frac{\partial \phi}{\partial y} = 0 \qquad (36)$$

In case of steady state unsaturated flow, $\frac{\partial S}{\partial h} = 0$, giving F13 = 0, and Equation (35) becomes:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + F1 \cdot \left(\frac{\partial \phi}{\partial x}\right)^2 + F2 \cdot \frac{\partial \phi}{\partial x} + F1 \cdot \left(\frac{\partial \phi}{\partial y}\right)^2 - F3 \cdot \frac{\partial \phi}{\partial y} = 0. \qquad (37)$$

Equation (35) is of the nonlinear parabolic type, Equation (36) is of the linear elliptic type, while Equation (37) is of the nonlinear elliptic type (although the classification of a nonlinear partial differential equation is usually given at a point for a particular solution as it depends on the coordinates of the point and the solution). For classification of partial differential equations see Petrovsky (1950) or Garabedian (1964).

Assumptions in the Formulation

In the above derivation the following assumptions are made:

1. Darcy's law applies in both the saturated and the unsaturated zones. In drainage problems, the low flow velocities and the range of unsaturation are usually within the range of applicability of Darcy's law.

2. The water is continuously connected throughout the system. This is true for the range of unsaturation encountered in drainage problems.

3. The flow of air in the porous medium takes place instantaneously

and under very small gradients (due to the low viscosity of air) such that the energy dissipated in this flow is negligible. The air in the unsaturated zone is assumed to exist at atmospheric pressure always. Despite some cases reported in the literature where air pressure build-ups and significant effects on the flow did occur during infiltration (Orlob and Radhakrishna, 1958; Van Phuc and Morel-Seytoux, 1972; Linden and Dixon, 1973), it is believed that it is reasonable and safe to assume that air will move freely into and out of the unsaturated zone in drainage problems. Entrapped air will tend to be removed by the permeating water (Swartzendruber, 1969). Experiments by Bloomsberg and Corey (1964) showed that entrapped air is removed through solution and difusion even in stagnant water.

This assumption simplifies the problem to a one-phase flow problem.

4. The flow is assumed isothermal. For shallow water tables the temperature variations in the system are usually small.

5. The flow of the liquid phase of water only is considered. Water vapor flow is negligible compared to the liquid flow.

6. There are no osmotic gradients that affect the flow.

7. There are no interactions between the water and the porous medium that affect the flow.

8. There are no biological effects that affect the flow.

9. The water is treated as a continuum. Strictly speaking the water is composed of discrete molecular entities, but the dimensions

of drainage problems are so large compared to the dimensions of these entities or their mean free paths that assumption of a continuum is justified.

10. The water is homogeneous in nature at all points in the system.

11. The water is incompressible (ρ = constant).

12. The porous medium exhibits no swelling, shrinkage or consolidation and the solid particles do not move as the flow takes place.

13. The functions that describe the flow are assumed continuous with continuous derivatives (Jeppson, 1972).

The Boundary and Initial Conditions

The boundaries are shown in Figure 1 with circled numbers to indicate the different segments with different boundary conditions.

Boundary (1) - (2)

$$\phi = y \, \mathrm{Sin}^2 \alpha + Y_1 \mathrm{Cos}^2 \alpha \qquad (38)$$

where Y_1 and α are as indicated in Figure 1, α being the general angle of slope of the bed. For the derivation of Equation (38) see Appendix B.

The assumptions on this boundary are:

a. Uniform flow across this boundary with equipotential lines normal to the general slope of the bed. This implies that the flow across the boundary is not affected by the drain.

b. Steady flow across the boundary. This implies that the flow across this boundary is not affected by recharge or discharge downstream.

Theoretically both of these assumptions may be true only at an infinite distance from the drain and recharge point, but for all practical purposes, the local effects of the drain or recharge can be considered insignificant if this boundary is kept at a reasonable distance from the drain, (which distance is inversely proportional to the slope of the bed).

Boundary (2) - (3)

$$\frac{d\phi}{dn} = 0 \qquad (39)$$

This is a zero flux boundary. Point (3) is sufficiently removed from point (2) such that any recharge beyond point (3) will not affect the boundary condition set for (1) - (2).

Boundary (3) - (4)

$$\frac{d\phi}{dn} = \frac{1}{K_o K_r} \cdot q(x,t) \cdot \text{Cos}\,\Theta \qquad (40)$$

or

$$\phi = y \qquad (41)$$

where:

$q(x,t)$ = Specified flux at the surface

Θ = Angle of surface with the horizontal

Equation (40) comes from Darcy's law:

$$-K_o K_r \frac{d\phi}{dn} = - q(x,t) \cdot \text{Cos}\ \Theta$$

where the minus sign preceeding q(x, t) is because this quantity is by definition measured opposite to the direction of the outward normal $\vec{n}$. Cos Θ enters the equation because q(x, t) is normally measured as a vertical flux (normal to the projection of the surface on the horizontal plane). The program will use condition (40) unless the specified flux cannot be accomodated with the existing hydraulic gradients in the system, whence the surface becomes saturated and ponding starts after which condition (41) is used. In Equation (41), H is assumed equal to zero which means that ponding is only of negligible thickness and any excess water is removed by surface drainage.

Boundary (4) - (5)

$$\frac{d\phi}{dn} = 0 \qquad (42)$$

The same conditions as in boundary (2) - (3) apply here. Point (4) is sufficiently far from point (5) so that recharge between points (3) and (4) will not affect the boundary condition on (5) - (6).

Boundary (5) - (6)

$$\phi = Y2 \qquad (43)$$

Where Y2 is as indicated in Figure 1. The potential (ϕ) on this boundary is assumed in static equilibrium with the water table at point (6). Y2 is not known a priori, but emerges as part of the solution on boundary (6) - (7).

Boundary (6) - (7)

$$\frac{\partial \phi}{\partial x} = -\operatorname{Sin} \alpha \cdot \operatorname{Cos} \alpha \tag{44}$$

For the derivation of Equation (44) see Appendix B. It is assumed that this boundary is sufficiently removed downstream of the drain such that uniform flow is re-established along this boundary.

Boundary (7) - (1)

$$\frac{d\phi}{dn} = 0 \tag{45}$$

This boundary is a streamline along the impermeable bed, with equipotential lines normal to the bed.

The initial condition

The initial condition for the transient problem is taken as the steady state solution with no local recharge. This is a reasonable starting condition as it reflects conditions in the field at the start of an irrigation season after a prolonged period (say six months) of no irrigation and no (or negligible) rainfall. This is the prevailing condition in many arid and semi-arid irrigated areas. This will enable us to model a complete cycle of water table build-up with irrigation and water table recession with drainage. After the transient solution is started it could be terminated and picked up again at any time step to continue the modeling if this is desired, the previous time step solution serving as an initial condition to the new time steps.

SOLUTION

Treatment of the Problem Domain

The domain shown in Figure 1 is enclosed in a rectangle with the sides in the x and y directions and the two vertical sides along boundaries (1) - (2) and (5) - (7). This rectangle is then divided by a rectangular grid mesh, thus creating nodes inside the domain and nodes outside it. A square mesh is known to give faster convergence, but in drainage problems which are usually long and shallow the rectangular mesh may be the optimum compromise. The dimensions of the rectangles of the mesh (Δx and Δy) could be chosen at will by specifying the number of rows and number of columns of the two dimensional grid. Motivated by the belief that, for numerical solutions, reporting failures is as important as reporting successes, I should report that when the rectangles were chosen with (Δx) much larger than (Δy) divergence was experienced mainly at the irregular boundaries.

The two irregular boundaries (2) - (5) and (7) - (1) were simplified by making them linear segments that go through nodes by running them only along horizontal sides or along diagonals of the rectangles. When the actual boundary intersected a vertical column of the mesh the point of intersection is moved up or down to the nearest node in that column. This gave a domain which was composed of complete rectangles and half rectangles (right angle triangles). This method of simplification is not serious at the impermeable bed which, under most field conditions, is never known precisely and in most cases is more or less a diffused boundary. On the land surface accurate surveys are possible, but it is believed that the effect of the above simplification is small and only local.

Drains were simulated at specified nodes by specifying a zero pressure head (H) at these nodes. To identify the different types of nodes in the mesh a two dimensional array of integer code numbers was computed and stored to correspond to the nodes of the mesh. Points exterior to the domain were assigned a code number of one, interior points a code number of two, drain nodes a code number of three and points on boundaries (2) - (5) and (7) - (1) were labeled with code numbers from four to 15, as illustrated in Table 1.

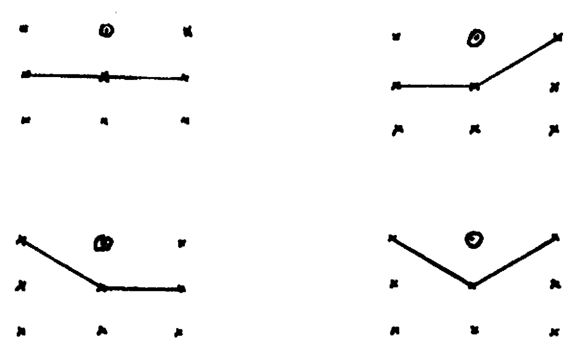
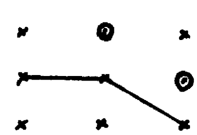
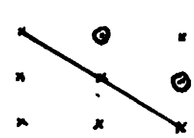
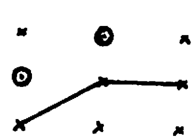
Finite Difference Operators

In the following discussion the nodes will be identified by the subscript (i) for position along the x-axis, the subscript (j) for position along the y-axis and the superscript (n) for position along the time axis.

The steady state

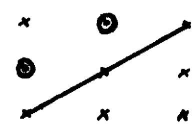
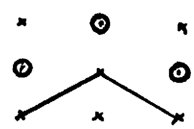
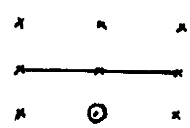
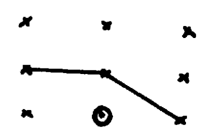
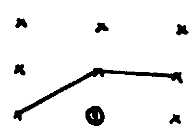
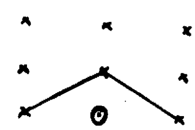
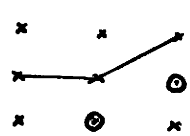
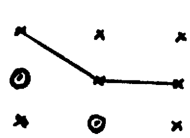
The steady state condition was solved to provide an initial condition for the transient solution. Central differences for the space variables and the five point scheme were used, thus resulting in a second order truncation error $[0(\Delta x)^2 + 0(\Delta y)^2]$. This procedure applied to Equation (36) gives:

Table 1. Calculation codes

Code number	Node type**
1	Exterior node
2	Interior node
3	Drain
4	
5	
6	
7	

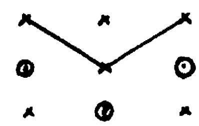
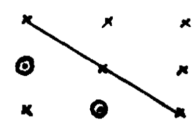
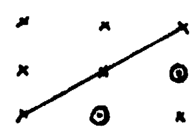
** Circled nodes are the imaginary nodes (exterior) that are needed in the five-point scheme for the central node (i, j).

Table 1. (Continued)

Code number	Node type **
8	
9	
10	 
	 
11	
12	

** Circled nodes are the imaginary nodes (exterior) that are needed in the five-point scheme for the central node (i, j).

Table 1. (Continued)

Code number	Node type**
13	
14	
15	

** Circled nodes are the imaginary nodes (exterior) that are needed in the five-point scheme for the central node (i, j).

$$\frac{1}{(\Delta x)^2}\left[\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}\right] + \frac{1}{(\Delta y)^2}\left[\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}\right]$$

$$+ \frac{F2_{i,j}}{2\Delta x}\left[\phi_{i+1,j} - \phi_{i-1,j}\right] - \frac{F3_{i,j}}{2\Delta y}\left[\phi_{i,j+1} - \phi_{i,j-1}\right] = 0 \qquad (46)$$

With algebraic manipulation this gives an explicit linear equation of $\phi_{i,j}$ in terms of the four surrounding nodes:

$$\phi_{i,j} = G4\left[F7_{i,j}\,\phi_{i+1,j} + F8_{i,j}\,\phi_{i-1,j} + F9_{i,j}\,\phi_{i,j+1} + F10_{i,j}\,\phi_{i,j-1}\right] \qquad (47)$$

where:

$$G4 = \frac{.5}{1 + \left(\frac{\Delta x}{\Delta y}\right)^2}$$

$$F7 = 1 + \frac{\Delta x}{2} F2$$

$$F8 = 1 - \frac{\Delta x}{2} F2$$

$$F9 = \left(\frac{\Delta x}{\Delta y}\right)^2 - \frac{(\Delta x)^2}{2\Delta y} F3$$

$$F10 = \left(\frac{\Delta x}{\Delta y}\right)^2 + \frac{(\Delta x)^2}{2\Delta y} F3$$

Similar differencing applied to Equation (37) gives:

$$\frac{1}{(\Delta x)^2}\left[\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}\right] + \frac{1}{(\Delta y)^2}\left[\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}\right]$$

$$+ \frac{F2_{i,j}}{2\Delta x}\left[\phi_{i+1,j} - \phi_{i-1,j}\right] + F1_{i,j}\left[\frac{\phi_{i+1,j} - \phi_{i-1,j}}{2\Delta x}\right]^2 - \frac{F3_{i,j}}{2\Delta y}\left[\phi_{i,j+1} - \phi_{i,j-1}\right]$$

$$+ F1_{i,j}\left[\frac{\phi_{i,j+1} - \phi_{i,j-1}}{2\Delta y}\right]^2 = 0 \qquad (48)$$

which gives:

$$\phi_{i,j} = G4\Big[F7_{i,j}\,\phi_{i+1,j} + F8_{i,j}\,\phi_{i-1,j} + F9_{i,j}\,\phi_{i,j+1} + F10_{i,j}\,\phi_{i,j-1}$$

$$+ F11_{i,j}\left(\phi_{i+1,j} - \phi_{i-1,j}\right)^2 + F12_{i,j}\left(\phi_{i,j+1} - \phi_{i,j-1}\right)^2\Big] \qquad (49)$$

where:

$$F11 = \frac{F1}{4}$$

$$F12 = \left(\frac{\Delta x}{\Delta y}\right)^2 \cdot \frac{F1}{4}$$

<u>The unsteady state</u>

For the transient case the Crank-Nicolson method of differencing was chosen. This method evaluates the space derivatives centrally in time as well as in space, by taking the average of the space differences at time (n) and at time (n+1). This results in a second order approximation in space and time with a truncation error of the order of

$[0(\Delta x)^2 + 0(\Delta y)^2 + 0(\Delta t)^2]$. This is an advantage over the first order approximation of the implicit method which gives a truncation error of the order of $[0(\Delta x)^2 + 0(\Delta y)^2 + 0(\Delta t)]$. For linear problems the Crank-Nicolson method gives faster convergence and larger time steps can be used due to the smaller truncation error. It is believed by workers in the field of numerical solutions to partial differential equations that these properties of the method hold also for non linear problems. The method is unconditionally stable for linear problems. Nothing comparable to this can be said about nonlinear problems.

It is interesting to note that in the literature there are two ways reported for implementing the Crank-Nicolson scheme in nonlinear problems. The first method multiplies the average of the nonlinear coefficient evaluated at (n) and (n+1) time levels, by the average of the differences at (n) and (n+1) time levels (Forsythe and Wasow, 1960; Douglas, 1961; Remson, Hornberger and Molz, 1971). The second method takes the average of the two products of the nonlinear coefficient evaluated at (n) time level multiplied by differences at (n) time level and the nonlinear coefficient evaluated at (n+1) time level multiplied by differences at (n+1) time level (Richtmyer, 1957; Jeppson, 1972). An example may best illustrate the difference. Suppose we want the Crank-Nicolson scheme for the expression:

$$a(x, y, t, u) \frac{\partial u}{\partial x}$$

Obviously we seek an approximation to:

$$a^{n+\frac{1}{2}} \left(\frac{\partial u}{\partial x}\right)^{n+\frac{1}{2}}$$

where:

$$a^{n+\frac{1}{2}} = a \ (x, y, t^{n+\frac{1}{2}}, U^{n+\frac{1}{2}})$$

The first method gives:

$$\left[\frac{A^n + A^{n+1}}{2}\right]\left[\frac{1}{2}\left(\frac{U^n_{i+1,j} - U^n_{i-1,j}}{2\Delta x} + \frac{U^{n+1}_{i+1,j} - U^{n+1}_{i-1,j}}{2\Delta x}\right)\right]$$

The second method gives

$$\frac{1}{2}\left[A^n \ \frac{U^n_{i+1,j} - U^n_{i-1,j}}{2\Delta x} + A^{n+1} \ \frac{U^{n+1}_{i+1,j} - U^{n+1}_{i-1,j}}{2\Delta x}\right]$$

To test which of the two expressions is a better approximation, and at the suggestion of Dr. Roland W. Jeppson and Dr. James D. Watson (Professor of Mathematics at Utah State University), both expressions were expanded in Taylor's series. Both were found to be second order approximations to:

$$A^{n+\frac{1}{2}} \left(\frac{\partial u}{\partial x}\right)^{n+\frac{1}{2}}$$

The second method (Jeppson, 1972) was chosen for this study as it is easier in computation. It is important to note that when the nonlinear coefficient is associated with the time derivative, the first method is used always (see Jeppson, 1972).

Applying the Crank-Nicolson scheme to Equation (35) gives:

$$\left\{ \begin{aligned} & F7_{i,j}\,\phi_{i+1,j} + F8_{i,j}\,\phi_{i-1,j} + F9_{i,j}\,\phi_{i,j+1} + F10_{i,j}\,\phi_{i,j-1} \\ & -2\left[1+\left(\frac{\Delta x}{\Delta y}\right)^2\right]\phi_{i,j} + F11_{i,j}(\phi_{i+1,j}-\phi_{i-1,j})^2 \\ & + F12_{i,j}(\phi_{i,j+1}-\phi_{i,j-1})^2 \end{aligned} \right\}^{n+1} + B^n_{i,j}$$

$$= G3.\ E5\ (F\ 131 + F\ 132)\ (\phi^{n+1}_{i,j} - \phi^n_{i,j}) \qquad (50)$$

where:

$$E5 = \frac{(\Delta x)^2}{\Delta t}$$

$$F\,13^{n+\frac{1}{2}} = \tfrac{1}{2}\ [F\ 131 + F\ 132]$$

$$F\ 131 = (F13)^n$$

$$F\ 132 = (F13)^{n+1}$$

$$B^n_{i,j} = \left\{ \begin{aligned} & F7_{i,j}\,\phi_{i+1,j} + F8_{i,j}\,\phi_{i-1,j} + F9_{i,j}\,\phi_{i,j+1} \\ & + F10_{i,j}\,\phi_{i,j-1} - 2\left[1+\left(\frac{\Delta x}{\Delta y}\right)^2\right]\phi_{i,j} \\ & + F11_{i,j}(\phi_{i+1,j}-\phi_{i-1,j})^2 + F12_{i,j}(\phi_{i,j+1}-\phi_{i,j-1})^2 \end{aligned} \right\}^{n}$$

For saturated flow, F11, F12, F13 and $B^n_{i,j}$ are all zero and Equation (50) becomes similar to Equation (47).

Treatment of the Boundary Conditions

Boundary (1) - (2)

The values of ϕ at nodes along this boundary were evaluated using Equation (38) where Y1 is read as part of the data. These values are then left static for the remainder of the solution.

Boundaries (5) - (6) and (6) - (7)

A central differencing is used for equation (44) resulting in:

$$\phi_{i+1,j} \doteq \phi_{i-1,j} - 2\Delta x \cdot \text{Sin}\,\alpha \cdot \text{Cos}\,\alpha \qquad (51)$$

A column of imaginary nodes is created at (i+1) position where (i) is the boundary, and values of ϕ at these imaginary nodes are computed using equation (51). The boundary nodes are then treated as interior nodes. Solution for ϕ at this boundary is started at the bottom node and worked up through the column of nodes. H is computed from ϕ at each node and when H is found negative for a node the values of ϕ for all the nodes above it are made equal to the value of ϕ at the node. This corresponds to the static equilibrium condition of boundary (5) - (6).

Boundaries (2) - (5) and (7) - (1)

These are the normal flux boundaries. Imaginary nodes are created as shown in Table 1 for each type of boundary geometry. Values of ϕ at these imaginary nodes are computed from ϕ values at neighboring real nodes according to the equations in Table 2. For the derivation of these equations see Appendix C. The boundary nodes are then treated as internal nodes. One more constraint was imposed on nodes along boundary (3) - (4) where the flux is not zero. Whenever the value of H at these nodes exceeded zero the value of ϕ was readjusted to give an H equal to zero. This simulates saturation but no ponding at the surface.

Treatment of Heterogeneity

As discussed before, only heterogeneity in the saturated hydraulic conductivity is considered in this study. There are methods of calculating the distribution of values of K_o from measurements of the distribution of ϕ values in a saturated heterogeneous domain of interest, (see King, 1974 and the references therein). These methods utilize the method of characteristics in partial differential equations to calculate K_o at any point along a streamline (or stream tube) from measurements of ϕ gradients and K_o at few base points in the domain. These methods are not yet of widely spread use and there are still some complications in applying them to field situations (King, 1974). For the general drainage design problem (and for the present study as well) it is not expected that there will be enough available data (especially on the distribution of ϕ) to allow application of the above methods. Characterization of heterogeneity will rather be made by the conventional method of taking measurements of K_o at different points in a field.

For the present study K_o will be assumed to vary linearly with x and with depth (T) according to the relation:

Table 2. Equations for normal flux boundaries.

Code number	Equations**
Upper boundary	
4	$\phi_{i,j+1} = \phi_{i,j-1} + E3\,\frac{q_i}{K}$
5	$\phi_{i,j+1} = \phi_{i,j-1} + E3\,\frac{q_i}{K}$
	$\phi_{i+1,j} = E10\,\phi_{i+1,j-1} + E9\,\phi_{i,j-1} + E8\,\frac{q_i}{K}$
6	$\phi_{i,j+1} = E10\,\phi_{i,j} + E9\,\phi_{i-1,j} + E8\,\frac{q_i}{K}$
	$\phi_{i+1,j} = E10\,\phi_{i+1,j-1} + E9\,\phi_{i,j-1} + E8\,\frac{q_i}{K}$
7	$\phi_{i,j+1} = \phi_{i,j-1} + E3\,\frac{q_i}{K}$
	$\phi_{i-1,j} = E10\,\phi_{i-1,j-1} + E9\,\phi_{i,j-1} + E8\,\frac{q_i}{K}$
8	$\phi_{i,j+1} = E10\,\phi_{i,j} + E9\,\phi_{i+1,j} + E8\,\frac{q_i}{K}$
	$\phi_{i-1,j} = E10\,\phi_{i-1,j-1} + E9\,\phi_{i,j-1} + E8\,\frac{q_i}{K}$
9	$\phi_{i,j+1} = \phi_{i,j-1} + E3\,\frac{q_i}{K}$
	$\phi_{i+1,j} = E10\,\phi_{i+1,j-1} + E9\,\phi_{i,j-1} + E8\,\frac{q_i}{K}$
	$\phi_{i-1,j} = E10\,\phi_{i-1,j-1} + E9\,\phi_{i,j-1} + E8\,\frac{q_i}{K}$

Table 2. (Continued)

Code number	Equations**
Lower boundary	
10	$\phi_{i,j-1} = \phi_{i,j+1}$
11	$\phi_{i,j-1} = \phi_{i,j+1}$ $\phi_{i+1,j} = E10\,\phi_{i+1,j+1} + E9\,\phi_{i,j+1}$
12	$\phi_{i,j-1} = \phi_{i,j+1}$ $\phi_{i-1,j} = E10\,\phi_{i-1,j+1} + E9\,\phi_{i,j+1}$
13	$\phi_{i,j-1} = \phi_{i,j+1}$ $\phi_{i+1,j} = E10\,\phi_{i+1,j+1} + E9\,\phi_{i,j+1}$ $\phi_{i-1,j} = E10\,\phi_{i-1,j+1} + E9\,\phi_{i,j+1}$
14	$\phi_{i-1,j} = E10\,\phi_{i-1,j+1} + E9\,\phi_{i,j+1}$ $\phi_{i,j-1} = E10\,\phi_{i,j} + E9\,\phi_{i+1,j}$
15	$\phi_{i+1,j} = E10\,\phi_{i+1,j+1} + E9\,\phi_{i,j+1}$

Table 2. (Continued)

Code number	Equations **
15	$\phi_{i,j-1} = E10\,\phi_{i,j} + E9\,\phi_{i-1,j}$

** $E3 = 2\Delta y$

** $E8 = \Delta y$

** $E9 = (\frac{\Delta y}{\Delta x})^2$

** $E10 = 1 - (\frac{\Delta y}{\Delta x})^2$

$$K_o = C1 + C2x + C3T \qquad (52)$$

The coeffecients C1, C2 and C3 are found by fitting the actual K_o measurements to Equation (52) by a least squares method. (For least sqares method see Kreider et al, 1966.) The matrix equation that results in the least squares treatment has a symmetric positive definite matrix (see Appendix D). The equation is solved by two subroutines (Decompose and Solve) adapted from Weaver (1967). Once the coefficients of Equation (52) are known the values of K_o at the nodes are calculated and stored in a two-dimensional array that corresponds to the mesh nodes. Since Equation (52), especially in extrapolation of data, may give values that may be too high or too low, two constraints $K_o$max and $K_o$min are imposed on the values calculated by Equation (52). Also $\frac{\partial k_o}{\partial x}$ and $\frac{\partial K_o}{\partial T}$ are calculated.

Method of Solution

The steady state

Equations (47) and (49) for the saturated and the unsaturated zones respectively, when applied at the nodes of the mesh, result in a system of simultaneous algebraic equations. This system is solved by the SOR iterative procedure. The solution in each iteration proceeds systematically from the leftmost column of nodes to the right and from the bottom node up in each column. Exterior nodes and drain nodes are skipped. At each of the interior and boundary nodes, H is evaluated and if the node is found saturated Equation (47) is applied. If the node is found unsaturated Equation (49) is applied. Saturation is arbitrarily defined by the user as a node is considered saturated if H at the node is equal or larger than Z of Equation (27) where Z is specified by the user. In this study Z was taken to be zero. Starting from an initial educated guess for the values of ϕ at the nodes each sweep through the nodes (one iteration) will improve the values of ϕ at the nodes toward the solution. This procedure is iterated until the sum of the absolute values of the improvements at all the nodes is less than a specified small value. This value was specified at 0.001 foot for this study. The solution should converge starting from practically any initial guess, but a close initial guess greatly cuts down the time needed for convergence. A close initial guess can be obtained using any one of the approximate steady state equations referred to in the review of literature (for example

Keller and Robinson, 1959). In the present study convergence was obtained from an initial guess of the position of the water table much above the solution and from one below the final solution.

Equation (49) gives rise to a system of nonlinear algebraic equations because of the dependence of $F11_{i,j}$ and $F12_{i,j}$ on $\phi_{i,j}$. These equations are linearized by evaluating $F11_{i,j}$ and $F12_{i,j}$ from known values of $\phi_{i,j}$ at the previous iteration. Thus if we introduce the iteration index m, Equation (49) at the advanced iteration will be:

$$\phi_{i,j}^{m+1} = G4 \Big[F7_{i,j} \cdot \phi_{i+1,j}^{m} + F8_{i,j} \cdot \phi_{i-1,j}^{m+1} + F9_{i,j} \cdot \phi_{i,j+1}^{m}$$

$$+ F10_{i,j} \cdot \phi_{i,j-1}^{m+1} + F11_{i,j}^{m} (\phi_{i+1,j}^{m} - \phi_{i-1,j}^{m+1})^2$$

$$+ F12_{i,j}^{m} (\phi_{i,j+1}^{m} - \phi_{i,j-1}^{m+1})^2 \Big] \qquad (53)$$

With the iteration index, Equation (47) will look like Equation (53) without the terms containing $F11_{i,j}$ and $F12_{i,j}$.

After the value of $\phi_{i,j}^{m+1}$ is found by Equation (53) it is over-relaxed according to Equation (54) below.

$$\overline{\phi_{i,j}^{m+1}} = \phi_{i,j}^{m} + W(\phi_{i,j}^{m+1} - \phi_{i,j}^{m}) \qquad (54)$$

Where $\overline{\phi_{i,j}^{m+1}}$ is the over-relaxed value of $\phi_{i,j}^{m+1}$ and W is an over-relaxation factor. As mentioned before, there are methods of calculating an optimum value of W for linear problems with certain simple geometries (a square or a rectangle). These optimum values give the fastest convergence to the solution. For nonlinear problems with irregular geometries there are no such methods. Even in the linear cases with simple geometries, sometimes the computer time spent in calculating an optimum value of W is more than the computer time saved by its use (Forsythe and Wasow,1960). For the present study few values of W were tried and the number of iterations and time required for convergence were observed. This resulted in a choice of W = 1.5 for the saturated nodes and W = 1.0 for the unsaturated nodes.

The unsteady state

The same SOR iterative procedure and the same method of linearization which worked for the steady state were tried for the unsteady state Equation (50) (which is nonlinear in the unsaturated zone). This procedure did not converge and showed undamped oscillation at the upper flux boundary. The reason for that was, most understandably, the method of linearization. At one iteration the value of $\phi_{i,j}^{m}$ is very low, K_r is very small, the normal gradient needed to effect the specified flux must necessarily be very large resulting in a very high value of ϕ at the imaginary nodes and a saturated $\phi_{i,j}^{m+1}$. In the next iteration $\phi_{i,j}^{m+1}$ is saturated, $K_r = K_o$ which allows the specified flux at a small normal gradient and the resulting $\phi_{i,j}^{m+2}$ is too small. This kept oscillating.

Because of the above difficulty the method was changed to a scheme of two nested iterations. The inner iteration is a Newton iteration to solve the nonlinear equation at a node with the neighboring nodes fixed, and the outer iteration is the usual SOR iteration.

The Newton iteration could be described by:

$$\left(\phi_{i,j}^{n+1}\right)^{K+1} = \left(\phi_{i,j}^{n+1}\right)^{K} - \frac{FN}{FNP} \tag{55}$$

where:

K = The iteration index of the Newton iteration

$$FN = f\left(\phi_{i,j}^{n+1}\right)^{K}$$

$$FNP = \left(\frac{\partial FN}{\partial \phi_{i,j}^{n+1}}\right)^{K}$$

From Equation (50) we get:

$$FN = F7_{i,j} \cdot \phi_{i+1,j}^{n+1} + F8_{i,j} \cdot \phi_{i-1,j}^{n+1} + F9_{i,j}\, \phi_{i,j+1}^{n+1} + F10_{i,j} \cdot \phi_{i,j-1}^{n+1}$$

$$- (2 + 2E2 + G5)\, \phi_{i,j}^{n+1} + F11_{i,j}^{n+1} \left(\phi_{i+1,j}^{n+1} - \phi_{i-1,j}^{n+1}\right)^2$$

$$+ F12_{i,j}^{n+1} \left(\phi_{i,j+1}^{n+1} - \phi_{i,j-1}^{n+1}\right)^2 + G5\, \phi_{i,j}^{n} + B_{i,j}^{n} = 0 \tag{56}$$

$$FNP = F7_{i,j} \cdot F17 + F8_{i,j}\, F18 + F9_{i,j}\, F19 + F10_{i,j}\, F20 - F25\, \phi_{i,j}^{n+1}$$

$$- (2 + 2E2 + G5) + 2F11^{n+1}_{i,j} \left(\phi^{n+1}_{i+1,j} - \phi^{n+1}_{i-1,j}\right)(F17 - F18)$$

$$+ F21\left(\phi^{n+1}_{i+1,j} - \phi^{n+1}_{i-1,j}\right)^2 + 2F12^{n+1}_{i,j}\left(\phi^{n+1}_{i,j+1} - \phi^{n+1}_{i,j-1}\right)(F19 - F20)$$

$$+ F22\left(\phi^{n+1}_{i,j+1} - \phi^{n+1}_{i,j-1}\right)^2 + F25\,\phi^{n}_{i,j} \qquad (57)$$

where:

$$E2 = \left(\frac{\Delta x}{\Delta y}\right)^2$$

$$G5 = G3 \cdot E5 \cdot F13^{n+\frac{1}{2}} = G3 \cdot E5\,(F131 + F132)$$

$$F17 = \frac{\partial}{\partial\phi_{i,j}}(\phi_{i+1,j})$$

$$F18 = \frac{\partial}{\partial\phi_{i,j}}(\phi_{i-1,j})$$

$$F19 = \frac{\partial}{\partial\phi_{i,j}}(\phi_{i,j+1})$$

$$F20 = \frac{\partial}{\partial\phi_{i,j}}(\phi_{i,j-1})$$

$$F21 = \frac{\partial}{\partial\phi_{i,j}}(F11_{i,j})$$

$$F22 = \frac{\partial}{\partial\phi_{i,j}}(F12_{i,j})$$

$$F25 = \frac{\partial G5}{\partial\phi_{i,j}}$$

In all of the above expressions the variables are taken at the (n+1) time level. Table 3, where,

$$F = \frac{\partial}{\partial \phi_{i,j}} \left(\frac{q_i}{K_o K_r} \right)$$

shows the values of F17, F18, F19 and F20 for the different calculation code numbers. These values are derived from the equations in Table 2. The other derivatives indicated are:

$$F = - \frac{G1 \cdot q_i}{K_o} (h)^{\tau-1}$$

$$F21 = \frac{G1}{4} \cdot Kr \cdot (h)^{\tau-2} [G1 \cdot K_r \cdot (h)^{\tau} + 1 - \tau]$$

$$F22 = E2 \cdot F21$$

$$F25 = \frac{G3 \cdot E5}{K_o} \cdot \frac{Se^2}{K_r} \cdot (h)^{\lambda-2} [2G2 \cdot Se \cdot (h)^{\tau} - G1 \cdot K_r \cdot (h) + 1 - \lambda]$$

This method of solution also did not work, especially at the upper flux boundary. This brought into focus the warnings of Douglas (1961) and Remson, Hornberger and Molz (1971) about the unworkability of using Crank-Nicolson averaging process at a boundary with a normal derivative boundary condition. For such cases both authors suggested differencing the boundary condition completely at the advanced time level.

The implicit differencing scheme was then used for the nodes at the normal derivative boundaries and the Crank-Nicolson scheme was retained for the interior nodes. This worked well in the solution, but

Table 3. Derivatives for Newton iteration

Code number	F17	F18	F19	F20
2	0	0	0	0
4	0	0	F. E3	0
5	F. E8	0	F. E3	0
6	F. E8	0	E10 + F. E8	0
7	0	F. E8	F. E3	0
8	0	F. E8	E10 + F. E8	0
9	F. E8	F. E8	F. E3	0
10	0	0	0	0
11	0	0	0	0
12	0	0	0	0
13	0	0	0	0
14	0	0	0	E10
15	0	0	0	E10

doubt is shed on the order of the global truncation error now, and whether the method as such would retain the second order truncation error (in time) of the Crank-Nicolson scheme, (Jeppson, 1974, verbal discussion).

The implicit operator can be obtained by differencing Equation (35) completely at the advanced time level, or from Equation (50) by setting $B^n_{i,j} = 0$, F13 = F132 and G5 = G3 · E5 · F132. The expressions for FN and FNP are similar to Equations (56) and (57) respectively, with $B^n_{i,j} = 0$, F13 = F132 and G5 as described above. The derivatives indicated for the Crank-Nicolson are not changed.

Since the values of $\phi_{i,j}$ computed in a Newton iteration are not final values, this iteration is not carried to a high degree of improvement. The iterations are terminated after three iterations or when the improvement in any iteration is less than 0.01 whichever comes first. The program however is flexible in these indeces as they are read as data.

For the saturated nodes Equation (35) simplifies to the linear Equation (36) and the usual SOR iteration (outer iteration only) is used with Equation (47).

THE COMPUTER PROGRAM

Description

The program is written in Fortran IV language. A listing of the program is shown in Appendix A. It consists of the following four main blocks and two subroutines.

1. Setting the problem

Data for the geometry of the problem and elevations of the land surface and the bed at different points are read. The geometry of the problem is set, divided by a mesh with pre-specified numbers of nodes in the x-direction and the y-direction, and calculation codes are computed and stored for each node.

2. Setting the heterogeneous saturated conductivity

Point data of coupled values of x, T and K_o are read for several measured points. The program sets the matrix equation for the coefficients of Equation (52) and the matrix equation is solved by the two subroutines Decompose and Solve, as described in Appendix D. There is an option of by-passing this block for homogeneous soils, as the program was constructed to handle both cases. This is done by reading in a code for heterogeneity (KCODE) as a data input. If KCODE = 0, the domain is specified as homogeneous, K_o is read and $\frac{\partial K_o}{\partial x}$ and $\frac{\partial K_o}{\partial T}$ are set equal to zero and the block is by-passed.

3. The steady state solution

The initial guess of the distribution of ϕ at the nodes, a one dimensional array of the values of q_i ($i = 1, 2$ number of columns of the mesh) and values for the over-relaxation parameters W1 and W2 are read. The solution then proceeds as described before in the section on method of solution.

4. The unsteady state solution

The over-relaxation parameters W3 and W4 and the number of time steps are read. For each time step the following data are read:

1. Magnitude of the time step
2. An extrapolation code number:

   0: No extrapolation of the results of the previous time step into the present step.

   1: Extrapolate.
3. A printing code number.

   0: No printing required, 1: Print
4. A code for the flux q:

   0: $q_i = 0$ is set for zero flux

   1: q_i is read

The solution then proceeds as described earlier.

The output of the program consists of the calculation code array, K_o array, the coefficients of the equation of heterogeneity, and for the steady state and each time step the values of ϕ and H at each node. The position of the water table is then easily drawn by interpolation (or extrapolation) from the values of H at the nodes.

Results and Discussion

A hypothetical drainage problem on sloping land was used in developing the program. The geometry of this problem is shown in Figure 2, where the broken lines are the actual land surface and impermeable bed boundaries while the stepped boundaries are these same boundaries as simplified by the computer. The soil was assumed heterogeneous and hypothetical hydraulic conductivity data were supplied. These data when fitted by the least squares method gave:

$$K_o = 0.32711 - 0.00001x - 0.01109T \qquad (58)$$

$$0.02 \leq K_o \leq 0.5$$

Where K_o is in feet per hour and x and T are in feet. the constraints on K_o are used because the least squares method may give minus values or unreasonably high values for K_o when the data are extrapolated.

The soil parameters required in Equations (27), (29), and (31) were arbitrarily (but reasonably) set as follows:

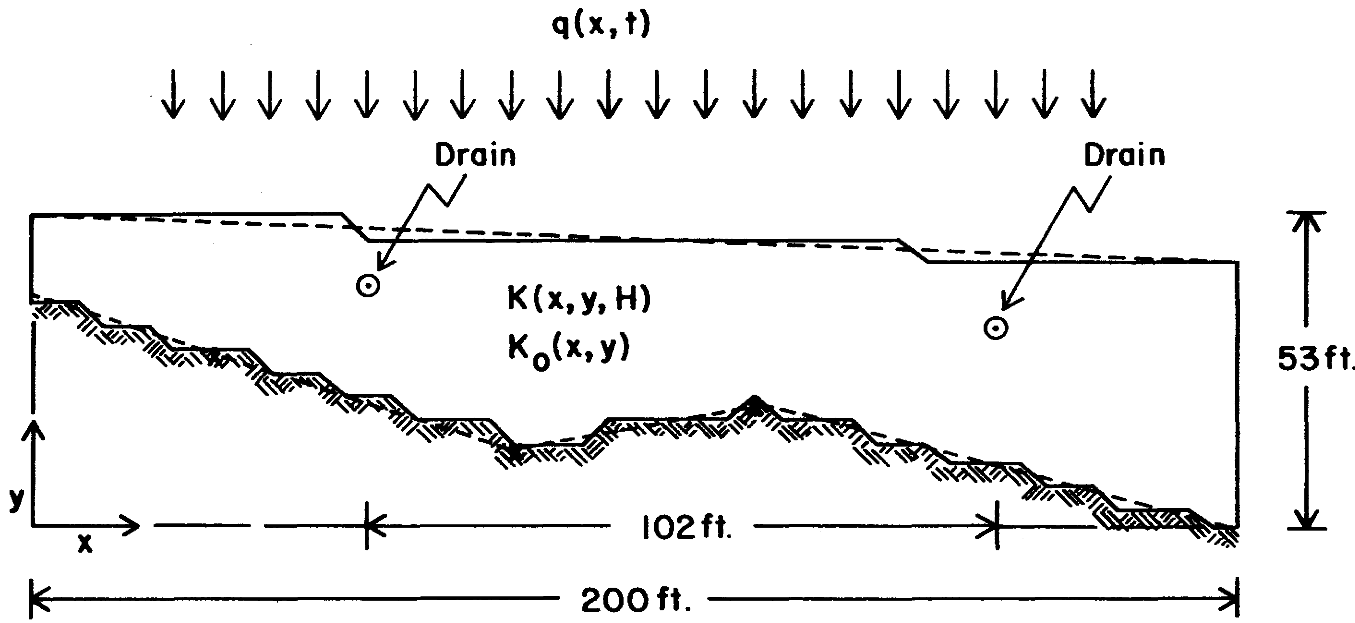


Figure 2. Geometry of the hypothetical problem.

$A = 1.0, \quad B = 1.0, \quad C = 1.0, \quad D = 1.0$

$h_b = 1.5$ ft, $\tau = 6.5$, $\lambda = 1.5$, $S_r = 0.15$

$\eta = 0.35$, $Z = 0$ ft

The general slope of the impermeable bed was about 19 percent.

Two runs with two different rates of recharge were run for this problem. One run had a very high flux rate to impose a severe condition on the program. The other run was more in conformity with the usual sprinkler irrigation practice as far as the application rate was concerned. Table 4 shows the rates and total amounts of recharge. Water was applied for five hours.

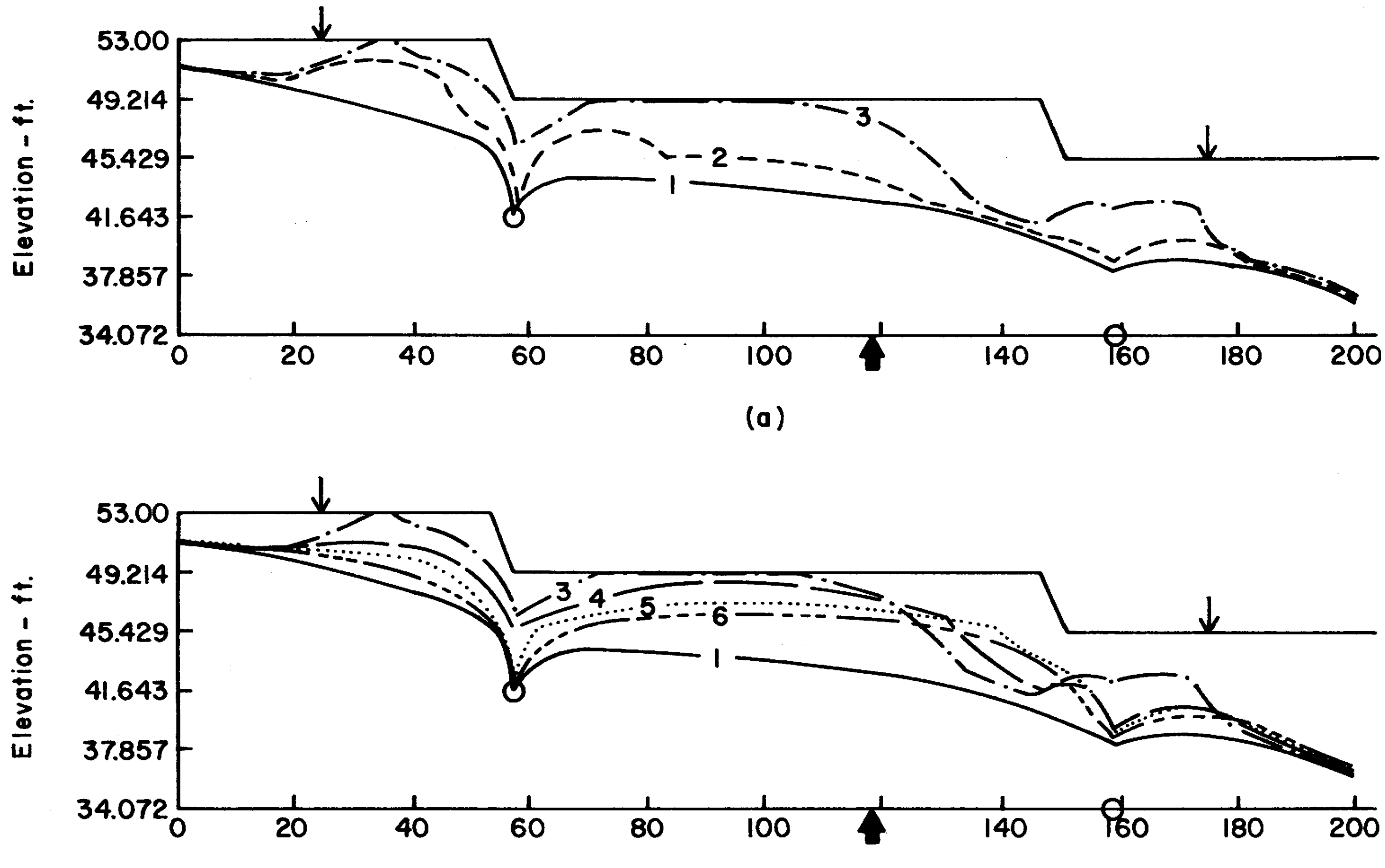
Table 4. Rates of recharge for the hypothetical problem

Node column	Run 1		Run 2	
	Rate ft/hour	Total ft	Rate ft/hour	Total ft
1 - 6	0	0	0	0
7	0.05	0.25	0.01	0.05
8	0.08	0.40	0.016	0.08
9 - 41	0.10	0.50	0.02	0.10
42	0.09	0.45	0.018	0.09
43	0.08	0.40	0.016	0.08
44	0.05	0.25	0.01	0.05
45 - 50	0	0	0	0

Figure 3a shows the water table buildup for run 1. The water table mounds and depressions seem exaggerated because of the distorted (enlarged) vertical scale. It is interesting to notice in this figure that the water table in some parts of the region (from $x \approx 130$ to $x \approx 145$) did not rise much while in others it was raised greatly. The probable reason for this is that in this region, having the maximum unsaturated thickness above the water table, most of the recharge is still in transient storage in the unsaturated zone at five hours. Figure 3b shows the water table recession for run 1. The water table is receding in some parts, but is still building up in the region $x \approx 130$ to $x \approx 145$ even at 39.45 hours after recharge stopped. This could be explained by the fact that this region is receiving water from the transient storage plus water from higher lands upslope after the recharge stops.

Figures 4a and 4b show the water table build-up and recession respectively for the run with one fifth the rate of recharge (run 2).

Figure 5 shows the water table build-up and part of the recession for the point midway between the two drains, for both run 1 and run 2. Curves A and B are for run 1. The difference between the two curves is that in curve A the position of the water table is found by linear interpolation between the uppermost saturated node and the lowest unsaturated node, while in curve B the position is found by linear extrapolation from the uppermost saturated node to zero H using $\frac{\partial H}{\partial y} = -1$. Obviously, both methods are only approximations, as the distribution of H is not linear especially when there is infiltration and more so across



Recharge stopped after 5 hours

1: Steady state with no local recharge

2: At 2.0 hours after recharge began

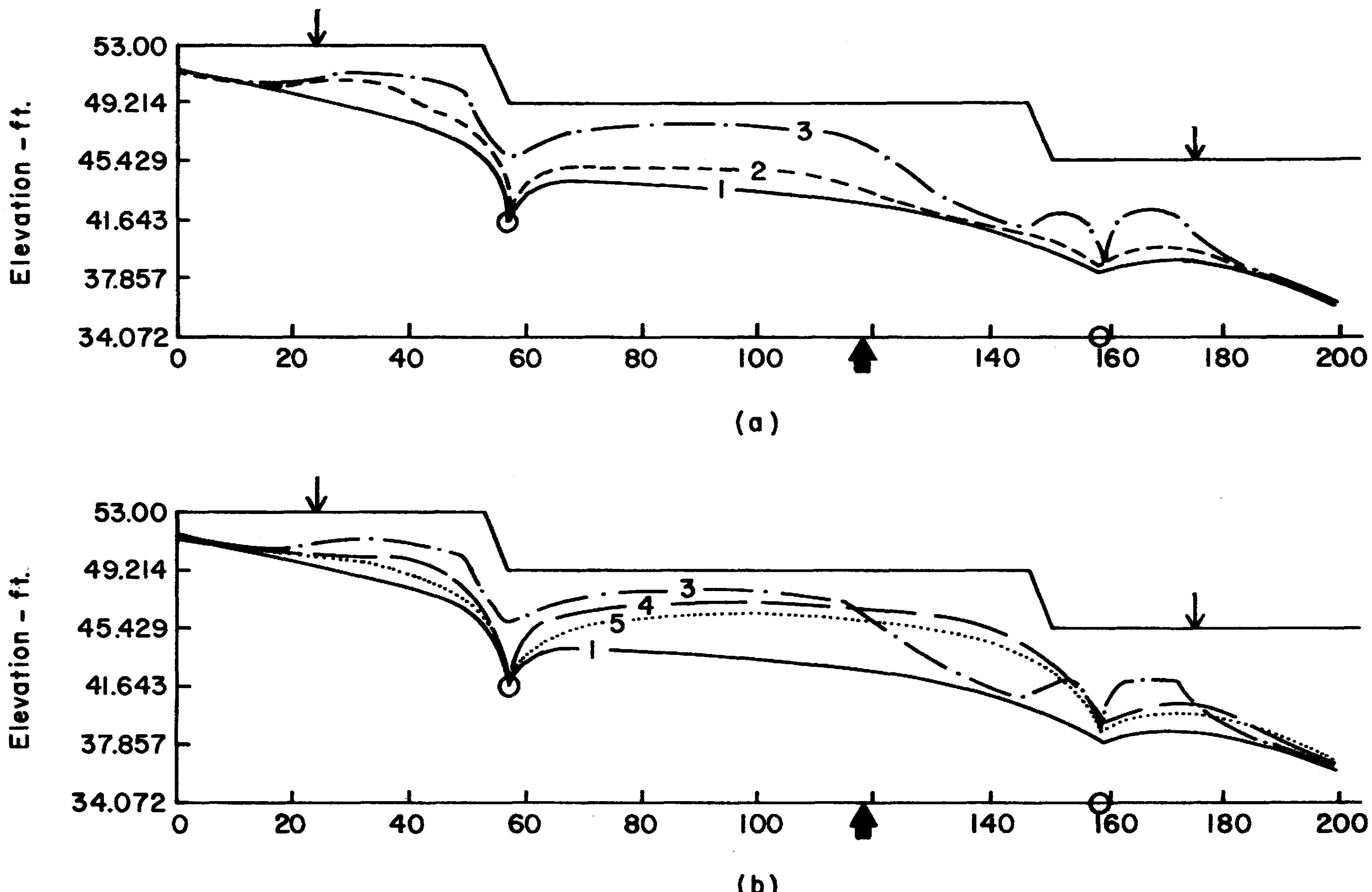
3: At 5.0 hours

4: At 11.45 hours

5: At 44.45 hours

6: At 186.45 hours

Figure 3. Water table response in hypothetical run 1. (Higher recharge rate)



Recharge stopped after 5 hours

1: Steady state with no local recharge

2: At 2.0 hours after recharge began

3: At 5.0 hours

4: At 44.45 hours

5: At 186.45 hours

Figure 4. Water table response in the hypothetical run 2. (Lower recharge rate)

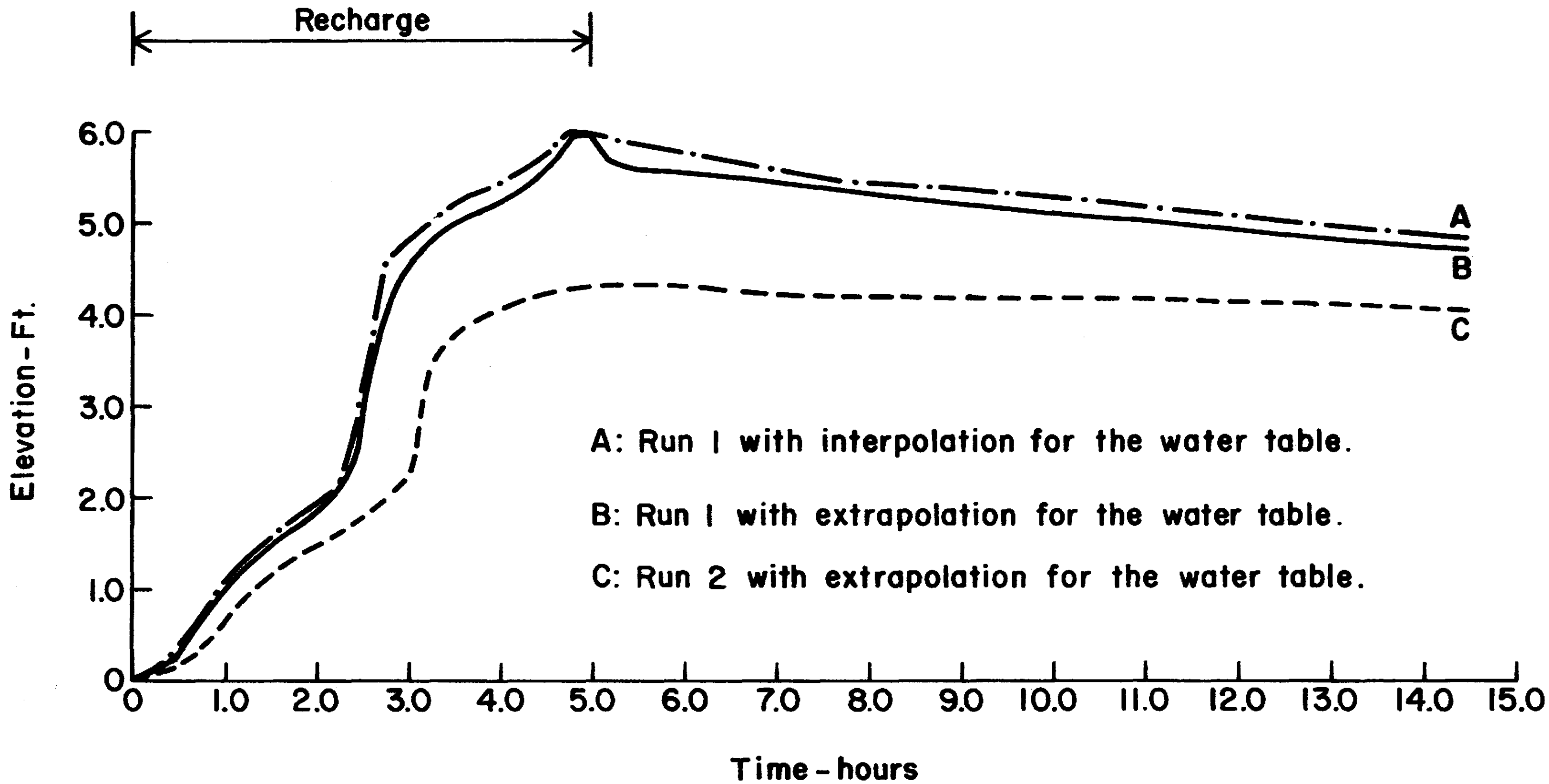


Figure 5. Water table build-up at the mid point between drains.

a wetting front. Due to the large mesh size and because interpolation would raise the water table instantaneously when the upper unsaturated node is wetted (which is not physically the case), the water position for all the other figures was found by extrapolation as described above. Figure 5 shows that for both runs, and for a certain period, the rate of rise of the water table is much larger than the rate of rise before or after that period. This period was from t = 2.25 hours to t = 3.0 hours for run 1 and from t = 3.0 hours to t = 3.5 hours for run 2. This can be explained by assuming that this was the time when extra seepage from upslope reached the midpoint. The saturated thickness of flow is quickly built up to pass that extra seepage. This is supported by the lag of 0.75 hours in run 2 for this to happen, and by noticing that the magnitude of the buildup for this seepage was smaller in run 2 than in run 1, which is to be expected physically. In both runs the very early response of the water table to recharge is due to the large mesh size.

Figure 6 shows the water table recession with time at the midpoint between the two drains. Time is started from t = 5.0 hours for run 2 but from t = 5.25 hours for run 1 because in this case the land surface at the midpoint was saturated at t = 5.0 hours. The curves are plotted on semilogarthmic paper with the two dimensionless parameters

$\frac{y}{y_o}$ vs. $\beta \left(\frac{t}{L^2}\right)$

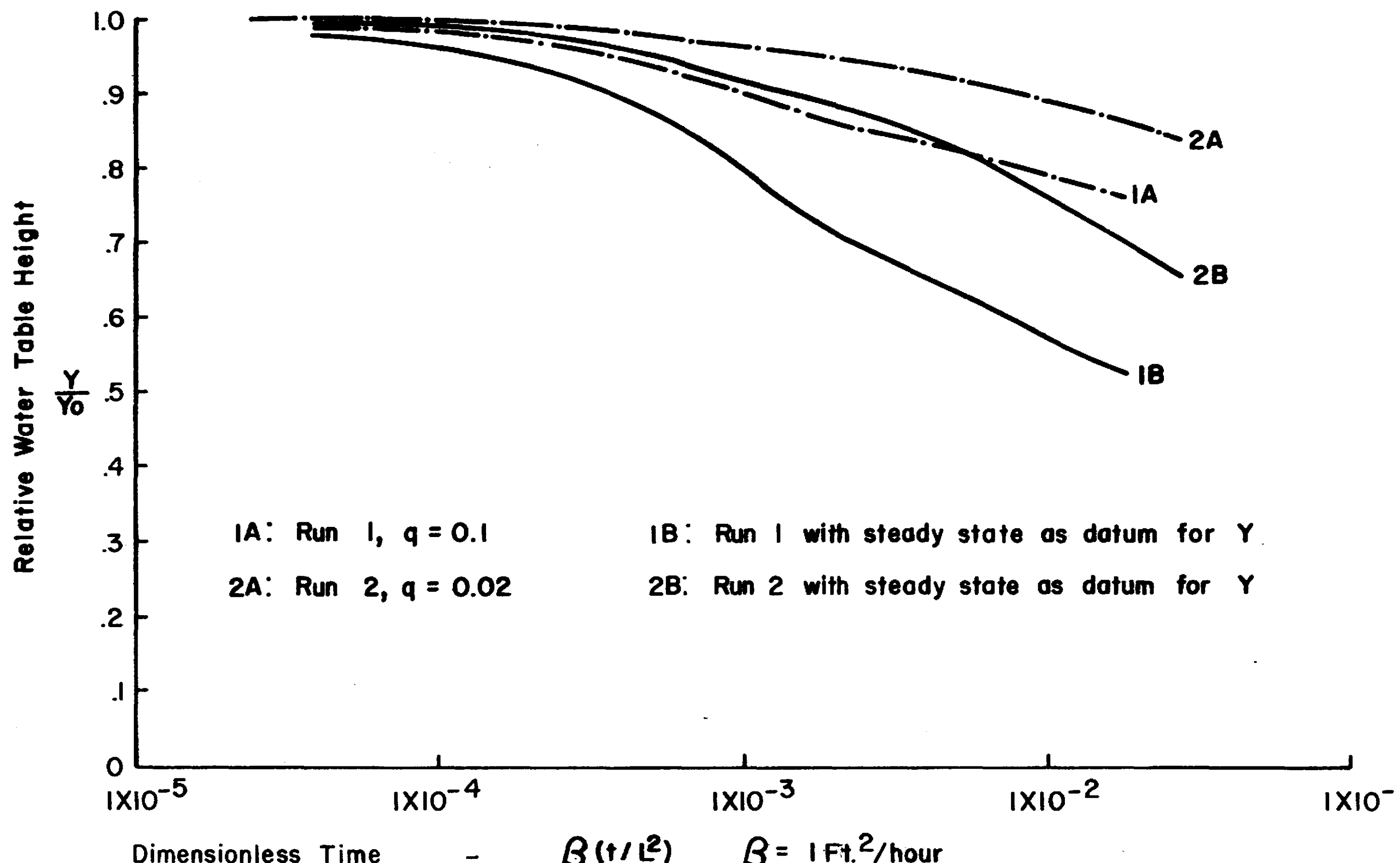


Figure 6. Water table recession for the hypothetical problem.

Where:

y = Height of the water table at the midpoint above the straight line joining the two drains.

y_o = y at time zero

β = A parameter with dimentions ft^2/hour

$\beta = 1\ ft^2$/hour in these curves.

These curves are presented in this form for a later comparison with some approximate mathematical solutions for the falling water table in flat lands. In curves 1A and 2A in this figure the datum for y and y_o is the midpoint of the straight line joining the two drains. In curves 1B & 2B the datum was taken as the position of the water table at the midpoint in the case of steady state with no local recharge. Two things to note about these curves are that they are very flat and that the curves for run 2 (lower volume and rate of recharge) are higher than the curves for run 1. The flatness of these curves is, most probably, due to the fact that a part of the recharge was still in transit through the unsaturated zone at the time of termination of recharge. The percolation of this water to the saturated zone below will slow down the water table recession. In fact this percolation in run 2, from $t = 0$ to $t = 0.25$ hours after stopping the recharge, was so great that it produced a rise in the water table instead of a recession. This effect was more pronounced in run 2 because more of the recharge was in transit in the unsaturated zone (thicker unsaturated zone) in this run than in run 1. This gave a flatter and higher curve for run 2. Another factor in the flatness of the curves is the slope. These factors will be discussed in more detail in a later section when the numerical solution is compared with some approximate mathematical solutions.

Testing of the Program

For proper testing of the program, the best thing would be a controled field experiment designed and executed for that purpose. This however would be too costly and time consuming. Available data were searched for something suitable for a test of the program. No suitable data were found for the rising water table where very frequent measurements of the water table elevations are required during the rise. Some good data for the falling water table were available for the Hullinger farm near Vernal, Utah. The data are those of Khalil-Ur-Rehman (1971) who recorded measurements of the water table depths daily for six days with hourly measurements in the early phases of the water table recession. Fortunately, data on the saturated hydraulic conductivity, other soil parameters and depths to the impermeable layer were also available for this farm.

Figure 7 shows the domain of the testing problem (with the vertical scale greatly enlarged). This is a section across drains 6 and 5 in the Hullinger farm. Again, the broken lines give the actual boundaries while the stepped boundaries are the computer boundaries. Data for the saturated hydraulic conductivity and elevations of land surface and impermeable layer were taken from unpublished data by Dr. Larry G. King.

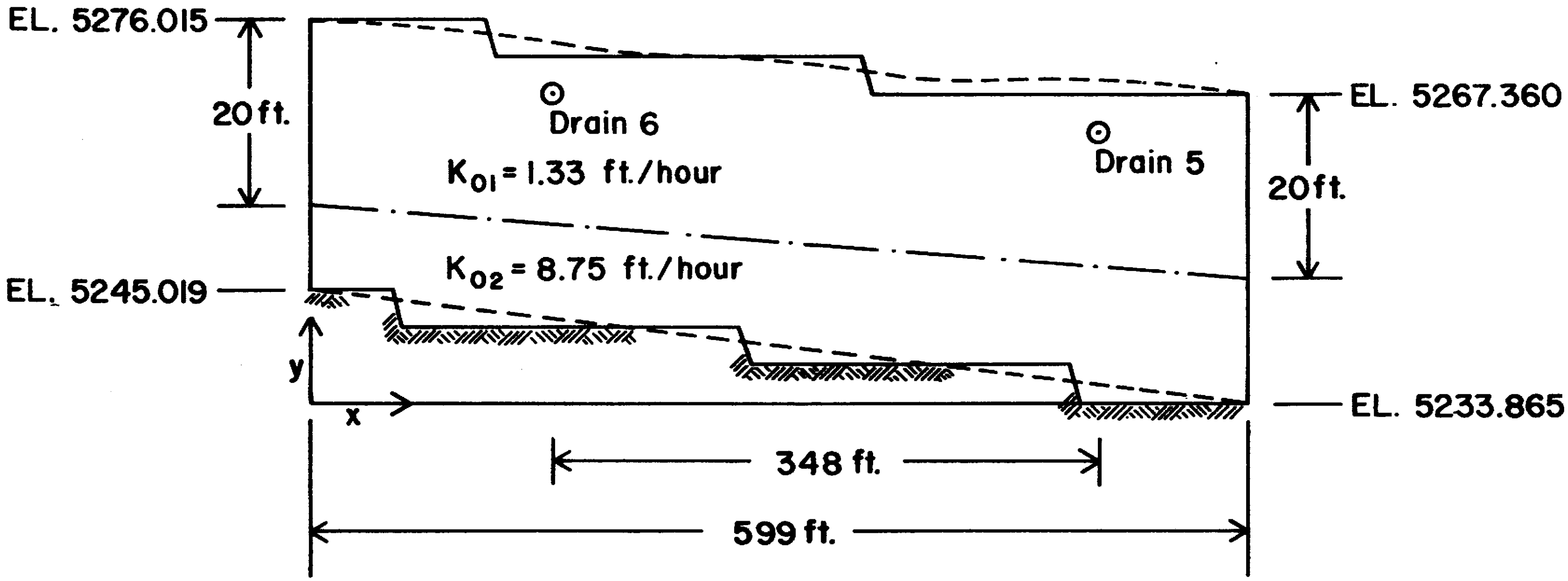


Figure 7. Hullinger farm testing problem.

Data on water table positions shown in Table 6 in Appendix E were taken from Table 22 of Khalil-Ur-Rehman (1971). Table 7 in Appendix E shows the relationship of K, Θ and H. This table is a reproduction of Table 51 of King and Hanks (1973) which is based on laboratory results by Andrade (1971).

The soil is a two-layered system as shown in Figure 7. The program however changes it into a one-layered heterogeneous system with

$$K_o = 0.09846 + 0.00011\ x + 0.30020\ T \tag{59}$$

$$1.33 \leq K_o \leq 8.75$$

Where K_o is in feet per hour and x, T are in feet. The general slope of the bed is about 2 percent.

From Table 7 in Appendix E, η is found, and using the methods of Brooks and Corey (1964), Sr, Pb, and τ are found. With these values known, fitting of the data to Equations (27) and (29) gave the values of A, B, C, D, and τ in these equations. The following values of these parameters were found for the Hullinger farm soil:

$A = 1.0$ $B = 1.0$ $C = 4.0$ $D = 4.0$

$h_b = 0.46$ ft $\tau = 2.81$ $\lambda = 0.527$ $Sr = 0.13$

$\eta = 0.48$ $Z = 0$ ft.

The above values of τ and λ are not any more related to each other by the Brooks-Corey (1964) relationship (Equation 17). However, to check the data they were fitted to Brooks-Corey (1964) equations (Equations 3, and 16) and the resulting values of τ and λ were in excellent agreement with Equation (17), (λ = 0.28 and τ = 2.81). This gave confidence in the data. Figures 8 and 9 show the data of Table 7 and the predictions of equations (27) and (29) with the above values of the soil parameters. In both figures the divergence of the table data from the predictions at high suction heads was disregarded because the data in this range were only extrapolations of Andrade's (1971) laboratory results. A second reason for the neglect of this divergence is the fact that in the drainage problem we are not likely to deal with such large suction heads (the divergence was for heads of 40 meters and above or 131 feet and above). A third reason for neglecting the divergence in the effective saturation curve (Figure 9) was the recommendation of Brooks and Corey (1964) to neglect this divergence at high heads as the value of S_e becomes very sensitive to the choice of the value of S_r at high heads.

The falling water table case on the Hullinger farm was solved by the program starting from an initial condition very close to the initial condition of Khalil-Ur-Rehman (1971). Differences in the two conditions, however, were inevitable because of the changes the computer program introduces on the geometry of the problem and because the initial distribution of values of ϕ at the nodes could only be grossly approximated from our knowledge of the initial water table position. The fall of the water table was simulated for a period of 349.55 hours (14.55 days). Figures 10 and 11 show the results of the numerical solution together

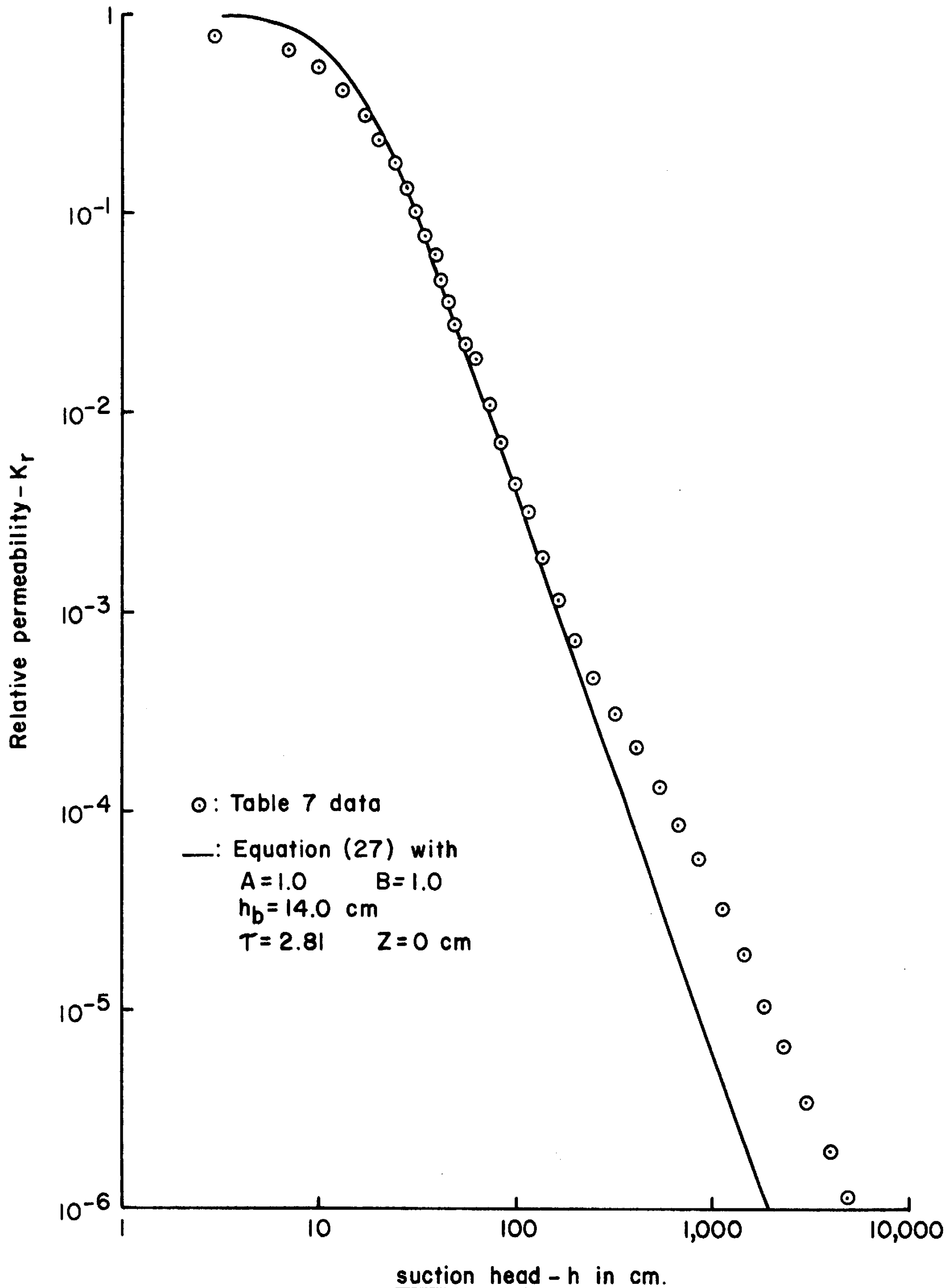


Figure 8. $K_r - h$ relationship for the Hullinger farm soil.

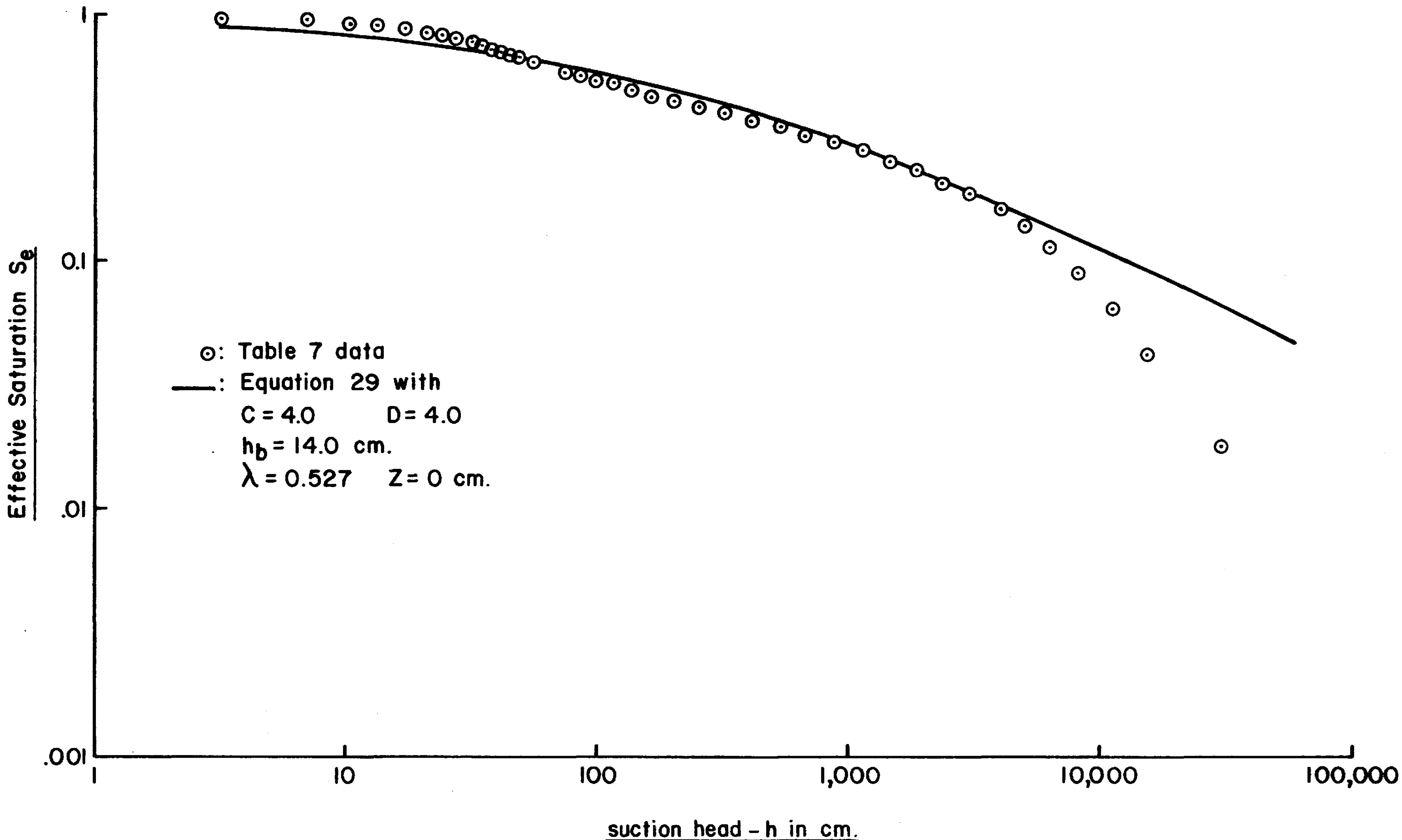


Figure 9. S_e – h relationship for the Hullinger farm soil.

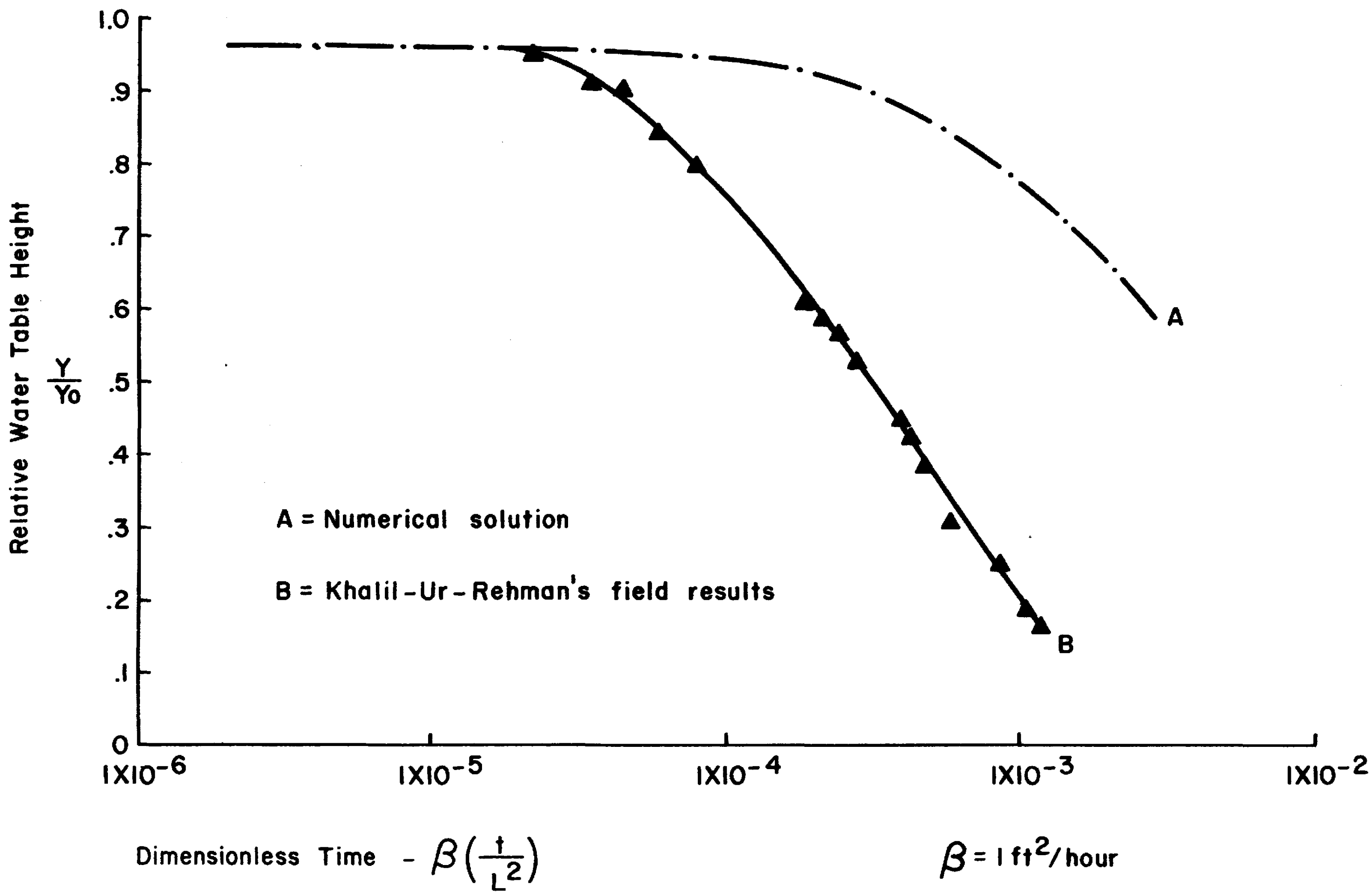


Figure 10. Water table recession for the Hullinger farm, Drains 5,6.

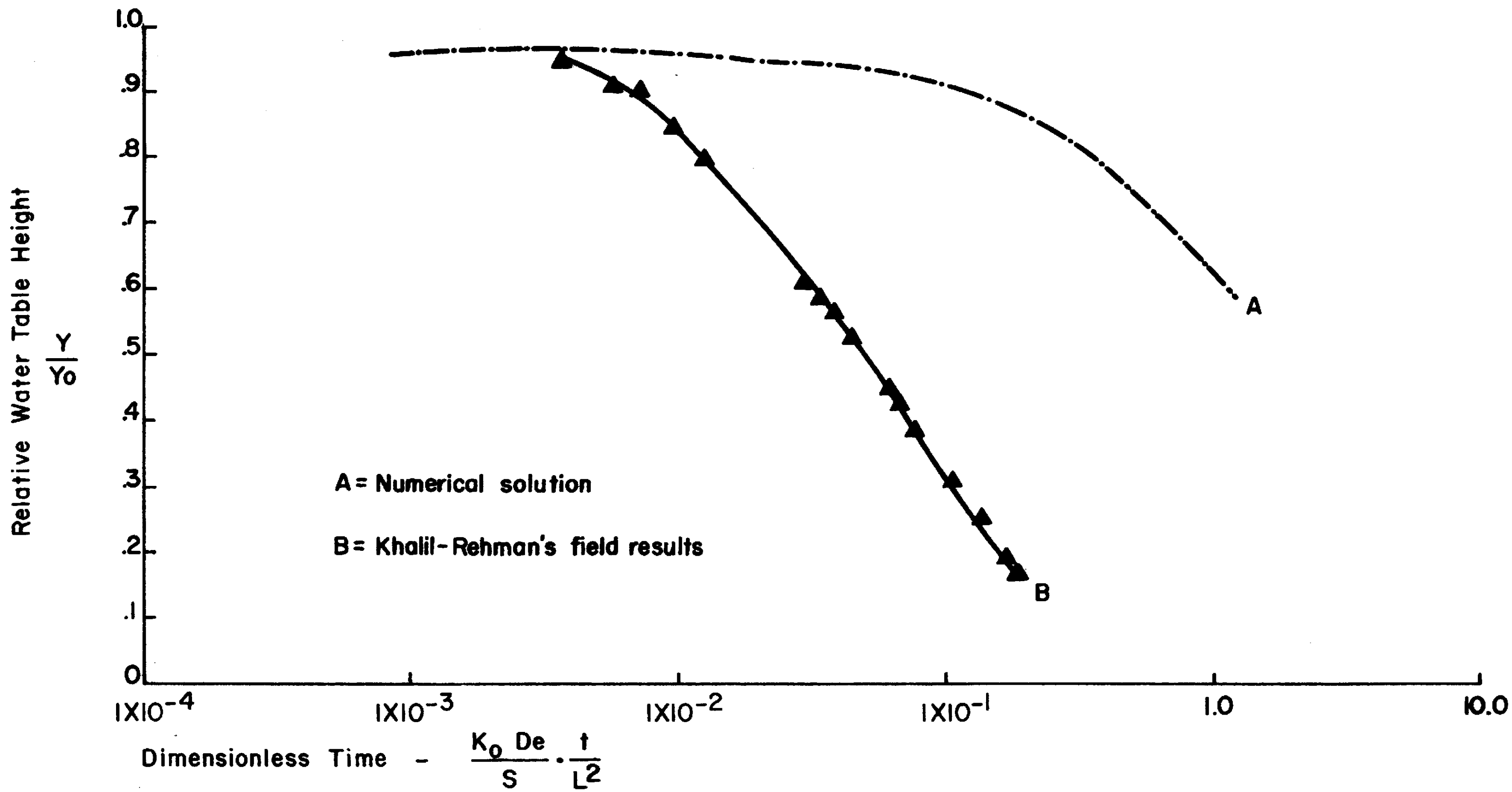


Figure 11. Water table recession for the Hullinger farm, Drains 5, 6

with Khalil-Ur-Rehman's (1971) field results for the recession of the water table at the mid point between the drains. Both curves in Figure 10 were plotted with $\beta = 1\ ft^2/hour$. In Figure 11 β was calculated separately for each curve from $\beta = \frac{K_o D_e}{S}$. For the numerical solution the following values were used:

K_o = 5.08 ft/hour (average K_o)

D_e = 29.29 ft.

$S = \eta - S_r = 0.35$

Khalil-Ur-Rehman (1971) used the following values for his experiment:

K_o = 1.339 ft/hour

D_e = 19.08 ft

S = 0.26

Figure 10 shows that the recession of the water table in the numerical solution was much slower than the results of the field experiment. This difference could be due to a combination of the following factors:

a. Inaccurate characterization of K_o and heterogeneity in the model.

b. The presence of natural drainage in the third dimension. This was observed by previous investigations on the Hullinger farm (King, in a verbal communication). This, of course, will give faster actual recession than the model will predict.

c. Effects of heterogeneity in, or inaccurate characterization of the soil parameters for the unsaturated flow.

d. Differences between the two initial conditions as discussed above. y_o for the model was 3.025 ft. While in Khalil-Ur-Rehman's (1971) experiment y_o was 2.29 ft.

Limitations of the Program

In addition to the limitations dictated by the assumptions in the formulation of the model as the one-phase assumption and the neglect of air pressure buildup in the medium and other assumptions which were discussed earlier in this dissertation, the program has another important limitation. This is that the model cannot simulate the phenomenon of infiltration as accurately as it should be. This limitation is introduced by the large size of the mesh which is dictated by the usually large size of the drainage problem. While infiltration simulation may need a mesh size of one inch or less, such a small size is neither needed nor economically possible for the rest of the domain of the drainage problem which may be several hundred feet in length.

COMPARISON OF THE RESULTS WITH SOME APPROXIMATE THEORY

Some researchers (Dumm,1954, 1968; Brooks, 1961; Jenab, Bishop, and Peterson, 1969; van Schillfgaarde, 1963, 1965; Moody, 1966) investigated the transient case of the falling water table in flat lands by solving the linearized form of the Boussinesq equation:

$$\frac{\partial}{\partial x}\left(K_o h \frac{\partial h}{\partial x}\right) = S \frac{\partial h}{\partial t} \tag{60}$$

Where:

h = Thickness of saturated flow

x = Horizontal coordinate. h and x are shown in Figure 12.

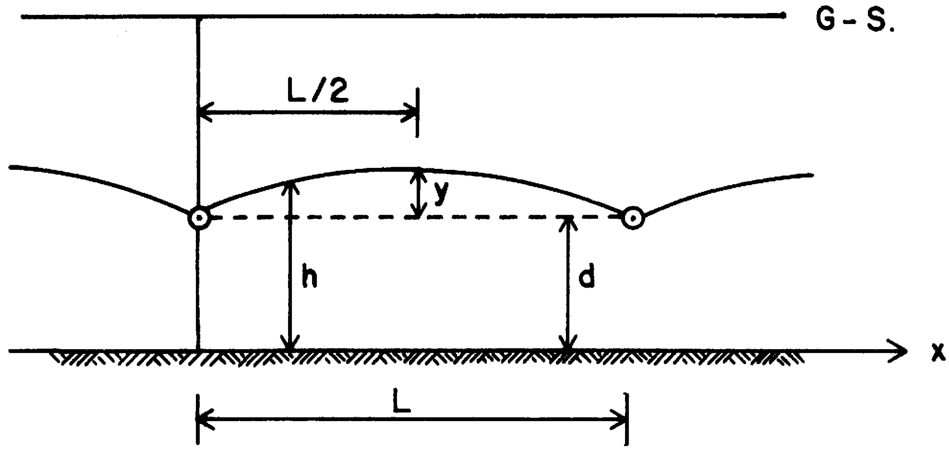
K_o = Saturated Hydraulic Conductivity

S = Drainable porosity or specific yield.

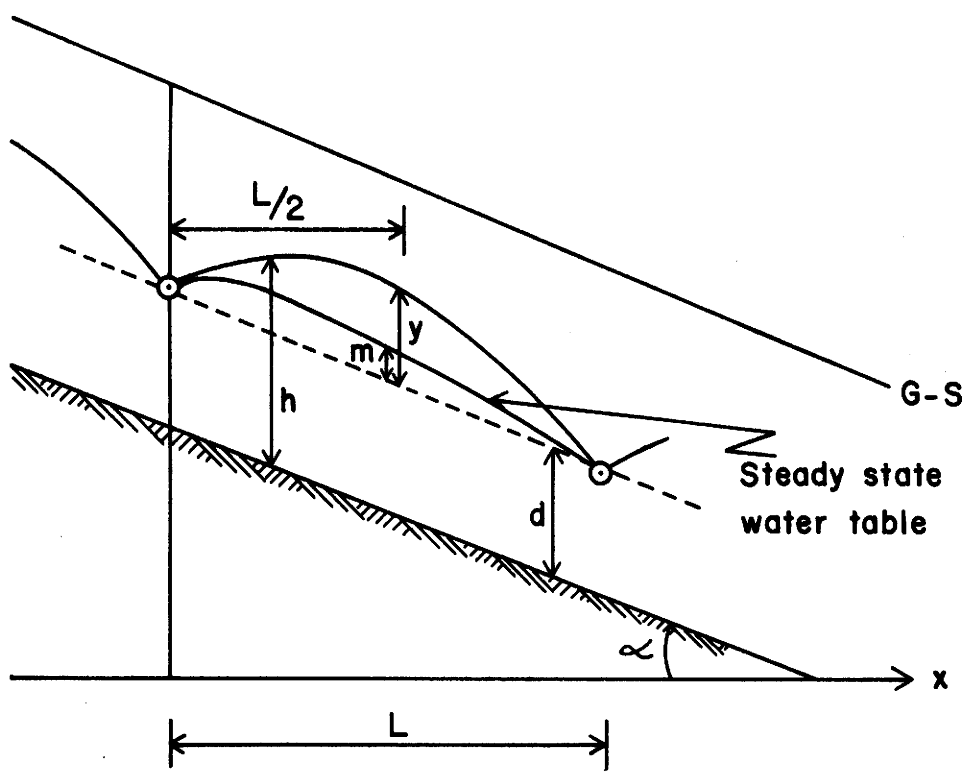
t = Time

By considering only homogeneous soils (K_o = constant) and by approximating the variable h in the brackets of the left hand side of Equation (60) by a constant average value of h called D_e, Equation (60) is linearized into:

$$K_o D_e \frac{\partial^2 h}{\partial x^2} = S \frac{\partial h}{\partial t} \tag{61}$$



(a) Flat land



(b) Sloping land

Figure 12. Definition of drainage parameters.

Where:

D_e = The average depth of flow

$= d + \frac{y_o}{2}$ or $d_e + \frac{y_o}{2}$

d = Depth below drains to the barrier

d_e = Depth below drains as modified by Hooghoudt's effective depth theory

y_o = Height of the water table at the midpoint between the drains above the line joining the two drains at time zero.

Some of these solutions for the falling water table were reviewed by Khalil-Ur-Rehman (1971) and Sabti (1974). Many of these solutions were presented as recession curves of the water table at the midpoint plotted on semi-logarithmic paper with the two dimensionless parameters $\frac{y}{y_o}$ and $\beta \frac{t}{L^2}$

Where:

y = Height of the water table at the midpoint above the drains at time t

L = Spacing of the drains

$$\beta = \frac{K_o D_e}{S}$$

Figure 13 shows some of these theoretical curves. This figure was taken from Sabti (1974).

Although the approach of the above theory is quite different from

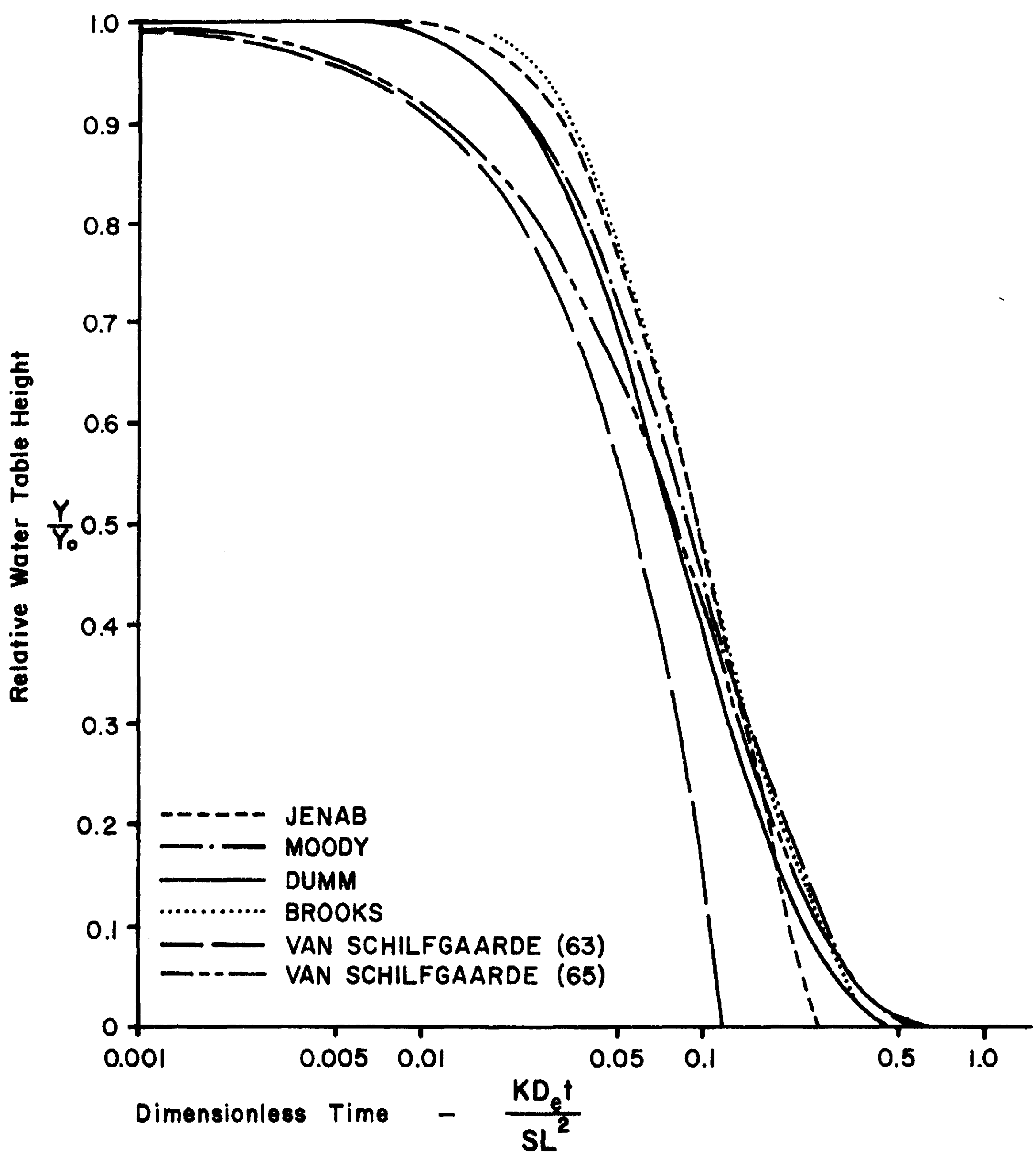


Figure 13. Theoretical recession curves for the midpoint between drains.

the approach of the model in the present study, it was felt that a qualitative comparison of the results of the two may give an insight into the applicability of this theory to drains in sloping heterogeneous lands. The formulation in the present model is closer to the physical process of drainage than the above theory and it may be desirable to indicate the differences between the two at this point. Firstly, the theory treats homogeneous soils only while the present model treats both the homogeneous and the heterogeneous cases. In nature, heterogeneity of the soil is the rule. Secondly, the theory is based on the Dupuit-Forchheimer assumptions which are not used in the present model. The model uses the more accurate hydrodynamic theory. Thirdly, the theory was developed for flat lands while the present model was developed for sloping lands. Fourthly, the theory considers saturated flow only and assumes instantaneous and complete desaturation as the water table falls beyond a point. The present model is closer to the natural process as it considers both the saturated and the unsaturated flows and the time variability of desaturation.

Khalil-Ur-Rehman's (1971) field results on the Hullinger farm did not agree with the theory. His recession curves were flatter than the theoritical curves. Sabti (1974) investigating drainage on the same farm found different degrees of correlation (from good to none) between his field results and the theoretical curves. Both workers mentioned slope as a main possible reason for the divergence of the field results from the theory.

In comparing the results of the hypothetical problem with the theory it was impossible to define a value for $\beta = \frac{K_o D_e}{S}$ for plotting the curves of Figure 6 because K_o, D_e and S were all variables. That was the reason for plotting these curves with $\beta = 1$ ft^2/hour. Since the use of any other value of β will only displace the curves horizontally without affecting their slopes it is possible to compare these curves with the theoretical curves of Figure 13. It is obvious that the curves of the present model are much flatter than the theoretical curves, indicating a slower rate of recession.

Figure 14 shows a comparison of the numerical curve and the field curve for the Hullinger farm with some of the theoretical recession curves. Again both the numerical prediction and the actual curve were flatter than the theoretical curves. The theoretical curves predict a much faster rate of recession than the numerical solution. Some of the factors that might contribute to this difference are slope, unsaturated flow, heterogeneity and method of linearizing the nonlinear Boussinesq equation.

1. Method of linearizing the Boussinesq equation. In getting Equation (61) from Equation (60) the depth of flow is assumed constant and equal to the average of the values of the variable depth ($D_e = d_e + \frac{y_o}{2}$). This underestimates the flow depth in the early stages of recession and over-estimates this depth for the latter stages. This will result in a theoretical recession curve which is steeper than the actual.

2. Effect of the unsaturated flow. Equations (60) and (61) give:

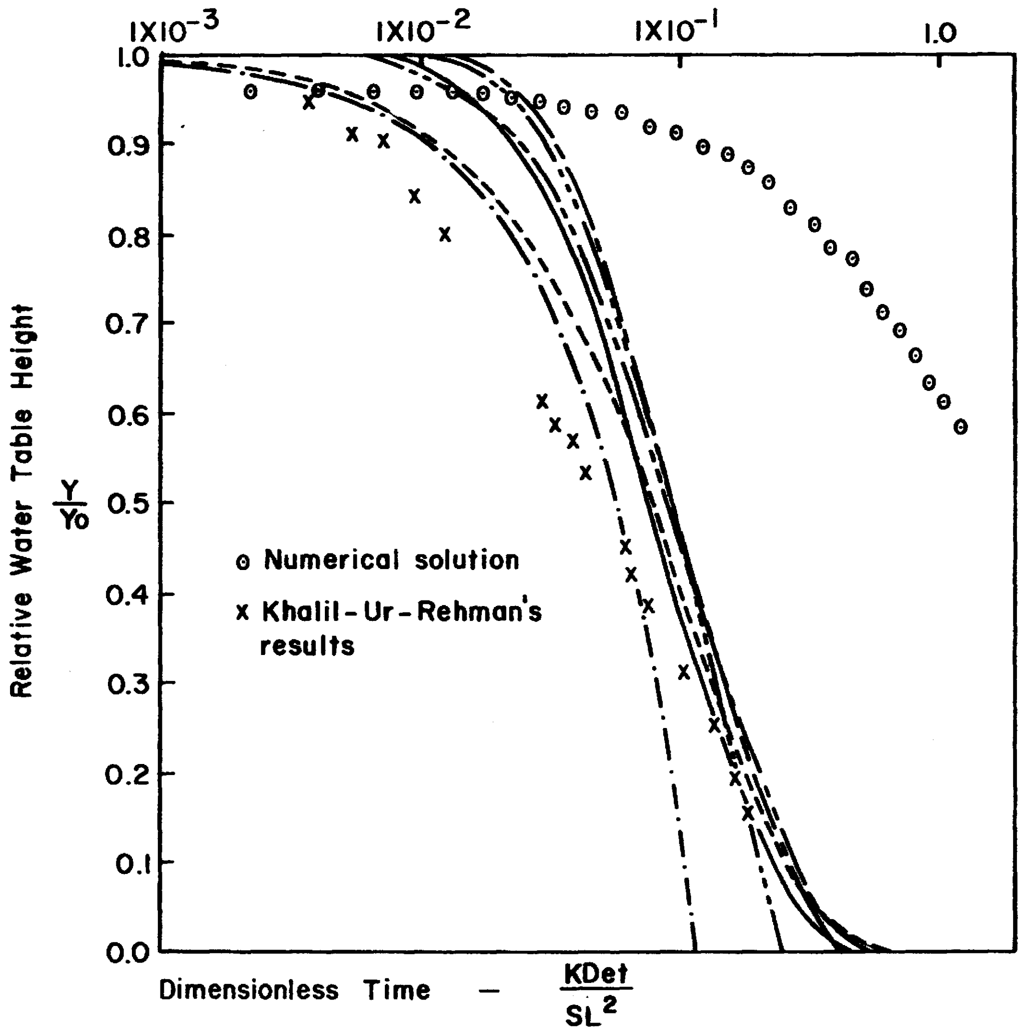


Figure 14. Hullinger farm numerical and field results compared with the theoritical curves.

$$\frac{\partial h}{\partial t} = \frac{K_o D_e}{S} \frac{\partial^2 h}{\partial x^2} \quad (62)$$

Since we are dealing with the falling water table, the right hand side of Equation (62) is negative.

If we include the percolation from the unsaturated region into the saturated region in the formulation of the Boussinesq equation, and recognize that desaturation is not complete and instantaneous as the water table falls, then Equation (60) will become:

$$\frac{\partial}{\partial x} (K_o h \frac{\partial h}{\partial x}) + i(t) = S_o \frac{\partial h}{\partial t} \quad (63)$$

Which gives after linearization:

$$\frac{\partial h}{\partial t} = \frac{K_o D_e}{S_o} \frac{\partial^2 h}{\partial x^2} + \frac{i(t)}{S_o} \quad (64)$$

Where:

$i(t)$ = Rate of percolation into the saturated zone

S_o = The fraction of the specific yield that is drained at the onset of desaturation (less than S)

Since the second term on the right hand side of Equation (64) is positive, it tends to make $\frac{\partial h}{\partial t}$ less negative which means a slower rate of recession. This effect will depend on the magnitudes of $i(t)$ and S_o, and in run 2 of the hypothetical problem between times $t = 0$ and $t = 0.25$ hours it must have been larger than the absolute value of the negative first term on the right hand side of Equation (64) changing $\frac{\partial h}{\partial t}$ to a positive value and resulting in a water table rise. This also could be one of the reasons for the difference between the field results of Sabti (1974) and those of Khalil-Ur-Rehman for the same farm. Sabti started his measurements directly after stopping irrigation (i(t) is still high), while Khalil-Ur-Rehman started his measurements after some time of stopping the irrigation (giving time for i(t) to become small).

3. Effect of the slope. As mentioned earlier the approximate theory was developed for flat lands, where the midpoint between the drains is a water divide. In sloping lands the midpoint is not a water divide. Actually the water divide in sloping lands was observed to be close to the upper drain in the steady state with no recharge case, to shift downslope as the water table rose and to shift back upslope as the water table fell. In all cases the midpoint was downslope of the water divide. This means that the section at the midpoint receives water from higher land between it and the upper drain as well as from lands farther up beyond the upper drain. This extra flow needs an added thickness of flow at the midpoint, thus slowing the water table recession there. The presence of this seepage, even in the steady state with no recharge in some problems (as in the hypothetical problem) makes it erroneous to use the midpoint of the line joining the two drains as a datum for measuring y and y_o. As seen in Figure 12, $\lim_{t \to \infty} y$ is zero for flat lands, but may be larger (m in Figure 12) in sloping lands.

This seepage and this choice of datum for y and y_o will make the recession curve flatter.

Investigation of the Boussinesq equation for sloping lands may shed some light on the behavior of the water table in such cases. This equation for sloping lands is:

$$\frac{\partial}{\partial x} [K_o \, h \, \text{Cos}\, \alpha \, (\frac{\partial h}{\partial x} + \tan \alpha)] = S \frac{\partial h}{\partial t} \qquad (65)$$

which gives after linearization:

$$\frac{\partial h}{\partial t} = \text{Cos}\, \alpha \cdot \frac{K_o D_e}{S} \frac{\partial^2 h}{\partial x^2} + \text{Sin}\, \alpha \, \frac{K_o}{S} \frac{\partial h}{\partial x} \qquad (66)$$

Where α is the slope angle. It is interesting to note that the presence of the second term on the right hand side of Equation (66) can account for the lop-sided shape of the water table and the shifting of the water devide in sloping lands. $\frac{\partial h}{\partial x}$ is negative for the part of the water table down-slope of the water divide and positive for the part upslope of the water divide. This results in a water table recession for the down slope part which is faster than that for the upslope part. Investigating Equation (66) at the midpoint between the drains, it is noted that $\frac{\partial h}{\partial x}$ is always negative there, but its absolute value decreases with time as the water devide shifts upslope. This means a slower recession at the midpoint as time passes and a flat recession curve. The magnitude of these effects of course depends on the magnitude of the slope angle α.

It is difficult to compare the predictions of Equation (66) with those of the flat land Equation (62). Whether Equation (66) will give a slower or a faster recession at the midpoint will depend on the net result of the decrease in the negative value of the first term of the right hand side of Equation (66) and the increase in negative value introduced by the second term there.

4. Effect of heterogeneity. If we formulate the Boussinesq equation for heterogeneous soil (flat land case) and if we assume K_o to vary with x and y, y being measured from the impermeable bed upwards, then the equation would be

$$\frac{\partial}{\partial x}\left[\left(\int_{y=0}^{y=h(x)} K_o(x, y)dy\right) \frac{\partial h}{\partial x}\right] = S \frac{\partial h}{\partial t} \quad (67)$$

Expanding Equation (67), using Leibnitz's rule for differentiation under the integral sign, and neglecting the term containing $\left(\frac{\partial h}{\partial x}\right)^2$ we get:

$$\frac{\partial h}{\partial t} = \frac{1}{S}\left[\left(\int_o^h K_o(x, y)dy\right) \frac{\partial^2 h}{\partial x^2} + \left(\int_o^h \frac{\partial K_o}{\partial x} dy\right) \frac{\partial h}{\partial x}\right] \quad (68)$$

Since it is difficult to investigate Equation (68) when K_o varies with both x, y, hetergeneity with one coordinate at a time will be considered.

a. K_o varies with y alone. Say $K_o = C_1 + C_3$ y. Equation (68) becomes

$$\frac{\partial h}{\partial t} = \frac{1}{S}\left[\left(C_1 + \tfrac{1}{2} C_3 h\right) h \frac{\partial^2 h}{\partial x^2}\right]$$

or to put it in a form similar to Equation (62)

$$\frac{\partial h}{\partial t} = \frac{K_{oav} \, D_e}{S} \frac{\partial^2 h}{\partial x^2} \tag{69}$$

where:

K_{oav} = average K_o for the flow section. It is the average only because of the assumed linear relation of K_o to y. If it is a nonlinear relation, then the definite integral of Equation (68) should be evaluated.

If K_o increases with depth, K_{oav} increases with time and a steeper recession curve results. A flat curve results if K_o decreases with depth. In practice it is believed that the effect of this factor is not significant because h varies over a small range compared to its magnitude.

b. K_o varies with x alone. Say $K_o = C_1 + C_2 x$.

Equation (68) becomes:

$$\frac{\partial h}{\partial t} = \frac{1}{S} \left[(C_1 + C_2 x) \, h \frac{\partial^2 h}{\partial x^2} + C_2 h \frac{\partial h}{\partial x} \right]$$

or

$$\frac{\partial h}{\partial t} = \frac{K_o(x) \, D_e}{S} \frac{\partial^2 h}{\partial x^2} + \frac{C_2 h}{S} \frac{\partial h}{\partial x} \tag{70}$$

The second term on the right hand side of Equation (70) will act to change the shape of the water table and to shift the water divide even in flat lands. If C_2 is positive (K_o increases with x) the water divide shifts back and $\frac{\partial h}{\partial x}$ at the midpoint becomes negative. If C_2 is negative (K_o

decreases with x) the water divide shifts forward and $\frac{\partial h}{\partial x}$ at the mid-point becomes positive. In both cases this second term in Equation (70) is negative. Its effect as time passes is difficult to evaluate as its value starts from zero ($\frac{\partial h}{\partial x}$ = 0 at the midpoint to start with) and then increases in the negative direction with time and then decreases as the water divide shifts farther away in the latter stages of drainage.

From the results of the numerical solutions presented, the field results for the Hullinger farm and the above theoretical discussion it can be safely concluded that the approximate analytical solutions developed for homogeneous flat lands are not applicable to heterogeneous sloping lands mainly because of slope, heterogeneity and the unsaturated flow.

SUMMARY AND CONCLUSION

The objectives of the present study were to develop a computer program to solve the transient tile drainage problem in heterogeneous sloping lands, and to test this program for an actual problem where data were available. The surface of the soil and the impermeable bed were to be of a general shape and slope.

A program was written in Fortran IV language to solve this problem. Finite difference formulation was used with a general rectangular mesh specified by input data. The saturated and the unsaturated parts of the soil mass were treated as one integrated composite system, and the flow, whether saturated or unsaturated was considered. The program can treat homogeneous or heterogeneous media. Heterogeneity with respect to the saturated hydraulic conductivity only was considered. This heterogeneity was characterized by specifying, as data, the measured values of hydraulic conductivity at several points and fitting these data to a linear relationship between K_o, x and depth using the least squares method of approximation.

The program solves the case of steady state with no local recharge first, to provide an initial condition for the unsteady state. The steady state solution was obtained using the successive over-relaxation iterative method.

The Crank-Nicolson difference scheme was used for the unsteady state except at the normal flux boundaries where the implicit difference scheme was used. The solution was obtained by using a combined method of few Newton inner iterations and the successive over-relaxation outer iterations. Both the rising water table with local recharge and the falling water table after the recharge stops were simulated. The results showing the water table response with time for a hypothetical solution are presented.

The program was tested with an actual drainage problem on the Hullinger farm, Vernal, Utah, for which data on the soil and water table positions for the falling water table case were available. Results of the numerical solution of this problem are presented and compared with the results of a previous field experiment. The two sets of results did not coincide most probably because of differences between the computer model and the actual field conditions.

The results of the numerical solution were compared qualitatively with some approximate analytical solutions for the falling water table in homogeneous flat lands. The numerical solution gave flatter recession curves for the water table at the midpoint between the drains. Some of the probable reasons for flat curves were discussed. It was concluded that these analytical solutions were not applicable to drainage of heterogeneous sloping lands.

It is concluded that the finite difference method of numerical solution can be used very effectively in solving the drainage problem in sloping heterogeneous lands taking into consideration both the saturated and the unsaturated flows.

RECOMMENDATIONS FOR FUTURE RESEARCH

It is recommended that the following areas of research be investigated:

1. The possibility of using the finite difference solutions for layered soils.

2. The possibility of using the numerical solutions to develop some design criteria for drainage systems on sloping heterogeneous lands.

3. The effects of the heterogeneity of the soil parameters other than K_o on the drainage solutions.

4. The possibility of using graded mesh sizes with small sizes above the drains to simulate the infiltration process more accurately.

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APPENDIXES

Appendix A.

Program Listing

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@FOR,IS MAIN,MAIN
      DIMENSION XT(10),ELT(10),XB(10),ELTB(10),DEPTH(10),ELB(10)
      DIMENSION NNB(100),NNT(100)
      DIMENSION Y1(100),Y2(100),JB(100),JT(100)
      COMMON A(3,3),B(3),C(3),ID(10),DD(10),JD(10),Q(100),H(100),YP(100)
      COMMON QT(102)
      COMMON BN(102,18),F131(102,18),PH(102,18)
      COMMON PHI(102,18)
      COMMON NCAL(102,18),SK(102,18),NX,NY,DELX,DELY,NX1,NY1,AC,HB,TAW
      COMMON BA,Z,AF,TAMP,BP,AITA,SR,EPR,W,WZ,MAX,E,E2,E3,E7,E8,E9
      COMMON XLBOUN,E10,G1,G2,G3,G4,DELKY,DELKT,SINALP,COSALP,SINTH
      COMMON COSTH,MD
C*************************************************************************
C*****DRAINAGE OF HETEROGENEOUS SLOPING LAND
C*****FINITE DIFFERENCE SOLUTION
C*****FAHD NATUR
C*************************************************************************
C*****SETTING THE PROBLEM
C*****READ DATA
C*****SL=LENETH IN X DIRECTION
C*****XT(I)=DISTANCE OF SURFACE SURVEY POINTS FROM ORIGION
C*****ELT(I)=ELEVATION OF SURFACE SURVEY POINTS
C*****XB(I)=DISTANCE OF IMPERMEABLE BOUNDARY POINTS FROM ORIGION
C*****ELTB(I)=ELEVATION OF SURFACE AT IMPERMEABLE BOUNDARY POINTS
C*****DEPTH(I)=DEPTH TO IMPERMEABLE BOUNDARY
C*****NT=NUMBER OF SURFACE SURVEY POINTS
C*****NB=NUMBER OF IMPERMEABLE BOUNDARY POINTS
C*****NX= NUMBER OF NODES IN THE X-DIRECTION
C*****NY= NUMBER OF NODES IN THE Y-DIRECTION
C
      READ(5,500)NT,NB,NX,NY
  500 FORMAT(4I5)
      NX1=NX-1
      NY1=NY-1
      READ(5,501)SL
      DO 100 I=1,NT
  100 READ(5,501)XT(I),ELT(I)
  501 FORMAT(3F10.5)
      DO 101 I=1,NB
  101 READ(5,501)XB(I),ELTB(I),DEPTH(I)
C*****WRITE  NT,NB,NX,NY,XT(I),ELT(I),XB(I),ELTB(I),DEPTH(I)
      WRITE(6,1001)
 1001 FORMAT(1H1,'DRAINAGE PROBLEM/SLOPING HETEROGENEOUS LAND')
      WRITE(6,1002)
 1002 FORMAT(1H0,'   DISPLAY OF GEOMETRY DATA  ')
      WRITE(6,1003)
 1003 FORMAT(1H0,'         NT          NB          NX          NY ')
      WRITE(6,1004)NT,NB,NX,NY
 1004 FORMAT(4(5X,I5))
      WRITE(6,1005)
 1005 FORMAT(1H0,'      XT(I)            ELT(I)')
      DO 900 I=1,NT
  900 WRITE(6,1006)XT(I),ELT(I)
 1006 FORMAT(2(5X,F10.5))
      WRITE(6,1007)
 1007 FORMAT(1H0,'      XB(I)            ELTB(I)            DEPTH(I)')
      DO 901 I=1,NB
  901 WRITE(6,1008)XB(I),ELTB(I),DEPTH(I)
 1008 FORMAT(3(5X,F10.5))
      WRITE(6,1017)
 1017 FORMAT(1H0,'     SL=')
      WRITE(6,1018)SL
 1018 FORMAT(5X,F10.5)
C***************************************************************************
C*****CALCULATE*ELB(I)=ELEVATION OF IMPERMEABLE BOUNDARY POINTS
C*****ELBO=ELEVATION OF IMPERMEABLE BOUNDARY AT ORIGION
C*****ELTO= ELEVATION OF SURFACE AT ORIGION
C*****ELBF=ELEVATTON OF IMPERMEABLE BOUNDARY AT END
C*****ELTF=ELEVATION OF SURFACE AT END
C
      DO 102 I=1,NB
  102 ELB(I)=ELTB(I)-DEPTH(I)
      ELTO=ELT(1)-XT(1)*(ELT(2)-ELT(1))/(XT(2)-XT(1))
      ELBO=ELB(1)-XB(1)*(ELB(2)-ELB(1))/(XB(2)-XB(1))
      ELTF=ELT(NT)+(SL-XT(NT))*(ELT(NT)-ELT(NT-1))/(XT(NT)-XT(NT-1))
      ELBF=ELB(NB)+(SL-XB(NB))*(ELB(NB)-ELB(NB-1))/(XB(NB)-XB(NB-1))
C*****DISPLAY*ELB(I),ELTO,ELBO,ELTF,ELBF
      WRITE(6,1009)
 1009 FORMAT(1H0,'        ELTO           ELBO             ELTF             ELBF
     $')
      WRITE(6,1010)ELTO,ELBO,ELTF,ELBF
 1010 FORMAT(4(5X,F10.5))
      WRITE(6,1011)
 1011 FORMAT(1H0,'     ELB(I)=  ')
      WRITE(6,1012)(ELB(I),I=1,NB)
 1012 FORMAT(10(2X,F10.5))
C*****SEARCH FOR HIGHEST ELT AND LOWEST ELB
      ELTM=ELTO
      IF(ELTM.GT.ELTF) GO TO 103
      ELTM=ELTF
  103 DO 104 I=1,NT
      IF(ELTM.GT.ELT(I)) GO TO 104
      ELTM=ELT(I)
  104 CONTINUE
      ELBM=ELBO
      IF(ELBM.LT.ELBF) GO TO 105
      ELBM=ELBF
  105 DO 106 I=1,NB
      IF(ELBM.LT.ELB(I)) GO TO 106
      ELBM=ELB(I)
  106 CONTINUE
      WRITE(6,1019)
 1019 FORMAT(1H0,'     ELBM                ELTM')
      WRITE(6,1020)ELBM,ELTM
 1020 FORMAT(2(5X,F10.5))
C*****CALCULATE  D=THE DEPTH
      D=ELTM-ELBM
C*****NORMALIZE SPACE DIMENSIONS
      DO 107 I=1,NT
  107 ELT(I)=ELT(I)-ELBM
      DO 108 I=1,NB
  108 ELB(I)=ELB(I)-ELBM
      ELTO=ELTO-ELBM
      ELTF=ELTF-ELBM
      ELBO=ELBO-ELBM
      ELBF=ELBF-ELBM
      WRITE(6,1021)
 1021 FORMAT(1H0,'  THE DEPTH=   ')
      WRITE(6,1022)D
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 1022 FORMAT(5X,F10.5)
      WRITE(6,1023)
 1023 FORMAT(1H0,'   NORMALIZED GEOMETRY')
      WRITE(6,1024)
 1024 FORMAT(1H0,'      ELTO              ELTF')
      WRITE(6,1025)ELTO,ELTF
 1025 FORMAT(2(5X,F10.5))
      WRITE(6,1026)
 1026 FORMAT(1H0,'      ELBO              ELBF')
      WRITE(6,1025)ELBO,ELBF
      WRITE(6,1005)
      DO 905 I=1,NT
  905 WRITE(6,1006)XT(I),ELT(I)
      WRITE(6,1027)
 1027 FORMAT(1H0,'     XB(I)             ELB(I)')
      DO 906 I=1,NB
  906 WRITE(6,1006)XB(I),ELB(I)
C*****CALCULATE Y1(I)=Y-COORDINATE OF IMPERMEABLE BOUNDARY AT NODE I
C*****Y2(I)=Y-COORDINATE OF SURFACE AT NODE I
      DELX=SL/NX1
      DELY=D/NY1
      R=DELX/DELY
      R2=R*R
      Y1(1)=ELBO
      Y2(1)=ELTO
      Y1(NX)=ELBF
      Y2(NX)=ELTF
      DO 109 I=2,NX1
      X=DELX*FLOAT(I-1)
      IF(X.LT.XB(1)) GO TO 110
      IF(X.GT.XB(NB)) GO TO 111
      DO 112 J=2,NB
      IF(X.LT.XB(J)) GO TO 113
  112 CONTINUE
  110 Y1(I)=ELBO+X*(ELB(1)-ELBO)/XB(1)
      GO TO 118
  111 Y1(I)=ELB(NB)+(X-XB(NB))*(ELBF-ELB(NB))/(SL-XB(NB))
      GO TO 118
  113 Y1(I)=ELB(J-1)+(X-XB(J-1))*(ELB(J)-ELB(J-1))/(XB(J)-XB(J-1))
  118 IF(X.LT.XT(1)) GO TO 114
      IF(X.GT.XT(NT)) GO TO 115
      DO 116 J=2,NT
      IF(X.LT.XT(J)) GO TO 117
  116 CONTINUE
  114 Y2(I)=ELTO+X*(ELT(1)-ELTO)/XT(1)
      GO TO 109
  115 Y2(I)=ELT(NT)+(X-XT(NT))*(ELTF-ELT(NT))/(SL-XT(NT))
      GO TO 109
  117 Y2(I)=ELT(J-1)+(X-XT(J-1))*(ELT(J)-ELT(J-1))/(XT(J)-XT(J-1))
  109 CONTINUE
C*****DISPLAY Y1(I) AND Y2(I)
      WRITE(6,1013)
 1013 FORMAT(1H0,'Y2(I) AND Y1(I) ')
      WRITE(6,1014)(Y2(I),I=1,NX)
 1014 FORMAT(15(1X,F7.3))
      WRITE(6,1014)(Y1(I),I=1,NX)
C*****PUT BOUNDARIES AT NODES
      DO 119 I=1,NX
      JB(I)=(Y1(I)/DELY)+1
      JT(I)=(Y2(I)/DELY)+1
      YT1=FLOAT(JB(I)-1)*DELY
      YT2=FLOAT(JT(I)-1)*DELY
      IF((Y1(I)-YT1).LT..5*DELY) GO TO 140
      JB(I)=JB(I)+1
  140 IF((Y2(I)-YT2).LT..5*DELY) GO TO 119
      JT(I)=JT(I)+1
  119 CONTINUE
      DO 9499 I=1,NX
      JB(I)=JB(I)+1
 9499 JT(I)=JT(I)+1
      NY=NY+1
      NY1=NY1+1
      WRITE(6,1028)
 1028 FORMAT(1H0,' J NUMBER OF BOUNDARY NODES  JT(I) AND JB(I)')
      WRITE(6,1029)(JT(I),I=1,NX)
      WRITE(6,1029)(JB(I),I=1,NX)
 1029 FORMAT(15(3X,I5))
C***********************************************************************
C******SETTING THE CALCULATION CODE
C*****READ  ND=NUMBER OF DRAINS
C*****SET  NCAL=1  FOR POINTS OUTSIDE THE FLOW REGION
C*****NCAL=2  FOR ALL INTERIOR POINTS THAT ARE NOT DRAINS
C*****NCAL=3  FOR DRAIN POINTS
C******NCAL=90  FOR ALL SURFACE BOUNDARY POINTS
C*****NCAL=100  FOR ALL IMPERMEABLE BOUNDARY POINTS
      READ(5,502)ND
  502 FORMAT(I5)
      IF(ND.EQ.0) GO TO 120
      DO 121 K=1,ND
      READ(5,503)ID(K),DD(K)
  503 FORMAT(I5,F10.5)
      I=ID(K)
      NDD=DD(K)/DELY
      DDT=DELY*FLOAT(NDD)
      IF((DD(K)-DDT).LT.0.5*DELY) GO TO 150
      JD(K)=JT(I)-NDD-1
      GO TO 121
  150 JD(K)=JT(I)-NDD
  121 CONTINUE
  120 DO 122 I=1,NX
      DO 122 J=1,NY
      IF(J.LT.JB(I)) GO TO 123
      IF(J.GT.JT(I)) GO TO 123
      IF(J.EQ.JB(I)) GO TO 124
      IF(J.EQ.JT(I)) GO TO 125
      IF(ND.EQ.0) GO TO 126
      DO 127 K=1,ND
      IF(I.EQ.ID(K).AND.J.EQ.JD(K)) GO TO 128
  127 CONTINUE
  126 NCAL(I,J)=2
      GO TO 122
  128 NCAL(I,J)=3
      GO TO 122
  123 NCAL(I,J)=1
      GO TO 122
  124 NCAL(I,J)=100
      GO TO 122
  125 NCAL(I,J)=90
  122 CONTINUE
C*****SET CALCULATION CODE AT THE BOUNDARIES
      DO 129 I=2,NX1
      IP=I+1
      IM=I-1
      DO 129 J=1,NY
```

```
      IF(NCAL(I,J).EQ.100) GO TO 130
      IF(NCAL(I,J).EQ.90) GO TO 131
      GO TO 129
  130 IF(NCAL(IM,J).EQ.100.AND.NCAL(IP,J).EQ.100) GO TO 10
      IF(NCAL(IM,J).EQ.100.AND.NCAL(IP,J).EQ.2)  GO TO 10
      IF(NCAL(IM,J).EQ.2.AND.NCAL(IP,J).EQ.100) GO TO 10
      IF(NCAL(IM,J).EQ.2.AND.NCAL(IP,J).EQ.2) GO TO 10
      IF(NCAL(IM,J).EQ.100.AND.NCAL(IP,J).EQ.1) GO TO 11
      IF(NCAL(IM,J).EQ.1.AND.NCAL(IP,J).EQ.100) GO TO 12
      IF(NCAL(IM,J).EQ.1.AND.NCAL(IP,J).EQ.1) GO TO 13
      IF(NCAL(IM,J).EQ.1.AND.NCAL(IP,J).EQ.2)  GO TO 14
      IF(NCAL(IM,J).EQ.2.AND.NCAL(IP,J).EQ.2) GO TO 15
   10 NNB(I)=10
      GO TO 129
   11 NNB(I)=11
      GO TO 129
   12 NNB(I)=12
      GO TO 129
   13 NNB(I)=13
      GO TO 129
   14 NNB(I)=14
      GO TO 129
   15 NNB(I)=15
      GO TO 129
  131 IF(NCAL(IM,J).EQ.90.AND.NCAL(IP,J).EQ.90) GO TO 4
      IF(NCAL(IM,J).EQ.90.AND.NCAL(IP,J).EQ.2)  GO TO 4
      IF(NCAL(IM,J).EQ.2.AND.NCAL(IP,J).EQ.90) GO TO 4
      IF(NCAL(IM,J).EQ.2.AND.NCAL(IP,J).EQ.2) GO TO 4
      IF(NCAL(IM,J).EQ.90.AND.NCAL(IP,J).EQ.1)  GO TO 5
      IF(NCAL(IM,J).EQ.2.AND.NCAL(IP,J).EQ.1) GO TO 6
      IF(NCAL(IM,J).EQ.1.AND.NCAL(IP,J).EQ.90) GO TO 7
      IF(NCAL(IM,J).EQ.1.AND.NCAL(IP,J).EQ.2) GO TO 8
      IF(NCAL(IM,J).EQ.1.AND.NCAL(IP,J).EQ.1) GO TO 9
    4 NNT(I)=4
      GO TO 129
    5 NNT(I)=5
      GO TO 129
    6 NNT(I)=6
      GO TO 129
    7 NNT(I)=7
      GO TO 129
    8 NNT(I)=8
      GO TO 129
    9 NNT(I)=9
  129 CONTINUE
      DO 132 I=2,NY1
      DO 135 J=1,NY
      IF(NCAL(I,J).EQ.100) GO TO 133
      IF(NCAL(I,J).EQ.90)  GO TO 134
      GO TO 135
  133 NCAL(I,J)=NNB(I)
  135 CONTINUE
  134 NCAL(I,J)=NNT(I)
  132 CONTINUE
C***********************************************************************
C*****DISPLAY CALCULATION CODE
      WRITE(6,1015)
 1015 FORMAT(1H0,'              CALCULATION CODE         ')
      N2=0
 1200 N2=N2+20
      N1=N2-19
      IF(N2.GT.NX)  N2=NX
      WRITE(6,1201)
 1201 FORMAT(1H0,'    CALCULATION CODE')
      DO 136 INDEX=1,NY
      J=NY+1-INDEX
  136 WRITE(6,1016)(NCAL(I,J),I=N1,N2)
 1016 FORMAT(1H ,20I5)
      IF(N2.LT.NX)  GO TO 1200
C***********************************************************************
C*****PART  2  DISTRIBUTION OF SATURATED HYDRAULIC CONDUCTIVITY
C*****A LEAST SQUARE METHOD IS USED ASSUMING LINEAR VARIATION OF
C     CONDUCTIVITY WITH  X  AND DEPTH OF THE FORM K=C1+C2Y+C3D
C*****KCODE=1  HETEROGENEOUS SOIL
C*****KCODE=0  HOMOGENEOUS SOIL
      READ(5,9933)KCODE
 9933 FORMAT(I5)
      IF(KCODE.EQ.0)  GO TO 9930
C*****
C*****READ DATA/  NPK=NUMBER OF MEASUREMENTS OF CONDUCTIVITY
C                 XK=X  COORDINATE OF CONDUCTIVITY MEASUREMENTS
C                 DK= DEPTH OF CONDUCTIVITY MEASUREMENTS
      READ(5,504)NPK
  504 FORMAT(I5)
      DO 142 I=1,3
      B(I)=0.0
      DO 142 J=1,3
  142 A(I,J)=0.0
C*****DISPLAY DATA
      WRITE(6,1042)
 1042 FORMAT(1H0,'    NPK=')
      WRITE(6,1043)NPK
 1043 FORMAT(5X,I5)
      READ(5,505)FKMIN,FKMAX
      WRITE(6,1044)
 1044 FORMAT(1H0,'           FKMIN            FKMAX')
      WRITE(6,1031)FKMIN,FKMAX
 1031 FORMAT(3(5X,F10.5))
      WRITE(6,1030)
 1030 FORMAT(1H0,'              XK               DK              FK')
      DO 143 I=1,NPK
      READ(5,505)XK,DK,FK
  505 FORMAT(3F10.5)
      A(1,2)=A(1,2)+XK
      A(1,3)=A(1,3)+DK
      A(2,2)=A(2,2)+XK*XK
      A(2,3)=A(2,3)+XK*DK
      A(3,3)=A(3,3)+DK*DK
      B(1)=B(1)+FK
      B(2)=B(2)+XK*FK
      B(3)=B(3)+DK*FK
  143 WRITE(6,1031)XK,DK,FK
      A(1,1)=NPK
      A(2,1)=A(1,2)
      A(3,1)=A(1,3)
      A(3,2)=A(2,3)
C*****DISPLAY MATRIX OF NORMAL EQUATIONS OF LEAST SQUARE
      WRITE(6,1032)
 1032 FORMAT(1H0,'    MATRIX  A   AND VECTOR  B')
      DO 907 I=1,3
  907 WRITE(6,1033)(A(I,J),J=1,3),B(I)
 1033 FORMAT(4(5X,F16.5))
      NM=3
      CALL DECOMP(NM,$1000)
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      GO TO 182
 1000 WRITE(6,1040)
 1040 FORMAT(1H1,'    ALGORITHM FAILS')
      GO TO 2000
C*****DISPLAY DECOMPOSED MATRIX
  182 WRITE(6,1034)
 1034 FORMAT(1H0,'   DECOMPOSED MATRIX')
      DO 908 I=1,3
  908 WRITE(6,1033)(A(I,J),J=1,3)
      CALL SOLVE(NM)
      WRITE(6,1035)
 1035 FORMAT(1H0,'   COEFFECIENTS VECTOR  C')
      WRITE(6,1036)
 1036 FORMAT(1H0,'            C(1)            C(2)             C(3)')
      WRITE(6,1031)(C(I),I=1,3)
C******FIND CONDUCTIVITY AT NODES OF DOMAIN
      DO 144 I=1,NX
      DO 144 J=1,NY
      IF(NCAL(I,J).EQ.1) GO TO 145
      X=FLOAT(I-1)*DELX
      JTEMP=JT(I)
      DK=FLOAT(JTEMP-J)*DELY
      SK(I,J)=C(1)+C(2)*X+C(3)*DK
      IF(SK(I,J).LT.FKMIN) GO TO 151
      IF(SK(I,J).GT.FKMAX) GO TO 152
      GO TO 144
  151 SK(I,J)=FKMIN
      GO TO 144
  152 SK(I,J)=FKMAX
      GO TO 144
  145 SK(I,J)=0.0
  144 CONTINUE
C******DISPLAY HYDRAULIC CONDUCTIVITY AT NODES
      WRITE(6,1037)
 1037 FORMAT(1H0,'  SATURATED CONDUCTIVITY AT NODES')
      N2=0
 1207 N2=N2+10
      N1=N2-9
      IF(N2.GT.NX)  N2=NX
      WRITE(6,1206)
 1206 FORMAT(1H0,'   SATURATED CONDUCTIVITY')
      DO 909 INDEX=1,NY
      J=NY+1-INDEX
  909 WRITE(6,1038)(SK(I,J),I=N1,N2)
      IF(N2.LT.NX)  GO TO 1207
      DELKX=C(2)
      DELKT=C(3)
      GO TO 9934
 9930 READ(5,9931)SATK
 9931 FORMAT(F10.5)
      DELKX=0.0
      DELKT=0.0
      DO 9932  I=1,NX
      DO  9932  J=1,NY
 9932 SK(I,J)=SATK
 9934 CONTINUE
C******READ SOIL PARAMETERS
      READ(5,510)AC,HB,TAW,BA,Z
  510 FORMAT(5F10.5)
      READ(5,301)AP,TAWP,BP,AITA,SR
  301 FORMAT(5F10.5)
      WRITE(6,1054)
 1054 FORMAT(1H0,'        SOIL PARAMETERS')
      WRITE(6,1051)
 1051 FORMAT(1H0,'          AC              HB             TAW
     $BA            Z')
      WRITE(6,1052)
 1052 FORMAT(1H ,'  DIMENSIONLESS          FT.      DIMENSIONLESS  DIMENSI
     $ONLESS         FT.          ')
      WRITE(6,1053)AC,HB,TAW,BA,Z
 1053 FORMAT(1H0,5(5X,F10.5))
      WRITE(6,303)
  303 FORMAT(1H ,'            AP             TAWP              BP
     $   AITA            SR')
      WRITE(6,304)
  304 FORMAT(1H ,'  DIMENSIONLESS  DIMENSIONLESS  DIMENSIONLESS  DIMENSI
     $ONLESS  DIMENSIONLESS ')
      WRITE(6,305)AP,TAWP,BP,AITA,SR
  305 FORMAT(1H0,5(5X,F10.5))
      READ(5,511)ERR,W,W2,MAX
  511 FORMAT(3F10.5,I5)
      WRITE(6,1055)
 1055 FORMAT(1H0,'              ERR             W              W2          MAX
     $')
      WRITE(6,1056)ERR,W,W2,MAX
 1056 FORMAT(1H0,3(5X,F10.5),5X,I5)
      E=DELX/DELY
      E2=E*E
      E3=2.*DELY
      E8=DELY
      E9=1./E2
      E10=1.-E9
      TV=HB**TAW
      G1=TAW/(AC*TV)
      G2=TAWP/(AP*HB**TAWP)
      G3=G2*AITA*(1.-SR)
      G4=.5/(1.+E2)
      TANA=ELBO/SL
      ALPHA=ATAN(TANA)
      HYP=ELBO*ELBO+SL*SL
      HYP1=SQRT(HYP)
      SINALP=ELBO/HYP1
      COSALP=SL/HYP1
      TATA=DELY/DELX
      THETA=ATAN(TATA)
      HYP3=DELX*DELX+DELY*DELY
      HYP2=SQRT(HYP3)
      SINTH=DELY/HYP2
      COSTH=DELX/HYP2
      E7=2.*DELX*SINALP*COSALP
      WRITE(6,1057)
 1057 FORMAT(1H0,'  GENERAL SLOPE OF BED  AS A FRACTUON')
      WRITE(6,1059)TANA
 1059 FORMAT(1H0,5X,F10.5)
C*******************************************************************************
C*****STEADY STATE SOLUTION
      WRITE(6,1050)
 1050 FORMAT(1H1,'         STEADY STATE SOLUTION')
C*****KLBOUN=1, LOWER BOUNDARY  UNIFORM FLOW
C*****KLBOUN=0, LOWER BOUNDARY  FIXED HEAD
      READ(5,9933)KLBOUN
C*****  INITIAL  GUESS
C*****READ ELEVATION OF WATER TABLE AT STATIC BOUNDARY
C*****ELW=ELEVATION OF WATER TABLE AT UPPER BOUNDARY
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C*****ELW2= ELEVATION OF WATER TABLE AT LOWER BOUNDARY
      READ(5,1060)ELW
 1060 FORMAT(8F10.5)
C*****SET STATIC BOUNDARY CONDITION AND INITIAL GUESS FOR THE REST OF
C*****THE DOMAIN
      YW=ELW-ELBM+DELY
      I=1
      DO 161 J=1,NY
      IF(NCAL(I,J).EQ.1) GO TO 170
      PHI(1,J)=FLOAT(J-1)*DELY*SINALP*SINALP+YW*COSALP*COSALP
      GO TO 161
  170 PHI(1,J)=1.
  161 CONTINUE
      IF(KLBOUN.EQ.0)   GO TO 9935
      GO TO 9936
 9935 READ(5,1060)ELW2
      YW2=ELW2-ELBM+DELY
 9936 READ(5,898)(YP(I),I=2,NX)
      DO 9498  I=2,NX
 9498 YP(I)=YP(I)+DELY
  898 FORMAT(8F10.5)
      DO 880 I=2,NX
      DO 881 K=1,ND
      IF(I.EQ.ID(K))  GO TO 883
      GO TO 881
  883 YP(I)=FLOAT(JD(K)-1)*DELY
      DO 884 J=1,NY
      IF(NCAL(I,J).EQ.1) GO TO 885
      PHI(I,J)=YP(I)
      GO TO 884
  885 PHI(I,J)=1.
  884 CONTINUE
      GO TO 886
  881 CONTINUE
      DO 887 J=1,NY
      IF(NCAL(I,J).EQ.1) GO TO 888
      PHI(I,J)=FLOAT(J-1)*DELY*SINALP*SINALP+YP(I)*COSALP*COSALP
      GO TO 887
  888 PHI(I,J)=1.
  887 CONTINUE
  886 CONTINUE
  880 CONTINUE
      IF(KLBOUN.EQ.0)  GO TO 9937
      GO TO 9939
 9937 DO 9938  J=1,NY
 9938 PHI(NX,J)=YW2
 9939 CONTINUE
C*****DISPLAY INITIAL GUESS
      WRITE(6,1061)
 1061 FORMAT(1H0,'   INITIAL GUESS FOR PHI')
      N2=0
 1212 N2=N2+10
      N1=N2-9
      IF(N2.GT.NX)  N2=NX
      WRITE(6,1211)
 1211 FORMAT(1H0,'  INITIAL GUESS FOR  PHI')
      DO 912 INDEX=1,NY
      J=NY+1-INDEX
  912 WRITE(6,1038)(PHI(I,J),I=N1,N2)
      IF(N2.LT.NX)  GO TO 1212
 1038 FORMAT(1H ,10(3X,F8.4))
      READ(5,9001)NS
C*****NS= NUMBER OF STEADY SOLUTIONS DESIRED
 9001 FORMAT(I5)
      DO 9000  KOUNT=1,NS
      WRITE(6,1065)
 1065 FORMAT(1H0,'STEADY STATE SOLUTION WITH RECHARGE  Q(I)=   FT/HR')
C*****READ RECHARGE RATES
      READ(5,1060)(Q(I),I=1,NX)
      WRITE(6,1038)(Q(I),I=1,NX)
      NCT=0
  860 SUM=0.0
      DO 810 I=2,NX1
      IM=I-1
      IP=I+1
      DO 811 J=2,NY
      JM=J-1
      JP=J+1
      IF(NCAL(I,J).EQ.1)  GO TO 811
      IF(NCAL(I,J).EQ.3)  GO TO 811
      F2=DELKX/SK(I,J)
      F3=DELKT/SK(I,J)
      F5=F2*DELX/2.
      F6=F3*E*DELX/2.
      F7=1.+F5
      F8=1.-F5
      F9=E2-F6
      F10=E2+F6
      HC=PHI(I,J)-FLOAT(J-1)*DELY
      NCA=NCAL(I,J)
      IF(HC.LT.Z) GO TO 813
      CK=SK(I,J)
      GO TO 700
  813 HD=ABS(HC)
      HE=HD/HB
      RK=AC/(BA+HE**TAW)
      CK=RK*SK(I,J)
      TAWM=TAW-1.
      F1=G1*(HD**TAWM)*RK
      F11=F1/4.
      F12=E2*F11
  700 GO TO(811,821,811,822,823,824,825,826,827,828,829,830,831,832,833)
     $,NCA
  822 PHI(I,JP)=E3*Q(I)/CK+PHI(I,JM)
      GO TO 821
  823 PHI(I,JP)=E3*Q(I)/CK+PHI(I,JM)
      PHI(IP,J)=E10*PHI(IP,JM)+E9*PHI(I,JM)+E8*Q(I)/CK
      GO TO 821
  824 PHI(I,JP)=E10*PHI(I,J)+E9*PHI(IM,J)+E8*Q(I)/CK
      PHI(IP,J)=E10*PHI(IP,JM)+E9*PHI(I,JM)+E8*Q(I)/CK
      GO TO 821
  825 PHI(I,JP)=E3*Q(I)/CK+PHI(I,JM)
      PHI(IM,J)=E10*PHI(IM,JM)+E9*PHI(I,JM)+E8*Q(I)/CK
      GO TO 821
  826 PHI(I,JP)=E10*PHI(I,J)+E9*PHI(IP,J)+E8*Q(I)/CK
      PHI(IM,J)=E10*PHI(IM,JM)+E9*PHI(I,JM)+E8*Q(I)/CK
      GO TO 821
  827 PHI(I,JP)=E3*Q(I)/CK+PHI(I,JM)
      PHI(IP,J)=E10*PHI(IP,JM)+E9*PHI(I,JM)+E8*Q(I)/CK
      PHI(IM,J)=E10*PHI(IM,JM)+E9*PHI(I,JM)+E8*Q(I)/CK
      GO TO 821
  828 PHI(I,JM)=PHI(I,JP)
      GO TO 821
  829 PHI(I,JM)=PHI(I,JP)
```

```
      PHI(IP,J)=E10*PHI(IP,JP)+E9*PHI(I,JP)
      GO TO 821
  830 PHI(I,JM)=PHI(I,JP)
      PHI(IM,J)=E10*PHI(IM,JP)+E9*PHI(I,JP)
      GO TO 821
  831 PHI(I,JM)=PHI(I,JP)
      PHI(IP,J)=E10*PHI(IP,JP)+E9*PHI(I,JP)
      PHI(IM,J)=E10*PHI(IM,JP)+E9*PHI(I,JP)
      GO TO 821
  832 PHI(IM,J)=E10*PHI(IM,JP)+E9*PHI(I,JP)
      PHI(I,JM)=E10*PHI(I,J)+E9*PHI(IP,J)
      GO TO 821
  833 PHI(IP,J)=E10*PHI(IP,JP)+E9*PHI(I,JP)
      PHI(I,JM)=E10*PHI(I,J)+E9*PHI(IM,J)
  821 VC=PHI(I,J)
      V1=PHI(IM,J)
      V2=PHI(IP,J)
      V3=PHI(I,JM)
      V4=PHI(I,JP)
      IF(HC.LT.Z)  GO TO 701
      PHT=G4*(F7*V2+F8*V1+F9*V4+F10*V3)
      DIF=PHT-PHI(I,J)
      SUM=SUM+ABS(DIF)
      PHI(I,J)=PHI(I,J)+W*DIF
      GO TO 702
  701 PHT=G4*(F7*V2+F8*V1+F9*V4+F10*V3+F11*(V2-V1)*(V2-V1)+F12*(V4-V3)*(
     $V4-V3))
      DIF=PHT-PHI(I,J)
      SUM=SUM+ABS(DIF)
      PHI(I,J)=PHI(I,J)+W2*DIF
  702 HC=PHI(I,J)-FLOAT(J-1)*DELY
      IF(HC.LT.0.0)  GO TO  811
      GO TO (811,811,811,703,703,703,703,703,703,811,811,811,811,811,811
     $),NCA
  703 PHI(I,J)=FLOAT(J-1)*DELY
  811 CONTINUE
  810 CONTINUE
      IF(KLBOUN.EQ.0) GO TO 848
      I=NX
      IM=I-1
      IP=I+1
      DO 837 J=2,NY
      JM=J-1
      JP=J+1
      IF(NCAL(I,J).EQ.1)  GO TO 837
      IF(NCAL(I,J).EQ.3)  GO TO 837
      F2=DELKX/SK(I,J)
      F3=DELKT/SK(I,J)
      F5=F2*DELX/2.
      F6=F3*E*DELX/2.
      F7=1.+F5
      F8=1.-F5
      F9=E2-F6
      F10=E2+F6
      IF(NCAL(I,J).EQ.90) GO TO 838
      IF(NCAL(I,J).EQ.100) GO TO 841
      GO TO 840
  838 PHI(I,JP)=PHI(I,JM)
      IF(NCAL(IM,J).EQ.1)  GO TO 839
      GO TO 840
  839 PHI(IM,J)=PHI(I,J)+E7/2.
      PHI(IP,J)=PHI(I,J)-E7/2.
      GO TO 843
  841 PHI(I,JM)=PHI(I,JP)
      IF(NCAL(IM,J).EQ.1)  GO TO 842
      GO TO 840
  842 PHI(IM,J)=PHI(I,J)+E7/2.
      PHI(IP,J)=PHI(I,J)-E7/2.
      GO TO 843
  840 PHI(IP,J)=PHI(IM,J)-E7
  843 VC=PHI(I,J)
      V1=PHI(IM,J)
      V2=PHI(IP,J)
      V3=PHI(I,JM)
      V4=PHI(I,JP)
      HC=PHI(I,J)-FLOAT(J-1)*DELY
      IF(HC.LT.Z) GO TO 844
      PHT=G4*(F7*V2+F8*V1+F9*V4+F10*V3)
      DIF=PHT-PHI(I,J)
      SUM=SUM+ABS(DIF)
      PHI(I,J)=PHI(I,J)+W*DIF
      GO TO 849
  844 HD=ABS(HC)
      HE=HD/HB
      RK=AC/(HE**TAW+BA)
      CK=RK*SK(I,J)
      TAWM=TAW-1.
      F1=G1*(HD**TAWM)*RK
      F11=F1/4.
      F12=E2*F11
      PHT=G4*(F7*V2+F8*V1+F9*V4+F10*V3+F11*(V2-V1)*(V2-V1)+F12*(V4-V3)*(
     $V4-V3))
      DIF=PHT-PHI(I,J)
      SUM=SUM+ABS(DIF)
      PHI(I,J)=PHI(I,J)+W2*DIF
  849 HC=PHI(I,J)-FLOAT(J-1)*DELY
      IF(HC.LT.0.0) GO TO 846
      IF(NCAL(I,J).EQ.90) GO TO 845
      GO TO 837
  845 PHI(I,J)=FLOAT(J-1)*DELY
  837 CONTINUE
      GO TO 848
  846 JP=J+1
      DO 847 K=JP,NY
      IF(NCAL(I,K).EQ.1) GO TO 847
      IF(NCAL(I,K).EQ.3) GO TO 847
      PHT=PHI(I,K-1)
      DIF=PHT-PHI(I,K)
      SUM=SUM+ABS(DIF)
      PHI(I,K)=PHI(I,K-1)
      PHI(IP,K)=PHI(I,K)-E7/2.
  847 CONTINUE
  848 NCT=NCT+1
      IF(SUM.GT.ERR.AND.NCT.LT.MAX)  GO TO 860
      WRITE(6,1070)
 1070 FORMAT(1H ,'     NCT=  ')
      WRITE(6,1071)NCT
 1071 FORMAT(5X,I5)
      WRITE(6,3000)
 3000 FORMAT(1H ,'   SUM=')
      WRITE(6,3001)SUM
 3001 FORMAT(F12.5)
      WRITE(6,1068)
 1068 FORMAT(1H1,'  STEADY STATE H  ')
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      N2=0
  736 N2=N2+10
      N1=N2-9
      IF(N2.GT.NX)  N2=NX
      WRITE(6,738)
  738 FORMAT(1H0,'  STEADY STATE  H')
      DO 720 INDEX=1,NY
      J=NY+1-INDEX
      DO 721  I=N1,N2
      IF(NCAL(I,J).EQ.1) GO TO 722
      H(I)=PHI(I,J)-FLOAT(J-1)*DELY
      GO TO 721
  722 H(I)=1.
  721 CONTINUE
  720 WRITE(6,1038)(H(I),I=N1,N2)
      IF(N2.LT.NX)  GO TO 736
 9000 CONTINUE
C*************************************************************************
C*****THE UNSTEADY STATE SOLUTION
      READ(5,320)MAX2
  320 FORMAT(2I5)
      WRITE(6,300)
  300 FORMAT(1H1,' THE UNSTEADY STATE SOLUTION')
      READ(5,9012)W3,W4
 9012 FORMAT(2(F10.5))
      READ(5,9014)NWET,EPNWT
 9014 FORMAT(I5,F10.4)
      WRITE(6,9013)
 9013 FORMAT(1H0,'   W3            W4')
      WRITE(6,9012)W3,W4
C*****READ NUMBER OF TIME STEPS
      READ(5,306)NTIM
  306 FORMAT(I5)
      TIME=0.0
      DO 201  I=1,NX
  201 QT(I)=0.0
      DO 202 KOUNT=1,NTIM
      READ(5,307)DELT,RT,RQ,KEXT,KPRINT,KQ
  307 FORMAT(3F10.5,3I5)
C*****DELT= TIME STEP
C*****RT= RATIO OF TIME STEP TO PREVIOUS TIME STEP
C*****RQ= RATIO OF RECHARGE RATE TO PREVIOUS RECHARGE RATE
C*****KEXT= EXTRAPOLATION CODE, 1  EXTRAPOLATE,  0  NO EXTRAPOLATUON
C*****KPRINT= PRINTING CODE,  1  PRINT,  0  NO PRINTING
C*****KQ= RECHARGE CODE,  1  NON-ZERO RECHARGE,  0  ZERO RECHARGE
      IF(KQ.NE.0)   GO TO 330
      DO 331  I=1,NX
  331 Q(I)=0.
      GO TO 332
  330 READ(5,308)(Q(I),I=1,NX)
  308 FORMAT(8F10.5)
  332 DO 203  I=1,NX
  203 QT(I)=QT(I)+Q(I)*DELT
      TIME=TIME+DELT
      E5=DELX*DELX/DELT
      DO 204  I=2,NX
      IP=I+1
      IM=I-1
      DO 205 J=2,NY
      JP=J+1
      JM=J-1
      IF(NCAL(I,J).EQ.3)  GO TO 206
      HC=PHI(I,J)-FLOAT(J-1)*DELY
      IF(HC.LT.Z)  GO TO 207
  206 F131(I,J)=0.0
      BN(I,J)=0.0
      GO TO 205
  207 IF(NCAL(I,J).EQ.2)  GO TO 315
      BN(I,J)=0.
      F131(I,J)=0.
      GO TO 205
  315 F2=DELKX/SK(I,J)
      F3=DELKT/SK(I,J)
      F5=F2*DELX/2.
      F6=F3*E*DELX/2.
      F7=1.+F5
      F8=1.-F5
      F9=E2-F6
      F10=E2+F6
      HD=ABS(HC)
      HE=HD/HB
      RK=AC/(BA+HE**TAW)
      CK=RK*SK(I,J)
      TAWM=TAW-1.
      F1=G1*(HD**TAWM)*RK
      F11=F1/4.
      F12=E2*F11
      SE=AP/(BP+HE**TAWP)
      TAWPM=TAWP-1.
      F131(I,J)=SE*SE*(HE**TAWPM)/CK
      BN(I,J)=F7*PHI(IP,J)+F8*PHI(IM,J)+F9*PHI(I,JP)+F10*PHI(I,JM)+F11*(
     $PHI(IP,J)-PHI(IM,J))*(PHI(IP,J)-PHI(IM,J))+F12*(PHI(I,JP)-PHI(I,JM
     $))*(PHI(I,JP)-PHI(I,JM))-(2.+2.*E2)*PHI(I,J)
  205 CONTINUE
  204 CONTINUE
      IF(KEXT.EQ.0)  GO TO 208
      DO 210 I=2,NX
      IF(I.EQ.NX.AND.KLBOUN.EQ.0)  GO TO 210
      DO 211 J=2,NY
      IF(NCAL(I,J).EQ.1)  GO TO 212
      IF(NCAL(I,J).EQ.3) GO TO 212
      DIF=PHI(I,J)-PH(I,J)
      PH(I,J)=PHI(I,J)
      PHI(I,J)=PHI(I,J)+DIF*RT
      GO TO 211
  212 PH(I,J)=PHI(I,J)
  211 CONTINUE
  210 CONTINUE
      GO TO 299
  208 DO 213 I=1,NX
      DO 213 J=2,NY
  213 PH(I,J)=PHI(I,J)
  299 NCT=0
  214 SUM=0.0
      DO 215 I=2,NX1
      IP=I+1
      IM=I-1
      DO 216 J=2,NY
      JP=J+1
      JM=J-1
      IF(NCAL(I,J).EQ.1)  GO TO 216
      IF(NCAL(I,J).EQ.3)  GO TO 216
      F2=DELKX/SK(I,J)
      F3=DELKT/SK(I,J)
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      F5=F2*DELX/2.
      F6=F3*E*DELX/2.
      F7=1.+F5
      F8=1.-F5
      F9=E2-F6
      F10=E2+F6
      NCA=NCAL(I,J)
      HC=PHI(I,J)-FLOAT(J-1)*DELY
      IF(HC.LT.Z)  GO TO 217
  468 CK=SK(I,J)
      F132=0.0
      F11=0.0
      F12=0.0
      GO TO 218
  217 PP=PHI(I,J)
      IF(KOUNT.NE.1)  GO TO 470
      IF(KQ.EQ.0)   GO TO  470
      IF(NCT.EQ.0)  GO TO 472
      GO TO 470
  472 GO TO (216,470,216,471,471,471,471,471,471,470,470,470,470,470,470
     $),NCA
  471 PHI(I,J)=.5*(PHI(I,J)+FLOAT(J-1)*DELY)
  470 NTT=0
      FF1=G1*Q(I)/SK(I,J)
  467 HC=PHI(I,J)-FLOAT(J-1)*DELY
      IF(HC.GE.Z)  GO TO 468
      HD=ABS(HC)
      HE=HD/HB
      RK=AC/(BA+HE**TAW)
      CK=RK*SK(I,J)
      TAWM=TAW-1.
      F1=G1*(HD**TAWM)*RK
      F11=F1/4.
      F12=E2*F11
      SE=AP/(BP+HE**TAWP)
      TAWPM=TAWP-1.
      F132=SE*SE*(HE**TAWPM)/CK
      F13=F132+F131(I,J)
      G5=G3*F13*E5
      GO TO(216,452,216,454,455,456,457,458,459,460,461,462,463,464,465)
     $,NCA
  452 F17=0.
      F18=0.
      F19=0.
      F20=0.
      GO TO 466
  454 FF=-FF1*(HD**(TAW-1.))
      F17=0.
      F18=0.
      F19=E3*FF
      F20=0.
      PHI(I,JP)=E3*G(I)/CK+PHI(I,JM)
      GO TO 466
  455 FF=-FF1*(HD**(TAW-1.))
      F17=E8*FF
      F18=0.
      F19=E3*FF
      F20=0.
      PHI(I,JP)=E3*Q(I)/CK+PHI(I,JM)
      PHI(IP,J)=E10*PHI(IP,JM)+E9*PHI(I,JM)+E8*Q(I)/CK
      GO TO 466
  456 FF=-FF1*(HD**(TAW-1.))
      F17=E8*FF
      F18=0.
      F19=E10+E8*FF
      F20=0.
      PHI(I,JP)=E10*PHI(I,J)+E9*PHI(IM,J)+E8*Q(I)/CK
      PHI(IP,J)=E10*PHI(IP,JM)+E9*PHI(I,JM)+E8*Q(I)/CK
      GO TO 466
  457 FF=-FF1*(HD**(TAW-1.))
      F17=0.
      F18=E8*FF
      F19=E3*FF
      F20=0.
      PHI(I,JP)=E3*Q(I)/CK+PHI(I,JM)
      PHI(IM,J)=E10*PHI(IM,JM)+E9*PHI(I,JM)+E8*Q(I)/CK
      GO TO 466
  458 FF=-FF1*(HD**(TAW-1.))
      F17=0.
      F18=E8*FF
      F19=E10+E8*FF
      F20=0.
      PHI(I,JP)=E10*PHI(I,J)+E9*PHI(IP,J)+E8*Q(I)/CK
      PHI(IM,J)=E10*PHI(IM,JM)+E9*PHI(I,JM)+E8*Q(I)/CK
      GO TO 466
  459 FF=-FF1*(HD**(TAW-1.))
      F17=E8*FF
      F18=F17
      F19=E3*FF
      F20=0.
      PHI(I,JP)=E3*Q(I)/CK+PHI(I,JM)
      PHI(IP,J)=E10*PHI(IP,JM)+E9*PHI(I,JM)+E8*Q(I)/CK
      PHI(IM,J)=E10*PHI(IM,JM)+E9*PHI(I,JM)+E8*Q(I)/CK
      GO TO 466
  460 F17=0.
      F18=0.
      F19=0.
      F20=0.
      PHI(I,JM)=PHI(I,JP)
      GO TO 466
  461 F17=0.
      F18=0.
      F19=0.
      F20=0.
      PHI(I,JM)=PHI(I,JP)
      PHI(IP,J)=E10*PHI(IP,JP)+E9*PHI(I,JP)
      GO TO 466
  462 F17=0.
      F18=0.
      F19=0.
      F20=0.
      PHI(I,JM)=PHI(I,JP)
      PHI(IM,J)=E10*PHI(IM,JP)+E9*PHI(I,JP)
      GO TO 466
  463 F17=0.
      F18=0.
      F19=0.
      F20=0.
      PHI(I,JM)=PHI(I,JP)
      PHI(IP,J)=E10*PHI(IP,JP)+E9*PHI(I,JP)
      PHI(IM,J)=E10*PHI(IM,JP)+E9*PHI(I,JP)
      GO TO 466
  464 F17=0.
      F18=0.
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      F19=0.
      F20=E10
      PHI(IM,J)=E10*PHI(IM,JP)+E9*PHI(I,JP)
      PHI(I,JM)=E10*PHI(I,J)+E9*PHI(IP,J)
      GO TO 466
  465 F17=0.
      F18=0.
      F19=0.
      F20=E10
      PHI(IP,J)=E10*PHI(IP,JP)+E9*PHI(I,JP)
      PHI(I,JM)=E10*PHI(I,J)+E9*PHI(IM,J)
  466 F21=(G1/4.)*(HD**(TAW-2.))*RK*(G1*RK*(HD**TAW)+1.-TAW)
      F22=E2*F21
      F25=(G3*E5*SE*SE/CK)*(HD**(TAWP-2.))*(2.*G2*SE*(HD**TAWP)-G1*RK*(H
     $D**TAW)+1.-TAWP)
      VC=PHI(I,J)
      V1=PHI(IM,J)
      V2=PHI(IP,J)
      V3=PHI(I,JM)
      V4=PHI(I,JP)
      FN=F7*V2+F8*V1+F9*V4+F10*V3-(2.+2.*E2+G5)*VC+F11*(V2-V1)*(V2-V1)+F
     $12*(V4-V3)*(V4-V3)+BN(I,J)+G5*PH(I,J)
      FNP=F7*F17+F8*F18+F9*F19+F10*F20-(2.+2.*E2+G5)-F25*VC+F21*(V2-V1)*
     $(V2-V1)+2.*F11*(V2-V1)*(F17-F18)+F22*(V4-V3)*(V4-V3)+2.*F12*(V4-V3
     $)*(F19-F20)+F25*PH(I,J)
      DIT=FN/FNP
      PHI(I,J)=PHI(I,J)-DIT
      NTT=NTT+1
      DIFT=ABS(DIT)
      IF(NTT.LT.NWFT.AND.DIFT.GT.ERNWT)   GO TO 467
      HC=PHI(I,J)-FLOAT(J-1)*DELY
      IF(HC.LT.0.)  GO TO 310
      GO TO (216,310,216,311,311,311,311,311,311,310,310,310,310,310,310
     $),NCA
  311 PHI(I,J)=FLOAT(J-1)*DELY
  310 DIF=PHI(I,J)-PP
      SUM=SUM+ABS(DIF)
      PHI(I,J)=PP+W4*DIF
      GO TO 281
  218 F13=F132+F131(I,J)
      G5=G3*F13*E5
      G6=1./(2.+2.*E2+G5)
      GO TO (216,226,216,228,229,230,231,232,233,220,221,222,223,224,225
     $),NCA
  228 PHI(I,JP)=E3*Q(I)/CK+PHI(I,JM)
      GO TO 226
  229 PHI(I,JP)=E3*Q(I)/CK+PHI(I,JM)
      PHI(IP,J)=E10*PHI(IP,JM)+E9*PHI(I,JM)+E8*Q(I)/CK
      GO TO 226
  230 PHI(I,JP)=E10*PHI(I,J)+E9*PHI(IM,J)+E8*Q(I)/CK
      PHI(IP,J)=E10*PHI(IP,JM)+E9*PHI(I,JM)+E8*Q(I)/CK
      GO TO 226
  231 PHI(I,JP)=E3*Q(I)/CK+PHI(I,JM)
      PHI(IM,J)=E10*PHI(IM,JM)+E9*PHI(I,JM)+E8*Q(I)/CK
      GO TO 226
  232 PHI(I,JP)=E10*PHI(I,J)+E9*PHI(IP,J)+E8*Q(I)/CK
      PHI(IM,J)=E10*PHI(IM,JM)+E9*PHI(I,JM)+E8*Q(I)/CK
      GO TO 226
  233 PHI(I,JP)=E3*Q(I)/CK+PHI(I,JM)
      PHI(IP,J)=E10*PHI(IP,JM)+E9*PHI(I,JM)+E8*Q(I)/CK
      PHI(IM,J)=E10*PHI(IM,JM)+E9*PHI(I,JM)+E8*Q(I)/CK
      GO TO 226
  220 PHI(I,JM)=PHI(I,JP)
      GO TO 226
  221 PHI(I,JM)=PHI(I,JP)
      PHI(IP,J)=E10*PHI(IP,JP)+E9*PHI(I,JP)
      GO TO 226
  222 PHI(I,JM)=PHI(I,JP)
      PHI(IM,J)=E10*PHI(IM,JP)+E9*PHI(I,JP)
      GO TO 226
  223 PHI(I,JM)=PHI(I,JP)
      PHI(IP,J)=E10*PHI(IP,JP)+E9*PHI(I,JP)
      PHI(IM,J)=E10*PHI(IM,JP)+E9*PHI(I,JP)
      GO TO 226
  224 PHI(IM,J)=E10*PHI(IM,JP)+E9*PHI(I,JP)
      PHI(I,JM)=E10*PHI(I,J)+E9*PHI(IP,J)
      GO TO 226
  225 PHI(IP,J)=E10*PHI(IP,JP)+E9*PHI(I,JP)
      PHI(I,JM)=E10*PHI(I,J)+E9*PHI(IM,J)
  226 PHT=G6*(F7*PHI(IP,J)+F8*PHI(IM,J)+F9*PHI(I,JP)+F10*PHI(I,JM)+F11*(
     $PHI(IP,J)-PHI(IM,J))*(PHI(IP,J)-PHI(IM,J))+F12*(PHI(I,JP)-PHI(I,JM
     $))*(PHI(I,JP)-PHI(I,JM))+G5*PH(I,J)+BN(I,J))
      HC=PHT-FLOAT(J-1)*DELY
      IF(HC.LT.0.)  GO TO 312
      GO TO (216,312,216,313,313,313,313,313,313,312,312,312,312,312,312
     $),NCA
  313 PHT=FLOAT(J-1)*DELY
  312 DIF=PHT-PHI(I,J)
      SUM=SUM+ABS(DIF)
      HC=PHT-FLOAT(J-1)*DELY
      IF(HC.LT.0.0)  GO TO 234
      PHI(I,J)=PHI(I,J)+W3*DIF
      GO TO 281
  234 PHI(I,J)=PHI(I,J)+W4*DIF
  281 HC=PHI(I,J)-FLOAT(J-1)*DELY
      IF(HC.LT.0.0)  GO TO 216
      GO TO (216,216,216,280,280,280,280,280,280,216,216,216,216,216,216
     $),NCA
  280 PHI(I,J)=FLOAT(J-1)*DELY
  216 CONTINUE
  215 CONTINUE
      IF(KLBOUN.EQ.0)  GO TO 253
      I=NX
      IP=I+1
      IM=I-1
      DO 240 J=2,NY
      JP=J+1
      JM=J-1
      IF(NCAL(I,J).EQ.1)  GO TO 240
      IF(NCAL(I,J).EQ.3)  GO TO 240
      F2=DELKX/SK(I,J)
      F3=DELKT/SK(I,J)
      F5=F2*DELX/2.
      F6=F3*E*DELX/2.
      F7=1.+F5
      F8=1.-F5
      F9=F2-F6
      F10=E2+F6
      IF(NCAL(I,J).EQ.90)  GO TO 241
      IF(NCAL(I,J).EQ.100)  GO TO 245
      GO TO 243
  241 PHI(I,JP)=PHI(I,JM)
      IF(NCAL(IM,J).EQ.1)  GO TO 242
      GO TO 243
```

```
  242 PHI(IM,J)=PHI(I,J)+E7/2.
      PHI(IP,J)=PHI(I,J)-E7/2.
      GO TO 244
  245 PHI(I,JM)=PHI(I,JP)
      IF(NCAL(IM,J).EQ.1) GO TO 246
      GO TO 243
  246 PHI(IM,J)=PHI(I,J)+E7/2.
      PHI(IP,J)=PHI(I,J)-E7/2.
      GO TO 244
  243 PHI(IP,J)=PHI(IM,J)-E7
  244 HC=PHI(I,J)-FLOAT(J-1)*DELY
      IF(HC.LT.Z)  GO TO 247
      CK=SK(I,J)
      F11=0.0
      F12=0.0
      F132=0.0
      GO TO 248
  247 HD=ABS(HC)
      HE=HD/HB
      RK=AC/(BA+HE**TAW)
      CK=RK*SK(I,J)
      TAWM=TAW-1.
      F1=G1*(HD**TAWM)*RK
      F11=F1/4.
      F12=E2*F11
      SE=AP/(BP+HE**TAWP)
      TAWPM=TAWP-1.
      F132=SE*SE*(HE**TAWPM)/CK
  248 F13=F132+F131(I,J)
      G5=G3*F13*E5
      G6=1./(2.+2.*E2+G5)
      PHT=G6*(F7*PHT(IP,J)+F8*PHI(IM,J)+F9*PHI(I,JP)+F10*PHI(I,JM)+F11*(
     $PHI(IP,J)-PHI(IM,J))*(PHI(IP,J)-PHI(IM,J))+F12*(PHI(I,JP)-PHI(I,JM
     $))*(PHI(I,JP)-PHI(I,JM))+G5*PH(I,J)+BN(I,J))
      DIF=PHT-PHI(I,J)
      SUM=SUM+ABS(DIF)
      HC=PHT-FLOAT(J-1)*DELY
      IF(HC.LT.0.0)  GO TO 249
      PHI(I,J)=PHI(I,J)+W3*DIF
      GO TO 250
  249 PHI(I,J)=PHI(I,J)+W4*DIF
  250 HC=PHI(I,J)-FLOAT(J-1)*DELY
      IF(HC.LT.0.0)  GO TO 251
      IF(NCAL(I,J).EQ.90) GO TO 252
      GO TO 240
  252 PHI(I,J)=FLOAT(J-1)*DELY
  240 CONTINUE
      GO TO 253
  251 JP=J+1
      DO 254 K=JP,NY
      IF(NCAL(I,K).EQ.1) GO TO 254
      IF(NCAL(I,K).EQ.3) GO TO 254
      PHI(I,K)=PHI(I,K-1)
      PHI(IP,K)=PHI(I,K)-E7/2.
  254 CONTINUE
  253 NCT=NCT+1
 6011 IF(SUM.GT.ERR.AND.NCT.LT.MAX2) GO TO 214
      IF(SUM.GT.ERR)   GO TO 323
      GO TO 324
  323 WRITE(6,325)
  325 FORMAT(1H0,'  SOLUTION  DIVERGED')
      GO TO 2000
  324 CONTINUE
      IF(KPRINT.EQ.0) GO TO 260
      WRITE(6,350)
  350 FORMAT(1H1,'   UNSTEADY STATE STEP')
      WRITE(6,351)
  351 FORMAT(1H0,'        DELT                TIME')
      WRITE(6,352)DELT,TIME
  352 FORMAT(2(5X,F10.5))
      WRITE(6,353)
  353 FORMAT(1H0,'    Q(I)= ')
      WRITE(6,9011)(Q(I),I=1,NX)
 9011 FORMAT(1H ,10(3X,F8.4))
      WRITE(6,354)
  354 FORMAT(1H0,'   QT(I)=  ')
      WRITE(6,9011)(QT(I),I=1,NX)
      WRITE(6,6002)
 6002 FORMAT(1H ,'     NCT=  ')
      WRITE(6,6003)NCT
 6003 FORMAT(5X,I5)
      WRITE(6,6000)
 6000 FORMAT(1H ,'   SUM=')
      WRITE(6,6001)SUM
 6001 FORMAT(F12.5)
      WRITE(6,355)
  355 FORMAT(1H0,'  UNSTEADY STATE  PHI')
      N2=0
  361 N2=N2+10
      N1=N2-9
      IF(N2.GT.NX)  N2=NX
      WRITE(6,360)
  360 FORMAT(1H0,'  UNSTEADY STATE PHI')
      DO 356  INDEX=1,NY
      J=NY+1-INDEX
  356 WRITE(6,9011)(PHI(I,J),I=N1,N2)
      IF(N2.LT.NX)  GO TO 361
      WRITE(6,362)
  362 FORMAT(1H0,'   UNSTEADY  STATE  H')
      N2=0
  376 N2=N2+10
      N1=N2-9
      IF(N2.GT.NX)  N2=NX
      WRITE(6,377)
  377 FORMAT(1H0,'  UNSTEADY STATE  H')
      DO 363  INDEX=1,NY
      J=NY+1-INDEX
      DO 364 I=N1,N2
      IF(NCAL(I,J).EQ.1)  GO TO 365
      H(I)=PHI(I,J)-FLOAT(J-1)*DELY
      GO TO 364
  365 H(I)=1.
  364 CONTINUE
  363 WRITE(6,9011)(H(I),I=N1,N2)
      IF(N2.LT.NX)  GO TO 376
  260 CONTINUE
  202 CONTINUE
 2000 STOP
      END
@FOR,IS SUB1,SUB1
      SUBROUTINE DECOMP(N,*)
      COMMON A(3,3)
C******
C******THIS SUBROUTINE DECOMPOSES A POSITIVE DEFINITE MATRIX
```

```
      DO 800 I=1,N
      DO 800 J=1,N
      SUM=A(I,J)
      LIMIT=I-1
      IF(LIMIT.EQ.0)  GO TO 801
      DO 802 K=1,LIMIT
  802 SUM=SUM-A(K,I)*A(K,J)
  801 IF(J.NE.I) GO TO 803
      IF(SUM.LE.0.0)  RETURN 1
      TT=SQRT(SUM)
      TEMP=1./TT
      A(I,J)=TT
      GO TO 800
  803 A(I,J)=SUM*TEMP
  800 CONTINUE
      RETURN
      END
@FOR,IS SUB2,SUB2
      SUBROUTINE SOLVE(N)
      COMMON U(3,3),B(3),C(3)
C*******
C******THIS SUBROUTINE SOLVES THE DECOMPOSED MATRIX EQUATION
      DO 808 I=1,N
      DO 808 J=1,N
      IF(I.NE.J) GO TO 808
      U(I,J)=1./U(I,J)
  808 CONTINUE
      DO 804 I=1,N
      SUM=B(I)
      LIMIT=I-1
      IF(LIMIT.EQ.0)  GO TO 804
      DO 805 K=1,LIMIT
  805 SUM=SUM-U(K,I)*C(K)
  804 C(I)=SUM*U(I,I)
      DO 807 INDEX=1,N
      I=N+1-INDEX
      SUM=C(I)
      L=I+1
      M=N+1
      IF(L.EQ.M)  GO TO 807
      DO 806 K=L,N
  806 SUM=SUM-U(I,K)*C(K)
  807 C(I)=SUM*U(I,I)
      RETURN
```

Appendix B

Derivation of Equations of Boundaries

(1) - (2) and (6) - (7)

Reference is made to Figure 15 which shows uniform flow on a bed sloping with an angle α.

Boundary (1) - (2)

$$\frac{d\phi}{ds} = 0 = \frac{\partial\phi}{\partial x} \cdot \frac{dx}{ds} + \frac{\partial\phi}{\partial y} \cdot \frac{dy}{ds} = \frac{\partial\phi}{\partial x} \cdot \text{Sin}\ \alpha + \frac{\partial\phi}{\partial y} \cdot \text{Cos}\ \alpha$$

$$\frac{d\phi}{dn} = \frac{d}{dn}\left(y + \frac{P}{\rho g}\right) = \frac{dy}{dn} = -\text{Sin}\ \alpha$$

Since P is constant along the normal $\vec{n}$.

also we have

$$\frac{d\phi}{dn} = \frac{\partial\phi}{\partial x} \cdot \frac{dx}{dn} + \frac{\partial\phi}{\partial y} \cdot \frac{dy}{dn} = \frac{\partial\phi}{\partial x} \cdot \text{Cos}\ \alpha - \frac{\partial\phi}{\partial y} \cdot \text{Sin}\ \alpha$$

Solving simultaneously the two equations:

$$\frac{\partial\phi}{\partial x} \text{Sin}\ \alpha + \frac{\partial\phi}{\partial y} \text{Cos}\ \alpha = 0 \qquad \text{(a)}$$

$$\frac{\partial\phi}{\partial x} \text{Cos}\ \alpha - \frac{\partial\phi}{\partial y} \text{Sin}\ \alpha = -\text{Sin}\ \alpha \qquad \text{(b)}$$

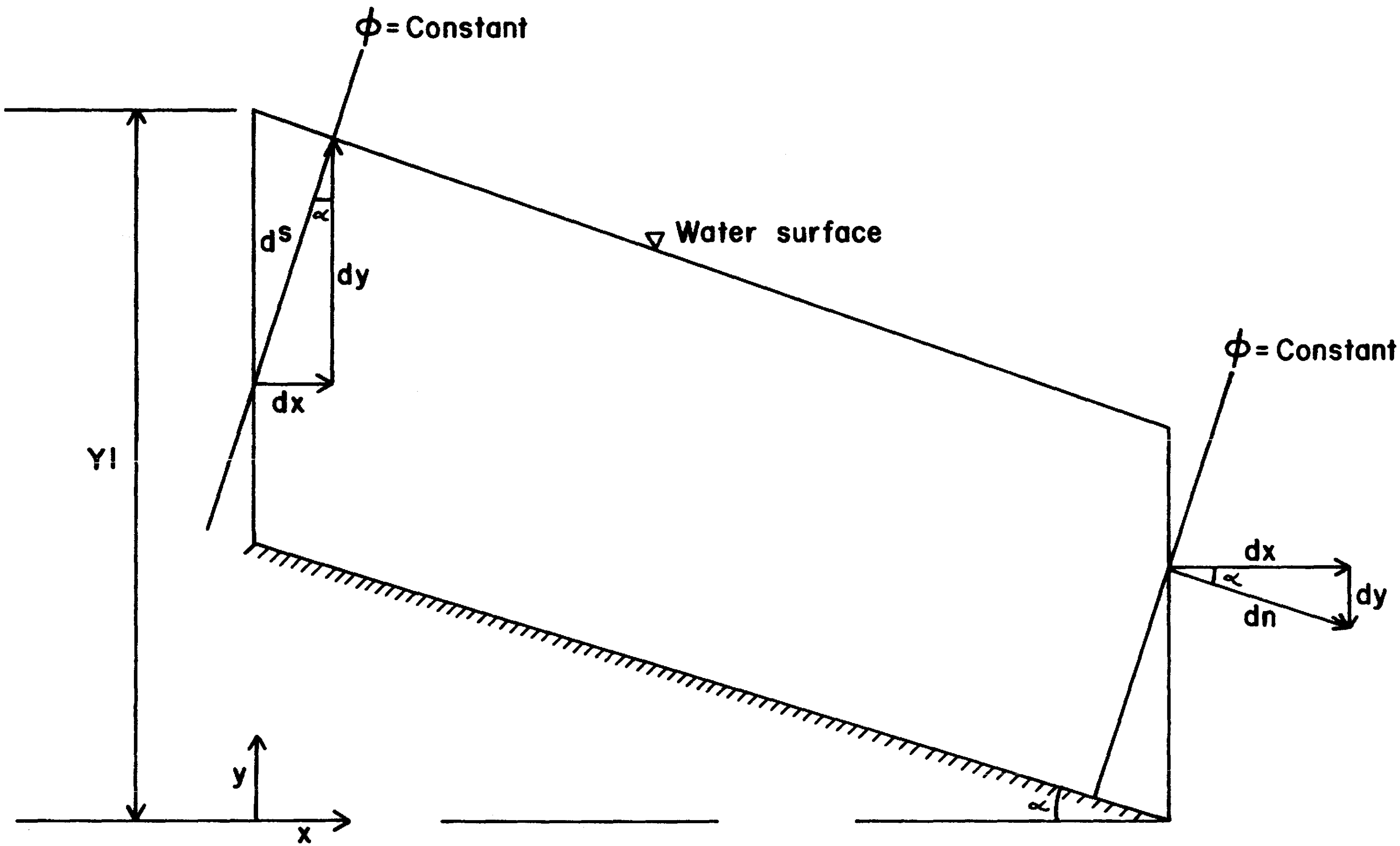

Figure 15. Derivation of equations of boundaries (1)-(2) and (6)-(7).

gives:

$$\frac{\partial \phi}{\partial y} = \mathrm{Sin}^2 \alpha \qquad \text{(c)}$$

integrating equation (c) gives:

$$\phi = y\ \mathrm{Sin}^2 \alpha + C$$

with $\phi = Y1$ when $y = Y1$

This gives:

$$C = Y1 \cdot \mathrm{Cos}^2 \alpha,$$

and:

$$\phi = y\ \mathrm{Sin}^2 \alpha + Y1\ \mathrm{Cos}^2 \alpha \qquad (38)$$

<u>Boundary (6) - (7)</u>

Again solving Equations (a) and (b) simultaneously gives:

$$\frac{\partial \phi}{\partial x} = -\mathrm{Sin}\ \alpha\ \mathrm{Cos}\ \alpha \qquad (44)$$

Appendix C

Derivation of Equations for the Imaginary Points at Irrigular Boundaries

Horizontal Segments

On those segments the normal direction is in the y-direction so that $\frac{d\phi}{dn} = \frac{\partial \phi}{\partial y}$. Equation (40) becomes

$$\frac{\partial \phi}{\partial y} = \frac{1}{K} q_i \text{ Cos } \Theta \qquad (a)$$

Where Θ is the angle between horizontal and the segment. Since $\Theta = 0$ and $\text{Cos } \Theta = 1$ Equation (a) above with central differences give:

$$\phi_{i,j+1} = \phi_{i,j-1} + 2\Delta y \frac{q_i}{K} \qquad (b)$$

Equations (39), (42) and (45) give:

$$\frac{\partial \phi}{\partial y} = 0$$

which results in:

$$\phi_{i,j+1} = \phi_{i,j-1} \qquad (c)$$

For the upper boundary and:

$$\phi_{i,j-1} = \phi_{i,j+1} \qquad \text{(d)}$$

for the lower boundary.

Sloping segments--upper boundary

Reference is made to Figure 16a and b where the circled points are the imaginary points. Fox's method as described by Forsythe and Wasow(1960)and Remson, Hornberger and Molz (1971). In Figure 16a N is normal to the sloping segment and intersects the side at ϕ_4

$$\phi_4 = \phi_1 + \frac{m}{\Delta x}(\phi_3 - \phi_1)$$

$$m = \Delta x - \Delta y \tan \Theta$$

$$\tan \Theta = \frac{\Delta y}{\Delta x}$$

$$\frac{d\phi}{dn} = \frac{\phi_5 - \phi_4}{N} = \frac{q_i}{K} \text{ Cos } \Theta$$

This gives:

$$\phi_5 = \phi_4 + N \text{ Cos}\Theta \frac{q_i}{K} = \phi_4 + \Delta y \frac{q_i}{K}$$

Which results in:

$$\phi_5 = \frac{1}{E2}\phi_1 + (1 - \frac{1}{E2})\phi_3 + \Delta y \frac{q_i}{K} \qquad \text{(e)}$$

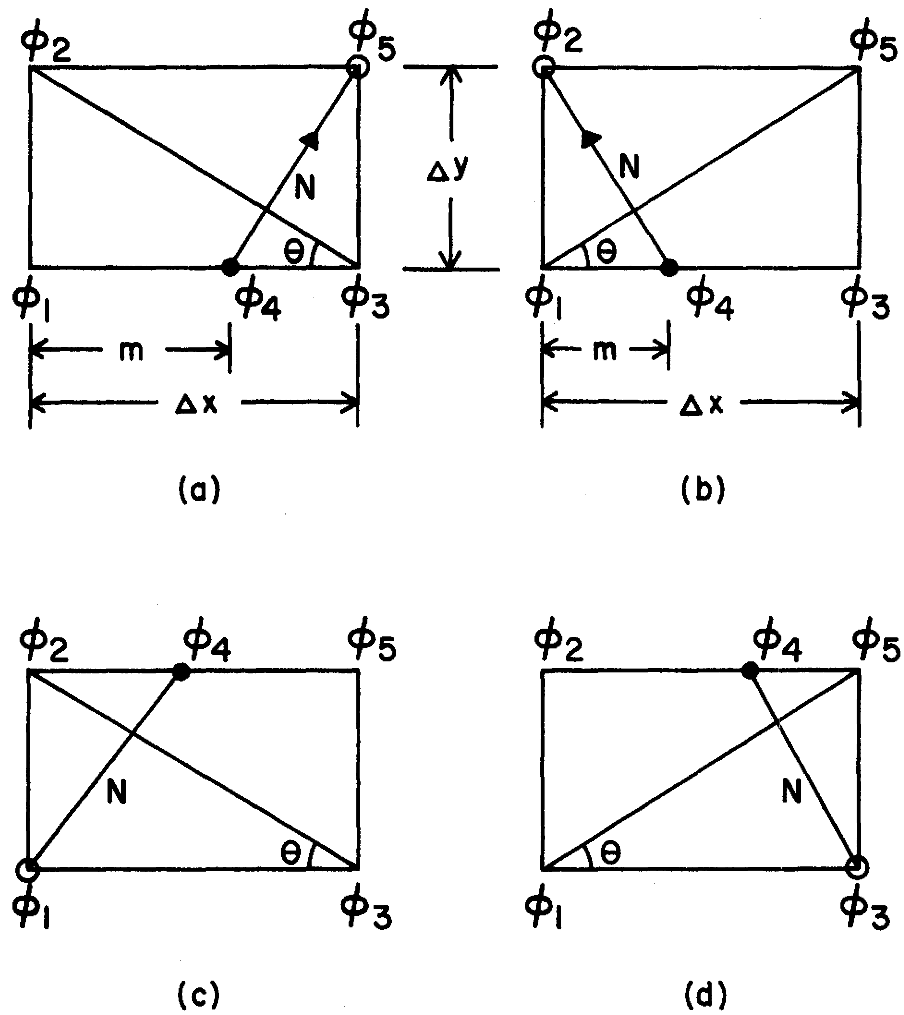

Figure 16. Imaginary nodes at irregular boundaries

Where:

$$E2 = \left(\frac{\Delta x}{\Delta y}\right)^2$$

Similar treatment of Figure 16b gives

$$\phi_2 = \frac{1}{E2}\,\phi_3 + \left(1 - \frac{1}{E2}\right)\phi_1 + \Delta y\,\frac{q_i}{K} \qquad \text{(f)}$$

<u>Sloping segments--lower boundary</u>

Treatment similar to the above for Figure 16c gives:

$$\phi_1 = \phi_4$$

$$\phi_4 = \phi_2 + \frac{1}{E2}\left(\phi_5 - \phi_2\right)$$

and

$$\phi_1 = \frac{1}{E2}\,\phi_5 + \left(1 - \frac{1}{E2}\right)\phi_2 \qquad \text{(g)}$$

similarly, Figure 16d gives

$$\phi_3 = \frac{1}{E2}\,\phi_2 + \left(1 - \frac{1}{E2}\right)\phi_5 \qquad \text{(h)}$$

Combinations of Equations (b), (e) and (f) and Equations (d), (g) and (h) give Equations for all the different types of segments (Codes 4 - 15).

Appendix D

Least Squares Fitting for Heterogeneous Conductivity

Suppose we have n number of tripples of data measurements:

$$(x_i, T_i, Ko_i) \qquad 1 = 1, 2 \qquad n \qquad n \geq 3$$

Where:

x = the x - coordinate of a point

T = the depth from surface to the point

K_o = Hydraulic conductivity at the point

We want to fit the data to a function of the form:

$$K_o = C1 + C2x + C3T \qquad \text{(a)}$$

The best approximation of the n-dimensional vector:

$$\vec{K}_o = (Ko_1, Ko_2, \ldots Ko_n) \qquad \text{(b)}$$

in the three-dimensional subspace spanned by the three linearly independant vectors:

$$\vec{I} = (1, 1, \ldots.) \quad \text{n elements}$$

$$\vec{x} = (x_1, x_2 \ldots x_n) \qquad \text{(c)}$$

$$\vec{T} = (T_1, T_2, \cdot T_n)$$

is the normal projection of $\vec{K}_o$ onto this subspace. This normal projection $\vec{K}_o'$ will be a linear combination of the three base vectors such that

$$\vec{K}_o' = C1\ \vec{1} + C2\ \vec{x} + C3\ \vec{T} \qquad \text{(b)}$$

This will minimize the square of the distance

$$\left\| \vec{K}_o - \vec{K}_o' \right\|^2$$

and hence the name of the method. From the properties of projections we know that the vector $(\vec{K}_o - \vec{K}_o')$ will be normal to all vectors in the subspace. Hence we have:

$$\begin{aligned} (\vec{K}_o - \vec{K}_o') \cdot \vec{1} &= 0 \\ (\vec{K}_o - \vec{K}_o') \cdot \vec{x} &= 0 \\ (\vec{K}_o - \vec{K}_o') \cdot \vec{T} &= 0 \end{aligned} \qquad \text{(c)}$$

or:

$$\begin{aligned} (C1\ \vec{1} + C2\ \vec{x} + C3\ \vec{T}) \cdot \vec{1} &= \vec{K}_o \cdot \vec{1} \\ (C1\ \vec{1} + C2\ \vec{x} + C3\ \vec{T}) \cdot \vec{x} &= \vec{K}_o \cdot \vec{x} \\ (C1\ \vec{1} + C2\ \vec{x} + C3\ \vec{T}) \cdot \vec{T} &= \vec{K}_o \cdot \vec{T} \end{aligned} \qquad \text{(d)}$$

This gives

$$C1\, n + C2 \sum_{i=1}^{n} x_i + C3 \sum_{i=1}^{n} T_i = \sum_{i=1}^{n} K_{oi}$$

$$C1 \sum_{i=1}^{n} x_i + C2 \sum_{i=1}^{n} x_i^2 + C3 \sum_{i=1}^{n} x_i T_i = \sum_{i=1}^{n} x_i K_{oi} \qquad \text{(e)}$$

$$C1 \sum_{i=1}^{n} T_i + C2 \sum_{i=1}^{n} x_i T_i + C3 \sum_{i=1}^{n} T_i^2 = \sum_{i=1}^{n} T_i K_{oi}$$

or in matrix form:

$$\begin{bmatrix} n & \Sigma x_i & \Sigma T_i \\ \Sigma x_i & \Sigma x_i^2 & \Sigma x_i T_i \\ \Sigma T_i & \Sigma x_i T_i & \Sigma T_i^2 \end{bmatrix} \begin{bmatrix} C1 \\ C2 \\ C3 \end{bmatrix} = \begin{bmatrix} \Sigma K_{oi} \\ \Sigma x_i K_{oi} \\ \Sigma T_i K_{oi} \end{bmatrix} \qquad \text{(f)}$$

or

$$A \vec{C} = \vec{B} \qquad \text{(g)}$$

Where A is the symmetric positive definite matrix shown. Equation (g) is solved by decomposing the matrix A into two triangular matrices, one upper and the other lower (one is the transpose of the other) such that:

$$A = U^t U$$

Where

$$U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

and

$$U^t = \begin{bmatrix} U_{11} & 0 & 0 \\ U_{12} & U_{22} & 0 \\ U_{13} & U_{23} & U_{33} \end{bmatrix}$$

Equation (g) becomes:

$$U^t \, U \, \vec{C} = \vec{B} \qquad \text{(h)}$$

or

$$U^t \, \vec{Y} = \vec{B} \qquad \text{(i)}$$

Where

$$\vec{Y} = U \, \vec{C} \qquad \text{(j)}$$

Equation (i) is solved easily for $\vec{Y}$ by a forward sweep and then Equation (j) is solved for $\vec{C}$ by a backward sweep. The program sets the matrix Equation (g) and the two subroutines Decompose and Solve do the solution.

Appendix E

Data for Hullinger Farm

Table 5. Piezometer locations and surface elevations for the Hullinger farm test**

Piezometer number	Station feet	Surface elevation feet
1	0	5276.60
2	99	5274.72
3	129	5274.07
4	144	5273.86
Drain No. 6	149	5273.85
5	154	5273.85
6	169	5273.41
7	199	5273.05
8	249	5272.21
9	324	5271.07
10	399	5270.25
11	479	5269.13
12	494	5269.05
Drain No. 5	499	5269.01
13	504	5268.97
14	519	5268.60
15	599	5267.53

**Source: Dr. Larry G. King, unpublished data

Table 6. Water table positions in Khalil-Ur-Rehman's experiment**

Date	Time	Depth to the Water table - Feet							
Oct. 1970		Piezometer Number							
		1	2	3	4	5	6	7	8
8	8:30	4.25	3.37	3.11	3.09	3.20	2.88	2.78	2.36
	11:10	4.31	3.49	3.29	3.40	3.49	3.05	2.90	2.48
	12:45	4.35	3.54	3.39	3.49	3.59	3.15	3.01	2.55
	13:55	4.35	3.58	3.41	3.52	3.62	3.16	3.02	2.66
	15:45	4.43	3.70	3.51	3.66	3.72	3.30	3.15	2.72
	18:00	4.48	3.78	3.60	3.68	3.79	3.38	3.27	2.82
9	7:15	4.70	4.04	3.89	3.96	4.06	3.70	3.60	3.20
	10:25	4.72	4.07	3.92	3.98	4.10	3.75	3.67	3.26
	14:00	4.75	4.12	4.00	4.01	4.15	3.80	3.71	3.32
	18:00	4.82	4.19	4.03	4.06	4.18	3.85	3.78	3.38
10	7:45	4.97	4.33	4.20	4.20	4.33	4.00	3.95	3.59
	12:00	4.98	4.38	4.24	4.24	4.37	4.06	4.01	3.66
	18:00	5.05	4.39	4.30	4.30	4.43	4.13	4.08	3.70
11	17:08	5.19	4.60	4.40	4.44	4.57	4.30	4.24	3.87
12	17:08	5.30	4.74	4.56	4.57	4.69	4.42	4.37	4.01
13	16:40	5.41	4.85	4.69	4.70	4.83	4.55	4.50	4.13
14	8:20	5.51	4.93	4.79	4.78	4.93	4.64	4.58	4.22

**Source: Table 22 of Khalil-Ur-Rehman (1971)

Table 6. Continued **

Date Oct. 1970	Time	Depth to the Water table - Feet						
		Piezometer Number						
		9	10	11	12	13	14	15
8	8:30	1.99	2.13	2.26	2.42	2.45	2.24	2.46
	11:10	2.10	2.25	2.39	2.92	2.95	2.48	2.57
	12:45	2.20	2.35	2.63	3.10	3.12	2.67	2.76
	13:55	2.21	2.42	2.70	3.19	3.21	2.76	2.85
	15:45	2.35	2.58	2.88	3.33	3.35	2.69	3.00
	18:00	2.45	2.68	2.99	3.41	3.43	3.03	3.12
9	7:15	2.88	3.10	3.40	3.69	3.72	3.38	3.45
	10:25	2.94	3.16	3.40	3.72	3.73	3.43	3.49
	14:00	2.99	3.23	3.48	3.77	3.79	3.47	3.54
	18:00	3.06	3.29	3.53	3.81	3.84	3.51	3.59
10	7:45	3.25	3.48	3.70	3.93	3.96	3.65	3.74
	12:00	3.31	3.52	3.71	3.95	4.00	3.69	3.77
	18:00	3.40	3.59	3.79	3.99	4.04	3.74	3.81
11	17:08	3.57	3.76	3.91	4.11	4.14	3.87	3.93
12	17:08	3.70	3.89	4.01	4.19	4.24	3.96	4.02
13	16:40	3.84	3.99	4.18	4.27	4.31	4.04	4.10
14	8:20	3.91	4.08	4.18	4.33	4.36	4.12	4.17

**Source: Table 22 of Khalil-Ur-Rehman (1971)

Table 7. Soil properties used for computations made. Mesa sandy clay loam soil, Hullinger farm.*

Water Content Θ	Hydraulic Conductivity, K (cm/hr)	Pressure Head h (cm)
.00	$1.0\ (10^{-9})$	$-2\ (10^{6})$
.01	$2.0\ (10^{-9})$	$-1.3\ (10^{6})$
.02	$3.4\ (10^{-9})$	$-8.5\ (10^{5})$
.03	$1.0\ (10^{-8})$	$-4.2\ (10^{5})$
.04	$1.7\ (10^{-8})$	$-2.2\ (10^{5})$
.05	$3.0\ (10^{-8})$	$-1.15\ (10^{5})$
.06	$5.4\ (10^{-8})$	$-5.8\ (10^{4})$
.07	$9.2\ (10^{-8})$	$-3.0\ (10^{4})$
.08	$1.6\ (10^{-7})$	$-1.5\ (10^{4})$
.09	$2.7\ (10^{-7})$	$-1.1\ (10^{4})$
.10	$4.8\ (10^{-7})$	$-8.0\ (10^{3})$
.11	$7.5\ (10^{-7})$	$-6.2\ (10^{3})$
.12	$1.5\ (10^{-7})$	$-4.9\ (10^{3})$
.13	$2.5\ (10^{-6})$	$-4.0\ (10^{3})$
.14	$4.5\ (10^{-6})$	$-3.0\ (10^{3})$
.15	$8.7\ (10^{-6})$	$-2.35\ (10^{3})$
.16	$1.4\ (10^{-5})$	$-1.85\ (10^{3})$
.17	$2.5\ (10^{-5})$	$-1.45\ (10^{3})$
.18	$4.5\ (10^{-5})$	$-1.12\ (10^{3})$

Table 7. (Continued)

Water Content Θ	Hydraulic Conductivity, K (cm/hr)	Pressure Head, h (cm)
.19	7.5 (10^{-5})	−8.7 (10^{3})
.20	1.1 (10^{-4})	−6.7 (10^{2})
.21	1.7 (10^{-4})	−5.3 (10^{2})
.22	2.7 (10^{-4})	−4.1 (10^{2})
.23	4.0 (10^{-4})	−3.2 (10^{2})
.24	6.1 (10^{-4})	−2.5 (10^{2})
.25	9.5 (10^{-4})	−2.0 (10^{2})
.26	1.5 (10^{-3})	−1.65 (10^{2})
.27	2.4 (10^{-3})	−1.35 (10^{2})
.28	3.5 (10^{-3})	−1.15 (10^{2})
.29	5.5 (10^{-3})	−9.9 (10)
.30	9.0 (10^{-3})	−8.5 (10)
.31	1.4 (10^{-2})	−7.4 (10)
.32	2.1 (10^{-2})	−5.5 (10)
.33	2.8 (10^{-2})	−5.6 (10)
.34	3.5 (10^{-2})	−4.8 (10)
.35	4.6 (10^{-2})	−4.5 (10)
.36	6.0 (10^{-2})	−4.1 (10)
.37	7.9 (10^{-2})	−3.8 (10)
.38	1.0 (10^{-1})	−3.4 (10)

Table 7. (Continued)

Water Content Θ	Hydraulic Conductivity, K (cm/hr)	Pressure Head h (cm)
.39	1.3 (10^{-1})	−3.112 (10)
.40	1.7 (10^{-1})	−2.731 (10)
.41	2.3 (10^{-1})	−2.413 (10)
.42	3.1 (10^{-1})	−2.096 (10)
.43	4.1 (10^{-1})	−1.715 (10)
.44	5.4 (10^{-1})	−1.335 (10)
.45	6.9 (10^{-1})	−1.016 (10)
.46	8.8 (10^{-1})	−6.985
.47	1.03	−3.175
.48	1.30	− .0000

**Source: King and Hanks (1973)

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