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Equivalent Fraction Learning Trajectories for Students with Mathematical Learning Difficulties When Using Manipulatives

Arla Westenskow

EQUIVALENT FRACTION LEARNING TRAJECTORIES FOR STUDENTS

WITH MATHEMATICAL LEARNING DIFFICULTIES WHEN

USING MANIPULATIVES

by

Arla Westenskow

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ABSTRACT

Equivalent Fraction Learning Trajectories for Students with Mathematical Learning Difficulties When Using Manipulatives

by

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Utah State University, 2012

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This study identified variations in the learning trajectories of Tier II students when learning equivalent fraction concepts using physical and virtual manipulatives. The study compared three interventions: physical manipulatives, virtual manipulatives, and a combination of physical and virtual manipulatives. The research used a sequential explanatory mixed-method approach to collect and analyze data and used two types of learning trajectories to compare and synthesize the results. For this study, 43 Tier II fifth-grade students participated in 10 sessions of equivalent fraction intervention.

Pre- to postdata analysis indicated significant gains for all three interventions. Cohen *d* effect size scores were used to compare the effect of the three types of manipulatives—at the total, cluster, and questions levels of the assessments. Daily assessment data were used to develop trajectories comparing mastery and achievement changes over the duration of the intervention. Data were also synthesized into an iceberg learning trajectory containing five clusters and three subcluster concepts of equivalent fraction understanding and variations among interventions were identified. The syntheses favored the use of physical manipulatives for instruction in two clusters, the use of virtual manipulatives for one cluster, and the use of combined manipulatives for two clusters.

The qualitative analysis identified variations in students' resolution of misconceptions and variations in their use of strategies and representations. Variations favored virtual manipulatives for the development of symbolic only representations and physical manipulatives for the development of set model representations. Results also suggested that there is a link between the simultaneous linking of the virtual manipulatives and the development of multiplicative thinking as seen in the tendency of the students using virtual manipulative intervention to have higher gains on questions asking students to develop groups of three or more equivalent fractions. These results demonstrated that the instructional affordances of physical and virtual manipulatives are specific to different equivalent fraction subconcepts and that an understanding of the variations is needed to determine when and how each manipulative should be used in the sequence of instruction.

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PUBLIC ABSTRACT

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This study identified variations in the equivalent fraction learning of students with mathematical learning difficulties when using physical and virtual manipulatives. The study compared three interventions: physical manipulatives, virtual manipulatives, and a combination of physical and virtual manipulatives. The research used a mixed-method approach to collect and analyze data. Two types of learning trajectories were used to compare and synthesize the result. For this study, 43 fifth-grade students with mathematical learning difficulties participated in 10 sessions of equivalent fraction intervention.

Pre- to postdata analysis indicated significant gains for all three interventions. Effect size scores were used to compare the effects of the three types of manipulative intervention at the total, cluster, and questions levels of the assessments. Daily assessment data were used to develop trajectories comparing mastery and achievement changes over the duration of the intervention. Data were also synthesized into an iceberg learning trajectory containing five clusters and three subcluster concepts of equivalent fraction understanding. The syntheses favored the use of physical manipulatives for instruction in two clusters, the use of virtual manipulatives for one cluster and the use of combined manipulatives for two clusters.

The qualitative analysis indentified variations in students' resolution of misconceptions and students' use of strategies and representations. Variations favored virtual manipulatives for the development of students' understanding of representations using only symbols. Physical manipulatives were favored for students' understanding of set model representations. Results also suggested that the ability of students using virtual manipulatives to see the link between their manipulation of the objects and simultaneous changes in the symbolic representations of the building of equivalent fraction groups. Students using virtual manipulatives tended to have higher gains on questions that asked students to develop groups of three or more equivalent fractions. The results of this study demonstrated that the instructional benefits of physical and virtual manipulative instruction are specific to the different equivalent fraction subconcepts and that an understanding of the variations is needed to determine when and how each manipulative should be used in the sequence of instruction.

DEDICATION

I dedicate this work to my friends and family who have given me both physical and emotional support throughout my Ph.D. education experience. I thank you for your understanding and your encouraging support. I thank my parents and grandparents for their strong belief in the importance of education and for making my education possible. Special thanks to Marilyn, Mom, and Marion, who began this experience with me and whose loving support and influence live on in my heart.

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Arla Westenskow

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CHAPTER I

INTRODUCTION

"Children who have fallen in the gap" is a term used to describe children who struggle with learning mathematical concepts but have not received the support or additional instruction needed to help them overcome their difficulties. These are children (referred to as children or students with mathematical learning difficulties) who do not respond adequately to regular classroom instruction, but also do not qualify for special education services. Due to recent changes in educational funding, many school systems are beginning to place a greater emphasis on providing intervention support for children with mathematical learning difficulties. However, one of the difficulties for designers and implementers of intervention programs has been the limited amount of research concerning effective instruction specific to intervention settings. This study focused on the use of physical and virtual manipulatives in intervention settings. Both types of manipulatives have been shown to be effective in regular education settings, but there has been little research evaluating their use in intervention settings and even less research which can be used to guide teachers and curriculum designers as to when the two manipulatives can be used most effectively during intervention. The purpose of this study was to identify variations in the learning trajectories of students with mathematical difficulties when learning equivalent fraction concepts during instruction using virtual and physical manipulatives.

Background of the Problem

Research on intervention for students struggling with mathematical learning difficulties has steadily, but slowly evolved throughout the past 100 years. By the early 1900s, the opportunity for education was made available to almost all children. However, during the first half of the century, little attention was given to the individual differences and needs of children (Eisner, 1994). Very little money and few programs were directed for the intervention of students who did not respond to regular education classroom instruction. The first unified movement of diversifying instruction to meet the needs of children came as a result the Cold War. The Russian launching of Sputnik in the 1960s, created a fear in the United States that the nation would be overpowered if the abilities of United States' mathematicians and scientist were substandard to those of communist countries allowing other countries to produce more advanced war technology. As a result of this fear, heavy emphasis was placed on building a large pool of highly educated mathematicians and the first practice of diversification, tracking students by ability, began.

With today's globalization of markets, the advancement of technology and the overwhelming spread of information through the World Wide Web, a flexible comprehension of mathematics by all people is becoming increasingly more important and therefore a greater emphasis is being placed on educating students of all abilities (Woodward, 2004). In 1975, the Education for All Handicapped Children Act (EHA) calling for free and appropriate public education for all children with disabilities was passed. EHA was reauthorized in 1990 to become the Individuals with Disabilities

Education Act (IDEA). This law set the criteria for determining which students could be considered learning disabled and who would receive special education services. In recent years, the movement to ensure that all children have adequate mathematics skills and knowledge has been reinforced by the No Child Left Behind Act (NCLB). One of the main goals of NCLB is that all children will become proficient at mathematics.

Yet, in the United States a large number of students still fail to acquire the needed mathematical skills. In the latest international mathematics study, United States students achieved the ranking of only 18th in a study of 25 countries (Frykholm, 2004). Each year, a large percentage of college students are required to enroll in remedial mathematics courses because they lack sufficient skills needed for beginning mathematics courses. Within the average classroom, it is estimated that a large number of students are functioning below grade level in mathematics (Din, 1998). Almost three million students in the public school system have been classified with learning disabilities and are receiving special education services and the number of students classified as learning disabled has increased by 22% over the last 25 years (Singapogu & Burg, 2009).

The failure of so many students to learn adequate mathematical skills has caused educational and government leaders to reevaluate their policies and practices of intervention. Until 2004, the United States government funded only the traditional remediation form of intervention: programs in which intervention is provided for students only when it is determined that the student is academically at least two years behind his/her peers and the student is diagnosed with a learning disorder or moderate to severe mental retardation (D. Fuchs, Compton, L. S. Fuchs, Bryant, & Davis, 2008a). This system has been referred to as the “wait to fail” approach (D. Fuchs et al., 2008a). As a result, intervention literature has primarily focused on students with diagnosed learning disorders. However, concern over the rising number of students needing mathematical remediation has recently caused a shift in focus towards earlier interventions. Supporters of earlier interventions believe that providing students with effective intervention earlier in their schooling will, for most students, prevent the need for more intense intervention later.

In 2004, in support of the shift towards earlier intervention, Congress passed the Individuals with Disabilities Education Improvement Act (P.L. 108-446), giving states and districts the right to redirect a proportion of their funding from the traditional remediation process to supporting classroom intervention. A number of states chose to respond to the opportunity and are now beginning to initiate changes in procedures and programs. Many have chosen to adopt the Response to Intervention (RtI) approach which targets earlier intervention for students having mathematical difficulties. Although RtI has been successfully used in the field of reading, research and program implementation of RtI intervention in mathematics is still in its infancy. Because of the emphasis of past funding on traditional intervention, intervention research has been heavily influenced by special education policies and research and has focused primarily on behavior analysis, direct instruction, peer mediated instruction and cognitive behavior modification (Gersten, Clarke, & Mozzocco, 2007). It is only in the last few years that the literature has begun to focus on the development of early intervention practices in the classroom (Gersten et al., 2009). Both research designers and program implementers have identified the lack of research concerning intervention appropriate materials and tools as one of the factors limiting the implementation of early intervention in the field of mathematics (D. Fuchs et al., 2008a; Glover & DiPerna, 2007).

Problem Statement

The purpose of this study was to identify variations in the learning trajectories of students with mathematical difficulties when learning equivalent fraction concepts during instruction using virtual and physical manipulatives and to pilot instruments and protocol for use in future research. Physical manipulatives have been shown to be effective tools for use in developing student understanding when used in regular classroom instruction, however their use in intervention has been limited (Sowell, 1989). Research results indicate that the action of manipulating physical objects can aid students in the process of constructing and retention of new mathematical concepts. Teachers report that students are typically more engaged and motivated to complete assignments when using physical manipulatives. Literature has also begun to emerge supporting their effectiveness in special education instruction. However, very limited research has focused directly on the use of physical manipulatives in early intervention settings for students with mathematical learning difficulties.

Virtual manipulatives are an “interactive, web based, visual representation of a dynamic object that presents opportunities constructing mathematical knowledge” (Moyer, Bolyard, & Spikell, 2002, p. 373). Although research is still limited, a recent synthesis of empirical research indicates that virtual manipulatives are also effective tools of instruction when used in regular instruction (Moyer-Packenham,Westenskow, & Salkin, 2012). In addition to having many of the same representational advantages as physical manipulatives, many virtual manipulative applets are designed specifically to aid students in linking concrete, semiconcrete and symbolic representations. Although there have been several studies assessing the use of virtual manipulatives with students having learning disabilities, there are no known studies which specifically target their use in early intervention settings.

A small number of studies (N = 26) have examined the effectiveness of combining the use of physical and virtual manipulatives for instruction. An effect size analysis of these studies resulted in a moderate effect size when the combined use of virtual and physical manipulatives was compared to traditional instruction, and a lesser, but still moderate effect when compared to the use of physical or virtual manipulatives alone (Moyer-Packenham et al., 2012). Several researchers report that the affordances of each type of manipulative produce variations in learning unique to the type of manipulative (Izydorczak 2003; Moyer, Niezgoda, & Stanley, 2005; Takahashi, 2002). This indicates that it would be incorrect to suggest that one manipulative is always more effective than the other. Instead research is needed comparing the effectiveness of each manipulative as used in specific settings to teach specific mathematical concepts thus aiding designers and implementers of curriculum to maximize the efficiency and effectiveness of the manipulatives.

The underlying differences of the learning trajectories of students with mathematical learning difficulties compared with the learning trajectories of students without learning difficulties has been debated in the literature. Some argue that the differences are in the amount of time students need to master concepts and that all students follow the same basic trajectories. Others argue that differences are more a result of how the students learn. It is more likely that the learning trajectories of students with mathematical learning difficulties differ in both time and direction. These differences in learning make it necessary for research on manipulative use to be conducted specifically for students with mathematical learning difficulties.

One of the most difficult mathematical topics for students with mathematical learning difficulties has been the study of fractions. Fractions do not follow the same rules which children have established and used in their study of whole numbers. The study of fractions is, for most students, the first time they experience numbers that can be represented by more than one name and that represent a relationship between two discrete quantities rather than a specific quantity (Smith, 2002; Van de Walle, 2004). Yet fractions are the foundation for many mathematical concepts (e.g., ratios, proportions, percents, decimals, rational numbers) and fraction mastery is essential for the future development of students' mathematical understandings (Chan & Leu, 2007). One of the basic fraction concepts to be mastered is understanding fraction equivalence. Until students have a conceptual understanding of equivalence they will not be able to grasp the concept of fraction arithmetic (Arnon, Nesher, Nirenburg, 2001; Smith, 2002). This study examined the effects of virtual and physical manipulatives usage on the development and resolution of students' misconceptions and errors in four areas of fraction understanding difficulties identified by Chan and Leu: (a) difficulty understanding that a fractional amount can be represented by an infinite set of names; (b) difficulty focusing on the need for all parts of the fraction to represent equal sizes; (c) difficulty identifying the whole and its relationship to the parts; and (d) difficulty simplifying and expanding fractions to represent equivalent fractions.

Research Questions

To identify and describe variations in the effects of virtual and physical manipulatives in the intervention of four areas of fraction difficulties, this study used a mixed methods approach with data collected during intervention instruction of fifth-grade students. The overarching research question and subquestions guiding this study were as follows.

1. What variations occur in the learning trajectories of students with mathematical learning difficulties that are unique to the use of different instructional manipulatives for intervention (virtual, physical or a combination of virtual and physical manipulatives) in the learning of equivalent fraction concepts?

   a. What are the variations of achievement, mastery, retention, and resolutions of errors in students' development of equivalent fraction concepts and skills?
   b. What are the variations in learning trajectories showing changes in student achievement over time?
   c. What are the variations in patterns of daily lesson achievement, retention and work completion?
   d. What are the variations in the strategies developed and used by students?
   e. What are the variations in students' use of representations?

Definition of Terms

The following terms are defined for this study.

An *affordance* is a design feature that determines how the object will be used (Norman, 1988)

Distracters are irrelevant or incomplete components of manipulatives which must be ignored by the student (Behr, Lesh, Post, & Silver, 1983).

Intervention is the additional instruction and activities needed to meet a student's individual circumstances and instructional needs (L. S. Fuchs & D. Fuchs, 2006).

Learning trajectories are a mapping of the progression of learning of mathematical concepts and skills (Clements & Sarama, 2004).

Misconceptions are previously learned incorrect mathematical conceptions which inhibit learning (Vosniadou & Vamvakoussi, 2006).

Multiplicative thinking is thinking of a fraction number as multiplicative groups (Ball, 1993).

Partitioning is sectioning into equal shares (Lamon, 1996).

Students with mathematical learning difficulties are those students who have not responded to Tier I intervention and have not been identified as needing Tier III intervention (L. S. Fuchs, 2005).

Physical manipulatives are concrete objects which students use to visually and tactilely explore abstract concepts (McNeil & Jarvin, 2007).

A *representation* is a configuration of signs, characters, icons, or objects that represent something else (Goldin, 2003).

Virtual manipulatives are interactive, Web-based, visual representations of dynamic objects that allow users opportunities to construct mathematical knowledge (Moyer et al., 2002).

CHAPTER II

LITERATURE REVIEW

The range of students' mathematical abilities in a regular elementary classroom tends to be very diverse with as much as five to seven years difference in mathematical ability (Brown, Askew, Millet, & Rhodes, 2002). Students not only differ in ability, but also each student uniquely differs in culture, learning preferences, motivation, past experiences and a variety of other characteristics. As a result, each student in a classroom understands and responds to group instruction differently and the instruction is more effective for some students than others. Thus some students will require intervention. Intervention is the additional instruction and activities needed to meet a student's individual circumstances and needs. An important component in effective intervention is the selection of methods and instructional materials. Yet, there has been relatively little research identifying which methods and tools are most effective in intervention settings. This study targeted the use of two instructional manipulatives (virtual and physical manipulatives) in the intervention instruction of equivalent fractions for students with mathematical learning difficulties. This chapter reviews the literature relevant to students with mathematical learning disabilities, intervention, fraction instruction, and virtual and physical manipulatives. The literature was searched in the databases of ERIC, PsychInfo, Google Scholar, and Digital Dissertation Index using search terms including intervention, mathematical learning difficulties, remediation, learning disorders, response to intervention, fractions, representations, manipulatives, virtual manipulatives, physical manipulatives, computer manipulatives, dynamic manipulatives, and rational numbers.

Reference lists of articles found were also manually searched for further references.

Conceptual Framework

To synthesize the relevant research, a conceptual framework of the relationships and elements of the intervention process was developed (see Figure 1). Four main elements affect the outcomes of the intervention process: the intervention goals, the student, the mathematical concept, and the environment. Each intervention is a unique reflection of the four elements and therefore differs in the order, frequency and difficulty of mathematical concepts presented; the activity level of the participants; the intervention duration; and the types of methods, models, and manipulatives used. As a result of the intervention process, students' mathematical attitudes, beliefs and understandings are changed. Interpretation of changes requires an evaluation of the four contributing elements.

Intervention Goals

The goals set for intervention instruction are a reflection of both what is considered necessary for each student's success and what is considered to be achievable by the student. The goals selected for the intervention will determine how focused instruction will be on varying elements, such as the development of the student's attitudes, problem solving abilities, retention, and conceptual and procedural understanding. How the intervention designer perceives the characteristics of the student, the environment and the mathematical topic influences their selection of goals of each intervention setting and thereby influences their selection of factors such as the depth at

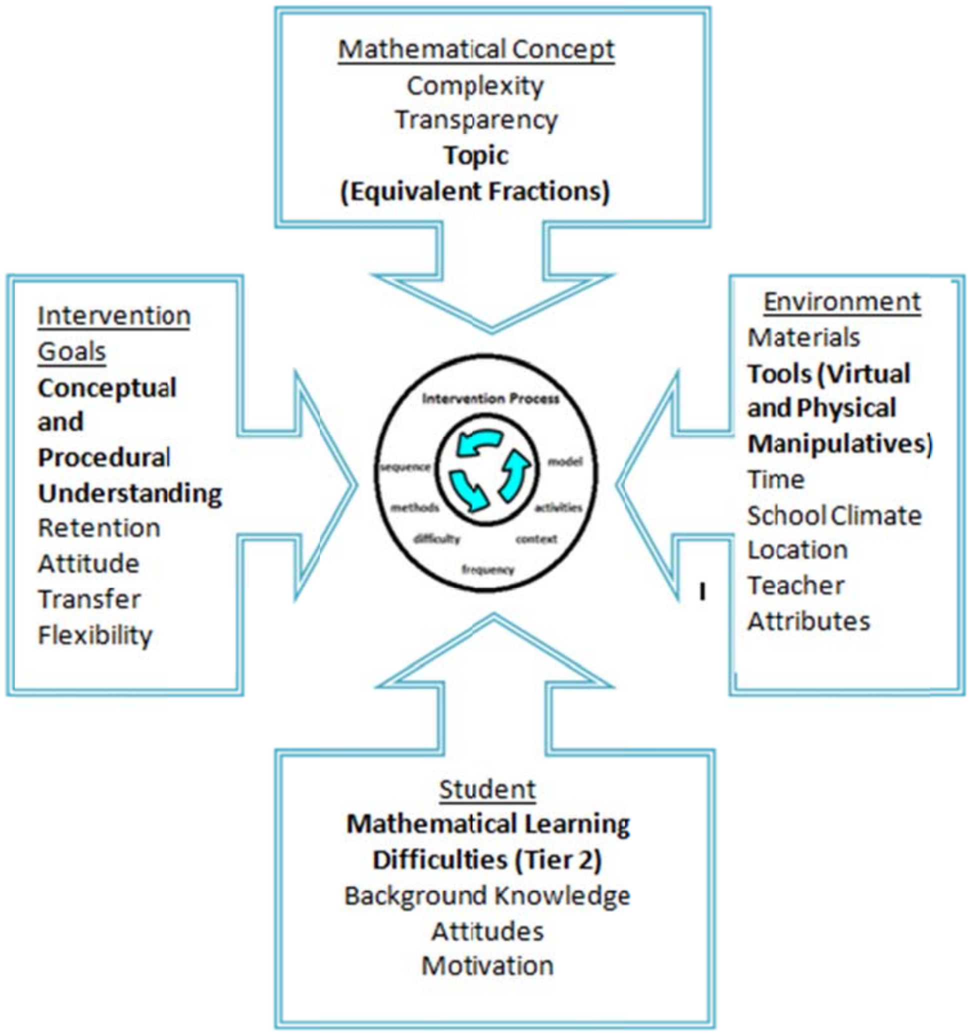


Figure 1. Conceptual framework of four factors influencing the intervention process.

which the mathematical topics will be taught and how, when and where intervention will take place.

The Student

Two main causes for students needing intervention are lack of opportunity to learn and individual characteristics which inhibit learning. Lack of opportunity may be caused by a teacher's inability to provide effective instruction or it may stem from circumstances in which the student has not received instruction because of absences, moving, or the topic not being taught. Individual characteristics inhibiting learning can be abilities (e.g., memory, organizational, attention, etc.), attitudes (e.g., anxiety, beliefs), motivation, self-efficacy, and cultural factors such as language and customs. Most often, a student's need for intervention stems from a combination of these elements. A student's characteristics affect their response to instruction, but instruction may also affect the development and influence of the individual characteristics. The challenge of intervention instruction is the selection of appropriate goals and environmental characteristics which, when matched with the student's individual characteristics, make learning each mathematical topic possible.

Mathematical Concept

The selection of the mathematics concept is typically determined by district, state or national standards. Varying characteristics of mathematics concepts which influence the intervention processes are the complexity of the concept, the transparency of representational models and algorithms, and the students' familiarity with the concept. These characteristics influence the intervention designer's selection of methods, materials and manipulatives to be used in the intervention process.

Environment

Environmental factors are the influences of the context on the intervention process and include teacher characteristics, school climate, setting characteristics and instructional methods. Research has shown that teachers' mathematical ability, attitudes and knowledge affect student learning (Bolyard & Moyer-Packenham, 2008). A school's learning culture affects the goals, motivation, and attitudes of both teachers and learners (Okpala, Smith, Jones, & Ellis, 2000). Factors such as schedules and space availability can effect when, where and in what types of groupings the intervention will take place. The availability of materials and manipulatives can also influence methods and style of instructional presentation.

This study focused specifically on the items in bold lettering in the conceptual framework in Figure 1 (conceptual and procedural knowledge, students with mathematical learning difficulties, equivalent fractions and virtual and physical manipulatives). The study investigated the conceptual and procedural mastery of fraction equivalence through the use of physical and virtual manipulatives for students with mathematics learning difficulties. Each of the four sections of the following literature review focuses on one of the four intervention process elements: intervention goal (i.e., conceptual and procedural knowledge), students with mathematical learning difficulties, mathematical concepts (i.e., equivalent fractions), and environmental tools (i.e., physical manipulatives and virtual manipulatives).

Intervention Goals

Throughout the past century, there has been a gradual development of intervention research, theory and practice. Each phase of the development has seen important changes in the purpose or goals of the intervention practices as people have reevaluated who should receive intervention and when, where and how intervention should be administered.

Early 1900s Goals of Intervention Instruction

Up until the middle of the 1900s, mathematics education focused primarily on preparing students for work in the industrial world, a world in which the average person needed only strong computational skills to be successful. The growth of technology in the 1900s significantly changed people's life styles and their educational needs. These changes have been mirrored in the focus of the development of mathematics education in this century and subsequently in the research and development of intervention instruction (Gersten et al., 2007).

It was with the "new math" movement of the 1950s that the first dramatic shift in mathematics education began. The drive for the new math movement began with the Soviet's launching of Sputnik, which was seen by many in the United States to be a demonstration of the Russians' superior mathematical and science advancements over those of the United States. In response, the U.S. government funded extensive spending programs for research in the field of mathematics education with particular funding attention to the ultimate goal of producing a more scientifically oriented society. Although this funding was primarily focused on the development of high achieving students, some funding was also given to intervention projects of low achieving students. As a result, the concept of learning disabilities was developed in the 1960s. Intervention goals at this time centered mostly on the development of perceptual motor skills and were not content specific (Woodward, 2004).

Late 1900s Goals of Intervention Instruction

In the 1970s the new-math movement was followed by the “back-to-basics” movement and intervention at this time became more individualized and content specific. During this period the influences of behaviorism, with a heavy emphasis on task analysis, began to take a strong hold on education. Mathematics instruction was typically a carefully planned progression of basic skills which were taught in explicit, step by step approaches. Mastery was determined by two elements: accuracy and efficiency (Lampert, 1990).

In 1975, Congress passed the Educational of All Handicapped Children Act of 1975 (Miller, McCoy, & Litcher, 2000; Woodward, 2004). Prior to the passing of this act, many students with disabilities had been denied education opportunities, but this act established the right of all children to receive appropriate education. During this time the goal of mathematics intervention was generally the development of the procedural knowledge students needed to perform the basic life skill tasks. This laid the foundation for most of the research that has been conducted in the field of mathematics intervention (Woodward, 2004). This strong focus on procedural mastery was reinforced by the research work of Pellegrino and Goldman (1987), which reported that a students’ inability to automatically recall facts was a strong predictor of mathematics learning disabilities and Hasselbring, Goin, and Bransford’s (1988) research, which found that by the age of 1, typically achieving students could recall, on average, three times more basic facts than students with mathematical learning disabilities. As a result of these influences, until the 2000s, mathematical intervention instruction and research focused almost exclusively on students with learning disabilities receiving basic fact and algorithm instruction (Woodward, 2004).

Recent Changes in Goals of Intervention Instruction

In the 1900s, information processing and cognitive construction theories became increasingly popular and instructional methods used in mathematics education gradually began to change (Schoenfeld, 2004). Spurring on these changes was the poor showing of U.S. students in two international studies: the Second International Mathematics and Science Study (SIMSS) and the Third International Mathematics and Science Study (TIMSS). These reports revealed the lack of strong mathematical conceptual knowledge of United States students when compared with other students in developed countries (Frykholm, 2004).

In 2000, the National Council of Teachers for Mathematics (NCTM) published its *Principles and Standards for School Mathematics* which is based primarily on constructivist learning theories and evaluation of concept mastery began to focus on both conceptual and procedural understanding. However, the shift towards a more cognitive construction approach of instruction methods has occurred much slower in the field of intervention practices. Typically intervention research and practices have been considered to be under the domain of special education practices which, in most situations, is still predominately influenced by behaviorism (Miller & Mercer, 1997).

In recent years, results of several studies have supported the use of constructivist practices for intervention. Two studies indicated that students with mathematical learning difficulties not only tended to have difficulties with recall but tend to also have more immature strategies for solving problems (Fletcher, Huffman, Bray, & Grupe, 1998; Geary, 1990). Dowker's (2005) research reported a positive correlation between people's mathematical abilities and the number of strategies people used when solving and resolving mathematical problems. Dowker explained:

> Development consists not of the replacement of a single immature strategy or by a single more mature strategy but of the discovery of increasingly more mature strategies, which co-exist for a long time with immature strategies, before gradually supplanting them. (p. 22)

These results suggest that intervention should focus not just on procedural understanding, but also on the development of flexible conceptual understanding. Several other research studies indicate that students with disabilities perform better in schools in which NCTM suggested practices of cooperative learning and the active manipulation of materials are used (Peetsma, Vergeer, Roeleveld, & Karsten, 2001; Rosa, 2002).

Until 2004, the United States government funded only intervention with students who qualified for special education services (D. Fuchs et al., 2008a). This system, used since the 1970's, is sometimes referred to as a "wait to fail" approach because students were not eligible for intervention services until they were academically at least two grade levels behind. The number of students with identified mathematical learning disabilities in the U.S has increased over 200% in the last 10 years and there is growing concern that students with mathematical difficulties are falling in a gap between regular classroom instruction and special education remediation and are not receiving the support they need (VanDerHeyden, Witt, & Barnett, 2006).

In response to these concerns, in 2004, Congress passed the Individuals with

Disabilities Education Improvement Act (P.L. 108-446) giving states the right to redirect a proportion of their funding from the traditional remediation processes to early classroom intervention. A number of states have adopted the “response to intervention” (RtI) approach in which students needing intervention are identified, not by types of disabilities, but by their levels of response to the intervention process (D. Fuchs et al., 2008a). The most commonly used model is the three-tiered design (L. S. Fuchs, D. Fuchs, & Hollenbeck, 2007). Tier I is research proven effective instruction presented in the regular education classroom setting. It is expected that at least 80% of the students will master the concepts taught in Tier I (D. Fuchs et al., 2008a).

Tier II intervention provides additional assistance for students who did not reach mastery through Tier I. Tier II intervention is content specific and is typically conducted by the classroom teacher or a mathematics coach. Students who do not respond to Tier II intervention are identified as nonresponders and are referred for Tier III intervention in special education settings designed to give specialized ongoing individual instruction.

The goals of Tier II intervention vary according to whether the intervention is preliminary, concurrent, or remedial (D. Fuchs et al., 2008a). Students receiving preliminary intervention are identified before implementation of the instructional unit and the goal of the intervention instruction is the development of the prerequisite knowledge and skills needed by the students for the unit of study. Students receiving concurrent intervention are identified from the results of daily assessments and assignments given during the instructional unit and the goal of intervention is to support scaffolding of new learning presented during classroom instruction. Students receiving remedial intervention are identified as not having mastered the concept on posttests and the goal of intervention is mastery of the unit concepts.

Research Needs

Although research investigating the processes of Tier II intervention is still in its infancy, preliminary research has been positive (L. S. Fuchs, 2005; Gersten, Jordan, & Flojo, 2005; Glover & DiPerna, 2007; VanDerHeyden et al., 2006). Yet, developers of RtI programs have reported that the lack of available Tier II instructional materials and lack of knowledge of effective use of instructional tools is limiting program implementation and research (e.g., L. S. Fuchs et al., 2008b; Gersten et al., 2009; Glover & DiPerna, 2007).

Teachers have also expressed the need for additional research in intervention. In 2008, to address the problem of linking research to teacher practices, NCTM brought together a group of 60 mathematics educators who examined 350 questions that over 200 teachers had identified as questions they would like to have answered by research. These 350 questions were then aggregated into seven areas from which 10 theme questions were identified. Three of the 10 theme questions that emerged in this process are relevant to this study: (a) What interventions work with helping students who are having difficulties in mathematics? (b) How can technology be used to facilitate student learning? and (c) What are the frameworks of student thinking development (Arbaugh, Ramirez, Knuth, Kranendonk, & Quander, 2010).

Summary

In the past century, intervention policies, instructional methods and research have gradually evolved and developed. Students who struggle with mathematics are receiving intervention earlier and students who previously did not qualify for intervention are now receiving intervention. The goals of intervention have become more focused on developing both conceptual and procedural understanding. Yet, additional research is needed to guide the planning and implementation of effective intervention instruction.

Students with Mathematical Learning Difficulties

Defining Mathematical Learning Difficulties

The second element affecting the intervention process is the student. When describing students who have difficulty learning mathematics, researchers have used a variety of definitions and terms. Some of the more commonly used terms are mathematical disabilities, mathematical learning disabilities, dyscalculia and mathematical learning difficulties (Mazzocco, 2007). The first three terms are typically used to describe the same population, students who have been or could be identified as having a disability and qualify to receive special education services. These terms imply a disorder that is inherent rather than a disorder resulting from environmental influences. A disorder affects learning in multiple mathematical topics (Gersten et al., 2007). It is generally estimated that approximately 6% of children have this type of mathematical disability (Dowker, 2005; Gersten et al., 2005). In RtI literature, these students are identified as nonresponders and receive Tier III intervention (L. S. Fuchs, 2005).

In contrast, the term mathematical learning difficulty encompasses students whose learning difficulties may be environmental and specific to one or two topics. The term is often used to describe all children below the 35^{th} percentile on a mathematical achievement test. It implies, not necessarily a disability, but low mathematical performance (Gersten et al., 2005). In RtI literature, students with mathematical learning difficulties are those students who have not responded to Tier I intervention and have not been identified as needing Tier III intervention (L. S. Fuchs, 2005). This is the definition which was used in this study.

How Students Differ

An important mediating factor in intervention is the question of whether people with mathematical learning difficulties differ in degree or differ in kind of learning (Dowker, 2005). Differing in degree implies that all students follow the same general paths in learning, but that students with mathematical learning difficulties require more and longer learning sessions. An example of a study supporting the difference in degree theory is that of Staszewski (1988) in which students were taught methods of fast calculations, a skill believed by many to be possible for only students of higher abilities to master. The students received over 300 hours of instruction within a three year period. By the end of the third year, students, regardless of ability, were able to accurately calculate within 30 seconds five digits by two digits multiplication problems. This study implies that given adequate time all students can master computation skills.

In contrast, results from a series of studies conducted by Dowker (2005) indicate that differences in learning of students with varying abilities is not just a difference in the time required to learn concepts, but also a difference in the number of strategies developed and used by students. The greater the mathematical ability, the greater the number of strategies the person is able to use to solve problems. In a study which compared estimation abilities of college students with those of mathematicians, the most striking difference identified was the variety of methods used by the mathematicians who rarely used traditional algorithms. Their deeper understanding of concepts allowed them greater flexibility in their uses of problem solving approaches (Dowker, 1992).

There are research findings indicating that the learning trajectories of lower achieving students differ from higher achieving students because the students differ in how they process information and use strategies. Sheffield (1994) compiled a list of characteristics identifying children with high mathematical abilities as students who could more easily perceive and generalize patterns and relationships: were more curious and aware of quantitative information: could reason both inductively and deductively: could more effectively transfer learning to new situations: were more creative: and were more persistent with difficult problems. In contrast, research results indicate that children with low mathematical abilities tend to have less positive identifying characteristics. Desoete, Roeyers, and Buysee's (2001) and Lucangeli and Cornoldi's (1997) results indicated that low-achieving students tend to be more inaccurate in mathematical tasks and in evaluating and predicting the correctness of their responses. Garrett, Mazzocco, and Baker (2006) found that students with mathematical learning difficulties were less effective in evaluating the correctness of whether their solutions were accurate. Mazzocco (2007) identified the following terms used in research studies to describe children with mathematical learning difficulties: poor retrieval skills, inadequate mathematics skills and procedures, poor selection of strategies and use of immature strategies, slowed response time, inaccurate calculations, and poor recognition of mathematics principles.

In recent years, literacy research results have indicated that the lack of phonetic awareness is a strong predictor of reading disabilities (Mazzocco, 2005). Theorist have questioned if there may be some similar underlying cause of mathematics difficulties which would explain differences in learning. However, the learning of mathematics is more complex with different sets of strategies needed for different types and topics of problems, making it difficult to identify one underlying cause of differences (Dowker, 2005; Mazzocco, 2007). The question of whether students differ in degree or kind is complex. It may be that students differ in degree because their optimum learning trajectories do not match the instructional methods being used and it takes them longer to perform and learn from the instructional tasks. But it may also be that students differ in kind because they learn at a different pace than students without mathematical learning difficulties and therefore appear to have different learning trajectories because they are forced into learning facts and concepts in unnatural sequences. Dowker (2005) explained:

> However, most difficulties in arithmetic, like most difficulties in learned subjects, lie on a ‘normal’ continuum between extreme talent and extreme weakness; and are due not to brain damage but to a mismatch between an individual’s pattern of cognitive strengths and weaknesses and the way that s(he) is taught. (p 11)

Summary

The term mathematical learning difficulties is used in the literature to describe students who, due to differences in environment or learning characteristics, do not respond adequately to regular classroom instruction at a specific time or setting and have not qualified for special education services. The question of whether students of varying ability differ in kind or in degree is an important, but unresolved question. This research study was built upon the assumption that students differ not only in degree, but also kind and that students with mathematical learning difficulties may respond differently to instruction incorporating physical and virtual manipulatives than would students not having mathematical difficulties. It is only through further study that we will discover why some programs do not work well with certain populations (Hiebert, 2003).

Mathematics Content of Equivalent Fractions

The third element of the intervention process is mathematical content, which for this study was equivalent fractions. The Common Core State Standards (2010) suggested that fourth-grade students should develop the ability to recognize two equivalent fractions, generate sets of equivalent fractions and be able to decompose fractions into unit fractions.

For many elementary and middle school students the study of fractions becomes a bottle neck in their mathematical education (Wu, 2005). A strong understanding of fractions is important because fractions are the basis for ratios, proportions, percents, and decimals and students with a weak understanding of fractions are hindered in their understanding of more advanced concepts of geometry, algebra, statistics and calculus (Behr et al., 1983; Chan & Leu, 2007). In their clearinghouse document, *Assisting* *students struggling with mathematics: Response to intervention (RTI) for elementary and middle schools,* Gersten and colleagues (2009) suggested that because of the importance of fraction understanding, intervention for students in grades four through eight should focus on the development of the key concepts of rational numbers. This suggestion is aligned with the NCTM (2006) curriculum focal points and the National Mathematics Advisory Panel (2008) call for U.S. curriculum to provide in-depth coverage of the key topics of numbers to kindergarten through fifth grade and rational numbers from fourth through eighth grades (NCTM, 2006). Yet, Moss (2005) reported that analysis of textbooks indicated the time allotted to teaching fractions was short, with as little as only one lesson developing the concept of equivalent fractions. In a typical elementary classroom 85% of the time is spent on teaching computation or rote procedures while less than 15% is spent on conceptual understanding (Niemi, 1995). This section will discuss the literature describing the difficulties students experience in developing equivalent fraction concepts.

Difficulties with Learning Equivalent Fraction Concepts

The three main indicators of equivalent fraction mastery are the ability to: rename fractions into their simplest forms, generate sets of equivalent fractions, and determine fraction equivalence (Van de Walle, 2004). Chan and Leu (2007) identified five main cognitive difficulties students experience in developing equivalent fraction concepts: (a) conceptualizing fractions as a quantity, (b) partitioning into equal subparts, (c) identifying the unit or whole, (d) building sets of equivalent fractions, and (e) representation model distractions. Chan and Leu tested 2,612 Taiwanese fifth and sixth graders and identified groups of students experiencing each of these five cognitive difficulties. They suggested that for intervention purposes, students could be grouped by the type of difficulty they seemed to be experiencing. The next section will focus on each of these five difficulties.

Conceptualizing fractions as a quantity. For most students, the study of fractions is the first encounter they have with numbers which have multiple names and with numbers being used to represent the relationship between two discrete quantities instead of one discrete quantity (Smith, 2002; Van de Walle, 2004). Most children, at first, attempt to apply whole number rules to fractions and are thereby hindered in their ability to interpret fractions correctly (Arnon et al., 2001; Hecht, Vagi, & Torgesen, 2007; Smith, 2002). In developing fractional understanding students must first develop the understanding that the fraction represents a relationship, the numerator represents the number of parts and that the denominator represents how many parts are in a whole (Smith, 2002). Until this is developed students see fractions only as a pair of whole numbers. The predominance of this misconception is demonstrated in the research of Behr and Post (1992) in which only 24% of 13 year olds estimated the sum of 12/18 + 7/8 to be about one or two. Twenty-eight percent answered 19 and 27% answered 21.

Adding to the complexity of understanding the part whole relationship is the need for students to maintain, while operating on fractions, a conceptualization of the whole. Maintaining conceptualization of the whole is much easier for students when working with region models, such as ¾ of a circle, than when students work with sets, such as ¾ of a group of people (Behr et al., 1983). Students must also develop an understanding that fractions describe the part whole relationship, but not the size of the whole. One half of the small pizza may be less than one third of a larger pizza.

Partitioning into equal subparts. A second cognitive difficulty students experience in developing their fraction equivalence understanding is developing the skill of partitioning into equal subparts (Smith, 2002). Partitioning is defined as "determination of equal shares" (Lamon, 1996). Piaget, Inhelder, and Szeminska (1960) asked children to equally divide an imaginary cake between dolls and observed a developmental sequence in students' learning to partition. In a later study, Pothier and Sawada (1983) identified five stages in the development of students' partitioning behaviors. In stage one, students are able to partition objects into halves and they tend to think of all partitions, regardless of the number of partitions, as halves (Ball, 1993). Sometimes when an object or line is split into sections, the children in this stage see each part as a new whole and not as a fractional part of the original whole. In stage two, students learn to use successive halving to get fourths, eighths, sixteenths, and so forth. When asked to split seven candy bars between three children, only six of the 17 first-grade students initially split the extra candy bar into thirds (Empson, 1995). In stage three, students learn to partition into other even numbered partitions such as sixths and tenths. Typically this is followed by a gap, until students reach stage four in which they overcome their tendency to always half objects and learn to divide objects into thirds , fifths, sevenths, and so forth. Finally, in stage five, students learn to use multiplicative thinking and are able to partition objects into sections that are a product of two odd numbers, such as ninths and fifteenths (Behr & Post, 1992).

Identifying the unit or whole. The third cognitive difficulty many students experience in mastering equivalent fractions is learning to identify the whole and learning to conserve their conceptualization of the whole as models are partitioned. In a study conducted by Kamii and Clark (1995), during individual interviews, 120 fifth- and sixth-grade students were shown two identical paper rectangles. The students watched as the researcher cut the two rectangles in half, one vertically and the other diagonally. When asked the fractional size of each of the pieces all students responded that the pieces were one half. Yet, when asked if the vertically and diagonally cut pieces were the same size only 44% of the fifth graders and 51% of the sixth graders reported that the two parts were the same size. Operationally they knew it was one half and that ½ = ½, but they responded to their perceptual interpretations that one of the pieces was larger than the other. Next the students were shown two more identical rectangles. This time the researcher folded the first rectangle in fourths and cut off a one-fourth strip. The second rectangle was cut into eight strips. The students were asked to show how many of the one-eighth strips would be needed to make the same amount as the three fourths. Only 13% of fifth graders and 32% of the sixth graders got the right answer (Kammi & Clark, 1995). These two examples illustrate the difficulty students have conserving the relationship of the parts to the whole. Research also indicates that most students are not able to visualize nested equal-sized partitions until about fourth grade (Grobecker, 2000). Models similar to the one in Figure 2 are often used to teach equivalence, to illustrate, for example, that three twelfths of the circle (c, d, and e) is equivalent to one fourth of the

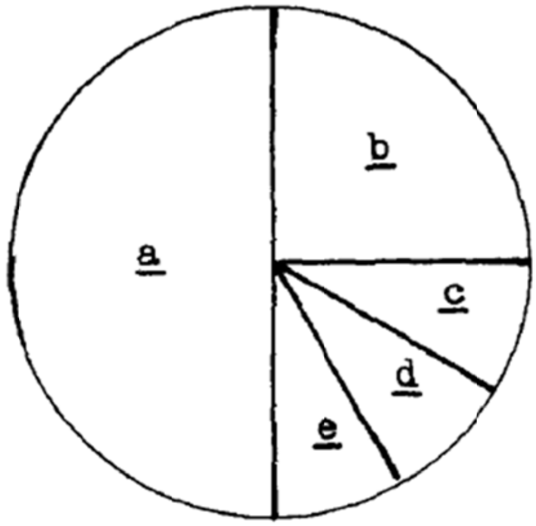


Figure 2. Equivalent fraction model.

circle. Yet, if students have not yet developed conservation of the whole they do not think of three twelfths and one fourth in relation to the whole. Instead they focus on the three twelfths as being three parts of one fourth and "lose" the other three fourths of the circle. The three twelfths becomes three thirds and these students become confused when they are asked to identify the equivalent fractions in the picture (Kamii & Clark, 1995).

Building sets of equivalent fractions. A fourth cognitive difficulty students experience in mastering equivalent fractions is developing an understanding that for each fraction there is a set containing an infinite number of fractions equal to it and that each set defines a distinct rational number (Ni, 2001; Smith, 2002). The process of developing sets of equivalent fractions is for many students the first step in overcoming their tendency to view fractions in terms of whole number rules (Lamon, 1999).

Students also must learn to use multiplicative, instead of additive, thinking when developing equivalent fractions (Kamii & Clark, 1995). Multiplicative thinking means students need to think of a fraction number not just as objects, but as multiplicative groups (Ball, 1993). Many children find the transition of reasoning from additive to multiplicative difficult and will initially seek to solve problems and discover patterns using additive principles (Kent, Arnosky, & McMonagle, 2002; Moss, 2005). When examining a set of equivalent fractions (e.g., 1/3, 2/6, 3/9…) students often first focus on what is added (e.g., one to the numerator and three to the denominator) rather than on the pattern of multiplying the denominator and numerator of the unit fraction by one integer (Moss, 2005). Repeated addition thinking involves only one level of successive thinking (e.g., 3+ 3 + 3 =9) whereas multiplicative thinking requires the student to focus on two levels simultaneously (e.g., one 3 is 3, two 3s is 6, three 3s is 9; Kamii & Clark, 1995). To become fluent in working with equivalent fractions students need to not only see the multiplicative relationship of numerator and denominator between fractions but also the multiplicative relationship between the numerator and denominator of a single fraction (e.g., 24/48 is equivalent to 1/2 because 24 x 2 = 48).

Representational model distractions. The final cognitive difficulty identified by Chan and Leu (2007) is representational model distractions. Fraction representational models are drawings, diagrams, symbols and manipulatives which support the development of children's conceptual understanding and strategies in solving fraction problems (Empson, 2002). The typical representations used in fraction instruction in elementary grades are usually of three types: geometric regions, sets of discrete objects or number lines (Behr & Post, 1992). The geometric region model is the most commonly used model (Witherspoon, 1993). Research results indicate model type affects student learning. In an Australian study, fourth-, sixth- and eighth-year students were asked to identify the fraction modeled by fraction pie, set, and region models of two-fifths. Results are show in Table 1 (Jigyel & Afamasaga-Fuata'I, 2007).

The variability in model results indicates there were perceptual features of the models which limited or enhanced students' ability to identify the fraction represented. Irrelevant components become distracters which must be ignored by the student. Behr and colleagues' (1983) research results demonstrated that as the degree of completeness and consistency of models decreased, students made more and more errors. They suggested that children tend to think that all the conditions presented in the model are relevant and therefore tend to accept rather than ignore distractions. For example in equivalent fraction models many students have a hard time ignoring extra distracting lines. In a rectangle model in Figure 3 students may have difficulty ignoring the horizontal line and understanding that the three columns each represent one-third of the model (Behr & Post, 1992).

Another aspect of model distracters is the difficulties some students have transferring between model types. Witherspoon (1993) asked fifth-grade students to

Table 1

Model Effect

		Percent of students correctly identifying two-fifths		
Type of model	Number	Fourth year	Sixth year	Eighth year
Fraction pie	21	90.0	83.3	90.5
Set model	12	54.5	66.7	47.6
Region model	22	72.7	83.3	57.1

Figure 3. Rectangle model.

illustrate the statements, “There are eight marbles. One fourth of the marbles are white.” One student colored in a small one-eighth portion on each of four circles. Another student drew a large circle and divided it into eight sections. The author’s description of the students’ confusion while drawing the models suggested that the students knew the model was not working, but they did not know what to do to solve the problem.

However, this does not mean that model incompleteness and distracters should always be avoided. Distracters and incomplete representations can help the child as they learn to identify what is relevant and irrelevant and mentally restructure incomplete models. In Martin and Schwartz’s (2002) fraction addition research, treatment groups were taught three identical lessons with each group using either fraction tiles or fraction circles. Fraction circles have a constant and well defined whole (the size of the circle). When students use fraction tiles, they must visualize what the whole is. After the three lessons, students were tested as they solved problems using both manipulatives. Both groups performed equally well on questions using the manipulative they learned with, however the fraction tile group was able to transfer their knowledge to the fraction circles and was significantly more accurate when using fraction circles than were the fraction circle group when using fraction tiles. The researchers concluded that the fraction tile model was more effective because it did not have the conceptualization of the whole built in and therefore students learned to focus on the whole as part of their work. Also, as students develop stronger conceptual understanding, a good representation can be a model that causes a certain amount of confusion thus creating cognitive disequilibrium which challenges students to rethink or restructure their understandings (Behr et al., 1983).

Summary

The concepts of equivalent fractions are important building blocks for fraction computation and a number of other mathematical topics. Yet, many students have difficulty developing equivalent fraction understanding, in part because many students must learn to overcome their tendency to inappropriately apply previous learning to fractions. Five cognitive difficulties many students have in developing equivalent fraction understanding have been identified in the literature: (a) conceptualizing fractions as a quantity, (b) partitioning into equal subparts, (c) identifying the unit or whole, (d) building sets of equivalent fractions, and (e) representation model distractions.

Environment: Physical and Virtual Manipulatives

The fourth component of intervention instruction is the environmental features. This component includes the teacher, school, setting and instructional characteristics. Intervention settings have several characteristics that differ from regular classroom instruction. Tier II intervention is often conducted in small group settings under the supervision of a math coach, teacher, paraprofessional, or volunteer. While the intervention setting offers more opportunity for individualization and immediate feedback, there are fewer opportunities for learning from peer modeling and discussions.

Also because of time limitations, materials, and space, instructors providing intervention often tend to use instruction which is more direct and explicit with little time given to problem solving activities. One possible result of differences between the classroom setting and the intervention settings is that instructional methods that have been proven to be effective in classroom instruction may not be found to be as effective in intervention settings. Also emphasized even more in intervention settings are demands for efficiency of instruction. It becomes paramount that instructors are able to select the manipulative which most closely fits the goals of instruction, the needs of the students, and the demands of the mathematical topic. This section will discuss the literature addressing the use of physical and virtual manipulatives as tools of instruction. First will be a description of the theory of representation and the use of manipulatives in developing representational images. Next will be a discussion of the relevant literature concerning the effects of physical and virtual manipulatives on student achievement. This will be followed by comparison of effects of specific physical and virtual manipulative characteristics on student learning.

Developing Representations

Gersten and colleagues (2009) conducted an extensive review of RtI literature and made eight research-based recommendations for setting up effective RtI programs for mathematical interventions. The fifth recommendation read:

> Intervention materials should include opportunities for students to work with visual representations of mathematical ideas and interventionists should be proficient in the use of visual representations of mathematical ideas. (p. 30)

Gersten and colleagues (2009) explained that the problems mathematically-at-risk students struggle most with is their lack of ability to connect the abstract symbols of mathematics to various visual representations. The researchers suggested that the “occasional and unsystematic” presentations of representations in the typical classroom is not enough to facilitate learning for students with mathematical difficulties and that intervention instruction must place strong emphasis on a systematic scaffolding of students’ representational models.

External representations (e.g., manipulatives, drawings, mathematical tables, etc.) are used to aid students in their development of internal representations (Behr et al., 1983). Students’ internal representations can be in the form of: (a) verbal/syntactic images in a person’s natural language, (b) mental images, (c) formal notation as students mentally manipulate numbers, and (d) affective images including emotions, attitudes, beliefs and values (Goldin & Shteingold, 2001). A student’s conceptual understanding of mathematical concepts rests in the power and flexibility of their internal representations and it is believed that students with mathematical learning difficulties often experience difficulties because they have developed only partial internal systems of representations (Goldin & Shteingold, 2001). The purpose of using manipulatives is to help students develop the internal representations necessary to give meaning to symbolic representations (Baroody, 1989).

Using Manipulatives to Develop Representations

Physical manipulatives are concrete objects which students use to explore mathematical concepts (McNeil & Jarvin, 2007). Virtual manipulatives are an “interactive, web based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge” (Moyer et al., 2002, p. 373). Advocacy for the use of manipulatives centers on a number of learning theories. Piagetian theory suggests that children learn best by actively manipulating objects and reflecting on the results of their physical actions (Baroody, 1989). The theories of Piaget, Bruner and Montessori are built upon the concept that students must develop and build knowledge from concrete to abstract and that the more experience students have with the concrete, the greater will be their conceptual understanding (McNeil & Jarvin, 2007).

The impact of student learning through manipulative use is demonstrated in two studies conducted by Martin and Schwartz (2005). In both studies, they compared the learning of students who manipulated objects to a control group who did not manipulate the objects. In the first study, both groups received the same instruction, but the treatment group manipulated fraction pies and tiles while the other group made marks on pictorial representations. Students moving the objects solved significantly more problems and tried more strategies than those students who did not manipulate objects. In the second study, children performed better when they physically rearranged the objects to find the solution than when the objects were prearranged for them. The authors suggested that physically moving the pieces helped the children to let go of their previously held whole number understandings.

Through their research and a review of the literature, Martin and Schwartz (2005) identified four levels in which the physical action of manipulating objects supports student thinking and learning: induction, offloading, repurposing, and physically distributed learning. Induction occurs when students, through the use of manipulatives, use inductive reasoning to change their understandings (Martin & Schwartz, 2005). For example, laying two one-eighth fraction pieces onto a one-fourth piece helps students understand the size relationships of the fractions and through reflection they begin to interiorize and visualize the mathematical concept of equivalence (Arnon et al., 2001). At the offloading level, students use the objects to keep track of elements, freeing up internal memory and making learning easier and more efficient (Cary & Carlson, 1999; Martin & Schwartz, 2005). Students functioning at the repurposing level change their environment enabling them to more efficiently implement their understanding (Martin & Schwartz, 2005). In physically distributed learning, both student understanding and the manipulative are changed so that the development of new ideas is distributed from both the physical adaptation and the individual. For example, when solving one fourth of eight, the student may think one fourth only as one fourth of one whole object but as the student puts eight objects into four groups, the student learns to reinterpret the two objects as a group of one, thereby overcoming their whole quantity interpretation of one. In physically distributed learning, the learning is situated in both the student understanding and in the action of manipulation (Martin & Schwartz, 2005).

Research reports from the Rational Number Project, a project in which the development of representations through the use of manipulatives received a heavy emphasis, indicated that students using manipulatives significantly outperformed other students taught using the more symbolic approach (Cramer, Post, & delMas, 2002). They reported four ways in which the use of manipulatives helped students understand fractions: (a) they helped students develop mental images of fraction meaning, (b) they helped students understand fractional size, (c) they gave students a reference when justifying their answers, and (d) students were less apt to resort to the misconceptions developed from applying whole number rules to fractions.

Effectiveness of Physical and Virtual Manipulatives

A large number of research studies have examined the effectiveness of using physical manipulatives in mathematical instruction. Three meta-analysis reports were identified which summarize the results of these studies. Suydam and Higgins (1977) evaluated 23 studies conducted during 1930 to 1970. Eleven of the studies showed significant differences in student achievement favoring the use of manipulatives, two studies favored not using manipulatives and in the remaining 10 studies no significant differences were found between use and nonuse of manipulatives. The researchers also reported that, for students of all age groups and ability groups, the majority of studies reported that students tested higher when using manipulatives than when using other methods of instruction. The majority of studies involving fraction instruction reported significant differences favoring the use of manipulatives.

Parham's (1983) meta-analysis from 64 studies conducted between 1960 to 1982 obtained 171 effect size scores comparing the use of manipulatives with nonuse on student achievement. The averaged mean effect size was 1.03, indicating a large effect size favoring manipulative use. Parham, however, expressed concern that the effect size may be inflated by study quality. Fifty-three of the 64 studies were unpublished studies and although Parham had already eliminated one-third of the studies of poorer quality, analyses indicated that the mean effect size from 40 studies that did not show evidence of equivalency in ability of treatment groups was 0.99, while the effect size was only 0.38 for the remaining studies that either showed evidence of group equality or used random assignment of students to treatment groups. This difference was significant.

Sowell (1989) calculated effect size scores from 60 studies in which the use of manipulatives was compared with other instructional methods. The studies were separated into two main categories, those studies using specific objectives and those using broad objectives. Only the two categories of studies which yielded significant mean effect sizes were studies, of at least one year duration, using broad objectives (0.29) and specific objectives (1.89). The effect sizes of studies of shorter durations were not significant and results were mixed. A comparison of 13 studies of retention, when compared with traditional instruction, produced an effect score of 0.38. These results indicate that when used for over a year, physical manipulatives are effective tools for mathematics instruction.

Moyer-Packenham and colleagues (2012) conducted a meta-analysis evaluating the effect of virtual manipulatives on student learning. The analysis of 82 effect scores obtained from 32 studies yielded a moderate average effect size of 0.35 when compared with the use of other methods of instruction. When virtual manipulatives were used alone as the primary tool of instruction and was compared with instruction using physical manipulatives and with traditional classroom instruction, the averaged effect scores were a small effect of 0.15 (38 effect scores) and moderate of 0.75 (18 effect scores) respectively. The researchers also conducted an analysis of effect size scores in relation to subject matter that resulted in a moderate averaged effect score of 0.53 (11 effect scores) when using virtual manipulatives for teaching fractions as compared to other types of instruction. From the studies, 26 effect size scores were identified in which instruction combining the use of virtual and physical manipulatives was compared with other instructional methods. When virtual manipulatives were used in combination with physical manipulatives and compared with all other forms of instruction effect scores produced a moderate effect of 0.33. Results of the meta-analysis indicate that virtual manipulatives are an effective in teaching mathematical concepts, that there may be an advantage to combining the use of virtual and physical manipulatives in instruction and that virtual manipulatives are effective in teaching fractions.

Use of Manipulatives in Instructing Students with Mathematical Difficulties

Five studies were identified in which the use of physical manipulatives with students with mathematical learning difficulties was investigated. Butler, Miller, Crehan, Babbit, and Pierce (2003) assigned 50 sixth-, seventh-, and eighth-grade students with mild to moderate mathematical disabilities to two treatment groups. Both groups received identical equivalent fraction instruction with the exception that one group used physical manipulatives during the first three of ten lessons. This group scored significantly higher on all five subtests and significantly higher overall. Witzel, Mercer, and Miller (2003) compared algebra posttest scores of 34 matched pairs of sixth- and seventh-grade students with mathematical learning disorders. Those students who had received treatment involving physical manipulatives significantly outperformed those involved in traditional instruction. Results of Moch's (2001) study with 15 fifth-grade students, of which one third were students requiring special services. Cass, Cates, Smith, and Jackson's (2003) study with three fourth-grade students with learning disabilities, and Maccini and Hughes' (2000) study with six adolescents with learning disabilities reported that students' scores and understanding improved after instruction with manipulatives.

A search of the literature identified seven studies in which the use of virtual manipulatives with students of differing mathematical abilities was investigated (Moyer-Packenham et al., 2012). Both Drickey's (2000) research with 219 sixth-grade students and Kim's (1993) research with 35 kindergarten students compared students of different ability levels and found no significant difference. However, in a third study Moreno and Mayer's (1999) analysis did indicate that sixth-grade students with high mathematical and spatial abilities benefit more from virtual manipulative instruction than those with low abilities. In their study, they used the same integer applet for the experimental and control group, except that the applet of the experimental group also included symbolic representation. Although posttest-score analysis indicated there was not a significant difference between the groups, when students were further grouped by ability, comparison of symbolic linked and nonsymbolic linked applets of students with high ability produced an effect size of 1.11 while gain scores of the low ability student produced an effect size of -0.47. Moreno and Mayer also grouped the students according to spatial and memory abilities. Students with high spatial abilities had, on average, gain scores which were six times greater than the gain scores of the students with low spatial abilities, but results comparing differences in memory were not statistically significant. Results of this study indicate that there may be difference in the effectiveness of certain virtual manipulatives among students of differing abilities.

Moyer-Packenham and Suh (2012) compared gains for low, average and high ability groups of fifth-grade students who used virtual manipulates in the study of fractions. Results of paired samples *t* tests indicated that although all three groups achieved gains, the gains were significant for only the low achieving group. Similarly, Lin, Shao, Wong, Li, and Niramitranon (2011) and Hativa and Cohen (1995) found low achieving sixth- and fourth-grade students (respectively) made greater gains than did higher achieving students when participating in instruction using virtual manipulatives.

Suh, Moyer, and Heo (2005) observed virtual manipulative use of 46 fifth graders who had been grouped into high, average and low ability instructional groups for fraction instruction. Researchers observing the different classrooms reported that the high achievement group was more efficient and used more mental processes for finding answers, while the low groups tended to be more methodical and followed each step of the program. The low groups were observed to also be more dependent on using the visual models to scaffold between the pictorial and symbolic.

Three studies were identified that investigated virtual manipulative use with students receiving special education services. All three reported positive effects and two of the studies reported students using virtual manipulatives outperformed students who did not use manipulatives (Guevara, 2009; Hitchcock & Noonan, 2000; Suh & Moyer-Packenham, 2008).

One concern that has been expressed about the use of virtual manipulatives with students with mathematical learning difficulties is that students' frustrations with computer manipulation may cause cognitive overload (Highfield & Mulligan, 2007; Sorden, 2005). The cognitive overload theory of John Sweller (Clark, Nguyen, & Sweller, 2006) suggested that a person's working memory is limited to five to nine items at one time. Once a person has reached cognitive overload they become limited in their ability to absorb new information. Concern has been expressed that when computer manipulation utilizes part of the working memory, less memory is available for processing the concepts. Others, however, suggest that the use of virtual manipulatives can lessen the cognitive demands through off loading and dual coding. An element in the distributed learning theory of Martin and Schwartz (2005) is the use of manipulatives to off-load information. The manipulatives hold the information for the user, freeing their memory and reducing cognitive overload. Dual coding theory suggests that the use of more than one mode produces an additive affect, increasing memory effectiveness (Clark & Paivio, 1991). The use of the linked dual modes in virtual manipulatives further enhances the users' cognitive abilities (Moreno & Mayer, 1999; Suh & Moyer-Packenham, 2007).

In summary, results of studies investigating the use of manipulatives with students of differing abilities indicate that, although there are variations, students of all abilities may benefit from the use of manipulatives.

Comparison of Physical and Virtual Manipulative Characteristics

Although evidence indicates that overall the use of manipulatives improves student achievement, individual results are still mixed (McNeil & Jarvin, 2007). One of the variables mediating the effectiveness of instruction involving manipulatives is characteristics of the individual manipulatives. The next section will discuss the literature relating to the structure, representations, constraints, distracters and usage affordances of both virtual and physical manipulatives.

Structure. Manipulatives can be used in both problem solving activities and in explicit guided instruction (Martin & Swartz, 2005; McNeil & Jarvin, 2007). Although most physical manipulatives do not have defined structures which guide students in usage, a number of the virtual manipulatives do and some applets are designed to teach specific mathematical skills and concepts by guiding students through explicit steps (Heal, Dorward, & Cannon, 2002; Suh & Moyer, 2007). These applets typically have features which give students instant feedback. Clements, Battista, and Sarama (2001) and Highfield and Mulligan (2007) indicated that as a result of feedback, students were more experimental in developing representations, making conjectures and in testing their ideas. Although the applet feedback has been identified as an affordance (e.g., Deliyianni, Michael, & Pitta-Pantazi, 2006; Highfield & Mulligan; 2007; Izydorczak, 2003; Steen, Brooks, & Lyon, 2006; Suh et al., 2005) research on the effects of applet feedback on student learning is limited.

Concern has been expressed that the supports built into some virtual manipulatives can allow students with mathematical learning difficulties to develop rote procedures (Izydorczak, 2003). If the supports allow students to complete procedures without reflecting on the connections between their actions and the mathematical concepts, the use of manipulatives becomes mechanical and students fail to develop understanding (Martin & Schwartz, 2002; Moyer, 2001). Students can become locked into what Sayeski (2008) calls “search space” in which they lock into using only one method and will not back track or seek to take different approaches to find solutions. Rather than experimenting or trying to fix mistakes by changing their conceptual thinking, the students simply hit reset, new problem or the help button (Izydorczak, 2003).

Linking representations. Another important difference between most physical and virtual manipulatives is the degree and manner in which representations are linked. A few physical manipulatives, such as fraction tiles, typically have symbolic representations written on the pieces, but most physical manipulatives and some virtual manipulatives (e.g., pattern blocks) do not have features connecting the object representations to the symbolic. In contrast many virtual manipulatives are designed specifically to support students in linking abstract symbolic representations to more concrete visual images (Bolyard, 2006; Heal et al., 2002). As students interact with the objects in these virtual manipulative applets, they can relate changes in the concrete representation to changes in the symbolic representation as a result of their actions (Moyer et al., 2005). In interviews conducted by Haistings (2009), students reported that they preferred an applet that contained both symbolic and pictorial representations over an applet with only pictorial representations because the problem was written for them on the screen, they did not have to keep recounting the number of blocks, they could confirm if they set up the problem correctly, they did not have to remember large numbers and they enjoyed seeing the numbers change when they lassoed blocks. These comments would indicate that the students made strong symbolic-pictorial links.

Amplification and constraints. Manipulative objects have built in constraints and amplifications which can limit or enhance their use in different settings (Behr et al., 1983). Takahashi's (2002) observations demonstrate the effects manipulative amplification and constraints can have on students' learning as they developed formulas of area. Takahashi reported that the virtual manipulative applets required students to perform the tasks step-by-step. This took more time, but also focused students' attention on the characteristics of the geometric shapes. In contrast those using the physical geoboards focused more on visually counting the squares. When calculating the area of shapes which had to be transformed into other shapes for area calculation (e.g., triangles are transformed into rectangular shapes), the students using the physical geo-boards still relied more on counting the squares, while those using the virtual geo-boards were more apt to look for equivalent area transformations and then use the formulas they had developed.

Some virtual manipulatives have been specifically designed to amplify mathematical concepts (Dorward & Heal, 1999; Moyer-Packenham, Salkind, & Bolyard, 2008; Suh, 2010). A review of research literature identified three processes of amplifying mathematical concepts that affected student learning: (a) requiring specific actions, (b) demonstrating simultaneous changes, and (c) focusing student attention or constraining on specific aspects or characteristics of objects, concepts, or procedural fluency (Moyer-Packenham et al., 2012). For example, Beck and Huse (2007) reported that students spinning a virtual spinner observed how visually the changes in the computer bar graph which decreased as the number of spins increased, it amplified the differences between experimental and theoretical probability for the students.

Distracters. As reported in the preceding section describing cognitive difficulties of fraction learning, each manipulative contains cognitive distracters which students must learn to ignore. Some researchers have expressed concern that there are features of virtual manipulative applets which can make their use less effective for students with mathematical learning difficulties. Highfield and Mulligan (2007) and Izydorczak (2003) reported that to some children the ability to change languages and the color and shape of objects was a distraction. The students became so focused on altering the features of their applets that they failed to learn the concept or to complete the assigned mathematical tasks.

Ease of use. Manipulatives also vary in the degree and ease with which they can be manipulated. Physical manipulatives can be physically handled and manipulated by students while virtual manipulatives are not physically handled, but are manipulated through the use of the computer mouse. If characteristics of the manipulative object make manipulation too difficult, students will not reach a level of automaticity in its use and effectiveness of the manipulative will be limited and may even be detrimental to student learning (Boulton-Lewis, 1998). Both advantages and disadvantages of the ease of use of physical and virtual manipulatives have been reported. Haistings (2009) and Izydorczak

(2003) observed that students using physical manipulatives often “sloppily” stacked and arranged the physical manipulatives while those using virtual manipulatives were more organized and thus were more accurate in their answers. Kim (1993) reported that Kindergarten students using virtual manipulatives were more methodical and purposeful than those using physical manipulatives. Some researchers indicated that physical manipulatives were less cumbersome for students to manipulate than the virtual manipulatives and that while using them students completed tasks quicker (e.g., Baturo, Cooper, & Thomas, 2003; Haistings, 2009; Highfield & Mulligan, 2007; Hsiao, 2001; Izydorczak, 2003; Kim, 1993; Nute, 1997; Takahashi, 2002). Other researchers indicated that the virtual manipulative applets were easier to manipulate and that features such as cloning objects and rapid repetition of computer actions made it possible for students to complete more work (e.g., Beck & Huse, 2007; Clements & Sarama, 2002; Deliyianni et al., 2006; Izydorczak, 2003; Steen et al., 2006; Terry, 1995; Yuan, Lee, & Wang, 2010). Several researchers also reported that students using virtual manipulatives created a greater variety of responses than those using other methods of instruction (e.g., Clements & Sarama, 2007; Heal et al., 2002; Highfield & Mulligan, 2007; Moyer et al., 2005; Suh et al., 2005; Thompson, 1992).

Summary

Manipulatives are used in mathematics instruction to support students in their development of representations. Research results indicate that the use of physical manipulatives or virtual manipulatives generally has positive effects on student achievement. Studies comparing use of manipulatives with differing abilities suggests that instruction using manipulatives is effective for both high and low achieving students, but that there may be differences in how students of differing abilities use and learn from the manipulatives. Although limited, the research indicates that there may be advantages to combining the use of physical and virtual manipulatives in instruction. Comparisons of physical and virtual manipulatives suggests that each manipulative has distinct affordances and limitations. As suggested by Behr and colleagues (1983), while a manipulative may be used to illustrate effectively one concept it may in fact impede a student's learning when used to illustrate another concept. They suggested that research needs to be designed that will identify which manipulative will facilitate specific mathematical learning. To take advantage of the affordances of manipulatives and to produce higher student achievement it is a necessary to identify and compare the learning effects of different manipulatives used to teach specific mathematical concepts. The purpose of this research study was to identify variations in the learning trajectories of students with mathematical learning difficulties when learning equivalent fraction concepts during instruction using virtual and physical manipulatives.

CHAPTER III

METHODS

Research Design

The purpose of this research study was to identify variations in the learning trajectories of students with mathematical learning difficulties when learning equivalent fraction concepts during instruction using virtual and physical manipulatives. This study also served as a pilot study used to validate study instruments and protocol for future research. The overarching research question and subquestions guiding the study were as follows.

1. What variations occur in the learning trajectories of students with mathematical learning difficulties that are unique to the use of different instructional manipulatives for intervention (virtual manipulatives, physical manipulatives or a combination of virtual and physical manipulatives) in the learning of equivalent fraction concepts?

   a. What are the variations of achievement, mastery, retention, and resolutions of errors in students' development of equivalent fraction concepts and skills?
   b. What are the variations in learning trajectories showing changes in student achievement over time?
   c. What are the variations in patterns of daily lesson achievement, retention and work completion?
   d. What are the variations in the strategies developed and used by students?
   e. What are the variations in students' use of representations?

This study used a sequential explanatory mixed methods approach of triangulating evidence from both quantitative and qualitative data in answering each of the research questions (Creswell, Plano Clark, Gutmann, & Hanson, 2003). Building from constructivist epistemology, this study, through the observations of student learning in the environment of virtual and physical manipulatives, describes how students' building of equivalent fraction understanding is affected by manipulative use. The research for the study was conducted during a three month time frame in four public schools. The research activities included Tier II intervention for fifth-grade students who did not demonstrate mastery of equivalent fractions concepts; concepts which the Common Core and the Utah state core suggest should be mastered in fourth grade. Data were collected from equivalent fraction tests, lesson assessments, instructors' logs and lesson artifacts (activity sheets, explore papers, and videotapes). Data analysis focused on the development of learning trajectories to develop models of the progress students make while constructing equivalent fraction understanding using virtual and physical manipulatives. Clements and Sarama (2004) explained the concept and use of learning trajectories.

> We conceptualize learning trajectories as descriptions of children's thinking and learning in a specific mathematical domain and a related conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain. (p. 83)

In this study, learning trajectories were used as a framework for building an understanding of how students' development of equivalent fraction understanding was influenced by manipulative types. Data were analyzed using a transitional conceptions perspective; not only were pre and post intervention data analyzed, but the data were also analyzed at the individual lesson and student levels, thus making it possible to identify the effects of different manipulatives on the spectrum of students' understanding as they developed equivalent fraction concepts as well as determining the overall effect on student learning and learning trajectories (Shaughnessy, 2007).

Participants and Setting

Forty-three fifth-grade students from four schools participated in this study. The concept of equivalent fractions is introduced in third grade and expanded in fourth grade. After completing fourth grade it is expected that students have developed the basic equivalent fraction understanding needed as a base for understanding more advanced fraction concepts. As a preliminary intervention, only students who had not yet participated in fifth-grade fraction instruction in the regular classroom were selected for participation. The equivalent fractions pretest was given to all fifth-grade students in the participating classrooms of the four schools. In total, 182 students completed the test. Student scores ranged from 5% to100% correct with a mean score of 51.1%. Students who scored below 40% and who were identified by teachers as having math learning difficulties were identified for participation in the study. Teachers also requested that eight other students be allowed to participate. These were students who had in the past experienced difficulty learning mathematical concepts. Seven of the students scored between 42% and 46% correct. One student scored 57%. Because this research was designed to target Tier II intervention, students who were receiving special education services in mathematics were not included. At the time of the study there were, in the four participating schools, four students receiving special education assistance for mathematics. This was 2.2% of the students. IRB and school district approval was obtained to conduct the study. Permission forms were obtained from the participating students and their parents. Of the 52 students invited to participate, 45 returned permission slips. After the second lesson, the parents of two students expressed concern that their students were not completing their regular classroom activities and opted to have their students removed from the intervention. In total, 43 students completed the intervention instruction. One student was not available for the delayed posttesting.

Participants were assigned to one of three intervention groups through a stratified selection process based on pretest scores. For each school, the three qualifying students with the highest, second highest, and third highest pretest scores were assigned to groups one, two and three, respectively. Students with the fourth, fifth and sixth highest scores were assigned to groups two, three and one respectively until all students were assigned to one of the three groups. Groups were then randomly assigned to one of three interventions: physical manipulatives alone (PM group), virtual manipulatives alone (VM group) or physical and virtual manipulative combined group (CM group). Instructional groups consisted of two to four students per group.

Procedures

The study consisted of three phases: preintervention, intervention, and data analysis. In the preintervention phase the researcher developed the lesson assessments and the equivalent fraction tests. Also during this time the necessary IRB approval and district approval were obtained. During the intervention phase, a pretest was administered to all fifth-grade students in the participating schools. Students scoring below the established criterion level were invited to participate in the study (with the exception of students receiving special education services). The intervention consisted of 10 instructional lessons. Data were gathered during the intervention lessons from assessments, instructor logs, activity sheets and videotaping. At the conclusion of the tenth instructional lesson, students completed a posttest. Three to four weeks after the final instructional lesson, participants completed a delayed posttest. In the final phase, data collected during the intervention was analyzed and results were synthesized to develop learning trajectories which were used to identify variations among student groups related to manipulative use.

Format of the Instructional Lessons

The instructional lessons followed the Rational Number Project: Initial Fraction Ideas Lessons (Cramer, Behr, Post, & Lesh, 2009) with adaptations to accommodate differences for the physical or virtual manipulatives. The Rational Number Project (RNP) is a series of 23 lessons designed and tested for fraction study of middle grades students. It was first published in 1997 and was revised and published again in 2009. The lessons were designed as an alternative to textbook instruction and have been used successfully by both regular and intervention settings. Each lesson provides students with hands on experience using concrete manipulatives. These lessons have been used with over 1,600 students and pilot testing results indicate the performance of students who used these lessons is significantly greater than those using traditional textbook lessons. Lessons 1 through 13 from the Rational Number Project were adapted and used in this study. Two additional activities, concept practice and lesson assessment, were added to each lesson. The lessons in this study consisted of four phases: (a) lesson pre assessment, (b) explore, (c) practice, and (d) lesson concept assessments. Each lesson lasted approximately 45 minutes. Students worked in groups of two to four students. The lesson sequence is summarized in Table 2.

During the lesson preassessment phase, students, using paper and pencil, answered two questions designed to assess if mastery of the previous lesson concept had been retained. In the explore phase of the lesson, the instructors guided students in discovery and discussion of the lesson concepts as outlined in the RNP lessons. Students verbally responded to questions and performed activities using the manipulative of their assigned intervention groups. In this and in the subsequent phases, when students failed to respond accurately to problems and instruction, the outlined instructional sequence was

Table 2

Lesson Sequence

Phase	Duration	Activity	Purpose
Lesson preassessment	5 minutes	2 review questions	Determine retention of previous lesson's concepts
Explore	20 minutes	Concept development and completion of activity sheets	Concept discovery and application
Practice	10 minutes	Practice	Practice of fraction skills
Assessments - Concept - Cumulative	10 minutes	10 questions	Determine concept mastery and cumulative fraction understanding

first repeated. If after repetition of instruction, students responded incorrectly, the instructor again repeated the instruction, using different words, word order, or pictorial representations. If students continued to respond incorrectly, the instructor returned to previous lesson concepts and repeated instruction scaffolding up to the concept causing the misunderstanding.

The RNP lessons include from one to six student activity sheets that were used in the explore phase of each lesson. Students solved problems using their assigned manipulative and recorded their answers on the activity sheets. Instructors attempted to provide immediate feedback to the students and when necessary retaught the concepts. Appendix A contains a table listing the lesson concepts and describing which activity sheets and manipulatives were used in each lesson.

In the practice phase, students were involved in approximately 10 minutes of additional practice, naming, comparing, and simplifying fractions and in finding multiple groups of equivalent fractions. For the VM and CM intervention groups, the computer presented the problems and gave feedback. For the PM intervention group, the instructor presented the problems and gave immediate feedback to students. Students in all three groups completed similar problems.

In the lesson assessment phase, the teacher first conducted a short discussion prompting students to summarize the lessons' concepts. Students then, without assistance, individually completed the lesson concept assessment (three questions) and the daily cumulative assessment (eight questions). Students were encouraged to use their assigned manipulative while responding to assessment questions.

Lesson Instructors

Instruction for each intervention group was conducted by the researcher and a second trained instructor. Both instructors had over 25 years of public school teaching experience. The second instructor received training from the researcher on the specific instructional procedures to be followed throughout the study. During the majority of the instructional lessons, the researcher and the second instructor taught in the same room and were able to synchronize the instruction and the duration of each instructional phase.

Manipulatives

Four types of physical manipulatives and six virtual manipulatives were used in the study. The next section contains descriptions of the physical manipulatives and their corresponding virtual manipulatives which were used during the explore and practice phases of the lessons. Appendix B contains tables summarizing the similarities and differences of each of the physical and virtual manipulative combinations.

Manipulatives used in explore activities. Two types of virtual and three types of physical manipulatives were used in the explore activities.

Physical fraction circles and virtual Fraction Pieces. The concrete fraction circles consist of eight different colored plastic circles partitioned into halves, thirds, fourths, fifths, sixths, eighths, tenths, and twelfths. Each set also contains one whole circle. None of the fraction pieces in the sets contain symbols. The virtual fraction circles are a web-based applet found on the National Library of Virtual Manipulatives (NLVM; found at http://nlvm.usu.edu; retrieved June 25, 2012). In this applet students can work with circular or square models. When students click on the pieces bins at the left of the applet, the pieces appear in the center of the work area (see Figure 4).

There are several distinct differences between the physical and virtual manipulatives. The physical fraction circles offer more flexibility in manipulation and movement, but the objects in the virtual applet can quickly be placed in and removed from the work area and colors of the objects can be changed, features which can help students organize their activities. As students bring the virtual fraction objects near to the whole region, the objects are automatically grabbed and clicked into position making it easier for students to build accurate models. Although when performing some movements, the computer's grabbing and clicking of objects makes manipulation of the objects more difficult. Students can also lasso pieces together so that multiple pieces can be held together and moved as one chunk. Another important difference is the option in the virtual manipulative applet to show fraction labels, thereby linking symbolic and pictorial representations.

Counters and pattern blocks. Counters, two-sided (red and yellow) round tokens, were used to represent fractions using a set model (see Figure 5). The corresponding

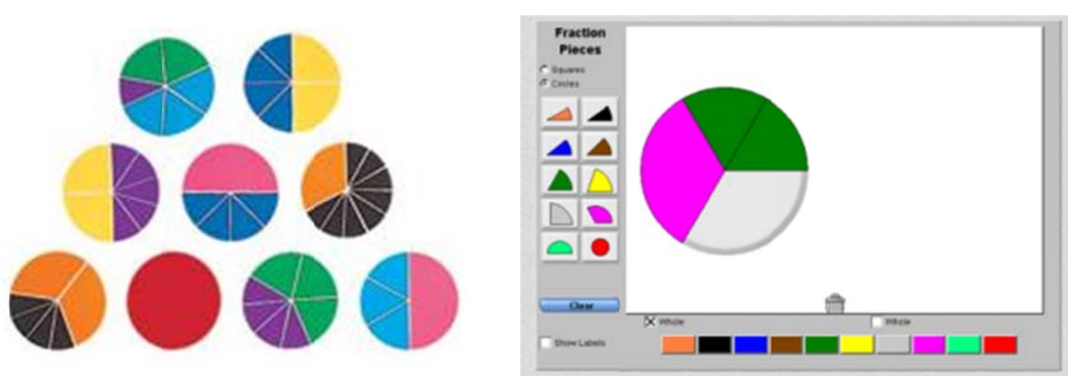


Figure 4. Fraction circles and virtual fraction pieces.

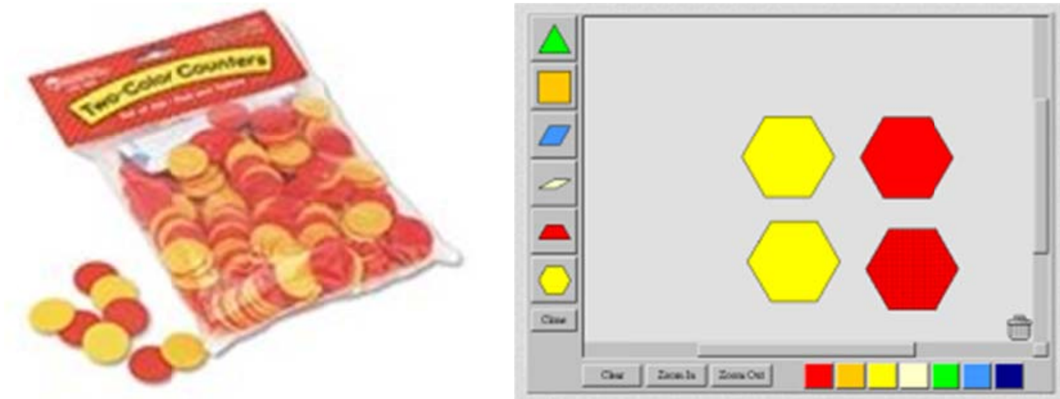


Figure 5. Two-color counters and pattern block applet.

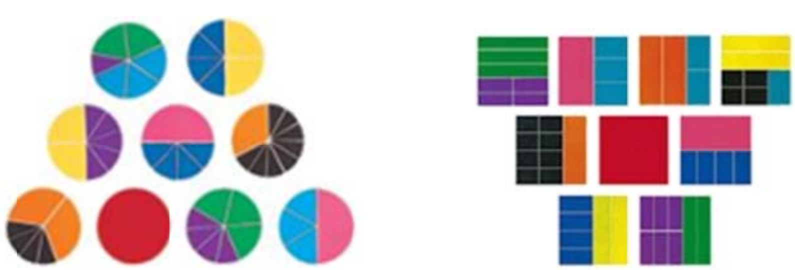
virtual applet was pattern blocks, a web-based applet from the NLVM site. When using the applet, students clicked on the shape bins to place blocks in the work area. Students could then drag the blocks to any location in the work space, making it possible for students to form groups of sets representing fractional amounts.

For the lessons used in this study, the differences between the manipulatives are limited to several features of the virtual pattern blocks applet, which can help users be more organized and efficient in their manipulation of objects. The virtual objects automatically snap together and can be lassoed and cloned to make the organization and duplication of sets easier.

Fraction strips. The fraction strips used in this study were long strips of paper used to represent the length model of fractions. Students folded the strips into halves, thirds, fourths, sixths and eighths. No alternative virtual manipulative was identified which could be used in the activity using fraction strips in introduction of Lesson 2. For this activity all three intervention groups used the physical fraction strips.

Manipulatives used for practice instruction. Student activities during the practice phase of the lessons involved the additional pairing of four virtual applets with the physical manipulatives of fraction circles and fraction squares. Fraction circles were the same as those used in the explore phases of the lessons. Fraction squares are a set of different colored squares divided into halves, thirds, fourths, fifths, sixths, eighths, ninths, and tenths, and twelfths (see Figure 6).

The corresponding virtual manipulative applets used in the practice phase were fractions-naming, fractions-equivalence, fractions-comparison from the NLVM site and equivalent fraction from the NCTM Illuminations web site (http://illuminations.nctm.org; retrieved June 25, 2012). In contrast to the open structure of the physical fraction circles and squares, the four virtual applets have guided procedure formats in which the computer presents to students a problem and then guides them through the process of solving it. The fractions-naming applet guides students in the naming the numerator and denominator represented in fraction models. The fractions-comparison applet guides students in the process of comparing fractions by finding common denominators and the fractions-equivalence and equivalent fraction applets guide students in the process of

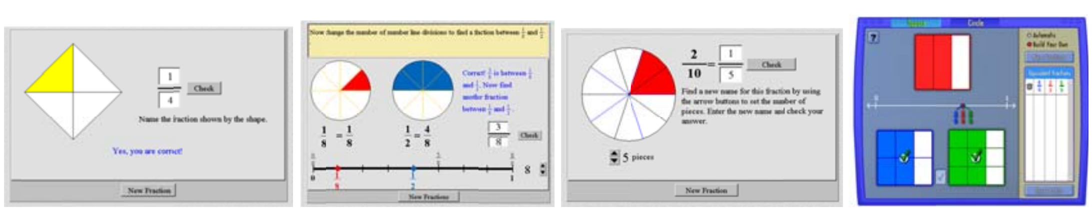


Fraction Circles Fraction Squares

Figure 6. Physical manipulatives used in practice phase of instruction.

finding equivalent fractions. While completing the processes the computer applets provide students with immediate feedback (see Figure 7).

While the physical manipulatives are not marked with symbolic representations, all four virtual manipulatives link pictorial and symbolic representations. The fractions-comparing and the equivalent fraction applets also links the symbolic and pictorial representations to number line representations. In all four virtual manipulatives, as students make changes in one representation they can see the corresponding changes in the other representations. The virtual applets also have the affordance of multiple accurate partitioning, as many as 20 (fractions-comparing) and 99 sections (fractions-equivalence and equivalent fractions) making it possible for students to accurately build and compare models. However, the many partitions can also be detractors as students must learn to focus only on partitions which are multiples or divisors of the target fraction denominator. Physical fraction squares and the virtual equivalent fraction applet have the affordance of partitioning in two directions, whereas the three NLVM applets present objects that can only be partitioned in one direction. In the fractions-equivalence original fractions and equivalent fractions are both displayed on the same region making



Naming Fractions-comparing Fractions-equivalence Equivalent fractions

Figure 7. Virtual manipulative applets used in practice phase of instruction.

it possible for students to “see” the equivalence. However, students must be able to distinguish both the original partitioning and the new partitioning which is within the original partitioning.

Instruments and Data Sources

Data were collected using the following instruments: equivalent fraction tests, lesson assessments, instructors’ logs and lesson artifacts (activity sheets, explore papers and video tapes of instructional lessons).

Equivalent Fraction Tests

There were three Equivalent Fraction Tests administered during the study: pretest, posttest and delayed posttest. Each equivalent fraction test consisted of three types of questions: open response, short response, and multiple choice. Each test contained 20 questions, four questions from each of the five fraction subtopics of: (a) modeling equivalence, (b) evaluating equivalence, (c) building an equivalent group, (d) solving equivalent sentences, and (e) simplifying fractions. Modeling equivalence questions assess students’ abilities to represent the concept of equivalent fractions through pictorial models. Evaluating equivalence questions assesses students’ ability to determine if two fractions are equivalent. Building an equivalent group questions assess students’ abilities to develop sets of multiple fractions representing the same amount. Solving equivalent sentences questions assess students’ abilities to identify a missing numerator or denominator in a pair of equivalent fractions and simplifying questions assess students’ ability to simplify fractions into their lowest forms. Appendix C contains a copy of the tests and a table showing the breakdown of questions by representational level (pictorial and symbolic only), question types (multiple choice, open response and short answer) and representation types (region or set). Questions on all three tests were similarly formatted with changes made only in the values used.

Validity of the questions was developed using a three stage process. First a pool of 60 potential questions based on research literature was developed by the researcher and an expert team of three mathematics specialist evaluated each question's content validity (Kane, 2001). In the second stage, to evaluate internal validity, the questions were administered to three students, a high-, a medium-, and a low-achieving student. After students completed each problem, the researcher asked the students to explain their reasoning processes to access if the question elicited the targeted equivalent fraction thinking. The questions which did not prompt students' equivalent fraction thinking were then refined. In the third stage, the multiple choice questions for each test were paired with the multiple choice questions from another test, resulting in three pilot tests. Pilot test A contained the questions from the pretest and the posttest, pilot test B contained the questions from the pretest and the delayed test and pilot test C contained questions from the posttest and the delayed test. The three pilot tests were then administered to a group of 81 students. Each question was answered by 54 to 56 students. An item response analysis of the questions was used to determine reliability and item difficulty (Hambleton, Swaminathan & Rogers, 1991). Table 3 contains the reliability and item difficulty level for each test.

Table 3

Reliability and Item Difficulty for Equivalent Fraction Tests

Variable	Pretest	Posttest	Delayed test
Reliability	0.74	0.76	0.74
Mean item difficulty	-0.1	-0.002	-0.005

Results were also used to establish the criterion for participation in the study. In the literature it is estimated that 80% of students will respond to Tier I instruction (D. Fuchs et al., 2008a). To allow for error the criterion for participation in this study was first set at the level at which 70% of the fifth-grade students (including those receiving special education services) scored above. Seventy percent of the students scored above 35% on the test. However, after consulting with the teachers of the participating schools, because the overall average on the pretest was so low, to capture all students needing intervention, the criteria for invitation to participate was set for 40%. This was approximately 33% of the students taking the pretest.

Lesson Assessments

Lesson assessments consisted of three types of assessments that occurred during each lesson: lesson preassessment, lesson concept assessment and the daily cumulative assessment. Figure 8 diagrams the three types of assessments.

The lesson preassessments were administered at the beginning of each daily instructional lesson and consisted of two questions from the previous lesson's activity sheets. These two questions were used to determine if the student had retained the previously learned concept. The lesson concept assessment was administered after the

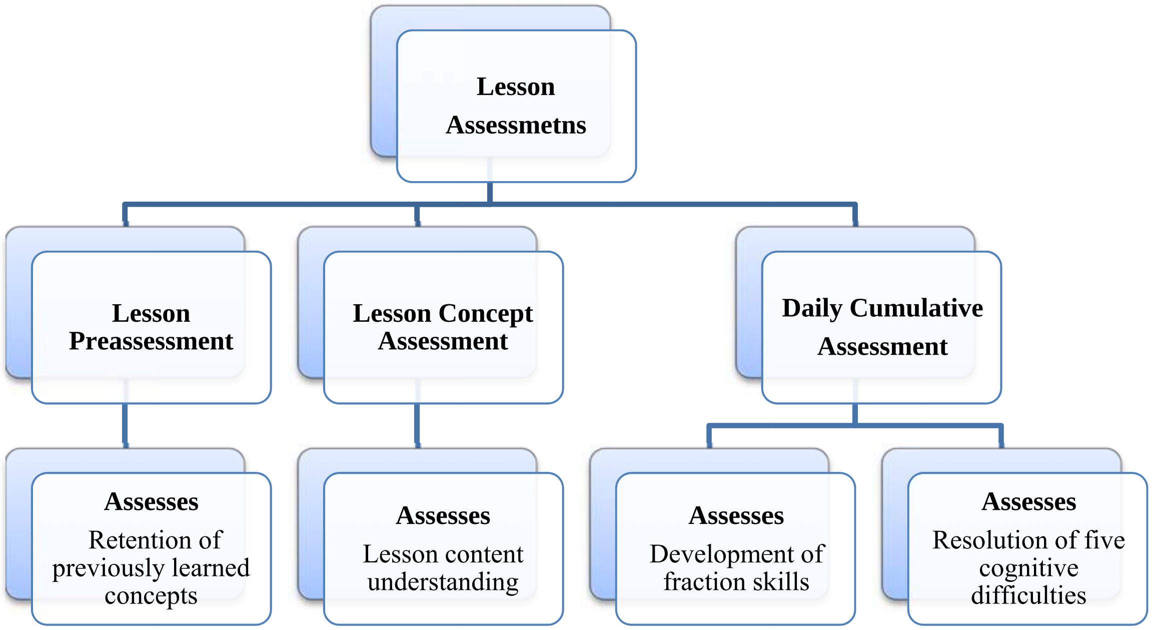


Figure 8. Lesson assessments.

practice phase of each lesson and consisted of three questions evaluating the student's understanding of the lesson concepts. The format of the three questions duplicated questions from the lesson activity sheets. The daily cumulative assessment was administered to students after they completed the lesson concept assessment. The daily cumulative assessment consisted of eight questions targeting eight fraction skills. These questions, developed from the research literature, were indicators of five areas of cognitive difficulty students experience in developing equivalent fraction understanding. The first five questions targeted skills not specific to equivalent fractions, indicative of fraction conceptual understanding. These skills were drawing fraction models, comparing fraction quantities, placement of fractions on number line, identifying a fraction between two other fractions, and partitioning sets. The final three questions of the quiz were specific to equivalent fraction understanding: identifying equivalent fractions in a region model, modeling equivalent fractions, and simplifying fractions (Appendix D contains an example of each of the three parts of the lesson assessments).

Instructors' Logs

The instructors kept a daily log detailing, for each student, the start and end time of each lesson phase and the number of problems the student completed in the explore and practice phases. The logs contained the instructors' suggestions for concepts which needed additional instruction or focus in subsequent lessons. Appendix E contains an example of the Instructors' Log Recording Sheet.

Lesson Artifacts

Lesson artifacts consisted of all activity sheets, explore papers and video tapes from each lesson. During the explore phases of each lesson, students completed one to three activity sheets. These activity sheets were collected and dated (Appendix F contains an example of a activity sheet). Instructions were videotaped. The video cameras were placed on a stationary stand and positioned to focus on the students. Seventy-two percent of the lessons were videotaped. Due to camera malfunctions and human errors (e.g., forgetting to turn on the video cameras, failure to observe a full memory card, low-battery power) there was no video recording for 10.8% of the lessons and short clips of 16.9% of the lessons. The percent of lessons for which complete videotaping was conducted was 84% of the physical manipulative lessons, 63.5% of the virtual manipulative lessons and 65% of the combined manipulative lessons. During instruction and assessments, students were encouraged to think-aloud and to discuss with the instructor and their peers their solutions and questions.

Data Analysis

Both quantitative and qualitative analyses were collected to answer the research questions. Quantitative analysis data were collected from equivalent fraction testing, lesson assessments, instructors' logs and summaries of lesson artifacts. These were used to identify quantitative differences in student achievement and to develop learning trajectories of students' understanding, including the resolution of students' errors and misconceptions. Because this study was designed to identify differences in achievement and learning trajectories and to validate instruments and analysis protocols for further research in studying fraction intervention, the primary focus of the quantitative data was a comparison of descriptive statistics, graphs and effect size scores. Statistical comparison were not used to establish significance, but as a method of identifying differences in the instructional effect of the manipulatives. Effect scores were calculated using the pooled Cohen *d* formula:

$$d = M_1 - M_2 / \sigma_{pooled}$$

$$\sigma_{pooled} = \sqrt{[(\sigma_1^2 + \sigma_2^{2)} / 2]}$$

M_1= mean for intervention group 1

M_2= mean for intervention group 2

σ_1= standard deviation of intervention group 1

σ_2= standard deviation of intervention group 2

Qualitative data from lesson artifacts (i.e., instructors' logs, activity sheets, explore papers and video tapes) and testing data (i.e. Equivalent Fraction Test open item responses) were collected and summarized to establish categories of misconceptions, errors, strategies and representations.

The analysis of data included two phases: (1) analysis of achievement data; and (2) analysis of lesson, strategy and representation data. Data from the analysis was used to develop learning trajectories. Tables 4 (shown below) and 6 (shown later in this chapter) summarize the techniques used in the first and second phases of analysis. These tables are followed by a description of the analysis in each phase.

Table 4

Data Analysis for Achievement Data

Variable	Data source	Analysis level	Data analysis
Sub 1(a). What are the variations of achievement, mastery, retention, and resolutions of errors in students' development of equivalent fraction concepts and skills?	Equivalent fraction tests	Summative 5 clusters Individual question	Paired samples *t* tests Comparison of effect sizes
	Daily cumulative Assessment	Summative 8 fraction concepts	Paired samples *t* tests Comparison of effect sizes Comparisons of scatter plots and trend lines of achievement and mastery
	Lesson artifacts	Summative	Open and axial coding Comparison of error scatter plots and trend lines
Sub 1(b). What are the variations in learning trajectories showing changes in student achievement over time?	Daily cumulative assessment	summative Individual question	Comparison of achievement and mastery trend lines
	Lesson artifacts	Summative	Comparison of error resolution trend lines

Note. Question asked, "What variations occur in the learning trajectories of students with mathematical learning difficulties that are unique to the use of different instructional manipulatives for intervention (virtual, physical or a combination of virtual and physical manipulatives) in the learning of equivalent fraction concepts?"

Phase 1: Analysis of Achievement Data

Sub 1(a) was: *What are the variations of achievement, mastery, retention, and resolutions of errors in students' development of equivalent fraction concepts and skills?* Achievement, mastery and retention data were collected from the Equivalent Fractions Test and the Daily Cumulative Assessment. Resolution of error data were collected from the Equivalent Fraction Tests, Daily Cumulative Assessments and lesson artifacts. The results of quantitative analysis of data collected was also qualitatively analyzed for the emergence of categories of variations related to the unique impacts of physical and virtual manipulatives on student achievement. In the following section, the analyses of each source of data will be described.

Equivalent Fraction Test analysis. Analysis of Equivalent Fraction Tests were conducted at three levels: summative, subtest, and individual questions. Student responses to all questions were evaluated as correct or incorrect and a score of five was assigned to all correct responses. Paired samples *t* tests were used to determine if each intervention produced significant changes in student understanding. From the statistics produced, Cohen *d* effect size scores were calculated from each student's pretest to posttest gain to determine the average amount of gain in scores. Gain scores were also used to calculate Cohen *d* effect size scores comparing differences among intervention groups. Finally a one-way ANOVA was conducted to determine if there were significant differences among the intervention gain scores. In a similar manner, an intervention mean effect size score of the gain between posttest and the delayed posttest was calculated and used to identify intervention effects relating to students' retention of concepts learned. At the subtest level, questions of the pre/post/delayed posttesting were categorized into the five equivalent fraction concept groups of *modeling* equivalence, *identifying* equivalence, building equivalent *groupings*, *solving* equivalent sentences and *simplifying* fractions. Paired samples *t* tests, one-way ANOVAs and Cohen *d* effects size scores were calculated to determine and compare student gains in achievement. At the individual question level, the gain in the percentages of students in each intervention group who answered the questions correctly were calculated and compared between the pre and post Equivalent Fraction Tests. Questions for which one intervention mean gain was 30 percentage points greater than another intervention group's gain were examined for differences in which manipulative type impacted students' response to the content of specific questions.

Students' incorrect responses on the post Equivalent Fraction Tests were also examined for differences related to intervention group. Incidences were identified in which the percentage of students either selecting incorrect responses of the multiple choice questions or responding with incorrect answers on the open response questions was greater than 20% difference between intervention groups. These questions were further examined for any possible differences in student errors related to intervention.

Daily Cumulative Assessment. Daily Cumulative Assessment scores were used to compare intervention effects through the development of three levels of student learning trajectories: individual fraction skills, concept skill clusters, and a summation of skill development. Responses to each question were evaluated using a rubric designed for that specific question (see Appendix G). Scores were totaled to obtain a test summation and clustered into concept skill groups (see Table 5). Results were analyzed using paired sample t tests to determine if the intervention was significant and Cohen *d* effect size scores to determine the magnitude of the intervention effectiveness for each intervention type. The gain scores of student achievement between pre and posttesting were calculated and analyzed using one-way ANOVAs to determine if the intervention groups differed significantly and using Cohen *d* effect size analysis to compare intervention group differences. Student scores in each intervention group were used to develop scatter plots and line plots for each fraction skill cluster (see Table 5). Each student score was graphed (*x* being the student score, *y* being lesson number) onto a scatter plot and a line of best fit was calculated. The slopes of the lines were compared to identify differences in the development of fraction skills throughout the duration of the intervention lessons.

Table 5

Daily Cumulative Assessment Questions for Learning Trajectories

Question content		Fraction skill learning trajectory	Cognitive difficulty group learning trajectory
1	Modeling fractions	Drawing fraction model	Partitioning Numerator/denominator relationship
2	Comparison	Compare fractional quantities	Numerator/denominator relationship Conceptualizing fraction quantity
3	Number Line	Placement of fractions on a number line	Numerator/denominator relationship Conceptualizing fraction quantity
4	Infinite number of fractions	Identify a fraction between two fractions	Infinite number of fractions
5	Fair shares	Partition sets	Partitioning
6	Identifying equivalent fractions	Identify equivalent fractions in a region model	Conceptualizing units/wholes Building equivalent sets
7	Equivalent Sets	Model equivalent fractions using pictorial representations	Building equivalent sets Conceptualizing of units/wholes
8	Simplifying fractions	Simplify fractions	Building equivalent sets

Intervention groups' average scores for each lesson were calculated and used to build line plots showing student trajectories of growth at the summative, cluster and question levels of analysis. These line plots were examined for differences which could be attributed to intervention effects.

Daily Cumulative Assessment data were also analyzed for differences in time required to reach mastery. When a student correctly answered a fraction skill question on two consecutive lessons and did not incorrectly answer the question in more than two-thirds of the subsequent lessons, the skill was considered mastered. For each question, a trend line graph comparing the mastery results for each intervention group was developed. The trend lines show the percentage of students which had reached mastery for each lesson.

Lesson artifacts analysis. During analysis, the activity sheets, instructors' logs and video tape from each lesson were viewed together as one lesson artifact unit. As lesson artifacts from an intervention lesson were viewed the researcher summarized data on the Lesson Summary Sheet and the Student Summary Sheet (see Appendix H for Lesson Summary Sheet and Student Summary Sheet templates). These data were analyzed using open and axial data coding to identify categories of variations not identified in the research literature (Stake, 1995; Strauss & Corbin, 1998). Analysis of the data from lesson artifacts focused on three areas: misconceptions and errors, representations, and alternative strategies.

Six categories of errors and five categories of misconceptions were identified from the literature as possible student errors and misconceptions (Appendix I contains descriptions of identified errors and misconceptions). During analysis of the lesson artifacts, occurrences of misconceptions and errors were coded and tallied using the Lesson Summary Sheet. For each type of error only one case was recorded per lesson per student. Observations of students' error responses from the Lesson Summary Sheet were entered onto the Student Summary Sheet. For each lesson the number of students exhibiting each error type was plotted in line plots and scatter plots. The lines of best fit were used to identify intervention group differences.

As the data of subquestion 1(a), concerning student achievement were analyzed, they were then synthesized by using an iterative process into the structure of a learning trajectory. Initially the data content determined the components of the learning trajectory, but as the learning trajectory emerged and findings were synthesized, new themes emerged. Additional data analyses were conducted which were then used to further shape the structure of the learning trajectories. In this manner both the data analysis and the learning trajectory were continually refined.

Sub 1(b) was: *What are the variations in learning trajectories showing changes in student achievement over time?* The artifacts used to answer sub 1(b) were the trend lines developed in the analysis of the achievement data. Two groups of trend lines were analyzed, the trend lines showing the continuous learning and mastery of the Daily Cumulative Assessment questions and the trend lines showing the resolution of errors (see Table 6). The trend lines of each intervention group were compared and variations of magnitude and sequence were identified.

Table 6

Data Analyses for Lesson Data

Variable	Data source	Analysis level	Data analysis
Sub 1(c). What are the variations in patterns of daily lesson achievement, retention and work completion?	Lesson concept assessment	Individual lesson	Comparison of effect sizes Comparisons of line plots
	Lesson pre assessment	Summative Individual lesson	Comparison of effect sizes Comparison of line plots
	Instructor logs	Summative Individual lesson	Comparison of problems completed Comparison of line plots
Sub 1(d). What are the variations in the strategies developed and used by students?	Equivalent fraction tests	Summative	Open and axial coding
	Lesson artifacts	Summative	Open and axial coding
Sub 1(e). What are the variations in students' use of representations?	Equivalent fraction tests	Summative	Open and axial coding
	Lesson artifacts	Summative	Open and axial coding

Note. Question asked was, "What variations occur in the learning trajectories of students with mathematical learning difficulties that are unique to the use of different instructional manipulatives for intervention (virtual, physical or a combination of virtual and physical manipulatives) in the learning of equivalent fraction concepts?"

Phases Two: Lesson, Strategies and Representations Data

To answer research questions 1 (c) through (e), quantitative and qualitative data was collected and analyzed from lesson concept assessments, lesson preassessment, equivalent fraction tests, daily cumulative assessments and lesson artifacts.

Sub 1(c) was: *What are the variations in patterns of daily lesson achievement, retention and work completion?* To identify variations related to manipulative type in students' performance during lesson activities, data of students' understanding and retention of lesson concepts taught and the number of problems completed was analyzed.

Understanding and retention. Student responses on the lesson preassessments and lesson concept assessments were scored using four point rubrics evaluating the amount of guidance students needed to correctly respond to the questions. For each intervention group students' scores for each lesson were averaged and the standard deviations calculated. A one-way ANOVA was conducted to determine if there were significance differences between intervention groups. Cohen *d* effect size scores were calculated to compare the magnitude of the effect of the types of manipulatives for each lesson and for the total average number of pre assessment questions answered correctly. Line plots of daily averaged scores were developed and compared for variations in trends.

Number of problems completed. The total number of problems completed by each student during the explore and practice phases of each lesson was recorded in the instructors' logs. Group averages were calculated and the results were placed on a line plot and used to compare intervention effects on the number of problems completed. A total number of problems completed by each student in all the lessons was also calculated and compared.

The processes of analysis were the same for Sub 1(d): *Are there variations in the strategies developed and used by students* and Sub 1(e): *What are the variations in students' use of representations?* Students' responses to the Equivalent Fraction Test questions and lesson artifacts were examined for differences in student strategies and representations related to intervention type. Identified differences were analyzed using open and axial data coding to determine categories of differences (Stake, 1995; Strauss & Corbin, 1998). While viewing the lesson artifacts, the researcher recorded on the Lesson Summary Sheet any variations in student strategies or representations. When a pattern of student variances was observed, responses were coded, tallied and compared.

CHAPTER IV

RESULTS

The purpose this study was to identifying variations in the learning trajectories of students with mathematical learning difficulties when learning equivalent fraction concepts during intervention instruction using physical and virtual manipulatives. This study used both quantitative and qualitative analyses to answer the research questions. Quantitative analysis was used to identify differences in student achievement. The student intervention groups were small and the quantitative results reported in this chapter should be interpreted as suggestive of trends and not as conclusive statements about the effectiveness of the types of manipulatives used. From the quantitative data, eight components of equivalent fraction learning emerged. An iceberg model of equivalent fraction understanding was developed and used to synthesize the findings. Qualitative analyses were used in the development of trend line learning trajectories showing patterns in student learning and resolution of errors and misconceptions. Variations in strategies and representations were identified.

The research question guiding this study was: What variations occur in the learning trajectories of students with mathematical learning difficulties that are unique to the use of different instructional manipulatives for intervention (virtual, physical, or a combination of virtual and physical manipulatives) in the learning of equivalent fraction concepts? There were five subquestions used to answer the research question. This chapter is divided into five sections based on the research questions. Section one contains results relevant to question 1(a), the identification of variations in student achievement, mastery, retention and resolution of errors. Section two, in response to subquestion 1(b), contains an overview of the continuous learning trajectories developed in section one and identifies variations in learning patterns which emerged from analysis of the trajectories. Section three contains results relevant to subquestion 1(c), lesson variations; section four contains results relevant to research question 1(d), variations in strategies; and section five contains results relevant to research question 1(e), variations in representations.

Throughout the chapter, abbreviations are used for the three intervention groups: (a) PM for physical manipulatives, (b) VM for virtual manipulatives, and (c) CM for the combined use of physical and virtual manipulatives. Abbreviations will also be used for the four assessment instruments: (a) EFT for equivalent fraction test, (b) DCA for daily cumulative assessment, (c) LCA for lesson concept assessment, and (d) LPA for lesson preassessment. For all comparisons among the intervention groups, one-way ANOVAs were conducted. Only one of the comparisons among intervention groups resulted in differences that were significant at the 95% level and this will be reported in the description of DCA question 5. The other results of the one way ANOVAs will not be reported in the following sections, but are summarized in a table in Appendix J.

Research Subquestion 1(a): Student Achievement, Mastery, Retention, and Error Resolution

Research subquestion 1(a) was: What are the variations of achievement, mastery, retention, and resolutions of errors in students' development of equivalent fraction concepts and skills? The sources of data for this question were the pre, post and delayed post EFT, DCA, and results which emerged from the misconception and error analyses of assessments and lesson artifacts. Because the data from these sources were used at multiple levels for multiple concepts, this section begins with an overview of the analyses processes. The overview is followed by an explanation of the iceberg learning trajectory used to synthesize the results of the three sources in the analyses. Next, findings for each concept of the iceberg learning trajectory are discussed and synthesized.

Overview of Analyses Processes

The analyses processes for each of the three sources are described in this section. Appendices K, L, and M contain tables summarizing the analysis of data from the EFT, DCA and misconception analysis, respectively. The results are described and variations identified in the corresponding content areas of the iceberg learning trajectory.

Equivalent fraction test (EFT). The EFT results were used to compare pre to posttest and post to delayed post variations in student achievement gains at the total test and concept clusters levels. Paired samples *t* tests were used to determine the pre and posttest means, standard deviations, gains and significance of the intervention for each intervention group. From these statistics, Cohen *d* effect sizes were calculated to determine differences in pre to post gain and post to delayed posttest differences within each intervention group. The gain scores of each intervention group were used to calculate Cohen *d* effect sizes to compare differences among intervention groups.

The pre to post EFT results were compared at the question level. The average gain from pre to posttest in the percent of students answering each question correctly was calculated for each intervention group. Eight questions were identified for which the differences between two intervention groups exceeded or were equal to 30% (see Appendix K).

Daily cumulative assessments (DCA). DCAs consisted of eight questions which were administered at the end of each lesson. The format of the questions remained constant for all ten lessons. Student progress measured through the DCAs was analyzed at the total test and individual question levels. Results at both levels were analyzed using paired samples *t* tests. Cohen *d* effect size analyses were conducted to determine differences between gains at the pre and post assessments and to compare effect size differences among intervention types. Scatter plots and trend line graphs of the data were developed and analyzed for differences.

Student responses to the eight questions on the DCA were analyzed for differences in the percent of students who reached question mastery and differences in the time required to reach mastery. When a student correctly answered a question during two consecutive lessons and continued to answer correctly for at least two-thirds of subsequent lessons, the question was considered mastered. For each DCA question, the percentage of students who had reached mastery was calculated, and a trend line trajectory was developed (see Appendix L for a summary of DCA results).

Misconceptions and errors. Two sources were used to identify student misconceptions and errors, lesson artifacts and EFTs. Types of errors were identified, tallied and comparisons were made to identify variations related to intervention type. The results of each source are reported in the next two sections.

Lesson artifacts. The lesson artifacts examined were: LPA, LCA, DCA, video tapings, instructors' logs, and students' written responses during explore and practice phases of the lessons. Errors were identified and instances of their occurrences tallied for each student. The errors were grouped into categories of student misconceptions. Seven categories of misconceptions with a total of 19 errors were identified (see Appendix M). In addition, three types of partitioning errors were identified. For each type of error, the number of observed cases by intervention group was tallied and charted. Only one case of each error type was reported for each student per lesson. Because of the limitations with the video recording and instructional differences, the number of errors should not be directly compared in general statements. Two examples of these limitations are the seating arrangements and the amount of problems solved. Seating arrangements for work with the PM tended to be students sitting around a table and the video recordings of these lessons included the conversation of all of the students. In contrast, in all but one of the intervention sites, computers were set up in straight lines and the video camera could only be focused on part of the students. Also, the rate at which each group of students solved problems differed, yet the amount of time given to the different phases of the lessons was fixed, therefore some problems were solved by some groups, but not by others. Some problems are more prone to elicit certain errors. Direct comparison of the number of errors could only have been made if all students completed the exact same problems on the computers. Reporting of the frequency of errors was done to establish trends, therefore, direct comparisons of numbers should not be made. To identify and compare variations in student error patterns, the frequency of error cases within each misconception group was totaled and used to develop trend lines and scatter plots.

EFT posttest error analysis. The second source used to compare variance in errors was student responses on the post EFT. The percentages of students in each intervention group which gave each type of incorrect response on the post EFT were calculated and compared. Fourteen incidences, with scores differing by more than 20% among intervention groups, were identified (see Appendix M).

Iceberg Learning Trajectory

An iceberg learning trajectory was used to synthesize the data from the three sources. The iceberg learning trajectory was developed and used by the Freudenthal Institute for planning and designing mathematical instruction (Webb, Boswinkel, & Dekker, 2008). In the iceberg model, the part of the iceberg above the water line represents the mastery of a skill or concept, the part of the iceberg below the water level represents the knowledge, understandings, and skills a student needs for mastery of the iceberg concept. The more basic the skill, the lower it is placed on the iceberg. The equivalent fraction iceberg model used in this study was developed through a synthesis of the literature and study results. From a review of the equivalent fraction literature, five clusters of equivalent fraction understanding were identified. The literature, also, described connections between students' ability to model and evaluate fractions and the development of equivalent fraction concepts. Through the analysis of findings of students' misconceptions and use of representations, additional layers of modeling, the importance of equivalent thinking and connections between the clusters emerged. The concepts and connections were synthesized using the iceberg model to form the equivalent fractions iceberg model used in this study.

The iceberg model of equivalent fractions contains four levels (see Figure 9). Level one is the visible tip of the iceberg or for this study, students' understanding of equivalent fractions as measured by the EFT and the DCA. The second level lies just beneath the water level and consists of the five cluster concepts of equivalent fractions which emerged from the literature: *modeling, identifying, grouping, solving,* and *simplifying*. The third level contains three basic understandings of fractions which are essential to an understanding and development of the five cluster concepts; *naming fractions, evaluating fraction values,* and *developing equivalence thinking.* These three basic fraction understandings each contain three skills or thought processes which make up the fourth level. To develop the skills of *naming* fractions, students need the skills of partitioning, labeling fractions, and building models of fractions. To develop *evaluating fraction values*, students need the skills of comparing, ordering and developing fractional

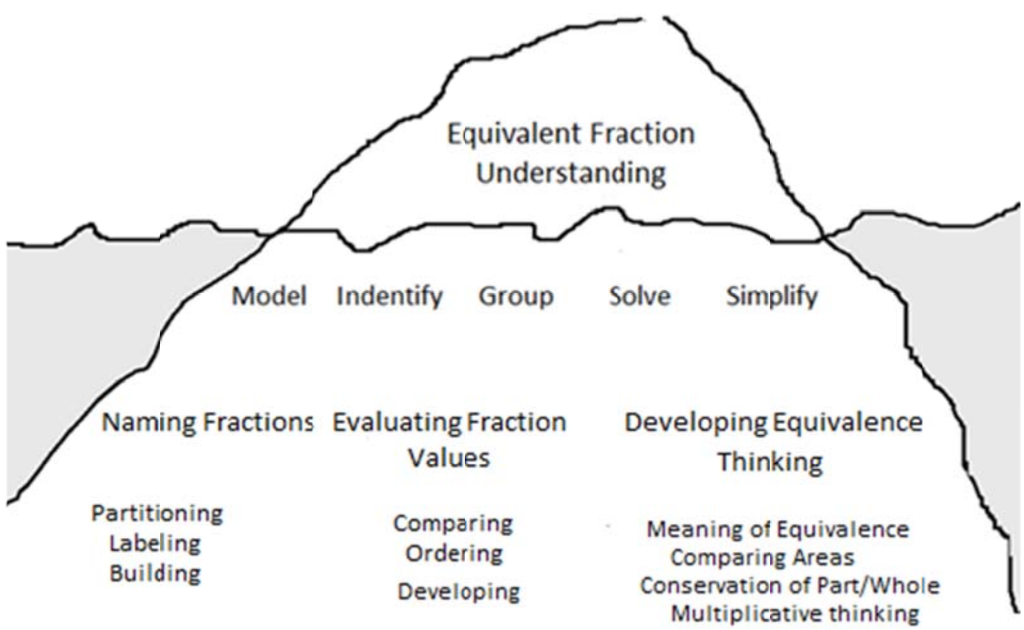


Figure 9. Iceberg trajectory of equivalent fraction learning.

amounts. To develop *equivalence thinking*, students need to develop the thought processes of meaning of equivalence, comparison of area, conservation of part-whole relationships and the development of multiplicative thinking. This model is not designed to be inclusive of all equivalent fraction understanding, but as a tool to synthesize the concepts examined in this study. In the next section, the findings of this study, pertinent to each level of the iceberg trajectory, are described. The description begins at the top with general equivalent fraction knowledge and descends to level three. Discussions of the skills and thought processes of level four will be addressed with the discussion of the level three general fraction understanding to which they contribute.

Level I: Equivalent fraction understanding. The sources of data used to analyze Level I of the iceberg model, gains in students' overall understanding of equivalent fractions, came from two sources, the EFT and the DCA.

Equivalent fraction test (EFT-total). Although, one way ANOVA indicated that pretest differences among the intervention groups were not significant, $F(2,43) = 1.69$, $p = .20$, there was a numerical difference with a range of 7.44 points (25.07-32.51). The CM group scored the highest and VM group scored the lowest. To limit the influence of the difference, gain scores were used for both paired samples t tests and Cohen d effect size calculations.

Paired samples t tests of gains from the pre to post EFT scores indicated that the PM, VM, and CM interventions were all significant at the 95% level (see Table 7). Using the findings from the paired samples t tests, Cohen d effect size scores were calculated. All three gains resulted in large effect size scores. PM intervention produced the greatest

Table 7

Comparison of Pre to Posttest Gains of EFT

	EFT pretest		EFT posttest		Pre to post			
Intervention type	*M*	*SD*	*M*	*SD*	*df*	*t*	*p*	Cohen *d*
PM	26.47	10.74	66.73	17.39	14	11.74	0.00	2.79
VM	25.07	8.72	59.79	22.57	13	6.66	0.00	2.03
CM	32.36	13.51	67.93	21.57	13	7.79	0.00	1.98

Note. $N = 43$.

positive effect size (2.79), followed by intervention using VM (2.03) and intervention using CM (1.98).

Table 8 shows the results of the study participants EFT gains in relation to the population from which the students were drawn. The averaged EFT posttest scores (64.9%) for the students in this study were 36.9 percentage points higher than the EFT pretest scores (27.9%). The averaged pretest score for all students in the classes involved in the intervention was 51.1%. After the intervention, 69.8% of the students in the study scored higher than the 51.1% averaged score of all students in the fifth-grade classes of the participating schools and 46.3% scored 75% or higher on the posttest. These findings suggest that all three of the interventions were effective in increasing students understanding of equivalent fractions.

An effect size comparison of the EFT pre to posttest gains among the intervention groups indicated there was a small to moderate effect favoring PM intervention when compared to VM ($d = 0.27$) and CM intervention ($d = 0.24$). Comparison of VM intervention and CM intervention resulted in a small effect size of 0.04 favoring CM intervention.

Table 8

Averaged Percent Correct for Intervention Students Compared to All Students

	Pretest percent correct		Posttest percent correct		
Students	*N*	Average	Average	Scored > 51.1%	Scored > 75%
Intervention	43	27.9	64.9	51.1	46.3
All classroom	183	51.1			

Comparison of the post to the delayed post EFT scores indicated that overall students retained their posttest achievement levels. Paired samples *t* tests of the difference between posttest and delayed posttest scores indicated that difference was not significant for any of the intervention groups (see Table 9). All three groups experienced only a slight decrease on delayed posttest averaged scores, 2.5% for the VM, 1.9% for the CM, and 1.7% for the PM intervention. Effect size scores comparing posttest to the delayed posttest resulted in small effect size scores of 0.12 or less for all three interventions.

Daily cumulative assessments (DCA-Total). Although, a one way ANOVA indicated that the pretest differences among the intervention groups was not significant, $F(2,43) = 0.01$, $p = .99$, there was a numerical difference with a range of 3.6 (25.07 - 32.51). PM groups scored the highest and VM groups scored the lowest. To limit the influence of the difference, gain scores were used for both paired samples *t* tests and Cohen *d* effect size comparison analyses.

Results of the DCAs were analyzed using paired samples *t* tests, effect size scores, scatter plots and trend lines. Paired samples *t* tests indicated the pre to posttest gains for all three intervention groups were significant at the 95% level (see Table 10).

Table 9

Summary of EFT Post to Delayed-Post Gains

	EFT Posttest		EFT delayed posttest		Post to delay			
Intervention type	*M*	*SD*	*M*	*SD*	*df*	*t*	*p*	Cohen *d*
PM	66.73	17.39	65.07	18.37	14	-0.47	0.65	-0.09
VM	59.79	22.57	57.29	20.75	13	-0.65	0.53	-0.12
CM	69.85	21.17	67.92	27.68	112	-0.54	0.60	-0.08

Note. $N = 42$.

Cohen *d* effect size calculations comparing pre and posttests gains yielded large effect size scores of 1.58, 1.53, and 1.81 for the PM, VM, and CM intervention groups, respectively. Cohen *d* effect size calculations among interventions yielded a moderate effect size favoring CM intervention when compared with PM (0.58) and VM (0.47) intervention. Comparison of PM and VM intervention yielded a small effect size score of 0.04 favoring VM.

To analyze the growth of student knowledge over the duration of the intervention, scatter plots were developed and the equations of the line of best fit for each intervention group were used to plot a comparison graph (see Figure 10). The greater slope of the line of best fit for the VM intervention ($y = 1.22x + 17.4$), when compared with the PM ($y = 1.00x + 22.38$) and the CM intervention ($y = 1.14x + 22.1$) suggests that the rate of growth was greater for the VM intervention.

The DCA-total data were used to develop a trend line of averaged intervention group scores over the duration of the ten lessons (see Figure 11). Although large differences are not seen relating to specific lessons, the line graph suggests that, although

Table 10

Summary of DCA-Total Paired Samples t *Test and Effect Size Analyses*

	DCA pretest		DCA posttest		Pre to post			
Intervention type	*M*	*SD*	*M*	*SD*	*df*	*t*	*p*	Cohen's *d*
PM	23.53	5.55	32.73	4.50	14	8.20	0.00	1.58
VM	19.93	5.90	29.36	6.43	13	8.63	0.00	1.53
CM	20.93	6.99	33.50	6.89	13	8.17	0.00	1.81

Note. $N = 43$.

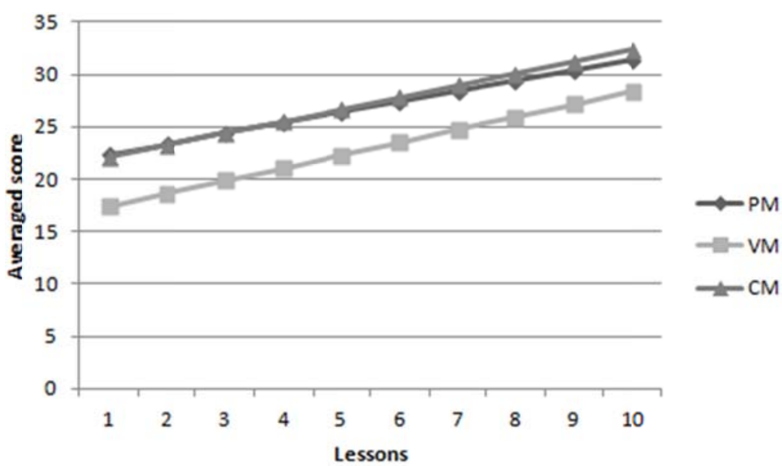


Figure 10. Trajectories of growth for total DCAs.

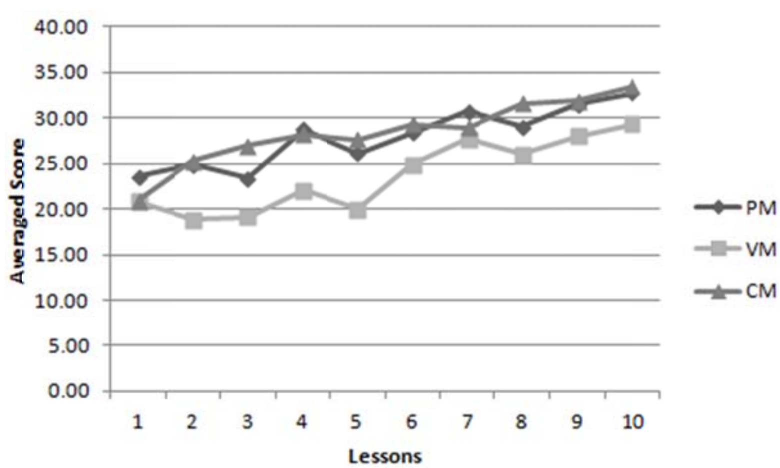


Figure 11. Trend line trajectory of student growth in DCA-total scores.

all three groups began at similar levels, the VM group tended to score slightly lower for the first five lessons. Beginning with lesson six that gap began to narrow.

In summary, the findings of both the EFT and DCA suggest that all three of the interventions were effective in increasing students' equivalent fractions achievement scores and that three weeks after the intervention students had retained their understandings. Analysis of EFT data favored PM intervention, while analysis of DCA data favored both CM and PM intervention. The effect sizes were small to moderate for EFT comparisons and moderate for DCA comparisons. The difference between findings may have been due in part to the differences in sources. EFT assessed only students' understanding of equivalent fractions. DCA assessed both general fraction understanding and equivalent fraction skills. Although the gains of the VM students were less than those of the CM and PM interventions, the scatter plots and trend lines suggested that the averaged VM students' rate of growth was greater than that of the CM and PM students and that during the last five lessons the difference among the intervention groups decreased. Synthesis of the findings suggested that, the variations among the three intervention types at the general level of equivalent fraction understanding were small.

Level II: Equivalent fraction clusters. A conceptual and procedural understanding of the equivalent fraction concepts consists of an understanding of the five clusters and skills: *modeling, identifying, grouping, solving,* and *simplifying*. To analyze Level II of the iceberg model, this section contains a description of each cluster and describes the findings of achievement, retention, mastery and error resolution related to each concept.

Modeling. The cluster of *modeling* is the ability to develop and to interpret models of equivalent fractions. The sources of data related to modeling are EFT *modeling* cluster questions, DCA question 6 (DCA-Q6), and tallies of Misconception 7 (Set Modeling) errors.

EFT modeling cluster. Paired samples *t* tests analyses of pre to posttest gains of the EFT *modeling* cluster indicated that the gain scores were significant at the 95% level for all three intervention groups (see Table 11). All pre to posttest gains yielded large Cohen *d* effect size scores. CM intervention produced the greatest positive effect size (1.45), followed by intervention using VM (1.34) and intervention using PM (1.22). Cohen *d* effect size comparisons among intervention groups yielded only small effects (VM to PM, d = 0.17; VM to CM, d = 0.20; and PM to CM, d = 0.01).

There were no differences of 30% or greater among the intervention groups for any of the individual questions of the EFT *modeling* cluster. Paired samples *t* tests of the post to delayed post EFT *modeling* cluster indicated that none of the differences were significant (see Table 12). PM and CM intervention retention differences yielded small effect sizes and the VM intervention differences in scores yielded a moderate effect size.

Table 11

Summary of EFT Modeling Cluster Analyses

	EFT pretest		EFT posttest		Pre to post			
Intervention type	*M*	*SD*	*M*	*SD*	*df*	*t*	*p*	Cohen's *d*
PM	7.00	3.48	12.87	5.48	14	3.12	0.01	1.22
VM	5.21	4.56	12.07	5.64	13	4.62	0.00	1.34
CM	4.50	1.91	10.29	5.30	13	4.21	0.00	1.45

Note. N = 43.

Table 12

Summary of EFT Modeling Cluster Post to Delayed-Post Results

	EFT posttest		EFT delayed post		Post to delay			
Intervention type	*M*	*SD*	*M*	*SD*	*df*	*t*	*p*	Cohen's *d*
PM	12.87	5.84	13.13	5.55	14	1.76	0.86	0.05
VM	12.07	5.64	13.57	4.97	13	0.86	0.40	0.28
CM	10.92	4.92	11.46	5.89	12	0.30	0.77	0.10

Note. $N = 42$.

Daily cumulative assessment question 6 (DCA-Q6). DCA-Q6 asked students to identify from a circle area region a set of equivalent fractions shown in the region. Students' responses were evaluated on a 6-point rubric that ranged from not identifying any fraction in the model to correctly identifying two equivalent fractions (see Appendix G). Paired samples *t* tests indicated that pre to post gain for all three intervention groups was significant at the 95% level (see Table 13). Cohen *d* effect size analyses of the pre to posttest gains yielded large effect size scores for each of the three groups, 1.98 for the VM group, 1.66 for the CM group and 0.97 for the PM group. An effect size comparison of the intervention groups yielded a moderate effect size of 0.67 favoring CM compared to PM intervention, a moderate effect size of 0.49 favoring VM compared to PM intervention, and a small effect size of 0.15 favoring CM compared to VM intervention.

To analyze the growth of student knowledge over the duration of the intervention, scatter plots were developed and the lines of best fit were compared (see Figure 12). The greater slope of the line of best fit for the CM intervention ($y = 0.26x + 2.68$), when compared with the PM ($y = 0.2x + 2.37$) and the VM intervention ($y = 0.2x + 1.86$) suggests that the rate of growth was slightly greater for the CM intervention.

Table 13

Summary of DCA-Q6 Analyses

	DCA pretest		DCA posttest		Pre to post			
Intervention type	*M*	*SD*	*M*	*SD*	*df*	*t*	*p*	Cohen's *d*
PM	2.60	1.92	4.33	1.63	14	4.25	0.00	0.97
VM	1.50	0.94	4.00	1.52	13	3.42	0.00	1.98
CM	2.21	1.89	4.93	1.33	13	1.57	0.00	1.66

Note. $N = 43$.

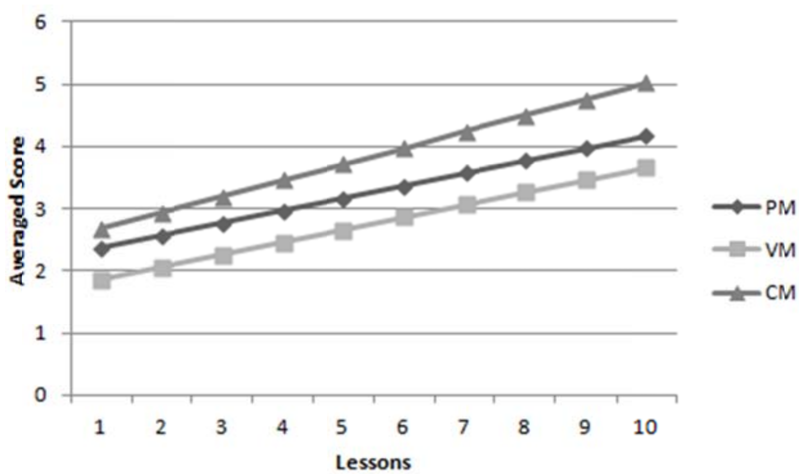


Figure 12. Trajectories of growth for DCA-Q6.

A trend line of student progression for question six over the duration of the ten lessons was developed (see Figure 13). The trend line indicates that although the CM group tended to score higher and the VM group tended to score lower, the growth of all three groups increased overall. Two variations, which may be related to the use of manipulatives, were observed. Both the VM and CM groups used the NLVM fractions-naming applet for practice during the first two lessons and both of these groups had an increase in scores at the end of lesson two whereas the PM students, using physical

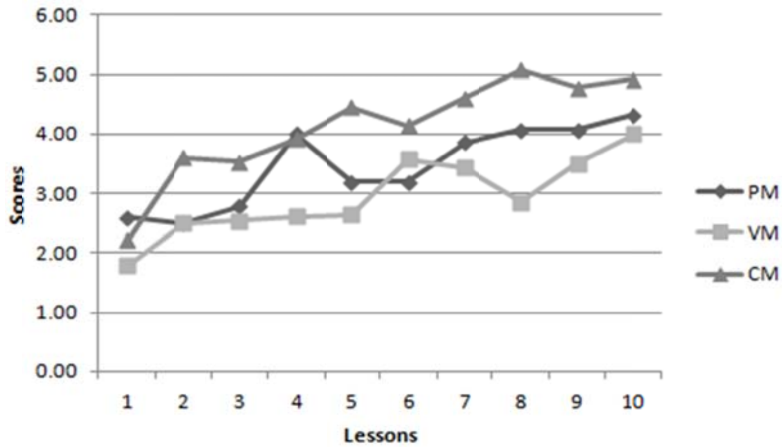


Figure 13. Trend line trajectory of student growth for DCA-Q6.

manipulatives, remained constant. This may indicate that features of the NLVM Fractions-naming applet had advantages for students learning to identify and name fractional amounts. The second variation is the decline of scores for the VM group during lessons 8, 9, and 10. During lessons 8, 9, and 10, the VM group used the pattern block applet. The VM and CM students, whose scores continued to gradually increase, used the two-colored counters. However, by lesson 10 the difference between the average scores of the three groups had narrowed.

A trend line showing the percentage of students who had reached mastery of DCA-Q6 for each intervention lesson was developed (see Figure 14). Twice as many CM students (42.9%) obtained mastery of DCA-Q6 than did VM students (21.4%).

Misconception 7 (set modeling errors). Misconception 7 emerged from qualitative analysis (see Appendix N for description of misconceptions and errors). It reads: Equivalent fractions of set models represent relationships other than the part/whole relationship. Four types of errors were observed in which students incorrectly named

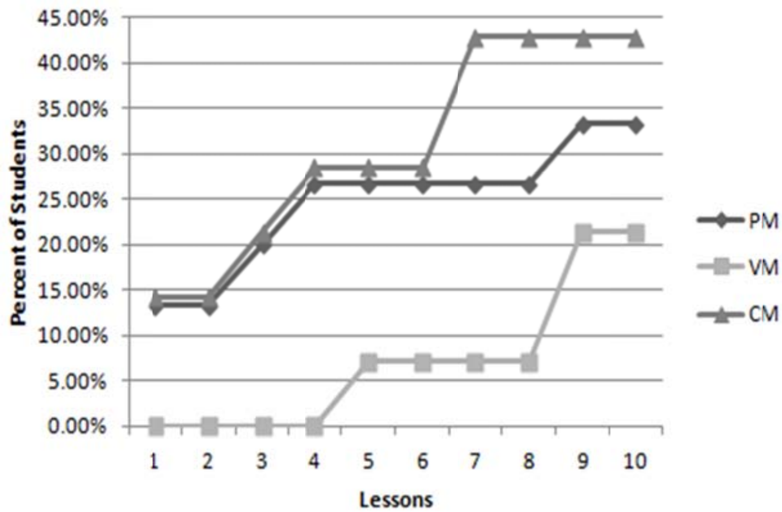


Figure 14. Trend line of percent of mastery for DCA-Q6.

fractions using set models. In Error 16 cases, making up 18.3% of the 93 Misconception 7 cases, students incorrectly used the numerator of the fraction as the number of groups the equivalent set would be partitioned into. In Error 17 cases, making up 23.7% of the Misconception 7 cases, students incorrectly used the numerator or denominator as the number of items in each group of the equivalent set. In Error 18 cases, making up 36.6% of the Misconception 7 cases, students used how many items in a group as the denominator of an equivalent fraction. In Error 19 cases, making up 21.5% of the Misconception 7 cases, students incorrectly used the numerator or denominator of the original fraction as the numerator or denominator of an equivalent fraction. Because all Misconception 7 errors occurred only in the last three lessons, trajectories were not developed. However, comparison of the number of cases observed does show a large difference between manipulative types. The two interventions, PM and CM groups, both used physical two-colored counters during the explore part of these three lessons. There were 45 cases of Misconception 7 errors observed with the PM group and 40 cases with the CM group, in contrast to 8 cases of Misconception 7 for the VM group which used the NLVM Pattern Blocks applet for the explore activities of Lessons 8, 9 and 10.

Fourteen cases were identified in which one intervention group differed from another intervention group by more than 20% in the number of students who selected incorrect answers on EFT questions. Seven of the 14 cases (50%) were Misconception 7 type errors. The averaged percent of students making Misconception 7 errors was 11.43% for the PM group, 22.71% for the VM group, and 15.29% for the CM group. Thus, although students of the VM intervention group made approximately 1/5 the number of Misconception 7 errors in the lessons as the CM and PM groups, they made almost twice the number of Misconception 7 errors on the EFT as were made by students of the other two groups.

In summary, results of the EFT indicated all three interventions were effective in increasing students' *modeling* achievement, and effect size comparisons among groups resulted in only small effects. These findings suggest that the variations in modeling achievement among interventions groups were minimal for the modeling cluster. DCA-Q6 focused on interpreting region models. Results indicate that CM intervention was favored in effect size comparisons of pre to posttest gains, question mastery, and rate of growth. Misconception 7 (set model errors) analyses examined errors made by students when developing set models of equivalent fractions. Students in the VM intervention group made fewer errors than the PM and CM intervention groups during the lessons, but made more Misconceptions 7 errors on the post EFT. A synthesis of the results suggests that there are advantages to CM interventions for *modeling* instruction, and that when solving set model representations the number of student errors varied in relation to the setting and the type of intervention.

Identifying. The concept *identifying* is the process of determining if two fractions are equivalent. The source of data was the EFT *identifying* cluster. Paired samples *t* tests analyses of pre to post gains of the EFT *identifying* cluster indicated that the gain scores were significant at the 95% level for only the PM intervention group (see Table 14). Cohen *d* pre to post effect size yielded a large effect ($d = 1.03$) for PM intervention and moderate effects for VM ($d = 0.77$) and CM ($d = 0.71$) intervention. Effect size comparisons among interventions yielded a moderate effect size favoring the PM intervention when compared to the VM ($d = 0.38$) and CM ($d = 0.46$) interventions.

There were differences of 20% or greater among the intervention groups for all of the individual questions of the EFT *identifying* cluster. Comparisons of the type of representation used in each of the EFT *identify* questions revealed differences related to the type of representation used in the questions. These differences are discussed in the last section of the chapter.

Table 14

Summary of EFT Identifying Cluster Analyses

	EFT pretest		EFT posttest		Pre to post			
Intervention type	*M*	*SD*	*M*	*SD*	*df*	*t*	*p*	Cohen's *d*
PM	6.60	5.54	12.00	4.93	14	2.40	0.03	1.03
VM	6.79	4.64	10.36	4.58	13	1.74	0.11	0.77
CM	10.00	4.39	13.21	4.64	13	2.09	0.06	0.71

Note. $N = 43$.

Paired samples *t* tests of the post to delayed post EFT *identifying* cluster indicated that none of the differences were significant (see Table 15). However, Cohen *d* effect size analysis of post to delayed test differences yielded a moderate negative effect size for PM intervention (d = - 0.33), no effect for VM intervention (d = 0) and a small positive effect for CM intervention (d = 0.19). These results indicate that the retention of concepts was less for the PM intervention.

In summary, paired samples *t* tests indicated that the intervention was only statistically significant for the PM intervention group on the posttest. Cohen *d* analyses of both pre to post differences and differences among interventions favored PM intervention. Although growth gains were greater for PM intervention, the CM intervention group's average posttest score was higher than those of PM and VM intervention, and retention of the identifying concepts was greater.

Grouping. *Grouping* is the development of equivalent fraction groups. The sources of data for this concept were gains of EFT *grouping* achievement and results of the DCA-Q7 analysis.

EFT grouping cluster. Paired samples *t* tests analyses of pre to post gains of the

Table 15

Summary of Identifying Cluster Post to Delayed-Post Analyses

	EFT posttest		EFT delayed		Post to delay			
Intervention type	*M*	*SD*	*M*	*SD*	*df*	*t*	*p*	Cohen's *d*
PM	12.00	4.93	10.33	5.16	14	-0.79	0.44	-0.33
VM	10.36	4.58	10.36	4.14	13	0.00	1.00	0.00
CM	13.08	4.80	14.23	7.03	12	1.00	0.34	0.19

Note. N = 42.

EFT *grouping* cluster indicated that the gain scores were significant at the 95% level for all three intervention groups (see Table 16). For the *grouping* cluster of the pretest, the averaged CM group score was almost double (6.79) the average scores of the PM (3.27) and VM (3.79) intervention groups. All three gains resulted in large Cohen *d* pre to post effect size scores. VM intervention produced the greatest effect size (d = 2.00), followed by intervention using PM (d = 1.60) and intervention using CM (d =1.07). Cohen *d* effect size comparisons among intervention groups yielded a moderate effect size favoring VM intervention compared to CM intervention (d = 0.54) and PM intervention (d = 0.35).

Three of the EFT *grouping* cluster questions resulted in greater than 20% differences in the gain of VM students answering correctly when compared to the gains of PM (Questions 7 and 18) and CM (Questions 8 and 18) students. In all four of the grouping questions the VM group scored higher than both the PM and CM groups.

Paired samples *t* tests of the post to delayed post EFT *grouping* cluster indicated that none of the differences between post and delayed post scores were significant (see Table 17). The PM and CM groups gains yielded small positive effect sizes and the VM group yielded a moderate negative effect size indicating that PM and CM students'

Table 16

Summary of EFT Grouping Cluster Analyses

	EFT pretest		EFT posttest		Pre to post			
Intervention type	*M*	*SD*	*M*	*SD*	*df*	*t*	*p*	Cohen's *d*
PM	3.27	3.20	10.87	5.90	14	5.21	0.00	1.60
VM	3.79	4.15	13.36	5.33	13	6.29	0.00	2.00
CM	6.79	6.39	13.36	5.87	13	3.73	0.00	1.07

Note. N = 43.

Table 17

Summary of EFT Grouping Cluster Post to Delayed-Post Results

	EFT posttest		EFT delayed		Post to delay			
Intervention type	*M*	*SD*	*M*	*SD*	*df*	*t*	*p*	Cohen's *d*
PM	11.00	5.95	12.27	4.85	14	0.86	0.40	0.23
VM	13.36	5.33	11.21	5.65	13	-1.28	0.22	-0.39
CM	13.69	5.94	14.92	6.08	12	0.76	0.46	0.20

Note. $N = 42$.

retention of *grouping* concepts at the time of the delayed test was greater than the retention of VM intervention students.

Daily cumulative assessment question 7. DCA-Q7 question provided a rectangular pictorial representation of a fraction and asked students to first identify the fraction pictured and then to name two equivalent fractions. Responses were evaluated on a 6-point rubric that ranged from incorrectly naming the fraction pictured to correctly naming the fraction and providing two equivalent fractions (see Appendix G). Paired samples *t* tests indicated that for all three interventions the pre to posttest gains of DCA-Q7 were significant at the 95% level (see Table 18). Cohen *d* effect size analyses of the pre to posttest gains yielded large effect size scores of 2.84 for the PM group, 1.86 for the VM group, and 1.67 for the CM group. An effect size comparison of the intervention groups yielded moderate effect sizes favoring PM when compared to the use of CM ($d = 0.66$) and VM ($d = 0.54$). Comparison of CM groups to VM groups yielded a small effect score of 0.05.

To analyze the growth of student knowledge over the duration of the intervention, scatter plots were developed and the lines of best fit were compared (see Figure 15). The

Table 18

Summary of DCA-Q7 Analysis

	DCA pretest		DCA posttest		Pre to post			
Intervention type	*M*	*SD*	*M*	*SD*	*df*	*t*	*p*	Cohen's *d*
PM	1.93	1.33	5.20	0.94	14	9.12	0.00	2.84
VM	1.64	0.63	4.14	1.79	13	3.24	0.00	1.86
CM	3.00	1.84	5.57	1.16	13	4.93	0.00	1.67

Note. *N* = 43.

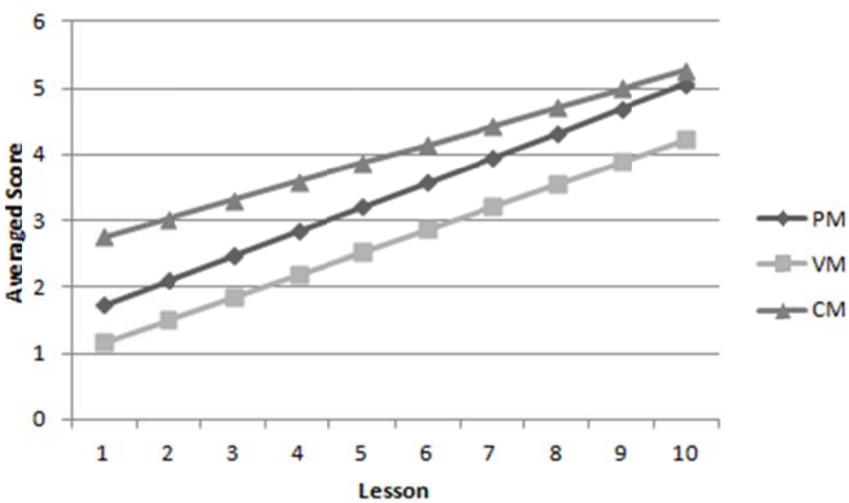


Figure 15. Trajectories of growth for DCA-Q7.

greater slope of the line of best fit for the PM intervention (y = 0.37x + 1.73), when compared with the VM (y = 0.34x + 1.17) and the CM intervention (y = 0.28x + 2.75) suggests that the rate of growth was slightly greater for the PM intervention.

Examination of trend lines of averaged intervention group scores for DCA-Q7 indicated that the trajectories for all three groups increased steadily throughout the duration of the lessons with the gaps between the intervention groups gradually narrowing until lesson 8 (see Figure 16). At lessons 8, 9, and 10, the lessons focused on

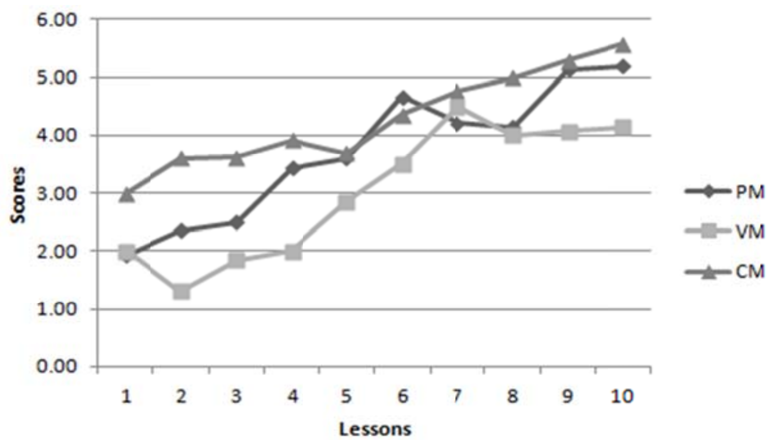


Figure 16. Trend line of averaged scores for DCA-Q7.

finding equivalent fractions of set models, and the gaps among the groups began to broaden again.

A trend line showing the percentage of students who had reached mastery of DCA-Q7 for each intervention lesson was developed (see Figure 17). The number of CM students (64.3%) who obtained mastery of DCA-Q7 was four times greater than the number of VM students (14.3%) who reached mastery. The percent of PM students obtaining mastery was 46.7%.

In summary, all analyses of the EFT *grouping* cluster, except retention results favored the use of VM intervention. In contrast, all comparisons for DCA-Q7, except for mastery, favored PM intervention. The questions of the DCA and EFT were examined for differences that could be related to the differences in VM and PM group performances but none were identified. Thus, the synthesis favored use of VM and PM interventions, but the higher pretest scores of the CM group on both the EFT *grouping* cluster and DCA-Q7 limit the strength of the comparisons.

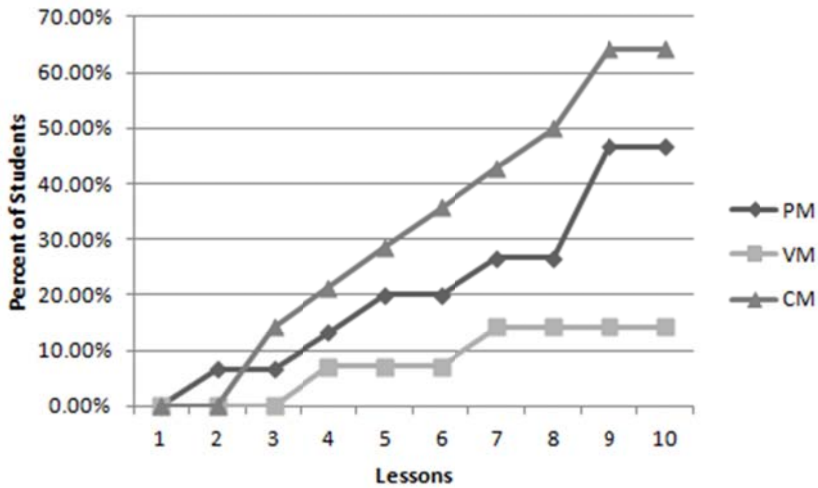


Figure 17. Trend line showing percentage of mastery of DCA-Q7.

Solving. *Solving* equivalent fraction sentences is the ability to determine the value of a missing numerator or denominator in an equivalent fraction sentence. The EFT was the only source of data for the solving cluster. Paired samples *t* tests of gain scores from the pre to posttest of the EFT *solving* cluster indicated that the pre to posttest gains were significant at the 95% level for all three intervention groups (see Table 19). All three pre to post gains resulted in large Cohen *d* pre to post effect size scores. PM intervention produced the greatest positive effect size (*d* = 3.44), followed by CM intervention (*d* =2.68) and PM intervention (*d* =1.90). Comparison among the three intervention groups resulted in a large effect size favoring PM groups when compared to VM groups (*d* =0.88) and a moderate effect size when compared to CM groups (*d* = 0.52).

Differences of 20% or greater among intervention groups were identified for all four of the EFT solving cluster questions. Similar to those found in the previous cluster, the differences identified were related to type of representation and will be discussed in the last section of this chapter.

Table 19

Summary of EFT Solving Cluster Analyses

	EFT pretest		EFT posttest		Pre to post			
Intervention type	*M*	*SD*	*M*	*SD*	*df*	*t*	*p*	Cohen's *d*
PM	4.00	4.31	17.00	3.16	14	13.67	0.00	3.44
VM	4.29	3.31	13.21	5.75	13	5.10	0.00	1.90
CM	5.36	4.14	16.43	4.13	13	9.28	0.00	2.68

Note. $N = 43$.

Paired samples *t* tests of the post to delayed post EFT *solving* cluster indicated that the difference was significant at the 95% level for the PM intervention (see Table 20), but not for the VM and CM interventions. PM and CM interventions posttest to delayed posttests decreases yielded moderate negative effect sizes, and the VM intervention yielded a small negative effect size.

In summary, the results of the EFT *solving* cluster pre to post gains favored the use of the PM intervention. Although the PM intervention students experienced the greater decrease in retention of the concepts from post to delayed posttesting, their averaged delayed score was still higher than that of the other two groups.

Simplifying. *Simplifying* is finding an equivalent fraction which is in its lowest terms. Data from three sources were analyzed for this concept: EFT *simplifying* cluster, DCA-Q8 and analysis of Misconception 4 (Partitioning/Simplifying Errors).

*EFT simplifying cluster***.** Paired samples *t* tests of the pre to posttest gains from the EFT *simplifying* cluster indicated that the gain was significant at the 95% level for all three interventions groups (see Table 21). Cohen *d* effect size analyses of the EFT

Table 20

Summary of EFT Solving Cluster Post to Delayed-Post Results

	EFT posttest		EFT delayed		Post to delay			
Intervention type	*M*	*SD*	*M*	*SD*	*df*	*t*	*p*	Cohen's *d*
PM	17.00	3.16	15.33	3.52	14	-2.65	0.02	-0.50
VM	13.21	5.75	12.50	5.46	13	-0.43	0.67	-0.13
CM	16.92	3.84	15.00	5.77	12	-1.81	0.10	-0.39

Note. $N = 42$.

Table 21

Summary of EFT Simplifying Cluster Analyses

	EFT pretest		EFT posttest		Pre to post			
Intervention type	M	SD	M	SD	*df*	*t*	*p*	Cohen's *d*
PM	5.60	4.97	13.87	6.37	14	5.43	0.00	1.45
VM	5.00	4.80	10.79	8.85	13	2.78	0.02	0.81
CM	5.00	5.55	14.64	7.03	13	4.87	0.00	1.52

Note. $N = 43$.

simplifying cluster pre to posttest gains yielded large effect size scores, with CM intervention yielding the largest effect size of 1.52, followed by 1.45 for PM intervention and 0.81 for VM intervention. Cohen *d* effect size comparisons among intervention groups favored CM groups compared to VM groups (d =0.48) and PM groups (d = 0.20).

Two questions of the EFT *simplifying* cluster, with a difference of 20% or greater among the intervention groups, were identified. Again these related to representation types and will be discussed in the last section. Paired samples *t* tests of the post to delayed post EFT *simplifying* cluster indicated that the decrease in post to delayed posttest scores was significant for the CM intervention, but not for the PM and VM interventions (see Table 22). Cohen *d* effect size analyses of post to delayed post results yielded small effect sizes for PM (*d* = 0.02) and VM (*d* = -0.15) interventions and a moderate effect size for the CM intervention (*d* = - 40). These results indicate that the CM intervention was not as effective as PM and VM intervention for the retention of *simplifying* concepts.

*Daily cumulative assessment question 8 (DCA-Q8)***.** DCA-Q8 asked students to simplify a given fraction. This question used only symbolic representations. Student responses were evaluated on a four point rubric which ranged from naming fractions which were not equivalent to naming a fraction that was reduced into lowest terms (see Appendix G). Paired samples *t* tests indicated that the gain for all three interventions was significant at the 95% level (see Table 23). Cohen *d* effect size analyses of pre to post gains yielded large effect sizes of 1.48 and 0.83 for the PM and CM intervention, respectively, and a moderate effect size of 0.63 for the VM intervention. Effect size comparisons of intervention groups yielded a moderate effect favoring PM groups when compared to VM groups (*d* = 0.46) and a small effect size when compared to CM groups (*d* = 0.25). Comparisons of CM to VM groups yielded a small effect size of 0.21.

Table 22

Summary of EFT Simplifying Cluster Post to Delayed-Post Results

	EFT posttest		EFT delayed post		Post to delay			
Intervention type	*M*	*SD*	*M*	*SD*	*df*	*t*	*p*	Cohen's *d*
PM	13.87	6.37	14.00	5.41	14	0.09	0.93	0.02
VM	10.79	8.85	9.64	6.64	13	-0.83	0.42	-0.15
CM	15.23	6.95	12.31	7.80	12	-2.61	0.02	-0.40

Note. *N* = 42.

Table 23

Summary of Analysis of DCA-Q8

	DCA pretest		DCA posttest		Pre to post			
Intervention type	*M*	*SD*	*M*	*SD*	*df*	*t*	*p*	Cohen's *d*
PM	0.93	0.46	2.53	1.46	14	4.77	0.00	1.48
VM	1.36	1.50	2.29	1.44	13	2.88	0.01	0.63
CM	1.71	1.64	3.00	1.47	13	3.35	0.01	0.83

Note. *N* = 43.

Scatter plots were developed and the lines of best fit were compared (see Figure 18). Comparison of the slopes of the lines of best fit indicated similar slopes for the PM (y = 0.17x+0.97) and CM (y = 0.16x+1.17) intervention which were greater than the slope of the VM (y = 0.11x+1.09) intervention. This suggests that the rates of growth were greater for the PM and CM interventions.

Analysis of trend lines showing averaged responses for DCA-Q8 indicated that the trajectories for the VM and CM groups were very similar (see Figure 19). The trajectory for the PM group had greater increases and decreases in scores for the first four lessons, but then remained at about the same level for the remaining lessons.

Trend lines showing the percentage of students who reached mastery of DCA-Q8 for each intervention lesson were developed (see Figure 20). The trend lines for the PM and CM intervention groups follow similar trajectories, but the trend line for the VM intervention indicates that fewer students reached mastery and that they tended to reach mastery later than the PM and CM intervention students. More CM students (42.9%) obtained mastery than did PM (33.3%) and VM (21.4%) students.

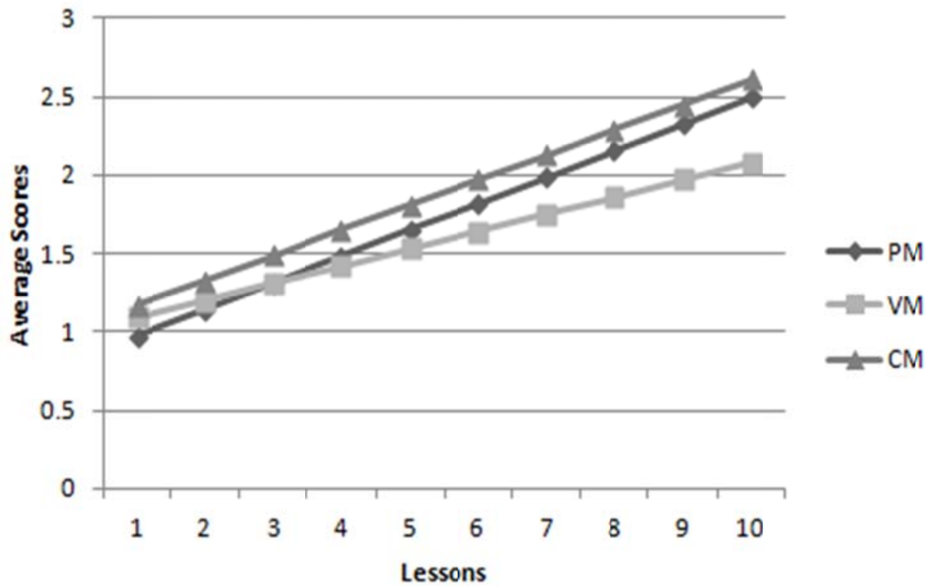


Figure 18. Trajectories of growth for DCA-Q8.

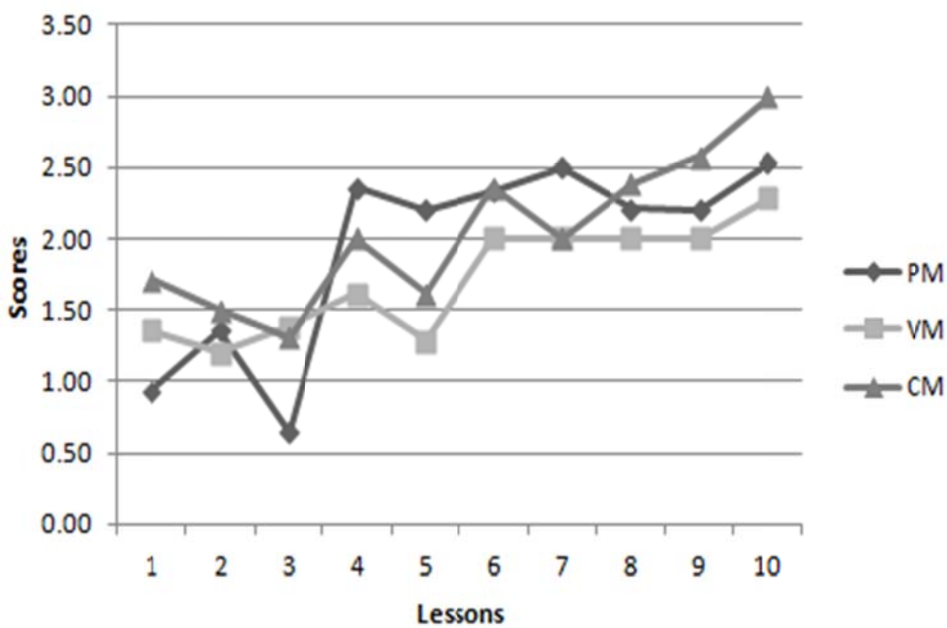


Figure 19. Trend line of averaged scores for DCA-Q8.

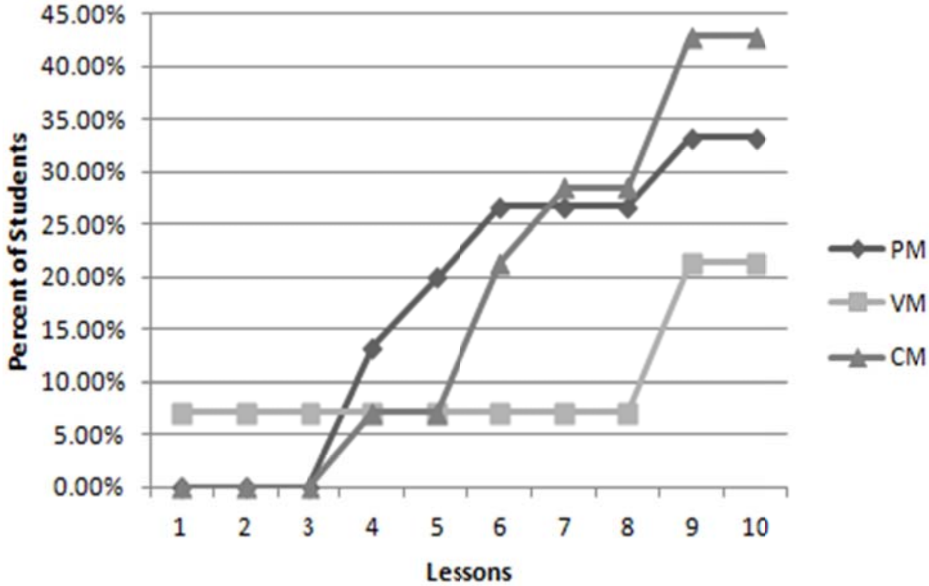


Figure 20. Trend line showing percentage of mastery of DCA-Q8.

Misconception 4 (partitioning/simplifying errors). The final source of comparison for *simplifying* fractions was the reduction of Misconception 4 type errors. Misconception 4 is the belief that partitioning and simplifying always produces halves. Misconception 4 was reflected in two types of errors. Error 7, which accounted for 71.8% of the 78 Misconception 4 errors, occurred when students responded to requests for equivalent fractions or simplified fractions with the incorrect response of ½. Error 8, which accounted for 28.2% of the cases, occurred when students were asked to simplify a fraction which was not divisible by two. Rather than determining a common factor, students incorrectly "halved" the numerator and the denominator.

The number of cases of errors for Misconception 4 observed in each lesson were totaled and plotted in scatter plots (see Figure 21). Ideally the scatter plots and trend lines of error analyses show a negative relationship, with the number of errors decreasing over

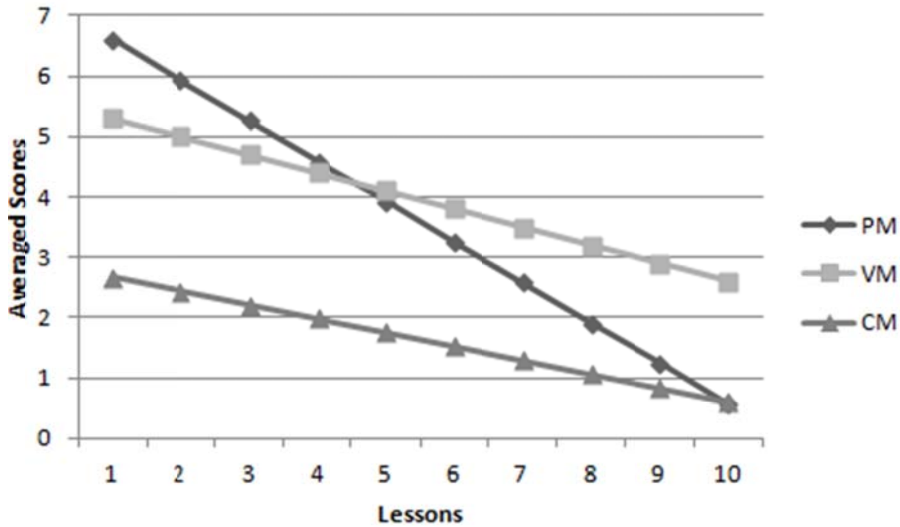


Figure 21. Trajectories of Misconception 4 cases.

the duration of the intervention. The greater slope of the line of best fit for the PM intervention ($y = -0.67x+6.6$) when compared to the VM ($y = -0.33x+5.3$) and the CM intervention ($y = -0.23x+2.67$) indicated that the PM students had the greatest rate of reduction of errors.

Figure 22 contains trend lines comparing the resolution of Misconception 4 for each intervention group. The PM group had a strong steady decrease in their number of errors while the VM intervention group, for the first six lessons, had dramatic rises and drops in the number of occurrences. However, by the last two lessons the resolution for the CM groups appeared to be complete and both the PM and CM groups had only one occurrence of Misconception 4 errors in each lesson.

In summary, the effect size calculations of the EFT *simplifying* cluster pre to post gains all favored CM intervention, but post to delayed posttest differences indicated that the retention of PM and VM students' scores was greater. Analysis of the percentage of

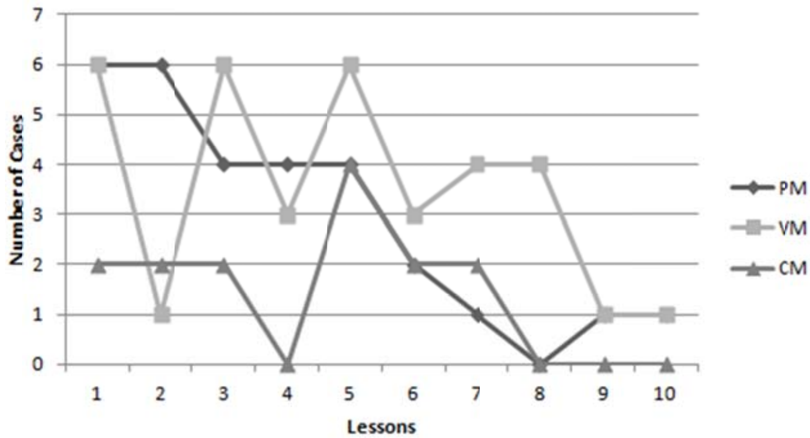


Figure 22. Trend line trajectories of Misconception 4.

students answering specific questions of the simplifying cluster suggested that, there may also have been variations in achievement which corresponded to the type of representations used in the questions. Analyses of DCA-Q8 indicated that the gain was greatest for the PM intervention, but that there was a linear relationship between lesson duration and scores for only the CM intervention students. Examination of the trend lines of DCA-Q8 shows that the increase in scores was erratic for the first five lessons for the PM interventions but by the last four lessons all three intervention groups had stabilized and their trajectories became more similar. During these last lessons, the CM group continued to steadily increase, while the VM and PM groups remained about the same. The opposite was observed in the resolution of Misconception 4 (partitioning/simplifying errors). The PM intervention trend line shows a steady continual decrease of errors while the CM and VM groups had more erratic trajectories. For all three groups the resolution of errors was almost complete. A synthesis of the results suggests that CM intervention produced a steadier gain of *simplifying* concepts, but there were areas of unexplained variations, and PM intervention produced a steadier resolution of Misconception 4 errors.

Levels III and IV: Basic understanding of fractions. Levels III and IV of the iceberg model contains three Level III concepts and 10 Level IV sub concepts that are basic to the development and understanding of equivalent fractions. The three concepts of Level III are: *naming fractions, evaluating fraction values,* and *developing equivalence thinking.* There are three to four Level IV subconcepts that are important for conceptual understanding of each of the Level III concepts. This section discusses the findings of the study related to each of the Level III concepts and their Level IV subconcepts.

Naming fractions. *Naming fractions,* of the iceberg model Level III, is the ability to give a symbolic representation to the part-whole relationship shown in concrete and pictorial models. It also involves the ability to develop models of symbolic fractions. Data were collected on three iceberg Level IV skills which contribute to students' understanding of *naming fractions*: *labeling fractions*, *partitioning* and *building models* of fractions.

Labeling fractions. *Labeling fractions* is the skill of identifying the part and the whole of a concrete or pictorial representation and writing that relationship in the proper symbolic form. In the study, although students' ability to *label fractions* was not specifically measured, it was a foundational skill students needed as they compared, described and modeled equivalent fractions. Although this is a skill which typically receives a strong focus of instruction in third and fourth grades, the emergence of Misconception 3 (misnaming errors) from the analysis of student errors, indicated that some students had not mastered the skill of labeling fractions. Misconception 3 was: *Fractions of regional models represent relationships other than the part-whole* *relationship of the model.* Coding of the data revealed four types of relationships students incorrectly focused on when naming fractions. These four types of relationships were: (a) Error 3, modeling fractions as arrays (18.8% of the 133 Misconception 3 cases); (b) Error 4, interchanging the numerator and denominator when naming fractions (10.5%); (c) Error 5, naming fractions as the relationship of shaded to unshaded (18.8%); and (d) Error 6, focusing on the number of sections and not the relationships of different sized sections of a whole (51.9%).

The number of cases of errors for Misconception 3 observed in each lesson were totaled and plotted in scatter plots (see Figure 23). The greater slope of the line of best fit for the PM intervention ($y = -1.35x+12.5$) when compared to the VM ($-0.71x+9.4$) and the CM intervention ($y = -0.66x+6.33$) indicated that the PM students had the greatest rate of reduction of errors. However, the comparison with the CM intervention could also be affected by the CM trajectory. As the trend lines in Figure 23 show, the CM intervention had fewer cases of Misconception 3 and the number of cases were reduced to none before the last lesson suggesting that the degree of the slope of best fit for the CM group may have been limited by the smaller number of error cases.

An analysis of the trend lines for Misconception 3 indicated that, although all three types of instruction appeared to effectively reduce errors, the VM trajectory does differ (see Figure 24). Resolution of errors for the VM intervention was slower, with the number of cases remaining almost constant for the first five lessons. The resolution of errors for the PM and CM groups was almost complete by lesson 8 while the VM group continued to have three to four error cases per lesson.

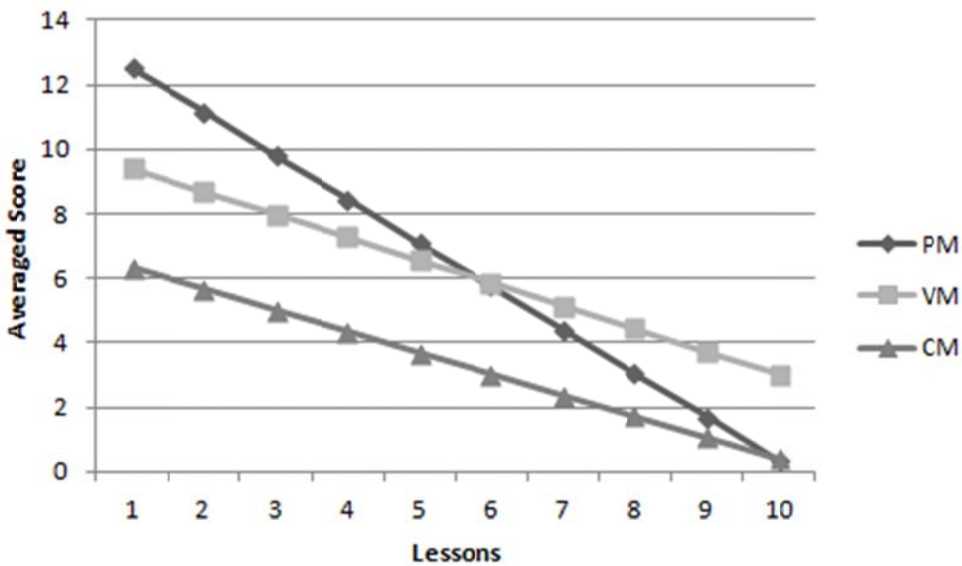


Figure 23. Trajectories of Misconception 3 cases.

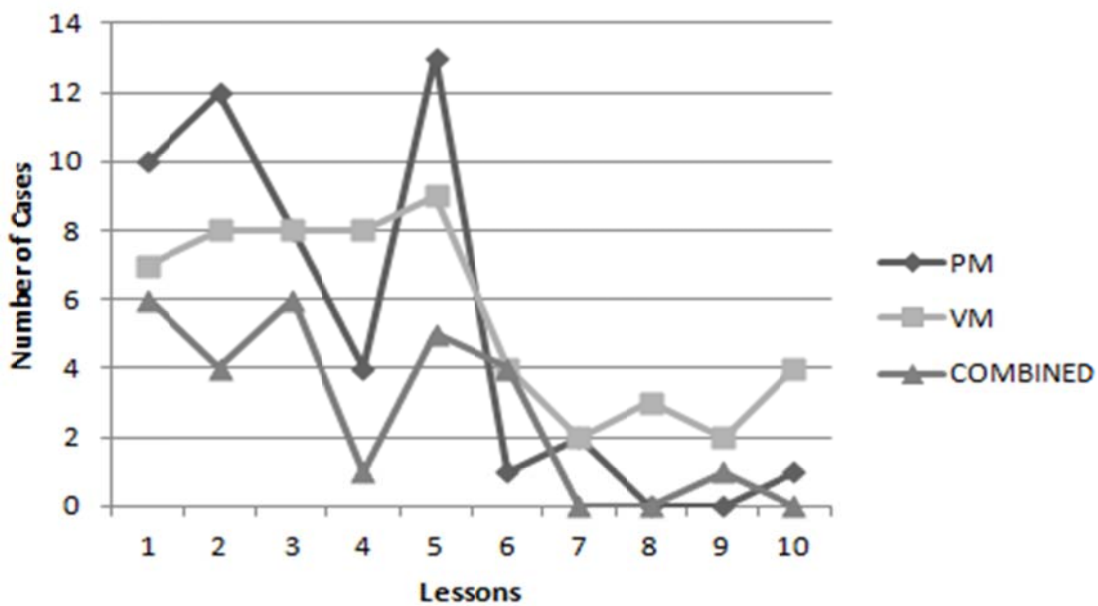


Figure 24. Trend line trajectories of Misconception 3.

One incidence of a Misconception 3 error type six was identified from the post EFT analyses of intervention group differences in the percent of incorrect responses. The Error 6 multiple choice selection was choosen by 28.6% of the VM intervention students as compared to 14.3% of the PM and 6.7% of the CM intervention students. Analysis of the data relating to the resolution of Misconception 3 suggested that, when compared with students of VM intervention, the rate of resolution of the errors was greater and more complete for students in the PM intervention group. For both the CM and PM interventions the resolution of the errors were complete.

Building models: *Building models* is the skill of building concrete models or drawing pictorial models to represent fractions. In the process of developing equivalent fraction understanding, students should be able to build accurate models which they can use to compare and partition. The purpose of DCA-Q1 was to assess students' ability to build models.

DCA-Q1 asked students to draw a model of a given fraction within a rectangular region. Responses were rated on a 6 point rubric (see Appendix G). Paired samples *t* tests indicated that the pre to posttest gains were significant for the VM and CM intervention, but not significant for the PM intervention (see Table 24). The Cohen *d* effect size analysis of pre to post gains yielded a large effect for the CM intervention (d = 1.05), and a moderate effect of for the VM (d = 0.62) and the PM (d = 0.46) interventions. An effect size comparison of the three intervention groups, yielded large effect sizes favoring CM groups when compared with PM groups (d = 1.36) and VM groups (d = 0.84) and a moderate effect (d = 0.39) favoring VM groups when compared to PM groups. The high

Table 24

Summary of Analysis of DCA-Q1

	DCA Pretest		DCA posttest		Pre to post			
Intervention type	*M*	*SD*	*M*	*SD*	*df*	*t*	*p*	Cohen's *d*
PM	5.00	1.65	5.60	0.83	14	1.42	0.18	0.46
VM	4.21	1.93	5.14	0.86	13	2.33	0.04	0.62
CM	4.29	1.98	5.79	0.43	13	3.07	0.01	1.05

Note. *N* = 43.

averages of all three groups' pretest scores indicated that many students had previously developed the ability to model fractions.

To analyze the growth of student knowledge over the duration of the intervention, scatter plots were developed and the lines of best fit were compared (see Figure 25). The greater slopes of the lines of best fit for the VM ($y = 0.14x + 3.87$) and CM interventions ($y = 0.13x+4.36$), when compared with the PM intervention ($y = 0.4x + 5.17$) suggests that the rate of growth was greater for the VM and CM intervention. However, the rate of growth of student knowledge was low for all three groups. In part this is likely due to the high number of students who had mastered the skill prior to the intervention.

Trend lines of the averaged intervention group scores show that the trajectory for DCA-Q1 had initial high scores and little gain for the PM and CM interventions (see Figure 26). In the rubric used in evaluating the student responses, a score of 5 was given if the students used the correct number of partitions and shading. A 6 was given if all the partitions were drawn accurately in equal proportion. All of the averaged scores of the PM group and all of the averaged scores of the CM group after lesson 3 were 5 or

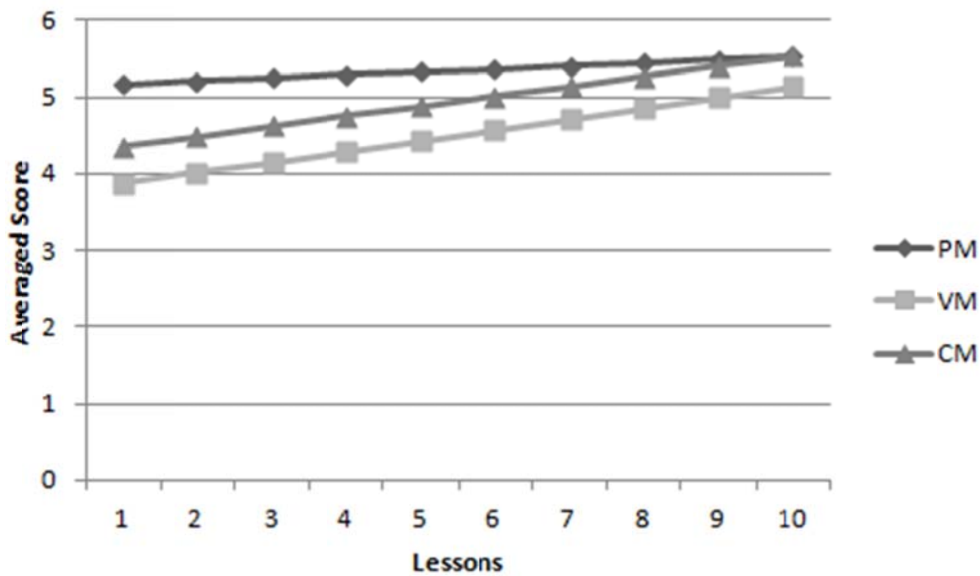


Figure 25. Trajectories of growth for DCA-Q1.

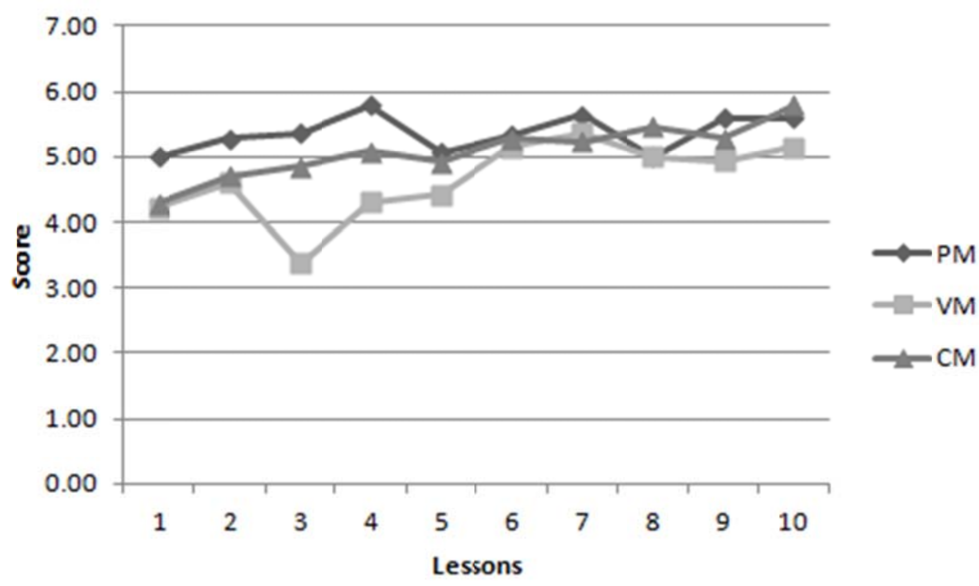


Figure 26. Trend line of averaged scores for DCA-Q1.

greater. This indicated that the majority of students correctly drew the models, but some were not accurate in their drawings. The VM intervention experienced a decrease in scores after the second lesson and then the scores slowly increased until the group was scoring slightly higher than the level the group had scored on the first two lessons.

The trend lines of student mastery for DCA-Q1 showed that the trajectories of mastery were similar for the PM and CM intervention groups, with 60.0% of the PM students and 53.3% of the CM students mastering the question (see Figure 27). Only 28.6% of the VM students reached mastery.

Results of the analyses of DCA-Q1 favored the CM intervention for both the magnitude of effect and the rate of growth. However, achievement levels and trajectories were similar for both the PM and CM interventions. Analysis was limited by the high number of students who, from the beginning of the intervention, scored between the 5- and 6-point range.

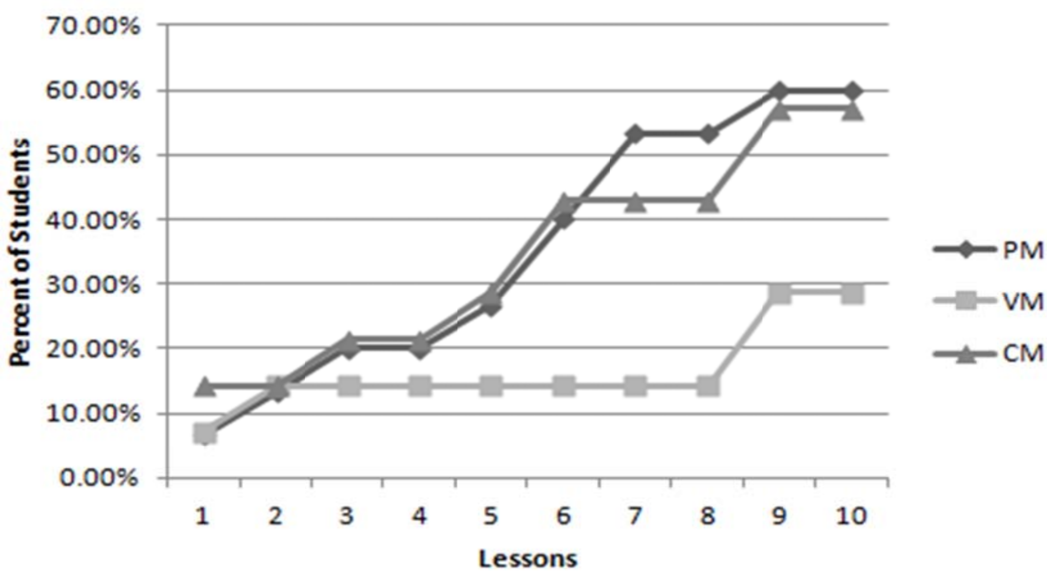


Figure 27. Trajectory showing percentage of mastery of DCA-Q1.

Partitioning. The ability to partition fraction models affects students' use of models in developing fraction concepts. When objects are not correctly partitioned, the results of visual comparisons made by the student become meaningless and may cause conceptual misunderstandings. Two sources provided data related to partitioning: DCA-Q5 and the analysis of student errors.

DCA-Q5 was a fair share question in which students were asked to divide a given number of pizzas with a given number of friends. Responses were evaluated using a six point rubric ranging from not drawing the correct number of pizzas to partitioning the models correctly and identifying the fractional amount each friend would receive (see Appendix G). Paired samples *t* tests indicated that the pre to posttest gain was significant for the VM and the CM intervention groups, but not the PM intervention group (see Table 25). Similarly, the effect size analyses yielded large effects for the VM and CM groups, but only a moderate effect for the PM intervention. Comparison of intervention groups yielded large effects favoring VM groups (d = 1.85) and CM groups (d = 1.31) when compared to the PM groups. The comparisons of VM and CM intervention yielded a small effect size score of 0.20 favoring the VM intervention. A one-way ANOVA comparison of gain scores indicated that the difference among groups was significant at the 95% level, $F(2, 43) = 3.87$, $p = 0.03$. However, the PM groups' pretest scores (4.27 out of 6 possible points) was considerably higher than those of the VM (2.21) and CM (2.43). On the posttest all three intervention groups scored within the range of 4.71 to 4.80. Although the PM groups made significantly less gains, the posttest achievement levels of the three groups were similar.

Table 25

Summary of DCA-Q5 Analyses

	Pretest		Posttest		Pre to post			
Intervention type	*M*	*SD*	*M*	*SD*	*df*	*t*	*p*	Cohen's *d*
PM	4.27	1.10	4.80	0.86	14	1.74	0.10	0.54
VM	2.21	2.01	4.79	1.31	13	3.56	0.00	1.52
CM	2.43	1.95	4.71	1.68	13	4.02	0.00	1.25

Note. $N = 43$.

To analyze the growth of student knowledge over the duration of the intervention, scatter plots of DCA-Q5 were developed and the lines of best fit were compared (see Figure 28). The greater slope of the line of best fit for the VM intervention (y = 0.28x + 1.85), when compared with the PM (y = 0.5x + 4.13), and the CM interventions (y = 0.18x + 2.98) suggests that the rate of growth was greater for the VM intervention.

Analyses of the trend lines of DCA-Q5 also suggested that, only for the VM group was the growth continuous over time (see Figure 29). The PM intervention trend line started high and remained almost constant showing little growth, and the CM intervention trend line showed an early dramatic increase and then remained fairly constant. The VM intervention trend line showed an initial decrease in scores and then the scores gradually increased to the level of the other two interventions. The three groups achieved similar scores for the last four lessons.

Trend lines of the percent of students who reached DCA-Q5 mastery showed that less than 15% mastered the question. The PM intervention students tended to reach mastery sooner than the students of the other two groups (see Figure 30).

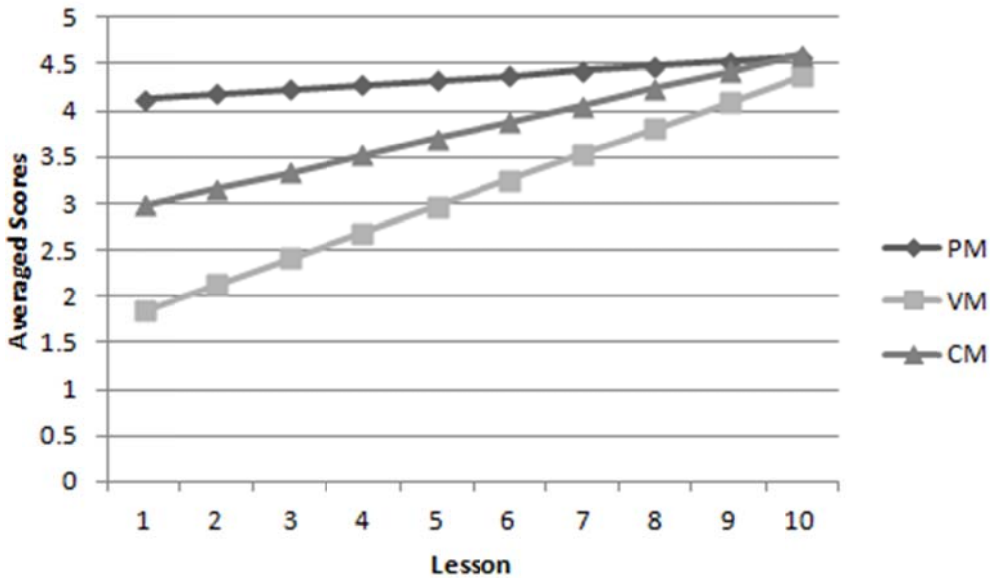


Figure 28. Trajectories of growth for DCA- Q5.

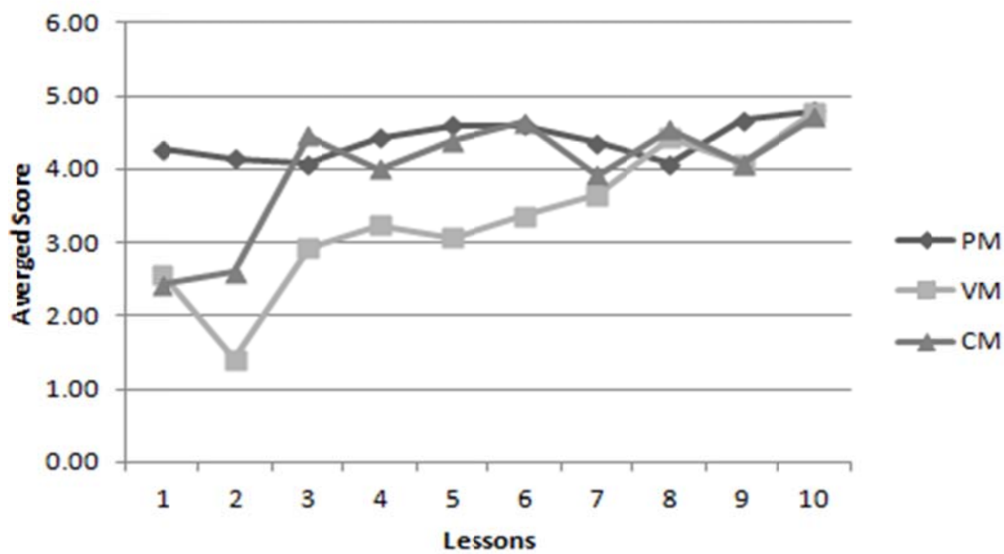


Figure 29. Trend line of averaged scores for DCA-Q5.

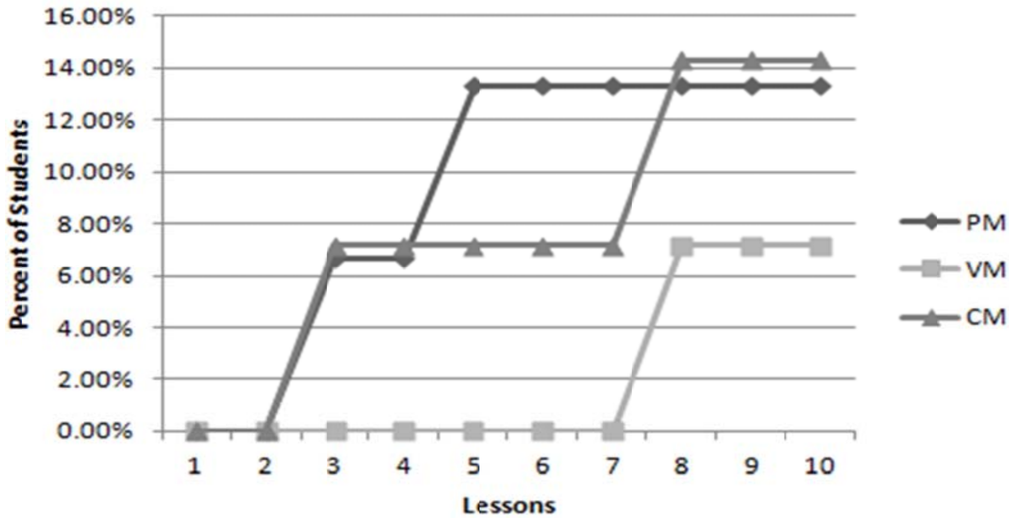


Figure 30. Trend line showing percentage of mastery of DCA-Q5.

For DCA-Q5, gain comparison analyses favored the VM intervention. Only the VM intervention showed a linear relationship between intervention duration and achievement. However, trend lines of achievement and mastery of DCA-Q5 show that the achievement levels were higher for the PM and CM intervention groups.

The second source of data for the *partitioning* sub concept was error the analysis. Practice sheets from the explore and practice phases of the lessons and students' responses on the daily DCAs, LCA, and LPA, were coded for students' errors made when drawing pictorial representations of fractions. Three codes were assigned: incorrect number of sections, partitioning a partition, and other errors. Incorrect number of sections occurred when students partitioned region models by drawing the same number of partition lines as was in the denominator of the fraction. Partitioning a partition occurred when students first partitioned a model into sections of lesser amounts than the denominator and then partitioned only part of the sections again to obtain the correct number of sections (e.g., a student partitions a circle into fourths and divides one of the fourths into half to model fifths.) Other partitioning errors were coded as other. Figure 31 shows three examples of other responses. The number of partitioning errors for each category is shown in Table 26.

Figure 32 shows the trajectories for each of the three errors. Comparison of the first to the last lesson of the trajectory showing a summary of all partitioning errors indicates that all three groups had a proportionally similar decrease in errors. However, the trajectory of the VM intervention after the first lesson had a rise in scores which continued to increase for the first three lessons. The VM intervention scores then began to steadily decrease until the final two lessons. Comparisons of the trajectories suggest.

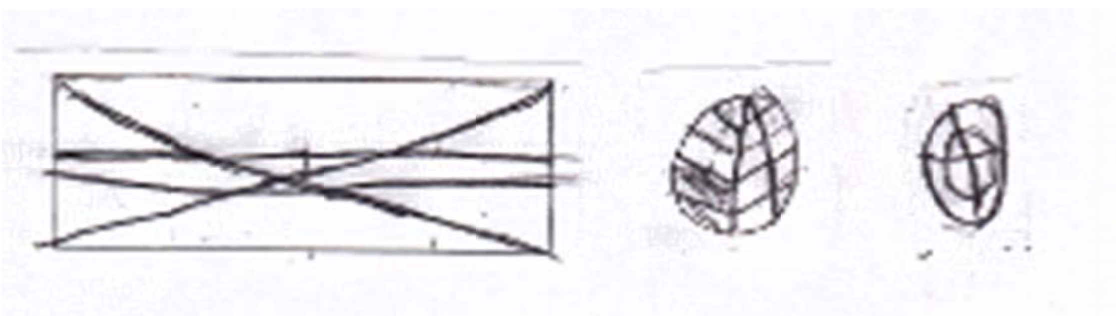


Figure 31. Student examples of “other” partitioning error responses.

Table 26

Analysis of Representation Partitioning Errors

	Number of cases		
Partitioning errors	PM	VM	CM
Incorrect number of sections	9	7	6
Partitioned a partition	14	10	42
Other errors	3	12	14

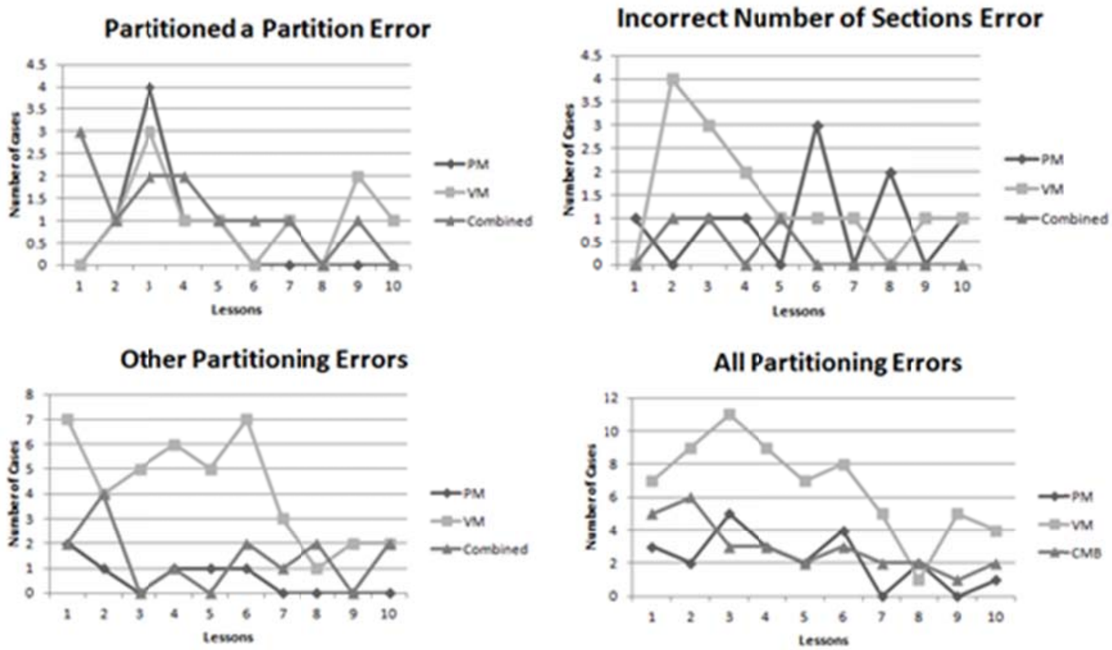


Figure 32. Trajectories of partitioning errors. Upper left: Partitioning a partition; upper right: Incorrect number of partitions; lower left: Other partitioning errors; lower right: All partitioning errors.

that there may be a relationship between manipulative type and resolution of specific errors. For example, resolution of the error of using an incorrect number of sections appeared to have been complete (no errors were observed for two or more lessons) for only the CM intervention. Resolution of partitioning a partition and other partitioning errors was complete for only the PM group.

In summary, the rate of error resolution of Misconception 3 (misnaming errors) was greater for the PM intervention, but was complete for both the PM and the CM interventions. CM interventions yielded the greatest gains and greatest rate of increase of gains for DCA-Q1. But for DCA-Q5, VM intervention yielded the greatest gains and was the only intervention with a linear relationship between intervention duration and student achievement. Error resolution of partitioning errors was complete for more PM and CM students. However, the analyses also indicate that a number of the students already were functioning at high levels of achievement, thus limiting comparisons.

Evaluating fraction values. For students to evaluate if two symbolic representations of fractions are equivalent, they need an understanding of the magnitude of fractions and the ability to compare fractions. From the literature, the three skills of comparing, ordering, and developing were identified and three corresponding DCA questions developed. DCA-2 asked students to compare three fractions, one of which was greater than ½. DCA-3 asked students to place two fractions on a number line and DCA-4 asked students to develop and place on the number line, between the two existing fractions, a new fraction. Table 27 contains a summary of the paired samples *t* tests. For DCA-Q2 and DCA-Q4 the intervention was not significant, indicating that students made only limited gains. Therefore, the analysis comparing the effects of the manipulative interventions did not reflect variations in learning and these questions were not analyzed further. Further analyses will be provided for DCA-Q3.

Daily cumulative assessment question 3 (*DCA-Q3).* DCA-Q3 asked students to place two given fractions on a number line. The difference between the student's placement of the fraction on the number line and the correct location was measured. Responses were evaluated on a six point rubric ranging from a total difference of greater than 20 centimeters to 0 centimeters (see Appendix G). Paired samples *t* tests indicated that the gains were significant for the PM and CM intervention at the 95% level (see Table 27). The Cohen *d* effect size analysis of pre to posttest gains yielded a large effect

Table 27

Summary of Data Analysis for DCA-Q2 and DCA-Q4

Question/ intervention type	DCA pretest *M*	DCA pretest *SD*	DCA posttest *M*	DCA posttest *SD*	*df*	*t*	*p*
DCA-Q2 PM	4.93	1.58	4.80	1.78	14	-0.13	0.71
DCA-Q2 VM	4.21	1.58	4.43	1.79	13	.366	0.72
DCA-Q2 CM	3.86	1.61	4.79	1.63	13	1.43	0.18
DCA-Q3 PM	3.47	1.61	5.00	1056	14	2.66	0.02
DCA-Q3 VM	4.00	1.52	4.14	1.75	13	0.30	0.77
DCA-Q3 CM	3.07	1.59	4.21	1.58	13	2.51	0.03
DCA-Q4 PM	0.40	0.51	0.47	0.52	14	0.44	0.67
DCA-Q4 VM	0.64	0.50	0.43	0.51	13	1.00	0.37
DCA-Q4 CM	0.36	0.50	0.50	0.52	13	1.47	0.17

size (d = 0.96) for the PM intervention, a moderate effect size (d = 0.72) for the CM intervention and a small effect size (d = 0.09) for the VM intervention. An effect size comparison among intervention groups yielded a large effect size favoring PM groups when compared to VM groups (d = 0.84) and a small moderate effect size when compared to CM groups (d = 0.25). Comparison of PM groups to CM groups yielded a moderate effect size of 0.60.

To analyze the growth of student knowledge over the duration of the intervention, scatter plots were developed and the lines of best fit were compared (see Figure 33). The greater slope of the line of best fit for the PM intervention (y = 0.14x + 3.35), when compared with the VM (y = 0.06x + 3.31) and the CM interventions (y = 0.05x + 3.67) suggests that the rate of growth was slightly greater for the CM intervention. However, the rate of growth was small for all three groups.

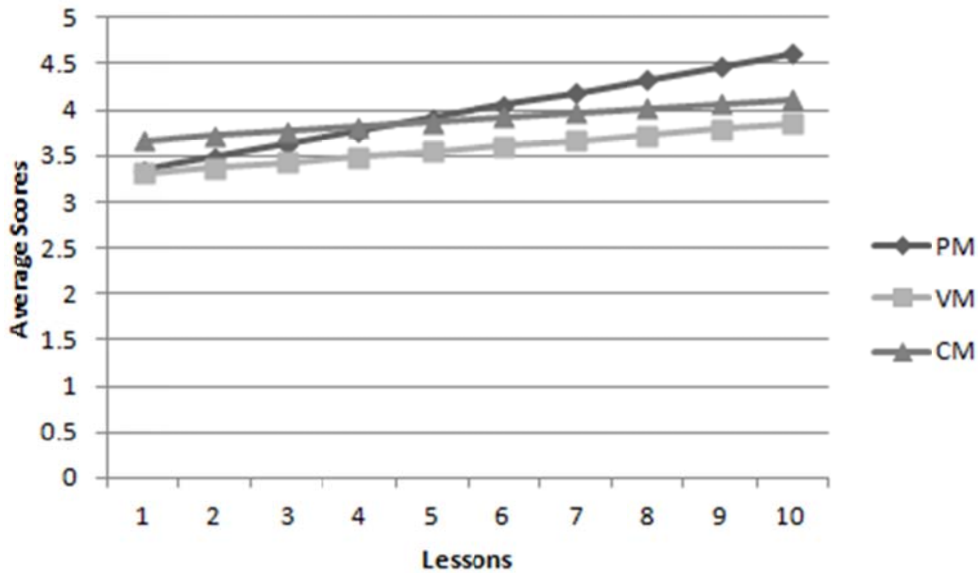


Figure 33. Trajectories of growth for DCA-Q3.

Trend lines of intervention groups averaged lesson scores for DCA-Q3 indicated that the trajectories were similar for the PM and CM groups with a gradual increase over time (see Figure 34). The plot of the VM group had lower scores overall and a drop in scores from lesson three to five. By the end of the intervention the VM groups had recovered to their beginning level of responses. This suggests that not only was the intervention measured by DCA-Q3 ineffective for this group, but may have had a negative effect on their learning.

Trend lines showing the percent of students who reached mastery of DCA-Q3 indicated that more than double the number of PM students reached mastery (40%) when compared with the VM (14.3%) and CM (14.3%) intervention students (see Figure 35).

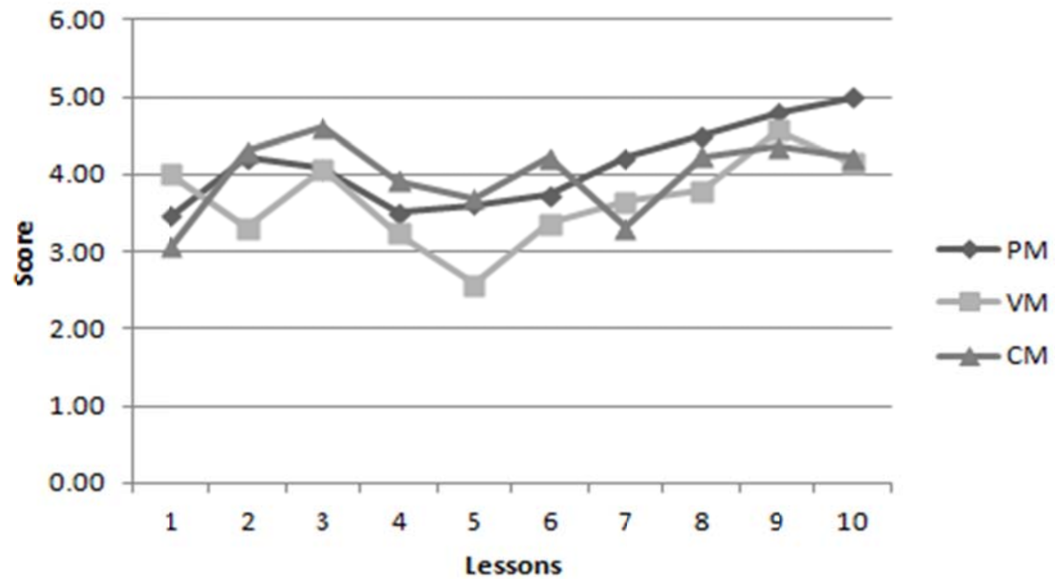


Figure 34. Trend line of averaged scores for DCA-Q3.

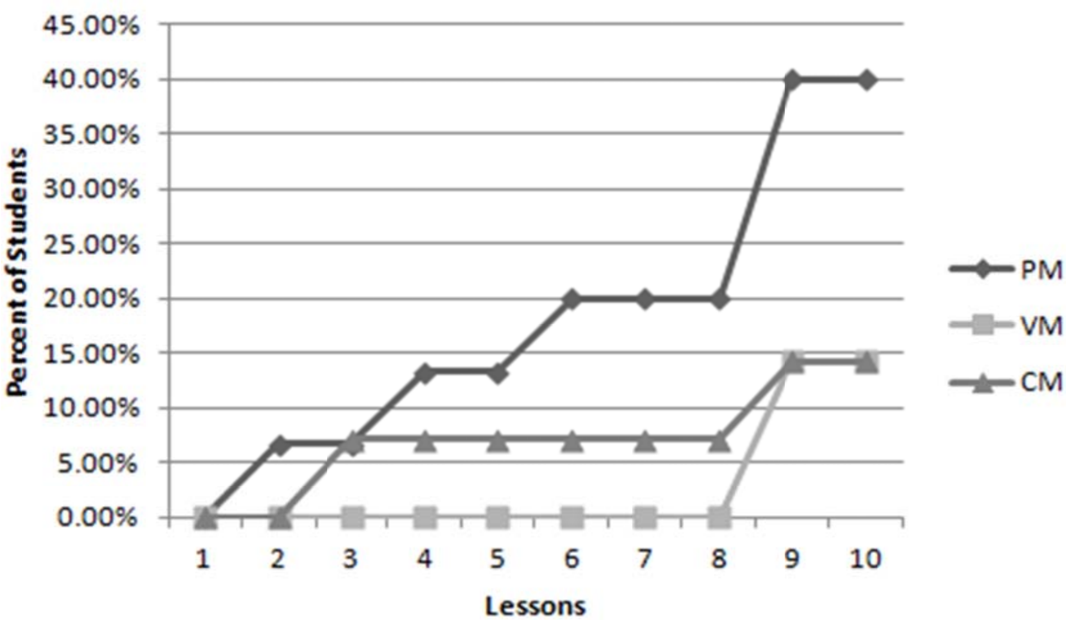


Figure 35. Trend line showing percentage of mastery of DCA-Q3.

Misconception 1 (whole number dominance). Another source of *evaluating fraction value* data emerged from the error analysis. Misconception 1 was: *Whole number dominance applies to fraction comparison*. This misconception was observed as students incorrectly applied the belief that because whole numbers of greater magnitude imply larger amounts, larger numbers in the denominators of fractions also imply larger amounts.

To compare the resolution of Misconception 1 during the duration of the intervention, scatter plots were developed and the equations of the line of best fit for each intervention group were used to plot a comparison graph (see Figure 36). The greater slope of the lines of best fit for the CM (y = -0.53x+6.8) and PM interventions (y = -0.51x+6.6) when compared to the VM intervention (y = -0.33x+5.6) indicated that the CM and PM students had the greatest rate of reduction of errors.

Figure 37 compares the error variations in students' learning trajectories among intervention groups. The trajectories for the three intervention groups are similar with

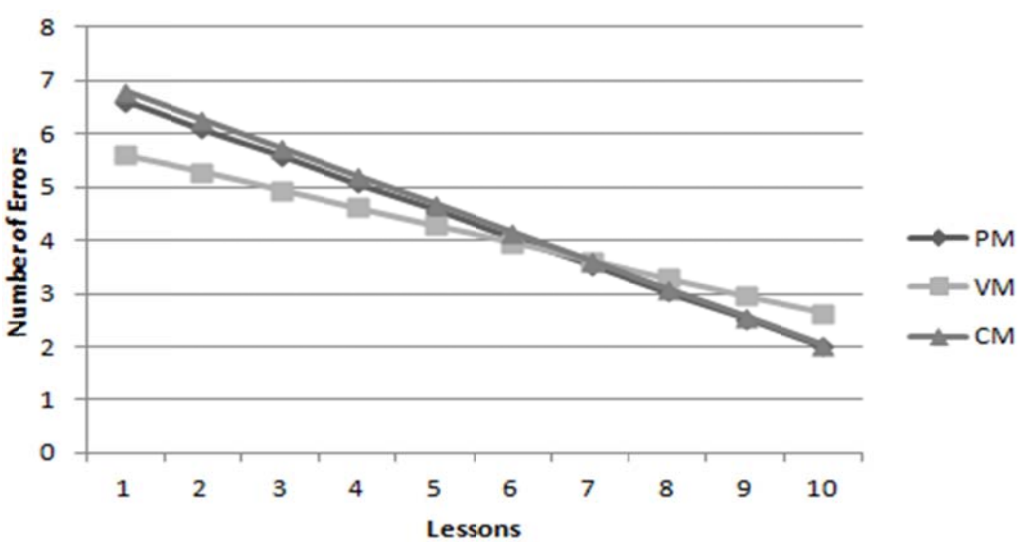


Figure 36. Trajectories of Misconception 1 cases.

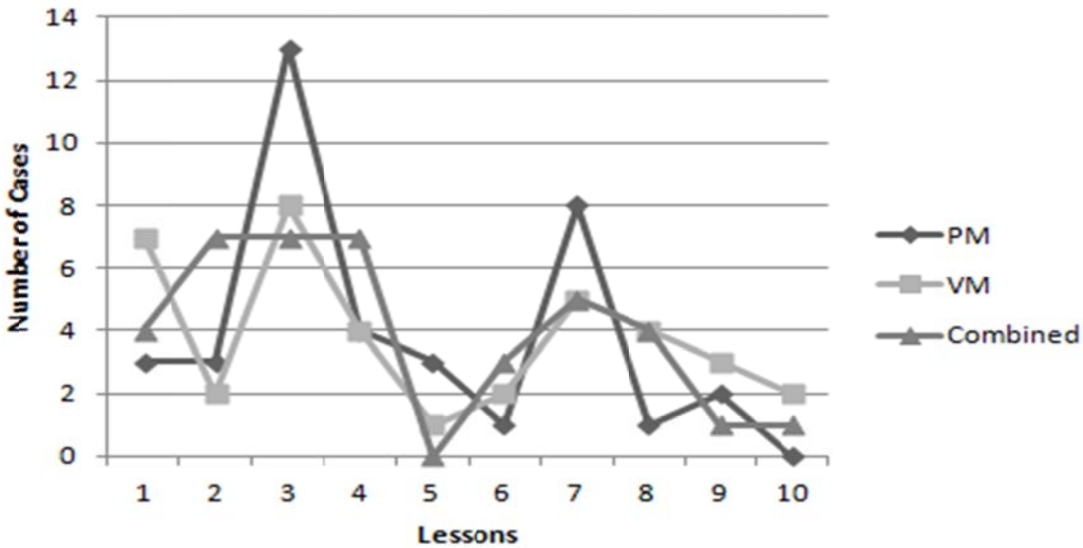


Figure 37. Trend line trajectories of Misconception 1.

higher peaks for the PM intervention. These peaks occur when the two comparing fractions lessons 3 and 7 were taught. Because these lessons focused on comparing fractions the students had more opportunities to make errors in these lessons and also in the subsequent lesson when retention of the concept was tested. For the other lessons of the intervention, the trajectories of the three intervention groups suggest that there was a slight decrease in the frequency of cases in which students were inappropriately applying whole number dominance to fraction comparisons.

In summary, two of the four sources of data for *evaluating faction values* did not yield significant gains in learning, indicating that either the intervention or the assessment for these two sub concepts was not effective. Therefore, data from these two sources was not used in the analysis. For the remaining two sources, although scatter plots indicated that there was not a strong correlation between intervention duration and DCA-Q3 scores, the paired samples *t* test results indicated that the intervention was effective for the PM and CM interventions and comparisons of gain and percent of students reaching mastery both favored PM intervention. Likewise, the rate and completeness of resolution of Misconception 1 (Whole Number Dominance) favored both the PM and the CM interventions.

Developing equivalence thinking. At the base of each of the five equivalent fraction concepts of *modeling, identifying, grouping, solving*, and *simplifying* is the students' development of equivalence thinking in relationship to fractions. Equivalence thinking requires students to develop a working understanding of: (a) the meaning of equivalence, (b) comparison of areas, (c) the conservation of the part-whole relationship, and (d) and the ability to think multiplicatively. Since it was through the qualitative coding of errors that the distinctiveness of equivalence thinking and the persistence of nonequivalence thinking emerged, none of the assessments specifically targeted students' development of equivalence thinking. Therefore, the sources of data relating to this concept were limited to error analysis. The four aspects of equivalence thinking are discussed in relation to the correlating misconceptions.

Meaning of equivalence. Thinking of equivalence in relation to fractions is not the same as thinking of equivalence of whole numbers. Fractions are part- whole relationships and that relationship is the focus for determining equivalence. As some students seek to understand equivalence of fractions they focus on incorrect relationships. Misconception 5 (equivalence meaning errors), which emerged from the analysis of lesson artifact student errors reads: *Equivalence denotes relationships other than equal amounts.* Four incorrect relationships that students used in developing an equivalent fraction were: (a) Error 9: Identifies equivalent fractions as being two fractions naming the relationship of the parts making up a whole (e.g., 1/3 = 2/3; 51.1% of the 247 Misconception 5 cases); (b) Error 10: Identifies equivalent fractions as the original fraction and a second fraction whose value is equal to one and contains numerals that were either in the original fraction or factors or multiples of the numerals in the original fraction (e.g., 6/8 = 6/6 or 8/8 – numerals from original fraction, 6/8 = 2/2 - factor or 2/3 = 4/4 or 6/6 – multiples; .20.6% of the 247 Misconception 5 cases); (c). Error 11: Identifies equivalent fractions as being a fraction and its reciprocal (e.g., 1/3 = 3/1; 15.4% of the 247 Misconception 5 cases); and (d) Error 12: Identifies equivalent fractions as being a fraction and a second fraction which is derived by determining the number of times a number will go into either the numerator or the denominator of the original fraction (e.g., 5/10 = 2/5 because 5 goes into 10 twice; 8.9% of the 247 Misconception 5 cases). Most of the Misconception 5 cases were from students' responses to the DCAs with only several instances during discussions.

The number of Misconception 5 error cases observed in each lesson were totaled and plotted in scatter plots (see Figure 38). To compare the resolution of errors over time scatter plots were developed and the lines of best fit compared. The greater slope of the line of best fit for the VM intervention (y = -1.35x+17.2) when compared to the PM (y = -1.16x+15.07) and the CM interventions (y = -0.83x+10.67) indicated that the VM students had the greatest rate of reduction of errors.

Figure 39 shows the trend line trajectories derived from the number of Misconception 5 error cases observed for each lesson. Over the first four lessons, the

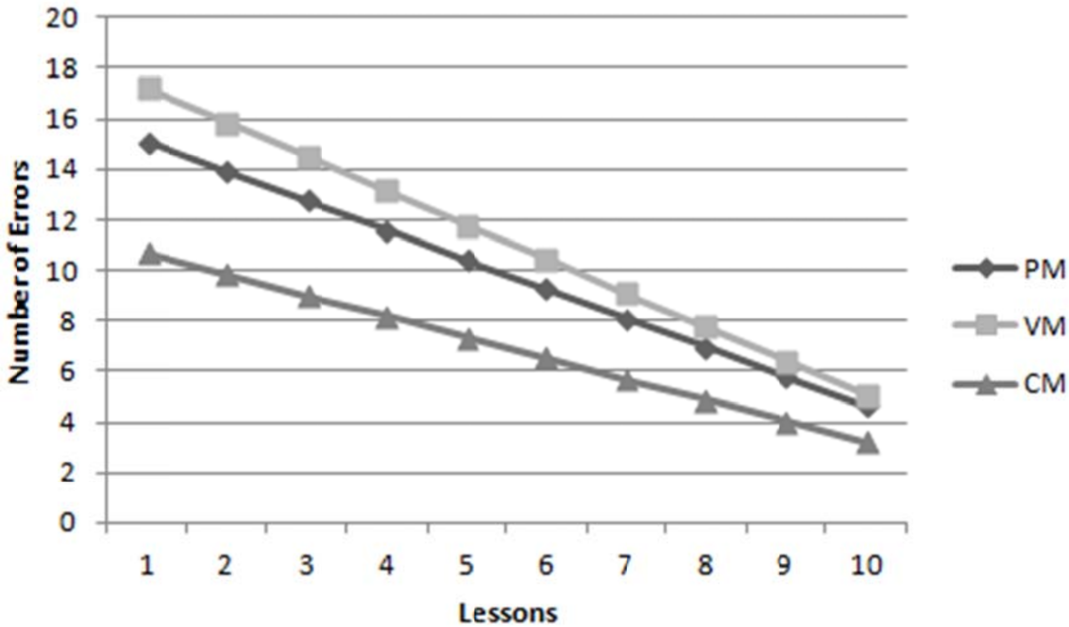


Figure 38. Trajectories of Misconception 5 cases.

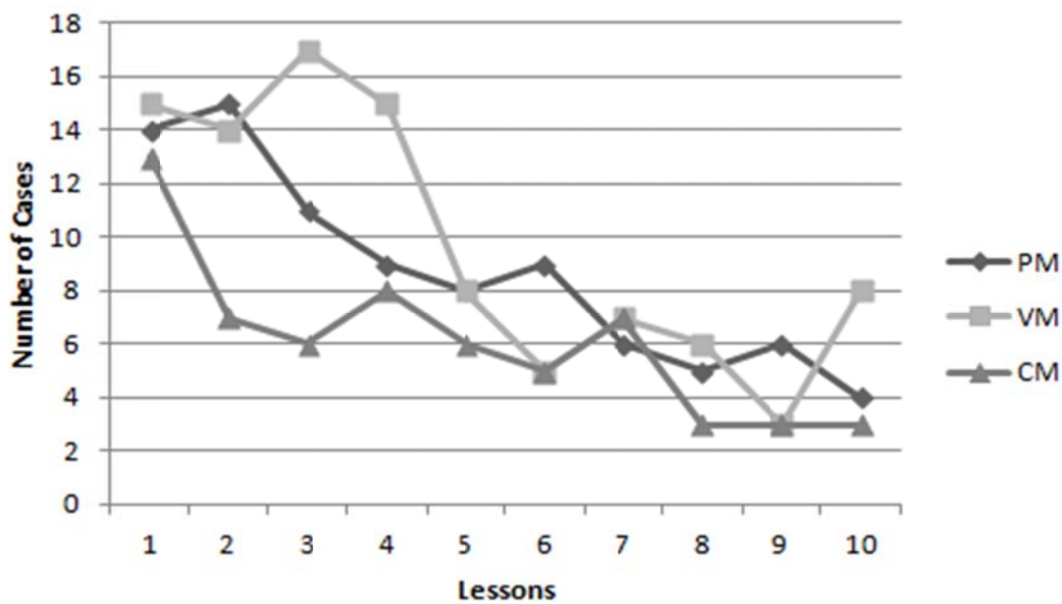


Figure 39. Trend line trajectories of Misconception 5.

number of error cases for the VM intervention group increased, while the PM and CM intervention groups decreased. After the fourth lesson all the trajectories follow approximately the same path until the last lesson when the VM cases increased again.

Comparison of areas. To visually determine if two fractions are equivalent, a person must determine if the areas of the fractions are equivalent. Sometimes, learners incorrectly focus on other model features. For example, when comparing ¾ and $\frac{11}{16}$ in Figure 40, students might respond that the fractions are not equivalent because the sizes of the shapes, the shape of the shaded area, or the number of parts are not the same.

The source of data for the *comparisons of areas* was the EFT. EFT Question 17 asked students to compare two models to determine if the models were equivalent (see Figure 40). They were asked to give the fraction names for the two models and to write a description of how they determined if the fractions were equivalent. The gain from pre to posttest was 26.7% for the PM group, 14.29% for the VM group and -14.29% for the CM group. Twenty-eight percent of the CM students responded with the correct fractions and answered that the fractions were equivalent. This suggests that the students were focusing on features other than the areas of the shaded parts.

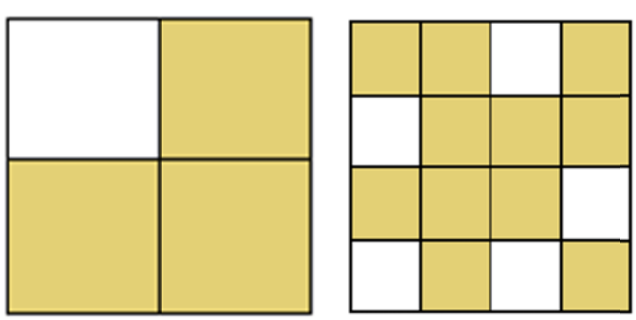


Figure 40. Pictorial representations for EFT Question 17.

Conservation of the part-whole relationship. A concept which beginning fraction students often struggle with is the understanding that partitioning or combining sections of a model, proportionally changes both the numerator and the denominator. Symbolically, to conserve the relationship of the fraction you must multiply or divide the numerator and the denominator by the same number to maintain the same proportional relationship. Error analysis of the lesson artifacts identified Misconception 6 (Incorrect Equivalent Sentences). It reads: *When developing equivalent fractions, numerators and denominators may vary independently of each other.* Three types of errors were observed in which students manipulated the numerators and denominators differently when developing equivalent fractions. The three manipulation errors were: (a) multiplying the numerator and denominator by different numbers (Error 13, 36.1% of the 147 Misconception 6 cases); (b) increasing or decreasing only the denominator or only the numerator (Error 14, 58.7% of the 147 Misconception 6 errors); and (c) multiplying a number in the numerator of the second fraction by either the numerator or the denominator of the original fraction (Error 15, 6.8% of the 147 Misconception 6 cases).

The cases of errors for Misconception 6 observed in each lesson were totaled and plotted in scatter plots (see Figure 41). The greater slope of the line of best fit for the PM intervention ($y = -0.41x+8.4$) when compared to the VM ($y = -0.12x+6.47$) and the CM interventions ($y = -0.05x+3.07$) indicated that the PM students had the greatest rate of reduction of errors for Misconception 6.

A trend line of Misconception 6 errors shows multiple increases and decreases in errors, with the differences in the number of error cases observed in lesson 10 being only

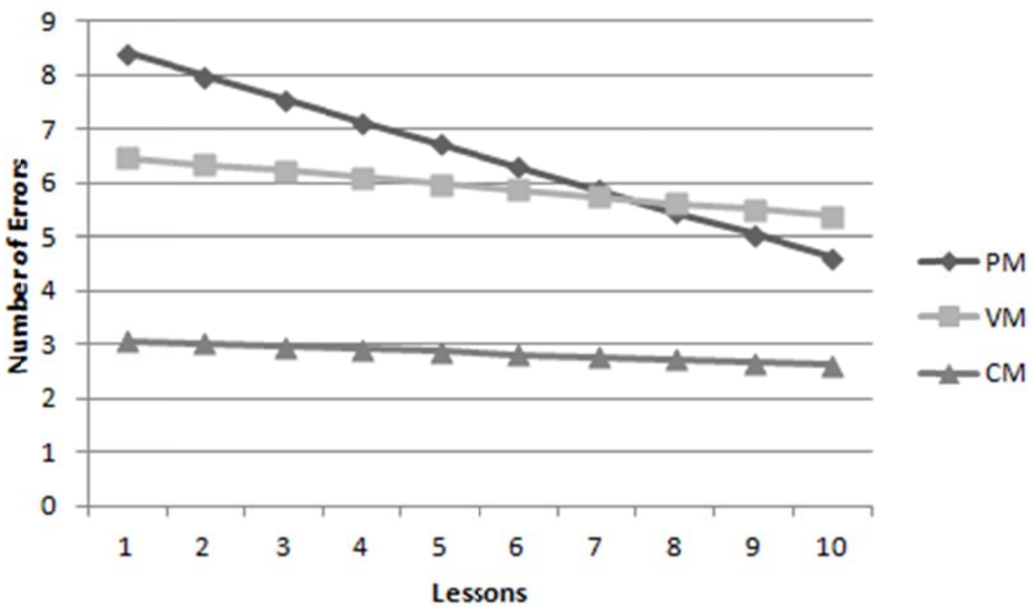


Figure 41. Trajectories of Misconception 6 cases.

one or two less or more than were observed in lesson one (see Figure 42). The trajectory for the PM group suggests that even though there does appear to be a correlation between the intervention and the reduction of errors, the higher rate of reduction occurred in the first five lessons only. This suggested that the intervention was not effective in the resolution of Misconception 6 errors for any of the intervention groups.

Multiplicative thinking. In lesson 2 students were asked to predict the partitioning of paper strips folded repeatedly in half. At the beginning of the activity all students used additive thinking to make their predictions. When students folded a strip of paper in half and folded it in half again, they could correctly predict that there would be four partitions. But when the students folded the paper strip in half again, almost all of the students predicted there would be six sections, not eight. The students were thinking additively. It was not until they had done this multiple times that they began to make predictions using

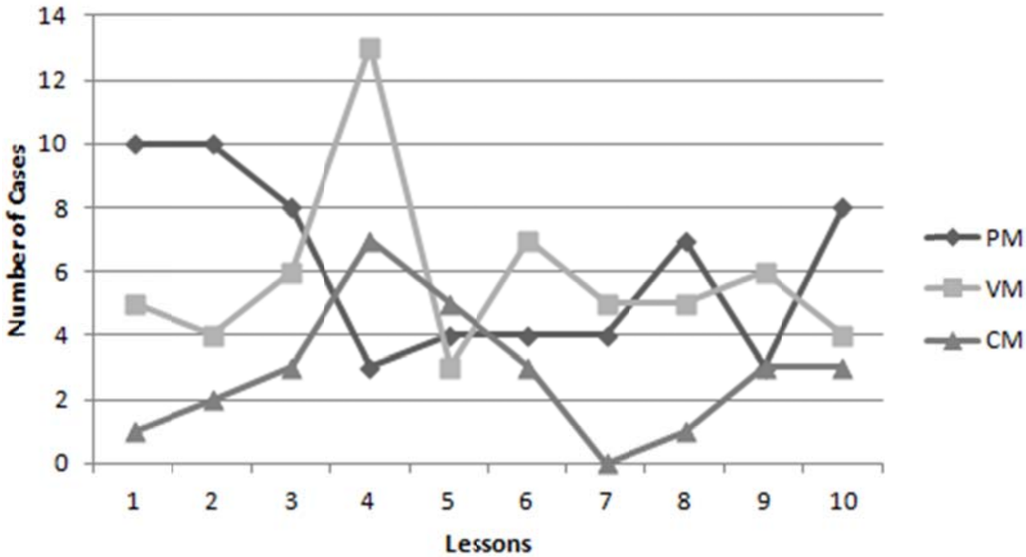


Figure 42. Trend line trajectories of Misconception 6.

multiplicative thinking. This ability to think multiplicatively is important in students' ability to make and to visualize the building of equivalent groups. The third type of errors affecting *equivalence thinking* was Misconception 2 (additive thinking) which reads: *Equivalent fractions can be formed by adding the same number to the numerator and the denominator of the original fraction.* Misconception 2 consisted of only one error, that of adding or subtracting, instead of multiplying or dividing, when developing or solving equivalent fraction sentences. The number of Misconception 2 cases was 78.

Analyses of the Misconception 2 scatter plots indicated that for VM and CM intervention groups, the relationship between lesson and the number of error cases appeared to be linear with the variation in error cases best explained in the VM intervention group (see Figure 43). To compare the resolution of errors over time scatter plots were developed and the lines of best fit compared. The greater slope of the line of best fit for the VM intervention (y = -0.44x+5.8) when compared to the PM (y = -0.22x

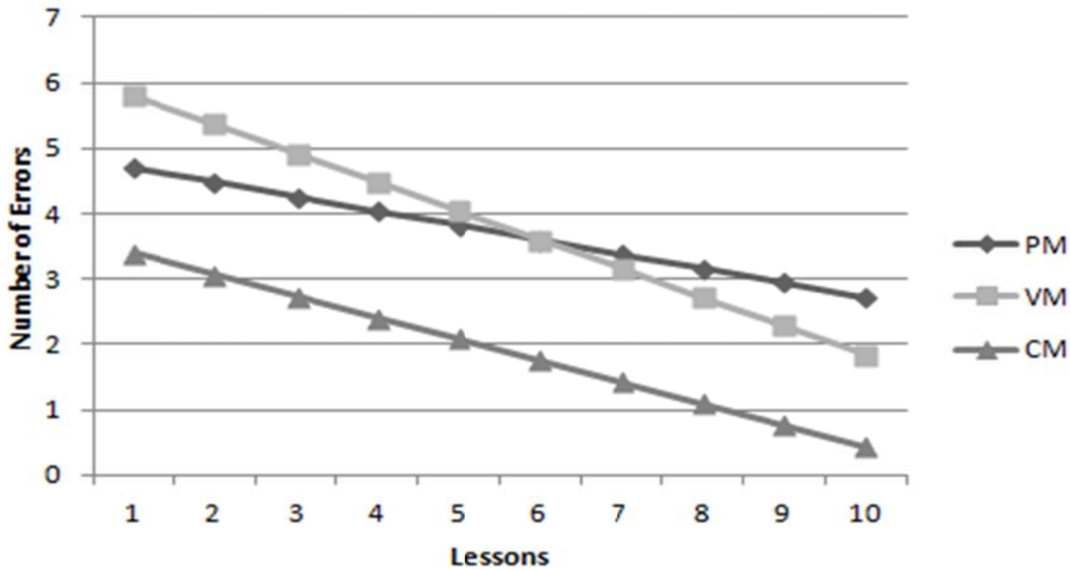


Figure 43. Trajectories of Misconception 2 cases.

+4.7) and the CM interventions (y = -0.33x+3.4) indicated that the PM students had the greatest rate of reduction of errors for Misconception 2. However, the comparison of the rate of resolution may not be valid because the number of occurrences for the PM and CM interventions tended to be low for all lessons except Lesson 2.

Figure 44 compares the variations in students' learning trajectories for Misconception 2 by intervention group. The trajectories for all three groups peaked on lesson 2. This was a lesson in which students learned to predict how many sections would be the result of repeated partitions. Instructors' comments indicated that almost all students first used additive thinking to predict the number of sections. This peak was especially strong for the PM group, but the observed number of cases returned to 0 for the next lesson. The trajectory of the VM group did continue to remain elevated and gradually decreased until lesson six.

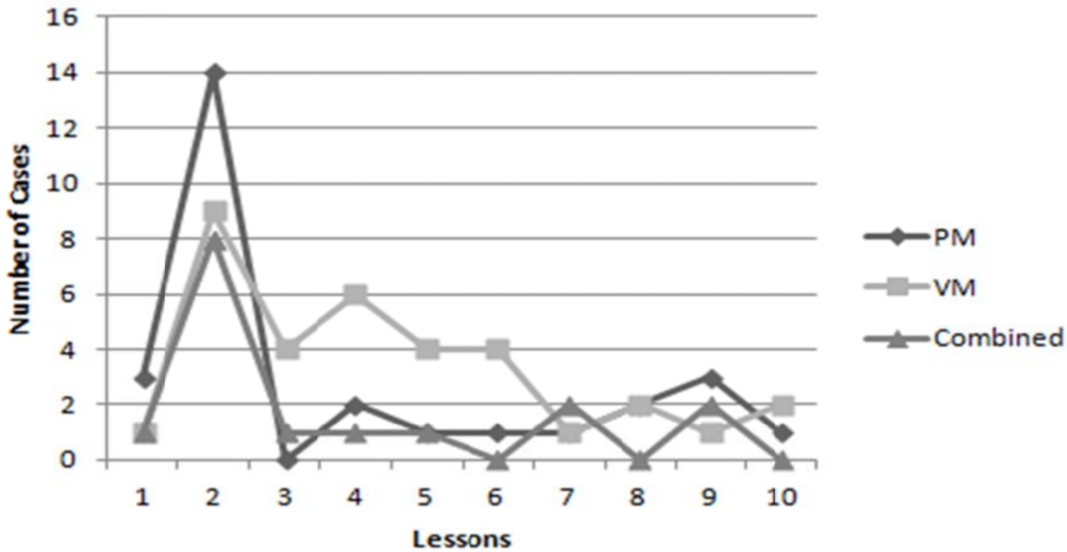


Figure 44. Trend line trajectories of Misconception 2.

In summary, four student misconceptions were identified which related to *developing equivalent thinking.* A comparison of the resolution of Misconception 2 (additive thinking) and Misconception 5 (equivalence meaning) by intervention groups suggested that the VM group had the greatest rate of resolution for both misconceptions. However, this was due in part to the greater number of cases observed for the VM intervention group. Comparisons of students' correct answers for question 17 of the EFT (comparing fraction areas) indicated that the PM intervention had the greatest gain. Analysis of Misconception 6 (incorrect equivalent sentences) indicated that there was little decrease in the occurrences of errors over the duration of the intervention.

Research Subquestion 1(b): Line Trajectories Showing Changes Achievement

The purpose of the line trajectory synthesis was to examine the trend lines developed in the study for any observable differences among intervention groups. Trajectories were divided into two groups. The first group was the trend lines which showed growth over time, including those showing data from the DCA (Figures 45 and 46). For this comparison, the data from the eight questions of the DCA were aggregated into the three concept clusters of partitioning, fraction value, and equivalence. Six observations resulted from an examination of the trend lines.

1. The PM and the CM groups tended to perform with similar increases and decreases, suggesting that use of these interventions had similar effects on student learning;

2. Both the PM and CM groups tended to score higher than the VM group. This difference was especially evident in the graphs of mastery;

3. As can best be observed in the four graphs of the DCA total score and clusters, the VM intervention scores decreased between the DCA pretest given before lesson 1 and the second DCA which was given at the end of lesson 2. The scores of the PM and CM interventions from lesson 1 to lesson 2 rose, suggesting that there may have been some type of factor influencing the performance of the VM students, but not that of the PM or CM students;

4. The baseline at the beginning of the mastery trajectories of the VM intervention tended to remain flat showing little difference for a longer period of time

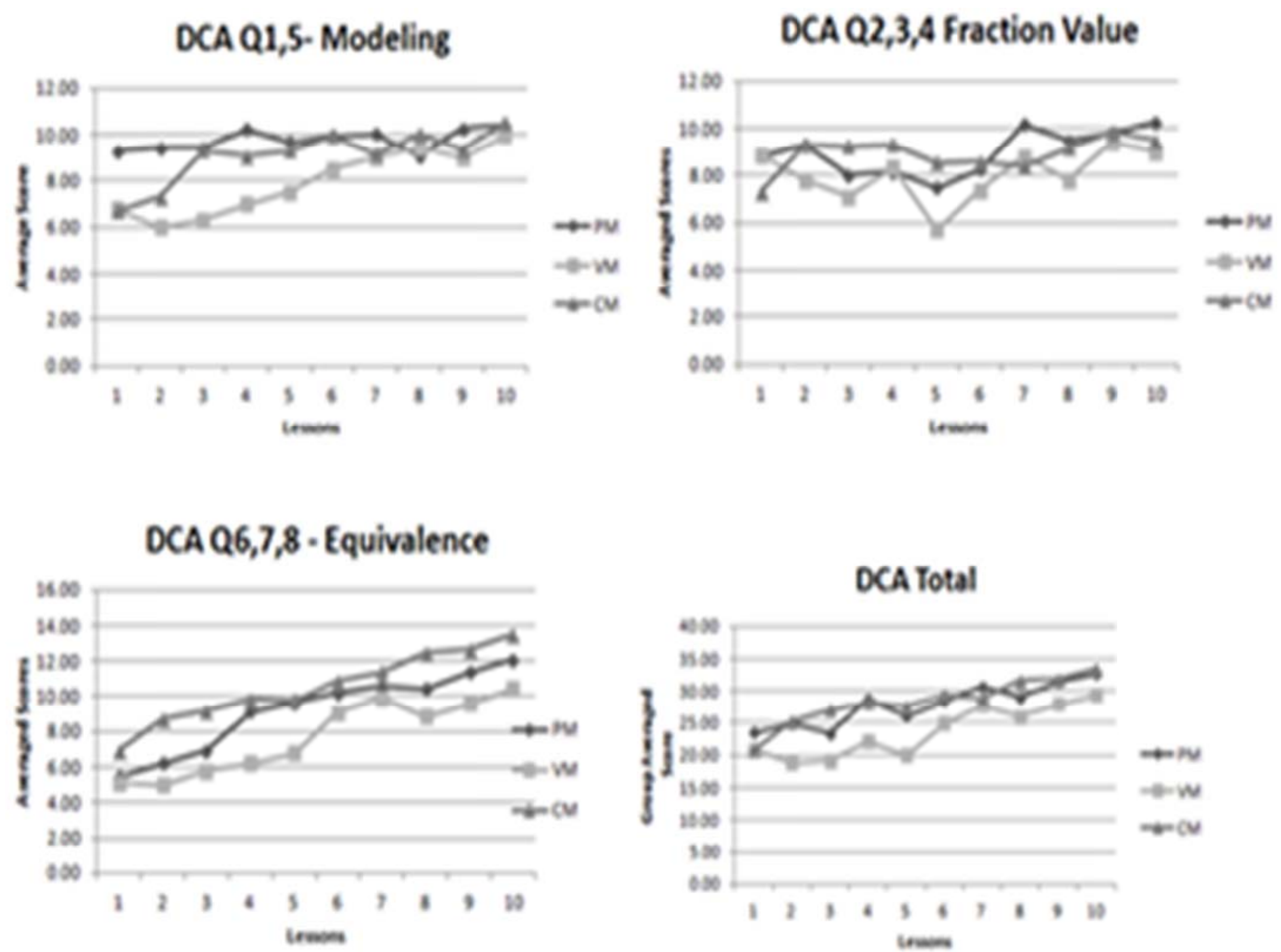


Figure 45. Trend lines showing growth over time.

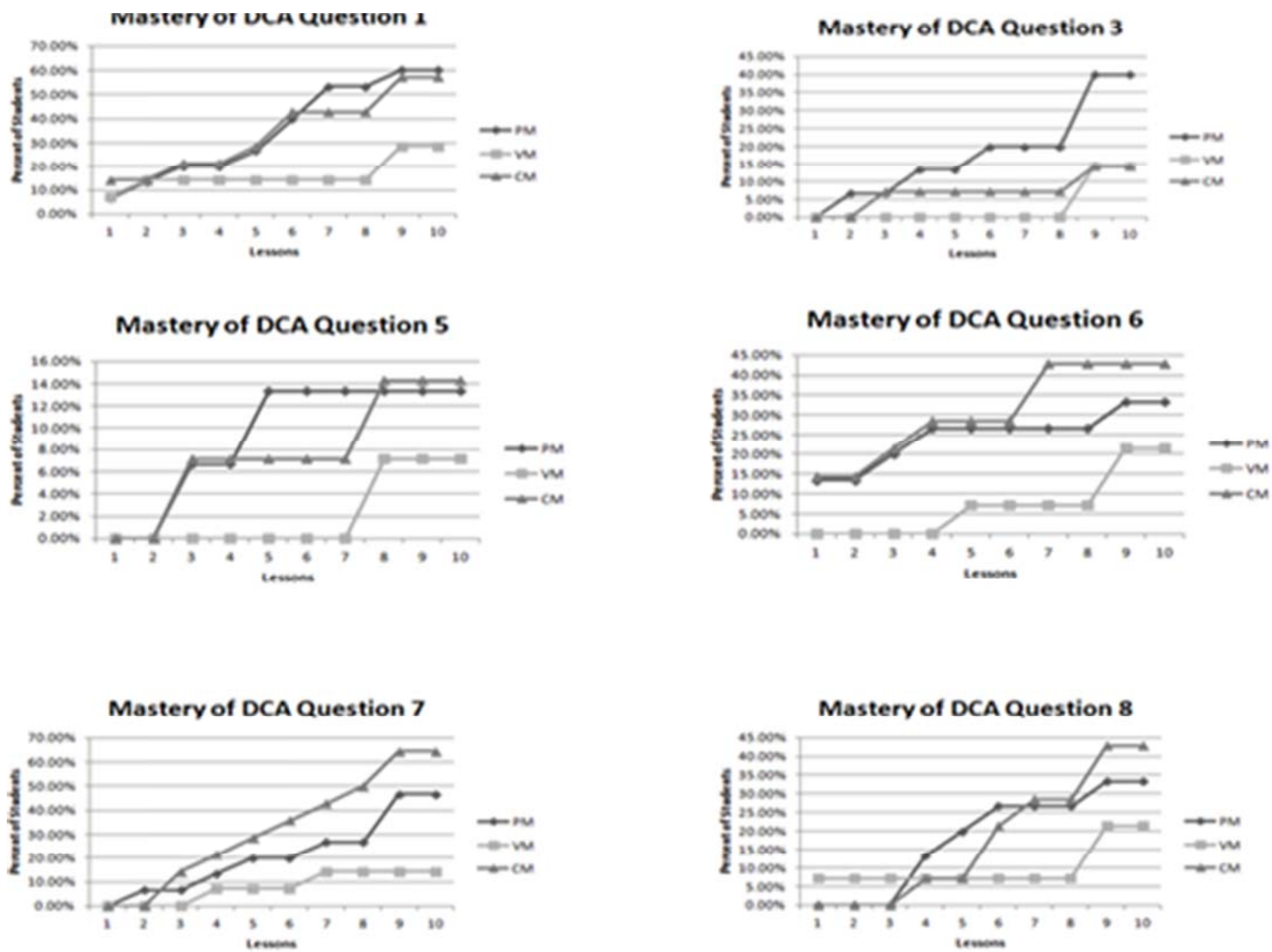


Figure 46. Trend line trajectories showing mastery over time.

than the trajectories of the other two manipulative groups. This suggests that the VM students obtained mastery at a slower rate than the other two intervention groups;

5. For the majority of the graphs, the range of difference between the groups began to decrease during lessons 4 to 7 and the trend line trajectories became more similar, suggesting that, as the duration of the intervention increases, the learning of the three groups became more similar; and,

6. The trend lines increased at a steadier rate, with less increases and decreases, during the last five lessons, suggesting that over time the students became more solidified in their knowledge. These six observations suggest that, there were variations in learning which may be related to intervention type, and that as the duration of the intervention increased, the variations in student learning decreased.

The second group of trajectories examined was the trajectories of students' misconceptions and students' errors of partitioning (see Figure 47). With the exception of Misconception 6 (incorrect equivalent sentences), for all three intervention groups, the number of errors was reduced over the time of the intervention. Again, as was observed in the previous group of trajectories, initially the trajectories show greater variance in rate and consistency, but over time these variances diminish and the trajectories become more similar. Again the VM intervention group progressed at a slower rate than the PM and the CM intervention groups, but the difference among the error rates began to narrow over the duration of the intervention.

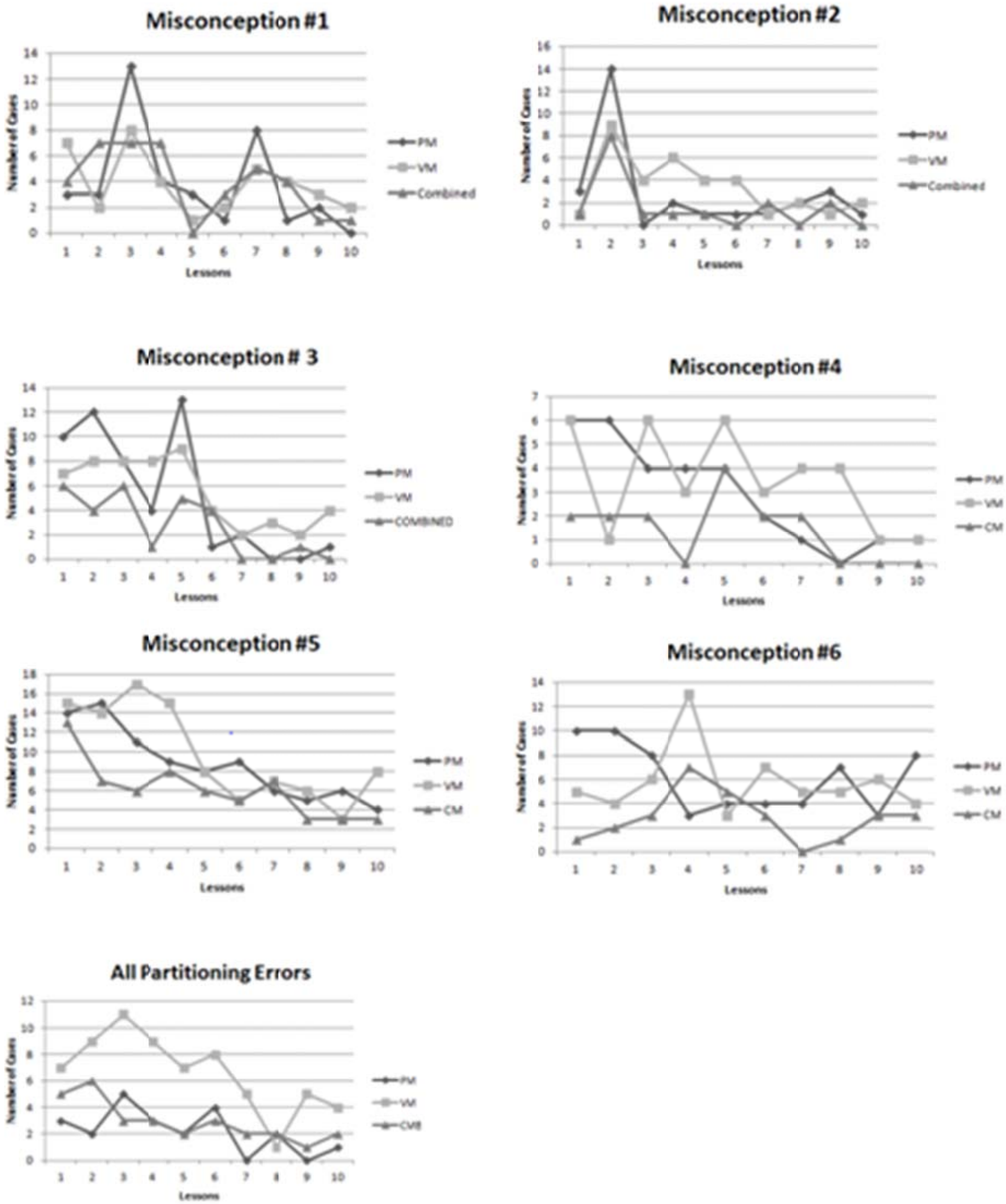


Figure 47. Trend lines showing resolution of errors.

Research Sub1uestion 1(c): Variations of Achievement in Lessons Activities

Research subquestion 1(c) was: What are the variations in patterns of daily lesson achievement, retention and work completion? The sources of data for this question are: LCA, which measured concept understanding; and the LPA, which measured retention and the number of problems completed in the explore and practice phases of the lessons. These measures were specific to each lesson and do not represent growth over time.

Lesson Concept Assessments

Each LCA administered at the end of the lesson consisted of three questions duplicating questions students had responded to during the explore phase of the lesson. Responses were scored on a 1- to 4-point rubric evaluating the level of guidance students received in answering the question. Students' scores were averaged for each intervention group (see Table 28). The Cohen *d* effect scores analysis yielded a moderate effect score favoring PM intervention when compared to VM intervention ($d = 0.74$) and a small effect when compared with CM intervention ($d = 0.07$). A moderate effect score of 0.50 favored the CM intervention compared to the VM interventions. A comparison of the CM and VM intervention yielded a moderate effect size of 0.50.

Next, intervention group averages for each lesson were compared using Cohen *d* effect size scores (see Table 29). Analyses yielded six large effect size differences. PM intervention was favored when compared with VM intervention for the concepts of: fraction names, developing groups, and identifying equivalent fractions by partitioning

Table 28

Summary of LCA Student Average Scores

	PM LCA scores			VM LCA scores			CM LCA scores		
Lessons	*N*	*M*	*SD*	*N*	*M*	*SD*	*n*	*M*	*SD*
1. Naming fractions	13	11.15	1.52	12	9.50	2.28	12	11.42	1.38
2. Fractions and wholes	15	9.53	2.53	10	8.50	2.36	11	10.91	1.92
3. Comparing	14	11.43	4.16	13	10.69	1.97	13	10.92	1.89
4. Equivalent groups	14	8.86	4.00	13	7.69	2.06	13	9.23	2.01
5. Equivalence in wholes	15	10.13	1.73	14	8.00	2.45	12	9.33	2.46
6. Equivalence by partitioning	15	11.47	1.06	14	10.43	1.45	14	11.93	0.27
7. Comparing fractions to ½	14	10.29	1.82	14	10.29	1.90	13	10.54	2.26
8. Set models	14	9.57	2.28	14	9.64	2.92	12	8.92	3.75
9. Equivalent set models	15	10.53	1.51	14	8.93	2.73	14	10.29	1.49
10. Simplify fractions	15	9.87	2.62	14	10.71	1.86	14	9.29	3.15
Total average	15	10.27	0.93	14	9.51	1.12	14	10.18	1.52

Note. $N = 43$.

Table 29

LCA Effect Size Comparisons by Intervention Groups

		Intervention comparisons		
Lesson	Concept	PM to VM	PM to CM	VM to CM
1	Naming fractions	0.85	-0.19	-1.02
2	Fractions and wholes	0.42	-0.61	-1.12
3	Comparing	0.23	0.16	-0.12
4	Equivalent groups	0.37	-0.12	-0.76
5	Equivalence in wholes	1.00	0.38	-0.54
6	Equivalence by partitioning	0.08	-0.59	-1.44
7	Comparing fractions to ½	0.00	-0.18	-0.18
8	Set models	-0.03	0.21	0.21
9	Equivalent set models	0.73	0.16	-0.62
10	Simplify fractions	-0.13	0.20	0.34
Total lessons		0.74	0.07	-0.50

region models. CM intervention was favored when compared with VM intervention for the concepts of: fraction names, identifying fractional amounts within region models, and identifying equivalent fractions by partitioning regions.

Finally, the intervention group averages were plotted on a line plot to determine if variations occurred among groups over the duration of the intervention (see Figure 48). The line plot indicates that the trajectories for the groups were similar, with the VM for tending to score slightly lower than the other two intervention groups for the first six lessons.

In summary, although the difference in averaged scores was not significant among intervention groups, there were moderate effect size differences favoring the PM and CM interventions when compared with VM intervention when all scores were totaled. Comparisons of individual lessons yielded six large effect size differences that indicate

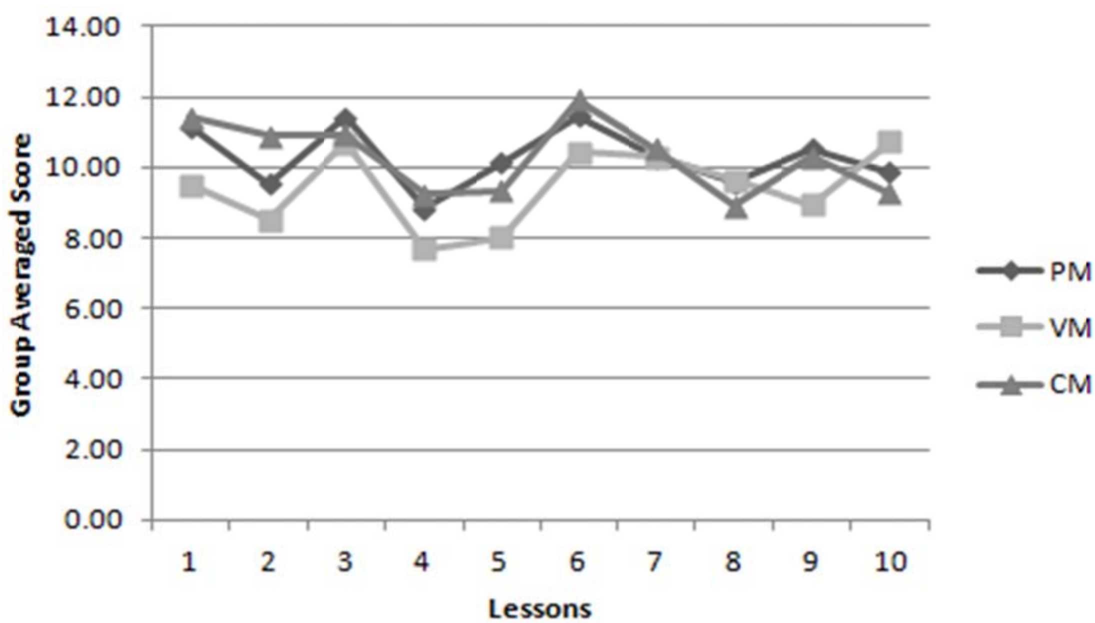


Figure 48. Line plot of LCA averaged scores.

that the students in the PM and CM interventions tended to score higher for specific lessons focusing on concepts of naming, grouping, identifying and partitioning. Over the duration of the intervention the LCA variations among the intervention groups tended to decrease.

Lesson Preassessment

The daily LPA was administered at the beginning of each lesson and consisted of two questions that duplicated questions students answered in the explore phase of the previous lesson. Responses were scored on a 1- to 4-point rubric that evaluated the amount of guidance students needed to correctly respond to the questions. Students' scores for each intervention group were averaged (see Table 30).

Cohen *d* effect size comparisons yielded a moderate effect score favoring the PM intervention when compared to the VM intervention (*d* = 0.63) and a small effect size

Table 30

Summary of Students' Averages on LPAs

	PM LPA scores			VM LPA scores			CM LPA scores		
Lesson	*N*	*M*	*SD*	*N*	*M*	*SD*	*N*	*M*	*SD*
1. Naming fractions	13	7.38	1.50	10	7.70	0.95	9	8.00	0.00
2. Fractions and wholes	15	6.93	1.39	12	6.83	1.41	13	6.54	1.85
3. Comparing	14	6.21	1.53	13	6.08	2.18	14	6.79	1.48
4. Developing equivalent groups	14	7.07	0.80	14	5.43	2.41	13	7.15	1.82
5. Equivalence in wholes	15	7.33	1.05	14	6.14	2.25	14	6.50	1.61
6. Equivalence by partitioning	15	7.86	0.53	14	7.71	1.07	14	8.00	0.00
7. Comparing fractions to ½	14	7.71	0.83	14	5.71	2.43	13	6.23	2.46
8. Set models	15	5.73	2.66	14	7.07	1.00	13	6.15	2.15
9. Equivalent set models	15	6.60	1.55	14	5.64	2.13	14	5.79	1.12
Average	15	6.98	0.63	14	6.48	0.88	14	6.79	0.78

Note. *N* = 43.

when compared to the CM intervention (*d* = 0.25). A moderate effect score of 0.38 favored the CM intervention when compared to the VM intervention.

Next Cohen *d* effect size comparisons of intervention group averages were calculated for each lesson concept (see Table 31). Four comparisons which yielded large effect size scores of 0.80 or higher were identified: (a) PM compared to VM intervention for the building of equivalent groups (*d* = 0.91), (b) CM compared to VM intervention for the building of equivalent groups (*d* = 0.81), (c) PM compared to VM intervention for comparing fractions to ½ (*d* = 1.10), and (d) PM compared to CM intervention for comparing fractions to ½ (*d* = 0.81).

A line plot of the LPA averages was developed to compare the trajectories of the three types of intervention (see Figure 49). The trajectories for all three groups were very similar, with differences being less than one and one half points.

Table 31

LPA Effect Size Comparisons by Intervention Groups

		Intervention comparisons		
Lesson	Concept	PM to VM	PM to CM	VM to CM
2	Naming fractions	-0.25	-0.58	-0.45
3	Fractions and wholes	0.07	0.24	0.18
4	Comparing	0.07	-0.39	-0.38
5	Developing equivalent groups	0.91[a]	-0.06	-0.81[a]
6	Equivalence in wholes	0.68	0.61	-0.18
7	Equivalence by partitioning	0.18	-0.37	-0.38
8	Comparing fractions to ½	1.10[a]	0.81[a]	-0.21
9	Set models	-0.67	-0.17	0.55
10	Equivalent set models	0.52	0.60	-0.09
Total lessons		0.79	0.24	-0.41

Note. $N = 43$.

[a] Large effect size

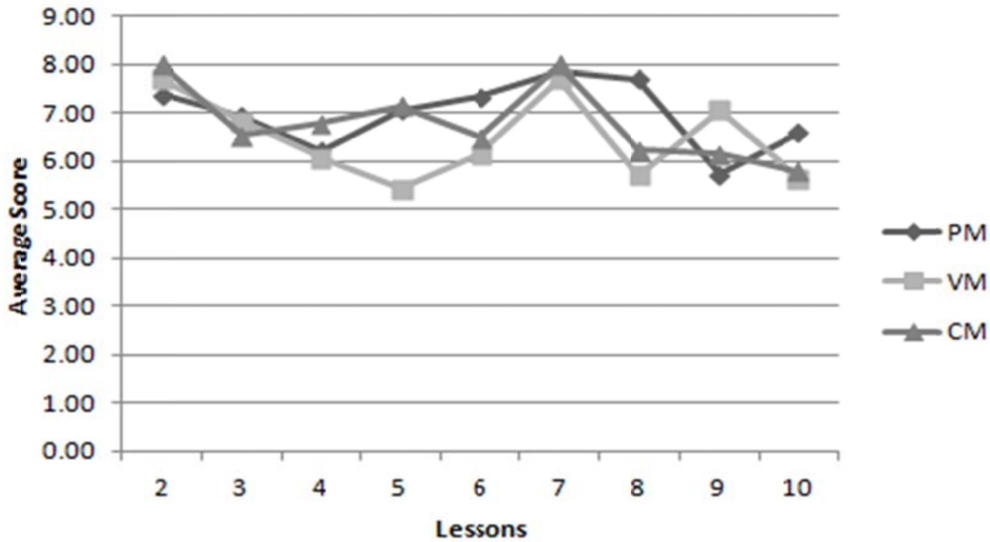


Figure 49. Line plot of LPA averaged scores.

Results of the comparison of intervention effects on the retention of lesson concepts indicated a moderately positive effect of using PM for intervention when compared with VM intervention and a smaller, but still moderate effect favoring the use of CM intervention when compared to VM intervention. The specific concepts which resulted in the largest effect differences were identifying equivalent fractions in fraction circles and comparing fractions to the benchmark of ½.

Problems Completed

To compare the amount of work completed by students, instructors recorded the total number of problems complete for each lesson. Two groups of problems were recorded and analyzed: the problems of the explore and practice phases of the lessons.

Problems completed in the explore phase. During the explore phase both the PM and the CM interventions used physical manipulatives while the VM group used virtual manipulatives. The number of problems completed in the explore phases of each lesson were recorded and an intervention group average was calculated (see Table 32). The average number of problems completed was: (a) 173.40 by the PM intervention group, (b) 162.41 by the VM intervention group, and (c) 157.58 by the CM intervention group. Effect size comparisons of the three intervention groups produced only small effect sizes.

The average number of problems completed was plotted on a line plot and examined. No variations related to the intervention groups were observed (see Figure 50).

Table 32

Average Number of Problems Completed During the Explore Phases

Intervention type	Number of explore problems completed in lessons										
	1	2	3	4	5	6	7	8	9	10	Total
PM	11.6	16.4	19.9	17.1	15.0	15.9	24.4	4.4	19.6	29.1	173.4
VM	8.4	15.7	24.2	10.9	13.8	19.1	20.6	3.0	17.3	29.3	162.4
CM	9.8	16.7	25.5	10.5	16.3	15.4	19.2	3.2	18.4	22.7	157.6

Note. $N = 43$.

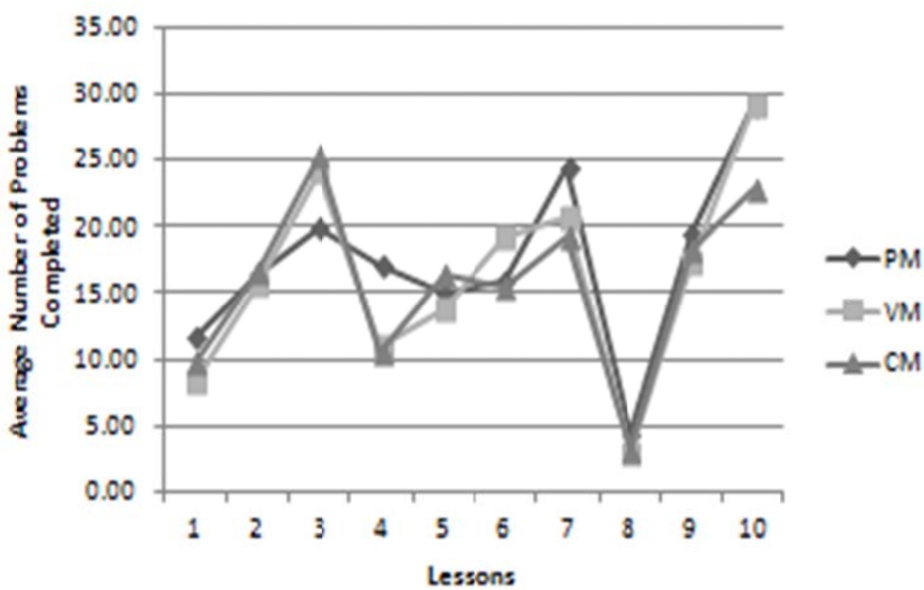


Figure 50. Line plot of the average number of explore problems completed.

Problems completed in the practice phase. The number of problems completed during the practice phase of each lesson was recorded and averages were calculated (see Table 33). During the practice phase of the lesson, both the VM group and the CM group used virtual manipulatives. The PM group used physical manipulatives. There was little variation in the averaged total number of problems completed by each intervention group. The average number of problems completed was 138.7 problems for the CM group, 131.97 for the PM group, and 127.9 problems for the VM group. Effect size comparisons of the three intervention groups produced only small effect sizes.

A line plot was developed to compare the number of problems solved by each intervention group (see Figure 51). The line plot suggests three variations in the average number of practice problems completed for each lesson. During the naming fractions practice (lessons 1and 2) the students using the NLVM Fractions – Naming applet completed more than twice as many problems as the students using the physical manipulatives.

During the practice phase of lessons 3, 4, and 5, the VM and the CM intervention groups used the Illuminations- Equivalent Fraction Applet and the line plot shows a slow

Table 33

Average Number of Problems Completed During the Practice Phases

Intervention type	Number of explore problems completed in lessons										
	1	2	3	4	5	6	7	8	9	10	Total
PM	9.38	9.3	8.0	6.5	14.5	26.5	26.9	7.2	6.9	16.9	132.0
VM	22.9	30.6	4.0	11.5	11.6	11.9	15.9	4.2	7.6	7.7	127.9
CM	22.8	23.0	8.1	13.7	18.2	14.5	17.0	4.9	5.6	10.8	138.7

Note. $N = 43$.

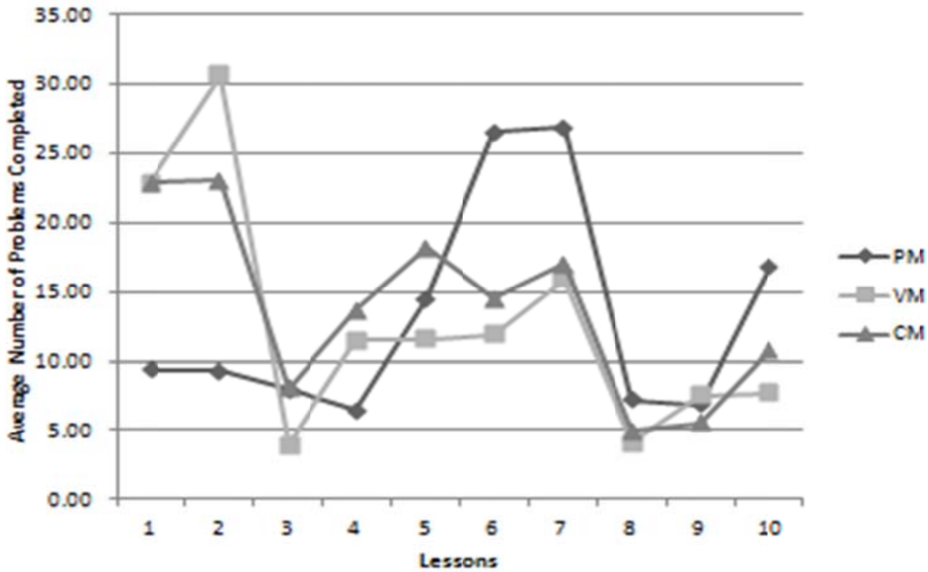


Figure 51. Line plot of the average number of practice problems completed.

steady increase in scores during these lessons. The PM group used pipe cleaners to partition fraction squares for this practice. At lesson 5, the line plot of the PM group shows a sharp rise in scores. This corresponds with log entries of the instructors indicating that at this point a number of the students using the fraction squares were no longer physically partitioning the squares, but were sometimes seen using their fingers to count sections of the squares as if they had partitioned them. The instructors suggested that many of the students appeared to have begun to internalize the partitioning process. In lessons six and seven, when the VM and CM students began using the NLVM Fraction-Equivalence applet, the instructors noted that many of the students became impatient with the processes of partitioning and the students began to calculate the answers without using the partitioning tool. For these two lessons, the line plots of the three intervention groups are similar in both the number of correct responses and in the trajectories of the lines.

The final variation occurred during the last three lessons. The virtual manipulative used during these lessons was the NLVM fractions-comparing. The applet guided students through the process of finding equivalent fractions with common denominators for the two fractions to be compared and then placement of the fractions on a number line. Students using physical manipulatives mirrored this process with the use of fraction squares and pipe cleaners. For lesson 8 and 9, the line plot shows a decrease in the number of problems completed by all three groups as they learned the new procedures. The instructors noted that during the practice portions of lessons 9 and 10, some of the PM students became frustrated with the manipulatives and began to find the common denominators by multiplying the original fractions by the number one in fraction form (e.g., $\frac{2}{2}, \frac{3}{3}, \frac{4}{4}$). By lesson 10, the instructors reported that a number of the students of the physical manipulative group were doing all of the problems without the use of the manipulatives. The line plot shows a rise in the number of problems completed by the PM group for lesson 10.

In summary, no statistically significant differences were found in concept achievement, concept retention or in the number of problems completed in the lessons. Effect sizes indicated moderate effects favoring PM and CM interventions. Six LCA and 4 LPA comparisons yielded large effect sizes favoring PM and CM interventions when compared with VM intervention for specific lesson concepts. Examination of line plots of the LCA and LPA results suggested that over the duration of the intervention lessons variations among the interventions tended to decrease. Line plots of the number of practice problems completed identified three variations which reflected the use of the different virtual manipulative applets.

Research Subquestion 1(d): Variations in Strategies

Research subquestion 1(d) was: What are the variations in the strategies developed and used by students? The Rational Number Project lessons provided strong guidance and structure for the explore phases of the lessons, as did also the tool guidance of the VM applets used in the practice phases. The instructors mirrored the applet guidance in their instruction of the PM intervention practice lessons. As a result of the strong structure and guidance, there were few observable variances in the strategies students used.

Students' answers to the open response questions of the post EFT and lesson artifacts were examined for variations in students' strategies. The post EFT open response items were coded to compare students' strategies used in solving a set model problem and their partitioning strategies for finding equivalent fractions. Another question was coded to compare differences in students' use of columns or rows in developing equivalent fraction representations. Only small variances were identified.

Identified variations were limited to three comparisons, the strategies used in building groups of equivalent fractions, strategies used in partitioning pizzas, and strategies used in modeling.

Strategies for Building Groups

Students' strategies for building groups of equivalent fractions during the daily practice phase were examined at the end of two lessons, the last day of the Illuminations -

Equivalent Fraction practice (lesson 5) and the last day of the NLVM Fraction - Equivalent practice (lesson 7). Responses were coded into three strategies: Strategy 1- doubling the original fraction twice (e.g., 2/3, 4/6, 8/12); Strategy 2 - multiplying the original fraction by 2/2 and then 3/3 (e.g., 2/3, 4/6, 6/9); and, Strategy 3 multiplying by fractions of one, other than 2/2, and 3/3 (e.g., 2/3, 10/15, 20/30). The results are summarized in Table 34.

In the first comparison, the PM students used Strategy 2 for 94.42% of the problems. In contrast the VM and the CM intervention students used Strategy 2 for only 55.88% and 48.12% of the problems, respectively. During the practice phase, the PM students had been guided to model each fraction using fraction squares and then to partition the model using pipe cleaners. In contrast, the VM and CM students using the computer applets partitioned the models by sliding over a bar which automatically partitioned the model into as many as 100 partitions. By the end of lesson 7, this trend had shifted. It was observed by the instructors, that in the seventh lesson, most of the PM group chose not to use the manipulatives and the percent of problems completed using

Table 34

Percent of Students Using Strategies for Building Groups of Equivalent Fractions

	Equivalent fractions (illuminations)				Fractions equivalent (NLVM)			
Intervention	*N* of problems	Doubles	Multiples	Other	*N* of problems	Doubles	Multiples	Other
PM	109	0.0	94.42	5.58	189	18.87	57.4	23.64
VM	104	3.67	57.88	38.45	93	8.10	79.37	12.54
CM	96	4.91	48.12	46.97	124	22.50	53.45	24.05

Note. $N = 43$.

Strategy 2 dropped to 57.49%. The CM and VM students had used the NLVM Fractions-Equivalent applet for practice in lessons 6 and 7. In this applet students were required to repeatedly click the partitioning button until the lines of the original and the new fractions merged together. It was observed by the instructors that many students appeared to become impatient with this method of partitioning, stopped using the virtual manipulative and began to mentally calculate the sets of fractions. For lesson 7, the percent of problems the PM and CM students solved using Strategy 3 decreased and the percent using Strategies 1 and 2 increased in comparison to lesson 5. For lesson 7 the percentages of the CM group were very similar to the percentages of the PM group while the VM group tended to use Strategy 2 more than the other two groups. These results suggest that at first, the use of the physical manipulatives limited the use of multiple strategies for building groups of equivalent fractions, but as practice continued the students began to use other strategies. The VM and CM groups initially used multiple strategies in building groups of equivalent fractions, but when students used a different applet there was a decrease in the variations of the types of strategies students used.

Strategies for Partitioning Pizza

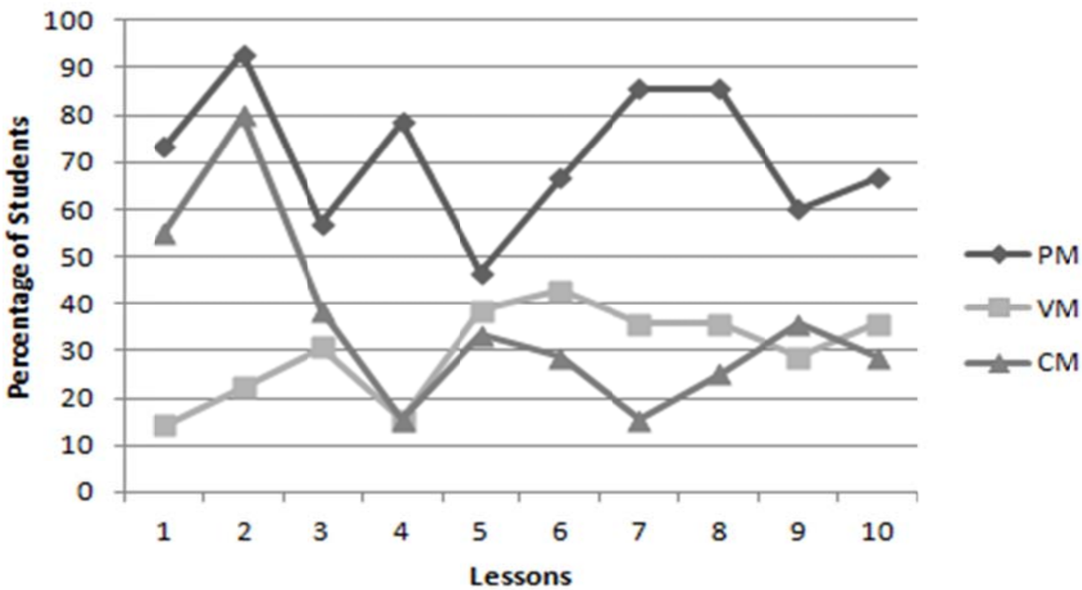
The data source for students' variations in strategies for partitioning pizza was DCA-Q2. The question asked students to divide a given number of pizzas evenly among a given number of friends. The difference in student strategies was the students' method for partitioning the pizzas. Some students divided each pizza by the number of friends while other students first distributed the possible number of the whole pizza to the friends and then partitioned the remaining pizzas. The strategy of first distributing the whole pizzas requires more steps and is a more complex strategy. A correct answer could be obtained using either method. Table 35 summarizes the use of the two strategies by treatment group and Figure 52 shows trend lines of the two strategies. For all three treatment groups, the percent of students developing the strategy of distributing the whole pizzas and partitioning the remaining pizzas gradually increased and the final percent of students using the strategy of partitioning remaining pizzas was similar. However, after lesson 2, there was a large drop in the percent of CM students who partitioned all pizzas and a large rise in the percent of students distributing the whole and partitioning the remaining pizzas. These results indicate that the CM intervention encouraged students to use the strategy of distributing the wholes and partitioning the pieces more than the PM and VM interventions.

Table 35

Percent of Students Using Pizza Partitioning Strategies for DCA-Q2

	Percent of students for each lesson									
Intervention/strategy	1	2	3	4	5	6	7	8	9	10
PM										
Every pizza	73.3	92.9	57.1	78.6	46.7	66.7	85.7	85.7	60.0	66.7
Remaining pizzas	20.0	7.1	28.6	21.4	40.0	33.3	14.3	14.3	40.0	33.3
VM										
Every pizza	14.3	22.2	30.8	15.4	38.6	42.9	35.7	35.7	28.6	35.7
Remaining pizzas	21.4	11.1	30.8	53.8	35.7	21.4	14.3	35.7	35.7	42.9
CM										
Every pizza	55.0	80.0	38.5	15.4	33.3	28.6	15.4	25.0	35.7	28.6
Remaining pizzas	0	0	46.2	69.2	58.3	57.1	46.2	58.3	35.7	50.0

Note. $N = 43$.



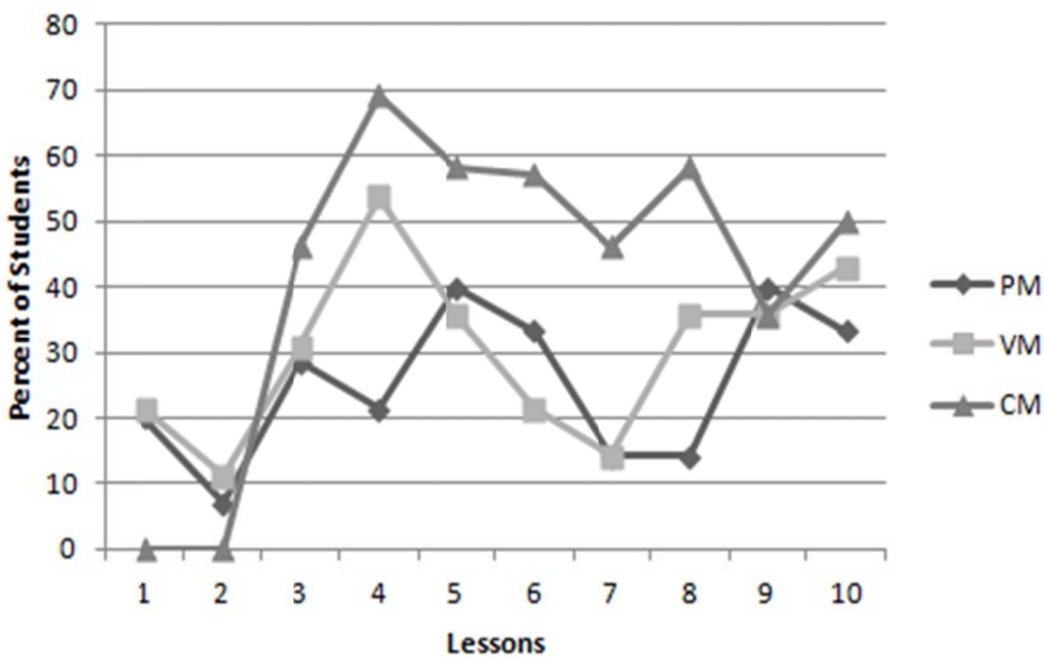


Figure 52. Trend lines of partitioning strategies for DCA-Q2. The first graph shows the percent of students who partitioned all pizzas. The second graph shows the percent of students who partitioned only the remaining pizzas.

Strategies for Partitioning

Variations were also identified in students' partitioning responses to an open response EFT question which asked students to model two fractions equivalent to 1/3. Responses were coded as to whether the students used only vertical partitions or if they used cross-sectioned partitions. Students using cross-sectioned representations used vertical lines to model 1/3 and then one or more horizontal lines to partition the 1/3 sections into equivalent fractions. It is suggested that the cross-sectioned representation is a more complex representation than the vertical lines only model. Cross-sectioning suggests that the student is using the model to develop equivalent fractions, whereas vertical only partitioning suggests that the student is first calculating the equivalent fraction and then drawing a model to match their answer. The percent of VM and PM intervention students using the cross-section representations was almost three times the percent of the CM intervention group using the strategy (see Table 36). This indicates that the PM and VM interventions tended to encourage the use of the cross-sectioned representation, while the CM intervention tended to encourage the use of vertical only partitioning.

Table 36

Variations in Methods Used to Partition Representations

	Percent of students in each intervention group using strategy		
Strategy	PM	VM	CM
Vertical partitions	40.0	33.3	78.6
Crossed partitions	60.0	66.6	21.4

Note. $N = 43$.

In summary, three variations in strategies were identified. For the comparison of grouping strategies, the variations among groups corresponded to differences in the use of manipulatives. In this comparison and in the comparison of partitioning pizzas, more of the CM students tended to use more complex and creative strategies. For the strategies used to partition a rectangle, the VM and PM groups both tended to use more complex strategies.

Research Subquestion 1(e): Variations in Representations

Research subquestion 1(e) was: Are there variations in the types of representations used by students and in the connections students make among representational models. Two sources were examined for variations in representations: the EFT and lesson artifacts. No variations within the lesson artifacts were identified. Two post EFT open response questions were coded to compare whether students explained their solutions using pictorial or symbolic representations. Again, the variances identified were too small to be meaningful.

The responses to questions on the pre and post EFT were examined to determine if the gain in the percent of correct responses varied according to whether questions used region, set, or symbolic-only representations. Each EFT test contained nine region, six set and five symbolic-only questions. Except for the values of the fractions, the wording and pictures on the pre and posttest questions were identical. The gain from pre to posttest in the percent of students responding correctly was calculated and averaged for each type of representation (see Table 37). For region representation questions, the percent of gain

Table 37

Average Gain for Representational Types of EFT Questions

		Gain of correct responses for each model		
Intervention type	*N*	Region	Set	Symbolic
PM	15	28.15	41.11	44.00
VM	14	30.16	20.24	54.29
CM	14	30.16	30.95	38.57

Note. $N = 43$.

was similar for all three intervention groups. For questions using set representations, the PM group had the highest gain, while for questions using symbolic-only representations the VM group had the greatest gains.

Because analyses of variations of individual questions for three of the clusters identified large differences related to the type of representation, the differences in gain were compared for the five equivalent clusters in Table 38. Eight cases, in which the gain in representation type between manipulative groups differed by more than 20% were identified. For the clusters of *identifying* and *solving* each type of representation had one manipulative intervention for which the gain was greater than 20% and the type of manipulative intervention with the highest gain was different for each representation. For *identifying*, the interventions with the greatest gains were: PM for the region, CM for the set, and VM for the symbolic only representations. For *solving*, the interventions with the greatest gains were: CM for the region, PM for the set, and VM for the symbolic only representations. These differences indicate that for these two clusters, there may be a relationship between the type of problems, the manipulative used and the gain in achievement.

Table 38

Cluster Gains by Types of Representations and Intervention

	Region percent gain			Set percent gain			Symbolic percent gain		
Variable	PM	VM	CM	PM	VM	CM	PM	VM	CM
Modeling	33.3	35.7	32.1	30.0	32.1	21.3	X	X	X
Identifying	30.0[a]	10.7	3.57	13.3	-7.14	35.7[a]	33.3	57.1[a]	21.4
Grouping	13.3	39.2[a]	21.4	X	X	X	40.0	46.4	32.1
Solving	53.3	42.8	78.6[a]	63.3[a]	28.6	39.3	60.0	78.6[a]	57.6
Simplifying	23.3	28.6	39.3	46.7[a]	7.14	28.6	46.7	42.8	50.0

[a] Differences greater than 20.0%, X- there were no question of this representation type.

In summary, although analyses were made to identify variations in representations in students' written responses of the EFT and lesson artifacts, none were identified. Analyses of the representations used in EFT questions and gains indicated that more students of the PM intervention groups tended to answer set model questions correctly than students in the other two groups, and more students in the VM intervention group tended to answer symbolic only questions correctly than students in the other two groups. Analyses of intervention type, equivalent fraction clusters, and the type of representation of the questions indicated that for the clusters of *identifying* and *solving* there may be relationships between students' gains and the type of manipulative used for each representation.

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CHAPTER V

DISCUSSION

This study focused on the use of physical and virtual manipulatives when used in the development of equivalent fraction understandings for students with mathematical learning difficulties during intervention. Effective intervention is a unique blending of student characteristics, intervention goals, mathematical content and the appropriate instructional environment. To design and teach effective intervention, teachers and curriculum designers need an understanding of the effective use of manipulative tools. As with the use of any tool, manipulatives (physical or virtual) are used most effectively when the user has an understanding of the affordances of the tool. A craftsman is able to use a variety of tools, sometimes selecting a specific tool for one job, sometimes using the tools interchangeably. Likewise, if a teacher or curriculum designer of intervention has an understanding of how the use of physical and virtual manipulatives affects student learning they are then able to make decisions that will maximize the effectiveness of how the tools are used during teaching and learning interactions with children. To make these decisions, teachers and designers need to know which manipulative affordances are most effective for instruction of each mathematics concept. They need an understanding of how virtual and physical manipulatives instruction differs and how best to use the manipulatives interchangeably. Because research investigating the use of manipulatives in intervention settings is limited, one goal of this study was to identify variations in student learning related to the types of manipulatives used during mathematics intervention instruction.

The purpose of this study was to identify variations in student learning related to the use of physical and virtual manipulatives when manipulatives were used in the intervention instruction of equivalent fraction concepts. This discussion of the results has five sections. The first section contains a summary and discussion of the identified variations in learning in relation to the study's research questions. The second section will describe three trends that emerged from the identification of variations. Section three contains implications of the findings for intervention instruction. Sections four and five contain the limitations of the study and ideas for future research, respectively.

Identified Variations in Learning in Relation to Research Questions

One research question with five subquestions guided this study. The main research question was: What variations occur in the learning trajectories of students with mathematical learning difficulties that are unique to the use of different instructional manipulatives for intervention (virtual, physical or a combination of virtual and physical manipulatives) in the learning of equivalent fraction concepts? The variations identified in relation to each of the five subquestions are summarized in the following section.

Subquestion 1(a). Variations in Achievement

Subquestion1(a) was: What are the variations of achievement, mastery, retention, and resolutions of errors in students' development of equivalent fraction concepts and skills? Data from the EFT, DCA, and error analyses of lesson artifacts were analyzed and synthesized to identify variations at the total test, concept clusters and basic fraction concepts levels. These findings are summarized in Table 39 highlighting the intervention

Table 39

Summary of Achievement Findings

Concept	Source	Favored	Magnitude	Growth/resolution
Total Test	EFT	PM	small comparisons	
	DCA	CM	moderate comparisons	CM greatest growth rate
Modeling	EFT	CM	small comparisons	
	DCA-Q6	CM	moderate to small	VM and CM greatest growth rate
Identifying	EFT	PM	moderate comparisons	
Grouping	EFT	VM	moderate comparison	
	DCA-Q7	PM	moderate comparison	PM greatest growth rate
Solving	EFT	PM	large to moderate	
Simplifying	EFT	CM	moderate comparisons	
	DCA-Q8	PM	moderate comparisons	PM and CM greatest growth rate
	Misc 4	PM		PM greatest slope of resolution
Naming				
Labeling	Misc 3	PM		PM greatest slope of resolution
Models	DCA-Q1	CM	large comparisons	CM greatest growth rate
Partitioning	DCA-Q5	VM	large to small	VM greatest growth rate
Evaluating				
Comparing	DCA-Q2		not effective	
	Misc 1	CM		CM greatest slope of resolution
Ordering	DCA-Q3	PM		
Developing	DCA-Q4		not effective	
Equivalence				
Meaning	Misc 5	VM		VM greatest slope of resolution
Area	EFT-17	PM CM	20% difference in gain 20% difference in incorrect responses	
Part/Whole	Misc 6	PM		PM greatest slope of resolution
Multiplicative thinking	Misc 2	VM		VM greatest slope of resolution

Note. Misc = Misconception.

strategy (PM, VM, or CM in the column labeled “Favored”) with the greatest growth or resolution for each mathematics concept.

Total test. Total test findings on both the EFT and DCA suggest that the PM, VM, and CM interventions were all effective in increasing students’ fractions achievement scores. The EFT assessed equivalent fraction understanding and comparison of EFT gains yielded small effect sizes favoring PM intervention when compared to VM and CM interventions. This indicates that differences in the effectiveness of instruction among the three interventions were minimal. In contrast, the DCA assessed both general fraction understanding and equivalent fraction skills and comparison of DCA gains yielded moderate effect sizes favoring CM intervention when compared to PM and VM interventions. However, analysis of the DCA scatter plots suggested that the VM students' rate of growth was greater than those of the CM and PM students. Analysis of the DCA trend line indicated that the VM group's DCA total test scores, in contrast to the steady growth of the CM and PM students, initially decreased, remained low for the first five lessons and then steadily increased for the remaining lessons. This indicates that, although the gains were similar for the three interventions, the pattern of growth varied. Three possible explanations for the variances observed in the VM group were considered: (a) Initially VM students' unique interactions with the virtual manipulative applets limited their focus on learning the mathematical concepts (e.g., demands of learning to manipulate the VMs or students' focus on VM features limiting their focus on mathematical concepts); (b) Unique affordances of the VM applets, such as simultaneous linking of symbolic to pictorial representations, required multiple experiences before the effects of the affordances could be observed in measurable student growth; and (c) The initial slower growth of the VM group could be due to differences in the ability of the students. Although, the three intervention groups pretest scores were not statistically different, the averaged VM students' scores on the pretests of both the EFT and the DCA were numerically lower than the averaged scores of the PM and CM groups. If the initial variation was due to differences in ability, the results would indicate that the VM intervention was successful in decreasing the influence of the differences in abilities. To obtain a more complete picture of differences among manipulative intervention types, results of the EFT and DCA were analyzed at the cluster and concept levels.

Five concept clusters. A review of the literature identified five sub concept clusters of equivalent fraction understanding: *modeling, identifying, grouping, solving* and *simplifying*. EFT, DCA and error analyses data were synthesized to identify variations at the cluster level. Because the EFT was previously piloted and analyzed for validity and because it assessed a wider range of questions, for this discussion, favoring of manipulatives for the concepts with mixed results will give preference to the EFT results. Thus, analyses of data indicated advantages of PM intervention for the two clusters of *identifying* and *solving*, CM intervention for the two clusters of *modeling* and *simplifying*, and VM intervention for the *grouping* cluster. The reason for this variance may be attributed to specific interactions between manipulative affordances and the development of concepts necessary for cluster mastery. For example, it may be that the processes of physically partitioning the physical fraction squares focused students' attention on partitioning fractions into two and three parts, developing their ability to identify equivalent fractions developed by doubling or tripling the numerator and denominators. Potential interactions are described in the second section of this chapter.

Basic fraction concepts. From a review of the literature and the qualitative analysis of this study, three basic fraction concepts with ten sub concepts were identified.

The three basic concepts were: *naming fractions, evaluating fraction value*, and *equivalence thinking*.

The basic fraction concept of *naming fractions* had three sub concepts: *labeling fractions*, *partitioning* and *building models*. For the sub concept of *labeling* fractions, scatter plot analysis indicated that PM students had the greatest rate of reduction of labeling errors. For the sub concept of *partitioning,* comparisons of DCA gains favored VM intervention. For the sub concept of *building models*, comparison of DCA gains favored CM intervention. Thus, for the basic skill of *naming fractions,* analyses suggested that there were advantages for the use of a different manipulative for each of the three sub concepts. These results suggest that the use of physical manipulatives tends to limit the errors students make in labeling fractions; the use of virtual manipulatives tends to encourage their development of partitioning skills; and, the use of both manipulatives best supports students in developing modeling concepts. The selection of the manipulative should match the focus of the lesson; labeling fractions, partitioning or modeling.

The basic fraction concept of *evaluating fraction values* had three sub concepts: *comparing, ordering and developing.* The results of the DCA gains suggest that intervention was effective only for the sub concept of *ordering* fractions on number lines. The results favored PM intervention, but the rate of growth for all three interventions was low. Because the instructional methods for evaluating fractions were not effective, the current results may not be accurate determiners of manipulative effectiveness for the *evaluating fraction value* concepts.

Although the intervention instruction was not effective for the development of the *comparing* fraction concept, scatter plot analyses indicated that the intervention was effective for the resolution of students' misconception of using whole number thinking when comparing fractions. CM and PM students had the greater rates of error reduction. The results suggest that while all three interventions effectively resolved inappropriate whole number thinking, the students developed other incorrect methods for evaluating fractions which were not detected or measured in the analysis. This limits the effectiveness of comparing effects of the three types of intervention.

For the basic fraction concept of *equivalence thinking*, four sub concepts were identified: *meaning of equivalence, comparison of area, conservation of part-whole relationships* and *multiplicative thinking.* Except for *comparison of area*, none of the testing instruments specifically measured *equivalence thinking*, therefore data for this concept came mainly from error analysis. Analysis indicated the PM group had the greatest rate of error resolution for the conservation of *part–whole relationship* errors and the gain of the PM group was more than the gains of the VM and CM groups for an equivalent fraction test question assessing students' *comparison of area* ability. The VM group had the greatest rate of error resolution for *meaning of equivalence* and for the resolution of the error of using additive instead of *multiplicative thinking*. These results indicate that there were advantages for using PM intervention for the development of the basic understanding that two equivalent fractions name the same amount of area. It may be that students benefit from the ability to physically manipulate the fraction objects when developing the basic understandings of equivalence. For the development of the more abstract concept of multiplicative thinking and for the resolution of *equivalency* misconceptions, results indicate advantages to the VM intervention. A possible explanation is that this reflects the virtual manipulative affordances of simultaneous linking of pictorial to multiple symbolic representations which supported students in their development of internal visualizations. It could also be that the ease with which students using the virtual manipulative applets in this study could produce a variety of equivalent fractions for the same area may have influenced their ability to apply the concepts of equivalency to a larger variety of situations. When using physical manipulatives, the partitioning of fractions is typically limited to equivalent fractions that are doubled or tripled the original fraction, whereas those using virtual manipulatives tended to develop a broader range of equivalent fractions. Although it is possible to develop doubled and tripled equivalent fractions using additive thinking (e.g., For two fractions equivalent to $\frac{1}{2}$, the numerators equal 1+1 and 2+1, and the denominators equal 2+2 and 4+2), multiplicative thinking is needed to develop groups of equivalent fractions that do not follow the double/triple pattern.

In summary, although total test EFT analyses, favored PM intervention and DCA analyses favored CM intervention, at the cluster and basic fraction concept levels, thirteen variations were identified. Figure 53 illustrates these variations using the iceberg model. These variations suggest that decisions of which manipulative to use when providing intervention instruction varies for each specific concept. Although there are curriculum, decisions about which manipulative to use for maximum effectiveness tended to vary in this group of students for each specific concept and sub concept. The demands

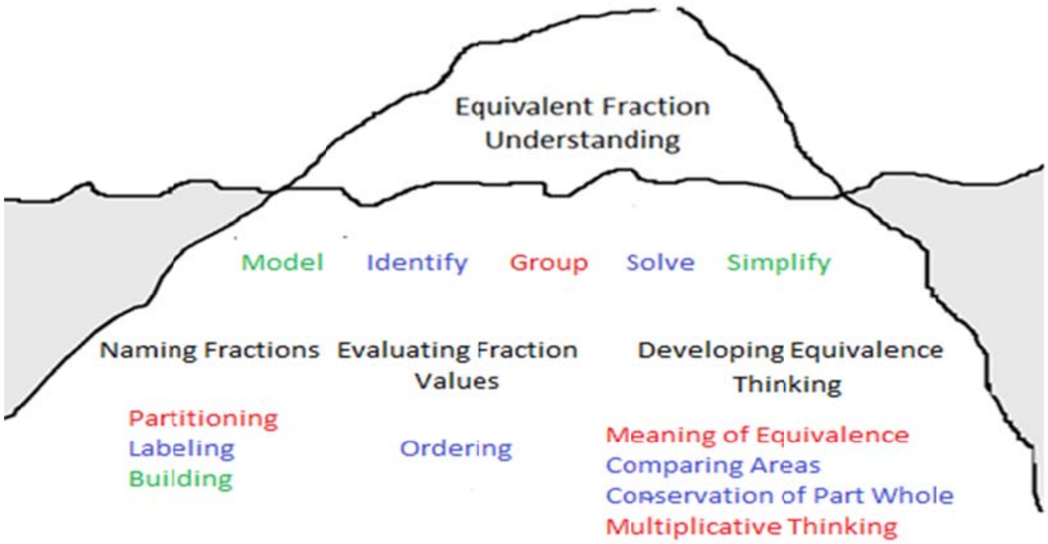

Figure 53. Favored manipulative for iceberg model concepts. Colors represent the manipulative favored: PM is blue, favoring identify, solve, labeling, ordering, comparing areas, and conservation of part whole; VM is red, favoring group, partitioning, meaning of equivalence and multiplicative thinking; and CM is green, favoring model, simplify, and building fractions.

of the mathematical topic, student characteristics, and the unique affordances of the manipulatives are all factors contributing to determining the effectiveness of each manipulative. Research can contribute to the decision making process by identifying trends of effective manipulative use and which manipulative tends to produce which results in specific settings. This study identified thirteen such trends with this group of students.

Subquestion 1(b): Variations in Learning Trajectories

Subquestion 1(b) was: What are the variations in learning trajectories showing changes in student achievement over time? Through analysis of the learning trajectories, five major variations in the rate and growth of achievement across the duration of the intervention lessons were identified: (a) During the first five intervention lessons, the VM groups tended to score lower than the PM and CM intervention groups; (b) During the final five intervention lessons, the differences among intervention groups tended to decrease; (c) More students of the PM and CM interventions mastered the DCA concepts than did VM students; (d) All student errors, except for Misconception 6 (Incorrect Equivalent Sentences) tended to be resolved by most of the students by the final sessions; and (e) The VM groups tended to have the highest rates of resolution, but they also tended to make more errors. A summary of the results for this question suggests that, for this group of students, the VM group trajectory of growth differed from that of the other two groups. But by the final lesson, VM students' scores and their resolution of errors tended to be similar to those of the other groups. As explained earlier, three possible explanations for the VM group's initial lower growth rates are: the types of student interactions with the virtual manipulative; the need for multiple experiences; and, differences in the abilities of students in each group. What is important for the development of intervention instruction is the pattern that the effectiveness of the VM intervention appeared to increase over time. This trend was reported by findings of Moyer-Packenham and colleagues' (2012) meta-analysis, that studies of longer duration tended to have higher effect size scores than those of shorter duration. What is not known from the present study is, if the intervention had continued, would further growth be similar to that of the other two interventions, or would VM students' growth continue to increase going beyond that of the other two groups?

Subquestion 1(c). Lesson Variations

Subquestion 1(c) was: What are the variations in patterns of daily lesson achievement, retention, and work completion? The variations identified are summarized in Table 40. Analyses for subquestion 1(c) focused on the identification of variations within the lessons by examining lesson concept retention, lesson concept understanding and the amount of work completed within lessons.

LPA assessed concept retention and LCA assessed concept development. Comparisons of the LPA and LCA scores and line plots indicated that although the scores of VM groups tended to be slightly lower, the trajectories of all three interventions were similar. The comparisons of the number of explore problems completed by the intervention groups also yielded trajectories with little variation among groups. These comparisons suggested that the concepts learned and retained and the amount of work completed were basically the same for all three intervention groups. Although the

Table 40

Summary of Processes of Learning Variations

Topic	Source	Favored	Finding
lesson concepts	LCA	PM and CM	consistently higher than VM
lesson retention	LPA	PM and CM	Consistently higher than VM
explore problems	Lesson artifacts	None	Small effect size
practice problems	Lesson artifacts	None	Small effect size
Strategies			
Grouping	Lesson artifacts	VM and CM	More complex ,greater variety
Fair shares	DCA	CM	More complex
Partitions	EFT	CM	More complex
Representations	EFT	VM	Symbolic only
	EFT	PM	Set models
	Misconception 7	VM	Less set model errors

analysis did not identify specific variations, the findings do support the premise that the instructional processes and students' opportunity to learn was basically the same for all three groups. In contrast, variations were identified in the number of practice problems completed. Three variations were identified, each which reflected the use of specific manipulative affordances. Two of these variations will be further discussed in a subsequent section of the discussion. The results of this question show that during the concept building (the explore phase of the lessons in this study) the amount of work students completed and students' understanding and retention of the specific concepts taught in these lessons were not dependent upon the type of manipulative intervention. However, the number of practice problems completed did vary according to concept and the manipulative objects or applets being used. This suggests that when planning practice activities, each activity should be examined to determine the most effective type of intervention.

Subquestion 1(d): Variations in Strategies

Subquestion 1(d) was: What are the variations in the strategies developed and used by students? Although the structure of the lessons and activities of the study were of a nature that did not encourage many variations in students' strategies, three variations were identified. In two incidences, partitioning pizzas and building equivalent groups, the CM groups tended to use a greater variety and more complex strategies than those used by the PM and VM groups. In the third variation, partitioning, the PM and VM groups used the more complex strategy. The results on strategy variations seem to indicate that CM intervention tended to encourage a greater variety of strategies and more complex thinking. It may be that the process of switching from one manipulative to another manipulative encouraged CM intervention students to observe and compare mathematical processes and understandings in a broader and more complex manner than those students exposed to only one type of manipulative. However, there are some situations in which the use of only one representation appears to lead students to the development of more complex strategies (e.g., partitioning, in this study). This suggests that selecting which manipulative to use requires an understanding of the four factors of the intervention process: student characteristics, intervention goals, mathematical content, and the affordances of the manipulatives.

Subquestion 1(e)

Subquestion1(e) was: What are the variations in students' use of representations? An analysis of the representations used in questions on the EFT indentified variations related to manipulative types. For questions using set model representations, the PM groups had the greatest gains. For questions using symbolic-only representations, the VM groups had the greatest gains. Gains for questions using region models were similar for all three groups.

Another identified variation in representations was the tendency of the PM and CM groups, during the three days of set model instruction, to make almost five times more set model errors each day than were made by the VM groups. Yet, when set model questions of the EFT were compared for intervention differences, the VM group consistently scored lower than the PM and CM groups. The VM group made twice the number of set model errors on the EFT as did the CM and PM groups. This suggests that the virtual manipulative applet constrained the making of errors, but it appears this limited learning. The increased number of errors made by the CM and PM groups may have encouraged more reflection about their understandings.

In summary, although lesson processes were similar for all three intervention groups, variations related to manipulative use were identified for 13 of the 15 clusters and sub concepts. Intervention was ineffective for the other two sub concepts. Analyses of growth trajectories suggested that variations in the scores of achievement, among the intervention groups, tended to decrease over time. Variations in the number of practice problems completed appeared to be related to features of the manipulatives being used. Variations related to the type of manipulatives and the complexities of three types of student strategies were identified and an analysis of EFT gains in relation to type of representations indicated that PM groups had the greatest gains on questions using set models while VM had the greatest gains on questions using symbols only. The importance of these results for the instructional intervention of equivalent fractions, is that the three types of manipulatives are effective instructional tools. Yet there are variations among the intervention types related to students' learning of specific concepts, number of practice problems completed, and the strategies and representations used by the students. Understandings of these variations can guide curriculum designers and implementers in their selection of manipulatives.

Trends

Once variations have been identified, the next step is to develop a deeper understanding of the variations. Although the main focus of this study was to identify variations, an examination of the four factors of the intervention setting (student characteristics, intervention goals, mathematical domains and environmental factors) in relation to trends in the variations made it possible to identify some connections among variations and to suggest some possible explanations for the variations. In this section the major trends for each of the intervention groups will be discussed.

Physical Manipulative Intervention

Figure 54 summarizes variations favoring use of the PM intervention. The seven concepts can be divided into two groups, those focusing on equivalency of two fractions and general fraction knowledge. Although a search of results did not reveal a plausible explanation for favoring of PM intervention for *ordering* or *labeling*, suggestions were identified for the other five concepts. These variations suggest that students of the PM groups tended to score higher than the VM and CM students when questions involved the ability to evaluate or develop the equivalency of two fractions. These were questions in

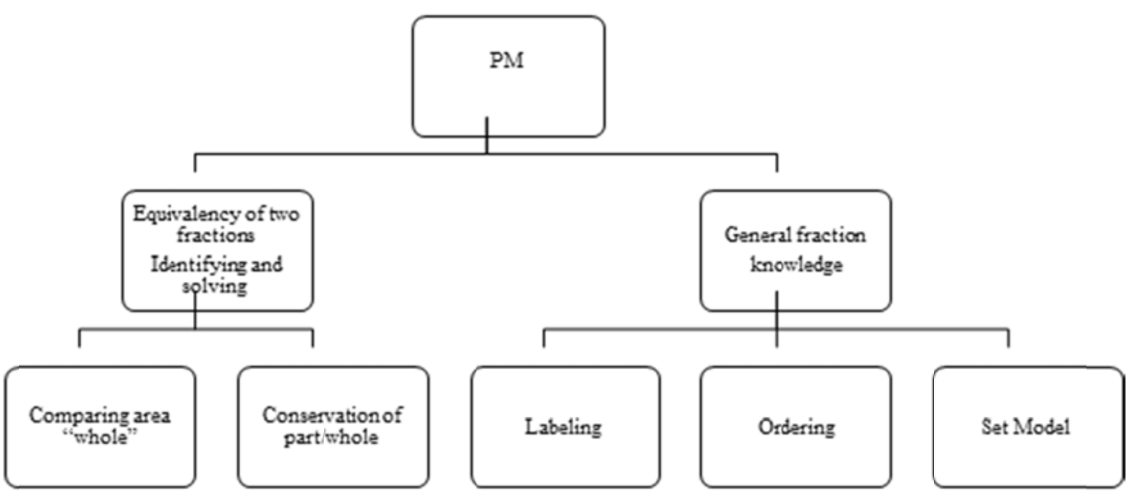

Figure 54. Concepts favoring PM intervention.

which students could use a fraction equal to one (e.g., $\frac{2}{2}$) to determine if two fractions were equivalent (*identifying*) or to complete an equivalent fraction sentence (*solving*). These concepts are built upon the general fraction knowledge concepts and skills of *comparing area* of two fractions *conserving the part whole relationship*. Three possible explanations for these findings are: tangibility, familiarity, or interactive affordances.

Tangibility. Several authors have suggested that students should use physical manipulatives when learning new mathematics concepts (Hunt, Nipper, & Nash, 2011; Swan & Marshall, 2010; Takahashi, 2002). This is built upon the theory that when learning new concepts, students learn best when they can tactilely explore the concrete objects (McNeil & Jarvin, 2007). The students' early experiences manipulating the physical objects becomes the basis for later conceptual learning (Skemp, 1987). Theories of embodiment suggest that body movement and sensory input plays an important role in the development of mathematical ideas (Lakoff & Nunez, 2000). Martin and Schwartz's (2005) theory of physically distributed learning suggests that students' ideas are developed and challenged as they physically interact with the manipulatives. *Identifying* and *solving* are the basic concepts of equivalent fraction understanding. It may be that in forming these basic ideas there were advantages in the students manipulating tangible three dimensional objects as compared to the virtual objects.

Familiarity. Students develop at an early age an understanding of how physical objects can be manipulated, making the offloading of memory and the processes of cognition easier (Manches, O'Malley, & Benford, 2010). The use of manipulatives is most effective when the learner is thinking, not so much about the tool, as about the mathematical concept (Boulton-Lewis, 1998). If a manipulative is too interesting, it becomes less likely that the students will be able to think of the manipulative as a representation of something else (Uttal, Scudder, DeLoache, 1997). Both fraction circles and fraction squares are simple objects and students quickly became familiar with them.

In contrast, the virtual tools used in the study were new to the students and initially the students had to focus, not only on the mathematical concepts, but also on learning to manipulate the virtual objects. Several authors have expressed concern that the additional load of computer manipulation may initially limit the learning of mathematical concepts (Baturo et al., 2003; Haistings, 2009; Highfield & Mulligan, 2007; Izydorczak, 2003; Takahashi, 2002). Cognitive load theory suggests that a person's working memory is limited to five to nine items at a time and that once the limit is reached, the person is limited in their ability to retain new information (Clark et al., 2006). Thus the novelty of the virtual manipulatives may have initially limited the ability of the students to retain knowledge and could in part explain why the VM group tended, for the first five lessons, to have lower rates of growth than the PM group.

Interactive affordances. Although both the tangibility and familiarity may in part explain why the use of the PM intervention was favored for the basic concepts of equivalency of two fractions, additional findings of this study suggest that a third effect, the effect of the type of interactions students had with the manipulatives, may have also had a large influence on student learning. Two examples of the effect of students interactions with the manipulatives will be given.

During the practice phases of lessons 3 through 10, PM students used pipe cleaners to partition fraction squares. As described in the results section, these students initially used the strategy of doubling and tripling fractions to develop equivalent fractions 94.4 % of the time. By lesson 6 many of the PM students were no longer using the manipulatives, suggesting that they had internalized the process of doubling and tripling. In contrast, the VM and CM students tended to initially use several different strategies and tended to continue to use the manipulatives tools for a longer duration. The virtual applet allowed the students to perform different types of interactions and to use different strategies whereas; the physical fraction squares limited students to only one strategy. By focusing the students' attention to the use of the doubling and tripling strategy, the students became more proficient in its use. This is the basic concept strategy used in the concepts of *identifying, solving, comparison of area* and *conservation of the part/whole relationship.*

The type of interactions students had with the different manipulatives may explain the advantage of physical manipulatives for teaching set models. When solving set model problems, PM and CM students using physical tokens tended to make five times more errors than VM students using virtual manipulatives. However, the PM and CM groups scored higher on EFT posttest questions which used set model representations. The errors made during the intervention were mistakes made as students moved the tokens to set up each of the set models problems. The students using virtual and physical tokens used the same procedure, except that the VM students cleared their screens after each problem and started each problem with new pieces. In contrast the PM students rearranged the tokens from the previous problem and in the process of regrouping the tokens they tended to make more errors. It appears that through correcting their errors they developed a deeper understanding of set models. Martin and Schwartz (2005) reported similar findings, reporting that children performed better on posttests, when they physically rearranged the objects to find practice solutions than when the objects had been prearranged for them. The authors suggested that physically moving the pieces helped the children to let go of their previously held whole number understanding.

These two examples suggest that the interaction of students with the affordances and constraints of each tools' features can create, within the same type of activity, very different learning processes and that the variations can affect achievement. Olive and Labato's (2008) results summarizing five projects involving the use of technology in teaching fraction concepts also described the importance of students' interactions with the features of applets.

> The nature of what students learn about rational numbers appears to be related to the match between the affordances and constraints of the technology and the mental operations involved in constructing rational numbers; if such links are missing then a greater demand is placed on the teacher and the non-technology activities. (pp. 30-31)

To effectively blend the use of physical and virtual manipulatives in intervention instruction, research describing interactions students have with the affordances and constraints of the manipulatives is needed.

Virtual Manipulative Intervention

Figure 55 summarizes the concepts favoring VM intervention. A possible link between the four concepts of *equivalent grouping*, *meaning of equivalence, symbolic representations*, and *multiplicative thinking* was identified, however, no suggestions were

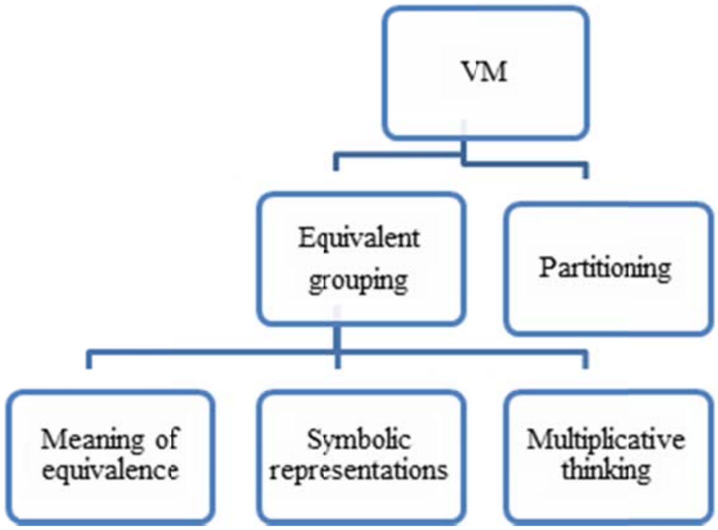

Figure 55. Concepts favoring VM intervention.

identified for the favoring VM intervention for *partitioning*. When posttest EFT questions were analyzed by representation type, the VM group scored higher gains for the symbolic only questions. Since the symbolic only questions are typically considered to be more abstract and, therefore, require greater understanding and ability, and because the VM group tended to score slightly lower on the EFT questions, the higher scoring of the VM group on symbolic only questions was unexpected. In the analysis of the EFT, the CM groups, who also used virtual manipulatives during the practice phase of the lessons, scored lower than the PM group. This indicates that the factor contributing to the variation most likely occurred during the explore phase of the VM lessons. During the explore phase of the first seven lessons, students used fraction circles. A main difference between the virtual and the physical fraction circles was the simultaneous linking of the symbolic and pictorial representations in the virtual fraction circles. It has been suggested that one of the challenges for students with mathematical difficulties is their ability to connect abstract symbols of mathematics to representations (Baroody, 1989; Gersten et al., 2009). The continuous simultaneous linking of the virtual objects to the symbolic representations may explain why the VM groups tended to have the greater gains in solving symbolic only questions. The affordance of simultaneous linking between pictorial and symbolic representations has been similarly identified in other research studies (e.g., Baturo et al., 2003; Clements et al., 2001; Suh & Moyer-Packenham, 2008; Takahashi, 2002).

Simultaneous linking may also explain the finding of analyses favoring the VM intervention for instruction of *grouping*. Since PM intervention was favored for concepts of equivalence of two fractions, it could be expected that PM intervention would have also been found to be favored for finding sets of three or more equivalent fractions. Yet, analysis of the EFT data indicated that the VM group had the greatest increase in gains on all four of the grouping questions. Several researchers have suggested that *grouping* is a difficult skill for students because it requires multiplicative thinking (Kamii & Clark, 1995; Kent et al., 2002; Moss, 2005). Although the growth of multiplicative thinking was not measured directly, error analysis did indicate that VM students experienced the greatest reduction of the use of additive thinking in situations requiring multiplicative thinking. The higher *grouping* scores and greater reduction of additive thinking errors suggest that use of the virtual manipulatives encouraged multiplicative thinking with this group of students. It is hypothesized that there is a link between simultaneous linking, symbolic understanding, multiplicative thinking and grouping: (a) Simultaneous linking of symbols deepens students' understanding of symbols and develops increased flexibility in their use of symbols; (b) A more flexible use of symbols strengthens students' ability to develop the more complex thinking of multiplicative thinking; and (c) The use of multiplicative thinking increases students' ability to develop groups of equivalent fractions (see Figure 56). Further research is needed to determine if the use of virtual manipulatives encourages the development of the multiplicative thinking needed for equivalent fraction understanding.

Combined Manipulatives Intervention

Figure 57 summarizes the concepts favoring CM intervention. CM intervention was favored for the equivalent fraction clusters of *modeling* and *simplifying* and the general fraction knowledge cluster of *modeling*.

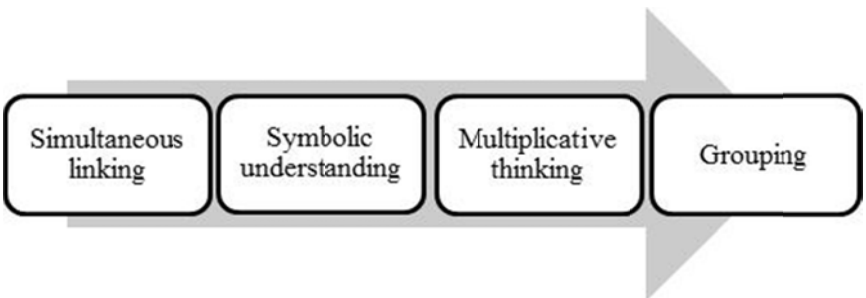

Figure 56. Grouping chain of development.

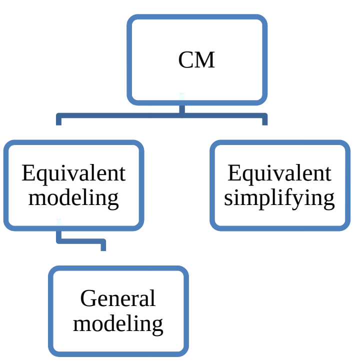

Figure 57. Concepts favoring CM intervention.

Modeling questions differed from other questions of the EFT in that students were asked to identify and build models representing and unlike other questions of the test, none could be solved using only procedural skills. One possible explanation of why CM intervention was favored for *modeling* instruction is that the CM group, through exposure to multiple representations, may have developed a deeper conceptual understanding. During the practice phases of the lessons the CM and VM intervention groups found equivalent fractions using models in three different applets. During the explore phases the PM and CM groups used tokens and fraction circles. Each manipulative had distinct features. As students transferred learning from one model to another, the unique features of the new models both challenged students' understandings and supported students in the development of new concepts (Kiczek, Maher, & Speiser, 2001). Friedlander and Taback (2001) found that presenting problems using different models increased students' flexibility. Other studies report that as students interact with manipulatives, they experience visual proof of their solutions and build understanding of mathematical concepts (Durmus & Karakirik, 2006; Moyer et al., 2002, 2008). For this group of students, there were five findings which suggest that the CM group may have developed a deeper conceptual understanding of fractions than the other two groups.

1. Effect size gain scores of the DCA favored CM intervention with moderate effect sizes when compared to VM and PM intervention.

2. More CM students mastered the skills for four of the six DCA skills than did students of the other two groups.

3. Analyses of the misconception and error line plots indicated that the CM

groups had the least number of errors and the most complete resolution of errors for five of the seven misconceptions.

4. Analysis of variations in strategies indicated that for two of the three variations, the CM groups used the more complex strategies.

5. Of the five equivalent fraction clusters, *simplifying* requires the most complex understanding. Simplifying involves not only an understanding of partitioning to develop equivalency, but also the ability to reverse partitioning and an understanding of unit fractions. Results of the EFT simplifying cluster favored CM intervention.

These results suggest that exposure to multiple representations may have encouraged students of the CM group to develop a deeper conceptual understanding, but further research is needed to determine if these variations were specific only to this group of students or if the trends will generalize to other groups of students.

Implications

One goal of research is to inform practice. The use of the iceberg model to synthesize results of this study aggregated the findings into components which could be directly applied to classroom intervention. Trends indicated advantages for the use of a specific type of manipulative intervention. If further research shows that these trends generalize to other groups, this information can be used in planning curriculum.

Findings suggest there may be advantages to using symbolic linking throughout the intervention process for the development of multiplicative thinking. The virtual fraction circles, all of the virtual applets used during the practice phases of these lessons, and other web based fraction applets have the affordance of simultaneous linking.

In the study, instances were identified when the use of a manipulative limited errors, but also limited the development of learning. In these instances the question for planning intervention instruction becomes how to balance the affordances of the manipulatives with their limitations. For example, when teaching set models, one method could be to start intervention instruction using virtual manipulative to limit errors while instruction focuses on the development of basic concepts. Then as students develop the concepts, physical manipulatives could be used for practicing the skills. Another method would be to start instruction using physical manipulatives while encouraging reflection and group discussion about any errors made. Then students could use the virtual manipulatives to practice the developed procedures. The most effective balance of manipulative use depends upon the students and the goals of the intervention.

Limitations

As with all studies, there were limitations that affect generalizing these data. The three main limitations were sample characteristics, ability differences and physical arrangements.

The size of the intervention groups were small. When groups are of small size the variations in students' abilities and characteristics have a more profound effect on comparison results. The small sample size also lessened the probability that the differences would be statistically significant. Also, all schools were located within one city and within one school district. All the schools used similar mathematics textbooks and assessments meaning that the majority of the students probably had similar third and fourth grade fraction instruction. There may be understandings or student errors that are a result of similar textbook use. Also the population of this group of 43 students was a white middle class with limited diversity. It may be that findings would differ for more diverse student populations. Future studies could include populations from a variety of schools with a variety of population characteristics.

Although the pretest scores of the three groups were not statistically different, there were numerical differences. The CM group scored higher on the EFT pretest and the PM group scored higher on the DCA pretest. Although most analyses compared gain scores, it is possible that, even though the pretest differences were small, differences in student's abilities within each group may in part explain the variations identified in this study.

Physical arrangements may also have been a contributing factor in the results. Seating arrangements for the three intervention groups were different. In all four schools, the computers were standalone PCs that were placed in straight lines. Students were limited in communication to the students sitting on each side of them. Also because the instructors moved from student to student, involvement in student teacher conversations was typically limited to one or two students. In contrast, students using physical manipulatives sat around a table, with the teacher at the center. These students had the benefit of hearing all student and teacher conversations and they had continual opportunities to compare their responses with those of others. Through the increased communications students may have been prompted to increased reflection about the mathematical concepts, and may have experienced more times when their misconceptions were challenged.

Future Research

This was a foundational study, developed to identify variations and to pilot methods of fraction intervention instruction. The most important extension of the study would be an expansion with a larger more diverse population. Because of the small number of participants in each intervention, a replication study could help determine if the results were unique to this population, or if the variations are common to the larger population.

This intervention was designed as a preliminary intervention. The ultimate comparison of the affordances of the three types of intervention would be a measurement of how affordances affected the students' learning of new fractional concepts in regular classroom settings. Future tracking could consist of follow up classroom observations in the regular classroom and the use of instruments designed to measure variations in learning concepts taught in the classroom.

This study did not focus on the motivation or the attitude variations related to manipulative type. Yet, these are important factors of learning. Future research could also develop instruments to measure affective variations as well as achievement variations.

The main purpose of this study was to identify variations in student learning related to the type of manipulative used. The next step, after identifying variations is to develop understanding of how the affordances of manipulatives specifically influence student thinking. For example, at what point does a student change from additive thinking to multiplicative thinking and how does the use of a manipulative influence this change.

Another purpose of this study was to pilot the use of two types of learning trajectories in research comparing manipulative types. Both the iceberg and the line plot trajectories were used to identify variations. The iceberg model was used as a tool to identify components of equivalent fraction understanding and to synthesize the findings of the study. The line plot trajectories were used to compare the effects of manipulative use over time. The use of trajectories to identify the student achievement of the learning components and to track variations in learning present a picture of student learning. The trajectories developed in this study could be compared with similar trajectories developed for students without mathematical learning difficulties. Comparisons of the rate and direction in the trajectories of learning development and the occurrences and resolution of misconceptions could deepen our understandings of how students with fraction learning difficulties differ from students who do not experience difficulties in both rate and kind of learning. Knowing this would make it possible to determine if the students need different types of instruction or if they require only more learning experiences.

Conclusion

The question of which manipulative is the most effective, virtual or physical, has been researched in more than 30 studies. A recent meta-analysis of the studies examining this comparison indicated that the difference between physical and virtual manipulatives' effectiveness for increasing student achievement produced small effects favoring instruction with virtual manipulatives (Moyer-Packenham et al., 2012). The present study was built on the premise that the effectiveness of the manipulative type depends on many factors, including the characteristics of the domain, learner, environment and goals of the intervention. In this study, the results of the total equivalent fraction test, favored the use of PM intervention, suggesting that physical manipulatives were the best manipulative for teaching these students many concepts of equivalent fractions. Yet, theoretically, if the use of one manipulative was best for all aspects of equivalent fraction instruction, then the analyses of the sub concepts of equivalent fractions should consistently favor the use of physical manipulatives. But analyses of the equivalent fraction test subconcepts favored VM intervention for the concepts for *grouping*, PM intervention for the concepts of *identifying* and *solving*, and CM intervention for the concepts of *modeling* and *simplifying*. One explanation for these variations is that, the learning characteristics of the students in the three intervention groups differed and that the students in each intervention group would have experienced higher gains in the identified sub concepts regardless of which manipulative they used. Another explanation is that the effectiveness of the unique affordances of the manipulatives vary according to the sub concepts and that the most effective equivalent fraction instruction would utilize both physical and virtual manipulatives in a manner that takes advantage of the manipulative affordances unique to each topic and situation. The findings of this study support the second explanation. Through the use of two types of learning trajectories, qualitative data and quantitative data were synthesized to identify variations in the learning of equivalent fractions in the intervention setting. From the literature and data collected in this study, the iceberg model of equivalent fraction learning was developed. For the 15 subconcepts identified in the model, 13 variations suggesting advantages for the use of a specific manipulative were identified.

From analyses of lesson data, variations in the learning processes of the students were identified. The data includes descriptions of the interactions of students with manipulative features throughout a series of ten practice sessions. The descriptions illustrate how variations in students' learning processes are a reflection of students' interactions with the manipulatives. Some types of interactions appeared to encourage exploration while others appeared to encourage the development of procedural abilities. Some interactions prevented errors while others encouraged reflection on errors. These types of variations are important for designing and implementing intervention instruction.

Three variations in students' strategies related to manipulative type were identified. The degree of creativity and complexity of the strategies varied according to the types of manipulatives used. Variations in students' use of representations were also identified. Students using physical manipulatives tended to score higher on questions using set model representations. Students using virtual manipulatives tended to score higher on questions using symbolic only representations.

The variations and trends identified in this study point to the complicated issues instructors and curriculum developers must consider when developing intervention instruction. Each intervention setting is a unique blend of goals, environment, content and students. Effective mathematics intervention requires knowledge of each of the factors. An important goal of future research is the development of knowledge supporting intervention instruction. An increased understanding of the use of physical and virtual manipulatives in intervention instruction is one step in developing the knowledge needed.

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APPENDICES

Appendix A

RNP Lessons

Table A1

RNP Lessons

RNP lesson focus	PM treatment group: Manipulatives and activities	VM treatment group: Applets and activities	VM/PM treatment group: Applets/ manipulatives and activities
1. Students explore relationships among circle pieces, modeling and orally naming fractions amounts for ½, 1/3, and ¼	Explore: Fraction circles Apply: Activity Sheet A/Fraction Circles Practice: Fraction Squares/Circles	Explore: *Fraction Pieces* Apply: Activity Sheet 1A,2A/*Fraction Pieces* Practice: *Fractions – Naming*	Explore: Fraction Circles and *Fraction Pieces* Apply: Activity Sheet A/*Fraction Pieces* Practice: *Fractions -Naming*
2. Students use paper folding to model and name unit and non-unit fractions. Students compare the paper-folding model to fraction circles. Students record fractions in words: one-fourth, two-thirds.	Explore: Paper strips and Fraction Circles Apply: Activity Sheet A, B &L/Fraction Circles Practice: Fraction Squares/Circles	Explore: *Fraction Pieces* Apply: Activity Sheet A, B&L/ *Fraction Pieces* Practice*: Fractions- Naming*	Explore: Paper strips and Fraction Circles Apply: Activity Sheet A, B&L / *Fraction Pieces* Practice: *Fractions-Naming*
3. Students observe with circles that as the unit is divided into more and more equal parts, the unit parts become smaller	Explore: Fraction circles Apply: Activity Sheet A&B/ Fraction Circles Practice: Fraction Squares/Circles	Explore: *Fraction Pieces* Apply: Activity Sheet A&B/ *Fraction Pieces* Practice: *Fraction Comparison*	Explore: Fraction circles Apply: Activity Sheet A&B/ *Fraction Pieces* Practice: *Fraction Comparison*
4. This lesson reinforces the idea that as the number of parts the unit is divided into increases, the size of the parts decreases	Explore: Fraction strips Apply: Activity Sheet A&B/ Fraction strips Practice: Fraction Squares/Circles	Explore: *Fraction Tiles* Apply: Activity Sheet A&B/ *Fraction Pieces* Practice: *Fraction Comparison*	Explore: Fraction Strips Apply: Activity Sheet A&B/ *Fraction Pieces* Practice: *Fraction Comparison*
5. Students explore fraction equivalence by naming fractions equal to ½ with fraction circles and by finding other fraction equivalences with fraction circles	Explore: Fraction circles Apply: Activity Sheet A/Fraction circles Practice: Fraction Squares/Circles	Explore: *Fraction Pieces* Apply: Activity Sheet A/*Fraction Pieces* Practice*: Fraction Comparison*	Explore: Fraction circles Apply: Activity Sheet A/ *Fraction Pieces* Practice: *Fraction Comparison*

(*table continues*)

RNP lesson focus	PM treatment group: Manipulatives and activities	VM treatment group: Applets and activities	VM/PM treatment group: Applets/ manipulatives and activities
6. Students continue to explore equivalence with pictures and fraction circles	Explore: Fraction circles	Explore: *Fraction Pieces*	Explore: Fraction circles
	Apply: Activity Sheet A&B/ Fraction circles	Apply: Activity Sheet A &B/*Fraction Pieces*	Apply: Activity Sheet A&B/ *Fraction Pieces*
		Practice: *Equivalent Fractions*	Practice: *Equivalent Fractions*
	Practice: Fraction Squares/Circles		
7. Students explore equivalence ideas with paper folding	Explore: Fraction strips	Explore: *Fraction Tiles*	Explore: Fraction strips
	Apply: Activity Sheet A&C/Fraction strips Practice: Fraction Squares/Circles	Apply: Activity Sheet A&C/ *Fraction Tiles* Practice: *Equivalent Fractions*	Apply: Activity Sheet A&C/ *Fraction Tiles* Practice: *Equivalent Fractions*
8. Students are introduced to chips as a fraction model.	Explore: Chips and fraction strips	Explore: *Pattern Blocks*	Explore: Chips and Fraction strips
They learn to represent a given fraction using different	Apply: Activity Sheet A/Chips	Apply: Activity Sheet A/ *Pattern Blocks*	Apply: Activity Sheet A/ *Pattern Blocks*
sets of chips as a unit.		Practice: *Equivalent Fractions*	Practice: *Equivalent Fractions*
	Practice: Fraction Squares/Circles		
9. Students continue practicing showing fractions with chips.	Explore: Chips and fraction strips	Explore: *Pattern Blocks*	Explore: Chips and fraction strips
They determine several units that can be used to model a	Apply: Activity Sheet A/Chips	Apply: Activity Sheet A */Pattern Blocks*	Apply: Activity Sheet A /*Pattern Blocks*
fraction and what units can't		Practice: *Fraction Equivalence*	Practice: *Fraction Equivalence*
be used to model fractions	Practice: Fraction Squares/Circles		
10. Students explore fraction equivalence using chips	Explore: Chips	Explore: *Pattern Blocks*	Explore: Chips
	Apply: Activity Sheets A,B,C,D&E/ Chips	Apply: Activity Sheets A,B,C,D&E/ *Pattern Blocks*	Apply: Activity Sheets A,B,C,D&E/ *Pattern Blocks*
	Practice: Fraction Squares/Circles	Practice: *Fraction- Equivalence*	Practice: *Fraction Equivalence*

Note. Names of virtual manipulatives are written using italics.

Appendix B

Physical Manipulative and Applet Comparisons

Table B1

Similarities and Differences of Fraction Circle and Virtual Fraction Circles

Variable	Tool similarities	Distinct attributes
Learning structure	Open structure with no student guidance	
Representation links		PM: None VM: Optional pictorial/symbolic link
Model type	Region: Circles	
Feedback	None	
Affordances		PM: Flexibility of movement VM: Colors can be changed - No limit to the amount of pieces - Objects designed can be lassoed and become fixed - Region unit grabbing
Constraints	Preselected sizes	
Distracters		PM: None VM: Region unit grabbing Optional symbolic link

Table B2

Similarities and Differences of Chips and Pattern Blocks

Variable	Tool similarities	Distinct attributes
Structure type	Open structure with no student guidance	
Representation links	None	
Model type	Set	
Feedback	None	
Affordances		PM: None VM: Pieces click together - Groups can be lassoed - Groups can be cloned
Constraints	None	
Distracters	Students must conceptualize the whole	

Table B3

Similarities and Differences of Physical Manipulatives and Virtual Manipulatives Used in Practice Phases of Lessons

Variable	Tool similarities	Distinct attributes
Structure type		PM: Open structure with no student guidance VM: Guided structure
Representation links		PM: None VM: Pictorial/symbolic
Feedback		PM: None VM: Correct/incorrect
Model type		PM: Circle and square regions VM: Circle and square regions and number line models
Affordances		PM: None VM: Partitioning accuracy - Will partition up to 99 sections
Constraints	Fraction piece sizes limited to one size	PM: None VM: Can partition only one direction
Distracters		PM: None VM: Multiple partitions are viewed - Equivalent partitions are viewed on one shape

Appendix C

Equivalent Fraction Tests

Table C1

Pre/Post/Delayed Test Question Types

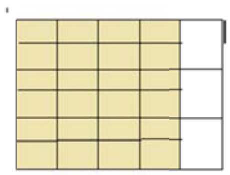
Content	Representation level	Type	Representation type
Modeling equivalence	4 Pictorial	2 multiple choice	1 Region and 1 Set
		2 open response	2 Region
Evaluating equivalence	3 Pictorial	1 multiple choice 1 matching 1 open response	1 Region 1 Region 1 Set
	1 Symbolic only	1 multiple choice	
Building equivalent groups	2 Pictorial	1 multiple choice 1 open response	1 Set 1 Region
	2 Symbolic only	1 multiple choice 1 open response	
Completing equivalent sentences	2 Pictorial	1 multiple choice 1 open response	1 Region 1 Set
	2 Symbolic only	2 short answer	
Simplifying fractions	2 Pictorial	1 multiple choice 1 open response	1 Set 1 Region
	2 Symbolic only	2 short answer	

Equivalent Fraction Pretest

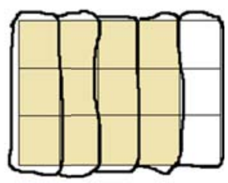
Name_______________________________________

1. Maria wants to show that $\frac{4}{5}$ equivalent to $\frac{12}{15}$. Circle the drawing that shows $\frac{4}{5}$ is equivalent to $\frac{12}{15}$
.

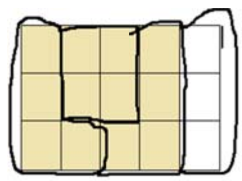
A



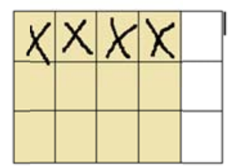
B



C



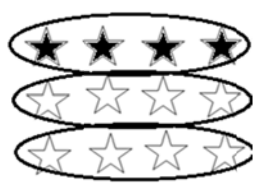
D



2. Sammy wants to show that $\frac{4}{12}$ is equivalent to $\frac{1}{3}$. Which drawing shows that $\frac{4}{12} = \frac{1}{3}$.

A

B



C

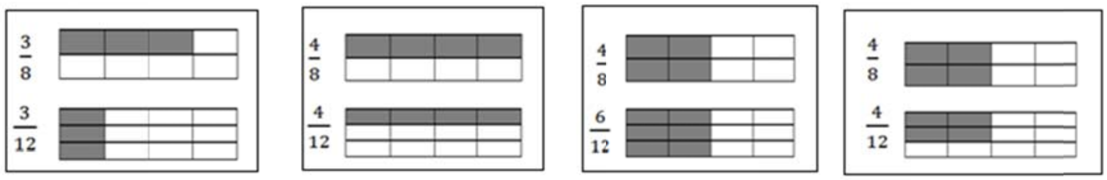
D

3. Draw lines to match the pictures with the correct fractions.

$\frac{3}{4}$ $\frac{6}{8}$ $\frac{12}{16}$

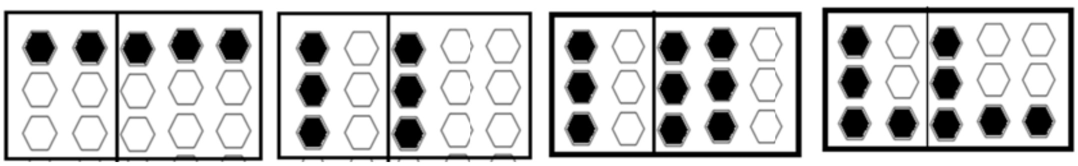
4. Circle the pair of drawings which show two equivalent fractions.

A B C D



5. Circle the pair of sets which shows that the two fractions are equivalent.

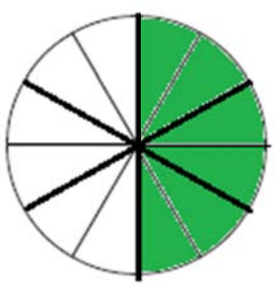
A B C D



6. Circle the statement that is correct,

A. $\frac{2}{3} = \frac{3}{2}$ B. $\frac{5}{6} = \frac{10}{12}$ C. $\frac{7}{9} = \frac{8}{9}$ D. $\frac{5}{7} = \frac{5}{8}$

7. What three equivalent fractions are shown in the circle?

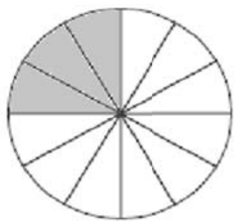


A. $\frac{1}{2}\ \frac{3}{4}\ \frac{6}{12}$ B. $\frac{1}{2}\ \frac{3}{2}\ \frac{6}{2}$ C. $\frac{1}{2}\ \frac{2}{2}\ \frac{3}{2}$ D. $\frac{1}{2}\ \frac{2}{4}\ \frac{3}{6}$

8. Which of the following groups show three equivalent fractions?

A. $\frac{1}{3}, \frac{2}{4}, \frac{3}{5}$ B. $\frac{1}{3}, \frac{2}{6}, \frac{4}{9}$ C. $\frac{1}{3}, \frac{2}{6}, \frac{3}{9}$ D. $\frac{1}{3}, \frac{3}{1}, \frac{3}{3}$

9. This picture shows that $\frac{1}{4} = \frac{3}{\square}$. What number can be put in the box to make the sentence true?



A. 9 B. 3 C. 4 D. 12

10. This picture shows that $\frac{6}{9} = \frac{\square}{3}$. What number can be put in the box to make the sentence true?

A. 3 B. 2 C. 1 D. 9

11. The box below shows that $\frac{10}{12}$ of the rectangle is shaded.

Which fraction is the simplified form of $\frac{10}{12}$?

A. $\frac{2}{6}$ B. $\frac{1}{4}$ C. $\frac{5}{12}$ D. $\frac{5}{6}$

12. $\frac{6}{10}$ of Liz's stars are black. Rename the fraction in its simplest form.

A. $\frac{3}{5}$ B. $\frac{4}{10}$ C. $\frac{1}{3}$ D. $\frac{1}{2}$

13. Write three fractions that are equivalent to $\frac{3}{5}$.

$\frac{3}{5} =$ ______ $\frac{3}{5} =$ ______ $\frac{3}{5} =$ ______

14. Fill in the missing numerator

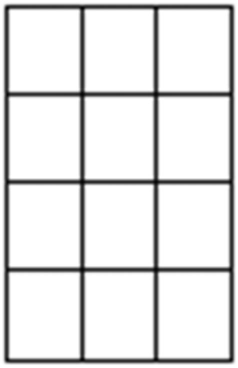
$$\frac{3}{4} = \frac{\square}{8}$$

15. Write $\frac{10}{15}$ in simplest form.

$$\frac{10}{15} = \frac{}{}$$

16. Using the box below show that $\frac{2}{3}$ is equivalent to $\frac{8}{12}$. Shade $\frac{8}{12}$ and then circle boxes to show $\frac{2}{3}$. Explain in words how your model shows that the two fractions are equivalent.

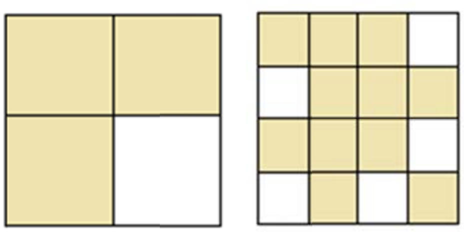
$\frac{8}{12} = \frac{2}{3}$



17. Sam said that the two squares below have the same fraction of shaded area.

Is Sam right or wrong?______________________

Write the fractions.

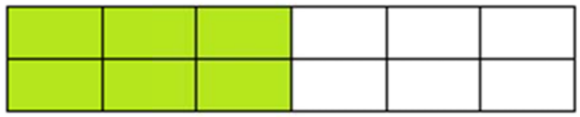


Fraction :_______ Fraction: _______

Explain why you think Sam is right or wrong.

18. This box shows that $\frac{1}{2} = \frac{6}{12}$

$\frac{1}{2} = \frac{6}{12}$



In the next two boxes show two <u>other</u> fractions which are also equivalent to ½.

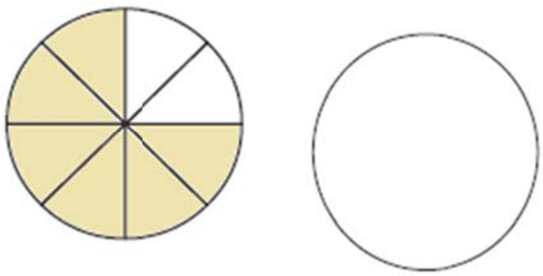
$\frac{1}{2} =$ ________

$\frac{1}{2} =$ __________

19. Nancy has 10 eggs. She colored $\frac{1}{5}$ of them blue. Draw a picture to show how many eggs Nancy colored blue.

20. The first circle shows $\frac{6}{8}$.
What is $\frac{6}{8}$ in simplest form? ___________

On the second circle draw and label a picture of the most simplified form of $\frac{6}{8}$.

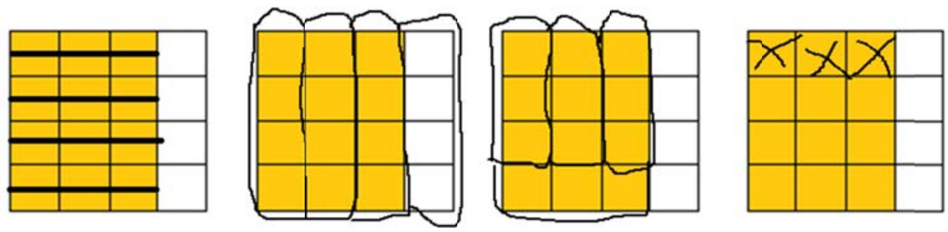


Equivalent Fraction Posttest 1

Name____________________________

1. Maria wants to show that $\frac{3}{4}$ is equivalent to $\frac{12}{16}$. Circle the drawing that shows $\frac{3}{4}$ is equivalent to $\frac{12}{16}$

A B C D



2. Sammy wants to show that $\frac{8}{12}$ is equivalent to $\frac{2}{3}$. Which drawing shows that $\frac{8}{12} = \frac{2}{3}$?

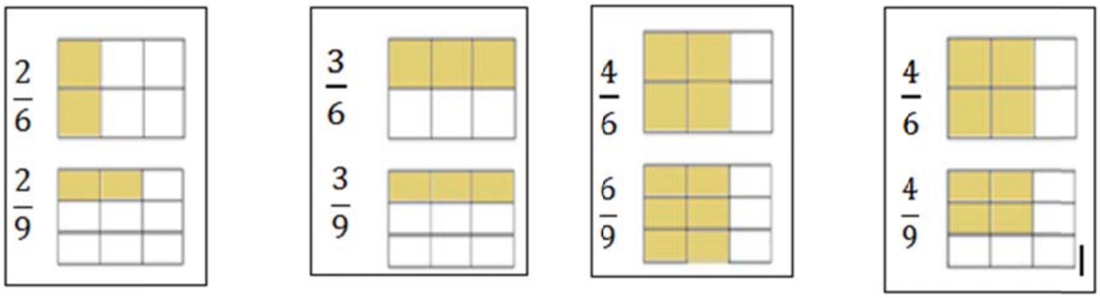
A B C D

3. Draw lines to match the pictures with the fractions.

$\frac{2}{3}$ $\frac{4}{6}$ $\frac{8}{12}$

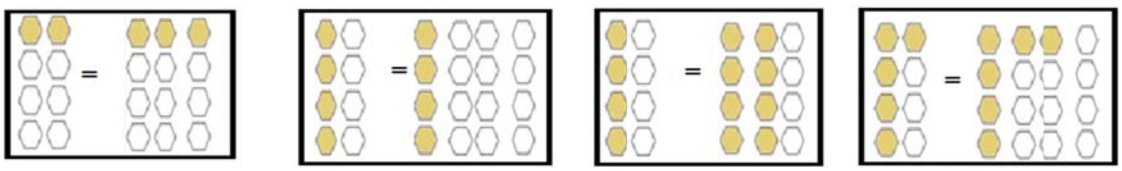
4. Circle the box with a pair of drawings which shows that the fractions are equivalent.

A B C D



5. Circle the box with a pair of sets which shows that the fractions are equivalent.

A B C D



6. Circle the statement that is correct.

A. $\frac{3}{4} = \frac{4}{3}$ B. $\frac{4}{5} = \frac{8}{10}$ C. $\frac{5}{7} = \frac{3}{7}$ D. $\frac{4}{7} = \frac{4}{9}$

7. What three equivalent fractions are shown in the circle?

A. $\frac{1}{4}\ \frac{2}{6}\ \frac{4}{16}$ B. $\frac{1}{4}\ \frac{2}{4}\ \frac{4}{4}$ C. $\frac{1}{4}\ \frac{2}{6}\ \frac{3}{16}$ D. $\frac{1}{4}\ \frac{2}{8}\ \frac{4}{16}$

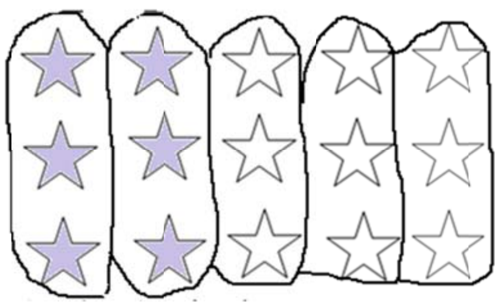
8. Which of the following groups show three equivalent fractions?

A. $\frac{1}{5}\ \frac{2}{6}\ \frac{3}{7}$ B. $\frac{1}{5}\ \frac{3}{10}\ \frac{6}{20}$ C. $\frac{1}{5}\ \frac{2}{10}\ \frac{3}{15}$ D. $\frac{1}{5}\ \frac{5}{1}\ \frac{5}{5}$

9. This picture shows that $\frac{3}{4} = \frac{9}{\ }$. What number can be put in the box to make the sentence true?

A. 9 B. 3 C. 4 D. 12

10. This picture shows that $\frac{6}{15} = \frac{\ }{5}$. What number can be put in the box to make the sentence true?



A. 6 B. 2 C. 3 D. 15

11. The box below shows that 8/10 of the rectangle is shaded.

Which fraction is the simplified form of 8/10?

B. $\frac{4}{7}$ B. $\frac{1}{3}$ C. $\frac{4}{10}$ D. $\frac{4}{5}$

12. $\frac{6}{8}$ of Ty's stars are black. Rename the fraction in its simplest form.

A. $\frac{3}{4}$ B. $\frac{1}{4}$ C. $\frac{3}{8}$ D. $\frac{1}{2}$

13. Write three fractions that are equivalent to $\frac{4}{5}$.

$$\frac{4}{5} = \frac{}{} \quad \frac{4}{5} = \frac{}{} \quad \frac{4}{5} = \frac{}{}$$

14. Fill in the missing numerator.

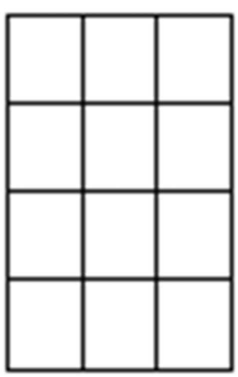
$$\frac{5}{6} = \frac{}{12}$$

15. Write $\frac{4}{10}$ in simplest form.

$$\frac{4}{10} = \frac{}{}$$

16. Using the box below show that $\frac{3}{4}$ is equivalent to $\frac{9}{12}$. Shade $\frac{9}{12}$ and then circle the boxes to show $\frac{3}{4}$. Explain in words how your model shows that the two fractions are equivalent.

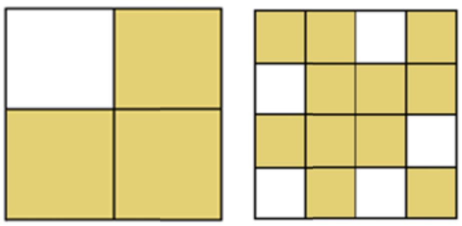
$$\frac{3}{4} = \frac{9}{12}$$



17. Sam said that the two squares below have the same fraction of shaded area.

Is Sam right or wrong? ___________________________

Write the fractions.



Fraction :_______ Fraction: _______

Explain why you think Sam is right or wrong.

____

____

18. This box shows that $\frac{1}{3} = \frac{4}{12}$

$\frac{1}{3} = \frac{4}{12}$

In the next two boxes make two <u>other</u> fractions which are also equivalent to ½.

$\frac{1}{3} = \frac{}{}$

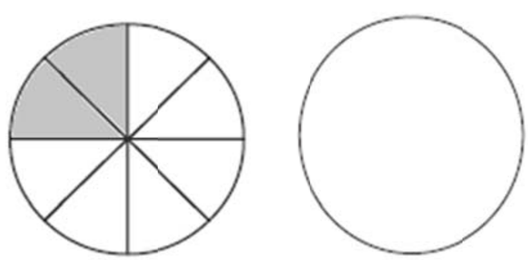
$\frac{1}{3} = \frac{}{}$

19. Nancy has 8 cup cakes. 1/4 of them are chocolate. Draw a picture to show how many cup cakes are chocolate.

20. The first circle shows $\frac{2}{8}$

What is $\frac{2}{8}$ in simplest form? ___________

On the second circle draw and label the simplified fraction for $\frac{2}{8}$.

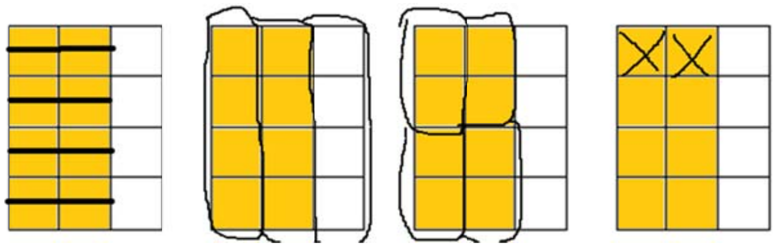


Equivalent Fraction Delayed Posttest

Name____________________

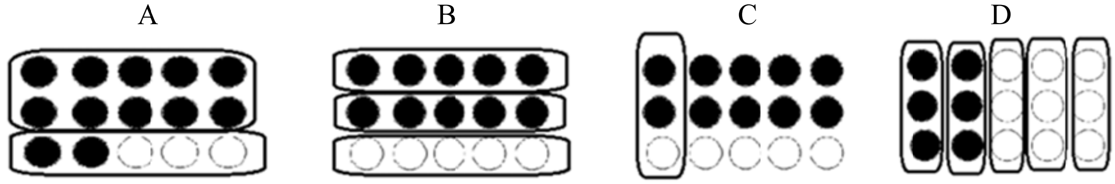
1. Maria wants to show that $\frac{2}{3}$ is equivalent to $\frac{8}{12}$. Circle the drawing shows $\frac{2}{3}$ is equivalent to $\frac{8}{12}$.

A B C D



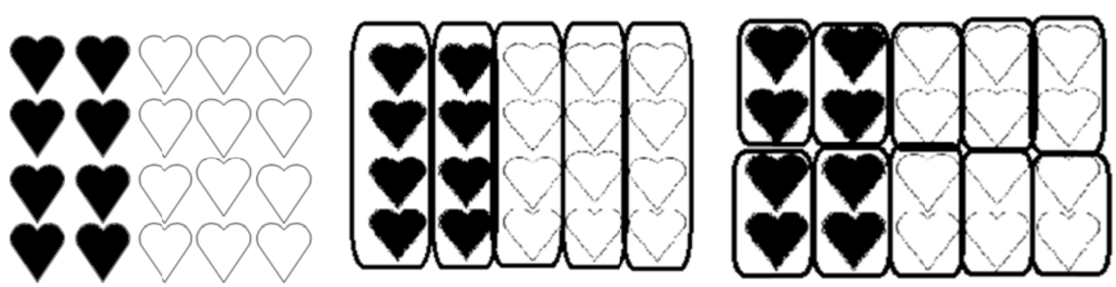
2. Sammy wants to show that $\frac{10}{15}$ is equivalent to $\frac{2}{3}$. Circle the drawing that shows $\frac{10}{15} = \frac{2}{3}$?

A B C D



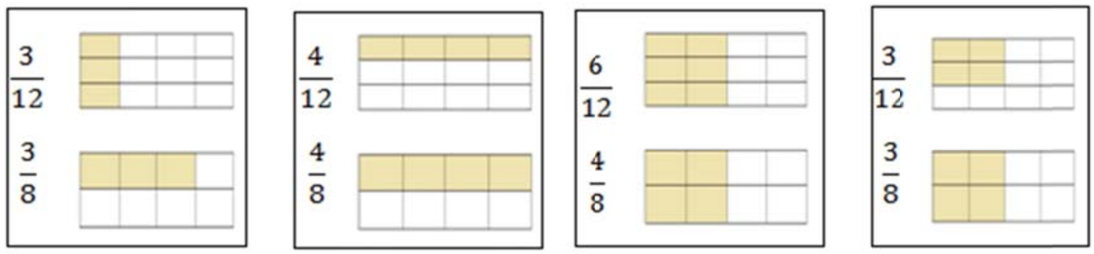
3. Draw a line to match each fraction with the correct picture.

$\frac{2}{5}$ $\frac{4}{10}$ $\frac{8}{20}$



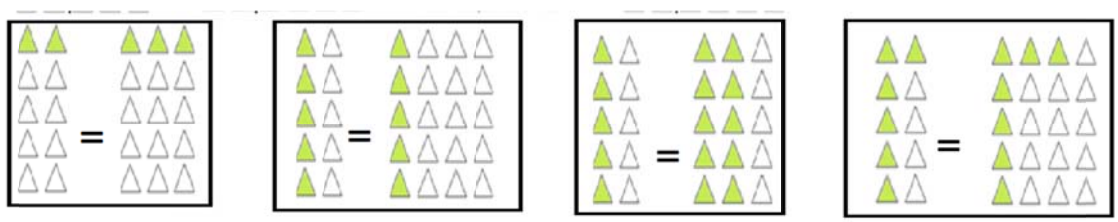
4. Circle the pair of drawings that shows that the fractions are equivalent.

A B C D



5. Circle the pair of sets that shows two equivalent fractions

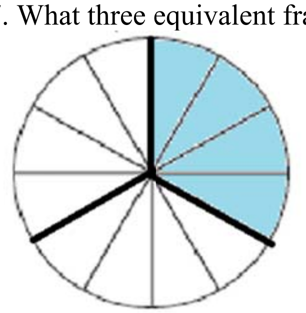
A B C D



6. Circle the statement that is correct?

A. $\frac{3}{5} = \frac{5}{3}$ B. $\frac{2}{3} = \frac{8}{12}$ C. $\frac{3}{9} = \frac{4}{9}$ D. $\frac{2}{3} = \frac{2}{5}$

7. What three equivalent fractions are shown in the circle?

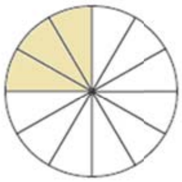


a. $\frac{1}{3}\ \frac{4}{4}\ \frac{4}{12}$ b $\frac{1}{3}\ \frac{2}{5}\ \frac{4}{7}$ c $\frac{1}{3}\ \frac{2}{3}\ \frac{3}{3}$ d $\frac{1}{3}\ \frac{2}{6}\ \frac{3}{9}$

8. Which of the following groups show three equivalent fractions?

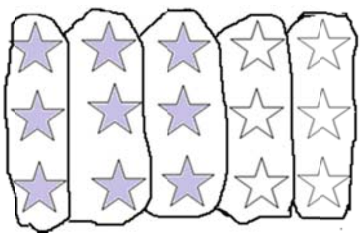
B. $\frac{1}{4}\ \frac{2}{5}\ \frac{3}{6}$ B. $\frac{1}{4}\ \frac{3}{8}\ \frac{9}{16}$ C. $\frac{1}{4}\ \frac{2}{8}\ \frac{3}{12}$ D. $\frac{1}{4}\ \frac{4}{1}\ \frac{4}{4}$

9. This picture shows that $\frac{1}{4} = \frac{\quad}{12}$. What number can be put in the box to make the sentence true?



A. 1 B. 12 C. 4 D. 3

10. This picture shows that $\frac{9}{15} = \frac{\quad}{5}$. What number can be put in the box to make the sentence true?



A. 9 B. 3 C. 2 D. 1

11. The box below shows that $\frac{6}{8}$ of the rectangle is shaded.

Which fraction is the simplified form of $\frac{6}{8}$?

C. $\frac{3}{5}$ B. $\frac{1}{4}$ C. $\frac{3}{8}$ D. $\frac{3}{4}$

12. 8/10 of Liz's stars are black. Rename the fraction in its simplest from.

A. $\frac{4}{5}$ B. $\frac{1}{2}$ C. $\frac{4}{10}$ D. $\frac{1}{3}$

13. Write three fractions that are equivalent to $\frac{2}{3}$.

$$\frac{2}{3} = \frac{\quad}{\quad} \quad \frac{2}{3} = \frac{\quad}{\quad} \quad \frac{2}{3} = \frac{\quad}{\quad}$$

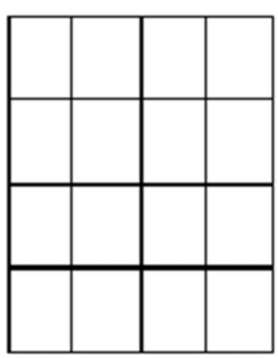
14. Fill in the missing denominator

$$\frac{3}{4} = \frac{9}{\quad}$$

15. Write $\frac{8}{12}$ in simplest form.

$$\frac{8}{12} = \frac{\quad}{\quad}$$

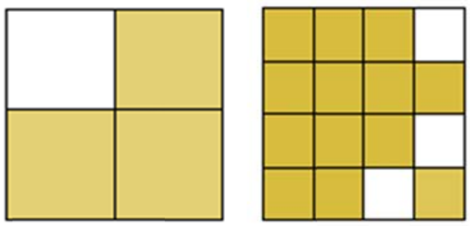
16. Using the box below show that $\frac{3}{4}$ is equivalent to $\frac{12}{16}$. Shade $\frac{12}{16}$ and then circle boxes to show $\frac{3}{4}$. Explain in words how your model shows that the two fractions are equivalent.



17. Sam said that the two squares below have the same fraction of shaded area.

Is Sam right or wrong?______________________

Write the fractions.



Fraction :_______ Fraction: _______

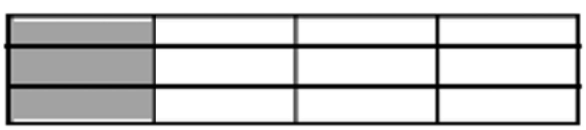
Explain why you think Sam is right or wrong.

____

____

18. This box shows that $\frac{1}{4} = \frac{3}{12}$

$\frac{1}{4} = \frac{3}{12}$



In the next two boxes make two other fractions which are also equivalent to ½.

$\frac{1}{4} = \frac{\ }{\ }$

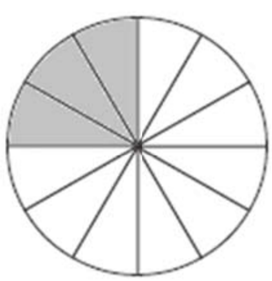
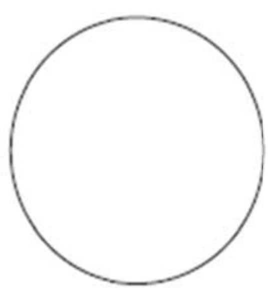
$\frac{1}{4} = \frac{\ }{\ }$

19. Nancy has 10 cup cakes. $\frac{1}{5}$ of them are chocolate. Draw a picture to show how many cup cakes are chocolate.

20. The first circle shows $\frac{3}{12}$

What is $\frac{3}{12}$ in simplest form? ___________

On the second circle draw and label the simplified fraction for $\frac{3}{12}$.

Appendix D

Lesson Assessments Samples

Lesson Concept Assessment
Lesson 2 (RNP 4)
Fractional Amounts

Name_____________________________

Pre-assessment:

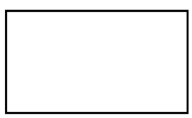
1. How many grey pieces equal one light green piece?_______

   One grey piece is _________ light green pieces.
   (fraction)

2. What color is ½ of a purple piece? _______________________

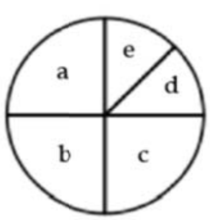
Concept Assessment

1. Liesel, Kami, Jesse, and J.P shared a pan pizza. J.P.s part looked like this. Add to the picture to show what the whole pizza looked like.



2. Circle all the pictures that show $\frac{3}{4}$.

3. Write the name for the fraction represented by d in the figure below. __________

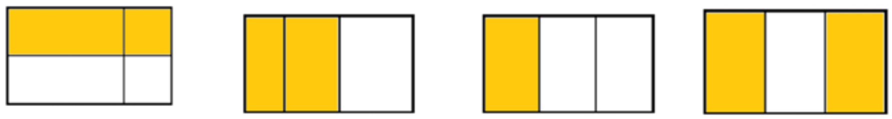


Lesson Concept Assessment
Lesson 3 (RNP 6)
Comparing Fractions Part I

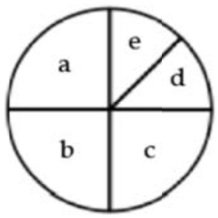
Name____________________________

Pre Assessment:

1. Circle the drawing which shows $\frac{2}{3}$.



2. Write the name for the fraction represented by e in the figure below. ________



Lesson Assessment

Circle the largest fraction.

1. $\frac{3}{7}$ $\frac{5}{7}$ $\frac{2}{7}$

2. $\frac{3}{5}$ $\frac{3}{7}$ $\frac{3}{8}$

3. Julie, Whitney and Manual shared a chocolate pie. Julie ate $\frac{2}{5}$. Manual ate $\frac{2}{6}$ and Manual ate $\frac{2}{4}$. Who ate the most pie?

Daily Cumulative Assessment

Pre Intervention

Name ______________________

1. Draw a picture of $\frac{5}{6}$.

2. Circle the largest fraction $\frac{2}{3}$ $\frac{1}{4}$ $\frac{3}{7}$

3. Place the two fractions on the number line $\frac{1}{3}$ $\frac{3}{5}$

0 1

4. Place and label another fraction between the two fractions.

5. You have 6 pizzas which you want to share with friends. Including yourself there are four people. How much pizza will each person receive? Draw your work.

6. What are two equivalent fractions represented in the circle?

___ = ___

7. Using the picture find three equivalent fractions for the shaded amount.

____ , ____ , ____

8. Simply the following into lowest terms.

$$\frac{3}{9}$$

Daily Cumulative Assessment
Lesson 2

Name ______________________

1. Draw a picture of $\frac{3}{8}$

2. Circle the largest fraction. $\frac{3}{4}$ $\frac{5}{12}$ $\frac{1}{7}$

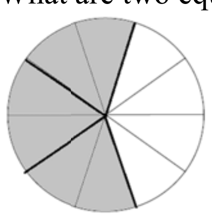
3. Place the two fractions on the number line $\frac{1}{2}$ $\frac{2}{3}$

0 1

4. Place and label another fraction between the two fractions.

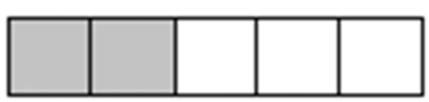
5. You have 8 pizzas which you want to share with friends. Including yourself there are three people. How much pizza will each person receive? Draw your work.

6. What are two equivalent fractions represented in the circle?



___ = ___

7. Using the picture find three equivalent fractions for the shaded amount



___ , ___ , ___

8. Simply the following into lowest terms.

$\frac{10}{15} =$

Appendix E

Instructor's Log

Instructor's Log

Date________________________ Instructor________________________

Lesson ____________________

Time	Lesson Activity	Notes
	Pre Assessment	
	Explore	
	Apply	
	Practice Session Applet:	
	Lesson Assessment	

Student				
PreAssess 1				
PreAssess 2				
Practice Problems Correct/ Problems Attempted				
Lesson Assess 1				
Lesson Assess 2				
Lesson Assess 3				
Ideas or concerns for next session				

Appendix F

Example of Activity Sheet

Name: ______________________

Lesson 10
Student Page A

Directions:
Work through each step:

1.		A. Fraction shaded: __________ B. Make into six equal-sized parts drawing in lines on picture. C. Fraction shaded now: ________
2.		A. Fraction shaded: __________ B. Make into 15 equal-sized parts drawing in lines on picture. C. Fraction shaded now: ________
3.		A. Fraction shaded: __________ B. Make into 12 equal-sized parts drawing in lines on picture. C. Fraction shaded now: ________
4.		A. Fraction shaded: __________ B. Make into six equal-sized parts drawing in lines on picture. C. Fraction shaded now: ________

Appendix G

Daily Cumulative Assessment Scoring Rubric

Daily Cumulative Assessment Scoring Rubric

Question 1

6	Correct number even distribution
5	Correct number uneven distribution
4	Correct partition but not shaded
3	One extra line
2	Numerator x denominator
1	other
0	No response

Question 2

6	Largest fraction - correct number line order
5	Largest fraction –incorrect number line
4	Second fraction –correct number line
3	Second fraction –incorrect number line
2	Lowest fraction- correct number line
1	Lowest fraction – incorrect number line or blank and correct number line
0	No response

Question 3

Centimeter distance from correct location

6	0-2
5	2.1-4

4 4.1-6

3 6.1-8

2 8.1-10

1 Greater than 10.1

0 No response

Question 4

Fraction between correct

1 Yes it is between

0 No it is not between or no response

Question 5

6 Model and answer correct

5 Model correct answer missing or incorrect

4 Correct number of pizzas -wrong partition or correct partition but wrong number of pizzas

3 Drew correct number of pizzas no partition or correct answer with no picture

2 Drew only one pizza correct partition

1 Drew only one pizza incorrect partition or drew wrong number of pizzas

0 No response

Question 6

6 Correct

5 Identified lowest but incorrectly counted or used equivalent not in picture

4 Identified highest but gave other equivalent

3 Identified highest but gave non equivalent

2 Identified shaded and non-shaded or flip flopped

1 Other non-related fractions

0 No response

Question 7

6 Correct

5 Three correct fractions – doubled all

4 Two correct fractions

2 One correct fraction

1 All fractions incorrect

0 No response

Question 8

4 Correct

3 Partial simplification

2 Higher Equivalent

1 Other

0 No Response

Appendix H

Lesson Summary Sheets

Student Summary Sheet

Student ________________ Instructor________________ Treatment group_________

Session	**No of Practice Problems**	**Pre-assessment**	**Concept**	**Cumulative**								
				1	2	3	4	5	6	7	8	Total
1												
2												
3												
4												
5												
6												
7												
8												

Session	**Misconceptions** **1. Whole number dominance** **2. Additive thinking** **3. Sees numbers as unrelated** **4. Ignores size of parts**	**Errors** **1. Multiply N/D Differently** **2. Add N/D** **3. Operates only with N or D** **4. Model N + D as the whole** **5. Adds same number to N/D**	**Partitioning Correct/Error (A, S)**											
			1	2	3	4	5	6	7	8	9	10	11	12
1														
2														
3														
4														
5														
6														
7														
8														

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SESSION SUMMARY SHEET

Session________________________ Instructor_______________________ Treatment Group______________

Student Name				
Misconceptions Description	Verbal –time___ Written-sheet___	Verbal –time___ Written-sheet___	Verbal –time___ Written-sheet___	Verbal –time___ Written-sheet___
Error Description	Verbal –time___ Written-sheet___	Verbal –time___ Written-sheet___	Verbal –time___ Written-sheet___	Verbal –time___ Written-sheet___
Linking of Representations	Verbal –time___ Written-sheet___	Verbal –time___ Written-sheet___	Verbal –time___ Written-sheet___	Verbal –time___ Written-sheet___
Use of Alternative Strategies	Verbal –time___ Written-sheet___	Verbal –time___ Written-sheet___	Verbal –time___ Written-sheet___	Verbal –time___ Written-sheet___

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Appendix I

Misconception and Error Codes

Coding of Errors

1. Multiply N/D differently

   Multiplies the numerator and the denominator by different numbers to find equivalent fractions.

2. Adds N/D within

   Adds or subtracts the numerator and denominator with in a fraction

3. Adds or subtraction numerators or denominators between fractions to determine equivalence

4. Adds same number to N and D to get equivalent fractions

   e.g., ¾ = 7/8 because you add 4 to both the numerator and denominator

5. Operates only with N or D

6. Performs an operation on only the numerator or denominator when finding equivalent fractions

7. Model N + D as the whole

   When modeling the whole they make the number of partitions or sets corresponding to the numerator and denominator added together

Coding of Misconceptions

1. Whole number dominance

   Size of the fraction is related to size of the numbers making up the fraction

2. Additive thinking

   Adding instead of multiplying when developing equivalent fractions, e. b. ½, 2/4, 3/6 add one to the top and 2 to the bottom

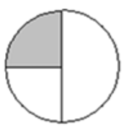
3. Sees numbers as unrelated

   Treats numbers in a fraction as unrelated to each other, e. g. Two thirds is a two and a three

4. N/D alone determines quantity

   Thinking that it is only the denominator that determines the size of the fraction

5. Ignores the size of the parts in relation to each other and the whole



Identifies the fraction in the picture as 1/3.

Appendix J

One Way ANOVAs

Table J1

One-Way ANOVAs

Source	*df*	*f*	*p*
EFT-Pre/Post	2	0.467	.631
Modeling	2	0.139	.871
Identifying	2	0.355	.703
Grouping	2	0.909	.411
Solving	2	2.382	.105
Simplifying	2	1.078	.350
EFT – Post/Delay	2	0.014	.986
Modeling	2	0.152	.859
Identifying	2	0.793	.459
Grouping	2	1.524	.231
Solving	2	0.289	.750
Simplifying	2	1.200	.312
DCA Total	2	2.207	.123
DCA Q1	2	1.085	.347
DCA Q2	2	1.021	.369
DCA Q3	2	1.977	.152
DCA Q4	2	1.327	.277
DCA Q5	2	3.870	.029
DCA Q6	2	0.860	.431
DCA Q7	2	1.200	.312
DCA Q8	2	0.941	.399
LCA	2	1.690	0.197
LPA	2	0.926	.410
N of Explore Problems	2	0.134	.875
N of Practice Problems	2	0.051	.951

Note. $N = 43$.

Appendix K

Analyses of Equivalent Fraction Test

Table K1

Comparison of Overall EFTs Results

	EFT pretest		EFT posttest		Pre to post			
Intervention type	*M*	*SD*	*M*	*SD*	*df*	*t*	*p*	Cohen's *d*
PM	26.47	10.74	66.73	17.39	14	11.74	0.00	0.79
VM	25.07	8.72	59.79	22.57	13	6.65	0.00	2.03
CM	32.36	13.51	67.93	21.57	13	7.79	0.00	1.98

Note. N = 43.

Table K2

Comparison of Equivalent Fraction Concept Test Results

	EFT pretest		EFT posttest		Pre to post			
Intervention type	*M*	*SD*	*M*	*SD*	*df*	*t*	*p*	Cohen *d*
Modeling								
PM	7.00	3.48	12.87	5.48	14	3.12	0.01	1.22
VM	5.21	4.56	12.07	5.64	13	4.62	0.00	1.34
CM	4.50	1.91	10.29	5.3	13	4.21	0.00	1.45
Identifying								
PM	6.60	5.54	12.00	4.93	14	2.40	0.03	1.03
VM	6.79	4.64	10.36	4.58	13	1.74	0.11	0.77
CM	10.00	4.39	13.21	4.64	13	2.09	0.06	0.71
Grouping								
PM	3.27	3.20	10.87	5.90	14	5.21	0.00	1.60
VM	3.79	4.15	13.36	5.33	13	6.29	0.00	2.00
CM	6.79	6.39	13.36	5.87	13	3.73	0.00	1.07
Solving								
PM	4.00	4.31	17.00	3.16	14	13.67	0.00	3.44
VM	4.29	3.31	13.21	5.75	13	5.10	0.00	1.90
CM	5.36	4.14	16.43	4.13	13	9.28	0.00	2.68
Simplifying								
PM	5.60	4.97	13.87	6.37	14	5.43	0.00	1.45
VM	5.00	4.80	10.79	8.85	13	2.78	0.02	0.81
CM	5.00	5.55	14.64	7.03	13	4.87	0.00	1.52

Table K3

Pre to Post Differences in the Percentage of Correct Student Answers

	Gain in percent of correct responses			
Questions	PM	VM	CM	Difference > 30%
1	20.00	28.57	14.29	
2	33.33	42.86	21.43	
3	26.67	21.43	21.43	
4	33.33	7.14	21.43	PM>VM
5	13.33	-7.14	35.71	CM>VM
6	33.33	57.14	21.43	VM>CM
7	20.00	50.00	42.86	VM>PM
8	13.33	21.43	00.00	
9	53.33	42.86	78.57	CM>VM
10	53.33	14.29	28.57	PM>VM
11	13.33	28.57	35.71	
12	46.67	7.14	28.57	PM>VM
13	66.67	71.43	64.29	
14	60.00	78.57	57.57	
15	46.67	42.86	50.00	
16	46.67	42.86	50.00	
17	26.67	14.29	-14.29	PM>CM
18	6.67	28.57	0	
19	73.33	42.86	50.00	PM>VM
20	33.33	28.57	42.86	

Table K4

Summary of Post EFT to Delayed Posttest EFT Differences

	EFT pretest		EFT posttest		Pre to post			
Intervention type	*M*	*SD*	*M*	*SD*	*df*	*t*	*p*	Cohen *d*
PM	66.73	17.39	65.07	18.37	14	-0.47	0.65	-0.09
VM	59.79	22.57	57.29	20.75	13	-0.65	0.53	-0.12
CM	69.85	21.17	67.92	27.68	112	-0.54	0.60	-0.08

Note. $N = 42$.

Table K5

Summary of Post to Delayed EFT Differences by Concepts

	EFT pretest		EFT posttest		Pre to post			
Intervention type	*M*	*SD*	*M*	*SD*	*df*	*t*	*p*	Cohen *d*
Modeling								
PM	12.87	5.84	13.13	5.55	14	1.76	0.86	+0.05
VM	12.07	5.64	13.57	4.97	13	0.86	0.40	+0.28
CM	10.92	4.92	11.46	5.89	12	0.30	0.77	+0.10
Identifying								
PM	12.00	4.93	10.33	5.16	14	-0.79	0.44	-0.33
VM	10.36	4.58	10.36	4.14	13	0.00	1.00	0.00
CM	13.08	4.80	14.23	7.03	12	1.00	0.34	+0.19
Grouping								
PM	11.00	5.95	12.27	4.85	14	+0.86	0.40	+0.23
VM	13.36	5.33	11.21	5.65	13	-1.28	0.22	-0.39
CM	13.69	5.94	14.92	6.08	12	+0.76	0.46	+0.20
Solving								
PM	17.00	3.16	15.33	3.52	14	-2.65	0.02	-0.50
VM	13.21	5.75	12.50	5.46	13	-0.43	0.67	-0.13
CM	16.92	3.84	15.00	5.77	12	-1.81	0.10	-0.39
Simplifying								
PM	13.87	6.37	14.00	5.41	14	+0.09	0.93	+0.02
VM	10.79	8.85	9.64	6.64	13	-0.83	0.42	-0.15
CM	15.23	6.95	12.31	7.80	12	-2.61	0.02	-0.40

Appendix L

Analysis of Daily Cumulative Assessment

Table L1

Summary of DCA Total Paired Samples t *Tests*

	EFT pretest		EFT posttest		Pre to post			
Intervention type	*M*	*SD*	*M*	*SD*	*df*	*t*	*p*	Cohen *d*
PM	23.53	5.55	32.73	4.50	14	8.20	0.00	1.82
VM	19.93	5.90	29.36	6.43	13	8.63	0.00	1.53
CM	20.93	6.99	33.50	6.89	13	8.17	0.00	1.81

Note. *N* = 43.

Table L2

Summary of DCA Questions Paired Samples t *Tests*

Question	Pretest	*SD*	Posttest	*SD*	*Df*	*T*	*p*	*ES*
Question 1								
PM	5.00	1.65	5.60	0.83	14	1.42	0.178	0.46
VM	4.21	1.93	5.14	0.86	13	2.33	0.037	0.62
CM	4.29	1.98	5.79	0.43	13	3.07	0.009	1.05
Question 2								
PM	4.93	1.58	4.80	1.78	14	0.13	0.709	-0.08
VM	4.21	1.53	4.43	1.79	13	0.37	0.720	0.13
CM	3.86	1.61	4.79	1.63	13	1.43	0.177	0.57
Question 3								
PM	3.47	1.64	5.00	1.56	14	2.66	0.019	0.96
VM	4.00	1.52	4.14	1.75	13	0.30	0.770	0.09
CM	3.07	1.59	4.21	1.58	13	2.51	0.026	0.72
Question 4								
PM	0.40	0.51	0.47	0.52	14	0.44	0.670	0.14
VM	0.64	0.50	0.43	0.51	13	1.00	0.336	-0.42
CM	0.36	0.50	0.50	0.52	13	1.47	0.165	0.27
Question 5								
PM	4.27	1.10	4.80	0.86	14	1.74	0.104	0.54
VM	2.21	2.01	4.79	1.31	13	3.56	0.001	1.52
CM	2.43	1.95	4.71	1.68	13	4.02	0.001	1.25
Question 6								
PM	2.60	1.92	4.33	1.63	14	4.25	0.001	0.97
VM	1.50	0.94	4.00	1.52	13	3.42	0.000	1.98
CM	2.21	1.89	4.93	1.33	13	1.57	0.000	1.66
Question 7								
PM	1.93	1.33	5.20	0.94	14	9.12	0.000	2.84
VM	1.64	0.63	4.14	1.79	13	3.24	0.000	1.86
CM	3.00	1.84	5.57	1.16	13	4.93	0.000	1.67
Question 8								
PM	0.93	0.46	2.53	1.46	14	4.77	0.000	1.48
VM	1.36	1.50	2.29	1.44	13	2.88	0.013	0.63
CM	1.71	1.64	3.00	1.47	13	3.35	0.005	0.83

Table L3

Percent of Students Who Mastered DCA Questions

	Daily cumulative assessment questions							
Intervention type	1	2	3	4	5	6	7	8
PM	60.0	53.3	40.0	40.0	13.3	33.3	46.7	33.3
VM	28.6	35.7	14.3	28.6	7.1	21.4	14.3	21.4
CM	53.3	53.3	14.3	50.0	14.3	42.9	64.3	42.9

Note. $N = 43$.

Appendix M

Misconception and Error Analyses

Student Misconceptions and Errors

Misconception 1: (Whole Number Dominance) Whole number dominance applies to fraction comparison.

Error 1: Compares fractions by comparing the numbers in the denominator as if comparing whole numbers (e.g., ¼ is greater than 1/3 because 4 is greater than 3).

Misconception 2: (Additive Thinking) Equivalent fractions can be formed by adding the same number to the numerator and the denominator of the original fraction.

Error 2: Adds or subtracts the same number to the numerator or denominator (e.g., ¾ = 5/6 because (3+2)/(4+2)=5/6).

Misconception 3: (Misnaming) Fractions of regional models represent relationships other than the part/whole relationship of the model.

Error 3: Models fractions as arrays (e.g., Draws as a model for the fraction 5/6 a five by six array).

Error 4: Interchanges numerator and denominator when naming fractions (e.g., Writes 1/3 as 3/1).

Error 5: Names a fraction by representing shaded/non shaded or nonshaded/ shaded (e.g., Writes 6/8 as 2/6 or 6/2).

Error 6: Incorrectly identifies fractional amount of the whole (e.g., Incorrectly identifies 1/6 section of a circle, which has been partitioned to show ½ and 3/6, as 1/5 instead of 1/6).

Misconception 4: (Partitioning/Simplifying) Partitioning and simplifying produces halves

Error 7: Responds to requests for equivalent fractions, not equal to ½, with ½ (e.g., 3/4 = ½ or 5/15 =1/2).

Error 8: Equates simplifying with dividing the numerator and denominator by two. When the fraction numerals are odd the student responds with either a decimal or the next whole number (e.g., 4/8 = 2/4 or 5/20 = 2.5/10 or 2/10 or 3/10)

Misconception 5: (Equivalence Meaning) Equivalence denotes relationships other than equal amounts

Error 9: Identifies equivalent fractions as being two fractions naming the relationship of the parts making up a whole (e.g., 1/3 = 2/3).

Error 10: Identifies equivalent fractions as the original fraction and a second fraction whose value is equal to one and contains numerals that were either in the original fraction or factors or multiples of the numerals in the original fraction (e.g., 6/8 = 6/6 or 8/8 – numerals from original fraction, 6/8=2/2-factor or 2/3 = 4/4 or 6/6/ - multiples) .

Error 11: Identifies equivalent fractions as being a fraction and its reciprocal (e.g., 1/3 = 3/1).

Error 12: Identifies equivalent fractions as being a fraction and a second fraction which is derived by determining the number of times a number will go into either the numerator or the denominator of the original fraction (e.g., 5/10 = 2/5 because five goes into 10 twice)

Misconception 6: (Incorrect Equivalent Sentences) When developing equivalent fractions, numerators and denominators may vary independently of each other.

Error13: Multiplies the numerator and denominator of the original fraction by different numbers (e.g., 3/4 = 9/16 because (3x3)/(4x4)=9/16) .

Error 14: Increases or decreases only the denominator or only the numerator of the original fraction (e.g., ¾ = 6/4 or 3/8).

Error 15: Multiplies the numerator of the original fraction by an arbitrary chosen number, which has been placed in the numerator of the new fraction, to obtain a new denominator (e.g., 3/4 = 2/8 because 2x4=8 or ¾ = 2/6 because 2x3=6).

Misconception 7: (Set Modeling) Fractions of set models represent relationships other than the part/whole relationship

Error 16: Identifying the numerator as being the number of groups in the set (e.g., when modeling ¾ they model 3 groups instead of four groups).

Error 17: Identifying as either the numerator or the denominator as being many items are in each group (e.g., given the fraction 3/4 they place three or four items in each set)

Error 18: When determining equivalent fractions using the set model, they interchange how many groups with how many in a group (e.g., When modeling what ¾ of 20 is, they make 5 groups).

Error 19: When determining equivalent fractions using the set model, they interchange the numerator of the new fraction with either the numerator or denominator of the original fraction (e.g., When asked to find 3/4 of 20, they respond with 3/20 or 4/20).

Table M1

Frequency of Student Error Types

Misconception/error	Number of observed cases		
	PM	VM	CM
1. Whole number dominance			
E1 compares whole numbers	38	38	39
2. Additive thinking			
E2 Adds or Subtracts	28	34	16
3. Misnaming			
E3 Arrays	4	19	2
E4 Reverses N and D	6	19	2
E5 Shaded/Un-shaded or Un-shaded/ Shaded	6	16	3
E6 Doesn't recognize whole	35	15	19
Total Errors	51	69	26
4. Partitioning/simplifying			
E7 Fraction=1/2	22	26	8
E8 Fraction= number of divisions/factor	7	9	6
Total Errors	29	35	14
5. Equivalence meaning			
E9 Shaded = un-shaded	58	50	28
E10 Fraction equivalent to one	19	17	15
E11 Reciprocal	3	22	13
E12 Fraction made of factors	7	9	6
Total	87	98	62
6. Incorrect equivalent sentences			
E13 Multiplies N and D by different numbers	21	20	12
E14 Operates on only N or D	35	34	15
E15 Multiplies N by another number to get D	5	4	1
Total	61	58	28
7. Set modeling			
E16 Uses N to determine the number of groups	9	2	6
E17 Uses N or D to determine how many in each group	9	3	10
E18 Uses how many in a group as the D	20	3	11
E 19 N or D of first fraction is used as N in second	7	0	13
Total	45	8	40

Note. Numerator (N), Denominator (D)

Appendix N

EFT Incorrect Responses

Table N1

Percent of Students' EFT Incorrect Responses

	Question 1 A-Error 14 D-Error 6			Question 2 C-Misc 7			Question 3 B Misc 7			Question 5 Misc 7			Question 7 Error 13		
	PM	VM	Com	PM	VM	Com	PM	VM	Com	PM	VM	Com	PM	VM	Com
A	13.3	0.0	35.7										46.7	21.4	35.7
B							6.7	28.6	7.1						
C				13.3	0.0	35.7				20.0	35.7	0.0			
D	6.7	28.6	14.3												
	Question 8 A-Error 13 B- Error 13			Question 10 A – Misc 7 C- Misc 7 D- Misc 7			Question 11 C - Error 14			Question 12 C- Misc 7			Question 17 C-Misc 5		
A	33.3	7.1	7.1	0.0	21.4	0.0									
B	20.0	28.6	42.9												
C				6.7	35.7	21.4	13.3	35.7	14.3	13.3	37.5	35.7	0.0	0.0	28.6
D				20.0	0.0	7.1									

CURRICULUM VITAE

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EDUCATION

Ph.D. May 2012
Education, Utah State University
Specialization: Curriculum and Instruction
Emphasis: Mathematics Education and Leadership

M.S April 1992
Special Education, University of Utah
Specialization: Mild and Moderate Disabilities
Thesis: *Benefits to Siblings of Children with Autism*

B.S. May 1974
Department of Communications, Brigham Young University
Utah State Professional Educator License Level 2 (1977)

EMPLOYMENT HISTORY

Adjunct Faculty

Utah State University at Logan, Utah (2009-present)

Director of TIME Clinic: Organized intervention program, developed materials, and tutored students
Taught 6 methods courses: EDLE 4060 – Teaching Mathematics and Practicum Level III
Relevant mathematics instruction in the elementary and middle-level curriculum; methods of instruction, evaluation, remediation, and enrichment. A field experience practicum is required.

University Supervisor and Instructor, Elementary Education Program (F2008, Sp2010)

Utah State University at Logan, Utah

Supervised student teachers in grades 2-6 and instructed bi-weekly group seminars.

Elementary School Teacher, Grades 1-6, (1978-2008)
Davis County School District at Farmington, Utah
Boulton Elementary, Bountiful, Utah
Kaysville Elementary, Kaysville, Utah

Co- Developer and Presenter for Davis Professional Development (1994-2008)
Davis County School District at Farmington, Utah
Co-instructed professional development workshops, developed teacher material and programs or co-instructed student groups for the following programs:
Davis Math Academy
Curriculum Standards
Curriculum Assessment Development
Discipline with Dignity
Student Assistance Program
Math Survival for Novice Teachers
Mathematics Core Enrichment

Cooperating Teacher (1999-2004)
University of Utah at Salt Lake City, Utah
Utah State University, at Logan, Utah
Supervised pre-service teachers during their student teaching experience.

Mathematics Instructor, Utah Core Academy (2004)
University of Utah at Logan, Utah
Number Sense Instructor in weeklong 6th grade teacher professional development workshops held throughout the state of Utah.

Part Time Hearing Impaired Seminary Instructor (1982-1983)
Church of Jesus Christ of Latter Day Saints at Bountiful, Utah
Instructor of weekly seminary class to Jr. High students with hearing limitations

Part Time Deaf/Blind Tutor (1973-1974)
American Fork Training School at American Fork, Utah
Week end tutor and caretaker for a teenage deaf-blind resident at training school.

Camp Recreation Leader and Speech Therapist (1970-1973)
Meadowwood Springs Speech Camp at Weston, Oregon
Instructor of camp recreation activities for children with speech disabilities and speech therapist instructing students with hearing limitations.

.Research

Research Interests:

- Mathematics professional development
- Mathematics early intervention
- Mathematics and technology

Research Projects

Equivalent fraction learning trajectories for students with mathematical learning difficulties when using manipulatives (2011-2012). Dissertation Research. Designed study, curriculum, tutored 5th grade mathematics fraction intervention and analyzed data. Utah State University.

Grades 3-4 Fractions and Virtual Manipulatives Mathematics Project (2009-present). Designed curriculum, taught 3rd and 4th grade mathematics fraction units and analyzed data. Utah State University.

Effects of Virtual Manipulatives on Mathematics Teaching and Student Achievement Literature Review (2010). Library search, data analysis and synthesis. Utah State University.

Analyzing Mathematics Teaching Anxiety Project (2009 – present). Method design, library search, data collection and analysis. Utah State University.

Evidence of Growth in Teacher Content Knowledge in Mathematics and Science Partnership Program Analysis (2008-2009). Document search and data analysis. Utah State University.

Student Teacher and First Year Teacher Mentoring (2005). Method design, data collection and analysis. University of Utah.

Publications

Journal Articles (Refereed)

Moyer-Packenham, P. S., & Westenskow, A. (in press). *Effects of virtual manipulatives on student achievement and mathematics learning*.

Brown, A. B., Westenskow, A., & Moyer-Packenham, P. S. (2012).Teaching Anxieties Revealed: Pre-service Elementary Teachers: Reflections on their Mathematics Teaching Experiences, *Teaching Education, 23*(4), 365-385.

Moyer-Packenham, P. S., & Westenskow, A. (2012). Processes and pathways: How do mathematics and science partnerships measure and promote growth in teacher content knowledge? *School Science and Mathematics. 112*(3), 133-146.

Westenskow, A., & Moyer-Packenham, P. S. (2011). Canine Conjectures: Using Data for Proportional Reasoning. *Mathematics Teaching in the Middle School*, *17*(1), 26-32.

Brown, A. B., Westenskow, A., & Moyer-Packenham, P. S. (2011). Elementary Pre-Service Teachers: Can They Experience Mathematics Teaching Anxiety Without Having Mathematics Anxiety?. *Issues in the Undergraduate Mathematics Preparation of School Teachers: The Journal*, *5*(teal_facpub), 1.

Marx, S., Gardner, J., Landon-Hayes, M., Sheridan, D., Westenskow, A., Johnson, K., Thurgood, L., (2009) Book review: Theory and educational research: Toward critical social explanation. *International Journal of Qualitative Studies in Education 23*(2),251-255.

Conference Proceedings (Refereed)

Moyer-Packenham, P.S., & Westenskow, A. (2012, Apr). Effects of virtual manipulatives on student achievement and mathematics learning. *Proceedings of the 2012 Annual Meeting of American Educational Research Association,* Vancouver, British Columbia, Canada.

Moyer-Packenham, P.S., & Westenskow, A. (2011, Sep). An initial examination of effect sizes for virtual manipulatives and other instructional treatments. *Proceedings of the 11th International Conference on Transformations and Paradigm Shifts in Mathematics Education*, Rhodes University, Grahamstown, South Africa.

Westenskow, A. (2011, September). Comparing the use of virtual manipulatives and physical manipulatives in equivalent fraction intervention instruction. *Proceedings of the 11th International Conference on Transformations and Paradigm Shifts in Mathematics Education*, Rhodes University, Grahamstown, South Africa.

Unpublished Manuscripts

Moyer-Packenham, P.S., Baker, J., Westenskow, A., Rodzon, K., Anderson, K., Shumway, J., Ng, D. & Jordan, K.(2012). *Comparing virtual manipulatives with other treatment modalities of mathematics instruction: hidden predictors of achievement.* Unpublished manuscript.

Moyer-Packenham, P.S., Baker, J., Westenskow, A., Rodzon, K., Anderson, K., Shumway, J., Ng, D. & Jordan, K.(2012). *Third and Fourth Results* Manuscript in preparation.

Shumway, J., Baker, J., Moyer-Packenham, P. S., Westenskow, A., Anderson, K. (2012). Comparing Virtual and Traditional Instruction in Fraction Concepts. Manuscript in preparation.

Wesstenskow, A., Moyer-Packenham, P.S., & Thurgood, J. (2012). Parental Perspectives in Tutoring Sessions. Manuscript in preparation.

Westenskow, A., Moyer-Packenham, P. S., Anderson, K, Shumway, J., Rodzon, K., & Jordan, K. (2012). *Modeling Fraction Error.* Manuscript in preparation.

Westenskow A. *The rhizomes of teachers' professional development pathways.* Manuscript in preparation.

Grants Funded

Graduate Research Assistant ($35,000). *Virtual Manipulatives Research Group: Effects of Multiple Visual Modalities of Representation on Rational Number Competence.* (2011-2012). Utah State University, Vice President for Research SPARC Funding. Lead PI – Patricia Moyer Packenham, Collaborating Faculty – Kerry Jordan, Dicky Ng, and Kady Schneiter. My role: design lesson plans for experimental classroom, teach experimental lessons at research sites, conduct data collection and analysis, participate in research team meetings, collaborated on publications and presentations focusing on using virtual manipulatives to teach rational number concepts.

Graduate Student ($600) *Professional Conference Awards.* Utah State University, Graduate Senate. (April, 2010). Travel grants award for presentations AERA and NCTM national conferences.

Graduate Student ($600) *Graduate Student Travel & Research Grant.* Utah State Univeristy, (Jan, 2010) Women & Gender Research Institute. Travel grant award for presentation at AMTE national conference.

Classroom Teacher ($1,000). *Toyota Education Grant.* (2005). Boulton Elementary, Bountiful, Utah. Designed and built mathematics lessons and tool kits for 5th and 6th grade mathematics lessons focusing on the integration of mathematics in other subject areas.

Classroom Teacher ($500). *Davis Education Foundation Grants.* (2000). Davis School District. Developed and distributed supplemental sixth-grade mathematics activities to Davis School District school.

Presentations

National/International

Anderson, K., Westenskow, A., & Moyer-Packenham, P. S. (2012, April). Teacher Resources for Using Virtual Manipulatives to Teach Fraction Concepts. . Presentation, Annual Meeting of the National Council of Teachers of Mathematics (NCTM), Philadelphia, PA.

Moyer-Packenham, P.S., & Westenskow, A. (2012, April). Effects and Affordances of Virtual Manipulatives on Students' Achievement. Research Paper Presentation, Annual Meeting of the National Council of Teachers of Mathematics (NCTM), Philadelphia, PA.

Moyer-Packenham, P. S., & Westenskow, A. (2012, April). Research on Teaching with Simulated Virtual Tools and Spaces. Roundtable session. American Educational Research Association (AERA) Annual Meeting, Vancouver, Canada.

Moyer-Packenham, P.S. & Westenskow, A. (2012, February). Connecting Research Results on the Effects of Virtual Manipulatives with Mathematics Teacher Development. Research Paper Presentation, *Annual Meeting of the Association of Mathematics Teacher Educators* (AMTE), Fort Worth, TX.

Moyer-Packenham, P.S. & Westenskow, A. (2011, November). A Meta-Analysis of the Effects of Virtual Manipulatives on Mathematics Learning and Student Achievement. Research Presentation, *Annual Meeting of School Science and Mathematics Association,*(SSMS) Colorado Springs, CO.

Moyer-Packenham, Jordan, K., Ng, D., Anderson, K., Baker, J., Rodzon, K., Shumway, J., & Westenskow, A.(2011, November). School Mathematics Research on Virtual Manipulatives: A Collaborative Team Approach. Research Presentation, *Annual Meeting of School Science and Mathematics Association* (SSMA), Colorado Springs, CO.

Westenskow, A. (2011, September). Comparing the use of virtual manipulatives and physical manipulatives in equivalent fraction intervention instruction. *Proceedings of the 11th International Conference on Transformations and Paradigm Shifts in Mathematics Education*, Rhodes University, Grahamstown, South Africa.

Moyer-Packenham, P. S., & Westenskow, A. (2011, September). An initial examination of effect sizes for virtual manipulatives and other instructional treatments. *Proceedings of the 11th International Conference on Transformations and Paradigm Shifts in Mathematics Education*, Rhodes University, Grahamstown, South Africa.

Brown, A. B., Westenskow, A., & Moyer-Packenham, P. S. (2011, January). *Analyzing Mathematics Teaching Anxiety: Assumptions, Findings and Implications for Mathematics Educators*. Research Paper Presentation, Annual Meeting of the Association of Mathematics Teacher Educators (AMTE), Irvine, CA.

Westenskow, A. (2010, November). *The rhizomes of teachers' professional development pathways.* Research Paper Presentation, Annual Conference of School Science and Mathematics Association (SSMA), Florida

Moyer-Packenham, P. S. & Westenskow, A. (2010, April). *Analyzing, Interpreting, and Connecting Data Relationships Using Virtual Manipulatives.* Presentation, 99th Annual Meeting of the National Council of Teachers of Mathematics (NCTM), San Diego, CA.

Moyer-Packenham, P. S., & Westenskow, A. (2010, April). *Process and Pathways: How Do Mathematics/Science Partnerships Measure and Promote Teacher Content Knowledge Growth?* Research Paper Presentation, American Educational Research Association (AERA) Annual Meeting, Denver, CO.

Westenskow, A., Brown, A. B., & Moyer-Packenham, P.S. (2010, January). *Reducing Pre-Service Teacher Anxieties for Teaching Elementary Mathematics.* Research Paper Presentation, Annual Meeting of the Association of Mathematics Teacher Educators (AMTE), Irvine, CA.

State/Regional

Westenskow, A. (2011), November). *Building equivalent fraction learning trajectories.* Annual Conference of the Utah Council of Teachers of Mathematics (UCTM), Magna, Utah

Westenskow, A. (2010, November). *Using data sets to explore proportional relationships.* Annual Conference of the Utah Council of Teachers of Mathematics (UCTM), Bountiful, Utah

Cloke, G., & Westenskow, A. (2000, November). *Using math games*. Workshop presentation at Utah Council of Teachers of Mathematics State Conference, Salt Lake City, Utah.

Westenskow, A *(2003, July) Ratios*. Workshop presentation at The Northern Utah Summer Math Institute at Bountiful, Utah.

Service and Leadership Activities

Liaison (2009-2010)	*Utah State University Graduate Senate Liaison*, Served as an information liaison and presided as session leader in research conference.
Course Developer (2005-2008)	*Davis Math Academy* Designed and presented mathematics professional development courses.
Committee Member (2004)	*Davis Curriculum Standards Alignment*. Developed curriculum guides for 1st year teachers.

Curriculum Designer (1998)	*Mathematics Core Enrichment*. Developed supplemental mathematics activities for 6th grade curriculum.

Awards

2012 AERA Best Paper Award
2011 Graduate Research Assistant of the Year, Utah State University
2009 Graduate Teaching Assistant of the Year, Utah State University
2006 Who's Who Among Teachers
2002 Davis County District Hall of Fame, Farmington, Utah
2002 Toyota Horizon Award, Farmington, Utah
2002 Who's Who Among Teachers
1995 Teacher Academy Fellow (Social Studies), Weber State University