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# Existence and Multiplicity Results on Standing Wave Solutions of Some Coupled Nonlinear Schrodinger Equations

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EXISTENCE AND MULTIPLICITY RESULTS ON STANDING WAVE SOLUTIONS OF SOME COUPLED NONLINEAR SCHRÖDINGER EQUATIONS

by

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# ABSTRACT


Existence and Multiplicity Results on Standing Wave Solutions of

Some Coupled Nonlinear Schrödinger Equations

by

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In this dissertation, we study the standing wave solutions of some coupled nonlinear Schrödinger equations (CNLS). Thanks to their wide applications in physics, such as nonlinear optics and Bose-Einstein condensates, CNLS have been extensively studied by many authors in recent years. However, there are still many questions that remain unanswered. In this work, we mainly focus on the existence and multiplicity of positive standing wave solutions of a few CNLS.

First, we consider a fully symmetric coupled nonlinear elliptic system with $N$ equations

$$(P1)\qquad \begin{cases} -\Delta u_j + u_j = \mu u_j^3 + \beta u_j \displaystyle\sum_{k\geq 1, k\neq j}^{N} u_k^2 & \text{in } \Omega, \\ u_j > 0 & \text{in } \Omega, \\ u_j = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mu > 0, \beta < 0$ and $j = 1, \cdots, N$. The domain $\Omega \subset \mathbb{R}^n$ is either bounded with smooth boundaries ($n = 1, 2, 3$), or unbounded and radially symmetric ($n = 2, 3$). System (P1) is

invariant under the action of $N$-th order cyclic group, which is generated by $\sigma : [H_0^1(\Omega)]^N \to [H_0^1(\Omega)]^N$

$$\sigma(u_1, u_2, \ldots, u_N) = (u_2, \ldots, u_N, u_1).$$

By introducing a $Z_N$-index and applying a Lusternik-Schnirelmann type theory, we find multiple $Z_N$-orbit solutions of $(P1)$, and describe the dependency of the quantity of standing wave solutions on the coupling constant $\beta$. Also, we extend these results to systems with more general exponents $1 < 2p - 1 < 2^* - 1$, where $2^* = 2n/(n-2)$ if $n \geq 3$ and $2^* = \infty$ if $n = 1, 2$.

Second, we consider the following asymmetric elliptic system

$$(P2) \qquad \begin{cases} -\Delta u - au = \mu_1 u^3 + \beta u v^2 & \text{in } \Omega, \\ -\Delta v - av = \mu_2 v^3 + \beta v u^2 & \text{in } \Omega, \\ u, v > 0 \text{ in } \Omega, \quad u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mu_1, \mu_2 \in \mathbb{R}$ are constants; $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, $n = 1, 2, 3$. The parameter $a$ is greater than the principal eigenvalue of $(-\Delta, \Omega)$ with zero Dirichlet boundary conditions. In this case, system $(P2)$ is indefinite. In certain ranges of $\beta$, determined by $\mu_1$ and $\mu_2$, we found local bifurcations of $(P2)$ with respect to a positive solution branch that can be explicitly expressed. Moreover, if $\Omega$ is radially symmetric or the spatial dimension is $n = 1$, then these local bifurcations become global bifurcations. Most of these global bifurcation branches in the product space $\mathbb{R} \times H_0^1(\Omega) \times H_0^1(\Omega)$ can be proved to be unbounded in the negative direction of $\beta$. Furthermore, some nonexistence results of positive solutions are also established for constants $\mu_1, \mu_2$ and $\beta$ satisfying certain conditions.

(107 pages)

# PUBLIC ABSTRACT


Existence and Multiplicity Results on Standing Wave Solutions of
Some Coupled Nonlinear Schrödinger Equations

Coupled nonlinear Schrödinger equations (CNLS) govern many physical phenomena, such as nonlinear optics and Bose-Einstein condensates. For their wide applications, many studies have been carried out by physicists, mathematicians and engineers from different respects. In this dissertation, we focused on standing wave solutions, which are of particular interests for their relatively simple form and the important roles they play in studying other wave solutions. We studied the multiplicity of this type of solutions of CNLS via variational methods and bifurcation methods.

Variational methods are useful tools for studying differential equations and systems of differential equations that possess the so-called variational structure. For such an equation or system, a weak solution can be found through finding the critical point of a corresponding energy functional. If this equation or system is also invariant under a certain symmetric group, multiple solutions are often expected. In this work, an integer-valued function that measures symmetries of CNLS was used to determine critical values. Besides variational methods, bifurcation methods may also be used to find solutions of a differential equation or system, if some trivial solution branch exists and the system is degenerate somewhere on this branch. If local bifurcations exist, then new solutions can be found in a neighborhood of each bifurcation point. If global bifurcation branches exist, then there is a continuous solution branch emanating from each bifurcation point.

We consider two types of CNLS. First, for a fully symmetric system, we introduce a new index and use it to construct a sequence of critical energy levels. Using variational methods and the symmetric structure, we prove that there is at least one solution on each one of these critical energy levels. Second, we study the bifurcation phenomena of a two-equation asymmetric system. All these bifurcations take place with respect to a positive solution branch that is already known. The locations of the bifurcation points are determined through an equation of a coupling parameter. A few nonexistence results of positive solutions are also given.

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CHAPTER 1

INTRODUCTION

In this dissertation, we study the coupled nonlinear Schrödinger equations (CNLS),

$$\begin{cases} -i\dfrac{\partial}{\partial t}\Phi_j = \Delta\Phi_j - V_j(x)\Phi_j + \mu_j|\Phi_j|^{2p-2}\Phi_j + \displaystyle\sum_{k\neq j}\beta_{jk}|\Phi_k|^p|\Phi_j|^{p-2}\Phi_j, \\ \\ \qquad \Phi_j = \Phi_j(x,t) : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{C}, \end{cases} \tag{1.0.1}$$

where the $V_j$'s are external potentials; $\mu_j$'s and $\beta_{jk} = \beta_{kj}$'s are constants, $j,k = 1,2,\cdots,N$. The nonlinear exponent is sub-critical, i.e. $1 < p < 2^* = \frac{2n}{n-2}$ if $n \geq 3$ and $p < \infty$ if $n = 1,2$.

In this chapter, we shall give a brief introduction to problem (1.0.1), including some background in physics and some known results in mathematics. Next, we will list the main theories and methods that will be used in later chapters, including *variational methods and bifurcation methods.* At the end of the introduction, a short summary of our main results will be given. The new results documented in this dissertation are:

(1) Introduce a $Z_N$-index and use it to study multiple standing wave solutions of CNLS.

(2) Establish a couple of bifurcation results and multiplicity results for an asymmetric and indefinite CNLS.

(3) Obtain some nonexistence results of positive standing wave solutions of an indefinite, possibly asymmetric, CNLS.

## 1.1 Background

### 1.1.1 From the viewpoint of physics

CNLS (also called Gross-Pitaevskii equations) govern many physical phenomena, such as nonlinear optics and Bose-Einstein condensates.

In nonlinear optics, the solution component $\Phi_j$ represents the $j^{th}$ component of a light beam in Kerr-like photo-refractive media. The Kerr effect, also called the quadratic electro-optic effect (QEO effect), is a change in the refractive index of a material in response to an applied electric field. The constant $\mu_j$ is for the self-focusing or self-defocusing effect of the $j^{th}$ component of the beam (self-focusing if $\mu_j > 0$ and self-defocusing if $\mu_j < 0$). Self-focusing is a nonlinear optical process. A medium whose refractive index increases with the electric field intensity acts as a focusing lens for an electromagnetic wave characterized by an initial transverse intensity gradient. Similarly, the self-defocusing effect works as a defocusing lens. The coupling constant $\beta_{jk}$ is for the interaction between the $j^{th}$ and the $k^{th}$ component of the beam. In a physical experiment [33], two dimensional photo-refractive screening solutions and a two dimensional self-trapped beam have been observed.

Another important application of (1.0.1) is to describe a low-temperature state of matter, Bose-Einstein Condensate (BEC). A BEC is a rare state (or phase) of matter in which a large percentage of bosons collapse into their lowest quantum state, allowing quantum effects to be observed on a macroscopic scale. The bosons collapse into this state in circumstances of extremely low temperature, near temperature values of absolute zero. The first BEC was produced by E. Cornell and C. Wieman using rubidium atoms. In the simplest case $N = 2$, system (1.0.1) describes Bose-Einstein double condensate, i.e., a binary mixture in two different hyperfine states $|1\rangle$ and $|2\rangle$. Physically, $\Phi_1$ and $\Phi_2$ are the corresponding condensates amplitudes. $\mu_1$, $\mu_2$ are the intraspecies scattering lengths and $\beta_{12} = \beta_{21}$ is interspecies scattering length. The sign of the scattering length $\beta_{12} = \beta_{21}$ determines whether the interactions of states $|1\rangle$ and $|2\rangle$ is repulsive or attractive (c.f. [20]). Recently, Bose-Einstein condensation of the triplet states has been observed (c.f. [43]).

### 1.1.2 From the viewpoint of mathematics

In this subsection, we summarize some known results about CNLS (1.0.1). Assume that all the external potentials are identically equal to zero, i.e. $V_j(x) \equiv 0$, $j = 1, \cdots, N$.

For relatively simple forms and also a wide of applications, extensive and intensive research on (1.0.1) has been done towards two special types of solutions: solitary wave solutions and standing wave solutions. A typical solitary wave solution takes the form

$$\Phi_j(x,t) = \exp\left(i[(a_j - \frac{1}{4}|\mathbf{v}|^2)t + (\frac{1}{2}\mathbf{v}\cdot x + m_j)]\right) u_j(x - \mathbf{v}t). \tag{1.1.1}$$

where the $u_j$'s are real valued functions of $x - \mathbf{v}t \in \mathbb{R}^n$ and $\mathbf{v}$ represents the wave speed; $a_j$'s are related to phase speed and the $m_j$'s determine the initial phases.

Restricting our attention to solutions of this form, (1.0.1) becomes

$$-\Delta u_j + a_j u_j = \mu_j |u_j|^{2p-2} u_j + \sum_{k\neq j} \beta_{jk} |u_k|^p |u_j|^{p-2} u_j,\ j = 1, ..., N, \tag{1.1.2}$$

When the traveling speed $\mathbf{v}$ equals zero, the solitary wave solutions are referred to as standing wave solutions. Now we briefly introduce a few established results about system (1.1.2), and also some systems with different nonlinear terms or coupling terms. Some of these results are obtained on $\mathbb{R}^n$, and some are obtained on bounded domains.

System (1.1.2) has been studied by many mathematicians. According to the relation between $a_j$'s and $\Lambda_1$, the principal eigenvalue of $(-\Delta, \Omega)$ with zero Dirichlet boundary condition, we can distinguish two cases, the definite case and the indefinite case.

In the definite case, the operator on the left-hand side of the system induces a norm in the product space. As a result, we can obtain the boundedness of a Palais-Smale sequence, defined in section 1.3, from the boundedness of its energy functional values. When the spatial dimension $n = 1$, the system (1.1.2) is integrable for some special values of parameters. There are many analytical and numerical results on solitary wave solutions, see [12, 22, 23, 24, 26] and references therein. In recent years, the studies of (1.1.2) have been extended to higher spatial dimension cases. A couple of existence and non-existence results via variational methods can be found in [5, 6, 9, 27, 30, 31, 34, 44]. According to a result of W.C. Troy [52], if $\Omega$ is a ball or $\mathbb{R}^n$ and $\beta_{jk} > 0$, then the positive standing wave solutions are radial. In [29], T.C. Lin and J.C. Wei studied bounded state solutions with $Z_p$ symmetry. Moreover, different classes of non-radial solutions have also been constructed on $\mathbb{R}^n$, for $\beta < 0$ and $|\beta|$ small in [29]. With similar assumptions, but for $\beta \leq -1$, non-radial solutions were found by J.C. Wei [55]. When the phase speeds of the two components are the same constant, the multiplicity of standing wave solutions of (1.1.2) can also be studied by using bifurcation methods and index theory. In [10], using the bifurcation method, T. Bartsch, Z.-Q. Wang and J.C. Wei studied the positive bound state solutions of system (1.1.2) with $N = 2$ and parameter $b = (\lambda_1, \lambda_2, \mu_1, \mu_2, \beta) \in \mathbb{R}^5$ in $\mathbb{R}^n$. In [8], T. Bartsch, E. N. Dancer and Z.-Q. Wang gave descriptions for the bifurcation phenomena of (1.1.2) ($N = 2$) with respect to a positive solution branch, which had explicit representation in terms of the unique positive solution of $-\Delta\omega + \omega = \omega^3$ on $\mathbb{R}^n$ or a ball. If system (1.1.2) is fully symmetric, the multiplicity of positive solutions can be derived by using a Lusternick-Schnirelmann type theory. In [18], E.N. Dancer, J.C. Wei and T. Weth studied (1.1.2) for $N = 2$. They used the $Z_2$-index, also called genus, to define multiple Lusternick-Schnirelmann levels and $Z_2$-invariant deformation flow. Multiple solutions were found in each level set. Moreover, a recent paper by Y. Sato and Z.-Q. Wang studied the multiplicity of standing wave solutions of a two-system with symmetry $\sigma(u, v) = (-u, v)$. They found multiple solutions of a 2-system with one positive component and a sign-changing component.

In addition to existence and multiplicity, the geometric properties of solitary wave solutions have been studied by many authors. For a fixed $N$, as the interspecific competition goes to infinity, the wave amplitudes $U_i$'s segregate, that is, their supports tend to be disjoint. This phenomenon is called phase separation. See [25, 32, 51] for experimental and theoretical studies from physical point of view. For mathematical research work in this aspect, one can consult [11, 13, 14, 37, 47, 56, 57]. In dealing with singular perturbed problems, a type of highly concentrated solutions arises, whose graphs display narrow peaks or spikes. These solutions are also called point-condensation solutions. For related results, one can consult [17, 28] for attractive cases and repulsive cases, respectively.

In the indefinite case, i.e. $a_j < -\Lambda_1$ for some $j = 1, 2, \cdots, N$ in system (1.1.2), the corresponding energy functional loses compactness. Boundedness on the energy functional $\mathcal{E}$ is not enough to derive boundedness of a P.S. sequence. Thus the variational methods are hard to apply under the same framework as the definite case. In [36], B. Noris and M. Ramos study the indefinite system for the fully symmetric case of (1.1.2) and $N = 2$ in a smooth bounded domain,

$\Omega \subset \mathbb{R}^3$. In [46], A. Szulkin and T. Weth considered an indefinite scalar equation with periodic potential. They used a generalized Nehari manifold, which was introduced by A. Pankov [39], and found infinitely many ground state solutions (on a modified Nehari manifold). A few phase separation results mentioned above are also derived for indefinite systems.

Linearly coupled systems also have important applications in physics. For instance, the propagation of solitons in nonlinear fiber couplers is described by a two-coupled nonlinear Schrödinger equations with linear coupling terms (c.f. [2]). For more results, see [19] and references therein. In [4], A. Ambrosetti, G. Cerami and D. Ruiz considered the existence of positive ground states and bound states of a linearly coupled nonlinear Schrödinger system. In [3], A. Ambrosetti extends the above results to $n = 1$.

### 1.2 Preliminaries

To study the standing wave solutions of (1.0.1), we use the variational methods and bifurcation methods. They are both widely used in studying elliptic partial differential equations. In this section, we give a brief introduction to these two approaches. We need some notation.

**Notation** Denote a general Banach space by $X$ and its dual space by $X^*$. The standard norm of $L^p$ functions on $\Omega$ is given by $|u|_p = \big(\int_\Omega |u|^p dx\big)^{1/p}$, $1 < p < \infty$. Also, let $\|\cdot\|$ be the standard norm on the Hilbert space $H_0^1(\Omega)$ or $H^1(\mathbb{R}^n)$, i.e.

$$\|f\| = \sqrt{\int_\Omega (|\nabla f|^2 + f^2)dx} \qquad \text{or} \qquad \|f\| = \sqrt{\int_{\mathbb{R}^n} (|\nabla f|^2 + f^2)dx}.$$

The meaning of $\|\cdot\|$ will be determined from the context. Denote the product space

$$\mathcal{H} = \overbrace{E \times E \times \cdots \times E}^{N},$$

where $E = H_0^1(\Omega)$ or $H^1(\mathbb{R}^n)$ or $H_{0,r}^1(\Omega)$. Accordingly, a vector $\mathbf{u} \in \mathcal{H}$ is given by

$$\mathbf{u} = (u_1, \cdots, u_N),$$

and the norm in $\mathcal{H}$ is

$$\|\mathbf{u}\|_{\mathcal{H}} = \sum_{j=1}^{N} \|u_j\| \qquad \text{for any } \mathbf{u} \in \mathcal{H}.$$

### 1.2.1 Variational methods

Variational methods are developed for finding weak solutions of differential equations or systems of differential equations that possess the variational structure. With this structure, the existence of weak solutions can be transformed to the existence of critical points of a corresponding energy functional. The basic idea of variational methods is generalized from solving optimization problems using calculus. One of the earliest applications involves finding a minimal surface with fixed boundary conditions. See [45] for a detailed introduction and more examples. In the area of scientific computing, variational methods are used to develop Finite Element Methods, providing a powerful tool for finding approximate solutions of partial differential equations. In this dissertation, we follow the framework that has been developed since Ambrosetti and Rabinowitz's pioneer work [7].

Now we use a simple example to explain the application of variational methods. Consider the following boundary value problem,

$$-u'' = f(u) \text{ in } (0,1) \quad \text{ and } \; u(0) = u(1) = 0, \tag{1.2.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. A solution $u \in C^2((0,1)) \cap C([0,1])$ that satisfies (1.2.1) pointwise is called a strong solution. The energy functional associated with (1.2.1) is

$$\mathcal{E}(u) = \frac{1}{2}\int_0^1 |u'|^2 dx - \int_0^1 F(u) dx, \tag{1.2.2}$$

where $F(t) = \int_0^t f(s) ds$. A function $u \in H_0^1([0,1])$ is a weak solution of (1.2.1), if

$$\langle \mathcal{E}'(u), \varphi \rangle := \int_0^1 u'\varphi' dx - \int_0^1 f(u)\varphi = 0, \qquad \forall \varphi \in C_0^\infty((0,1)).$$

Thus $u$ is a critical point of $\mathcal{E}$, i.e.

$$\mathcal{E}'(u) = 0. \tag{1.2.3}$$

The corresponding value $c = \mathcal{E}(u)$ is called a *critical value* of $\mathcal{E}$. On the other hand, equation (1.2.1) is called the *Euler-Lagrange equation* of problem (1.2.3). The critical points of $\mathcal{E}$ are weak solutions of (1.2.1). There is also a close relation between strong solutions and weak solutions. Assume that $u$ is a strong solution of (1.2.1). Multiplying both sides of (1.2.1) by a test function $\varphi \in C_0^\infty((0,1))$ and integrating by parts, we see that any strong solution must be a weak solution. Conversely, when the non-homogeneous term and boundary satisfy some regularity conditions, the differentiability of weak solutions can be improved and weak solutions can become strong solutions. See $L^2$ estimates, $L^p$ estimates and Schauder estimates [21] for details. A large number of differential equations become solvable by taking weak solutions into consideration.

The extreme values (local or global extrema) of the functional $\mathcal{E} \in C^1(X, \mathbb{R})$ are natural candidates of critical values. For example, if $c := \inf_{u \in X} \mathcal{E}(u) > -\infty$, then *Ekeland's variational principle* asserts that there exists a minimizing sequence $\{u_n\}_1^\infty \subset X$ such that

$$\mathcal{E}(u_n) \to c \quad \text{and} \quad \mathcal{E}'(u_n) \to 0 \ \text{ in } X^*. \tag{1.2.4}$$

If $X$ is a finite dimensional space, then the sequence $\{u_n\}_1^\infty$ must have a weakly convergent subsequence, and the limiting point is a critical point of $\mathcal{E}$.

If $X$ is an infinite dimensional space, a minimizing sequence in $X$ may not converge. A sequence satisfying (1.2.4) but not converging to any point in $X$ can be constructed. Thus we need to introduce a compactness condition.

**(PS)** Assume (1.2.4). If there is a subsequence $\{u_{n_k}\} \subset \{u_n\}$ and $u_0 \in X$ such that $u_{n_k} \to u_0$ as $k \to \infty$, then we say $\mathcal{E}$ satisfies the *Palais-Smale condition* or (PS) condition in short. Here the sequence $\{u_n\}$ is called a PS sequence.

If $\mathcal{E}$ satisfies (1.2.4) and the (PS) condition, then the infimum of $\mathcal{E}$ in $X$ gives rise to a critical point $u_0$. The (PS) condition is the most commonly used compactness condition in these problems.

Besides local extrema, *saddle points* are another type of critical points. Saddle points correspond to critical values that are characterized as minimax values over a suitable class of sets. The first and simplest minimax result is the *Mountain Pass Theorem*, which was established by Ambrosetti and Rabinowitz [7].

**Mountain Pass Theorem** *Assume that $\mathcal{E} \in C^1(X, \mathbb{R})$ satisfies the (PS) condition. If $\mathcal{E}(0) = 0$ and*

$(I_1)$ *there are constants $\rho, \alpha > 0$ such that $\mathcal{E}|_{\partial B_\rho} \geq \alpha$, and*

$(I_2)$ *there is an $e \in X \backslash B_\rho$ such that $\mathcal{E}(e) \leq 0$,*

*then $\mathcal{E}$ possesses a critical value $c \geq \alpha$. Moreover $c$ can be characterized as*

$$c = \inf_{g \in \Gamma} \max_{u \in g([0,1])} \mathcal{E}(u),$$

*where $\Gamma = \{g \in C([0,1], X) | g(0) = 0, g(1) = e\}$.*

The above theorem is a special case of the following more general minimax theorem [41].

**Saddle Point Theorem** *Assume that $X$ has direct sum decomposition $X = Y \oplus Z$, where $Y$ is a finite dimensional subspace of $X$. Suppose $\mathcal{E} \in C^1(X, \mathbb{R})$, satisfies the (PS) condition, and*

$(I_3)$ *there exists a bounded neighborhood of $0$ in $Y$, denoted by $D$, a constant $\alpha$ such that $\mathcal{E}|_{\partial D} \leq \alpha$, and*

$(I_4)$ *there is a constant $\beta > \alpha$ such that $\mathcal{E}|_Z \geq \beta$.*

*Then $\mathcal{E}$ has a critical value $c \geq \beta$. Moreover $c$ can be characterized as*

$$c = \inf_{S \in \Gamma} \max_{u \in S} \mathcal{E}(u),$$

*where $\Gamma = \{S = h(\overline{D}) | h \in C(\overline{D}, X)$ and $h = id$ on $\partial D\}$.*

To use the minimax method, a proper *linking structures* in $X$, i.e. $S \cap Z \neq \emptyset$ for all $S \in \Gamma$, must be constructed. We refer to [42] for various of applications of the Saddle Point Theorem.

Variational methods are developed based on the variation of the *level sets* of the energy functional $\mathcal{E}$. To be precise, denote the level set of $\mathcal{E}$ below level $a$ by

$$\mathcal{M}^a = \{u \in X | \mathcal{E}(u) \leq a\}.$$

Assume that $\mathcal{E}$ satisfies the (PS) condition and $b$ is a constant less than $a$. If there is no critical point in $\mathcal{M}^a \backslash \mathcal{M}^b$, then there exists a nontrivial pseudo-gradient vector field for $\nabla\mathcal{E}$ on $\mathcal{M}^a \backslash \mathcal{M}^b$. The higher level set $\mathcal{M}^a$ can retract to the lower level set $\mathcal{M}^b$ along a descending flow induced by the nontrivial vector field. Otherwise there is at least one critical point can be found in $\mathcal{M}^a \backslash \mathcal{M}^b$. These geometric descriptions are proved with a few of deformation lemmas. One can find more detailed discussions in [42, 58].

For some equations or systems of equations (see [58] for assumptions on the non-homogeneous term), the critical value defined by a minimax argument in the entire space is equivalent to the minimum or maximum of the restricted functional on the so-called *Nehari manifold*

$$\mathcal{N} = \{u \in X \backslash \{\theta\} | \mathcal{E}'(u)u = 0\}.$$

This is an infinite dimensional manifold and homomorphic to the unit sphere in $X$. The definition of $\mathcal{N}$ may be modified in order to find solutions of a particular form.

When a system possesses some symmetries, multiple solutions are often expected and found by using a symmetric version of variational methods. The Lusternik-Schnirelmann type theory, LS theory for short, is one important method for studying these types of multiplicity problems. The key step of applying LS theory is constructing a sequence of critical values by using an index associated with the symmetry. Let $\mathcal{E} : X \to \mathbb{R}$ be the energy functional that is invariant under the action of a group $G$, that is $\mathcal{E}(g\mathbf{u}) = \mathcal{E}(\mathbf{u})$ for any $\mathbf{u} \in X$ and $g \in G$. Define the index of a $G$-invariant closed subset $A$ of $X$ to be the smallest dimension $m$, such that there exists a continuous map $h : A \to \mathbb{C}^m \backslash \{0\}$ satisfying $h(gu) = gh(u)$ for any $g \in G$. Now we can define the level sets of $\mathcal{E}$ that are characterized by indices associated with the symmetric group $G$, i.e.

$$c_k := \inf\{c \in \mathbb{R} | \gamma(\mathcal{E}^c) \geq k\}.$$

Thus we obtain a sequence of *LS levels*. If these LS levels are below all symmetric energy levels, then by using a symmetric deformation flow, we can show that each $c_k$ defines a critical value of $\mathcal{E}$, and its corresponding critical point will give rise to a solution. In this process, $G$-index theory plays an important role. In particular, the well-known genus is an index induced by $Z_2$.

### 1.2.2 Bifurcation methods

For a class of differential equations, certain special solutions (depending on parameters) are relatively easy to find, and these solutions form a solution branch. On the other hand, it is difficult to find other nontrivial solutions directly. In this case, bifurcation methods provide an alternate approach to tackle this problem. For example, $x \equiv 0$ is a trivial solution of the two algebraic equations shown in Figure 1.1a and Figure 1.1b. Besides the trivial branch $x \equiv 0$, nontrivial solution branches also exist and form bifurcation branches.

In Figure 1.1a, there are two unbounded solution branches $x = \pm\sqrt{\lambda}$, $\lambda \geq 0$, emanating from the trivial solution branch $x \equiv 0$. In Figure 1.1b, infinitely many bifurcation branches emanate from trivial solution branch $x \equiv 0$ at $\lambda = k\pi$ for all integer $k$. They are both examples of global bifurcations, and they also show two possibilities for global bifurcations: unbounded bifurcation branch and bifurcation branches containing multiple bifurcation points. In the general case, the definition of bifurcation point is the following:

**Bifurcation Point** *Let $\mathcal{F} : X \to Y$ be a continuous map, where $X$ and $Y$ are both Banach spaces. Assume that*

*(i) $\gamma = \{\mathbf{u}(t) | a \leq t \leq b\} \subset X$ is a solution curve of $\mathcal{F}$;*

*(ii) $\mathbf{u}_0 = \mathbf{u}(t_0) \in \gamma$ and there exists a point $\mathbf{v}_\epsilon \in B_\epsilon(\mathbf{u}_0)$ satisfying $\mathcal{F}(\mathbf{v}_\epsilon) = 0$ for any $\epsilon > 0$,*

*then $\mathbf{u}_0$ is called a bifurcation point of $\mathcal{F}$ with respect to $\gamma$.*

Consider an operator that takes the form $G(\lambda, \mathbf{u}) = \mathbf{u} - \lambda L\mathbf{u} + H(\mathbf{u}) = 0$, where $L : \mathcal{H} \to \mathcal{H}$ is a linear operator, and $H(\mathbf{u}) = o(\|\mathbf{u}\|)$. Let $r(L)$ be the set of $\lambda \in \mathbb{R}$ such that there exists $\mathbf{v} \in X\backslash\{\mathbf{0}\}$ with $\mathbf{v} = \lambda L\mathbf{v}$. Denote

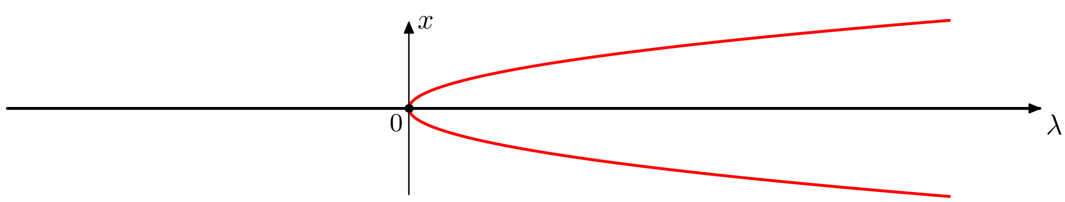
$$\mathcal{S} = \{(\lambda, \mathbf{u}) | G(\lambda, \mathbf{u}) = 0\} \subset \mathbb{R} \times X.$$

In order for $(\mu, 0)$ to be a bifurcation point, a necessary condition is $\mu \in r(L)$. If the equation $G(\lambda, \mathbf{u}) = 0$ possesses a variational structure, then $\mu \in r(L)$ is also a sufficient condition for $(\mu, 0)$ being a bifurcation point. More precisely, a local bifurcation theorem due to Rabinowitz [42] is stated as follows.

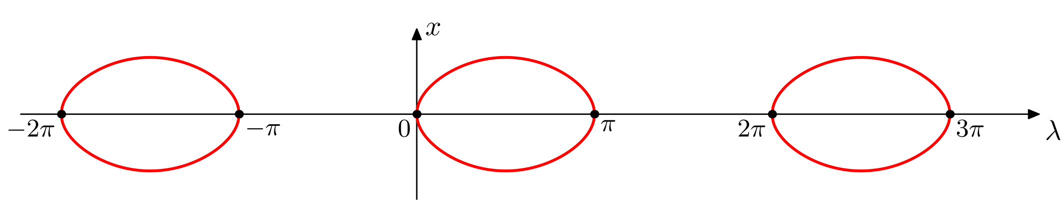
**Local Bifurcation** *Suppose $E$ is a real Hilbert space and $\mathcal{E} \in C^2(E, \mathbb{R})$ with $D\mathcal{E}'(\mathbf{u}) = L\mathbf{u} + H(\mathbf{u})$, where $L$ is a symmetric and compact operator. $H(\mathbf{u}) = o(\|\mathbf{u}\|)$ as $\mathbf{u} \to 0$. If $\mu \in \sigma(L)$ is an isolated eigenvalue of finite multiplicity, then $(\mu, 0)$ is a bifurcation point for*

$$G(\lambda, \mathbf{u}) = \mathbf{u} - \lambda L\mathbf{u} + H(\mathbf{u}) = 0.$$

*Moreover there is an $r_0 > 0$ such that*




(a) Bifurcation diagram of equation $x^3 - \lambda x = 0$




(b) Bifurcation diagram of equation $x^3 - x\sin\lambda = 0$

Figure 1.1: Two simple bifurcation diagrams

*(i) for each $r \in (0, r_0)$ there exist at least two distinct solutions $(\lambda_i(r), \mathbf{u}_i(r))$, $i = 1, 2$ of $G = 0$ having $\|\mathbf{u}_i\| = r$ and $|\lambda_i - \mu|$ small.*

*(ii) As $r \to 0$, $(\lambda_i(r), \mathbf{u}_i(r)) \to (\mu, 0)$.*

The following global bifurcation theorem is also due to Rabinowitz [40]. This theorem provides a sufficient condition for global bifurcation, and also gives descriptions for the two types of global bifurcation branches.

**Global Bifurcation** *If $\mu \in r(L)$ is of odd multiplicity, then $\gamma$ possesses a maximal subcontinuum $\gamma_\mu$ such that $(\mu, 0) \in \gamma_\mu$ and $\gamma_\mu$ either*

*(i) meets infinity in $\mathcal{S}$, or*

*(ii) meets $(\hat{\mu}, 0)$, where $\mu \neq \hat{\mu} \in r(L)$.*

See [40] for detailed proof and applications.

## 1.3 Summary of main results

We study the CNLS from two aspects:

(i) multiplicity of solutions of fully symmetric system with $N$ equations, and

(ii) bifurcations and multiple solutions of asymmetric systems with two equations.

In Chapter 2, we will consider a fully symmetric system. Denote the coefficient matrix on the right-hand side of (1.1.2) by

$$\Sigma = \begin{pmatrix} \mu_1 & \beta_{12} & \cdots & \beta_{1N} \\ \beta_{21} & \mu_2 & \cdots & \beta_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{N1} & \beta_{N2} & \cdots & \mu_N \end{pmatrix}, \tag{1.3.1}$$

and assume $\mu_j = \mu$, $\beta_{jk} = \beta$ for $j, k = 1, \cdots, N$. Moreover, assume $a_j = 1$ for $j = 1, \cdots, N$ (the essential requirement is $a_j \equiv a > \Lambda_1$, where $\Lambda_1$ is the principal eigenvalue of $(-\Delta, \Omega)$). Then the associated energy functional of $\mathcal{E} : \mathcal{H} \to \mathbb{R}$

$$\mathcal{E}(\mathbf{u}) = \frac{1}{2}\sum_{j=1}^{N}\int_\Omega (|\nabla u_j|^2 + |u_j|^2)dx - \frac{\mu}{4}\sum_{j=1}^{N}\int_\Omega |u_j|^4 dx - \frac{\beta}{2}\sum_{k\neq j}^{N}\int_\Omega |u_j|^2|u_k|^2 dx,$$

is invariant under the $N$-th order cyclic group $Z_N$. By virtue of this symmetry, we find a sequence of positive solutions of (1.1.2) when $\beta$ is in certain intervals.

**Theorem** *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain for $n = 1, 2, 3$ or an unbounded radially symmetric domain for $n = 2, 3$. Also assume $\mu_j = \mu$, $\beta_{jk} = \beta$, and $a_j = 1$ for $j, k = 1, \cdots, N$.*

*(a) If $\beta \leq -\dfrac{\mu}{N-1}$, then system* (1.1.2) *has an infinite sequence of $Z_N$-orbit solutions.*

*(b) For any positive integer $m$, there exists a $\beta_m \in (-\dfrac{\mu}{N-1}, 0)$, such that system* (1.1.2) *has at least $m$ $Z_N$-orbit solutions, if $\beta \in (-\dfrac{\mu}{N-1}, \beta_m)$.*

Chapter 2 consists of six sections. In section 2.1, we give an introduction of the system and present our main results. In section 2.2, we will prove several important properties of the corresponding variational problem. As a result, the existence of multiple solutions of (1.1.2) is equivalent to the existence of multiple critical points of the corresponding energy functional. In section 2.3, we introduce a new $Z_N$-index that corresponds to the $Z_N$-symmetry of the fully symmetric system. A $Z_N$-Borsuk-Ulam theorem by Wang [54] will be used to prove several properties of this index. In section 2.4, we construct a sequence of Lusternik-Schnirelmann levels, and prove the main theorem by using $Z_N$-index theory proved in section 2.3. In section 2.5, we generalize the multiplicity results to more general nonlinear terms. In the last section, in section 2.6, we will summarize the conclusions and give more comments.

In Chapter 3, we consider an asymmetric case of system (1.1.2) with two components. Let $a_1 = a_2 = -a$ in order to have a one-parameter trivial bifurcation branch $\mathcal{T}_\omega \subset \mathbb{R} \times H_0^1(\Omega) \times$

$H_0^1(\Omega)$. On the other hand, $\mu_1 \neq \mu_2$ is allowed; thus the two components of system (1.1.2) are not interchangeable. Denote the sequence of eigenvalues of

$$\begin{cases} -\Delta\phi = \Lambda\phi & \text{in } \Omega, \\ \quad \phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3.2}$$

by

$$0 < \Lambda_1 < \Lambda_2 \leq \cdots \leq \Lambda_m \leq \cdots . \tag{1.3.3}$$

We consider the indefinite case, i.e. $a$ is greater than $\Lambda_1$. Denote the positive non-degenerate solution of the scalar equation

$$\Delta\phi + \phi - \phi^3 = 0 \ \text{ in } \Omega, \qquad \text{and} \qquad \phi = 0 \text{ on } \partial\Omega,$$

by $\omega$. Then the local bifurcation parameter $\beta_k$ is determined by the $k$-th eigenvalue of eigenvalue problem

$$-\Delta\psi - \psi = \lambda_k \omega^2 \psi,$$

with zero Dirichlet boundary conditions. According to the values of $\mu_1$ and $\mu_2$, we study the bifurcation phenomena in four cases,

(a) Self-focusing case: $\mu_2 \geq \mu_1 > 0$;

(b) Self-defocusing case: $\mu_1 \leq \mu_2 < 0$;

(c) Mixed case with negative sum: $\mu_1 \leq -\mu_2 < 0 < \mu_2$;

(d) Mixed case with positive sum: $\mu_1 < 0 < -\mu_1 < \mu_2$.

The main results of this chapter are summarized as follows.

**Theorem** *There are infinitely many local bifurcations in the case (a), and finitely many local bifurcations in the rest of three cases. These bifurcations consist of positive standing wave solutions of* (1.1.2)*. If $n = 1$ or $\Omega$ is radially symmetric, then every local bifurcation parameter* *gives rise to a global bifurcation branch* $\mathcal{S}_k$ *with respect to* $\mathcal{T}_\omega$*. For any solution* $(u,v) \in \mathcal{S}_k$*, the weighted difference* $\sqrt{\mu_1 - \beta}u - \sqrt{\mu_2 - \beta}v$ *has precisely* $k-1$ *simple zeroes. Furthermore, the global bifurcation branches are unbounded in the negative direction of* $\beta$*, except possibly for finitely many of them in the case* $\mu_1 < 0 < -\mu_1 < \mu_2$*.*

Chapter 3 contains six sections. In section 3.1, we give a general introduction to the system and related results. A trivial solution branch will first be defined using the non-degenerate unique solution of a related scalar equation. Then definitions of local and global bifurcations will be given for system (1.1.2). According to the constants $\mu_j$, there are the four cases listed above. In section 3.2, we study the self-focusing case. A framework and some important auxiliary results will be established. In section 3.3, we study the local and global bifurcations for the self-defocusing case and the other two mixed cases will be discussed in section 3.4. In section 3.5, we summary the bifurcation results and make a few more comments. In the last section, we give a brief introduction to a problem that will be considered in the future.

# CHAPTER 2

# MULTIPLE POSITIVE SOLUTIONS OF A DEFINITE AND FULLY SYMMETRIC SYSTEM [1]

In this chapter, we will use the variational method and a Lusternik-Schnirelmann type theory to study the multiplicity of positive standing wave solutions of a fully symmetric CNLS.

## 2.1 Introduction

Assume $p = 2$, $a_j = 1$, $\mu_j = \mu > 0$ and $\beta_{jk} = \beta < 0$ for all $j, k = 1, \cdots, N$ in system (1.1.2), then the positive standing wave solutions satisfy

$$\begin{cases} -\Delta u_j + u_j = \mu u_j^3 + \beta u_j \sum_{k \neq j} u_k^2 & \text{in } \Omega, \\ u_j > 0 \text{ in } \Omega, u_j = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1.1}$$

Here the domain $\Omega \subset \mathbb{R}^n$ is bounded with smooth boundary if $n \leq 3$, or unbounded and radially symmetric if $n = 2, 3$. System (2.1.1) is fully symmetric, in the sense that if $\mathbf{u} = (u_1, \cdots, u_N) \in \mathcal{H}$ solves (2.1.1), then so does any permutation of $\mathbf{u}$. Let $G$ be a subgroup of the $N$-th order permutation group. We call the set $\{g\mathbf{u} | \mathbf{u} \text{ solves system (2.1.1)}, \ g \in G\}$ a $G$-orbit solution. Our goal is to find multiple solutions of (2.1.1) that possesses the symmetry associated with $N$-th order cyclic group.

In [18], E.N. Dancer, J.C. Wei and T. Weth studied system (2.1.1) for $N = 2$. They obtained infinitely many $Z_2$-orbit solutions when $\beta \leq -\mu$, and $m_\beta$ $Z_2$-orbit solutions when $\beta > -\mu$, where $m_\beta \to \infty$ as $\beta$ approaches $-\mu$ from the right-hand side. To take advantage of the symmetric structure of (2.1.1), the authors used a *Lusternik-Schnirelmann type theory* and the involution invariance of the associated Nehari manifold $\mathcal{M}$, i.e. for any $(u_1, u_2) \in \mathcal{M}$, there holds $\sigma(u_1, u_2) = (u_2, u_1) \in \mathcal{M}$. They estimated the minimum energy on the set of fixed points of $\sigma$ in $\mathcal{M}$, which was denoted by $c(\beta)$. It was shown that $c(\beta) \to \infty$ as $\beta$ approaches

[1] Coauthored by Rushun Tian and Zhi-Qiang Wang [48].

$-\mu$ from the right-hand side and there is no fixed point of $\sigma$ in $\mathcal{M}$ when $\beta \leq -\mu$. The later fact implies that no involution invariant solution orbit exists in $\mathcal{M}$ when $\beta \leq -\mu$. Next, the authors used the $Z_2$-index, or genus, to define a sequence of Lusternik-Schnirelmann (LS) type levels $c_k := \inf\{c \in \mathbb{R} | \gamma(\mathcal{M}^c) \geq k\}, k = 1, 2, \cdots$. With the estimate on $c(\beta)$, a sequence of $\sigma$-invariant subsets of $\mathcal{M}$ with increasing indices were constructed. Then an infinite sequence of LS levels $\{c_k\}_1^\infty$ were defined for $\beta \leq -\mu$, and a finite sequence $\{c_k\}_1^{m_\beta}$ were defined for $\beta > -\mu$, where $m_\beta \to \infty$ as $\beta \to (-\mu)^+$. By using the involution equivariant deformation flow, all $c_k$'s are proved to be critical values of the associated energy functional and each of them gives rise to at least one $Z_2$-orbit solution of (2.1.1).

It is natural to expect similar multiplicity results for system (2.1.1) with $N$ equations. In the generalization, two difficulties arise. The first one lies in choosing a group that compatible with the variational structure of the $N$-system. It is easy to see that system (2.1.1) is invariant under the the $N$-th order permutation group $S_N$, which seems to be a likely candidate for representing the symmetry. But it is actually hard to use a non-communicative group in the framework of variational methods. We find out that the $N$-th order cyclic group $Z_N$ can be employed. Let $\sigma : \mathcal{H} \to \mathcal{H}$ be the special permutation such that for any $\mathbf{u} = (u_1, u_2, \cdots, u_N) \in \mathcal{H}$,

$$\sigma(\mathbf{u}) = (u_2, u_3, \cdots, u_N, u_1). \tag{2.1.2}$$

Then $Z_N = \{Id, \sigma, \sigma^2, \cdots, \sigma^{N-1}\}$. Clearly, $Z_N$ is communicative. To use the invariant property of (2.1.1), we will introduce a $Z_N$-index and define a sequence of LS levels by using this index. The second difficulty is finding an upper bound for those LS levels. This upper bound must be independent of $\beta$. In [18], the authors constructed a sequence of finite dimensional spheres in $\mathcal{H}$ by using the positive parts and negative parts of some nonzero functions in $H_0^1(\Omega)$ as components. Because the positive part and negative part of the same function have separated supports, the restriction of energy functional on this sphere does not have the coupled terms, thus the influence of coupling coefficient $\beta$ was removed. If there are more than two equations, this construction does not work. We give a new and more general method to obtain $\beta$-independent upper bound of LS levels for arbitrary $N \geq 2$. Moreover, when $N$ is not prime, more iterative arguments are required for dealing with the nontrivial proper subgroups of $Z_N$.

Our main result is the following theorem:

**Theorem 2.1.1** *Assume that $\Omega$ is a bounded smooth domain ($n = 1, 2, 3$), or an unbounded and radially symmetric domain ($n = 2, 3$) in $\mathbb{R}^n$.*

*(a) If $\beta \leq -\frac{\mu}{N-1}$, then system* (2.1.1) *has an infinite sequence of $Z_N$-orbit solutions.*

*(b) For any positive integer $m$, there exists a constant $\beta_m \in (-\frac{\mu}{N-1}, 0)$, such that for any $\beta \in (-\frac{\mu}{N-1}, \beta_m)$, system* (2.1.1) *has at least $m$ $Z_N$-orbit solutions.*

**Remark 2.1.2** *Since each $S_N$-orbit contains at most $(N-1)!$ $Z_N$-orbit, the conclusions of Theorem 2.1.1 can also be stated for $S_N$-orbit solutions, i.e., for $\beta \leq -\frac{\mu}{N-1}$, there are infinitely many $S_N$-orbit solutions; and for any positive integer $m$, there exists a $\beta'_m \in (-\frac{\mu}{N-1}, 0)$, such that for $\beta \in (-\frac{\mu}{N-1}, \beta'_m)$, system (2.1.1) has at least $m$ $S_N$-orbit of solutions. Actually, we only need to choose $\beta'_m$ close enough to $-\frac{\mu}{N-1}$, such that there are at least $m(N-1)!$ $Z_N$-orbit solutions of (1.3).*

**Remark 2.1.3** *If $N$ is not a prime number, then we can get more information about the distribution of the solution orbits in terms of $\beta$. Actually, both conclusions of Theorem 2.1.1 require induction arguments in this case. The $Z_N$-orbit solution that consists of $N$ identical components can only be found on the right-hand side of $-\frac{\mu}{N-1}$. As $\beta$ moves left, the solutions of* (2.1.1) *tend to lose symmetry. But if a $Z_N$-orbit solution found on the left-hand side of $-\frac{\mu}{N-1}$ is invariant under the actions of some subgroup of $Z_N$, then its $Z_N$-index is infinity. In this case, we cannot claim the existence of infinitely many $Z_N$-orbit solutions directly. We must exclude all fixed points for every subgroup of $Z_N$ to confirm the existence of infinite many solution orbits.*

*The idea is reducing* (2.1.1) *to smaller systems in accordance with the subgroup symmetries. Roughly speaking, we study system* (2.1.1) *in a subspace of $\mathcal{H}$, which is the fixed space of a subgroup of $Z_N$. Then we obtain a smaller system since some equations in the $N$-system are identical now. Note that the number of equations in reduced system is the same as the number of* *elements in this subgroup. Considering solutions that are fixed by the subgroup with prime number elements, then for the reduced system we can use the conclusion of prime number equations. Repeating this procedure, we get a sequence of intervals. In each of these intervals, solution orbits with one subgroup symmetry disappear and some other type solution orbits become more and more as $\beta$ approaches the left endpoint. See the proof of Theorem 2.1.1 and Remark 2.4.8 for details.*

## 2.2 Variational structure

In this section, we will verify the variational structure of system (2.1.1), i.e. the equivalence of finding critical points of a modified energy functional and solving system (2.1.1). Moreover, we shall estimate the minima of the energy functional $\mathcal{E}$ invariant subsets of $Z_N$, and introduce the $Z_N$-symmetric deformation flow associated with $\mathcal{E}$.

Recall that $\mathcal{H}$ is a Hilbert space with inner product (notation given in Chapter 1)

$$(\mathbf{u}, \mathbf{v}) = \sum_{j=1}^{N} \int_{\Omega} \nabla u_j \nabla v_j + u_j v_j,$$

where $\mathbf{u} = (u_1, \cdots, u_N), \mathbf{v} = (v_1, \cdots, v_N) \in \mathcal{H}$. Denote $\mathbf{u}_j = (0, \cdots, u_j, \cdots, 0)$ with $u_j \in H_0^1(\Omega)$ or $H_{0,r}^1(\Omega)$. In order to get the positive solutions of (2.1.1), i.e. $u_j > 0$ for all $j = 1, \cdots, N$, we consider the modified system

$$\begin{cases} -\Delta u_j + u_j = \mu (u_j^+)^3 + \beta u_j \displaystyle\sum_{k \neq j}^{N} u_k^2, & \text{in } \Omega, \\ u_j = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2.1}$$

The energy functional of problem (2.2.1) is

$$\mathcal{E}(\mathbf{u}) = \frac{1}{2} \sum_{j=1}^{N} \|u_j\|^2 - \frac{\mu}{4} \int_{\Omega} \left( \sum_{j=1}^{N} |u_j^+|^4 \right) - \frac{\beta}{2} \int_{\Omega} \sum_{k \neq j}^{N} u_j^2 u_k^2. \tag{2.2.2}$$

By the Sobolev embedding theorems [1] and Proposition B.34 in [42] (see Appendix A. Lemma

III), $\mathcal{E}$ is a $C^2$ functional. We say a critical point $\mathbf{u}$ nontrivial, if $u_j \neq 0$ for all $j = 1, \cdots, N$.

**Lemma 2.2.1** *Every nontrivial critical point of $\mathcal{E}$ in $\mathcal{H}$ is a classical solution of* (2.1.1).

*Proof.* Let $\mathbf{u}$ be a nontrivial critical point of $\mathcal{E}$ in $\mathcal{H}$. Then for any test function $v_j \in H_0^1(\Omega)$, the inner product

$$(\nabla \mathcal{E}(\mathbf{u}), \mathbf{v}_j) = 0,$$

where $\mathbf{v}_j = (0, \cdots, 0, v_j, 0, \cdots, 0)$, and $j = 1, \cdots, N$. Thus $\mathbf{u}$ is a weak solution of system (2.2.1). Multiplying the $j$-th equation of (2.2.1) by $u_j^-$ and integrating over $\Omega$, we get

$$\int_\Omega |\nabla u_j^-|^2 + \int_\Omega (1 - \beta \sum_{k \neq j}^{N} u_k^2)|u_j^-|^2 = 0.$$

Since $\beta$ is negative, this equation implies that the $H^1$ norm of $u_J^-$ is zero. Therefore $u_j^- = 0$, or equivalently, $u_j \geq 0$ for $j = 1, \cdots, N$. By the standard elliptic regularity theory and the bootstrap arguments (see [21] for more details about implementing this procedure), each component $u_j$ of $\mathbf{u}$ is a $C^2$ function, thus $\mathbf{u}$ is a classical solution of (2.2.1). For the strict positivity, rewriting the $j$-th equation as

$$-\Delta u_j + (1 - \beta \sum_{k \neq j}^{N} u_k^2) u_j = \mu u_j^3 \geq 0,$$

and applying the Strong Maximum Principle, we must have $u_j \equiv 0$ or $u_j > 0$ for any $x \in \Omega$ $j = 1, \cdots, N$. The first possibility does not happen since $\mathbf{u}$ is nontrivial. So $\mathbf{u}$ is a classical solution of (2.1.1). ■

By Lemma 2.2.1, we can focus on finding nontrivial critical points of the functional $\mathcal{E}$. Consider the $N$-constraint Nehari manifold associated with system (2.2.1)

$$\mathcal{M} = \left\{ \mathbf{u} \in \mathcal{H} \middle| \nabla \mathcal{E}(\mathbf{u}) \mathbf{u}_j = 0, \text{ and } u_j \neq 0 \text{ for all } j = 1, \cdots, N \right\}. \tag{2.2.3}$$

Define functional $\mathcal{F} : \mathcal{H} \to \mathbb{R}^N$

$$\mathcal{F}(\mathbf{u}) = \begin{pmatrix} F_1(\mathbf{u}) \\ \vdots \\ F_N(\mathbf{u}) \end{pmatrix} = \begin{pmatrix} \|u_1\|^2 - \beta \int_\Omega u_1^2 \sum_{j\neq 1} u_j^2 - \mu \int_\Omega |u_1^+|^4 \\ \vdots \\ \|u_N\|^2 - \beta \int_\Omega u_N^2 \sum_{j\neq N} u_j^2 - \mu \int_\Omega |u_N^+|^4 \end{pmatrix}. \tag{2.2.4}$$

Then $\mathcal{M}$ can be expressed as

$$\mathcal{M} = \{\mathbf{u} \in \mathcal{H} : \mathcal{F}(\mathbf{u}) = (0, \cdots, 0), u_j \neq 0, j = 1, \cdots, N\}. \tag{2.2.5}$$

The following lemma shows the smoothness and non-degeneracy of this manifold, which ensure that the limiting point of a Palais-Smale sequence exists and stays on $\mathcal{M}$.

**Lemma 2.2.2** *Assume $\beta < 0$, then $\mathcal{M}$ is a $C^2$ manifold of co-dimension $N$.*

*Proof.* By Proposition B.34 [42](see Appendix A. Lemma III), $F_j : \mathcal{H} \to \mathbb{R}$ is a $C^2$ functional for any $j = 1, 2, \cdots, N$. Differentiating $\mathcal{F}$ at $\mathbf{u} = (u_1, \cdots, u_N) \in \mathcal{M}$, and applying the resulting operator to test functions $\mathbf{u}_j = (0, \cdots, u_j, \cdots, 0)$, $j = 1, \cdots, N$, one obtains

$$\nabla \mathcal{F}(\mathbf{u})(\mathbf{u}_1, \cdots, \mathbf{u}_N) = \begin{pmatrix} \partial_{u_1} F_1(\mathbf{u}) u_1 & \cdots & \partial_{u_1} F_N(\mathbf{u}) u_1 \\ \vdots & \ddots & \vdots \\ \partial_{u_N} F_1(\mathbf{u}) u_N & \cdots & \partial_{u_N} F_N(\mathbf{u}) u_N \end{pmatrix} =: T_{\mathbf{u}}, \tag{2.2.6}$$

where

$$\partial_{u_j} F_j(\mathbf{u}) u_j = 2\|u_j\|^2 - 2\beta \int_\Omega u_j^2 \Big( \sum_{k \neq j} u_k^2 \Big) - 4\mu \int_\Omega |u_j^+|^4 = -2\mu \int_\Omega |u_j^+|^4, \quad j = 1, \cdots, N,$$

$$\partial_{u_j} F_k(\mathbf{u}) u_j = -2\beta \int_\Omega u_j^2 u_k^2 = \partial_{u_k} F_j(\mathbf{u}) u_k, \qquad 1 \leq j \neq k \leq N.$$

Since $\mathbf{u} \in \mathcal{M}$, there holds $\|u_j\|^2 - \beta \int_\Omega u_j^2 \sum_{k\neq j} u_k^2 = \mu \int_\Omega |u_j^+|^4$ for each $j$. Due to the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$ and the fact $\beta < 0$, there exists $C > 0$ such that

$$\|u_j\|^2 \le \|u_j\|^2 - \beta \int_\Omega u_j^2 \sum_{k \ne j} u_k^2 = \mu \int_\Omega |u_j^+|^4 \le C \|u_j\|^4.$$

Thus $\|u_j\| \ge C^{-1/2} > 0$. With these estimates, we get

$$\mu \int_\Omega |u_j^+|^4 > -\beta \int_\Omega u_j^2 \sum_{k \ne j} u_k^2, \qquad j = 1, \cdots, N.$$

These inequalities imply that $T_{\mathbf{u}}$ is strictly diagonally dominant. Moreover, all the elements on the major diagonal of $T_{\mathbf{u}}$ are negative, thus $T_{\mathbf{u}}$ is negative definite and therefore non-degenerate. According to the Implicit Function Theorem and the fact that $F_j \in C^2$ for $j = 1, 2, \cdots, N$, the manifold $\mathcal{M}$ is $C^2$ smooth.

From the non-degeneracy of $T_{\mathbf{u}}$, the vectors $\nabla \mathcal{F}(\mathbf{u})\mathbf{u}_j$, $j = 1, \cdots, N$, are linearly independent. Thus $\mathcal{M}$ has codimension $N$. ■

The energy functional $\mathcal{E}$ is bounded from below on $\mathcal{M}$. Actually, for any $\mathbf{u} \in \mathcal{M}$,

$$\begin{aligned}
\mathcal{E}(\mathbf{u}) =& \frac{1}{2} \sum_{j=1}^N \|u_j\|^2 - \frac{\mu}{4} \int_\Omega \left( \sum_{j=1}^N |u_j^+|^4 \right) - \frac{\beta}{2} \int_\Omega \sum_{k \ne j} u_j^2 u_k^2 \\
=& \frac{1}{2} \sum_{j=1}^N \|u_j\|^2 - \frac{\mu}{4} \left( \sum_{j=1}^N \|u_j\|^2 - 2\beta \int_\Omega \sum_{k \ne j} u_j^2 u_k^2 \right) - \frac{\beta}{2} \int_\Omega \sum_{k \ne j} u_j^2 u_k^2 \\
=& \frac{1}{4} \sum_{j=1}^N \|u_j\|^2
\end{aligned} \tag{2.2.7}$$

by using (2.2.2), (2.2.4) and (2.2.5). This fact will be used to estimate the minimum of $H^1$ norm of $\mathbf{u} \in \mathcal{M}$ and minimum of energy functional on $\mathcal{M}$.

**Lemma 2.2.3** *Let $\mathcal{E}_{\mathcal{M}}$ be the restriction of $\mathcal{E}$ on $\mathcal{M}$.*

*(i) If $\mathbf{u}$ is a critical point of $\mathcal{E}_{\mathcal{M}}$, then $\mathbf{u}$ is a nontrivial critical point of $\mathcal{E}$.*

*(ii) $\mathcal{E}_{\mathcal{M}} : \mathcal{M} \to \mathbb{R}$ satisfies the Palais-Smale condition.*

*Proof* (i) Assume that $\mathbf{u}$ is a critical point of $\mathcal{E}_{\mathcal{M}}$, then there exist Lagrangian multipliers $\lambda_1, \lambda_2, \cdots, \lambda_N$ such that

$$\sum_{j=1}^{N} \lambda_j \nabla F_j(\mathbf{u}) = \nabla \mathcal{E}(\mathbf{u}).$$

To prove (i), it is sufficient to show $\lambda_j = 0$ for $j = 1, 2, \cdots, N$.

By definition (2.2.3), for any $\mathbf{u} \in \mathcal{M}$

$$\sum_{k=1}^{N} \lambda_k \nabla F_k(\mathbf{u})\mathbf{u}_j = (\nabla \mathcal{E}(\mathbf{u}), \mathbf{u}_j) = 0, \qquad j = 1, \cdots, N. \tag{2.2.8}$$

It is seen from (2.2.4) and (2.2.5) that

$$\partial_{u_j} F_j(\mathbf{u}) u_j = 2\|u_j\|^2 - 2\beta \int_\Omega u_j^2 \Big( \sum_{k \neq j} u_k^2 \Big) - 4\mu \int_\Omega |u_j^+|^4 = -2\mu \int_\Omega |u_j^+|^4, \quad j = 1, \cdots, N,$$

$$\partial_{u_j} F_k(\mathbf{u}) u_j = -2\beta \int_\Omega u_j^2 u_k^2 = \partial_{u_k} F_j(\mathbf{u}) u_k, \qquad 1 \le j \neq k \le N.$$

So (2.2.8) can be written as

$$T_{\mathbf{u}} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_N \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{2.2.9}$$

where

$$\begin{aligned} T_{\mathbf{u}} &= \begin{pmatrix} \partial_{u_1} F_1(\mathbf{u}) u_1 & \cdots & \partial_{u_1} F_N(\mathbf{u}) u_1 \\ \vdots & \ddots & \vdots \\ \partial_{u_N} F_1(\mathbf{u}) u_N & \cdots & \partial_{u_N} F_N(\mathbf{u}) u_N \end{pmatrix} \\ &= \begin{pmatrix} -2\mu \int_\Omega |u_1^+|^4 & \cdots & -2\beta \int_\Omega u_1^2 u_N^2 \\ \vdots & \ddots & \vdots \\ -2\beta \int_\Omega u_N^2 u_1^2 & \cdots & -2\mu \int_\Omega |u_N^+|^4 \end{pmatrix}. \end{aligned}$$

As it is shown in the proof of Lemma 2.2.2, $T_{\mathbf{u}}$ is non-degenerate on $\mathcal{M}$. Then we get $\lambda_1 = \cdots = \lambda_N = 0$ by solving the homogeneous linear system (2.2.9). Conclusion (i) is proved.

**(ii)** Let $\{\mathbf{u}^k\}_1^\infty = \{(u_1^k, \cdots, u_N^k)\}_1^\infty \subset \mathcal{M}$ be a Palais-Smale sequence of $\mathcal{E}_{\mathcal{M}}$. Then (2.2.7) implies that $\{\mathbf{u}^k\}_1^\infty$ is bounded in $\mathcal{H}$. Since bounded sequences in Hilbert space are weakly compact, there exists a subsequence of $\{\mathbf{u}^k\}_1^\infty$, for simplicity still denoted by $\{\mathbf{u}^k\}_1^\infty$, and $\mathbf{w} \in \mathcal{H}$, such that $\mathbf{u}^k \rightharpoonup \mathbf{w}$ as $k \to \infty$. By the compact embedding $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$ and Hölder inequality

$$|w_j^+|_4^4 = \lim_{k\to\infty} |(u_j^k)^+|_4^4 \geq \liminf_{k\to\infty} \|u_j^k\|^2 - \limsup_{k\to\infty} \beta \int_\Omega \sum_{l\neq j} (u_j^k)^2 (u_l^k)^2 > 0, \qquad j = 1, \cdots, N.$$

Thus $\mathbf{w}$ is nontrivial. For each $k \geq 1$, there exist Lagrange multipliers $\lambda_j^k$ such that

$$o(1) = \nabla \mathcal{E}_{\mathcal{M}}(\mathbf{u}^k) = \nabla \mathcal{E}(\mathbf{u}^k) - \sum_{j=1}^N \lambda_j^k \nabla F_j(\mathbf{u}^k), \quad \text{as } k \to \infty. \tag{2.2.10}$$

We want to show $\lambda_j^k \to 0$ as $k \to \infty$, for every $j = 1, \cdots, N$. Then (2.2.10) implies that $\mathbf{w}$ is a critical point of $\mathcal{E}$. Applying $\nabla \mathcal{E}_{\mathcal{M}}(\mathbf{u}^k)$ to $\mathbf{u}^k$ and using the boundedness of $\{\mathbf{u}^k\}_1^\infty$ in $\mathcal{H}$,

$$\begin{aligned}
o(1) &= \begin{pmatrix} \nabla \mathcal{E}(\mathbf{u}^k)\mathbf{u}_1^k - \left[\sum_{j=1}^N \lambda_j^k \nabla F_j(\mathbf{u})\right]\mathbf{u}_1^k \\ \vdots \\ \nabla \mathcal{E}(\mathbf{u}^k)\mathbf{u}_N^k - \left[\sum_{j=1}^N \lambda_j^k \nabla F_j(\mathbf{u})\right]\mathbf{u}_N^k \end{pmatrix} \\
&= -\begin{pmatrix} \left[\sum_{j=1}^N \lambda_j^k \nabla F_j(\mathbf{u})\right]\mathbf{u}_1^k \\ \vdots \\ \left[\sum_{j=1}^N \lambda_j^k \nabla F_j(\mathbf{u})\right]\mathbf{u}_N^k \end{pmatrix} \\
&= -\begin{pmatrix} \partial_{u_1^k} F_1 u_1^k & \cdots & \partial_{u_1^k} F_N u_1^k \\ \vdots & \ddots & \vdots \\ \partial_{u_N^k} F_1 u_N^k & \cdots & \partial_{u_N^k} F_N u_N^k \end{pmatrix} \begin{pmatrix} \lambda_1^k \\ \vdots \\ \lambda_N^k \end{pmatrix} \\
&= (-T_{\mathbf{w}} + o(1)) \begin{pmatrix} \lambda_1^k \\ \vdots \\ \lambda_N^k \end{pmatrix},
\end{aligned}$$

as $k \to \infty$.

By the weakly lower semi-continuity of $\|\cdot\|$,

$$\|w_j\|^2 - \beta \int_\Omega w_j^2 \sum_{l \neq j} w_l^2 \leq \mu \int_\Omega |w_j^+|^4, \qquad j = 1, \cdots, N.$$

Then by a similar proof for Lemma 2.2.2, $T_{\mathbf{w}}$ is negative definite. Therefore

$$\lambda_j^k \to 0, \text{ as } k \to \infty, \quad \text{for all } j = 1, \cdots, N.$$

For any $1 \leq j \leq N$, the functional $\nabla F_j(\mathbf{u}^k) \in \mathcal{H}^*$ is bounded, due to the boundedness of $\mathbf{u}^k$ and the continuity of $\nabla F_j$. Thus (2.2.10) implies $\nabla \mathcal{E}(\mathbf{u}^k) \to \mathbf{0}$ as $k \to \infty$. Now for any $\mathbf{v} \in \mathcal{H}$,

$$(\nabla \mathcal{E}(\mathbf{u}^k), \mathbf{v}) \to 0, \qquad \text{as} \quad k \to \infty.$$

On the other hand, we also have $\mathbf{u}^k \rightharpoonup \mathbf{w}$ as $k \to \infty$, so $(\nabla \mathcal{E}(\mathbf{w}), \mathbf{v}) = 0$ for any $\mathbf{v} \in \mathcal{H}$, i.e. $\mathbf{w}$ is a weak solution of (2.2.1). Multiplying the first equation by $w_1$ and integrating over $\Omega$, we get

$$\|w_1\|^2 = \mu |w_1^+|_4^4 + \beta \int_\Omega w_1^2 \sum_{j \neq 1} w_j^2 = \lim_{k \to \infty} \left( \mu |(u_1^k)^+|_4^4 + \beta \int_\Omega (u_1^k)^2 \sum_{j \neq 1} (u_j^k)^2 \right) = \lim_{k \to \infty} \|u_1^k\|^2.$$

The weak convergence and norm convergence indicate $u_1^k \to w_1$ strongly in $H_0^1(\Omega)$. Similarly, $u_j^k \to w_j$ as $k \to \infty$ in $H_0^1(\Omega)$ for $j = 2, \cdots, N$. So $\mathbf{u}^k \to \mathbf{w}$ as $k \to \infty$ in $\mathcal{H}$. Therefore $\mathcal{E}_{\mathcal{M}}$ satisfies the Palais-Smale condition. ■

Next, we consider the level sets of $\mathcal{E}$ on $\mathcal{M}$

$$\mathcal{M}^c = \{\mathbf{u} \in \mathcal{M} : \mathcal{E}(\mathbf{u}) \leq c\}, \qquad c \in \mathbb{R},$$

and the sets of critical points of $\mathcal{E}$ with critical value $c$

$$K_c = \{\mathbf{u} \in \mathcal{M} \mid \mathcal{E}(\mathbf{u}) = c, \nabla \mathcal{E}(\mathbf{u}) = 0\} = \{\mathbf{u} \in \mathcal{M} \mid \mathcal{E}_{\mathcal{M}}(\mathbf{u}) = c, \nabla \mathcal{E}_{\mathcal{M}}(\mathbf{u}) = 0\}.$$

It is easy to see that $\mathcal{M}, \mathcal{M}^c$ and $K_c$ are all invariant under the action of $S_N$. In particular, they are invariant under the action of $\sigma : \mathcal{H} \to \mathcal{H}$

$$\sigma(u_1, u_2, \cdots, u_N) = (u_2, \cdots, u_N, u_1).$$

Let

$$1 = q_0 < q_1 < q_2 < \cdots < q_a < N, \qquad \text{for some integer } a > 0$$

be all the distinct prime factors of $N$. Correspondingly, we have a sequence $N = N_0 > N_1 > N_2 > \cdots > N_a > 1$ defined as $N_b = N/q_b$ for $0 \le b \le a$. It is easy to see that

$$\begin{aligned}\sigma^{q_b}(u_1, \cdots, u_{q_b}, &u_{q_b+1}, \cdots, u_{2q_b}, \cdots\cdots, u_{(N-1)q_b+1}, \cdots, u_N) \\ &= (u_{q_b+1}, \cdots, u_{2q_b}, \cdots\cdots, u_{(N-1)q_b+1}, \cdots, u_N, u_1, \cdots, u_{q_b}).\end{aligned}$$

Denote the least energy on the sets of fixed points of $\sigma^{q_b}$ by

$$c^{q_b}(\beta) := \inf\{\mathcal{E}(\mathbf{u}) \bigm| \mathbf{u} \in \mathcal{M}, \sigma^{q_b}(\mathbf{u}) = \mathbf{u}\} \qquad b = 0, \cdots, a.$$

Set $c^{q_b}(\beta) = \infty$ if $\sigma^{q_b}$ has no fixed point on $\mathcal{M}$. The following lemma shows the dependence of $c^{q_b}(\beta)$ on $\beta$.

**Lemma 2.2.4** *$c^{q_b}(\beta) = \infty$ for $\beta \le -\frac{\mu}{N_b-1}$, and $\lim_{\beta \searrow -\frac{\mu}{N_b-1}} c^{q_b}(\beta) = \infty$ for $0 \le b \le a$.*

*Proof.* Assume $\beta \le -\frac{\mu}{N_b-1}$. If there is a fixed point $\mathbf{u}$ of $\sigma^{q_b}$ on $\mathcal{M}$, then $\mathbf{u}$ must take the form $\mathbf{u} = (u_1, \cdots, u_{N_b}, \cdots, u_1, \cdots, u_{N_b})$ and satisfy

$$\begin{aligned}\|u_j\|^2 &= (\mu + \beta(N_b - 1))|u_j^+|_4^4 + \beta(N_b - 1)|u_j^-|_4^4 + \beta N_b \int \sum_{k \ne j}^{q_b} u_k^2 u_j^2 \\ &\le (\mu + \beta(N_b - 1))|u_j^+|_4^4 + \beta(N_b - 1)|u_j^-|_4^4 \\ &\le (\mu + \beta(N_b - 1))|u_j^+|_4^4 \\ &\le 0.\end{aligned}$$

Thus $\|u_j\| = 0$ for all $j = 1, \cdots, q_b$, i.e. $\mathbf{u} = 0$. This is a contradiction, since $0 \notin \mathcal{M}$. So $\sigma^{q_b}$ has no fixed point on $\mathcal{M}$. By definition, $c^{q_b}(\beta) = \infty$.

Let $\mathbf{u} \in \mathcal{M}$ be a fixed point of $\sigma^{q_b}$ for any $0 \leq b \leq a$. If $-\frac{\mu}{N_b - 1} < \beta < 0$, then the first equation of $\mathcal{M}$ implies

$$\begin{aligned}
\|u_1\|^2 &= (\mu + \beta(N_b - 1))|u_1^+|_4^4 + \beta(N_b - 1)|u_1^-|_4^4 + \beta N_b \int_\Omega \sum_{k \neq 1}^{q_b} u_k^2 u_1^2 \\
&\leq (\mu + \beta(N_b - 1))|u_1^+|_4^4 + \beta(N_b - 1)|u_1^-|_4^4 \\
&\leq C(\mu + \beta(N_b - 1))\|u_1\|^4,
\end{aligned}$$

where $C > 0$ is the embedding constant of $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$. Since $C$ does not depend on $\beta$, we have $\|u_1\|^2 \geq \dfrac{\mu}{C(1 + (N_b - 1)\beta)}$. Then (2.2.7) implies

$$\mathcal{E}_{\mathcal{M}}(\mathbf{u}) \geq \|u_1\|^2 \geq \frac{1}{C(\mu + (N_b - 1)\beta)}.$$

It is easy to see that $\mathcal{E}_{\mathcal{M}}(\mathbf{u}) \to \infty$ as $\beta \searrow -\frac{\mu}{N_b - 1}$. Hence $c^{q_b}(\beta) \to \infty$ as $\beta \searrow -\frac{\mu}{N_b - 1}$. ■

In order to find critical points of $\mathcal{E}_{\mathcal{M}}$, we need the $\sigma^{q_b}$-equivariant deformation lemma, for any integer $0 \leq b \leq a$.

**Lemma 2.2.5** *Let $c \in \mathbb{R}$, and let $\mathcal{N} \subset \mathcal{M}$ be a relatively open and $\sigma^{q_b}$-invariant neighborhood of $K_c$, where $q_b$ is a factors of $N$. Then there exists $\epsilon > 0$ and a $C^1$-deformation $\eta : [0, 1] \times \mathcal{M}^{c+\epsilon} \backslash \mathcal{N} \to \mathcal{M}^{c+\epsilon}$ such that for any $\mathbf{u} \in \mathcal{M}^{c+\epsilon} \backslash \mathcal{N}$ and $t \in [0, 1]$,*

$$\eta(0, \mathbf{u}) = \mathbf{u}, \quad \eta(1, \mathbf{u}) \in \mathcal{M}^{c-\epsilon} \quad \textit{and} \quad \sigma^{q_b}[\eta(t, \mathbf{u})] = \eta(t, \sigma^{q_b}\mathbf{u}). \tag{2.2.11}$$

*Proof* Since $\mathcal{E}_{\mathcal{M}}$ satisfies the Palais-Smale condition, $K_c$ is relatively compact in $\mathcal{M}$. Note that $\mathcal{E}$ and $\mathcal{F}$ are $C^2$ functionals, then (2.2.10) implies that $\nabla \mathcal{E}_{\mathcal{M}}$ is $C^1$ smooth. There exists $\epsilon > 0$ and $\delta > 0$, such that

$$|\nabla \mathcal{E}_{\mathcal{M}}(\mathbf{u})| \geq \sqrt{\delta}, \qquad \text{for any } \mathbf{u} \in \mathcal{M}^{c+\epsilon} \backslash (\mathcal{M}^{c-\epsilon} \cup \mathcal{N}).$$

Consider the descending flow $\eta : [0,1] \times \mathcal{M}^{c+\epsilon}\backslash\mathcal{N} \to \mathcal{M}^{c+\epsilon}$ determined by the following initial value problem

$$\begin{cases} \dfrac{d\eta(t,\mathbf{u})}{dt} = -\dfrac{2\epsilon}{\delta}\nabla\mathcal{E}_{\mathcal{M}}(\eta(t,\mathbf{u})), \\ \eta(0,\mathbf{u}) = \mathbf{u}. \end{cases}$$

Claim: $\eta$ is a deformation satisfying all requirements of the lemma.

First, $\eta$ is a $C^1$ deformation since $\nabla\mathcal{E}_{\mathcal{M}}$ is $C^1$ vector field on $\mathcal{M}$. Next, if $\mathbf{u} \in \mathcal{M}^{c-\epsilon}$, then by the descending nature of the deformation flow, $\eta(1,\mathbf{u}) \in \mathcal{M}^{c-\epsilon}$. If $\mathbf{u} \in \mathcal{M}^{c+\epsilon}\backslash(\mathcal{M}^{c-\epsilon} \cup \mathcal{N})$, then

$$\mathcal{E}_{\mathcal{M}}(\eta(1,\mathbf{u})) = \int_0^1 -\frac{2\epsilon}{\delta}|\nabla\mathcal{E}_{\mathcal{M}}(\eta(t,\mathbf{u}))|^2 dt + \mathcal{E}_{\mathcal{M}}(\eta(0,\mathbf{u})) \leq -2\epsilon + \mathcal{E}_{\mathcal{M}}(\eta(0,\mathbf{u})) \leq c - \epsilon.$$

Using the fact that $\mathcal{E}_{\mathcal{M}}$ and $\nabla\mathcal{E}_{\mathcal{M}}$ are $\sigma^{q_b}$ invariant, we see that $\sigma^{q_b}\eta(t,\mathbf{u})$ and $\eta(t,\sigma^{q_b}\mathbf{u})$ satisfy the same Cauchy problem

$$\begin{cases} \dfrac{d\eta(t,\sigma^{q_b}\mathbf{u})}{dt} = -\dfrac{2\epsilon}{\delta}\nabla\mathcal{E}_{\mathcal{M}}(\eta(t,\sigma^{q_b}\mathbf{u})) = \sigma^{q_b}\dfrac{d\eta(t,\mathbf{u})}{dt}, \\ \eta(0,\sigma^{q_b}\mathbf{u}) = \sigma^{q_b}\mathbf{u} = \sigma^{q_b}\eta(0,\mathbf{u}). \end{cases}$$

Then by the uniqueness of Cauchy problem, we have $\sigma^{q_b}[\eta(t,\mathbf{u})] = \eta(t,\sigma^{q_b}\mathbf{u})$. Thus the claim holds and the lemma follows. ■

## 2.3 A $Z_N$-index

One of the most well-known index is perhaps the $Z_2$-index, or *genus*. This index is defined based on the Borsuk-Ulam Theorem, which asserts that any continuous odd map defined on a reflection-invariant domain with center at the origin must have a zero. To use the $Z_N$-symmetry, we also need to define a compatible index, which requires a simpler version of the $Z_N$-Borsuk-Ulam theorem by Z.-Q. Wang (Theorem 2, [53]).

**Proposition 2.3.1** *Let $T : \mathbb{C}^m \to \mathbb{C}^m$ be a $Z_N$-action given by*

$$T\mathbf{z} = T(z_1, \cdots, z_m) = (e^{i\frac{2\pi}{N}}z_1, \cdots, e^{i\frac{2\pi}{N}}z_m), \qquad \text{for any } \mathbf{z} = (z_1, \cdots, z_m) \in \mathbb{C}^m. \tag{2.3.1}$$

*Let $D \subset \mathbb{C}^m$ be a bounded open neighborhood of the origin $\theta$ and invariant under $T$. Assume that $f : \partial D \to \mathbb{C}^n$ is a continuous mapping satisfying*

$$f_j(T\mathbf{z}) = e^{i2\pi/N} f_j(\mathbf{z}), \qquad j = 1, \cdots, n. \tag{2.3.2}$$

*If $n < m$, then $\theta \in f(\partial D)$.*

Now we define an index based on the $Z_N$-symmetry.

**Definition 2.3.2** *Let $\sigma$ be the special permutation defined in* (2.1.2). *For any closed $\sigma$-invariant subset $A \subset \mathcal{M}$, define the $Z_N$-index $\gamma(A)$ to be the smallest $m \in \mathbb{N} \cup \{0\}$ such that there exists a continuous map $h : A \to \mathbb{C}^m \setminus \{0\}$ satisfying*

$$h(\sigma u) = Th(u), \tag{2.3.3}$$

*where $T$ is defined in* (2.3.1). *If there is no such a map, set $\gamma(A) = \infty$. Define $\gamma(\emptyset) = 0$.*

It is easy to see from the definition that if $A$ contains a fixed point of $\sigma^{q_b}$ for any $1 \le b \le a$, then $\gamma(A) = \infty$. Actually, assume $\mathbf{u} \in A$ satisfying $\sigma\mathbf{u} \neq \mathbf{u}, \sigma^{q_b}\mathbf{u} = \mathbf{u}$ (if $A$ contains fixed point of $\sigma$, then we already have $\gamma(A) = \infty$). Simple calculation shows

$$h(\mathbf{u}) = h(\sigma^{q_b}\mathbf{u}) = Th(T^{\sigma_b - 1}\mathbf{u}) = \cdots = T^{\sigma_b} h(\mathbf{u}).$$

Since $T^{\sigma_b} \neq Id : \mathbb{C}^m \to \mathbb{C}^m$, one has $h(\mathbf{u}) = 0$. This is a contradiction.

**Remark 2.3.3** *If $N$ is not a prime number, then we can define an index for each subgroup of $Z_N$. According to the decomposition of $N$, we denote the $Z_{N_b}$-index by $\gamma_{q_b}$. Then the $Z_{q_b}$-index of a $Z_{q_b}$-invariant set $A$ is the minimal dimension $m$ such that there exists a continuous map $A \to \mathbb{C}^m \setminus \{0\}$ satisfying*

$$h(\sigma^{q_b} u) = T^{q_b} h(u).$$

*On a closed $\sigma$-invariant subset $A \subset \mathcal{M}$, there are $a+1$ indices can be defined,*

$$\gamma_{q_0}(A),\ \gamma_{q_1}(A),\ \cdots,\ \gamma_{q_a}(A).$$

*If $q_d$ is a proper divisor of $q_b$ for some $0 \le d < b \le a$, then $\gamma_{q_d}(A) \ge \gamma_{q_b}(A)$ on any closed $\sigma^{q_d}$-invariant subset $A \subset \mathcal{M}$. Moreover, if $\gamma_{q_b}(A) > 0$, then $\gamma_{q_d}(A) = \infty$.*

The following lemma investigates important properties of the $Z_N$-index. For simplicity, we only consider $\sigma$-invariant sets and $Z_N$-index. Same results hold for $\sigma^{q_b}$-invariant sets and $Z_{q_b}$-index as well.

**Lemma 2.3.4** *Let $A, B \subset \mathcal{M}$ be closed and $\sigma$-invariant sets. Let $\gamma$ be the index that corresponds to group $Z_N$ and $T$ is given by (2.3.1).*

*(i) If $A \subset B$, then $\gamma(A) \le \gamma(B)$;*

*(ii) $\gamma(A \cup B) \le \gamma(A) + \gamma(B)$;*

*(iii) If $g : A \to \mathcal{M}$ is continuous and $\sigma$-equivariant, i.e. $g(\sigma(\mathbf{u})) = \sigma g(\mathbf{u})$ for all $\mathbf{u} \in A$ then*

$$\gamma(A) \le \gamma\overline{(g(A))}.$$

*Furthermore, if $A$ does not contain fixed point of $\sigma_{q_b}$ for any $0 \le b \le a$,*

*(iv) $\gamma(A) > 1$ implies that $A$ is an infinite set;*

*(v) if $A$ is compact, then $\gamma(A) < \infty$, and there exists a relatively open and $\sigma$-invariant neighborhood $\mathcal{N}$ of $A$ in $\mathcal{M}$ such that $\gamma(A) = \gamma(\overline{\mathcal{N}})$.*

*Finally,*

*(vi) if $S$ is the boundary of a bounded and $T$-invariant neighborhood of zero in a $m$-dimensional complex normed vector space and $\Psi : S \to \mathcal{M}$ is continuous map satisfying $\Psi(T\mathbf{u}) = \sigma(\Psi(\mathbf{u}))$, then $\gamma(\Psi(S)) \ge m$.*

*Proof* **(i)** Without loss of generality, assume $\gamma(B) = m < \infty$. By definition, there exists a continuous map $h : B \to \mathbb{C}^m \backslash \{0\}$ such that

$$h(\sigma(\mathbf{u})) = Th(\mathbf{u}) \qquad \text{for all } \ \mathbf{u} \in B.$$

The restriction of $h$ on $A$ is also a continuous map satisfying (2.3.3). Since $\gamma(A)$ is defined the minimal dimension such that (2.3.3) holds, then (i) follows from Definition 2.3.2.

**(ii)** Suppose $\gamma(A) = m_1$ and $\gamma(B) = m_2$. Then there exist continuous maps

$$\phi \in C(A, \mathbb{C}^{m_1} \backslash \{0\}), \qquad \text{and} \qquad \psi \in C(B, \mathbb{C}^{m_2} \backslash \{0\}),$$

both satisfying (2.3.3). By the Tietze Extension Theorem, there are continuous maps $\hat{\phi} \in C(\mathcal{H}, \mathbb{C}^{m_1})$ and $\hat{\psi} \in C(\mathcal{H}, \mathbb{C}^{m_2})$ such that $\hat{\phi}|_A = \phi$ and $\hat{\psi}|_B = \psi$. Replacing $\hat{\phi}, \hat{\psi}$ by

$$\frac{1}{N} \sum_{j=0}^{N-1} e^{-i\frac{2j\pi}{N}} \hat{\phi}(\sigma^j \mathbf{u}), \qquad \frac{1}{N} \sum_{j=0}^{N-1} e^{-i\frac{2j\pi}{N}} \hat{\psi}(\sigma^j \mathbf{u})$$

if it is necessary, we may assume that $\hat{\phi}, \hat{\psi}$ both satisfy (2.3.3). Set $\hat{h} = (\hat{\phi}, \hat{\psi})$, then

$$h := \hat{h}|_{A \cup B} \in C(A \cup B, \mathbb{C}^{m_1 + m_2} \backslash \{0\})$$

satisfies (2.3.3). According to Definition 2.3.2,

$$\gamma(A \cup B) \leq m_1 + m_2 = \gamma(A) + \gamma(B).$$

**(iii)** Without loss of generality, assume $\gamma\overline{(g(A))} = m < \infty$. By Definition 2.3.2, there exists a continuous map $\tilde{g} : \overline{(g(A))} \to \mathbb{C}^m \backslash \{0\}$, satisfying (2.3.3). Then the composite map $\tilde{g} \circ g : A \to \mathbb{C}^m \backslash \{0\}$ also satisfies (2.3.3) and is continuous. Therefore,

$$\gamma(A) \leq m = \gamma\overline{(g(A))}.$$

**(iv)** If $A \subset \mathcal{M}$ is a finite set, then there exists $m \in \mathbb{N}$ such that

$$A = \left\{ \mathbf{u}^1, \cdots, \mathbf{u}^m, \sigma\mathbf{u}^1, \cdots, \sigma\mathbf{u}^m, \cdots\cdots, \sigma^{N-1}\mathbf{u}^1, \cdots, \sigma^{N-1}\mathbf{u}^m \right\},$$

where each $\mathbf{u}^k \in \mathcal{M}$ $(k = 1, \cdots, m)$ is a $N$-vector. Since there is no fixed point of $\sigma_{q_b}$ for any $0 \le b \le a$, we can define map $h : A \to \mathbb{C}^1 \backslash \{0\}$ as

$$h(\sigma^j \mathbf{u}^k) = e^{i\frac{2(j+1)\pi}{N}}, \quad j = 0, \cdots, N-1, \; k = 1, \cdots, m.$$

It is easy to see that $h$ is continuous and satisfies (2.3.3), and therefore $\gamma(A) = 1$.

**(v)** Denote $\mathcal{B}_\rho(\mathbf{u}) = \{\mathbf{v} \in \mathcal{H} \mid \|\mathbf{v} - \mathbf{u}\|_{\mathcal{H}} \le \rho\}$. Since $\mathbf{0} \notin A$, there exists $\rho > 0$ such that $A \cap \mathcal{B}_\rho(\mathbf{0}) = \emptyset$. Because $A$ does not contain fixed point of $\sigma_{q_b}$ for any $0 \le b \le a$, there exists a cover of $A$

$$\left\{ \tilde{\mathcal{B}}_\rho(\mathbf{u}) = \bigcup_{j=0}^{N-1} \mathcal{B}_\rho(\sigma^j \mathbf{u}) \right\}_{\mathbf{u} \in A},$$

which does not contain any fixed point of $\sigma_{q_b}$ either. According to the compactness of $A$, this cover admits a finite sub-cover $\{\tilde{\mathcal{B}}_\rho(\mathbf{u}^1), \cdots, \tilde{\mathcal{B}}_\rho(\mathbf{u}^m)\}$. Moreover, by choosing $\rho > 0$ small enough, we may assume

$$\mathcal{B}_\rho(\sigma^k \mathbf{u}) \cap \mathcal{B}_\rho(\sigma^l \mathbf{u}) = \emptyset,$$

if $1 \le k \ne l \le m$. Let $\{\phi_k\}_1^m$ be a partition of unity on $A$ and subordinate to $\{\tilde{\mathcal{B}}_\rho(\mathbf{u}^k)\}_1^m$, i.e., $\phi_k \in C(A)$ with $\mathrm{supp}(\phi_k) \subset \tilde{\mathcal{B}}_\rho(\mathbf{u}^k)$, and $0 \le \phi_k \le 1, \sum_{k=1}^m \phi_k(\mathbf{u}) = 1$, for all $\mathbf{u} \in A$. Replacing $\phi_k$ by

$$\frac{1}{N} \sum_{j=0}^{N-1} \phi_k(\sigma^j \mathbf{u}), \quad 1 \le k \le m,$$

if it is necessary, we may assume that $\phi_k$ is $\sigma$-invariant. Then for each $k$, define $h_k : A \to \mathbb{C}$ as

$$h_k(\mathbf{u}) = \begin{cases} e^{i\frac{2j\pi}{N}} \phi_k(\mathbf{u}), & \text{if } \mathbf{u} \in \mathcal{B}_\rho(\sigma^j \mathbf{u}^k), \; j = 0, \cdots, N-1, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

It is easy to see that $h := (h_1, \cdots, h_m) : A \to \mathbb{C}^m \setminus \{\mathbf{0}\}$ is continuous and satisfies (2.3.3). By Definition 2.3.2, $\gamma(A) \leq m < \infty$.

Assume that $A$ is compact, $\mathbf{0} \notin A, \gamma(A) = m < \infty$, and $h \in C(A, \mathbb{C}^m \backslash \{\mathbf{0}\})$ is the corresponding map with property (2.3.3). By Tietze Extension Theorem, we may extend $h$ such that $h \in C(\mathcal{M}, \mathbb{C}^m)$. Since $A$ is compact, its image under continuous map $h$ is also compact. Then there exists a $T$-invariant open neighborhood $\widetilde{\mathcal{N}}$ of $h(A)$ that is compactly contained in $\mathbb{C}^m \backslash \{\mathbf{0}\}$. Define $\bar{\mathcal{N}} = h^{-1}(\overline{\widetilde{\mathcal{N}}})$. By construction, $\mathbf{0} \notin h(\bar{\mathcal{N}})$ and $\gamma(\bar{\mathcal{N}}) \leq m$. On the other hand, $\gamma(A) \leq \gamma(\bar{\mathcal{N}})$ holds by using (i). Hence $\gamma(A) = \gamma(\bar{\mathcal{N}})$.

**(vi)** If $\gamma(\Psi(S)) \leq m - 1$, then there exists a continuous map $h : \Psi(S) \to \mathbb{C}^{m-1} \backslash \{\mathbf{0}\}$ that satisfies (2.3.3). Hence the composition $h \circ \Psi : S \to \mathbb{C}^{m-1} \backslash \{0\}$ is continuous and satisfies (2.3.2). Since $m - 1 < m$, we get $0 \in h(\Psi(S))$ by using Proposition 2.3.1, but this contradicts with the definition of $h$. Therefore $\gamma(\Psi(S)) \geq m$. ■

## 2.4 Proof of the main theorem

In order to get $\beta$-independent estimates the $Z_N$-index, we need to construct a continuous map from a finite dimensional sphere to $\mathcal{M}$ that is compatible with the $Z_N$-symmetry. Precisely, we will define a finite dimensional space $\mathcal{C}^m$ that is isomorphic to $\mathbb{C}^m$. Denote the unit sphere in $\mathcal{C}^m$ by $\mathcal{S}^{2m-1}$. Then we need to construct a continuous map $\psi : \mathcal{S}^{2m-1} \to \mathcal{M}$ such that

$$\psi(e^{i\frac{2\pi}{N}}\mathbf{u}) = \sigma\psi(\mathbf{u}), \qquad \text{for all } \ \mathbf{u} \in \mathcal{S}^{2m-1}.$$

Proposition 2.4.1 gives the construction for bounded domains and Proposition 2.4.2 gives the construction for radially symmetric, possibly unbounded, domains.

**Proposition 2.4.1** *Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with $n \leq 3$. Then for any $m \geq 1$, there exists a $2m-1$ dimensional sphere $\mathcal{S}^{2m-1}$ and a continuous map $\psi : \mathcal{S}^{2m-1} \to \mathcal{M}$, such that*

$$\psi(e^{i\frac{2\pi}{N}}\mathbf{u}) = \sigma\psi(\mathbf{u}), \qquad \textit{for all } \ \mathbf{u} \in \mathcal{S}^{2m-1}.$$

*Proof* Consider the case $n = 2$ first. We use the polar coordinate system on $\Omega$. Without loss of generality, assume that the origin is an interior point of $\Omega$, then there exists $\rho_0 > 0$, such that

$$D := \{(\rho, t)|0 \le \rho < \rho_0, 0 \le t < 2\pi\} \subset \Omega.$$

Divide $D$ into $N$ parts $D_1, D_2, \cdots, D_N$, where

$$D_j = \left\{(\rho, t) \in D \middle| \frac{2\pi(j-1)}{N} \le t < \frac{2\pi j}{N}, 0 \le \rho < \rho_0\right\}, \qquad j = 1, 2, \cdots, N.$$

Let $0 < \rho_1 < \rho_2 < \cdots < \rho_m < \rho_{m+1} := \rho_0$ be a partition of $(0, \rho_0)$. Choose $m$ functions $U_1^1, \cdots, U_1^m \in H_0^1(\Omega)$, such that the support of $U_1^k$ is contained in $D_1 \cap \{(\rho, t)|\rho_k < \rho < \rho_{k+1}\}$, and $\|U_1^k\| = 1$. Define

$$U_j^k(\rho, t) = U_1^k(\rho, t - 2(j-1)\pi/N), \;\; \rho_k < \rho < \rho_{k+1}, \;\; \frac{2(j-1)\pi}{N} < t < \frac{2j\pi}{N}, j = 2, \cdots, N. \tag{2.4.1}$$

and let $\mathbf{u}^k(\rho, t) = \sum_{j=1}^N U_j^k(\rho, t), k = 1, \cdots, m$. According to the definition, $U_i^k U_j^l = 0$ if $i \ne j$ or $k \ne l$, $i, j = 1, \cdots, N$ and $k, l = 1, 2, \cdots, m$. Define $\mathcal{C}^m$ to be the space spanned by $\mathbf{u}^k$, i.e.

$$\mathcal{C}^m = \left\{\sum_{k=1}^m r_k e^{i\theta_k}\mathbf{u}^k \middle| r_k \in \mathbb{R}^+, \theta_k \in [0, 2\pi), \text{ and } \theta_k = 0 \text{ if } r_k = 0\right\}.$$

Clearly, $\mathcal{C}^m$ can be identified as a $2m$-dimensional linear subspace of $\mathcal{H}$, and the unit sphere in $\mathcal{C}^m$ can be represented as

$$\mathcal{S}^{2m-1} = \left\{\sum_{k=1}^m r_k e^{i\theta_k}\mathbf{u}^k \in \mathcal{C}^m \middle| \;\; \sum_{k=1}^m r_k^2 = 1\right\}. \tag{2.4.2}$$

Define map $\psi : \mathcal{S}^{2m-1} \to \mathcal{M}$

$$\psi\left(\sum_{k=1}^m r_k e^{i\theta_k}\mathbf{u}^k(\rho, t)\right) = \left(U_1^*(\rho, t), U_2^*(\rho, t), \cdots, U_N^*(\rho, t)\right), \tag{2.4.3}$$

where

$$U_j^*(\rho,t) = \frac{\sqrt{\mu}\|\sum_{k=1}^m r_k U_j^k(\rho,t-\theta_k)\|}{|\sum_{k=1}^m r_k U_j^k(\rho,t-\theta_k)|_4^2}\left|\sum_{k=1}^m r_k U_j^k\Big(\rho,t-\theta_k\Big)\right|, \qquad j=1,2,\cdots,N.$$

Then it is easy to see that $\psi$ is continuous (see Appendix A. Lemma I), $U_j^* \neq 0$ for $j=1,\cdots,N$, $U_i^*U_j^*=0$ for $i\neq j$, and

$$\begin{aligned}\psi\Bigg(e^{i\frac{2\pi}{N}}\sum_{k=1}^m r_k e^{i\theta_k}\mathbf{u}^k(\rho,t)\Bigg) &= \psi\Bigg(\sum_{k=1}^m r_k e^{i(\theta_k+\frac{2\pi}{N})}\mathbf{u}^k(\rho,t)\Bigg)\\ &= \Big(U_2^*(\rho,t),U_3^*(\rho,t),\cdots,U_N^*(\rho,t),U_1^*(\rho,t)\Big)\\ &= \sigma\psi\Bigg(\sum_{k=1}^m r_k e^{i\theta_k}\mathbf{u}^k(\rho,t)\Bigg).\end{aligned}$$

If $n=3$, we choose the cylindrical coordinates and define

$$D := \{(\rho,t,h)|0\le\rho<\rho_0, 0\le t<2\pi, |h|\le h_0\}\subset\Omega \quad \text{for some } h_0>0.$$

Then divide $D$ into $N$ parts along $t$ direction in the same manner as we did for $n=2$, and consider the functions of the form

$$\tilde{U}_j^k(\rho,t,h) = \phi^k(h)U_j^k(\rho,t),$$

where $\phi^k(h)$ is continuous function with $\text{supp}\phi\subset(-h_0,h_0)$, and $U_j^k$ is defined in (2.4.1). For $n=1$, we divide a subinterval of $\Omega$ into $N$ equal parts, i.e.

$$D=\bigcup_{j=1}^N D_j\subset\Omega \qquad \text{where } D_j=\left\{x+\frac{(j-1)L}{N}\,\middle|\, x\in D_1\right\} \text{ and } \ L=\text{length of } D.$$

Choose $m$ independent functions that have separated supports in $D_1$, and denote these functions by $U_1^k(x)$ for $k=1,\cdots,m$. For each fixed $k$, define $U_j^k(x)=U_1^k(x+(j-1)L/N)$. Then (2.4.2), (2.4.3) will give the corresponding construction of $\mathcal{S}^{2m-1}$ and $\psi$ for the case $n=1$ and 3. ■

**Proposition 2.4.2** *Let $\Omega$ be a radially symmetric domain in $\mathbb{R}^n, n = 2, 3$. Then for any $m \geq 1$, there exists a $2m-1$ dimensional sphere $\mathcal{S}^{2m-1}$ and a continuous map $\psi : \mathcal{S}^{2m-1} \to \mathcal{M}$, such that*

$$\psi(e^{i\frac{2\pi}{N}}\mathbf{u}) = \sigma\psi(\mathbf{u}), \qquad \textit{for all} \;\; \mathbf{u} \in \mathcal{S}^{2m-1}.$$

*Proof* Divide $\Omega$ into $N$ radially symmetric open subsets: $\Omega_j, j = 1, \cdots, N$ such that

$$\Omega = \bigcup_{j=1}^{N} \Omega_j, \quad \Omega_j \cap \Omega_k = \emptyset, \text{ if } j \neq k.$$

Let $\mathbb{S}^1$ be the unit circle and denote $\mathcal{O} = \mathbb{S}^1 \times \Omega$.

Let $r = |x|$ and consider functions in the form $U_j(t, r) = U_j(t, |x|)$. Fix $m \geq 1$. Choose $m$ functions $U^k(t, r) = \sum_{j=1}^{N} U_j^k(t, r), k = 1, \cdots, m$, such that for each $k$

(a) $U_j^k \in C^1(\mathcal{O})$;

(b) $\text{supp}U_j^k(\cdot, \cdot) \subset \mathbb{S}^1 \times \Omega_j$;

(c) $\text{supp}U_j^k(t + 2i\pi/N, \cdot) \cap \text{supp}U_j^k(t + 2l\pi/N, \cdot) = \emptyset$, for all $t$ and $1 \leq i \neq l \leq N$;

(d) for any $t$, $\sum_{j=1}^{N} |U_j^k(t, \cdot)|_4 \neq 0$.

Moreover, assume $\text{supp}U^k(\cdot, \cdot) \cap \text{supp}U^l(\cdot, \cdot) = \emptyset$ if $k \neq l$ (which can be accomplished by slicing each $\Omega_j$ into $m$ annuli), and

$$\|U^k(t, \cdot)\| = 1, \qquad \text{for any } t \in [0, 2\pi).$$

Then the following space can be identified as an $m$-dimensional complex space

$$\mathcal{C}^m = \left\{ \sum_{k=1}^{m} d_k e^{i\theta_k} U^k \middle| d_k \in \mathbb{R}^+, \theta_k \in [0, 2\pi), \text{ and } \theta_k = 0 \text{ if } d_k = 0, k = 1, \cdots, m \right\},$$

and the unit sphere in $\mathcal{C}^m$ is

$$\mathcal{S}^{2m-1} = \left\{ \sum_{k=1}^{m} d_k e^{i\theta_k} U^k \in \mathcal{C}^m \middle| \sum_{k=1}^{m} d_k^2 = 1 \right\}. \tag{2.4.4}$$

For any vector $Y = \sum_{k=1}^m d_k e^{i\theta_k} U^k$ in $\mathcal{S}^{2m-1}$, let $V(Y) : \mathcal{O} \to \mathbb{R}$ be

$$V(Y)(t,r) = \frac{\sqrt{\mu}\| \sum_{k=1}^m d_k \mathbf{u}^k(t-\theta_k, r)\|}{| \sum_{k=1}^m d_k \mathbf{u}^k(t-\theta_k, r)|_4^2} \Big| \sum_{k=1}^m d_k \mathbf{u}^k(t-\theta_k, r) \Big|.$$

Now define map $\psi : \mathcal{S}^{2m-1} \to \mathcal{M}$

$$\psi(Y) = \psi\Big( \sum_{k=1}^m d_k e^{i\theta_k} \mathbf{u}^k \Big) = \Big( V(Y)(t_1^*, \cdot), V(Y)(t_2^*, \cdot), \cdots, V(Y)(t_N^*, \cdot) \Big), \tag{2.4.5}$$

where for $j = 1, 2, \cdots, N$, $t_j^* = 2(j-1)\pi/N$. The construction of $V$ shows

$$V(Y)(t_i^*, \cdot)V(Y)(t_j^*, \cdot) = 0,$$

for $i \neq j$. Then $\psi$ is continuous (see Appendix A. Lemma I) and

$$\begin{aligned}
\psi\Big(e^{i\frac{2\pi}{N}} Y\Big) &= \psi\Big(e^{i\frac{2\pi}{N}} \sum_{k=1}^m d_k e^{i\theta_k} \mathbf{u}^k\Big) \\
&= \psi\Big(\sum_{k=1}^m d_k e^{i(\theta_k + \frac{2\pi}{N})} \mathbf{u}^k\Big) \\
&= \Big(V(e^{i\frac{2\pi}{N}} Y)(t_1^*, \cdot), V(e^{i\frac{2\pi}{N}} Y)(t_2^*, \cdot), \cdots, V(e^{i\frac{2\pi}{N}} Y)(t_N^*, \cdot)\Big) \\
&= \Big(V(Y)(t_1^* + \frac{2\pi}{N}, \cdot), V(Y)(t_2^* + \frac{2\pi}{N}, \cdot), \cdots, V(Y)(t_N^* + \frac{2\pi}{N}, \cdot)\Big) \\
&= \Big(V(Y)(t_2^*, \cdot), \cdots, V(Y)(t_N^*, \cdot), V(Y)(t_1^*, \cdot)\Big) \\
&= \sigma\psi\Big(\sum_{k=1}^m d_k e^{i\theta_k} \mathbf{u}^k\Big) = \sigma\psi(Y).
\end{aligned}$$

■

**Remark 2.4.3** *The construction of finite dimensional sphere $\mathcal{S}^{2m-1}$ and continuous map $\psi$ in Proposition 2.4.2 also works for bounded domains if $n = 2$ or $3$. Actually, we only need to select a ball that is entirely contained in $\Omega$, and consider this ball as the radial domain in Proposition 2.4.2.*

Next, we construct multiple critical points of $\mathcal{E}_{\mathcal{M}}$ by using the $Z_N$-index.

**Definition 2.4.4** *Define the Lusternik-Schnirelmann type level on $\mathcal{M}$ with $Z_{q_b}$-symmetry as*

$$c_k^{q_b} := \inf\{c \in \mathbb{R} | \gamma_{q_b}(\mathcal{M}^c) \geq k\},$$

*where $k = 1, 2, \cdots$ and $0 \leq b \leq a$.*

**Remark 2.4.5** *There is a sequence of Lusternik-Schnirelmann (LS) levels $\{c_k^{q_b}\}$ for each subgroup $Z_{N_b} \subset Z_N$. The following Lemmas are stated for group $Z_N$, but they can be easily modified for its subgroups. On the other hand, according to Remark 2.3.3, if $q_d | q_b$ then $c_k^{q_b} \geq c_k^{q_d}$ for any $k$. There is generally no comparison between $c_k^{q_b}$ and $c_k^{q_d}$ if $(q_b, q_d) = 1$, where $(m, n)$ denotes the largest common divisor of $m$ and $n$.*

We need some estimates on the index $\gamma$ of critical level sets. Similar as the previous section, the conclusions and proofs are given for the group $Z_N$ for simplicity. This assumption surely makes no difference if $N$ is a prime number.

**Lemma 2.4.6** *For $c < \min_{0 \leq b \leq a}\{c^{q_b}(\beta)\}$, the $Z_N$-index of $K_c$ is finite, i.e. $\gamma(K_c) < \infty$. And there exists $\epsilon > 0$ such that*

$$\gamma(\mathcal{M}^{c+\epsilon}) \leq \gamma(\mathcal{M}^{c-\epsilon}) + \gamma(K_c).$$

*Proof.* Since $\mathcal{E}_{\mathcal{M}}$ satisfies the Palais-Smale condition, the set $K_c$ is compact. By the definition of $c(\beta)$ and the assumption $c < \min_{0 \leq b \leq a}\{c^{q_b}(\beta)\}$, there is no fixed point of $\sigma^{q_b}$ in $K_c$ for $0 \leq b \leq a$. By Lemma 2.3.4 (v), $\gamma(K_c) < \infty$ and there exists a relatively open $\sigma$-invariant neighborhood $\mathcal{N}$ of $K_c$ such that $\gamma(\overline{\mathcal{N}}) = \gamma(K_c)$.

For $\epsilon > 0$ small, let $\eta : [0, 1] \times \mathcal{M}^{c+\epsilon} \to \mathcal{M}^{c+\epsilon}$ be the $C^1$-deformation given by Lemma 2.2.5. Then $\eta(1, \cdot)$ is a continuous and $\sigma$-equivariant map from $\mathcal{M}^{c+\epsilon} \backslash \mathcal{N}$ to $\mathcal{M}^{c-\epsilon}$. Using Lemma 2.3.4 (iii), we have $\gamma(\mathcal{M}^{c+\epsilon} \backslash \mathcal{N}) \leq \gamma(\mathcal{M}^{c-\epsilon})$, and therefore

$$\gamma(\mathcal{M}^{c+\epsilon}) \leq \gamma(\mathcal{M}^{c+\epsilon} \backslash \mathcal{N}) + \gamma(\overline{\mathcal{N}}) \leq \gamma(\mathcal{M}^{c-\epsilon}) + \gamma(K_c).$$

The proof is complete. ∎

**Lemma 2.4.7** *(i) For every $m$, $c_m < \infty$ is bounded independently of $\beta < 0$.*

*(ii)* $c_m \to c^*$ *as* $m \to \infty$, *where* $\min_{0\le b\le a}\{c^{q_b}(\beta)\} \le c^* \le \infty$.

*(iii) If* $c := c_m = c_{m+1} = \cdots = c_l < \min_{0\le b\le a}\{c^{q_b}(\beta)\}$ *for some* $l \ge m$, *then* $\gamma(K_c) \ge l - m + 1$.

*(iv) If* $c_m < \min_{0\le b\le a}\{c^{q_b}(\beta)\}$, *then* $K_{c_m} \neq \emptyset$, *and* $\mathcal{M}^{c_m}$ *contains at least* $m$ $Z_N$*-orbit critical points of* $\mathcal{E}_{\mathcal{M}}$.

*Proof.* **(i)** By Proposition 2.4.1, or 2.4.2 if $\Omega$ is radial, there exists continuous map $\psi : \mathcal{S}^{2m-1} \to \mathcal{M}$ satisfying (2.3.3), i.e.

$$\psi\Bigg(e^{i\frac{2\pi}{N}}\sum_{k=1}^{m} r_k e^{i\theta_k}\mathbf{u}^k(\rho,t)\Bigg) = \sigma\psi\Bigg(\sum_{k=1}^{m} r_k e^{i\theta_k}\mathbf{u}^k(\rho,t)\Bigg).$$

Then Lemma 2.3.4 (vi) implies $\gamma(\psi(\mathcal{S}^{2m-1})) \ge m$, and therefore $c_m \le \sup_{\mathbf{u}\in\mathcal{S}^{2m-1}} \mathcal{E}(\psi(\mathbf{u})) < \infty$. By the definition of $\psi$ and the construction of $\mathcal{S}^{2m-1}$, the value of $\sup_{\mathbf{u}\in\mathcal{S}^{2m-1}} \mathcal{E}(\psi(\mathbf{u}))$ does not depend on $\beta$. Hence (i) follows.

**(ii)** If the conclusion is not true, it must hold that $c^* < \min_{0\le b\le a}\{c^{q_b}(\beta)\}$ such that $c_m \to c^*$ as $m \to \infty$, since $\{c_m\}$ is a monotone increasing sequence. With similar argument as the proof of Lemma 2.4.6, there exists $\epsilon > 0$ corresponding to $c^*$, such that

$$\gamma(\mathcal{M}^{c^*+\epsilon}) \le \gamma(\mathcal{M}^{c^*-\epsilon}) + \gamma(K_{c^*}).$$

Choosing $m$ large such that $c_m > c^* - \epsilon$, then the above inequality and Lemma 2.4.6 imply $\gamma(\mathcal{M}^{c^*+\epsilon}) < \infty$. Now we choose $m' > \gamma(\mathcal{M}^{c^*+\epsilon})$ and then the corresponding $c_{m'} \ge c^* + \epsilon$, which is a contradiction. Thus $\min_{0\le b\le a}\{c^{q_b}(\beta)\} \le c^* \le \infty$.

**(iii)** By Definition 2.3.2,

$$\gamma(\mathcal{M}^{c-\epsilon}) \le m - 1 \;\text{ and }\; \gamma(\mathcal{M}^{c+\epsilon}) \ge l \;\text{ for all }\; \epsilon > 0.$$

Then $\gamma(K_c) \geq l - m + 1$ follows from Lemma 2.4.6.

**(iv)** If $c_m < \min_{0 \leq b \leq a}\{c^{q_b}(\beta)\}$, then we get $\gamma(K_{c_m}) \geq 1$ by choosing $l = m$ in (iii). Hence $K_{c_m}$ is not empty. If $c_1 < \cdots < c_m$, then $\mathcal{M}^{c_m}$ contains at least $m$ $Z_N$-orbits of critical points of $\mathcal{E}$. If $c_i = c_j$ for some $i < j \leq m$, then $\gamma(K_{c_i}) > 1$. By Lemma 2.3.4 (iv), $K_{c_i}$ is an infinite set. Hence in either case we have at least $m$ $Z_N$-orbit critical points of $\mathcal{E}_{\mathcal{M}}$. ■

Now we are ready to prove the main theorem.

**Proof of Theorem 2.1.1** Denote the sequence of all distinct factors of $N$ by

$$1 = q_0 < q_1 < q_2 < \cdots < q_a < N.$$

Clearly, if $a = 0$ then $N$ is prime. Also, denote $N_b = N/q_b$ for $0 \leq b \leq a$. We will apply mathematical induction to $a$.

**Part (a)** First, consider $a = 0$. In this case, $N$ is a prime number, thus $Z_N$ has no nontrivial proper subgroup. According to Lemma 2.2.4, $c^{q_0}(\beta) = c(\beta) = \infty$ for $\beta \leq -\frac{\mu}{N-1}$. Using Lemma 2.4.7 (i) and (ii), we get an increasing sequence of LS levels, namely $\{c_m\}_1^\infty$. Then Lemma 2.4.7 (iii) implies that $c_m$ is a critical value for every $m$, and there exists at least one critical point, denoted by $\mathbf{u}_m$, which corresponds to the critical value $c_m$. Choosing $\mathbf{u}_m \in K_{c_m}$ for every positive integer $m$, we obtain a sequence of nontrivial $Z_N$-orbit critical points of $\mathcal{E}_{\mathcal{M}}$, which are, according to Lemma 2.2.1, 2.2.3 nontrivial $Z_N$-orbit solutions of system (2.1.1).

Next, consider the cases $a = 1$, i.e. $N$ is the square of a prime number. When $\beta \leq -\frac{\mu}{N_1-1}$, Lemma 2.2.4 implies $\min\{c^{q_0}(\beta), c^{q_1}(\beta)\} = \infty$. Then using Lemma 2.4.7 (i)-(iii), we can define an infinite sequence of LS levels $\{c_m\}_1^\infty$, which yields an infinite sequence of $Z_N$-orbit solutions of system (2.1.1). Now we need to extend the existence of infinitely many $Z_N$-orbit solutions to interval $(-\frac{\mu}{N_1-1}, -\frac{\mu}{N-1}]$.

Consider solutions of system (2.1.1) in the form

$$(u_1, \cdots, u_{q_1}, u_1, \cdots, u_{q_1}, \cdots, u_1, \cdots, u_{q_1}), \tag{2.4.6}$$

which are clearly solutions of the reduced system too,

$$\begin{cases} -\Delta u_j + u_j = [\mu + \beta(N_1 - 1)]u_j^3 + \beta N_1 \sum_{k\neq j}^{q_1} u_k^2 u_j, \quad \text{in } \Omega, \\ u_j > 0 \text{ in } \Omega, u_j = 0 \text{ on } \partial\Omega, \quad j = 1, \cdots, q_1. \end{cases} \tag{2.4.7}$$

This system can be viewed as system (2.1.1) with $\widetilde{\mu} = \mu + \beta(N_1 - 1)$, $\widetilde{\beta} = \beta N_1$ and $\widetilde{N} = q_1$. Since $q_1$ is prime, we can use the conclusion for $a = 0$, i.e. if $\widetilde{\mu} > 0$ and $\widetilde{\beta} < -\frac{\widetilde{\mu}}{\widetilde{N} - 1}$, or equivalently

$$\mu + \beta(N_1 - 1) > 0 \qquad \text{and} \qquad \beta N_1 \leq -\frac{\mu + \beta(N_1 - 1)}{\widetilde{N} - 1}, \tag{2.4.8}$$

system (2.4.7) has an infinite sequence of $Z_{q_a}$-orbit solutions. According to (2.4.6), system (2.1.1) thus has an infinite sequence of $Z_N$-orbit solutions when (2.4.8) holds. Solve $\beta$ from (2.4.8), we get $-\frac{\mu}{N_1 - 1} < \beta \leq -\frac{\mu}{N_1\widetilde{N} - 1} = -\frac{\mu}{N_1 q_1 - 1} = -\frac{\mu}{N - 1}$. Part (a) is proved in this case.

Consider the case $a = 2$. If $(q_1, q_2) = 1$, then the proof is exactly the same as the case $a = 1$ (The difference is $N_1 \neq q_1$ now, but this is not important. We only need $q_2$ to be prime, which is satisfied here). If $(q_1, q_2) > 1$, then $q_2 = q_1^2$. For $\beta \leq -\frac{\mu}{N_2 - 1}$, Lemma 2.2.4 implies that $\min\{c^{q_0}(\beta), c^{q_1}(\beta), c^{q_2}(\beta)\} = \infty$, and then infinitely many $Z_N$-orbit solutions of system (2.1.1) can be found by using LS levels corresponding to $Z_N$-symmetry. In order to obtain infinitely many $Z_N$-orbit solutions for $-\frac{\mu}{N_2 - 1} < \beta \leq -\frac{\mu}{N - 1}$, we consider solution in the form,

$$(u_1, \cdots, u_{q_2}, u_1, \cdots, u_{q_2}, \cdots, u_1, \cdots, u_{q_2}). \tag{2.4.9}$$

For this type of solutions, system (2.1.1) is reduced to a system of $q_2$ equations

$$\begin{cases} -\Delta u_j + u_j = [\mu + \beta(N_2 - 1)]u_j^3 + \beta N_2 \sum_{k\neq j}^{q_2} u_k^2 u_j, \quad \text{in } \Omega, \\ u_j > 0 \text{ in } \Omega, u_j = 0 \text{ on } \partial\Omega, \quad j = 1, \cdots, q_2. \end{cases} \tag{2.4.10}$$

Let $\widetilde{\mu} = \mu + \beta(N_2 - 1)$ and $\widetilde{\beta} = \beta N_2$, then system (2.4.10) can be view as system (2.1.1) with

$q_2$ equations. Since $q_2$ has only one prime factor, we can use the conclusion for $a = 1$ here, i.e. system (2.4.10) has infinitely many $Z_{q_2}$-orbit solutions if $\widetilde{\mu} > 0$ and $\widetilde{\beta} \le -\frac{\widetilde{\mu}}{q_2-1}$. Solving for $\beta$, we see that if $-\frac{\mu}{N_2-1} < \beta \le -\frac{\mu}{N-1}$, then system (2.4.10) has infinitely many $Z_{q_2}$-orbit solutions. According to (2.4.9), system (2.1.1) also has infinitely many $Z_N$-orbit solutions. The Part (a) is also proved for $a = 2$.

Now assume that part(a) holds for $a = p - 1 > 1$, then we look at the case $a = p$. First, if $\beta \le -\frac{\mu}{N_p-1}$, then we get infinitely many $Z_N$-orbit solutions with increasing LS levels defined with $Z_N$-symmetry, by using Lemma 2.2.4 and Lemma 2.4.7. Next, we extend the existence of infinitely many solutions to $-\frac{\mu}{N_p-1} < \beta \le -\frac{\mu}{N-1}$. Consider solutions in the form,

$$(u_1, \cdots, u_{q_p}, u_1, \cdots, u_{q_p}, \cdots, u_1, \cdots, u_{q_p}), \tag{2.4.11}$$

then system (2.1.1) is reduced to a system of $q_p$ equations

$$\begin{cases} -\Delta u_j + u_j = [\mu + \beta(N_p - 1)]u_j^3 + \beta N_p \sum_{k \ne j}^{q_p} u_k^2 u_j, \quad \text{in } \Omega, \\ \\ u_j > 0 \text{ in } \Omega, u_j = 0 \ \text{ on } \partial\Omega, \quad j = 1, \cdots, q_p. \end{cases} \tag{2.4.12}$$

Let $\widetilde{\mu} = \mu + \beta(N_p - 1)$ and $\widetilde{\beta} = \beta N_p$. By the induction assumption, system (2.4.12) has infinitely many $Z_{q_p}$-orbit solutions if $\widetilde{\mu} > 0$ and $\widetilde{\beta} < -\frac{\widetilde{\mu}}{q_p-1}$. Solving for $\beta$, we see that if $-\frac{\mu}{N_p-1} < \beta \le -\frac{\mu}{N-1}$, then system (2.4.12) has infinitely many $Z_{q_p}$-orbit solutions. According to (2.4.11), system (2.1.1) also has infinitely many $Z_N$-orbit solutions. The Part (a) is also proved for $a = p$.

By mathematical induction, Part (a) holds for any positive integer $a$. Then the factorization of $N$ indicates that Part (a) also holds for any integer $N$.

**Part (b)** First, consider that $N$ is a prime number. By Lemma 2.4.7 (i), for any given positive integer $m$, there exists a LS level $c_m = c_m^{q_0} < \infty$ defined independently of $\beta$. On the other hand, Lemma 2.2.4 implies that $c(\beta) \to \infty$ as $\beta \searrow -\frac{\mu}{N-1}$. Therefore there exists $\beta_m > -\frac{\mu}{N-1}$ such that $c(\beta) > c_m$ for any $\beta \in (-\frac{\mu}{N-1}, \beta_m)$. According to Lemma 2.4.7 (iii), there is at least one critical point that corresponds to each $c_m$, which implies that there exist at least $m$ $Z_N$-orbit solution of system (2.1.1) for $\beta \in (-\frac{\mu}{N-1}, \beta_m)$.

Next, assume that $N$ is a composite number. Note that $q_1$, the smallest nontrivial factor of $N$, must be prime. Consider the solution in the form (2.4.6), then system (2.1.1) will reduce to a smaller system (2.4.7). Let $\widetilde{\mu} = \mu + \beta(N_1 - 1)$, $\widetilde{\beta} = \beta N_1$ and $\widetilde{N} = q_1$, then system (2.4.7) can be viewed as system (2.1.1) with prime number of equations, provided $\widetilde{\mu} > 0$. Thus by the conclusion proved for $N$ prime, for any given integer $m > 0$, there exists $\widetilde{\beta_m} > -\frac{\widetilde{\mu}}{q_1-1}$, such that system (2.4.7) has at least $m$ $Z_{q_1}$-orbit solution for any $\widetilde{\beta} \in (-\frac{\widetilde{\mu}}{q_1-1}, \widetilde{\beta_m})$. Note that in this interval, $\widetilde{\mu} > 0$ is automatically satisfied. Now we change back to the original notations, $\beta = \widetilde{\beta}/N_1$, $-\frac{\widetilde{\mu}}{(q_1-1)N_1} < \beta < \frac{\widetilde{\beta_m}}{N_1}$. Define $\beta_m = \widetilde{\beta_m}/N_1$, then we can restate the conclusion as: for any given integer $m > 0$, there exists $\beta_m > -\frac{\mu}{N-1}$ such that system (2.1.1) has at least $m$ solutions in the form (2.4.6) for all $\beta \in (-\frac{\mu}{N-1}, \beta_m)$. Thus Part (b) is proved. ■

**Remark 2.4.8** *Denote the sequence of distinct factor of $N$ as above. When $b$ changes from* 1 *to $a$, we obtain a sequence of numbers between $-\frac{\mu}{N-1}$ and $-\frac{\mu}{N_a-1}$ (for prime number $N$, the following discussion is unnecessary since $-\frac{\mu}{N-1} = -\frac{\mu}{N_a-1}$). Between two of these consecutive numbers, say $-\frac{\mu}{N_{b_1}-1} > -\frac{\mu}{N_{b_2}-1}$ with $b_1 < b_2$, one has the following facts:*

1. *solutions with $Z_{q_b}$-symmetry for $b \leq b_1$ does not exist;*

2. *solutions with $Z_{q_b}$-symmetry for $b > b_1$ may exist;*

3. *if $q_{b_2}$ is prime, then the number of solutions with $Z_{q_{b_2}}$-symmetry will be getting larger and larger as $\beta \searrow -\frac{\mu}{N_{b_2}-1}$.*

*In other words, when $\beta$ moves from $-\frac{\mu}{N-1}$ to the left-hand side of $-\frac{\mu}{N_a-1}$, the solution orbits tend to lose symmetry. On the right-hand side of $-\frac{\mu}{N-1}$, system* (2.1.1) *has totally symmetric solution orbits, i.e. solution orbits with $N$ identical components. On the left-hand side of $-\frac{\mu}{N_a-1}$, there is no solution orbit possessing symmetry of $Z_N$ or any of its subgroups.*

## 2.5 Generalization

Our methods can be used to study a more general version of system (2.1.1)

$$\begin{cases} -\Delta u_j + u_j = \mu|u_j|^{2p-2}u_j + \beta \sum_{k\neq j} |u_k|^p |u_j|^{p-2}u_j, & \text{in } \Omega, \\ u_j > 0 \text{ in } \Omega, u_j = 0 \text{ on } \partial\Omega, \quad j = 1, \cdots, N \end{cases} \tag{2.5.1}$$

where $\mu > 0$ is a constant and $\Omega \subset \mathbb{R}^n$ is smooth bounded domain for $n \geq 1$, or radially symmetric (possibly unbounded) domain for $n \geq 2$. The nonlinear exponent $p$ satisfies

$$1 < p < \frac{2^*}{2} = \begin{cases} \dfrac{n}{n-2}, & n \geq 3, \\ \infty, & n = 1, 2. \end{cases}$$

With obvious changes (of notations, essentially) of the proof of Theorem 2.1.1, we can obtain the following theorem. The details are omitted.

**Theorem 2.5.1** *(a) If $\beta \leq -\frac{\mu}{N-1}$, then system* (2.5.1) *has an infinite sequence of $Z_N$-orbit solutions.*

*(b) For any positive integer $m$, there exists a $\beta_m \in (-\frac{\mu}{N-1}, 0)$, such that for $\beta \in (-\frac{\mu}{N-1}, \beta_m)$, system* (2.5.1) *has at least $m$ $Z_N$-orbit solutions.*

Similar like Remark 2.4.8, we can obtain more information regarding the distribution of solutions when $N$ is composite. Details are omitted.

## 2.6 Summary

In this chapter, we use variational methods and the symmetric structure to find multiple solutions of system (2.1.1). A Nehari manifold $\mathcal{M}$ with co-dimension $N$ is defined. By restricting the energy functional $\mathcal{E}$ on $\mathcal{M}$, we exclude the trivial solution and semi-trivial solutions, but we still have all nontrivial solutions contained in $\mathcal{M}$.

While implementing the variation method, a critical point is usually located as a minimizer of the energy functional on some set. In order to find multiple solutions, we introduce a new $Z_N$-

index and use it to construct a sequence of Lusternik-Schnirelmann type levels. A minimizing sequence with certain symmetry will converges to a point that stays on the minimal level. In other words, the symmetry, or the $Z_N$-index, provides a way to divide the Nehari manifold into different levels. Then solutions are found as minimizers on those levels. The $Z_N$-index and the construction of continuous map from finite dimensional sphere to $\mathcal{M}$ with property (2.3.3) are the main novelties of this work.

# CHAPTER 3

# EXISTENCE AND BIFURCATION RESULTS ON POSITIVE SOLUTIONS OF SOME ASYMMETRIC ELLIPTIC SYSTEMS[1]

In this chapter, we will study the bifurcation phenomena and multiplicity of positive standing wave solutions of some two-equation asymmetric CNLS, including the self-focusing, self-defocusing and two mixed cases. The information obtained about local and global bifurcation structures is of independent interests. Moreover, we also obtain some nonexistence results of positive solutions.

## 3.1 Introduction

Consider the positive standing wave solutions of (1.1.2) with $N = 2$, $a_1 = a_2 = -a$ and $\beta_{12} = \beta_{21} = \beta$, then we get the following elliptic system

$$\begin{cases} -\Delta u - au = \mu_1 u^3 + \beta uv^2 & \text{in } \Omega, \\ -\Delta v - av = \mu_2 v^3 + \beta vu^2 & \text{in } \Omega, \\ u, v > 0 \text{ in } \Omega,\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1.1}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary and $n \leq 3$. $\mu_1$ and $\mu_2$ are real numbers. In contrast with the assumption on the system (2.1.1), $\mu_1$ and $\mu_2$ are not necessarily equal to each other. Thus system (3.1.1) is generally asymmetric. If $\mu_j > 0$, then the $j$-th component is called self-focusing; if $\mu_j < 0$, then $j$-th component is self-defocusing. Then there are four cases: focusing-focusing, defocusing-defocusing and two mixed cases. Note that the two mixed cases are not essentially equivalent, since their sum being positive or negative will result in different global bifurcation phenomena. For fixed $\mu_1$ and $\mu_2$, the coupling constant $\beta$ determines bifurcation points along a trivial solution branch of (3.1.1), and also determines some nonexistence intervals of positive solutions. Let $\Lambda_1$ be the principal eigenvalue of $(-\Delta, \Omega)$

[1] Coauthored by Rushun Tian and Zhi-Qiang Wang [49, 50].

with zero Dirichlet boundary condition.

Our work is mostly inspired by a recent paper of T. Bartsch, E.N. Dancer and Z.-Q. Wang [8]. Precisely, the authors studied system (3.1.1) in the definite case, i.e. $a < \Lambda_1$, and obtained an infinite sequence of bifurcations with respect to

$$\mathcal{T}_\omega := \left\{ (\beta, u_\beta, v_\beta) \middle| u_\beta = \left( \sqrt{\frac{\mu_2 - \beta}{\mu_1\mu_2 - \beta^2}} \right) \omega, v_\beta = \left( \sqrt{\frac{\mu_1 - \beta}{\mu_1\mu_2 - \beta^2}} \right) \omega, \beta > -\sqrt{\mu_1\mu_2} \right\},$$

where $\omega$ is the unique positive standing solution of scalar Schrödinger equation with cubic nonlinearity

$$-\Delta\omega + \omega = \omega^3 \ \text{ in } \Omega, \quad \omega = 0 \quad \text{on } \partial\Omega.$$

They first used the method of spectral analysis to find an infinite sequence of local bifurcations with respect to $\mathcal{T}_\omega$. In the case $\Omega$ being a radial domains or the spatial dimension $n = 1$, local bifurcations become global bifurcations, due to Rabinowitz's bifurcation results [40]. Then the authors studied global bifurcations by establishing a new Liouville type theorem and investigating the nodal property of a weighted difference between the two solution components along each bifurcation branch.

In this chapter, we consider the indefinite case of system (3.1.1), which is determined by $a > \Lambda_1$. In this case, the trivial solution branch $\mathcal{T}_\omega$ is expressed in terms of the positive solution of Cahn-Hillards equation (3.1.2), instead of the positive solution of the above Schrödinger type equation. Moreover, there are infinitely many bifurcations along $\mathcal{T}_\omega$ in the focusing-focusing case, and finitely many bifurcations along $\mathcal{T}_\omega$ in the other three cases. In [36], B. Noris and M. Ramos also studied indefinite system (3.1.1). They obtained an infinite sequence of solutions for any $\beta$ small enough under symmetric condition $\mu_1 = \mu_2 > 0$. Comparing with our result in the focusing-focusing case, they do not require the domain to be radial to obtain infinitely many solutions for $\beta < 0$ small. But on the other hand, they must have symmetric system, i.e. $\mu_1 = \mu_2 > 0$. Also, their method only deals with the spatial dimension $n = 3$.

Without loss of generality we may assume $\Lambda_1 < 1$ and take $a = 1$. Our main goal is to obtain the bifurcations of (3.1.1) with respect to a known solution branch that can be explicitly expressed. We now define this trivial branch.

### 3.1.1 A trivial solution branch

Let $\omega$ be the unique positive solution of the scalar equation

$$-\Delta\omega - \omega = -\omega^3 \ \text{ in } \Omega, \quad \text{and } \omega = 0 \text{ on } \partial\Omega, \tag{3.1.2}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, $n \leq 3$. For the existence and uniqueness of $\omega$, one may consult Theorem 2.5 [38] for detailed proof. By the standard regularity theory and the Strong Maximum Principle [21]

$$\omega \in C^2(\Omega) \cap C(\overline{\Omega}) \qquad \text{and} \qquad \frac{\partial\omega}{\partial\nu} > 0 \qquad \text{on } \partial\Omega, \tag{3.1.3}$$

where $\nu$ is outer normal vector on $\partial\Omega$. It is easy to see that

$$u_\beta = \left(\sqrt{\frac{\mu_2 - \beta}{\beta^2 - \mu_1\mu_2}}\right)\omega, \quad v_\beta = \left(\sqrt{\frac{\mu_1 - \beta}{\beta^2 - \mu_1\mu_2}}\right)\omega, \tag{3.1.4}$$

solve (3.1.1) for $\beta \in I$, which is an interval determined according to the values of $\mu_1$ and $\mu_2$ (solve $\beta$ such that the coefficients of $\omega$ in (3.1.4) are real) as follows:

$$I = \begin{cases} (-\infty, -\sqrt{\mu_1\mu_2}), & \text{in the case } 0 < \mu_1 \leq \mu_2, \\ (-\infty, \sqrt{\mu_1\mu_2}) \backslash (\mu_1, \mu_2), & \text{in the case } \mu_1 < \mu_2 < 0, \\ (-\infty, \mu_1], & \text{in the case } \mu_1 < 0 < \mu_2 \text{ or } \mu_1 = \mu_2 < 0. \end{cases} \tag{3.1.5}$$

Then we obtain a trivial solution branch of (3.1.1)

$$\mathcal{T}_\omega := \{(\beta, u_\beta, v_\beta) : u_\beta, v_\beta \text{ are given by (3.1.4)}, \ \beta \in I\} \subset \mathbb{R} \times \mathcal{H},$$

here $\mathcal{H} = H_0^1(\Omega) \times H_0^1(\Omega)$.

### 3.1.2 The notions of local and global bifurcations

The general description of bifurcation is given in Chapter 1, Section 1.2. We call

$$(\beta^*, u_{\beta^*}, v_{\beta^*}) \in \mathcal{T}_\omega$$

a local bifurcation point of (3.1.1) with respect to $\mathcal{T}_\omega$, or a local bifurcation point for short, if there exists a sequence of points $\{(\beta_m, u_m, v_m)\}_1^\infty \subset \mathbb{R} \times \mathcal{H}$, such that $(\beta_m, u_m, v_m)$ solves (3.1.1) and

$$(\beta_m, u_m, v_m) \to (\beta^*, u_{\beta^*}, v_{\beta^*}) \qquad \text{as } m \to \infty.$$

Accordingly, $\beta^*$ is called a local bifurcation parameter. Denote the set of all nontrivial solutions (non-trivial relative to $\mathcal{T}_\omega$) of (3.1.1) by

$$\mathcal{S} = \{(\beta, u, v) \in \mathbb{R} \times \mathcal{H} | (\beta, u, v) \text{ solves system (3.1.1)}\} \setminus \mathcal{T}_\omega.$$

Local bifurcation parameter $\beta_0$ becomes a global bifurcation parameter, if there exists a connected component $\mathcal{S}_0 \subset \mathcal{S} \cup \{(\beta_0, u_{\beta_0}, v_{\beta_0})\}$, such that $(\beta_0, u_{\beta_0}, v_{\beta_0}) \in \overline{\mathcal{S}}$ and one of the following situations occurs:

(i) $\mathcal{S}_0$ is unbounded in $\mathbb{R} \times \mathcal{H}$;

(ii) $\overline{\mathcal{S}_0} \cap \mathcal{T}_\omega \setminus \{(\beta_0, u_{\beta_0}, v_{\beta_0})\} \neq \emptyset$.

This definition was given by Rabinowitz's bifurcation theorem [40]. One can find more detailed discussions and applications in [42].

### 3.1.3 A related eigenvalue problem

Let $\omega$ be the unique ground state solution of (3.1.2). The following eigenvalue problem plays an important role in determining the bifurcation parameters,

$$-\Delta\phi - \phi = \lambda\omega^2\phi \ \text{ in } \Omega, \qquad \phi = 0 \text{ on } \partial\Omega. \tag{3.1.6}$$

Denote the sequence of distinct eigenvalues of (3.1.6) by

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_{k_0} < \cdots , \tag{3.1.7}$$

and the multiplicity of $\lambda_k$ by $n_k$. The non-degeneracy of $\omega$ implies that $\lambda_k \neq -3$ for any $k \in \mathbb{N}$. In fact, it is easy to see $\lambda_1 = -1$. Also as $k \to \infty$, we have $\lambda_k \to \infty$. Thus (3.1.6) has finitely many negative eigenvalues and infinitely many positive eigenvalues. Let $\lambda_{k^*+1}$ be the least eigenvalue greater than 0. If we assume more precisely $\Lambda_l < 1 \leq \Lambda_{l+1}$ for some $l$, where $\Lambda_1 < \Lambda_2 \leq \Lambda_3 \leq \ldots$ are the eigenvalues of $(-\Delta, \Omega)$ with zero Dirichlet boundary condition, then we have $\sum_{j=1}^{k^*} n_j \geq l$ ($\sum_{j=1}^{k^*} n_j \geq l+1$ if $1 = \Lambda_{l+1}$).

Besides the minimal eigenvalue, 1 is another important value that we need to compare with (3.1.7). Let $\lambda_{k_0+1}$ be the least eigenvalue of (3.1.6) that is greater than 1.

### 3.1.4 Main results

We study (3.1.1) from three perspectives. First, we will give a nonexistence result. The conclusions are given for a system slightly more general than (3.1.1). Second, we study the local bifurcations of (3.1.1) along $\mathcal{T}_\omega$. At last, when $n = 1$ or $\Omega$ is radial, we study the global bifurcation branches with respect to $\mathcal{T}_\omega$. As a result, the existence of multiple bifurcation branches implies the existence of multiple solutions of (3.1.1) when $\beta$ is in certain range.

Comparing with [8], here we can consider all real values of $\mu_1$ and $\mu_2$. Precisely, there are four cases,

a. $0 < \mu_1 \leq \mu_2$;

b. $\mu_1 \leq \mu_2 < 0$;

c. $\mu_1 \leq -\mu_2 < 0 < \mu_2$;

d. $\mu_1 < 0 < -\mu_1 < \mu_2$.

In the process of studying the nonexistence and bifurcation problems, we use the same framework for the four cases. But, on the other hand, the conclusions and proofs are still different in many places. To clearly show our results, we first summary the main results as three theorems, with detailed descriptions omitted. Then in the following sections, we will restate the complete version of the theorems for each case, and present the proofs accordingly. A few of results will be established independently of the four cases, so they will only be proved once and then reused when they are needed in the other cases. One may consider the arguments for the self-focusing case, the case we shall discuss first, as a framework.

To explain the local bifurcation theorem, we define two auxiliary functions:

$$f(\beta) = \frac{\beta^2 - 2\beta(\mu_1 + \mu_2) + 3\mu_1\mu_2}{\beta^2 - \mu_1\mu_2}, \tag{3.1.8}$$

and

$$g(\beta) = -4(\mu_1+\mu_2)\beta^3 + 2(\mu_1^2 + 10\mu_1\mu_2 + \mu_2^2)\beta^2 - 12\mu_1\mu_2(\mu_1+\mu_2)\beta + 2\mu_1\mu_2(\mu_1+\mu_2)^2. \tag{3.1.9}$$

These two functions, as we can see in later sections, are naturally derived from the definition and verification of bifurcations.

Our first result is

**Theorem A** *Let $\beta_k$ be a solution of $f(\beta) = \lambda_k$ for some $\lambda_k$ given in* (3.1.7)*. Then $\beta_k$ is a bifurcation parameter, if and only if $g(\beta_k) \neq 0$. In other words, the following set contains all bifurcation parameters of* (3.1.1) *with respect to $\mathcal{T}_\omega$*

$$\{\beta_k \in I \mid f(\beta_k) = \lambda_k,\ g(\beta_k) \neq 0,\ k \geq 1\}.$$

*Moreover, if the multiplicity of $\lambda_k$ is odd, then $\beta_k$ is a global bifurcation parameter, and the corresponding bifurcation branch $\mathcal{S}_k$ is a solution branch of* (3.1.1)*.*

**Remark 3.1.1** *Theorem A gives the necessary and sufficient conditions for the existence of local bifurcations, also global bifurcations with respect to $\mathcal{T}_\omega$, provided that the multiplicity of the bifurcation parameter is odd. The equation $f(\beta) = \lambda_k$ is used to find the k-th possible bifurcation* *parameter, and the inequality $g(\beta_k) \neq 0$ is used to verify that bifurcation indeed occurs at $\beta_k$.*

When $n = 1$ or $\Omega$ is radially symmetric, the multiplicity of every $\lambda_k$ is odd. Then we have the following theorem to describe the global bifurcation branches.

**Theorem B** *If $n = 1$ or $\Omega$ is radially symmetric. Then every bifurcation parameter found in Theorem A gives rise to a global bifurcation branch $\mathcal{S}_k$ with respect to $\mathcal{T}_\omega$ in the sense of Rabinowitz's theorem [40]. Moreover, if $(u, v) \in \mathcal{S}_k$ then the weighted difference $\sqrt{\mu_1 - \beta}u - \sqrt{\mu_2 - \beta}v$ has precisely $k-1$ simple zeroes. Except, possibly, finitely many branches found in the case $\mu_1 < 0 < -\mu_1 < \mu_2$, all other bifurcation branches are unbounded in the negative direction of $\beta$.*

About the nonexistence of positive solutions, our results are given for a more general indefinite system,

$$\begin{cases} -\Delta u - au = \mu_1 u^3 + \beta u v^2 & \text{in } \Omega, \\ -\Delta v - bv = \mu_2 v^3 + \beta v u^2 & \text{in } \Omega, \\ u > 0, v > 0 \ \text{ in } \Omega, \ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1.10}$$

where $\Omega \subset \mathbb{R}^n$, $n \leq 3$ is a bounded domain with smooth boundary.

**Theorem C** *Assume that $a \geq \Lambda_1, b \geq \Lambda_1$. System* (3.1.10) *does not have positive solutions, in any one of the following cases,*

*(i)* $0 \leq \mu_1 \leq \mu_2$, $\beta \geq -\sqrt{\mu_1 \mu_2}$, *and at least one of the last two inequalities holds strictly;*

*(ii)* $\mu_1 \leq \beta \leq \mu_2 \leq 0$, *and at least one of the first two inequalities holds strictly, or $a \neq b$;*

*(iii)* $\mu_1 \leq \min\{\beta, 0\}, 0 \leq \mu_2$, *and the first inequality holds strictly, or $b > \Lambda_1$.*

**Remark 3.1.2** *The conclusions of Theorem C are divided into three parts in order to match the four cases of $\mu_1$ and $\mu_2$.*

**Remark 3.1.3** *There are a few bifurcation diagrams in the following sections and the no positive solution regions are also labeled in these graphs. Note that, the conclusions of the theorems are* *more or less more general than the graphs, since we try to illustrate all conclusions in one graph for each case, which requires all assumptions must be satisfied at the same time.*

At last, for technical reasons, when we discuss the bifurcations, the following relaxed system will be considered first,

$$\begin{cases} -\Delta u - u = \mu_1 u^3 + \beta u v^2, & \text{in } \Omega \\ -\Delta v - v = \mu_2 v^3 + \beta v u^2, & \text{in } \Omega \\ \qquad u = v = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.1.11}$$

Later on, after the bifurcations are verified, the positivity of these bifurcation solutions will be recovered by using the Strong Maximum Principle. So we will get the bifurcations of original system (3.1.1).

## 3.2 The self-focusing case

As it was mentioned in Chapter 1, in the case (a), $0 < \mu_1 \leq \mu_2$, system (3.1.1) is called self-focusing. We assume $0 < \mu_1 \leq \mu_2$ throughout this section. For simplicity, let us prove the nonexistence result theorem first.

### 3.2.1 Nonexistence of positive solutions

In this case, the nonexistence theorem is stated as follows:

**Theorem 3.2.1** *Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^n$ $(n \leq 3)$. System* (3.1.10) *has no solution, if $\beta \geq -\sqrt{\mu_1 \mu_2}$, $a \geq \Lambda_1$, $b \geq \Lambda_1$ and at least one of the three inequalities holds strictly.*

*Proof.* Let $\phi_1$ be the eigenfunction of $(-\Delta, \Omega)$ that corresponds to $\Lambda_1$. Multiplying the first equation of (3.1.10) by $\mu_1^{-\frac{1}{4}}\phi_1$ and the second equation by $\mu_2^{-\frac{1}{4}}\phi_1$, adding the two equations together, and integrating over $\Omega$, we get the following estimates from the assumptions

$$\begin{aligned}&(\Lambda_1 - a)\int_\Omega \mu_1^{-\frac{1}{4}}\phi_1 u dx + (\Lambda_1 - b)\int_\Omega \mu_2^{-\frac{1}{4}}\phi_1 v dx \\ &= \int_\Omega \phi_1(\mu_1^{\frac{3}{4}}u^3 + \beta\mu_1^{-\frac{1}{4}}uv^2 + \beta\mu_2^{-\frac{1}{4}}u^2 v + \mu_2^{\frac{3}{4}}v^3)dx\end{aligned}$$

$$\begin{aligned}
&\geq \int_\Omega \phi_1(\mu_1^{\frac{3}{4}}u^3 - \mu_1^{\frac{1}{4}}\mu_2^{\frac{1}{2}}uv^2 - \mu_1^{\frac{1}{2}}\mu_2^{\frac{1}{4}}u^2v + \mu_2^{\frac{3}{4}}v^3)dx \\
&= \int_\Omega \phi_1\left[\mu_1^{\frac{1}{4}}u(\mu_1^{\frac{1}{2}}u^2 - \mu_2^{\frac{1}{2}}v^2) + \mu_2^{\frac{1}{4}}u(\mu_2^{\frac{1}{2}}v^2 - \mu_1^{\frac{1}{2}}u^2)\right]dx \\
&= \int_\Omega \phi_1(\mu_2^{\frac{1}{2}}v^2 - \mu_1^{\frac{1}{2}}u^2)(\mu_2^{\frac{1}{4}}v - \mu_1^{\frac{1}{4}}u)dx \\
&= \int_\Omega \phi_1(\mu_1^{\frac{1}{4}}u + \mu_2^{\frac{1}{4}}v)(\mu_1^{\frac{1}{4}}u - \mu_2^{\frac{1}{4}}v)^2dx.
\end{aligned}$$

The inequality in the third line is strict, if $\beta > -\sqrt{\mu_1\mu_2}$ and $u, v > 0$. This yields a contradiction, since the first expression is non-positive and the last expression is strictly positive. If $\beta = -\sqrt{\mu_1\mu_2}$, the right hand side is equal to zero only if $\mu_1^{\frac{1}{4}}u - \mu_2^{\frac{1}{4}}v = 0$. Now if $a > \Lambda_1$ or $b > \Lambda_1$, then the first expression is strictly negative. This is also get a contradiction. Thus the theorem is proved. ■

### 3.2.2 Local bifurcations

The next lemma gives the necessary conditions such that local bifurcations of system (3.1.11) with respect to $\mathcal{T}_\omega$ exist.

**Lemma 3.2.2** *All the possible bifurcation parameters of* (3.1.11) *are determined from the following equations about* $\beta$*:*

$$f(\beta) = \lambda_k, \qquad k = k_0 + 1,\ k_0 + 2,\ \cdots,$$

*where* $f$ *is defined as* (3.1.8)*,* $\lambda_k$ *is the* $k$*-th eigenvalue of* (3.1.6) *and* $k_0$ *is defined below* (3.1.7)*. Let* $\beta_k$ *be a solution of* $f(\beta) = \lambda_k$*, and let* $V_k$ *be the kernel space of the linearization of* (3.1.11) *at* $(\beta_k, u_{\beta_k}, v_{\beta_k})$*. Then* $\dim V_k = n_k$*, where* $n_k$ *is the multiplicity of* $\lambda_k$*.*

*Proof.* Linearizing (3.1.11) at $(\beta, u_\beta, v_\beta) \in \mathcal{T}_\omega$, we get

$$\begin{cases} -\Delta\phi - \phi = 3\mu_1 u_\beta^2\phi + \beta v_\beta^2\phi + 2\beta u_\beta v_\beta \psi, \\ -\Delta\psi - \psi = 2\beta u_\beta v_\beta \phi + 3\mu_2 v_\beta^2 \psi + \beta u_\beta^2 \psi. \end{cases} \tag{3.2.1}$$

To simplify notations, denote

$$
\begin{aligned}
A &= 3\mu_1 \frac{\mu_2-\beta}{\beta^2-\mu_1\mu_2} + \beta\frac{\mu_1-\beta}{\beta^2-\mu_1\mu_2} = \frac{3\mu_1\mu_2-2\beta\mu_1-\beta^2}{\beta^2-\mu_1\mu_2},\\
B &= 2\beta\sqrt{\frac{(\mu_1-\beta)(\mu_2-\beta)}{(\beta^2-\mu_1\mu_2)^2}} = \frac{2\beta\sqrt{(\mu_1-\beta)(\mu_2-\beta)}}{\beta^2-\mu_1\mu_2},\\
C &= 3\mu_2 \frac{\mu_1-\beta}{\beta^2-\mu_1\mu_2} + \beta\frac{\mu_2-\beta}{\beta^2-\mu_1\mu_2} = \frac{3\mu_1\mu_2-2\beta\mu_2-\beta^2}{\beta^2-\mu_1\mu_2}.
\end{aligned}
$$

Using (3.1.4) and the above notations of $A, B$ and $C$, we can rewrite system (3.2.1) as

$$
\begin{cases}
-\Delta\phi - \phi = (A\phi + B\psi)\omega^2,\\
-\Delta\psi - \psi = (B\phi + C\psi)\omega^2.
\end{cases} \tag{3.2.2}
$$

If $\phi$ and $\psi$ are linearly dependent, then (3.2.2) can be reduced to one equation. Precisely, let $\gamma$ be the solution of $A\gamma + B = B\gamma^2 + C\gamma$, then for $\beta \in I$ (the interval given by (3.1.5)), the equation has solutions

$$
\gamma_\pm = \frac{A - C \pm \sqrt{(C-A)^2 + 4B^2}}{2B}.
$$

Now multiplying the second equation of (3.2.2) by $\gamma_\pm$, then subtracting the resulting equation from the first equation, and at last replacing $B - C\gamma_\pm$ with $B\gamma_\pm^2 - A\gamma_\pm$, one can observe that $\phi - \gamma_\pm\psi$ solves

$$
-\Delta(\phi - \gamma_\pm\psi) - (\phi - \gamma_\pm\psi) = (A - B\gamma_\pm)\omega^2(\phi - \gamma_\pm\psi). \tag{3.2.3}
$$

Simple calculation shows $A - B\gamma_- = B\gamma_+ + C = f(\beta)$ and $A - B\gamma_+ = B\gamma_- + C = -3$. We use the constant $\gamma_+$ in (3.2.3). Since $-3$ is not an eigenvalue of (3.1.6), equation (3.2.3) implies that $\phi - \gamma_+\psi = 0$, i.e. $\phi = \gamma_+\psi$. Then $\psi$ solves

$$
-\Delta\psi - \psi = (B\phi + C\psi)\omega^2 = (B\gamma_+ + C)\omega^2\psi = f(\beta)\omega^2\psi. \tag{3.2.4}
$$

If (3.2.4) has nontrivial solution, then system (3.2.1) has nontrivial solution $(\gamma_+\psi, \psi)$. The linearized system (3.1.11) therefore has nontrivial kernel space, and then bifurcation may happen. On the one hand, (3.1.7) is an increasing sequence of eigenvalues with

$$\lambda_k \to \infty \text{ as } k \to \infty, \text{ and } \lambda_1 = -1,\ \lambda_{k_0} \le 1 < \lambda_{k_0+1}.$$

On the other hand, $f$ is an increasing homomorphism from $(-\infty, -\sqrt{\mu_1\mu_2})$ to $(1, \infty)$, as shown in Figure 3.1. Thus $f(\beta) = \lambda_k$ has a unique solution $\beta = \beta_k$ for every $k \ge k_0 + 1$. As we discussed above, (3.2.1) then has nonempty kernel space

$$V_k = \{(\gamma_+\psi, \psi) : \psi \text{ is an eigenfunction of (3.1.6) associated to } \lambda_k\}, \tag{3.2.5}$$

and a local bifurcation of (3.1.11) may occur at $(\beta_k, u_{\beta_k}, v_{\beta_k})$. Also it is easy to see from (3.2.5) that $\dim V_k = n_k$, where $n_k$ is the multiplicity of $\lambda_k$ as eigenvalue of (3.2.1). ■

Next we need to verify that the bifurcations indeed happen at the possible bifurcation points found in Lemma 3.2.2.

**Lemma 3.2.3** *Let $\beta_k$ be the solution of $f(\beta) = \lambda_k$, $k \ge k_0 + 1$. Then $\beta_k$ is a local bifurcation parameter if and only if $g(\beta_k) \neq 0$, where $g$ is defined as* (3.1.9).

*Proof.* Denote the energy functional associated with system (3.1.11) by

$$\mathcal{E}_\beta(u, v) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - u^2 + |\nabla v|^2 - v^2)dx - \frac{1}{4}\int_\Omega (\mu_1 u^4 + \mu_2 v^4)dx - \frac{\beta}{2}\int_\Omega u^2v^2 dx.$$

According to Sobolev embeddings and Proposition B34 [42], it is easy to see that $\mathcal{E}_\beta \in C^2(\mathcal{H}, \mathbb{R})$. Then by Theorem 8.9 [35], we only need to show that the Morse index of $\mathcal{E}_\beta$ changes at

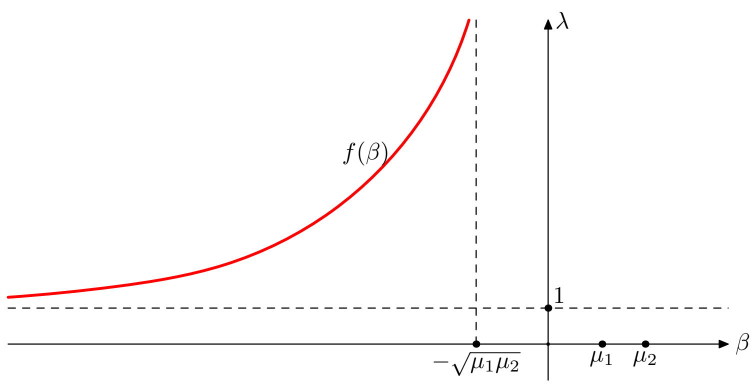



Figure 3.1: Graph of $f$ in the case $0 < \mu_1 \le \mu_2$

$(\beta_k, u_{\beta_k}, v_{\beta_k})$ for every bifurcation parameter $\beta_k$. The Hessian of $\mathcal{E}_\beta$ at $(u_\beta, v_\beta)$, $H_\beta : \mathcal{H}\times\mathcal{H} \to \mathbb{R}$, is represented as

$$\begin{aligned} H_\beta[(\phi,\psi)^2] &= \int_\Omega (|\nabla\phi|^2 - \phi^2 + |\nabla\psi|^2 - \psi^2) - \int_\Omega [3\mu_1 u_\beta^2 \phi^2 + 3\mu_2 v_\beta^2 \psi^2 \\ &\quad + \beta(v_\beta^2\phi^2 + 4u_\beta v_\beta \phi\psi + u_\beta^2\psi^2)] \\ &= \int_\Omega (|\nabla\phi|^2 - \phi^2 + |\nabla\psi|^2 - \psi^2) - \int_\Omega [A(\beta)\phi^2 + 2B(\beta)\phi\psi + C(\beta)\psi^2]\omega^2. \end{aligned} \tag{3.2.6}$$

When $\beta$ is close to $\beta_k$, the Taylor expansion of $H_\beta$ at $\beta = \beta_k$ is

$$H_\beta = H_{\beta_k} + (\beta - \beta_k)H'_{\beta_k} + o(|\beta - \beta_k|).$$

Thus on the produce space $V_k \times V_k$, $H'_{\beta_k}$ is represented as

$$H'_{\beta_k}[(\gamma_+\psi, \psi)^2] = -\int_\Omega [A'(\beta)\gamma_+^2 + 2B'(\beta)\gamma_+ + C'(\beta)]\omega^2\psi^2, \tag{3.2.7}$$

where $\psi$ is the eigenfunction of corresponding to $\lambda_k$. Now the conclusion is true if and only if $H'_{\beta_k}[(\gamma_+\psi, \psi)^2]$ is either positive definite or negative definite. To see that we need to take derivative of $A, B$ and $C$. Direction calculation shows

$$\begin{aligned} A'(\beta) &= \frac{-(2\mu_1 + 2\beta)(\beta^2 - \mu_1\mu_2) - (3\mu_1\mu_2 - 2\beta\mu_1 - \beta^2)2\beta}{(\beta^2 - \mu_1\mu_2)^2} \\ &= \frac{2\mu_1(\beta^2 - 2\beta\mu_2 + \mu_1\mu_2)}{(\beta^2 - \mu_1\mu_2)^2}, \\ B'(\beta) &= \frac{(2\mu_1\mu_2 - 3(\mu_1+\mu_2)\beta + 4\beta^2)(\beta^2 - \mu_1\mu_2) - 4\beta^2(\mu_1\mu_2 - (\mu_1+\mu_2)\beta + \beta^2)}{(\beta^2 - \mu_1\mu_2)^2\sqrt{(\mu_1 - \beta)(\mu_2 - \beta)}} \\ &= \frac{(\mu_1+\mu_2)\beta^3 - 6\mu_1\mu_2\beta^2 + 3\mu_1\mu_2(\mu_1+\mu_2)\beta - 2\mu_1^2\mu_2^2}{(\beta^2 - \mu_1\mu_2)^2\sqrt{(\mu_1 - \beta)(\mu_2 - \beta)}}, \\ C'(\beta) &= \frac{-(2\mu_2 + 2\beta)(\beta^2 - \mu_1\mu_2) - (3\mu_1\mu_2 - 2\beta\mu_2 - \beta^2)2\beta}{(\beta^2 - \mu_1\mu_2)^2} \\ &= \frac{2\mu_2(\beta^2 - 2\beta\mu_1 + \mu_1\mu_2)}{(\beta^2 - \mu_1\mu_2)^2}. \end{aligned}$$

Substituting the three expressions into (3.2.7) and noting $\gamma_+ = -\sqrt{(\mu_1-\beta)/(\mu_2-\beta)}$, we get

$$H'_{\beta_k}[(\gamma_+\psi,\psi)^2] = -\int_\Omega [A'(\beta)\gamma_+^2 + 2B'(\beta)\gamma_+ + C'(\beta)]\omega^2\psi^2 = \frac{g(\beta)}{(\beta^2-\mu_1\mu_2)^2(\beta-\mu_2)}\int_\Omega \omega^2\psi^2.$$

Clearly, $H'_{\beta_k}[(\gamma_+\psi,\psi)^2]$ is positive definite if and only if $g(\beta)<0$ and negative definite if and only if $g(\beta)>0$, for $\beta\in I$. Therefore if $g(\beta_k)\neq 0$, the Morse index of $\mathcal{E}_\beta$ changes at $(u_{\beta_k}, v_{\beta_k})$. According to Theorem 8.9 [35], $(\beta_k, u_{\beta_k}, v_{\beta_k})$ is a bifurcation point. ■

So far we have found the local bifurcations of system (3.1.11). To find the local bifurcations of system (3.1.1), we need to verify the positivity of bifurcation solutions that we find out in Lemma 3.2.2 and Lemma 3.2.3.

**Lemma 3.2.4** *Let $(\beta_k, u_{\beta_k}, v_{\beta_k})\in\mathcal{T}_\omega$ be a bifurcation point of* (3.1.11) *for any $k\geq k_0+1$. Then there exist a sequence of solutions of* (3.1.11)*, denoted by $\{(\beta_k^{(l)}, u_k^{(l)}, v_k^{(l)})\}\subset\mathbb{R}\times\mathcal{H}$, such that $(\beta_k^{(l)}, u_k^{(l)}, v_k^{(l)})\to(\beta_k, u_{\beta_k}, v_{\beta_k})$ as $l\to\infty$, and $u_k^{(l)}>0, v_k^{(l)}>0$. In other words, $(\beta_k, u_{\beta_k}, v_{\beta_k})$ is also a bifurcation point of* (3.1.1)*.*

*Proof.* By the definition of bifurcation point, there exist a sequence of solutions of (3.1.11) $\{(\beta_k^{(l)}, u_k^{(l)}, v_k^{(l)})\}\subset\mathbb{R}\times\mathcal{H}$ that satisfies

$$(\beta_k^{(l)}, u_k^{(l)}, v_k^{(l)})\to(\beta_k, u_{\beta_k}, v_{\beta_k}) \text{ in } \mathbb{R}\times\mathcal{H} \text{ as } l\to\infty.$$

Thus we only need to verify $u_k^{(l)}>0$ and $v_k^{(l)}>0$, at least for $l$ large enough.

Since $\Omega$ is bounded, then it follows from Sobolev embeddings, Hölder inequality, $L^p$ estimates and Schauder estimate (c.f. [21]) that $u_k^{(l)}, v_k^{(l)}\in C_0^1(\Omega)$ (see Appendix A. Lemma II). Then $u_k^{(l)}\to u_{\beta_k}$ and $v_k^{(l)}\to v_{\beta_k}$ in $H_0^1(\Omega)$ implies that

$$\|u_k^{(l)}-u_{\beta_k}\|_{C_0^1}\to 0,\quad \|v_k^{(l)}-v_{\beta_k}\|_{C_0^1}\to 0 \qquad \text{as } l\to\infty.$$

According to (3.1.3) and (3.1.4), there exists $L_1>0$ large enough such that

$$\frac{\partial u_k^{(l)}}{\partial\nu}\leq\frac{\partial\omega}{2\partial\nu}<0,\qquad \frac{\partial v_k^{(l)}}{\partial\nu}\leq\frac{\partial\omega}{2\partial\nu}<0 \quad \text{for } l\geq L_1.$$

Using the continuous differentiability of $u_k^{(l)}$ and $v_k^{(l)}$, there exists an open neighborhood $B$ of $\partial\Omega$ such that $u_k^{(l)}(x) > 0, v_k^{(l)}(x) > 0$ for all $l \geq L_1$ and $x \in B \cap \Omega$. On the other hand, there exists $L_2 > 0$ such that

$$\min_{x\in\Omega\setminus B} u_k^{(l)}(x) \geq \frac{1}{2}\min_{x\in\Omega\setminus B}\omega(x), \qquad \min_{x\in\Omega\setminus B} v_k^{(l)}(x) \geq \frac{1}{2}\min_{x\in\Omega\setminus B}\omega(x),$$

for all $l \geq L_2$. Choose $L = \max\{L_1, L_2\}$, then for $l \geq L$, the above arguments imply that $u_k^{(l)} > 0$ and $v_k^{(l)} > 0$ in $\Omega$. Thus we obtain a sequence of positive solutions to system (3.1.11) that converge to $(u_{\beta_k}, v_{\beta_k})$. The lemma is proved. ∎

**Theorem 3.2.5** *The trivial solution branch $\mathcal{T}_\omega$ exists on $(-\infty, -\sqrt{\mu_1\mu_2})$. There are infinitely many local bifurcations of* (3.1.1) *along $\mathcal{T}_\omega$ with $k \geq k_0 + 1$.*

*Proof.* In Lemma 3.2.2, all the possible bifurcation parameters are found. Now, according to Lemma 3.2.3, we only need to check $g(\beta_k)$ for $k \geq k_0 + 1$. Recall the definition of $g$ in (3.1.9), we have two estimates

$$g(0) = 2\mu_1\mu_2(\mu_1 + \mu_2)^2 > 0,$$

and

$$g'(\beta) = -12(\mu_1 + \mu_2)\beta^2 + 4(\mu_1^2 + 10\mu_1\mu_2 + \mu_2^2)\beta - 12\mu_1\mu_2(\mu_1 + \mu_2) < 0,$$

for all $\beta \in (-\infty, 0)$. Thus $g(\beta) > 0$ on $(-\infty, -\sqrt{\mu_1\mu_2})$. Therefore we have the fact that $\{\beta_k\}_{k_0+1}^{\infty}$ is a sequence of bifurcation parameters of system (3.1.11).

Next, by Lemma 3.2.4, there is a sequence of positive solutions of (3.1.11) converging to each bifurcation point. Thus the local bifurcations of (3.1.11) are also local bifurcations of (3.1.1). Hence the theorem is proved. ∎

### 3.2.3 Global bifurcations

By Theorem 3.2.5, we have obtained infinitely many local bifurcations of (3.1.1) in the self-focusing case. According to Rabinowitz's global bifurcation theorem, when the multiplicity $n_k$ is odd, the corresponding bifurcation is a global bifurcation, i.e. there is a continuous bifurcation branch emanating from $(\beta_k, u_k, v_k)$. In particular, if $n = 1$ or the domain $\Omega$ is radially symmetric, $n_k = 1$ for each $k$. Thus every local bifurcation is actually a global bifurcation. Next we study the property of global bifurcations of (3.1.1).

The following lemma extends the positivity of bifurcation solutions to the whole branch $\mathcal{S}_k$ for every $k \geq k_0 + 1$.

**Lemma 3.2.6** *Let $k \geq k_0 + 1$ and $\beta_k$ be a bifurcation parameter. If there exists a connected bifurcation branch $\mathcal{S}_k$ emanating from $\mathcal{T}_\omega$ at $(\beta_k, u_{\beta_k}, v_{\beta_k})$, then all solution pairs $(u, v)$ on $\mathcal{S}_k$ are strictly positive in $\Omega$.*

*Proof.* With the same arguments used in the proof of Lemma 3.2.4, the weak solutions of (3.1.11) found on $\mathcal{S}_k$ are $C^2$ functions. Using (3.1.3) and (3.1.4), all solutions on the trivial solution branch $\mathcal{T}_\omega$ are strictly positive, i.e. for any $(\beta, u_\beta, v_\beta) \in \mathcal{T}_\omega$,

$$u_\beta > 0, \quad v_\beta > 0 \quad \text{in } \Omega, \quad \text{and } \frac{\partial u_\beta}{\partial \nu} > 0, \qquad \frac{\partial v_\beta}{\partial \nu} > 0. \tag{3.2.8}$$

Here $\beta < \mu_1$ and $\nu$ is the outer normal vector on $\partial\Omega$. Fix a bifurcation point $(\beta_k, u_{\beta_k}, v_{\beta_k})$. Since $\mathcal{S}_k$ is connected, if $(\beta, u, v) \in \mathcal{S}_k$ is close enough to $(\beta_k, u_{\beta_k}, v_{\beta_k})$, then $(\beta, u, v)$ also satisfies (3.2.8). Actually, (3.2.8) can be continued along $\mathcal{S}_k$ and the following inequalities hold

$$\Delta u + (\mu_1 u^2 + \beta v^2)u = -u \leq 0, \qquad \Delta v + (\mu_2 v^2 + \beta u^2)v = -v \leq 0. \tag{3.2.9}$$

Therefore all solutions of (3.1.11) on $\mathcal{S}_k$ have two nonnegative components, i.e. for any $(\beta, u, v) \in \mathcal{S}_k$, $u \geq 0, v \geq 0$ in $\Omega$. Then by the Strong Maximum Principle, $u$ or $v$ is either strictly positive or identical to zero in $\Omega$.

Next, we exclude trivial solutions and semi-trivial solutions.

**Claim 1** $\mathcal{S}_k$ does not contain trivial solution $(0, 0)$. Otherwise if $(\hat{\beta}, 0, 0) \in \mathcal{S}_k$ for some $\hat{\beta}$, then $(\hat{\beta}, 0, 0)$ becomes a bifurcation point connecting $\mathcal{S}_k$ and trivial solution branch $\{(\beta, 0, 0) | \beta \in \mathbb{R}\}$.

Thus the linearized system at $(0,0)$

$$-\Delta\phi - \phi = 0, \qquad \Delta\psi - \psi = 0, \ \text{ in } \Omega, \quad \phi = \psi = 0 \ \text{ on } \partial\Omega,$$

has nonzero solution. Therefore 1 must be an eigenvalue of $(-\Delta, \Omega)$, and $\phi, \psi$ are corresponding eigenfunctions. Since the principal eigenvalue $\Lambda_1 < 1$, the eigenvalues corresponding to $\phi$ and $\psi$ are greater than or equal to $\Lambda_2$. Thus $\phi$ and $\psi$ are both sign-changing functions. But then the solutions of (3.1.11) on $\mathcal{S}_k$ and close enough to $(0,0)$ will have negative parts. This is a contradiction, so Claim 1 holds.

**Claim 2** $\mathcal{S}_k$ does not contain semi-trivial solutions $(U, 0)$ or $(0, V)$, where $U, V > 0$. It is enough to show that there is no solution of the form $(U, 0)$. We multiply the first equation of (3.1.11) by the first eigenfunction of $(-\Delta, \Omega)$ and integrate both sides of the equation over $\Omega$. Since the first eigenvalue $\Lambda_1 < 1$, a contradiction follows from the equations, if $U$ is positive. With the same argument, the branch does not contain semi-trivial solutions of the form $(0, V)$ either.

At last, combining the two claims, the lemma is proved. ■

In the sense of Rabinowitz's theorem [40], each global bifurcation branch is either unbounded, or contains multiple bifurcation points. In the following lemma, we use a result from ordinary differential equations to rule out the second possibility.

We call $\xi$ a zero of function $f$, if $f(\xi) = 0$ in the case $n = 1$, or if $f(\xi, \mathbf{0}) = 0$ in the cases $n = 2, 3$. Moreover, $\xi$ is a simple zero of $f$, if $x = \xi$ is of multiplicity 1.

**Lemma 3.2.7** *Assume that $n = 1$ or $\Omega$ is radial domain. Let $\beta_k$ be the $k$-th bifurcation parameter. Then for any solution $(u, v) \in \mathcal{S}_k$, the weighted difference $\sqrt{\mu_1 - \beta}u - \sqrt{\mu_2 - \beta}v$ has precisely $k - 1$ simple zeroes. In particular this property implies*

$$\overline{\mathcal{S}}_k \cap \mathcal{T}_\omega = \{(\beta_k, u_{\beta_k}, v_{\beta_k}) \mid f(\beta_k) = \lambda_k\}.$$

*And for each $k \geq k_0 + 1$, the above set contains only one bifurcation point.*

*Proof.* Fix $k \geq k_0 + 1$, then for $(\beta, u, v) \in \mathcal{S}_k$ close to $(\beta_k, u_{\beta_k}, v_{\beta_k})$ but $\beta \neq \beta_k$, we have

$$u = u_{\beta_k} + (\beta - \beta_k)\gamma_+(\beta_k)\psi_k + o(\beta - \beta_k) \quad \text{and} \quad v = v_{\beta_k} + (\beta - \beta_k)\psi_k + o(\beta - \beta_k).$$

Here $\psi_k$ is the $k$-th eigenfunction of (3.1.6) (c.f. [15]). Let $\alpha = [(\mu_1 - \beta)/(\mu_2 - \beta)]^{1/2}$, then the weighted difference

$$\alpha u - v = (\beta - \beta_k)(\alpha\gamma_+(\beta_k) - 1)\psi_k + o(\beta - \beta_k),$$

has precisely $k - 1$ simple zeroes provided $\beta$ is close to $\beta_k$. Here we use the facts that $\psi_k$ has precisely $k - 1$ simple zeroes [16] and that $\alpha\gamma_+(\beta_k) < 0$. Now $h = \alpha u - v$ solves, in radial coordinates, the equation

$$\begin{aligned} -h'' - \tfrac{N-1}{r}h' - h &= \alpha\mu_1 u^3 + \alpha\beta v^2 u - \mu_2 v^3 - \beta u^2 v \\ &= \big(\mu_1 u^2 + (\mu_1 - \beta)^{1/2}(\mu_2 - \beta)^{1/2} uv + \mu_2 v^2\big)h. \end{aligned}$$

This equation implies that all zeros of $h$ are simple. Otherwise the uniqueness of solution of the above equation yields $h \equiv 0$, hence $\alpha u = v$. Note that $(u, v)$ is positive solution of (3.1.11). By the uniqueness of positive solution of (3.1.2), we get $u = u_\beta, v = v_\beta$, which is a contradiction. Therefore $\alpha u - v$ has precisely $k - 1$ simple zeros for every

$$(\beta, u, v) \in \overline{\mathcal{S}_k} \backslash \{(\beta_k, u_{\beta_k}, v_{\beta_k})\}.$$

For different values of $k$, $\alpha u - v$ has different number of zeros. Thus two branches with different $k$ are disjoint. Moreover, the equation $f(\beta) = \lambda_k$ has a unique solution $\beta_k$ for each $k \geq k_0 + 1$. This one-to-one correspondence indicates that each bifurcation branch contains only one bifurcation point. The lemma is proved. ■

According to Rabinowitz's bifurcation theorem, Lemma 3.2.7 indicates that the global bifurcation branches must be unbounded in $\mathbb{R} \times \mathcal{H}$. Note that Lemma 3.2.7 is established independent of the different values of $\mu_1$ and $\mu_2$, so we will use it directly for other cases of $\mu_1$ and $\mu_2$.

In the next lemma, we want to show that each branch is bounded for $\beta$ contained in a compact set. Thus the bifurcation branches must be unbounded in the negative direction of $\beta$. Note that Theorem 3.2.1 implies that the bifurcation branches do not cross $\beta = 0$.

**Lemma 3.2.8** *Assume that $n = 1$ or $\Omega$ is radially symmetric. For fixed integer $k \geq k_0 + 1$, if there is a global bifurcation branch $\mathcal{S}_k$, then for $\beta$ contained in a compact subset of $I$,*

$$\{(\beta, u_\beta, v_\beta) \in \mathcal{S}_k | (\mu_1 - \beta)^{1/2} u_\beta - (\mu_2 - \beta)^{1/2} v_\beta \text{ has at most } k \text{ zeroes}, \ \beta \in I\}$$

*is bounded. Here the interval $I$ is defined in* (3.1.5).

*Proof.* Rewriting system (3.1.1) in radial variable $r = |x|$ for $n = 2$ or 3, we may assume that $u$ and $v$ are functions of $r$ with $r \in (a, b)$ and $0 \leq a < b < \infty$. If the conclusion is not true, then there exist an integer $k$ and a sequence $\{(\beta_m, u_m, v_m)\} \subset \mathcal{S}_k$ (radial, if $n \geq 2$) such that $\beta_m \to \beta \leq \mu_1$, $\|u_m\|_\infty \to \infty$ or $\|v_m\|_\infty \to \infty$ as $m \to \infty$. Set $\epsilon_m = \min\{\|u_m\|_\infty^{-1}, \|v_m\|_\infty^{-1}\}$ and choose $r_m$ such that $u_m(r_m) = \|u_m\|_\infty$ if $\|u_m\|_\infty \geq \|v_m\|_\infty$; otherwise, choose $r_m$ such that $v_m(r_m) = \|v_m\|_\infty$. Let

$$\tilde{u}_m(r) = \epsilon_m u_m(r_m + \epsilon_m r), \qquad \tilde{v}_m(r) = \epsilon_m v_m(r_m + \epsilon_m r).$$

Then $\{\tilde{u}_m\}_1^\infty, \{\tilde{v}_m\}_1^\infty$ are bounded in $L^\infty$ and satisfy the system

$$\begin{cases} -\tilde{u}_m'' - \dfrac{\epsilon_m(n-1)}{r_m + \epsilon_m r}\tilde{u}_n' - \epsilon_m^2 \tilde{u}_m = \mu_1 \tilde{u}_m^3 + \beta_m \tilde{v}_m^2 \tilde{u}_m, \\ -\tilde{v}_m'' - \dfrac{\epsilon_m(n-1)}{r_m + \epsilon_m r}\tilde{v}_n' - \epsilon_m^2 \tilde{v}_m = \mu_2 \tilde{v}_m^3 + \beta_m \tilde{u}_m^2 \tilde{v}_m, \end{cases} \tag{3.2.10}$$

on the scaled domain $(a - r_m)\epsilon_m^{-1} < r < (b - r_m)\epsilon_m^{-1}$. Along a subsequence, $(\tilde{u}_m, \tilde{v}_m)$ converge in $C^2_{loc}$ as $m \to \infty$ towards a solution $(u, v)$ of the following system

$$\begin{cases} -u'' - \dfrac{n-1}{c+r}u' = \mu_1 u^3 + \beta v^2 u, \\ -v'' - \dfrac{n-1}{c+r}v' = \mu_2 v^3 + \beta u^2 v, \\ \qquad u, v \geq 0. \end{cases} \tag{3.2.11}$$

Here $c \geq 0$ and $r > -c$. Set $a_m = \dfrac{a - r_m}{\epsilon_m}, b_m = \dfrac{b - r_m}{\epsilon_m}$, then $a_m, b_m$ converge in $[-\infty, \infty]$ and $u, v$ may defined in one of the following intervals: $(-c, \infty), (-\infty, c)$. By Lemma 3.2.6, $u > 0$ and $v > 0$ in $\Omega$. Since $0 < \mu_1 \leq \mu_2$, Theorem 2.6 [8] yields the fact that $(\mu_1 - \beta)^{1/2}u - (\mu_2 - \beta)^{1/2}v$ has infinitely many zeroes. This is a contradiction, since $(\mu_1 - \beta_m)^{1/2}u_m - (\mu_2 - \beta_m)^{1/2}v_m$ has at most $k$ zeroes for every $m$. Therefore $\mathcal{S}_k$ is bounded for $\beta$ in any compact subset of $I$. ∎

Combining Theorem 3.2.5, Lemma 3.2.6, Lemma 3.2.7 and Lemma 3.2.8, we get the global bifurcation diagram Figure 3.2 in the case $n = 1$ or $\Omega$ is radially symmetric.

**Theorem 3.2.9** *There is a global bifurcation branch of system* (3.1.1) *for every* $k \geq k_0 + 1$. *All these branches are unbounded in the negative direction of* $\beta$. *The bifurcation diagram is shown in Figure 3.2.*

**Remark 3.2.10** *Different from the other cases (b)-(d), there is no semi-trivial solution branches in this case.*

### 3.3 The self-defocusing case

In the previous section, we establish the local and global bifurcation results for the self-focusing case. In this section, we study the self-defocusing case, i.e. $\mu_1 \leq \mu_2 < 0$, which is an assumption made through out this section.

Recall the definition of $k_0$, which is determined by $\lambda_{k_0} \leq 1 < \lambda_{k_0+1}$. For simplicity, we assume $k_0 < 1$. Otherwise, we replace $k_0$ by the largest integer $\tilde{k}$ such that $\lambda_{\tilde{k}} < 1$.

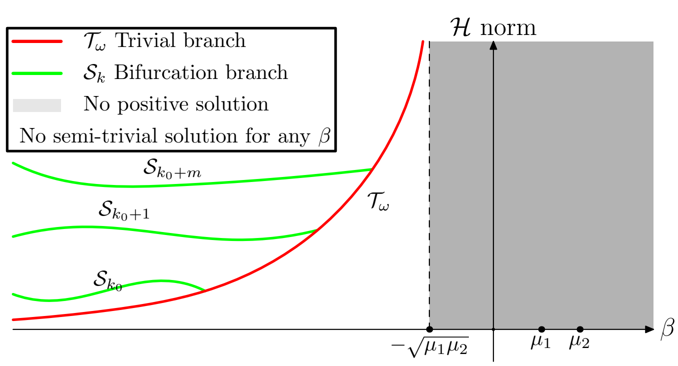



Figure 3.2: Bifurcation diagram in the case $0 < \mu_1 < \mu_2$

### 3.3.1 Nonexistence of positive solutions

In this case, we have the nonexistence result of positive solution of system (3.1.10) for $\beta$ in a finite interval.

**Theorem 3.3.1** *Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^n$ $(n \le 3)$. If $\mu_1 \le \beta \le \mu_2$, $b \ge a$ and at least one of the three equalities holds strictly, then* (3.1.10) *has no positive solution.*

*Proof.* Multiplying the first equation of (3.1.10) by $v$, then integrating over $\Omega$, we get

$$\int_\Omega (\nabla u \nabla v - auv)dx = \int_\Omega (\mu_1 u^3 v + \beta u v^3)dx. \tag{3.3.1}$$

Similarly, multiplying the second equation by $u$ and integrating over $\Omega$,

$$\int_\Omega (\nabla u \nabla v - buv)dx = \int_\Omega (\mu_2 u v^3 + \beta u^3 v)dx. \tag{3.3.2}$$

Subtracting (3.3.2) from (3.3.1), one obtains

$$0 \le (b-a)\int_\Omega uvdx = (\mu_1 - \beta)\int_\Omega u^3 vdx + (\beta - \mu_2)\int_\Omega uv^3 dx \le 0.$$

Since one or both of the above inequalities will hold strictly, if one or more inequalities in the assumptions hold strictly, we obtain a contraction if system (3.1.10) has solutions with two positive components. The theorem is proved. ■

### 3.3.2 Local bifurcations

Next, we study the local bifurcations with respect to $\mathcal{T}_\omega$.

**Lemma 3.3.2** *All the possible bifurcation parameters of* (3.1.11) *are determined from the following equations about $\beta$:*

$$f(\beta) = \lambda_k, \qquad k = 1, \cdots, k_0,$$

*where $f$ is defined as* (3.1.8) *and $\lambda_k$ is the $k$-th eigenvalue of* (3.1.6)*. Let $\beta_k$ be a solution of*

*$f(\beta) = \lambda_k$, and let $V_k$ be the kernel space of the linearization of* (3.1.11) *at* $(\beta_k, u_{\beta_k}, v_{\beta_k})$*. Then* $\dim V_k = n_k$*, where $n_k$ is the multiplicity of $\lambda_k$.*

*Proof.* The proof is similar to the proof of Lemma 3.2.2. Actually, we only need to consider the behavior of function $f$. In this case, $f$ is a monotone decreasing homomorphism from $I = (-\infty, \mu_1)$ to $(-\infty, 1)$, as shown in Figure 3.3. Comparing with (3.1.7), there are finite many possible bifurcation parameters determined from $f(\beta) = \lambda_k$ with $1 \leq k \leq k_0$. It is worth to mention that since the smallest eigenvalue of (3.1.6) is $-1$, there is no bifurcation points for $\beta$ on the lower branch of graph of $f$ other than $\mu_2$. Linearized system (3.1.11) at $(\beta_k, u_{\beta_k}, v_{\beta_k})$ is (3.2.1), and its kernel space can be represented as (3.2.5). Thus $\dim V_k = n_k$. ■

**Theorem 3.3.3** *The trivial solution branch $\mathcal{T}_\omega$ exists on $(-\infty, \sqrt{\mu_1\mu_2}) \backslash (\mu_1, \mu_2)$. There are finitely many local bifurcations of* (3.1.11) *along $\mathcal{T}_\omega$ with $1 \leq k \leq k_0$. And they are also local bifurcations of* (3.1.1)*, except at $k = 1$.*

*Proof.* By Lemma 3.3.2, there are $k_0 + 1$ possible bifurcation parameters, and $\mu_1$, $\mu_2$ are two of them. Actually, they are both corresponding to equation $f(\beta) = \lambda_1$. It is easy to see that when $\mu_1 < \mu_2$ there are two semi-trivial branches,

$$\mathcal{T}_1 = \{(\beta, \omega/\sqrt{-\mu_1}, 0) : \beta \in \mathbb{R}\} \quad \text{and} \quad \mathcal{T}_2 = \{(\beta, 0, \omega/\sqrt{-\mu_2}) : \beta \in \mathbb{R}\}.$$

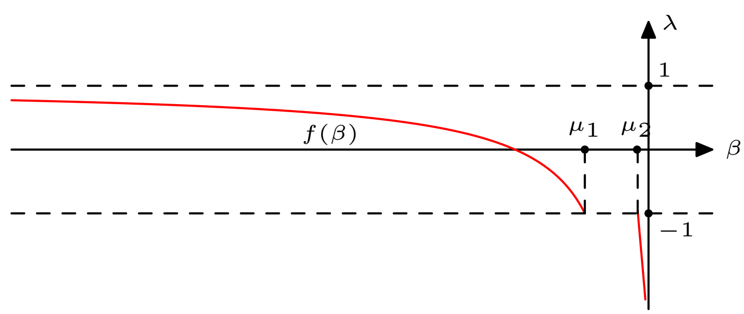



Figure 3.3: Graph of $f$ in the case $\mu_1 < \mu_2 < 0$

They meet with $\mathcal{T}_\omega$ at $(\mu_1, \omega/\sqrt{-\mu_1}, 0)$ and $(\mu_2, 0, \omega/\sqrt{-\mu_2})$, respectively. If $\mu_1 = \mu_2 =: \mu$, then the two semi-trivial solution branches $\mathcal{T}_1$ and $\mathcal{T}_2$ are connected by line segment

$$\mathcal{T}_\theta = \left\{ \left( \mu, \frac{\omega}{\sqrt{-\mu}} \cos\theta, \frac{\omega}{\sqrt{-\mu}} \sin\theta \right) : \theta \in \left(0, \frac{\pi}{2}\right) \right\} \subset \mathbb{R} \times \mathcal{H}.$$

$\mathcal{T}_\theta$ intersects $\mathcal{T}_\omega$ at $\theta = \pi/4$. See [18] for more detailed discussion.

Now we consider the other parameters, i.e. $\{\beta_k\}_2^k$. According to Lemma 3.2.3, we only need to check $g(\beta_k) \neq 0$ for $2 \leq k \leq k_0$, then $\{\beta_k\}_2^k$ are indeed bifurcation parameters. Lemma 3.3.2 implies that all these parameters are contained in interval $(-\infty, \mu_1)$. Direct calculation shows

$$g'(\beta) = -12(\mu_1 + \mu_2)\beta^2 + 4(\mu_1^2 + 10\mu_1\mu_2 + \mu_2^2)\beta - 12\mu_1\mu_2(\mu_1 + \mu_2) > 0,$$

for all $\beta < \mu_1$. Thus $g$ strictly increasing on $(-\infty, \mu_1)$. On the other hand, evaluating $g$ at the right endpoint, we get $g(\mu_1) = -2\mu_1(\mu_1 - \mu_2)^3 < 0$. So $g(\beta_k) < 0$ for $2 \leq k \leq k_0$. Therefore $\{\beta_k\}_2^k$ are bifurcation parameters of (3.1.11). Apply Lemma 3.2.4, $\{\beta_k\}_2^k$ are bifurcation parameters of system (3.1.1). ∎

### 3.3.3 Global bifurcations

When $n_k$ is odd, local bifurcations become global bifurcations, in particular when $n = 1$ or $\Omega$ is radially symmetric. In addition, we hope to prove that all global bifurcation branches are unbounded in the negative direction of $\beta$. So we need to verify Lemma 3.2.6 and Lemma 3.2.8 for $\mu_1 \leq \mu_2 < 0$.

**Lemma 3.3.4** *Let $\beta_k$ be a bifurcation parameter for any $2 \leq k \leq k_0$. If there exists a connected bifurcation branch $\mathcal{S}_k$ emanating from $\mathcal{T}_\omega$ at $(\beta_k, u_{\beta_k}, v_{\beta_k})$, then all solution pairs $(u, v)$ on $\mathcal{S}_k$ are strictly positive in $\Omega$.*

*Proof.* First, the Strong Maximum Principle implies that all solution on bifurcation branches are nonnegative. This can be shown by using the same arguments in the proof of Lemma 3.2.6. Then we need to exclude the trivial solution and semi-trivial solutions. It is equivalent to establish the two claims in the proof of Lemma 3.2.6. The first claim, which rules out the trivial solution, is independent of the values of $\mu_1$ and $\mu_2$, so the arguments used in Lemma 3.2.6 are still applicable in this case. The second Claim is used to remove the possibility of semi-trivial solutions, which requires some work.

It is enough to show that $\mathcal{S}_k$ does not contain semi-trivial solution in the form $(U, 0)$, and the possibility of semi-trivial solution in the form $(0, V)$ can be removed in the same way. Assume that $\mathcal{S}_k$ contains $(U, 0)$. The linearized system at $(U, 0)$

$$\begin{cases} -\Delta\phi - \phi = 3\mu_1 U^2\phi, \\ -\Delta\psi - \psi = \beta U^2\psi, \end{cases}$$

should have nontrivial kernel space. Using the first equation of (3.1.11) and the uniqueness of positive solution of (3.1.2), we get $U = (\sqrt{-1/\mu_1})\omega$, and the above two equations become

$$-\Delta\phi - \phi = -3\omega^2\phi, \qquad -\Delta\psi - \psi = -\frac{\beta}{\mu_1}\omega^2\psi. \tag{3.3.3}$$

The special form of $U$ implies that $(U, 0) \in \mathcal{T}_1$, which is the semi-trivial solution branch defined in Theorem 3.3.3. Since the principal eigenvalue of (3.1.6) is $-1$, from the first equation of (3.3.3) we get $\phi = 0$, and from the second equation of (3.3.3) we get $\beta \geq \mu_1$. Since $\mu_1 \leq \beta < 0$, the above system has nonzero solution only if $\beta/\mu_1 = -\lambda_k$ for $1 < k \leq k_0$. But the corresponding solution to the second equation will be a sign-changing function. For $(u, v) \in \mathcal{S}_k$ close to $(U, 0)$, $v$ can be viewed as a small perturbation of 0 that is dominated by $\psi$ (see the expansion in Lemma 3.2.7), thus it is also a sign-changing function. This contradicts with the fact that all solutions of (3.1.11) on $\mathcal{S}_k$ are nonnegative. Therefore $\mathcal{S}_k$ does not contain semi-trivial solution $(U, 0)$.

Follows from the two Claims, all solutions on the global bifurcation branches are positive. ■

Using Lemma 3.2.7, we can rule out the possibility that $\mathcal{S}_k$ contains multiple bifurcation points. Hence each global bifurcation branches must be unbounded. Now we show that $\mathcal{S}_k$ can only be unbounded in the negative direction of $\beta$.

**Lemma 3.3.5** *Assume that $n = 1$ or $\Omega$ is radially symmetric. For fixed integer $2 \leq k \leq k_0$, if* *there is a global bifurcation branch* $\mathcal{S}_k$*, then for* $\beta$ *contained in a compact subset* $B \subset I$*, which is defined in* (3.1.5)*, the following set*

$$\{(\beta, u_\beta, v_\beta) \in \mathcal{S}_k | (\mu_1 - \beta)^{1/2} u_\beta - (\mu_2 - \beta)^{1/2} v_\beta \text{ has at most } k \text{ zeroes}, \ \beta \in B\},$$

*is bounded in* $\mathbb{R} \times \mathcal{H}$.

*Proof.* Prove by contradiction. With the same contradiction assumptions in Lemma 3.2.8, we arrive at the limiting system (3.2.11). Since $\mu_1 \le \mu_2 < 0$, the Strong Maximum Principle implies $u \equiv 0$ and $v \equiv 0$. Actually, for every $m$, $\tilde{u}_m(a_m) = 0, \tilde{u}_m(b_m) = 0$, $\tilde{v}_m(a_m) = 0, \tilde{v}_m(b_m) = 0$. Without loss of generality, assume $\|u_m\|_\infty \ge \|v_m\|_\infty$, then $\tilde{u}_m$ achieves its maximum at 0, i.e. $\tilde{u}_m(0) = 1$ and $\tilde{u}_m'(0) = 0$. We consider the case that limiting interval is $(-c, \infty)$, and the other case $(-\infty, c)$ is similar. If $a_m \to -c < 0$, then

$$u(-c) = \lim_{m \to \infty} \tilde{u}_m(a_m) = 0.$$

Notice that the limiting system (3.2.11) implies

$$u'' + \frac{n-1}{c+r} u' \ge 0,$$

then by the Strong Maximum Principle and the assumption $\tilde{u}_m(a_m) = 0$ for any $m$, we get $u \equiv 0$, which is a contradiction since $u(0) = 1$. If $a_m \to 0$, the $C^2_{loc}$ convergence of implies $u'(0) = 0$. But we also have $u''(0) > 0$, which implies that $u$ achieves strict local minimum at $v = 0$, a contradiction. Thus the conclusion holds for the self-defocusing case. ∎

With all lemmas in hand, we can describe the global bifurcations for $\mu_1 \le \mu_2 < 0$. There are two semi-trivial bifurcation branches $\mathcal{T}_1$ and $\mathcal{T}_2$, which are defined in the same way as we discussed in Theorem 3.3.3. Combining Theorem 3.3.3, Lemma 3.3.4, Lemma 3.3.5 and Lemma 3.2.7, we obtain the following bifurcation theorem in the defocusing case.

**Theorem 3.3.6** *Assume that* $n = 1$ *or* $\Omega$ *is radially symmetric. Also assume* $2 \le k \le k_0$.

*(i) If $\mu_1 < \mu_2 < 0$, the trivial solution branch $\mathcal{T}_\omega$ exists on two separated intervals of $\beta$: $(-\infty, \mu_1] \cup [\mu_2, \sqrt{\mu_1 \mu_2})$. On the interval $(-\infty, \mu_1)$, there are $k_0 - 1$ bifurcation parameters, with $2 \leq k \leq k_0$. Each of them gives rise to a unbounded bifurcation branch of (3.1.1) when $n = 1$ or $\Omega$ is radially symmetric.*

*(ii) If $\mu_1 = \mu_2 < 0$, $\mathcal{T}_\omega$ is defined on $(-\infty, \mu_1)$, and there are $k_0 - 1$ local bifurcation parameters, which give $k_0 - 1$ unbounded global bifurcation branches when $n = 1$ or $\Omega$ is radially symmetric.*

The global bifurcation diagram in this case is shown in Figure 3.4

## 3.4 The two mixed cases

In this section we discuss the mixed case, i.e. $\mu_1 < 0 < \mu_2$. Here we distinguish two sub-cases: $-\mu_1 \geq \mu_2$ and $-\mu_1 < \mu_2$. Most of the results for these two sub-cases can be established with the same arguments, whereas, the latter case is more complicated and a new phenomenon appears. In this section, assume $\lambda_{k_0} \leq 1 < \lambda_{k_0+1}$.

We still start with a nonexistence theorem.

### 3.4.1 Nonexistence of positive solutions

**Theorem 3.4.1** *Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^n$ $(n \leq 3)$. If $\beta \geq \mu_1$, $b \geq a \geq \Lambda_1$ and at least one of the three inequalities holds strictly, then* (3.1.11) *has no positive solution.*

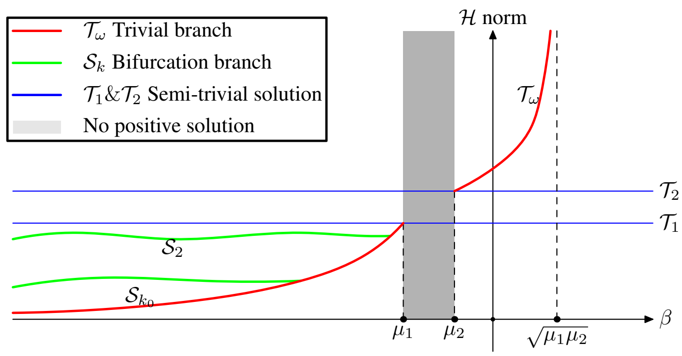



Figure 3.4: Bifurcation diagram in the case $\mu_1 < \mu_2 < 0$

*Proof.* If $\mu_1 \leq \beta \leq \mu_2$, the nonexistence of positive solution follows from the same argument as Theorem 3.3.1. If $\beta > \mu_2$, multiplying the second equation by $\phi_1$ and integrating over $\Omega$, we get

$$0 \geq (\Lambda_1 - b) \int_\Omega \phi_1 v dx \geq \mu_2 \int_\Omega v^3 \phi_1 dx \geq 0.$$

If any of the three inequalities in the assumption holds strictly, then there will be a contradiction, unless $v = 0$. Thus (3.1.10) has no positive solution when $\beta \geq \mu_1$. ∎

### 3.4.2 Local bifurcations

For fixed $\mu_1$ and $\mu_2$, there are finitely many local bifurcation parameters determined from the next lemma.

**Lemma 3.4.2** *All the possible bifurcation parameters of* (3.1.11) *are determined from the following equations about $\beta$:*

$$f(\beta) = \lambda_k, \qquad k \geq 1,$$

*where $f$ is defined as* (3.1.8) *and $\lambda_k$ is the $k$-th eigenvalue of* (3.1.6)*. Let $V_k$ be the kernel space of the linearization of* (3.1.11) *at $(\beta_k, u_{\beta_k}, v_{\beta_k})$. Then* $\dim V_k = n_k$.

*Proof.* The derivation of equations $f(\beta) = \lambda_k$, including linearization of (3.1.11) and then the reduction, is the same as the proof of Lemma 3.2.2. Therefore we only need to study the behavior of $f$ and compare with the sequence of eigenvalues (3.1.7), then the location and quantity of possible bifurcation parameters can be determined.

If $-\mu_1 > \mu_2 > 0$, it is easy to see from the expressions of $f$ and $f'$ that

$$f \text{ is strictly decreasing on } (-\infty, \mu_1), \quad \lim_{\beta \to \mu_1^-} f(\beta) = -1, \quad \lim_{\beta \to -\infty} f(\beta) = 1.$$

The graph of $f$ is shown in Figure 3.5. Therefore, the equation $f(\beta) = \lambda_k$ has unique solution for each $1 \leq k \leq k_0$ and has no solution if $k \geq k_0 + 1$.

If $-\mu_1 < \mu_2$, then there are possibly more bifurcation parameters. Define a constant $\xi$ as

$$\xi := \frac{2\mu_1\mu_2 + (\mu_1 - \mu_2)\sqrt{-\mu_1\mu_2}}{\mu_1 + \mu_2}.$$

It is easy to see that

$$\xi < \frac{2\mu_1\mu_2}{\mu_1 + \mu_2} < \mu_1, \quad f(\frac{2\mu_1\mu_2}{\mu_1 + \mu_2}) = 1, \quad f(\mu_1) = -1,$$

and

$$f'(x) > 0, \quad \text{on } (-\infty, \xi), \quad f'(x) < 0, \text{ on } (\xi, \mu_1), \quad \lim_{\beta \to -\infty} f(x) = 1.$$

Denote $I_1 = [\frac{2\mu_1\mu_2}{\mu_1+\mu_2}, \mu_1]$ and $I_2 = (-\infty, \frac{2\mu_1\mu_2}{\mu_1+\mu_2})$. On $I_1$, there exist $k_0$ bifurcation parameters

$$\mu_1 = \beta_1 > \beta_2 > \cdots > \beta_{k_0},$$

which correspond to $\lambda_1, \lambda_2, \ldots, \lambda_{k_0}$ respectively. On $I_2$, $f$ is concave down and achieves its global maximum at $\xi$. Moreover, $f$ approaches 1 as $\beta \to -\infty$ and as $\beta$ approaches the right endpoint $\frac{2\mu_1\mu_2}{\mu_1+\mu_2}$. Each $\lambda_k$ with $\lambda_k < f(\xi)$ and $k > k_0$ gives rise to two possible bifurcation parameters. The graph of $f$ is shown in Figure 3.6. Comparing with (3.1.7), there are at least $k_0$ possible bifurcation parameters on $I_1$, and possibly more on $I_2$.

Linearized system (3.1.11) at $(\beta_k, u_{\beta_k}, v_{\beta_k})$ is (3.2.1), and its kernel space can be expressed as (3.2.5). Thus $\dim V_k = n_k$. ■

**Remark 3.4.3** *In the case $\mu_2 > -\mu_1 > 0$, there may be more than $k_0$ global bifurcation*

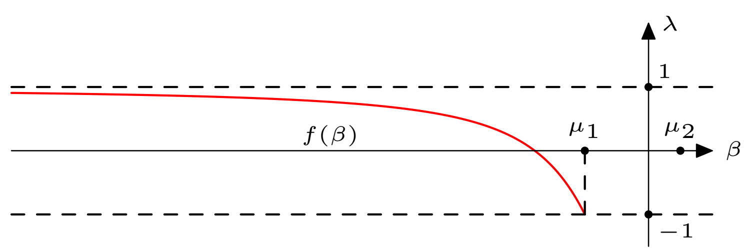



Figure 3.5: Graph of $f$ in the case $\mu_1 \leq -\mu_2 < 0 < \mu_2$

*branches. Define* $\xi$ *in the same way as the proof as Lemma 3.4.2, then it is easy to see* $f(\xi) > 1$*. If there exists* $m \geq 1$ *such that* $f(\xi) > \lambda_{k_0+m}$*, then for each* $i = 1, ..., m$ *there exist exactly two values of* $\beta$ *such that* $f(\beta) = \lambda_{k_0+i}$*. We denote them by* $\beta_{k_0+i}$ *and* $\beta'_{k_0+i}$*, which satisfy*

$$\beta'_{k_0+1} > \beta'_{k_0+2} > ... > \beta'_{k_0+m} > \xi > \beta_{k_0+m} > ... > \beta_{k_0+1}.$$

*This is also a sequence of possible bifurcation parameters.*

*Furthermore,* $f$ *achieves its maximum at* $\xi$ *and the maximum value is*

$$\begin{aligned} f_{max} &= \frac{5\mu_1^2\mu_2^2 - 2\mu_1^3\sqrt{-\mu_1\mu_2} + 2\mu_2^3\sqrt{-\mu_1\mu_2}}{\mu_1^2\mu_2^2 + 2\mu_1^2\mu_2\sqrt{-\mu_1\mu_2} - 2\mu_1\mu_2^2\sqrt{-\mu_1\mu_2} - 2\mu_1^3\mu_2 - 2\mu_1\mu_2^3} \\ &= \frac{5t^2 + 2\sqrt{-t} - 2t^3\sqrt{-t}}{t^2 - 2t\sqrt{-t} + 2t^2\sqrt{-t} - 2t - 2t^3} \qquad (t = \mu_2/\mu_1) \\ &= \frac{5s^4 + 2s + 2s^7}{s^4 + 2s^3 + 2s^5 + 2s^2 + 2s^6} \qquad (s = \sqrt{-t}). \end{aligned}$$

*It is easy to see* $f_{max} \to \infty$ *as* $s \to \infty$*. Therefore there are more and more bifurcation parameters can be solved from* $f(\beta) = \lambda_k$ *as* $\sqrt{-\mu_2/\mu_1} \to \infty$*.*

**Theorem 3.4.4** *The trivial solution branch* $\mathcal{T}_\omega$ *exists on* $(-\infty, \mu_1)$*. There is one semi-trivial solution branch* $\mathcal{T}_1 = \{(\beta, \omega/\sqrt{-\mu_1}, 0) : \beta \in \mathbb{R}\}$ *emanating from* $\mathcal{T}_\omega$*. And*

*(i) if* $-\mu_1 > \mu_2$*, there are* $k_0 - 1$ *local bifurcations with* $2 \leq k \leq k_0$*;*

*(ii) if* $-\mu_1 < \mu_2$*, denote* $I_1 = [\frac{2\mu_1\mu_2}{\mu_1\mu_2}, \mu_1]$ *and* $I_2 = (-\infty, \frac{2\mu_1\mu_2}{\mu_1\mu_2})$*. Then on* $I_1$*, there exist* $k_0$

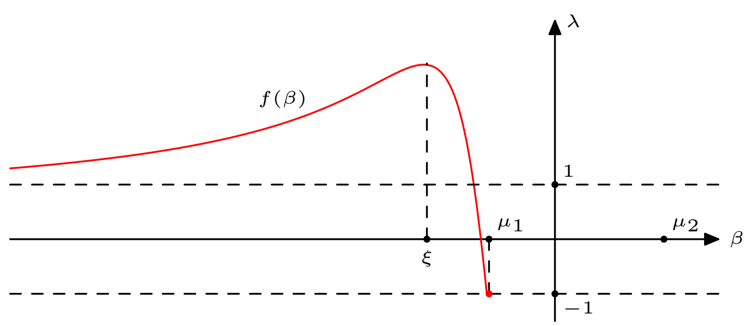



Figure 3.6: Graph of $f$ in the case $\mu_1 < 0 < -\mu_1 < \mu_2$

*bifurcation parameters (including $\mu_1$). On $I_2$, there exist $m$ pairs of bifurcation parameters, where the $m$ pairs of bifurcation points are characterized in Remark 3.4.3, provided $f(\xi) > \lambda_{k_0+m}$.*

*Proof.* Direct calculation verifies the existence of the semi-trivial bifurcation branch $\mathcal{T}_1$. Now we focus on nontrivial bifurcations.

According to Lemma 3.2.3, we need to check $g(\beta_k) \neq 0$ for all $\beta_k$ solved from Lemma 3.4.2 with $k \geq 2$. When $\mu_1 < 0 < \mu_2$, $g$ is a polynomial of degree 3 with negative leading coefficient and

$$g(\beta) = 2(\mu_1 + \mu_2 - 2\beta)\left[(\mu_1 + \mu_2)\beta^2 - 4\mu_1\mu_2\beta + \mu_1\mu_2(\mu_1 + \mu_2)\right].$$

A direct calculation shows that $g(\mu_1) < 0$ and $\xi$ is the only negative zero of $g$. Thus $g$ is negative on $(\xi, \mu_1]$ and positive on $(-\infty, \xi)$.

In both cases (i) and (ii), the bifurcation parameters $\{\beta_k\}_2^{k_0} \subset (\xi, \mu_1]$, thus $g(\beta_k) \neq 0$ and they are bifurcation parameters by using Lemma 3.2.3. In the subcase (ii), $g$ is not equal to zero except possibly at $\xi$. Thus there are another $m$ pairs of bifurcation parameters in this case, provided $f(\xi) > \lambda_{k_0+m}$.

At last, the positivity of solutions follows from Lemma 3.2.4. ∎

**Remark 3.4.5** *If $f(\xi) = \lambda_{k_\xi}$ for some $k_\xi > k_0$, then this $\lambda_{k_\xi}$ is a possible bifurcation parameter. But it cannot be verified by using Lemma 3.2.2 since $g(\xi) = 0$.*

### 3.4.3 Global bifurcations

When $N = 1$ or $\Omega$ is radially symmetric, all bifurcation parameters give rise to global bifurcation branches.

**Lemma 3.4.6** *Let $\beta_k$ be a bifurcation parameter and $k \geq 2$. If there exists a connected bifurcation branch $\mathcal{S}_k$ emanating from $\mathcal{T}_\omega$ at $(\beta_k, u_{\beta_k}, v_{\beta_k})$, then all solution pairs $(u, v)$ on $\mathcal{S}_k$ are strictly positive in $\Omega$.*

*Proof.* By the same arguments used in Lemma 3.2.6, we can show that all solutions on bifurcation branch $\mathcal{S}_k$, $k \geq 2$ are nonnegative. Now, we only need to verify the two claims given in Lemma 3.2.6. The proof of Claim I does not require the assumptions on $\mu_1$ and $\mu_2$, then it also holds for this case.

Consider Claim II, i.e. no semi-trivial solution on $\mathcal{S}_k$. First, we can rule out semi-trivial solution in the form $(U, 0)$ by using the same argument in the proof of Lemma 3.3.4. Second, system (3.1.11) does not have semi-trivial solution in the form $(0, V)$ either. Otherwise, multiplying the second equation of (3.1.11) by the first eigenfunction of $(-\Delta, \Omega)$ and integrating over $\Omega$, the left-hand side of the equation is negative but the right-hand side is positive. A contradiction. Thus $\mathcal{S}_k$ will not meet any semi-trivial solution.

The lemma follows from the two claims. ∎

According to Lemma 3.2.7, $\mathcal{S}_k$ does not contain multiple bifurcation points, therefore each global bifurcation branches must be unbounded. We now prove a lemma to show that $\mathcal{S}_k$ can only be unbounded in the negative direction of $\beta$.

**Lemma 3.4.7** *Assume that $n = 1$ or $\Omega$ is radially symmetric. For fixed integer $k \geq 2$, if there is a global bifurcation branch $\mathcal{S}_k$, then for $\beta$ contained in a compact subset $B \subset I$, which is defined in* (3.1.5),

$$\{(\beta, u_\beta, v_\beta) \in \mathcal{S}_k | (\mu_1 - \beta)^{1/2} u_\beta - (\mu_2 - \beta)^{1/2} v_\beta \text{ has at most } k \text{ zeroes}, \ \beta \in B\}$$

*is bounded.*

*Proof.* First, we can use the same blow-up assumptions and obtain the limiting system (3.2.11). If $\|v_m\|_\infty \leq \|u_m\|_\infty$, applying the Strong Maximum Principle to the first equation, we get $u \equiv 0$ then $v \equiv 0$, which is a contradiction. If $\|v_m\|_\infty \geq \|u_m\|_\infty$, then $u \equiv 0$, $v(0) = 1$ and

$$-v'' - \frac{N-1}{c+r} v' = \mu_2 v^3.$$

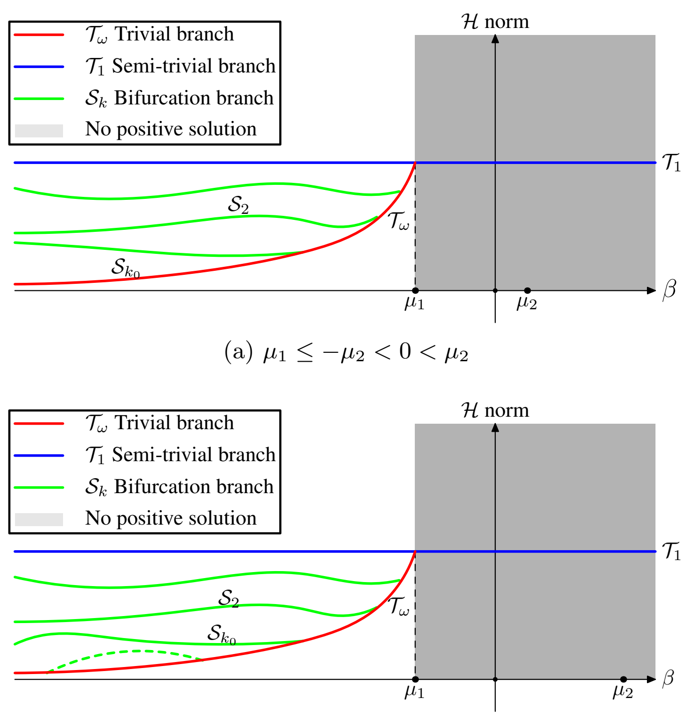
The above equation can be viewed as system (3.2.11) with $u \equiv 0$ and $v \geq 0$. Then according to Theorem 2.6 [8], $\sqrt{\mu_1 - \beta}u - \sqrt{\mu_2 - \beta}v$ has infinity many zeros. By the Strong Maximum Principle, we have $v \equiv 0$, which contradicts with $v(0) = 1$.

Therefore $\mathcal{S}_k$ is bounded for $\beta$ in any compact subset of $(-\infty, \mu_1)$. ∎

Now, we summary the global bifurcations of case (c) and (d) in the following theorems. Figure 3.7 illustrate the behaviors of bifurcation branches in the two mixed cases.

**Theorem 3.4.8** *Assume $n = 1$ or $\Omega$ is radially symmetric. If $\mu_1 < 0 < \mu_2$, there are $k_0$ global bifurcation branches of* (3.1.11)*. They are all unbounded in the negative direction of $\beta$. Except the semi-trivial bifurcation branch $\mathcal{T}_1$, the other bifurcation branches are also bifurcation branches of* (3.1.1)*.*

*Proof.* It follows from Theorem 3.4.4, Lemma 3.2.7 and Lemma 3.4.7. ∎




(a) $\mu_1 \leq -\mu_2 < 0 < \mu_2$



(b) $\mu_1 < 0 < -\mu_1 < \mu_2$

Figure 3.7: Bifurcation schematic diagrams in the case $\mu_1 < 0 < \mu_2$.

**Theorem 3.4.9** *Assume that $n = 1$ or $\Omega$ is radially symmetric. In the case (d) if $\lambda_{k_0+m+1} \geq f(\xi) > \lambda_{k_0+m}$ for some $m \geq 1$, then for each $i = 1, ..., m$, $\beta_{k_0+i}$ (resp. $\beta'_{k_0+i}$) is a bifurcation parameter and gives rise to a global bifurcation branch $\mathcal{S}_{k_0+i}$ (resp. $\mathcal{S}'_{k_0+i}$) with respect to $\mathcal{T}_\omega$. For $k = k_0 + 1, ..., k_0 + m$, any solution $(u, v) \in \mathcal{S}_k$ (or $\mathcal{S}'_k$) satisfies that $\sqrt{\mu_1 - \beta}u - \sqrt{\mu_2 - \beta}v$ has precisely $k - 1$ simple zeroes, and either $\mathcal{S}_k$ and $\mathcal{S}'_k$ are both unbounded in negative direction $\beta$ or they are actually the same bifurcation branch.*

*Proof.* According to Theorem 3.4.8, there are $m$ pairs of bifurcation parameters if $\lambda_{k_0+m+1} \geq f(\xi) > \lambda_{k_0+m}$ for some $m \geq 1$. If $n = 1$ or $\Omega$ is radially symmetric, then there are global bifurcation branches emanating from each of these bifurcation point. But different from the bifurcation parameters $\beta_k$ with $2 \leq k \leq k_0$, Lemma 3.2.7 cannot distinguish the global bifurcation branches emanating from $\beta_{k_0+i}$ and $\beta'_{k_0+i}$. Thus there may be bounded bifurcation branch connecting $\beta_{k_0+i}$ and $\beta'_{k_0+i}$.

If there is a method that can be used to distinguish $\mathcal{S}_{k_0+i}$ and $\mathcal{S}'_{k_0+i}$, then these two branches are unbounded in the negative direction of $\beta$ by the same arguments in the proof of Theorem 3.4.8. ∎

### 3.5 Summary

In this chapter, we use the bifurcation method to find multiple solutions of (3.1.1). The existence of trivial solution branch $\mathcal{T}_\omega$ is derived from the known solution $\omega$ of scalar equation (3.1.2). Note that the non-degeneracy of $\omega$ plays an important role in reducing the linearized system (3.2.1) to one equation. Then for the reduced equation, the sequence of eigenvalues of (3.1.7) induce the bifurcation parameters of (3.1.11) through an auxiliary function of $\beta$ with parameters $\mu_1$ and $\mu_2$. In verifying the bifurcations and the positivity of global bifurcation branches, the proofs are different for the cases characterized by $\mu_1$ and $\mu_2$. In the last case, $\mu_1 < 0 < -\mu_1 < \mu_2$, the boundedness of some global bifurcation branches cannot be determined by using Lemma 3.2.7.

Comparing with Chapter 2, this method requires less symmetry for the system. But, since this method relies on the reduction of linearized system to a scalar equation, it is hard to generalize the results to system with more than two components. And the global bifurcation results rely on the assumptions: $n = 1$ or $\Omega$ is radial.

### 3.6 Future work

Let $V : \mathbb{R}^n \to \mathbb{R}^+$ be a function satisfying

**(V)** $\inf_{\mathbb{R}^n} V(x) > 0$ and $V(x) \to \infty$ as $|x| \to \infty$.

Then consider the following system

$$\begin{cases} -\Delta u + V(x)u - \lambda u = \mu_1 u^3 + \beta u v^2 & \text{in } \mathbb{R}^n, \\ -\Delta v + V(x)v - \lambda v = \mu_2 v^3 + \beta v u^2 & \text{in } \mathbb{R}^n, \\ u, v > 0 \text{ in } \mathbb{R}^n, & u, v \in H_V^1(\mathbb{R}^n), \end{cases} \tag{3.6.1}$$

where $\lambda$ is a positive constant, $\mu_1, \mu_2, \beta \in \mathbb{R}$ and

$$H_V^1(\mathbb{R}^n) = \left\{ u \in H^1(\mathbb{R}^n) | \int_{\mathbb{R}^n} V(x) u^2 dx < \infty \right\}.$$

Denote the sequence of eigenvalues of $-\Delta + V(x)$ on $\mathbb{R}^n$ with Dirichlet boundary condition by

$$0 < \Lambda_1(V, \mathbb{R}^n) < \Lambda_2(V, \mathbb{R}^n) \leq \Lambda_3(V, \mathbb{R}^n) \leq \cdots .$$

If $\lambda > \Lambda_a(V, \mathbb{R}^n)$, then system (3.6.1) is indefinite. Now the question we want to answer is whether similar bifurcation results can be established for the system (3.6.1), with proper assumptions on $\mu_1, \mu_2$ and the coupling constant $\beta$ are satisfied.

# CHAPTER 4

## SUMMARY AND CONCLUSIONS

In this dissertation, we considered the standing wave solutions of coupled nonlinear Schrödinger equations (CNLS),

$$\begin{cases} -i\dfrac{\partial}{\partial t}\Phi_j = \Delta\Phi_j - V_j(x)\Phi_j + \mu_j|\Phi_j|^{2p-2}\Phi_j + \displaystyle\sum_{k\neq j}\beta_{jk}|\Phi_k|^p|\Phi_j|^{p-2}\Phi_j, \\ \quad \Phi_j = \Phi_j(x,t) : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{C}, \end{cases} \tag{4.0.1}$$

where standing wave solutions take the following form: $\Phi_j(x,t) = e^{ia_jt}u_j(x,t)$ and $u_j : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$. By imposing different assumptions on the coefficients, the reduced system of (4.0.1) with standing wave solutions can be classified into many categories.

A large number of established results on the standing wave solutions of (4.0.1) concern the existence or asymptotic behavior of solutions in the definite case. In contrast, results on multiplicity, in particular in the indefinite case, are relatively few. In this dissertation, we studied the multiplicity of standing wave solutions in following two cases: definite and fully symmetric case, in which system (4.0.1) becomes

$$\begin{cases} -\Delta u_j + u_j = \mu u_j^3 + \beta u_j \sum_{k\neq j} u_k^2 & \text{in } \Omega, \\ u_j > 0 \text{ in } \Omega, u_j = 0, j = 1, \ldots, N & \text{on } \partial\Omega, \end{cases} \tag{4.0.2}$$

and indefinite and asymmetric case, in which system (4.0.1) becomes

$$\begin{cases} -\Delta u - au = \mu_1 u^3 + \beta u v^2 & \text{in } \Omega, \\ -\Delta v - av = \mu_2 v^3 + \beta v u^2 & \text{in } \Omega, \\ u, v > 0 \text{ in } \Omega,\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.0.3}$$

For system (4.0.2), we used variational methods and a $Z_N$-symmetric structure to find multiple solutions. To find positive solutions, a Nehari manifold $\mathcal{M}$ with co-dimension $N$ was defined. By restricting the associated energy functional $\mathcal{E}$ on $\mathcal{M}$, we excluded the trivial solution and all semi-trivial solutions. A new $Z_N$-index were introduced in order to find multiple critical points. Then a Lusternik-Schnirelmann type of arguments were used to divide the Nehari manifold into a sequence of level sets, and minimizers of $\mathcal{E}$ found on these levels were proved to be critical points of $\mathcal{E}$, by using the corresponding $Z_N$-invariant deformation flow. Since $Z_N$ has proper subgroups when $N$ is not prime, some induction arguments are required.

For system (4.0.3), we used the bifurcation methods to find multiple solutions. Since (4.0.3) is indefinite, a PS sequence does not necessarily have a convergent subsequence. Thus variational methods do no apply directly. On the other hand, a trivial solution branch $\mathcal{T}_\omega$ of (4.0.3) can be derived from the non-degenerate positive solution $\omega$ of a scalar equation. We linearized (4.0.3) along $\mathcal{T}_\omega$, and found out that the linearized system had nonempty kernel spaces for some values of $\beta$. According to bifurcation theory, local bifurcations may happen and new solutions may exist in a neighborhood of these $\beta$'s. We verified local bifurcations at those values of $\beta$ by using Morse indices of the associated energy functional. When $n = 1$ or $\Omega$ is radial, unbounded global bifurcation branches emanate from $\mathcal{T}_\omega$ at most of the bifurcation points. Thus multiple solutions are found on these branches for every $\beta$ small enough.

The new results of this dissertation are summaries as follows:

(1) Introduce a $Z_N$-index and use it to study multiple standing wave solutions of CNLS.

(2) Establish a couple of bifurcation results and multiplicity results for an asymmetric and indefinite CNLS.

(3) Obtain some nonexistence results of positive standing wave solutions of an indefinite, possibly asymmetric, CNLS.

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APPENDICES

# APPENDIX A

## LEMMAS I, II, III

We verify a few assertions that are required in Chapter 2 and Chapter 3.

**Lemma I** *The map $\psi : \mathcal{S}^{2m-1} \to \mathcal{M}$ defined in Chapter 2* (2.4.3) *or* (2.4.5) *is continuous.*

*Proof.* The proofs for definition (2.4.3) and definition (2.4.5) are similar, so we only prove for the first case. Recall definition (2.4.3),

$$\psi\Big(\sum_{k=1}^{m} r_k e^{i\theta_k}\mathbf{u}^k(\rho,t)\Big) = \Big(U_1^*(\rho,t), U_2^*(\rho,t), \cdots, U_N^*(\rho,t)\Big),$$

where, according to the construction of $\mathbf{u}^k(\rho,t)$,

$$\begin{aligned}
U_j^*(\rho,t) &= \frac{\sqrt{\mu}\|\sum_{k=1}^m r_k U_j^k(\rho,t-\theta_k)\|}{|\sum_{k=1}^m r_k U_j^k(\rho,t-\theta_k)|_4^2}\left|\sum_{k=1}^m r_k U_j^k\Big(\rho,t-\theta_k\Big)\right| \\
&= \frac{\sqrt{\mu}\sum_{k=1}^m r_k\|U_j^k(\rho,t-\theta_k)\|}{\sum_{k=1}^m r_k^2|U_j^k(\rho,t-\theta_k)|_4^2}\left|\sum_{k=1}^m r_k U_j^k\Big(\rho,t-\theta_k\Big)\right| \\
&= \frac{\sqrt{\mu}\sum_{k=1}^m r_k}{\sum_{k=1}^m r_k^2|U_j^k(\rho,t-\theta_k)|_4^2}\left|\sum_{k=1}^m r_k U_j^k\Big(\rho,t-\theta_k\Big)\right| \\
&= \frac{\sqrt{\mu}\sum_{k=1}^m r_k}{\sum_{k=1}^m r_k^2|U_1^k|_4^2}\left|\sum_{k=1}^m r_k U_j^k\Big(\rho,t-\theta_k\Big)\right|.
\end{aligned}$$

For each $1 \le j \le m$, the facts that the supports of $U_j^k$ are separated and $\|U_j^k\| = 1$ are used.

Assume $z_j = \sum_{k=1}^m r_k^{(j)} e^{i\theta_k^{(j)}}\mathbf{u}^k(\rho,t) \in \mathcal{S}^{2m-1}$, $j = 1, 2$. Then

$$\begin{aligned}
\|\psi(z_1)-\psi(z_2)\|_{\mathcal{H}} &= \sum_{j=1}^N \|U_j^{*(1)}(\rho,t) - U_j^{*(2)}(\rho,t)\| \\
&= \sum_{j=1}^N \left\| C^{(1)}\left|\sum_{k=1}^m r_k^{(1)}U_j^{k(1)}(\rho,t-\theta_k^{(1)})\right| - C^{(2)}\left|\sum_{k=1}^m r_k^{(2)}U_j^{k(2)}(\rho,t-\theta_k^{(2)})\right|\right\| \\
&\le (C^{(1)}+C^{(2)})\sum_{j=1}^N\sum_{k=1}^m \left\| r_k^{(1)}U_j^k(\rho,t-\theta_k^{(1)}) - r_k^{(2)}U_j^k(\rho,t-\theta_k^{(2)})\right\| \\
&\le (C^{(1)}+C^{(2)})\sum_{j=1}^N\sum_{k=1}^m \left\| r_k^{(1)}e^{i\theta_k^{(1)}}U_j^k(\rho,t-\theta_k^{(1)}) - r_k^{(2)}e^{i\theta_k^{(2)}}U_j^k(\rho,t-\theta_k^{(2)})\right\|
\end{aligned}$$

$$= (C^{(1)} + C^{(2)}) \sum_{k=1}^{m} \|z_1 - z_2\|_{\mathcal{H}},$$

where $C^{(j)} = \dfrac{\sqrt{\mu}\sum_{k=1}^{m} r_k^{(j)}}{\sum_{k=1}^{m}[r_k^{(j)}]^2|U_1^k|_4^2}$ is a nonzero constant depending on $z_j$, $j = 1, 2$. The estimate show that $\psi$ is locally Lipschitz continuous, thus continuous. ■

**Lemma II** *Assume that $(u, v)$ is a $H_0^1(\Omega)$ solution of system*

$$\begin{cases} -\Delta u - au = \mu_1 u^3 + \beta u v^2 & \text{in } \Omega, \\ -\Delta v - bv = \mu_2 v^3 + \beta v u^2 & \text{in } \Omega, \\ u, v > 0 \text{ in } \Omega, \ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

*where $a, b, \mu_1, \mu_2$ are real numbers. $\Omega \subset \mathbb{R}^n$ is bounded, $n \leq 3$. Then $u, v \in C_0^1(\Omega)$.*

*Proof.* The lemma is proved by using the boot-strap arguments.

If $n = 1$ or $n = 2$, then by Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^\infty(\Omega)$ and Hölder inequality, it holds that

$$\mu_1 u^3 + \beta u v^2 \in L^\infty, \qquad \mu_2 v^3 + \beta v u^2 \in L^\infty.$$

According to the $L^p$ estimates, $u, v \in W^{2,\infty}(\Omega)$. Using Sobolev embedding $W^{2,\infty} \hookrightarrow C(\overline{\Omega})$, it is easy to see that

$$\mu_1 u^3 + \beta u v^2 \in C(\overline{\Omega}), \qquad \mu_2 v^3 + \beta v u^2 \in C(\overline{\Omega}).$$

Next, by Schauder estimate, we get $u, v \in C^2(\Omega) \cap C_0^1(\Omega)$.

If $n = 3$, we need more steps. By Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$ and Hölder inequality, it holds that

$$\mu_1 u^3 + \beta u v^2 \in L^2, \qquad \mu_2 v^3 + \beta v u^2 \in L^2.$$

According to $L^2$ estimates, we get $u, v \in W^{2,2}(\Omega)$. Again, using Sobolev embedding $W^{2,2}(\Omega) \hookrightarrow C^\alpha(\overline{\Omega})$ for $0 < \alpha < 1/2$. Consequently,

$$\mu_1 u^3 + \beta u v^2 \in C^\alpha(\overline{\Omega}), \qquad \mu_2 v^3 + \beta v u^2 \in C^\alpha(\overline{\Omega}).$$

At last, by Schauder estimate, there holds $u, v \in C^2(\Omega) \cap C_0^1(\Omega)$. ■

The following lemma is from [42] (Proposition B.34). Since it is frequently used in this work, we put it here for the readers' convenience.

**Lemma III** *Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ whose boundary is a smooth manifold. Let $p$ satisfy*

*($\hat{p}_1$) $p \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, and*

*($\hat{p}_2$) there are constants $a_1, a_2 > 0$ such that*

$$|p_\xi(x, \xi)| \leq a_1 + a_2 |\xi|^{s-1}$$

*where $0 \leq s < \frac{n+2}{n-2}$ and $n \geq 3$. If*

$$P(x, \xi) = \int_0^\xi p(x, t) dt$$

*and*

$$I(u) = \int_\Omega (\frac{1}{2} |\nabla u|^2 - P(x, u(x))) dx,$$

*then $I \in C^2(H_0^1(\Omega, \mathbb{R}))$.*

# APPENDIX B

# PERMISSION LETTERS

# Permission to reprint

**TMNA** <tmna@ncu.pl> Tue, Jan 29, 2013 at 9:20 AM
To: rushun.tian@aggiemail.usu.edu

Dear Rushun Tian,

Of course, we give permission to reprint.

Lech Górniewicz
(Managing Editor of TMNA)

> -- Treść oryginalnej wiadomości --
> Temat: Permission to reprint
> Data: Sun, 27 Jan 2013 15:51:43 -0700
> Nadawca: Rushun Tian <rushun.tian@aggiemail.usu.edu>
> Adresat: Mariusz.Czerniak@uni.torun.pl
>
> Dear Mgr Mariusz Czerniak,
>
> I am a Ph.D student of Prof. Zhi-Qiang Wang in the department of Mathematics and Statistics at Utah State University. I am preparing my doctoral dissertation now and hope to complete my degree in May of 2013.
>
> I would like to include the following joint paper with Prof. Zhi-Qiang Wang
> * Multiple solitary wave solutions of nonlinear Schrodinger systems, *Vol. 37, No. 2, June 2011, 203-223
> as a part of my doctoral dissertation. I wonder how could I obtain such a permission of agreement, so I can include it in the appendix of my dissertation? Thank you for your assistance.
>
> Best regards,
>
> Rushun Tian

With best regards
Mariusz Czerniak
Secretary of the Office of the TMNA
--
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January 29, 2013

Liwei Ning
Editorial Manager
American Institute of Mathematical Sciences
Springfield, MO 65801 USA

Dear Ms. Liwei Ning,

I am preparing my doctoral dissertation in the department of Mathematics & Statistics at Utah State University. I hope to complete my degree in May of 2013.

I am requesting the permission to include the following joint paper with Prof. Zhi-Qiang Wang as a chapter of my doctoral dissertation:

R.-S. Tian and Z.-Q. Wang, *Bifurcation results on positive solutions of an indefinite nonlinear elliptic system.* Disc. Cont. Dyna. Sys. - Series A **33** (2013), 335-344.

I will include a citation to the original publication on the first page of the chapter, as shown below. Copyright and permission information will be included in the appendix. Please indicate your approval of this request by signing in the space provided. Please let me know if you have any questions. Thank you for your assistance.

Rushun Tian

---

I hereby give permission to Rushun Tian to reprint the requested paper in his dissertation, with the following citation:

- R.-S. Tian and Z.-Q. Wang, *Bifurcation results on positive solutions of an indefinite nonlinear elliptic system.* Disc. Cont. Dyna. Sys. - Series A **33** (2013), 335-344.

Signed Liwei Ning

Data 1/30/2013

Rushun Tian
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Shair Ahmad
Managing Editor
Department of Mathematics
University of Texas at San Antonio
San Antonio, TX 78249 USA

Dear Shair Ahmad,

I am preparing my doctoral dissertation in the department of Mathematics & Statistics at Utah State University. I hope to complete my degree in May of 2013.

I am requesting the permission to include the following joint paper with Prof. Zhi-Qiang Wang as a chapter of my doctoral dissertation,

R.-S. Tian and Z.-Q. Wang, *Bifurcation results on positive solutions of an indefinite nonlinear elliptic system II.* Adv. Non. Stud. **13** (2013), 245-262.

I will include a citation to the original publication on the first page of the chapter, as shown below. Copyright and permission information will be included in the appendix. Please indicate your approval of this request by signing in the space provided. If you have any questions, please call me at the number above or send me an e-mail message at the above address. Thank you for your assistance.

Rushun Tian

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I hereby give permission to Rushun Tian to reprint the requested paper in his dissertation, with the following citation:

- R.-S. Tian and Z.-Q. Wang, *Bifurcation results on positive solutions of an indefinite nonlinear elliptic system II.* Adv. Non. Stud. **13** (2013), 245-262.

Signed Shair Ahmad Data 1/28/13

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## EDUCATION

- **Ph.D in Mathematics**. May 2013

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  Dissertation title: Multiplicity and stability results on solitary wave solutions of some coupled nonlinear Schrödinger equations

- **Master of Science in Mathematics**. Jul. 2008

  Capital Normal University, Beijing, China

  Thesis advisor: Jiabao Su GPA 3.8

  Thesis title: Weighted Sobolev embeddings and radial solutions of inhomogeneous quasi-linear elliptic equations

- **Bachelor of Science in Mathematics**. Jul. 2005

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  Thesis title: Contraction mappings principles and their applications

## PUBLICATIONS

1. *Stability of ground-state solutions for a Schrödinger system with power-type nonlinearities*, submitted (with N. Nguyen and Z.-Q. Wang)

2. *Global existence for a system of Schrödinger equations with power-type nonlinearities*, J. Math. Phys. **54**, 011503 (2013) (with N. Nguyen, B. Deconinck and N. Sheils)

3. *Bifurcation results on positive solutions of an indefinite nonlinear elliptic system II*, Adv. Nonlinear Stud., **13** (2013), 245-262 (with Z.-Q. Wang)

4. *Bifurcation results on positive solutions of an indefinite nonlinear elliptic system*, Discrete Contin. Dyn. S. - Ser. A,**33** (2013), 335-344 (with Z.-Q. Wang)

5. *Multiple solitary wave solutions of nonlinear Schrödinger systems*, Topol. Methods Nonlinear Anal., **37** (2011), 203 - 223 (with Z.-Q. Wang)

6. *Weighted Sobolev type embeddings and coercive quasilinear elliptic equations on* $\mathbb{R}^N$, Proc. Amer. Math. Soc., **140** (2012), 891-903 (with J.-B. Su)

7. *Weighted Sobolev embeddings and radial solutions of inhomogeneous quasilinear elliptic equations*, Comm. Pure Appl. Anal., **9** (2010), 885 - 904 (with J.-B. Su)

## HONORS AND AWARDS

- **Graduate Researcher of the Year - 2011**, Department of Mathematics & Statistics, Utah State University

- **Graduate Student Professional Conference Award**, October 2011, Graduate Student Senate, Utah State University

- **Graduate Student Travel Grant**, January 4 - 7, 2012, AMS-MAA Joint Mathematics Meetings, Boston, MA