# Full Isolation Number of Matrices: Some Extremal Results

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MAA Conference, Salt Lake City, Utah
March 24-25, 2012

**Isolated Set of Ones:** A set of nonzero entries of a (0,1)-matrix is an *isolated set* if no two entries belong to the same row, no two entries belong to the same column, and no two entries belong to a submatrix of the form $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

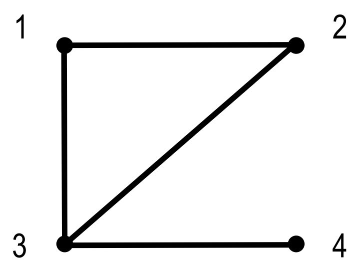
**Isolation Number:** The maximum size over all isolated sets.

Define $\iota(M)$ to be the isolation number of an $m \times n$ matrix, $M$. The isolation number of M is bounded by:

$$0 \leq \iota(M) \leq \min\{m, n\}.$$

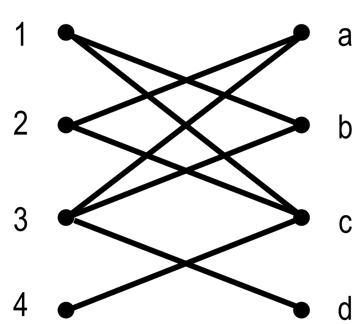
# Application to Graph Theory

Given a graph, $G$, construct the *adjacency matrix* $M$ of $G$ in the following way: Each vertex in $G$ is represented by a row and a column in $M$. Vertices $u, v \in G$ are adjacent if and only if $M_{u,v} = 1$. Otherwise $M_{u,v} = 0$.




| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 0 |
| 2 | 1 | 0 | 1 | 0 |
| 3 | 1 | 1 | 0 | 1 |
| 4 | 0 | 0 | 1 | 0 |

Given a bipartite graph, $B$, construct the *adjacency matrix* $M$ of $B$ in the following way: the vertices in one partite set are represented by rows in $M$ and the vertices in the other partite set are represented by columns in $M$. Again, vertices $u, v \in G$ are adjacent if and only if $M_{u,v} = 1$



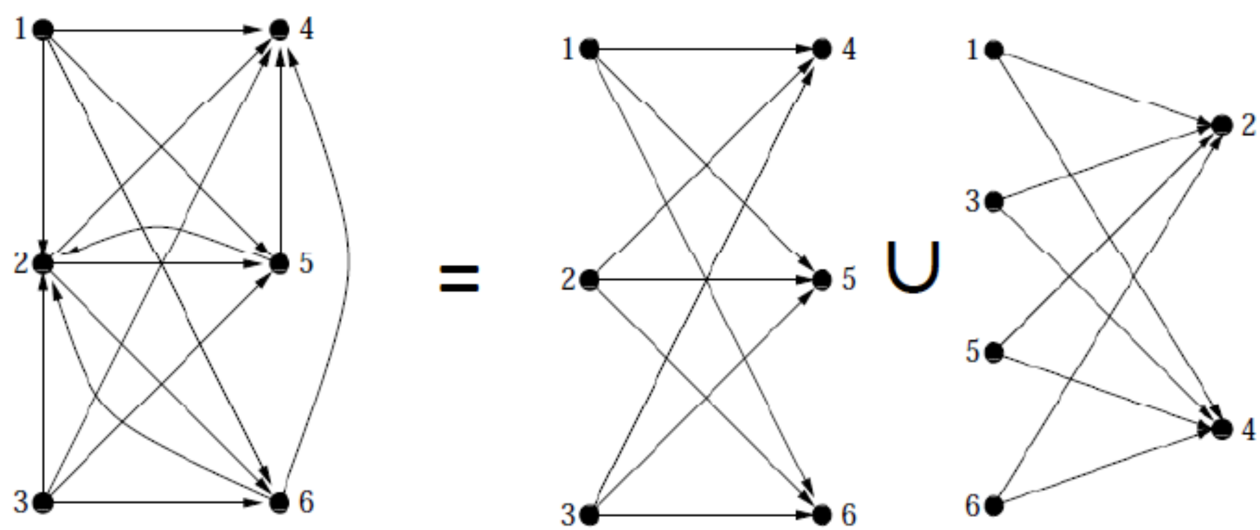

| | a | b | c | d |
|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 0 |
| 2 | 1 | 0 | 1 | 0 |
| 3 | 1 | 1 | 0 | 1 |
| 4 | 0 | 0 | 1 | 0 |

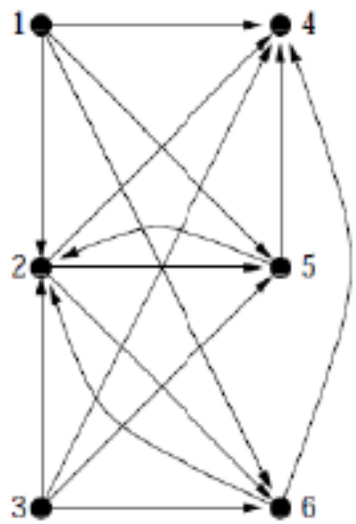
# Why research isolation number?

The isolation number of a matrix is the best-known lower bound for the Boolean rank of (0,1)-matrices and equivalently, the biclique cover number for graphs and directed graphs.

**Boolean Rank:** Let $\mathbb{B}$ be the binary Boolean algebra. The *Boolean rank* of a matrix $A$ in $M_{m,n}(\mathbb{B})$ is the smallest $k$ such that $A$ can be factored as an $m \times k$ times a $k \times n$ matrix.

**Biclique Cover Number:** the minimum number of bicliques that cover the edges of $G$.



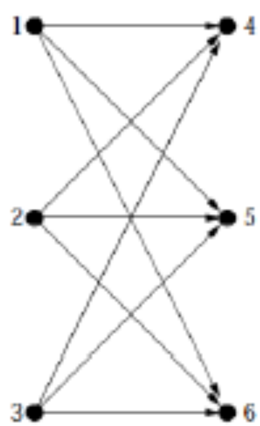


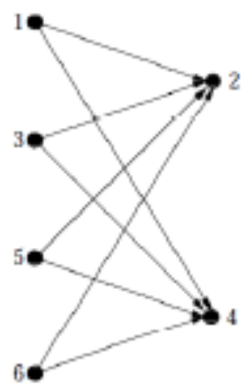

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}_{6\times 6}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}_{6\times 2} \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}_{2\times 6}$$

The *Boolean rank* of a matrix A can also be thought of as the minimum number of rank one matrices whose sum is A.

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$



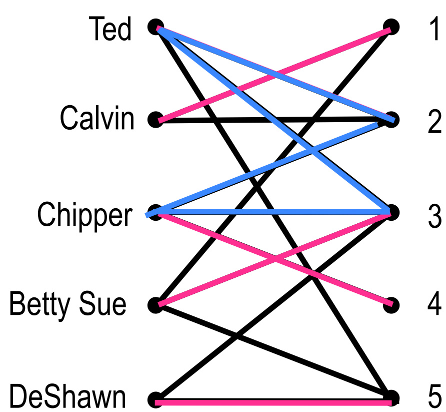

| | |
|---|---|
| 0 | 1 |
| 0 | 0 |
| 0 | 1 |
| 0 | 0 |
| 0 | 1 |
| 0 | 1 |

| | | | |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 |
| 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 0 | 0 |

Isolation number is a lower bound for Boolean rank since no two isolated ones can come from the same rank one matrix.

# Maximum Matching Problem (with an additional constraint)

Suppose you have a list of tasks to be accomplished. Each task can be done by a single employee, and no employee is to have more than one task. Additionally, each employee has a certain skill set which allows them to do only some of the tasks. You decide to solve this problem by constructing a bipartite graph with employees and tasks in separate sets. An edge between an employee and a task means that employee is capable of accomplishing that task.




| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Ted | 0 | 1 | 1 | 0 | 1 |
| Calvin | 1 | 1 | 0 | 0 | 1 |
| Chipper | 0 | 1 | 1 | 1 | 0 |
| Betty Sue | 1 | 0 | 1 | 0 | 0 |
| DeShawn | 0 | 0 | 1 | 0 | 1 |

You could do this by simply choosing ones not in the same row and column. But suppose your employees like to gripe and complain. Say you assign Ted task 2 and Chipper task 3. Ted doesn't like doing task 2 so he says, "Why don't you let me do task 3 and make Chipper do task 2? He is perfectly capable!" Now you wish you had chosen your assignments more wisely. If you choose so that the ones in your matrix are isolated, it will not be so easy to rearrange things, so you can tell Ted to bug off.

# Structure of nxn Matrices With Full Isolation Number

An $n \times n$ matrix with full isolation number can have no more than $\binom{n}{2} + n$ ones. In such a matrix, every pair of isolated ones will either be in a submatrix of the form $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

We can see this by permuting the rows and columns of the matrix so that the isolated ones are along the diagonal.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

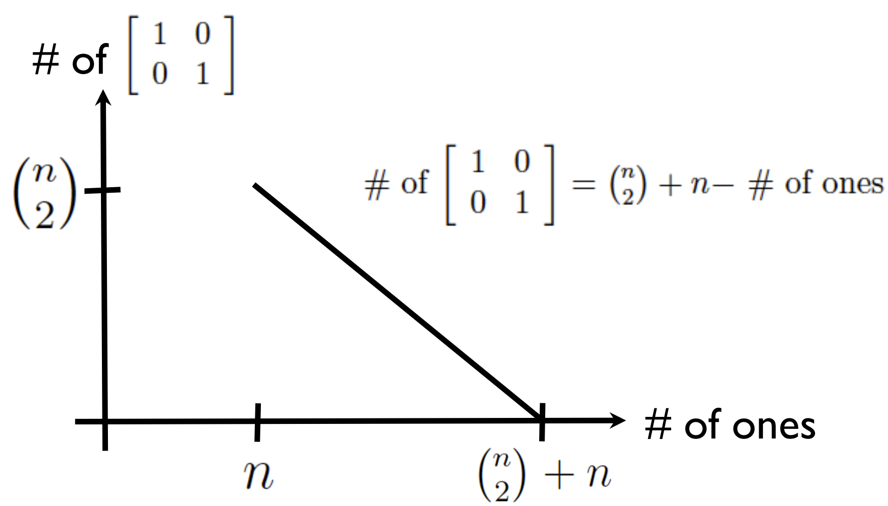
Changing any 0 entry in this matrix to a 1 will decrease the isolation number by one because you are making exactly one submatrix of the form $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Changing any 1 to a 0 will make exactly one submatrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ among the isolated ones. The number of $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ varies linearly with the number of ones in the matrix.

An $n \times n$ matrix with full isolation number can have no fewer than $n$ ones. In such a matrix, every pair of isolated ones will be in a submatrix of the form $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

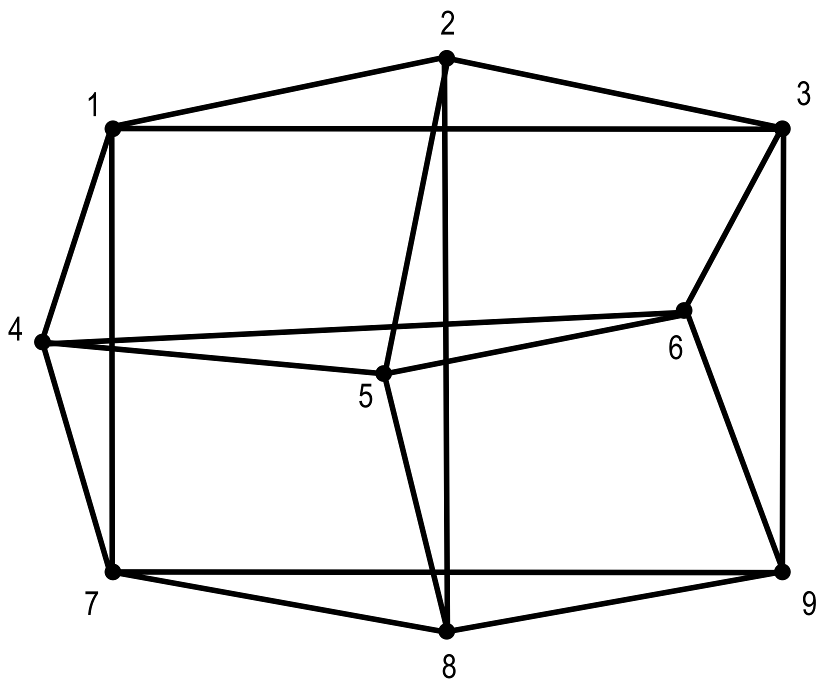
We can see this by permuting the rows and columns of the matrix so that the isolated ones are along the diagonal.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

By counting the number of 1 entries in a matrix with full isolation number we can know how many $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ submatrices exist among the isolated ones.



This example illustrates the previous results. Here we have a 9-vertex graph with its $9 \times 9$ adjacency matrix. This matrix has 36 ones. So according to a previous result, if the matrix has full isolation number we expect $9 = \binom{9}{2} + 9 - \binom{9}{2}$ submatrices of the form $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ among the isolated ones.




$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{bmatrix} 0 & \textcircled{1} & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & \textcircled{1} & 0 & 1 & 0 & 0 & 1 & 0 \\ \textcircled{1} & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & \textcircled{1} & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & \textcircled{1} & 0 & 1 & 0 \\ 0 & 0 & 1 & \textcircled{1} & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & \textcircled{1} & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & \textcircled{1} \\ 0 & 0 & 1 & 0 & 0 & 1 & \textcircled{1} & 1 & 0 \end{bmatrix} \end{array}$$

The technique employed for finding all 9 isolated ones was to start at the top, choosing the leftmost 1 which is isolated to all other ones previously chosen and also which uses as few $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ submatrices as possible with previously chosen ones. As expected, there are 9 isolated ones and 9 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ submatrices.

# Current Research and Further Directions

In a very recent paper by L. Beasley (Isolation number versus Boolean rank, Linear Algebra Appl., (2012) DOI: 10.1016/j.laa.2011.12.013) the problem of determining isolation number and analyzing its relationship with the Boolean rank, for example when is isolation number different from Boolean rank, was brought to the forefront of research.

Analyze the isolation number for special classes of graphs and digraphs:
- Restricted classes of tournaments
- Threshold graphs
- Interval graphs
- Line graphs

Analyze the complexity of restricted isolation number problems
- Bounding the degree
- Restricting special classes of graphs and digraphs