Mathematical hydraulics of surface irrigation

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ABSTRACT

The general hydrodynamic equations for a spatially varied unsteady flow in a prismatic open channel having an arbitrary cross-sectional shape can be derived from the equations of continuity and momentum. The assumptions based on the general concept of hydrodynamics and the theory of shallow water are introduced. The mathematical models in the surface irrigation can be formulated by these equations of motion with the appropriate initial and boundary conditions prescribed at the singularity point (the origin in the x, t-plane) and at $x = 0$. Therefore, the flow in the surface irrigation must be described by solving the boundary-value problem for the velocity and the depth of flow.

In the two-dimensional flow, the discontinuity at the origin in the x, t-plane is overcome by imposing a critical velocity and correspondingly, a critical depth at the initial state. Without considering all the channel slope, friction, and infiltration terms, the mathematical model becomes the model for a "centered simple wave," in which an exact solution, the Ritter solution, in the dam-breaking problem is already well-known. The Ritter solution satisfies the boundary condition at $x = 0$, where the discharge is always constant. However, even though an additional term, the channel slope, is considered, the modified Ritter solution no longer satisfies the same boundary condition. No exact solution seems possible with the present knowledge of mathematical techniques.

The theory of characteristics is presented and the method of finite-difference based on this basic theory with the fixed-time interval is introduced for the basic equations of motion in the surface irrigation. The trajectory of the wavefront is usually defined as a locus of zero celerity ($c = 0$) and forming an envelope of two different families of characteristics. Such an extremely complicated situation at the wavefront results in the possible failure of the present technique by simply using the finite-difference method unless some additional judicious assumptions at the wavefront, some type of the boundary-layer technique must be developed. This is the present status of the finite-difference method in the surface irrigation.

The more simplified approximation by the kinematic-wave method is presented, based on an additional assumption analogous to the Dupit-Forchheimer theory in the groundwater flow. The method possesses a potential applicability in the surface irrigation.

Inasmuch as no results are yet available, this study will be limited to a description of what is hoped to be accomplished and how it will be done.

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CHAPTER 1

INTRODUCTION

An increasing interest in the studies of flow characteristics in the area of surface irrigation in the recent decade has led to the establishment of a new Hydraulics of Surface Irrigation Committee in the American Society of Agricultural Engineers. Dr. Vaughn E. Hansen is its first chairman. Many researches* have been conducted all over the United States as well as here at Utah State University.

The flow phenomenon manifesting itself in surface irrigation, either furrow or border, is considered as an unsteady open-channel flow over a porous bed having a variable infiltration rate. An idealized mathematical model for this type of flow can be obtained through utilizing the customary concept of hydrodynamics and the theory of shallow water. The equations of continuity and momentum, derived for a spatially varied unsteady flow in a prismatic open channel having an arbitrary cross-sectional shape, form the basic equations of motion. With the help of the appropriate initial and boundary conditions prescribed, the formulated boundary-value problem for the flow in the surface irrigation must be solved for the values of velocity and depth at any location at any time. The exact solutions for this nonlinear-wave

* Proceedings of the ARS-SCS Workshop on Hydraulics of Surface Irrigation, ARS 41-43, 1960.

problem seem impossible with the present knowledge of mathematical techniques except that the well-known Ritter solution is obtainable for the most simplified version of the problem which is analogous to the classical dam-breaking problem.

The finite-difference method based on the theory of characteristics must be explored. Because of the discontinuity (singularity) at the origin in the x, t-plane and the complexity at the wavefront, the numerical solutions, even by using this method, seems hopeless unless some other judicious assumptions at the wavefront can be established first. The whole argument regarding this will be presented in the report and the present status of the studies in this direction will be discussed in detail.

Another approximate method such as the kinematic-wave method looks very promising. However, the reliable result from this method cannot be expected because of the unrealistic assumptions used. The method will be briefly demonstrated and the result will be schematically shown.

Simply using the equation of continuity to determine the trajectory of the wavefront is not included in this report. Most of the studies during the past years in this field falls in this category. The method appears adequate if one is only interested in obtaining the trajectories of the wavefront and the watertail.

This report is intended to give the present status of the theoretical studies in the hydraulics of surface irrigation and to shed light on the direction of the further studies in this area.

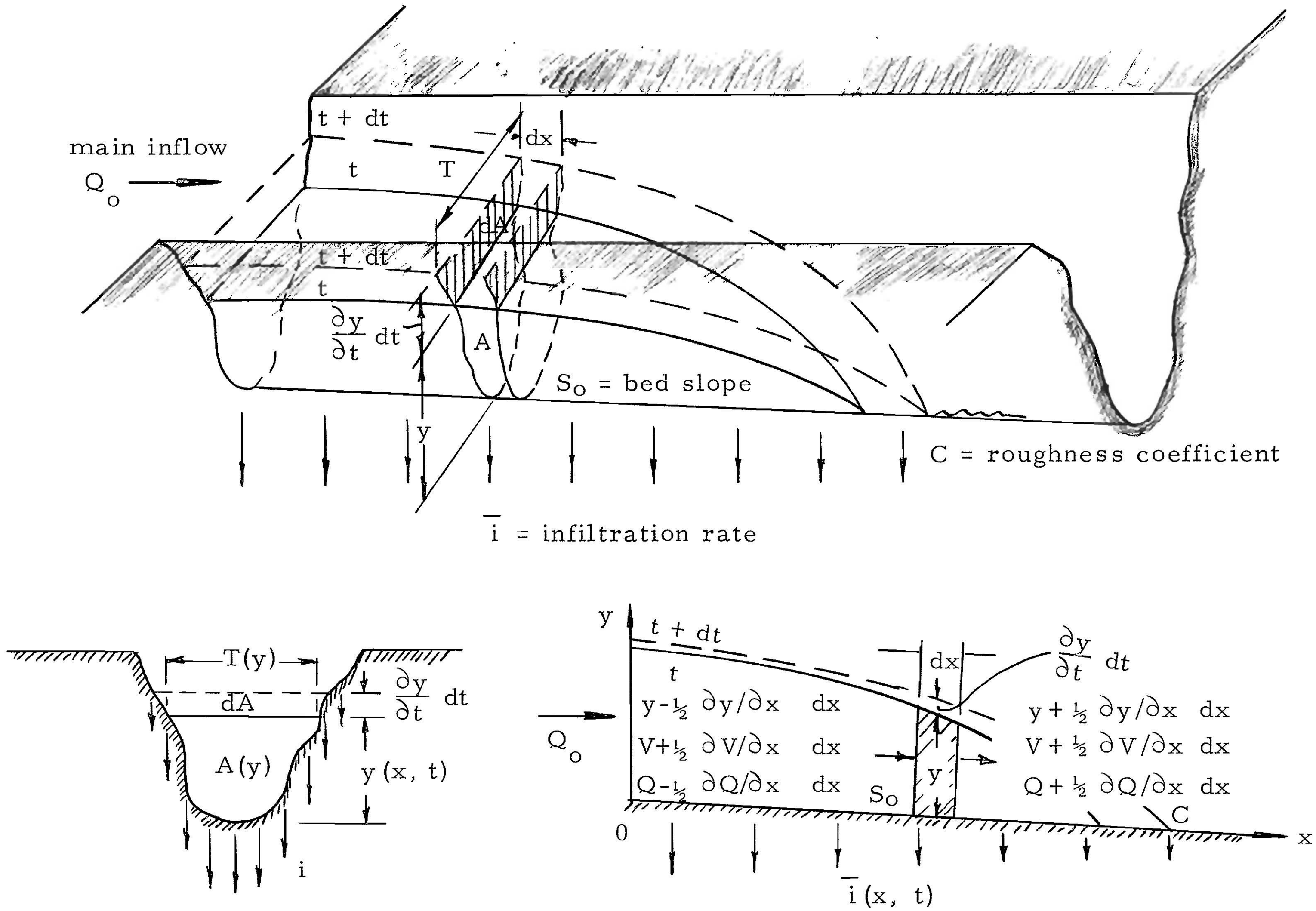
CHAPTER 2

DIFFERENTIAL EQUATIONS OF FLOW IN THE SURFACE IRRIGATION

In the surface irrigation, either furrow or border, when an inflow Q_o, is introduced from the upstream end and maintained at a constant rate, the water front of the flow starts to advance in the confined boundary of a given channel having an arbitrary cross-sectional shape. As shown in Fig. 1, the direction of the flow taken as the x-axis and the depth of flow, y, measured from the lowest bottom of the cross-section to the level of free surface is considered positive in the direction of increasing y.

At any instant time, t, the flow profile of the free surface describes a certain shape which can be formulated by a differential equation. At any point x in the longitudinal direction of the flow at this instant has a flow depth, y, velocity, V, discharge, Q, and cross-sectional area, A, which are all functions of the independent variables, x and t. The top width of the cross-section, i.e., the width of the free surface at the section is denoted by T and the infiltration rate, i, is evidently nonuniform on the perimeter of the cross-section considered. In order to simplify the formulation of the problem, a uniform infiltration rate, $\bar{i}$ (x, t), is assumed across the top width. Thus, the relation between i and $\bar{i}$ must be

(a) General view of channel flow



(b) Channel cross-section

(c) Channel profile

Fig. 1. Definition sketch of channel flow.

$$\bar{i} = \frac{1}{T} \int_{o}^{T} i \, dz$$

The channel is assumed to be straight enough so that its course can be thought of as a straight line (prismatic) without causing serious errors in the flow; hence, the bed slope, S_o, and the roughness coefficient, C, of the channel can be readily defined.

The differential equations of motion governing the flow are expressions of the laws of conservation of mass and momentum. In deriving them the following assumptions, in addition to those mentioned above, are made:

(1) Customary assumptions of hydrodynamics

a. The fluid is incompressible (density ρ = constant).

b. The fluid is frictionless (kinematic viscosity $\nu = 0$).

c. The flow is irrotational.

(2) Assumptions of shallow water theory

a. The depth of water is sufficiently small compared with some other characteristic length.

b. y-component of the acceleration of the water particles has a negligible effect on the pressure (i. e., assume a hydrostatic pressure distribution). Consequently, the vertical velocity component is zero and the horizontal velocity component is independent of y.

(3) Other assumptions

a. The channel is considered prismatic in the sense of infinitesimal distance, dx.

b. On the perimeter of the cross-section considered is a uniform shearing stress, τ_o.

c. Across the top width of the cross-section considered is the uniform infiltration rate, $\bar{i}$.

<u>Equation of Continuity</u>. The quantity of fluid flowing into the element considered in Fig. 1 in the infinitesimal time dt is

$$\left(Q - \frac{1}{2}\frac{\partial Q}{\partial x}dx\right) dt$$

and the amount flowing out the element is

$$\left(Q + \frac{1}{2}\frac{\partial Q}{\partial x}dx\right) dt + \bar{i}\ T \cos\theta\ dx\ dt$$

in which θ is the angle of inclination of the bed with respect to the horizontal surface. The amount of storage during dt is

$$\frac{\partial A}{\partial t}\cos\theta\ dt\ dx$$

Hence, equating that the amount of fluid flowing into the element equals the sum of the amount of fluid flowing out, and the amount of storage, one readily has

$$\left(Q - \frac{1}{2}\frac{\partial Q}{\partial x}dx\right) dt = \left(Q + \frac{1}{2}\frac{\partial Q}{\partial x}dx\right) dt + \bar{i}\ T\cos\theta\ dx\ dt + \frac{\partial A}{\partial t}\cos\theta\ dt\ dx$$

which can be simplified to yield

$$\frac{\partial A}{\partial t}\cos\theta + \frac{\partial Q}{\partial x} = -\bar{i}\,T\cos\theta \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (1)$$

This is the equation of continuity. Since

$$A = DT$$

in which D is defined as a hydraulic depth in the open-channel hydraulics,

$$Q = VA\cos\theta$$

and

$$dA = T\,dy \qquad A = A(y), \qquad T = T(y)$$

Eq. 1 can be rearranged* to give

$$\frac{\partial y}{\partial t} + V\frac{\partial y}{\partial x} + D\frac{\partial V}{\partial x} = -\bar{i} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (2)$$

<u>Equation of Momentum.</u> In addition to the assumptions mentioned above, the following assumptions are necessary to derive the equation of momentum.

(1) The momentum efflux of lateral outflow (infiltration) is so small that it is ignored without causing serious errors.

(2) The shear is equal to that of uniform flow of the same depth and velocity on a surface having the frictional slope, S_f.

(3) The angle of inclination of the bed with respect to the horizontal surface, θ, is so small that it is considered as

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* $\frac{\partial A}{\partial t} = \frac{\partial A}{\partial y}\frac{\partial y}{\partial t} = T\frac{\partial y}{\partial t}$, $\frac{\partial A}{\partial x} = \frac{\partial A}{\partial y}\frac{\partial y}{\partial x} = T\frac{\partial y}{\partial x}$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(VA) = V\frac{\partial A}{\partial x} + A\frac{\partial V}{\partial x} = VT\frac{\partial y}{\partial x} + DT\frac{\partial V}{\partial x}$$

$$\sin\theta \approx S_o$$

$$\text{and } \cos\theta \approx 1$$

The flow of the fluid element considered in Fig. 2 is subject to forces such as gravity, pressure, boundary shear. The resultant of these forces in the x-direction must equal the time rate of the increase of momentum flux within the element plus the net flux of momentum out of the element in unit time, dt; namely

$$\Sigma dF = \frac{dM}{dt} \quad \dots \quad (3)$$

in which F = force on the element in the x-direction

M = momentum flux of the element in the x-direction

$= \beta \rho A \cos\theta \; V \, dx$ (where β = momentum coefficient).

The time rate of increase of momentum within the element is

$$\frac{dM}{dt} = \frac{\partial M}{\partial x}\frac{dx}{dt} + \frac{\partial M}{\partial t} \quad \dots \quad (4)$$

Since $V = \frac{dx}{dt}$, Eq. 4 becomes

$$\frac{dM}{dt} = V\frac{\partial M}{\partial x} + \frac{\partial M}{\partial t} \quad \dots \quad (5)$$

in which $\frac{\partial M}{\partial x}$ = excess of momentum flux leaving the element over that entering the element, and

$\frac{\partial M}{\partial t}$ = time rate of increase of momentum within the element.

Hence, if β is assumed constant,*

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* Since $\frac{dx}{dt} = V$, $dx = V\,dt$. (Note that $\frac{\partial x}{\partial t} = 0$ and $\frac{\partial t}{\partial x} = 0$)

$$\frac{\partial}{\partial x}(A\,dx) = \frac{\partial}{\partial x}(A\;V\,dt) = A\frac{\partial}{\partial x}(V\,dt) + \frac{\partial A}{\partial x}dx$$

$$= A\frac{\partial V}{\partial x}dt + \frac{\partial A}{\partial x}dx$$

since $\quad \frac{\partial(dt)}{\partial x} = 0$

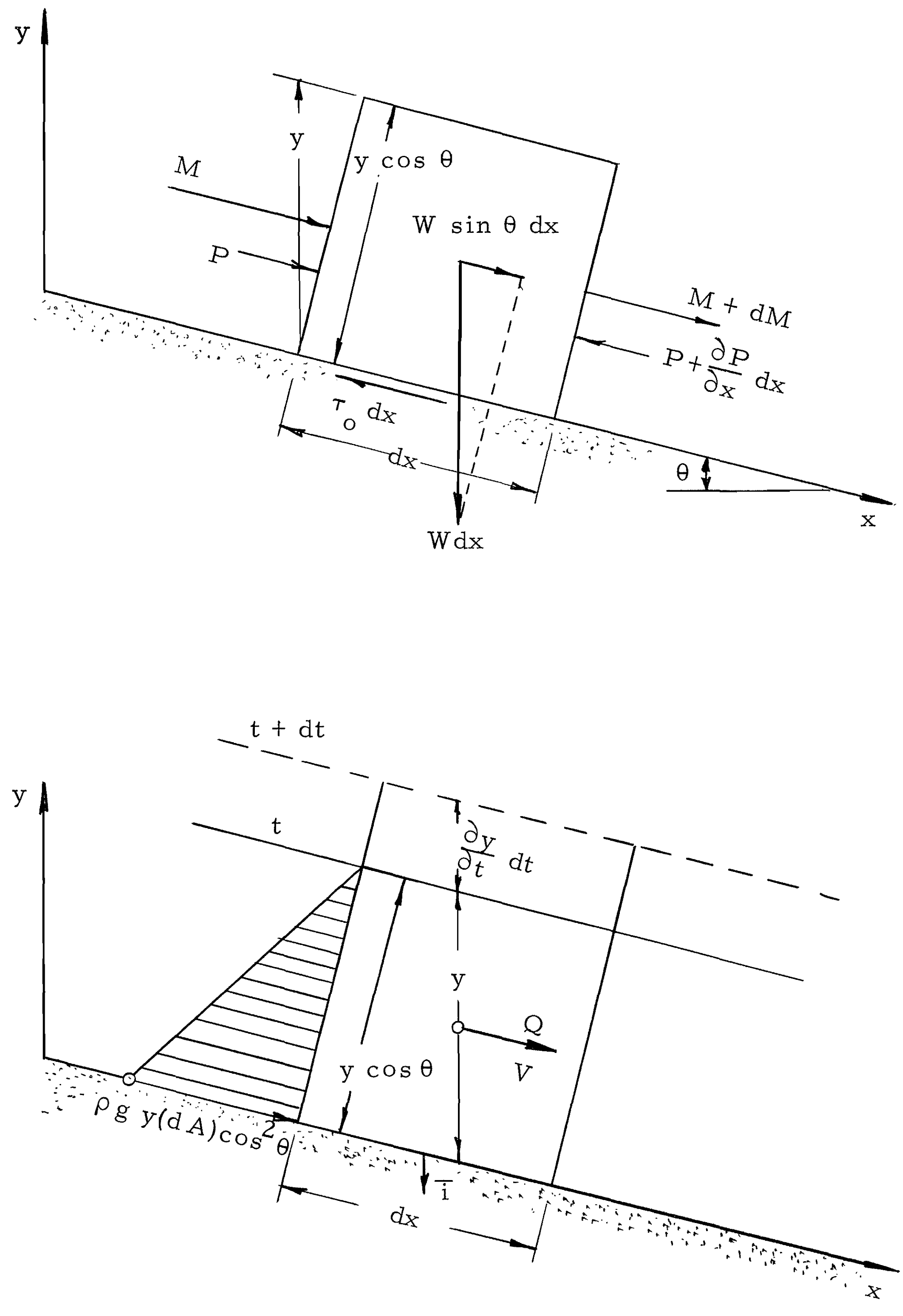

Fig. 2. Schematic diagram of a fluid element under external forces and momentum flux.

$$\frac{\partial M}{\partial x} = \frac{\partial}{\partial x} (\beta \rho A \cos \theta \ V\, dx) = \beta\rho \frac{\partial}{\partial x} (A \cos \theta \ V\, dx)$$

$$= \beta\rho A \cos \theta \frac{\partial V}{\partial x}\, dx + \beta\rho V \frac{\partial}{\partial x} (A \cos \theta\, dx)$$

$$= \beta\rho A \cos \theta \frac{\partial V}{\partial x}\, dx + \beta\rho V (A \cos \theta \frac{\partial V}{\partial x}\, dt + \frac{\partial A}{\partial x} \cos \theta\, dx)$$

$$= 2\ \beta\rho A \cos \theta \frac{\partial V}{\partial x}\, dx + \beta\rho V \frac{\partial A}{\partial x} \cos \theta\, dx$$

$$\frac{\partial M}{\partial t} = \frac{\partial}{\partial t} (\beta\rho A \cos \theta \ V\, dx) = \beta\rho \frac{\partial}{\partial t} (A \cos \theta \ V\, dx)$$

$$= \beta\rho A \cos \theta \frac{\partial V}{\partial t}\, dx + \beta\rho V \frac{\partial}{\partial t} (A \cos \theta\, dx)$$

$$= \beta\rho A \cos \theta \frac{\partial V}{\partial t}\, dx + \beta\rho V \frac{\partial A}{\partial t} \cos \theta\, dx$$

Therefore, from Eq. 5,

$$\frac{dM}{dt} = V\ (2\, \beta\rho A \cos \theta \frac{\partial V}{\partial x}\, dx + \beta\rho V \frac{\partial A}{\partial x} \cos \theta\, dx)$$

$$+ (\beta\rho A \cos \theta \frac{\partial V}{\partial t}\, dx + \beta\rho V \frac{\partial A}{\partial t} \cos \theta\, dx)$$

$$= 2\, \beta\rho A \cos \theta\ V \frac{\partial V}{\partial x}\, dx + \beta\rho V^2 \frac{\partial A}{\partial x} \cos \theta\, dx + \beta\rho A \cos \theta \frac{\partial V}{\partial t}\, dx$$

$$+ \beta\rho V \frac{\partial A}{\partial t} \cos \theta\, dx$$

Since $\dfrac{\partial A}{\partial x} = T \dfrac{\partial y}{\partial t}$ and $\dfrac{\partial A}{\partial t} = T \dfrac{\partial y}{\partial t}$

$$\frac{dM}{dt} = \beta\rho\, (2\, A\, V \frac{\partial V}{\partial x} + V^2\ T \frac{\partial y}{\partial x} + A \frac{\partial V}{\partial t} + V\, T \frac{\partial y}{\partial t}) \cos \theta\, dx \qquad \text{. . . . . (6)}$$

Also, using the equation of continuity, Eq. 2, to express the second term in the right side of Eq. 6, one has

$$V^2 T\frac{\partial y}{\partial x} = VT(-\bar{i} - \frac{\partial y}{\partial t} - D\frac{\partial V}{\partial x})$$

$$= -VT\bar{i} - VT\frac{\partial y}{\partial t} - VTD\frac{\partial V}{\partial x}$$

substitutes into Eq. 6, whence

$$\frac{dM}{dt} = \beta\rho(DTV\frac{\partial V}{\partial x} + DT\frac{\partial V}{\partial t} - VT\bar{i})\cos\theta\, dx \qquad (7)$$

Forces on the element are: (1) the component of the weight of the fluid in the direction of motion (x-axis), $W \sin\theta\, dx$, (2) the pressure force in the direction of flow, $-\frac{\partial P}{\partial x}dx$, and(3) the boundary shearing resistance force in the direction of movement, $-\tau_o\, dx$. Therefore, the resultant force on the element is

$$\Sigma dF = W \sin\theta\, dx - \frac{\partial P}{\partial x}dx - \tau_o\, dx \qquad (8)$$

in which W = weight of the element per unit length

= $\rho g A \cos\theta$ (where g = gravitational acceleration)

P = static pressure force on a normal section of the element = $\rho g \int_o^y y \cos\theta\, d(A\cos\theta)$

τ_o = boundary shear on the element per unit length

= $\rho g S_f A \cos\theta$

Since* $\frac{\partial P}{\partial x} = \rho g \frac{\partial}{\partial x} \int_o^y y \cos\theta \, d(A \cos\theta) = \rho g \cos^2\theta \frac{\partial(\bar{y}A)}{\partial x}$

$= \rho g \cos^2\theta \, A \frac{\partial y}{\partial x}$

Eq. 8 thus becomes

$$\Sigma d F = \rho g A \cos\theta \, S_o \, dx - \rho g A \cos^2\theta \frac{\partial y}{\partial x} dx - \rho g S_f A \cos\theta \, dx$$

$$= \rho g D T \cos\theta \, (S_o - \cos\theta \frac{\partial y}{\partial x} - S_f) \, dx \quad . \quad . \quad . \quad . \quad (9)$$

Substituting Eqs. 7 and 9 into Eq. 3, one has, for small θ

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + \frac{g}{\beta} \frac{\partial y}{\partial x} - \frac{V}{D} \bar{i} = \frac{g}{\beta} (S_o - S_f) \quad . \quad . \quad . \quad . (10)$$

Since the flow is assumed to be turbulent, the value of the momentum coefficient, β, for fully developed, steady uniform open-channel flow usually stays within the range of 1.03 and 1.07; however, in the present study, β is considered as unity for mathematical simplicity. Thus Eq. 10 becomes

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + g \frac{\partial y}{\partial x} - \frac{V}{D} \bar{i} = g (S_o - S_f) \quad . \quad . \quad . \quad . (11)$$

This is the equation of momentum. The two differential equations, Eqs. 2 and 11, which serve to determine two unknown functions, the depth $y(x, t)$ and the velocity $V(x, t)$, are generally the basic equations for the study of flows in the surface irrigation. The inspection of

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* $\int_o^y y \, d A = \bar{y} A$, where $\bar{y}$ is the depth of the centroid of the cross-sectional area, A, from the free surface. $d(\bar{y}A) = [A(\bar{y} + dy) + T(dy)^2/2] - \bar{y} A = A \, dy$, by assuming $(dy)^2 = 0$

Eqs. 2 and 11 will reveal that they essentially confirm the forms of the equations appearing in Stoker (1957).

Solving for $\frac{\partial y}{\partial x}$ from Eqs. 2 and 11, one obtains

$$\frac{\partial y}{\partial x} = \frac{1}{1 - \frac{V^2}{gD}} \left[S_o - S_f + \frac{2V}{gD} \bar{i} + \frac{1}{g} \left(\frac{V}{D} \frac{\partial y}{\partial t} - \frac{\partial V}{\partial t} \right) \right] \quad \ldots\ldots\ldots \quad (12)$$

which expresses the slope of free surface at any location, x, in the direction of flow at any instant time, t. This is the dynamic equation for unsteady spatially varied flow in the surface irrigation.

The dynamic equation, Eq. 12, can be organized into a meaningful form by introducing the following notations used by Chow (1959). The discharge, Q, of uniform flow in a prismatic channel of small slope is

$$Q = VA = CA_n R_n^m S_o^n \quad \ldots\ldots\ldots \quad (13)$$

R_n is the hydraulic radius of the channel section and in which the conveyance of the channel section, K_n, at a normal depth, y_n, is defined such that

$$K_n = CA_n R_n^m \quad \ldots\ldots\ldots \quad (14)$$

in which A_n is the cross-sectional area under consideration corresponding to a normal depth, y_n, and K_n expresses a measure of the carrying capacity of the channel section for a uniform flow. Thus Eq. 13 becomes

$$Q = K_n S_o^{\,n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Similarly, the conveyance of the channel section, K, at an actual flow depth, y, corresponding to the frictional slope, S_f, can be defined such that

$$Q = K S_f^{\,n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

in which $K = CAR^m$ (R = hydraulic radius of the channel section corresponding to the actual depth, y). From Eqs. 15 and 16, one may write

$$S_o = \left(\frac{Q}{K_n}\right)^{1/n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$$S_f = \left(\frac{Q}{K}\right)^{1/n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The exponents, m and n, in Eq. 13 which hold in the following cases are:

(i) Laminar flow : $m = 2$ and $n = 1$

(ii) Turbulent flow : If the Chezy formula is used; $m = 1/2$ and $n = 1/2$; however, if the Manning formula is used, $m = 2/3$ and $n = 1/2$.

Another parameter introduced is the section factor for critical flow computation, Z, in which one defines

$$Z = A\sqrt{D} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

When the discharge, Q, is given, the critical section factor, Z_c, at a critical depth, y_c, is defined such that

$$Z_c = \frac{Q}{\sqrt{g}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

in which $Z_c = A_c \sqrt{D_c}$, where A_c and D_c are the cross-sectional area and the hydraulic depth respectively corresponding to a critical depth under consideration.

With these definitions it can readily be shown that, using the Chezy formula,

$$\frac{S_f}{S_o} = \left(\frac{K_n}{K}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

$$\frac{V^2}{gD} = \left(\frac{Z_c}{Z}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Consequently, Eq. 12 can be arranged to yield

$$\frac{\partial y}{\partial x} = \frac{S_o}{1 - \left(\frac{Z_c}{Z}\right)^2} \left[1 - \left(\frac{K_n}{K}\right)^2 + \frac{2}{g} \left(\frac{K_n}{Z}\right)^2 \frac{\bar{i}}{V} \right.$$

$$\left. + \frac{1}{g} \left(\frac{K_n}{Z}\right)^2 \left(\frac{1}{V} \frac{\partial y}{\partial t} - \frac{D}{V^2} \frac{\partial V}{\partial t} \right) \right] \quad . \quad . \quad . \quad . \quad . \quad (23)$$

This is the general dynamic equation for unsteady spatially varied flow, in which K, K_n, Z, and Z_c are all functions of $y(x, t)$. For a wide open channel flow, simply $A = y$, $R = y$, and $D = y$, if the Chezy equation is used,

$$K_n = CA_nR_n^{1/2} = Cy_n^{3/2}$$

$$K = CAR^{1/2} = Cy^{3/2}$$

$$Z = A\sqrt{D} = y^{3/2}$$

and $$Z_c = A_c\sqrt{D_c} = y_c^{3/2}$$

Hence, Eq. 23 can be reduced to

$$\frac{\partial y}{\partial x} = \frac{S_o}{1 - \left(\frac{y_c}{y}\right)^3} \left[1 - \left(\frac{y_n}{y}\right)^3 + \frac{2C^2}{g}\left(\frac{y_n}{y}\right)^3 \frac{\bar{i}}{V} \right.$$

$$\left. + \frac{C^2}{g}\left(\frac{y_n}{y}\right)^3 \left(\frac{1}{V}\frac{\partial y}{\partial t} - \frac{y}{V^2}\frac{\partial V}{\partial t}\right)\right] \quad . \quad . \quad . \quad . \quad . \quad (24)$$

which becomes identical with the dynamic equation for overland flow developed by Chen and Hansen (1966). This is the dynamic equation for open channel flows in the border irrigation.

The dynamic equations for other geometrically simple cross-sectional shapes such as a rectangle and trapezoid, for example, can readily be derived from Eq. 23 if all the magnitudes of K, K_n, Z, and Z_c can be expressed by their own definitions.

Example 1: Rectangular cross-sectional shape

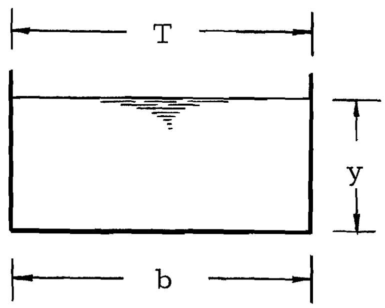

Fig. 3. Rectangular shape.

$$A = by$$

$$R = \frac{by}{b + 2y}$$

$$T = b$$

$$D = y$$

$$Z = by^{3/2}$$

$$K = \frac{C(by)^{3/2}}{(b + 2y)^{1/2}}$$

Hence, Eq. 23, after substitution of these quantities, yields

$$\frac{\partial y}{\partial x} = \frac{S_o}{1 - \left(\frac{y_c}{y}\right)^3} \left[1 - \left(\frac{b + 2y}{b + 2y_n}\right)\left(\frac{y_n}{y}\right)^3 + \frac{2C^2}{g}\left(\frac{b}{b + 2y_n}\right)\left(\frac{y_n}{y}\right)^3 \frac{\bar{i}}{V} \right.$$
$$\left. + \frac{C^2}{g}\left(\frac{b}{b + 2y_n}\right)\left(\frac{y_n}{y}\right)^3 \left(\frac{1}{V}\frac{\partial y}{\partial t} - \frac{y}{V^2}\frac{\partial V}{\partial t}\right)\right] \quad . \; . \; . \; . \; . \quad (25)$$

This is the dynamic equation for open channel flows in the furrow irrigation with a rectangular cross-sectional shape. However, for b being infinite, Eq. 25 becomes identical with Eq. 24, because

$$\frac{b + 2y}{b + 2y_n} \rightarrow 1$$

$$\frac{b}{b + 2y_n} \rightarrow 1$$

Another interesting feature of Eq. 25 can be obtained by setting $b = 0$. In this case, Eq. 25 simply becomes*

$$\frac{\partial y}{\partial x} = \frac{S_o}{1 - \left(\frac{y_c}{y}\right)^3} \left[1 - \left(\frac{y_n}{y}\right)^2\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

in which all y, y_n, and y_c are still functions of x and t. In the case of steady flow, Eq. 26 remains the same form so that it becomes comparible with Eq. 27. For $y < y_n$

$$1 - \left(\frac{y_n}{y}\right)^2 > 1 - \left(\frac{y_n}{y}\right)^3$$

and for $y > y_n$

$$1 - \left(\frac{y_n}{y}\right)^2 < 1 - \left(\frac{y_n}{y}\right)^3$$

hence the distinct difference of characteristics of flow profiles between $b = \infty$ and $b = 0$ for steady flow can be depicted as shown in Fig. 4. The inspection of Fig. 4 reveals that either positive or negative value of dy/dx for $b = 0$ is greater than that for $b = \infty$. This shows the apparent effect of side wall on the flow profiles.

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* For a two-dimensional steady flow, the gradually-varied-flow equation for a negligible infiltration rate ($i = 0$) is deduced from Eq. 24 as

$$\frac{dy}{dx} = \frac{S_o}{1 - \left(\frac{y_c}{y}\right)^3} \left[1 - \left(\frac{y_n}{y}\right)^3\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

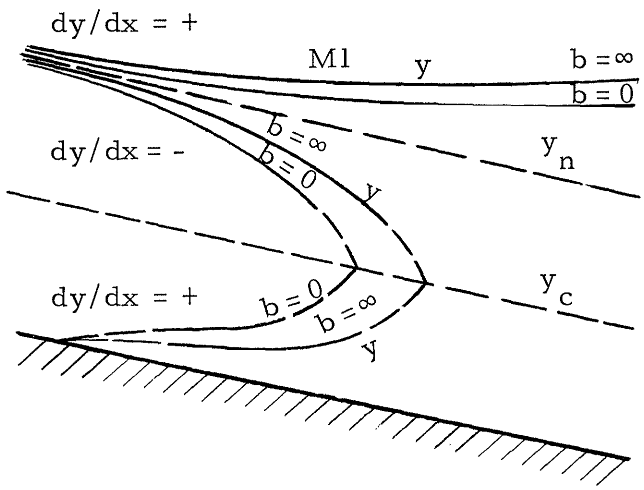

Fig. 4. Characteristics of flow profiles.

Example 2: Trapezoidal cross-sectional shape.

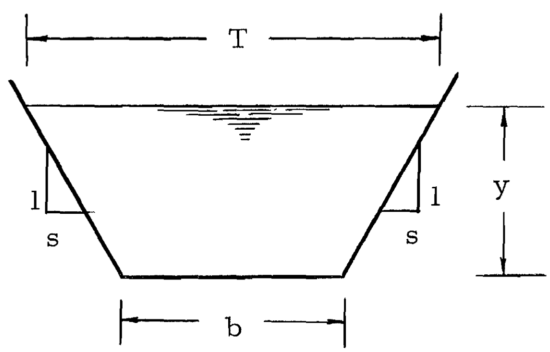

Fig. 5. Trapezoidal shape.

$$A = (b + sy)\,y$$

$$R = \frac{(b + sy)y}{b + 2y\sqrt{1 + s^2}}$$

$$T = b + 2\,sy$$

$$D = \frac{(b + sy)y}{b + 2sy}$$

$$Z = \frac{[(b + sy)y]^{1/2}}{\sqrt{b + 2sy}}$$

$$K = \frac{C(b + sy)^{3/2} y^{3/2}}{\sqrt{b + 2y\sqrt{1 + s^2}}}$$

in which s is the side slope of the cross-section. Hence one obtains

$$\left(\frac{Z_c}{Z}\right)^2 = \left(\frac{b + sy_c}{b + sy}\right)^3 \left(\frac{b + 2sy}{b + 2sy_c}\right)\left(\frac{y_c}{y}\right)^3$$

$$\left(\frac{K_n}{K}\right)^2 = \left(\frac{b + sy_n}{b + sy}\right)^3 \left(\frac{b + 2y\sqrt{1 + s^2}}{b + 2y_n\sqrt{1 + s^2}}\right) \left(\frac{y_n}{y}\right)^3$$

$$\left(\frac{K_n}{Z}\right)^2 = C^2 \left(\frac{b + sy_n}{b + sy}\right)^3 \left(\frac{b + 2sy}{b + 2y_n \sqrt{1 + s^2}}\right)\left(\frac{y_n}{y}\right)^3$$

which can be substituted into Eq. 23 to yield the dynamic equation for open channel flows in furrow irrigation with a trapezoidal cross-sectional shape. From these values, the same reasoning for $b = \infty$ will also give the dynamic equation for a wide open channel flow, because $b = \infty$

$$\left(\frac{Z_c}{Z}\right)^2 \rightarrow \left(\frac{y_c}{y}\right)^3$$

$$\left(\frac{K_n}{K}\right)^2 \rightarrow \left(\frac{y_n}{y}\right)^3$$

$$\left(\frac{K_n}{Z}\right)^2 \rightarrow C^2 \left(\frac{y_n}{y}\right)^3$$

However, for b = 0,

$$\left(\frac{Z_c}{Z}\right)^2 \rightarrow \left(\frac{y_c}{y}\right)^5$$

$$\left(\frac{K_n}{K}\right)^2 \rightarrow \left(\frac{y_n}{y}\right)^5$$

$$\left(\frac{K_n}{Z}\right)^2 \rightarrow C^2 \frac{s}{\sqrt{1+s^2}} \left(\frac{y_n}{y}\right)^5$$

which can be substituted into Eq. 23 to yield the dynamic equation for open channel flows in furrow irrigation with a triangular cross-sectional shape; that is

$$\frac{\partial y}{\partial x} = \frac{S_o}{1 - \left(\frac{y_c}{y}\right)^5} \left[1 - \left(\frac{y_n}{y}\right)^5 + \frac{2C^2}{g} \frac{s}{\sqrt{1+s^2}} \left(\frac{y_n}{y}\right)^5 \frac{\bar{i}}{V}\right.$$

$$\left. + \frac{C^2}{g} \frac{s}{\sqrt{1+s^2}} \left(\frac{y_n}{y}\right)^5 \left(\frac{1}{V}\frac{\partial y}{\partial t} - \frac{y}{V^2}\frac{\partial V}{\partial t}\right)\right] \quad . \quad . \quad . \quad (28)$$

An identical dynamic equation (Eq. 25) can also be obtained by setting $s = 0$ to all the expressions of K, K_n, Z, and Z_c in a trapezoidal cross-sectional shape. Similarly, the dynamic equations for other cross-sectional shapes can be attained by finding K, K_n, Z, and Z_c which, in turn, are substituted into Eq. 23.

CHAPTER 3

MATHEMATICAL MODELS IN THE SURFACE IRRIGATION

The hydrodynamic equations of motion in the surface irrigation were already formulated by the equation of momentum (Eq. 11) and the equation of continuity (Eq. 2) that will be reproduced here for clarity.

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + g\frac{\partial y}{\partial x} - \frac{V}{D}\bar{i} = g(S_o - S_f) \quad . \quad . \quad . \quad . \quad (11)$$

$$\frac{\partial y}{\partial t} + V\frac{\partial y}{\partial x} + D\frac{\partial V}{\partial x} = -\bar{i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

These two equations must be solved simultaneously for y and V with the prescribed boundary condition at the upstream end ($x = 0$) and the initial condition at $t = 0$. The customary boundary condition at the upstream end is $Q(0, t) = V(0, t)\,A(y) = V(0, t)\,A[y(0, t)] = Q_o$ being constant and the initial condition when $t = 0$ is $y(x, 0) = 0$ and $V(x, 0) = 0$. Knowing these boundary and initial values for V and y, Eqs. 11 and 2 for V and y can be solved. However, the inspection of Eqs. 11 and 2 reveals that obtaining the exact solution may not be so easy because these two equations are composed of nonlinear nonhomogeneous partial differential equations of the first order.

In two-dimensional flows, without the lateral outflow terms, $-\frac{V}{D}\bar{i}$ in Eq. 11 and $-\bar{i}$ in Eq. 2, and the frictional loss term, $-gS_f$, in Eq. 11, the exact solution is obtainable in terms of the complete elliptic integral of the second kind by means of a non-Newtonian reference frame as exhibited by Dressler (1958). As the moving forward wave front comprises always the same particle having velocity V with $y = 0$, the trajectory of this particle forms a characteristic curve there, provided these two terms are ignored. With initial conditions known at the origin ($x = 0$), the trajectory of this wave front can be obtained from the integration of the relation along the characteristic curve. This characteristic curve in the surface irrigation is named an "advance curve." However, the integration is quite difficult if two additional terms, which have been omitted in Dressler's analysis, are considered.

In the two-dimensional horizontal flow ($S_o = 0$), if the aforementioned two terms are again ignored, the well-known Ritter solution is obtained with the forward (positive) wave advancing at uniform velocity, V_w, which is equal to $3\sqrt{g y_{co}}$. Thus y_{co} is a constant, $V_w^2/9g$, for this particular case. The depth and the velocity at the site of a sudden release of the impounding water are a critical depth, y_{co}, and a critical velocity, v_{co}, respectively. Therefore, a singularity exists at the origin where $x = 0$ and $t = 0$. The discharge q, at the section is always a constant, $q_o = v_{co} y_{co} = (g y_{co}^3)^{1/2}$. However, this is not the case when the channel bed has the slope, S_o. In other words, the trajectory of the forward wave front for this special case of the two-dimensional flow can generally be shown to yield the relation

$$x = V_w t + \frac{1}{2} S_o g t^2 \quad , \ldots\ldots\ldots\ldots \quad (29)$$

which simply becomes, for $S_o = 0$

$$x = V_w t \quad \ldots\ldots\ldots\ldots\ldots \quad (30)$$

as a particular solution mentioned previously.

In the two-dimensional flow, Eqs. 11 and 2 can be simplified by expressing two-dimensional velocity and discharge by $v = V$ and $q = Q$ respectively.

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + g\frac{\partial y}{\partial x} - \frac{v}{y}\,\bar{i} = g\,(S_o - S_f) \quad \ldots\ldots \quad (31)$$

$$\frac{\partial y}{\partial t} + v\frac{\partial y}{\partial x} + y\frac{\partial v}{\partial x} = -\,\bar{i} \quad \ldots\ldots\ldots\ldots \quad (32)$$

from which Eq. 24 can be derived. Without the lateral outflow $(\bar{i} = 0)$, Eqs. 31 and 32 thus become

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + g\frac{\partial y}{\partial x} = g\,(S_o - S_f) \quad \ldots\ldots\ldots \quad (33)$$

$$\frac{\partial y}{\partial t} + v\frac{\partial y}{\partial x} + y\frac{\partial v}{\partial x} = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (34)$$

These equations form the basic dynamic equations for uniformly progressive wave flow in a prismatic channel. Dressler (1949) and Chow (1959) have used different approaches to develop the same profile equation for this wave flow. Let U be this uniform velocity. Then introducing a new coordinate variable, ξ,

$$\xi = x - U t \qquad (35)$$

one thus has*

$$\frac{\partial y}{\partial \xi} + \frac{(v - U)}{g} \frac{\partial v}{\partial \xi} = S_o - S_f \qquad (36)$$

$$(v - U) \frac{\partial y}{\partial \xi} + y \frac{\partial v}{\partial \xi} = 0 \qquad (37)$$

Solving this set of equations yields

$$\frac{\partial y}{\partial \xi} = \frac{S_o - S_f}{1 - \frac{(v - U)^2}{g y}} \qquad (38)$$

Furthermore, Eq. 37 indicates that

$$\frac{\partial}{\partial \xi} (v - U) y = 0 \qquad (39)$$

Hence $\quad (v - U) y = \text{constant} \qquad (40)$

Let us define again, using the Chezy formula

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* Since $\frac{\partial \xi}{\partial x} = 1$ and $\frac{\partial \xi}{\partial t} = - U$, two dependent variables, $v(x, t)$ and $y(x, t)$, can be transformed to two new dependent variables, $v(\xi)$ and $y(\xi)$, by letting $\frac{\partial y}{\partial x} = \frac{\partial y}{\partial \xi} \frac{\partial \xi}{\partial x} = \frac{\partial y}{\partial \xi}$, $\frac{\partial y}{\partial t} = \frac{\partial y}{\partial \xi} \frac{\partial \xi}{\partial t} = - U \frac{\partial y}{\partial \xi}$, $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial \xi} \frac{\partial \xi}{\partial x} = \frac{\partial v}{\partial \xi}$, and $\frac{\partial v}{\partial t} = \frac{\partial v}{\partial \xi} \frac{\partial \xi}{\partial t} = - U \frac{\partial v}{\partial \xi}$.

$$q(\xi) = v(\xi)\, y(\xi)$$

$$S_f = \frac{q^2(\xi)}{C^2 y^3(\xi)}$$

$$S_o = \frac{q^2(\xi)}{C^2 y_n^3(\xi)}$$

$$\frac{S_f}{S_o} = \left[\frac{y_n(\xi)}{y(\xi)}\right]^3$$

$$q^* = [v(\xi) - U]\; y(\xi) \qquad (q^* = \text{constant}) \quad . \quad . \quad . \quad . \qquad (41)$$

$$g y_c^*(\xi) = [v_c^*(\xi) - U]^2$$

Since

$$[v(\xi) - U]\; y(\xi) = [v_c^*(\xi) - U]\; y_c^*(\xi) = q^*$$

Hence

$$q^{*2} = [v(\xi) - U]^2\, y^2(\xi) = [v_c^*(\xi) - U]^2\, y_c^{*2}(\xi) = g y_c^{*3}(\xi)$$

in which $y_c^*(\xi)$ can be shown constant because

$$[v_c^*(\xi) - U]^2 = \frac{q^{*2}}{y_c^{*2}(\xi)} = g y_c^*(\xi)$$

Therefore, $y_c^*(\xi) = \left(\frac{q^{*2}}{g}\right)^{1/3}$ = constant because q^* = constant.

Denote $y_c^*(\xi) = y_c^*$. Using all of these definitions and relations yields the profile equation from Eq. 38.

$$\frac{\partial y(\xi)}{\partial \xi} = \frac{S_o}{1 - \left[\frac{y_c^*}{y(\xi)}\right]^3} \left\{ 1 - \left[\frac{y_n(\xi)}{y(\xi)}\right]^3 \right\} \quad . \; . \; . \; . \; . \quad (42)$$

As the relation, Eq. 41,

$$(v(\xi) - U)\, y(\xi) = q^*$$

holds for any value of $y(\xi)$, the constant q^* must be a zero because $y(\xi)$ has a zero value for this type of flow advancing on the dry bed. Hence

$$v(\xi)\, y(\xi) = U\, y(\xi)$$

Consequently,

$$q(\xi) = U\, y(\xi)$$

or $$v(\xi) = U$$

Knowing the foregoing relation, one can easily derive

$$\frac{\partial y}{\partial \xi} = S_o - S_f \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (43)$$

from Eq. 36; however, one may alternatively obtain Eq. 43 from Eq. 42 by using the following relations. Since $q^* = 0$,

$$y_c^* = 0$$

and

$$y_n^3(\xi) = \frac{U^2 y^2(\xi)}{S_o C^2}$$

Thus $$\left[\frac{y_n(\xi)}{y(\xi)}\right]^3 = \frac{U^2}{S_o C^2 y(\xi)}$$

Substituting all those relations into Eq. 42, it becomes

$$\frac{\partial y}{\partial \xi} = S_o \left[1 - \frac{U^2}{S_o C^2 y(\xi)} \right] \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (44)$$

which is equivalent to Eq. 43. Therefore, the integration of Eq. 44 yields

$$\xi = \frac{y}{S_o} + \frac{U^2}{S_o^2 C^2} \ln \left(y - \frac{U^2}{S_o C^2}\right) + A_1 \quad . \; . \; . \; . \; . \quad (45)$$

in which A_1 is an integration constant. Choose the tip of the wave front, where $\xi = 0$ and $y = 0$, as the origin, the value of A_1 can be evaluated

$$A_1 = - \frac{U^2}{S_o^2 C^2} \ln \left(- \frac{U^2}{S_o C^2} \right)$$

substituted into Eq. 45, thus

$$\xi = \frac{y}{S_o} + \frac{U^2}{S_o^2 C^2} \ln \left(1 - \frac{S_o C^2 y}{U^2} \right) \quad . \; . \; . \; . \; . \; . \; . \quad (46)$$

This is the equation of the free surface profile for a uniformly progressive wave flow on the dry surface with the uniform advancing velocity, U. Since $\xi = x - Ut$, Eq. 46 can alternatively be written as

$$x = \frac{y}{S_o} + \frac{U^2}{S_o^2 C^2} \ln \left(1 - \frac{S_o C^2 y}{U^2} \right) + Ut \quad . \; . \; . \; . \quad (47)$$

Equations 46 and 47 are schematically plotted in Fig. 6. Furthermore, when $U = v_n$, Eq. 47 can be simplified as

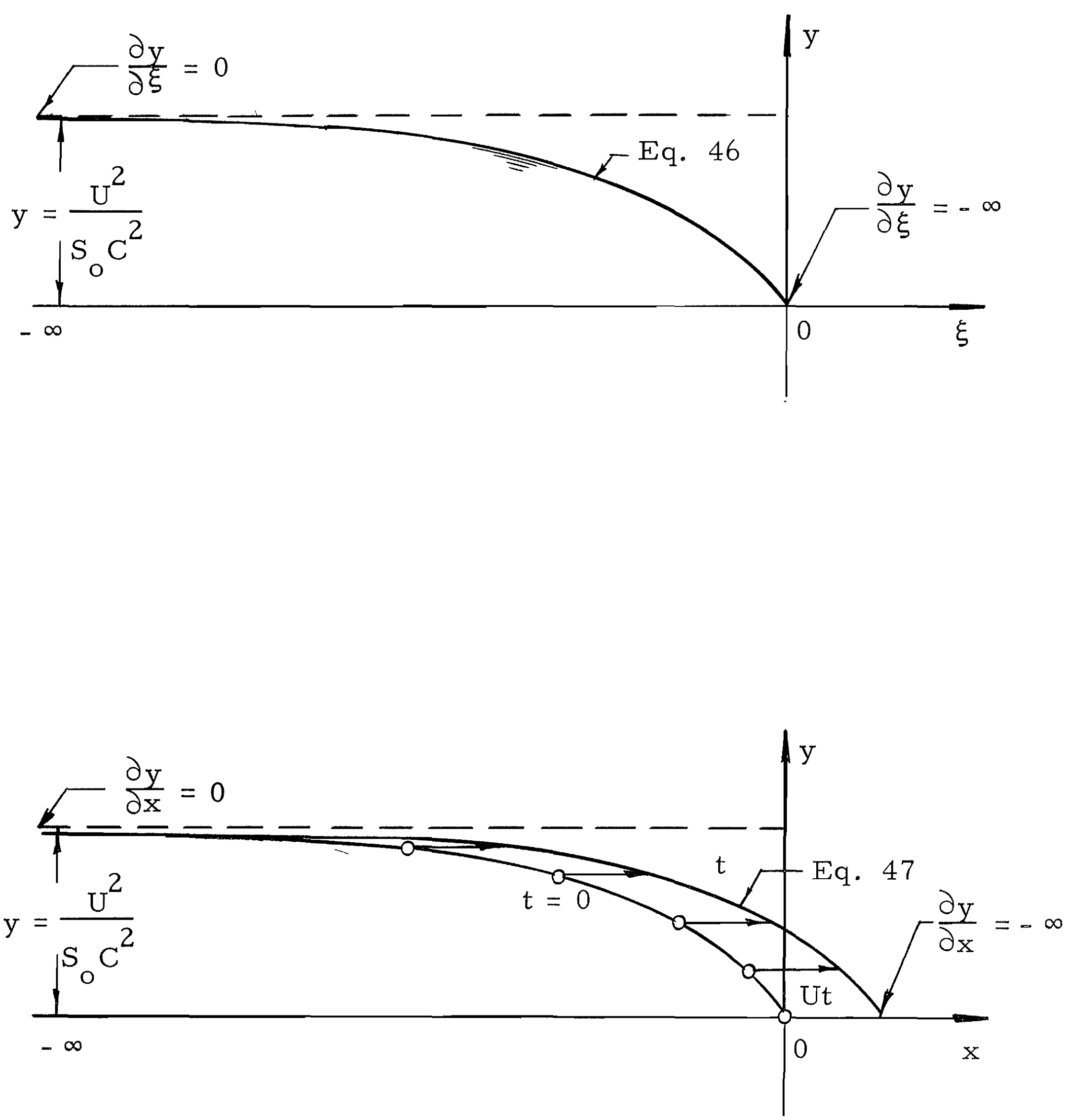


Fig. 6. Uniformly progressive wave flows on the dry surface.

$$x = \frac{y_n}{S_o}\left[\frac{y}{y_n} + \ln\left(1 - \frac{y}{y_n}\right)\right] + v_n t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (48)$$

in which $y_n = \left(\frac{q^2}{C^2 S_o}\right)^{1/3}$, the normal depth of the channel located at the negative infinite ξ or x. The flow with the free surface-profile equation (Eq. 48) having the tip of the waterfront at the origin ($x = 0$) starts to advance uniformly in the direction of the positive x-axis with the uniform velocity equal to the normal velocity, $v_n = C\sqrt{S_o y_n}$. Although Eq. 47 or Eq. 48 satisfies the set of the equations of motion (Eqs. 33 and 34), it obviously does not satisfy the boundary conditions for the case of flow in the surface irrigation. The boundary condition at the upstream end ($x = 0$) in the border irrigation must be $q(0, t) = q_o$, in which q_o is a constant flow rate. Since we already showed that

$$q^* = 0$$

knowing the relation

$$q(x - Ut) = Uy(x - Ut)$$

yields at $x = 0$

$$q(0 - Ut) = Uy(0 - Ut) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (49)$$

The inspection of Fig. 6 reveals that $y(0 - Ut)$ is changing with respect to t while the wave front is progressing with a uniform velocity U in the direction of the positive x-axis. Correspondingly, from Eq. 49, one can readily see that $q(0 - Ut)$ is changing with t also. Therefore, the solution (Eq. 47 or Eq. 48) for the uniformly progressive flow that satisfies the set of the equations of motions (Eqs. 33 and 34) but not the boundary condition, $q(0, t) = q_o$, cannot be used as the solution applicable to the case of flow in the border irrigation. The experimental result from Tinney and Bassett (1961) already verified this vital point. However, in the past, approaches using the simple volume-balance relationship by most of the researchers in the surface irrigation result in the distance of advance being linearly related with the time of advance provided there is no lateral outflow $(\bar{i} = 0)$. This improper result may be caused by the injudicious assumption (such as flow depth $y(x, t)$ = constant) used to make the equation of continuity (Eq. 34) integrable, as indicated by the writer (1966). Since the mathematical model of the surface irrigation including frictional-slope and lateral-outflow terms is extremely difficult to solve, thus far the exact solution to this type of boundary-value problem is not available. In what follows, the method of characteristics will be studied, whence the possible solution on the wave front trajectory can be numerically obtained and qualitatively analyzed.

CHAPTER 4

THEORY OF CHARACTERISTICS

Continuing to confine to the cases of two-dimensional overland flow and introducing for convenience celerity, $c = \sqrt{gy}$, at once the momentum and continuity equations (Eqs. 31 and 32) can be expressed as

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + 2c\frac{\partial c}{\partial x} - \frac{g v \bar{i}}{c^2} = g(S_o - S_f) \quad . \quad . \quad . \quad . \quad (50)$$

$$2\frac{\partial c}{\partial t} + 2v\frac{\partial c}{\partial x} + c\frac{\partial v}{\partial x} = -\frac{g\bar{i}}{c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

in which $S_f = \frac{g}{c^2}\left(\frac{v}{c}\right)^2$ is not a constant. These equations are next added, then subtracted, to obtain the following equivalent pair of differential equations:

$$2\left[(c+v)\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right]c + \left[(c+v)\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right]v$$

$$+ \frac{g\bar{i}}{c}\left(1 - \frac{v}{c}\right) - gS_o + gS_f = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

$$-2\left[(-c+v)\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right]c + \left[(-c+v)\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right]v$$

$$- \frac{g\bar{i}}{c}\left(1 + \frac{v}{c}\right) - gS_o + gS_f = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

The inspection of Eqs. 52 and 53 reveals that the functions c and v in Eq. 52 are both subject to the differentiation operator $\left[(c+v)\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right]$

along curves satisfying the differential equation $\frac{dx}{dt} = c + v$ and similarly, the functions c and v in Eq. 53 are both subject to the differentiation operator $\left[(-c + v)\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right]$ along curves satisfying the differential equation $\frac{dx}{dt} = -c + v$. Eqs. 52 and 53 can be written

$$\left[(v + c)\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right](v + 2c) + \frac{g\bar{i}}{c}\left(1 - \frac{v}{c}\right) - gS_o + gS_f = 0 \qquad \text{(54)}$$

$$\left[(v - c)\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right](v - 2c) - \frac{g\bar{i}}{c}\left(1 + \frac{v}{c}\right) - gS_o + gS_f = 0 \qquad \text{(55)}$$

There are two sets of curves, C^+ and C^-, called characteristics, which are the solution curves of the ordinary differential equations

$$C^+ : \frac{dx}{dt} = v + c \qquad \text{(56)}$$

$$C^- : \frac{dx}{dt} = v - c \qquad \text{(57)}$$

Therefore, there exist the relations from Eq. 54, along a characteristic curve C^+

$$\frac{d}{dt}(v + 2c) + \frac{g\bar{i}}{c}\left(1 - \frac{v}{c}\right) - gS_o + gS_f = 0 \qquad \text{(58)}$$

and similarly, from Eq. 55, along a characteristics curve C^-

$$\frac{d}{dt}(v - 2c) - \frac{g\bar{i}}{c}\left(1 + \frac{v}{c}\right) - gS_o + gS_f = 0 \qquad \text{(59)}$$

The characteristics for this solution in the region of motion are still unknown. The characteristics pattern in the x, t-plane for these relations is rather complicated. Two families of characteristics determined by Eqs. 56 and 57 are no longer distinct at the tip of the wave front because $c = \sqrt{gy} = 0$. As defined, the trajectory of the wave front is the locus where $c = 0$. Hence Eqs. 56 and 57 are both satisifed by the wave front curve, both characteristics directions C^+ and C^- coincide with the direction of the wave front curve. Therefore, the wave front curve cannot be a characteristic curve itself because neither Eq. 58 nor Eq. 59 can hold on this curve. Evidently, it must be an envelope for C^+ and C^- curves. Without considering two terms: the lateral outflow, $\bar{i}$, and the slope of channel, S_o, the foregoing fact has been discussed by Dressler (1952).

If the frictional slope, S_f, and the lateral outflow, $\bar{i}$, are not considered, it can be readily seen from Eqs. 56, 57, 58, and 59 that the relations can be integrated for the wave front tip.

$$\frac{dx}{dt} = v$$

$$dv = g S_o \, dt$$

Hence

$$v = g S_o t + A_1$$

$$\frac{dx}{dt} = g S_o t + A_1$$

$$x = A_1 t + \frac{1}{2} g S_o t^2$$

in which $A_1 = V_w = 3\sqrt{g\,y_{co}}$ prescribed previously as an initial condition of the boundary-value problem because of the implication that particles at the tip of the wave front always remain at the tip. Furthermore, if the channel slope, S_o, is ignored in addition to S_f and $\bar{i}$, the trajectory equation of the wave front simply becomes

$$x = 3\sqrt{g\,y_{co}\,t}$$

For this relation, the Ritter solution is

$$v = \frac{2}{3}\left(\frac{x}{t} + \frac{3}{2}\sqrt{g\,y_{co}}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

$$c = \frac{1}{3}\left(3\sqrt{g\,y_{co}} - \frac{x}{t}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

in which $y_{co} = (q_o^2/g)^{1/3}$ and q_o is a given flow rate at $x = 0$. The Ritter solution in an exact solution which is obtainable from the simplified forms of Eqs. 50 and 51 such as

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + 2c\frac{\partial c}{\partial x} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

$$2\frac{\partial c}{\partial t} + 2v\frac{\partial c}{\partial x} + c\frac{\partial v}{\partial x} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

In another type of unsteady nonlinear wave-problem, Dressler (1958) has successfully obtained the exact solution for the water wave in a sloping channel produced by sudden release of the triangular wedge of water initially at rest behind a vertical wall. The corresponding equations of motion for this type of problem are

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + 2c\frac{\partial c}{\partial x} = g\,S_o \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

$$2\frac{\partial c}{\partial t} + 2v\frac{\partial c}{\partial x} + c\frac{\partial v}{\partial x} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

By means of a non-Newtonian reference frame, every such wave problem for a sloping channel can be replaced by an associated problem for a horizontal channel. Introducing a new coordinate variable, ξ, moving with a prescribed acceleration, $\frac{1}{2}\,g\,S_o\,t^2$; namely

$$\xi = x - \frac{1}{2}\,g\,S_o\,t^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

and differentiating with respect to t

$$\frac{d\xi}{dt} = \frac{dx}{dt} - g\,S_o\,t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

with the given notations that $w = \frac{d\xi}{dt}$ and $v = \frac{dx}{dt}$, thus one defines from Eq. 67 that

$$w = v - g\,S_o\,t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

Then, transforming Eqs. 64 and 65 with t and c remaining unchanged yields*

---

* $\frac{\partial \xi}{\partial x} = 1, \quad \frac{\partial \xi}{\partial t} = -\,g\,S_o\,t$

$$\frac{\partial v}{\partial t} = \frac{\partial\,(w + g\,S_o\,t)}{\partial t} = \frac{\partial\,(w + g\,S_o\,t)}{\partial \xi}\,\frac{\partial \xi}{\partial t} + \frac{\partial\,(w + g\,S_o\,t)}{\partial t}$$

$$= -\,g\,S_o\,t\,\frac{\partial w}{\partial \xi} + \frac{\partial w}{\partial t} + g\,S_o \qquad \text{(cont.)}$$

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial \xi} + 2c\frac{\partial c}{\partial \xi} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

$$2\frac{\partial c}{\partial t} + 2w\frac{\partial c}{\partial \xi} + c\frac{\partial w}{\partial \xi} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

Eqs. 69 and 70 have exactly the same forms as Eqs. 62 and 63, for which the Ritter solutions are still applicable; therefore

$$w = \frac{2}{3}\left(\frac{\xi}{t} + \frac{3}{2}\sqrt{g\,y_{co}}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

$$c = \frac{1}{3}\left(3\sqrt{g\,y_{co}} - \frac{\xi}{t}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

Consequently, substituting Eqs. 66 and 68 for Eqs. 71 and 72 yield the exact solution of Eqs. 64 and 65:

$$v = \frac{2}{3}\left(\frac{x}{t} + g\,S_o\,t + \frac{3}{2}\sqrt{g\,y_{co}}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

$$c = \frac{1}{3}\left(3\sqrt{g\,y_{co}} - \frac{x}{t} + \frac{1}{2}\,g\,S_o\,t\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

in which y_{co} is still defined such a constant as $(q_o^2/g)^{1/3}$. The trajectory equation of the wave front can be obtained by putting $c = 0$ in Eq. 74, which is exactly the form of Eq. 29.

$$* \quad \frac{\partial v}{\partial x} = \frac{\partial (w + g\,S_o\,t)}{\partial x} = \frac{\partial w}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{\partial w}{\partial \xi}$$

$$\frac{\partial c}{\partial t} = \frac{\partial c}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial c}{\partial t} = -\,g\,S_o\,t\,\frac{\partial c}{\partial \xi} + \frac{\partial c}{\partial t}$$

$$\frac{\partial c}{\partial x} = \frac{\partial c}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{\partial c}{\partial \xi}$$

$$x = 3\sqrt{g\,y_{co}}\,t + \frac{1}{2}\,g\,S_o\,t^2 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (29)$$

The characteristic curves, C^+ and C^-, can thus be obtained from Eqs. 73 and 74.

$$C^+ \; : \; \frac{dx}{dt} = \frac{1}{3}\left(\frac{x}{t} + \frac{5}{2}\,g\,S_o\,t + 6\sqrt{g\,y_{co}}\right) \quad . \; . \; . \; . \; . \quad (75)$$

$$C^- \; : \; \frac{dx}{dt} = \frac{x}{t} + \frac{1}{2}\,g\,S_o\,t. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (76)$$

At the tip of wave front where Eq. 29 is satisfied, the characteristic curves for this trajectory can be obtained by substituting Eq. 29 into Eqs. 75 and 76.

$$C^+ \text{ for } c = 0 \; : \; \frac{dx}{dt} = 3\sqrt{g\,y_{co}} + g\,S_o\,t \quad . \; . \; . \; . \; . \; . \quad (77)$$

$$C^- \text{ for } c = 0 \; : \; \frac{dx}{dt} = 3\sqrt{g\,y_{co}} + g\,S_o\,t \quad . \; . \; . \; . \; . \; . \quad (78)$$

Eqs. 77 and 78 are exactly the same form as the one obtained by differentiating Eq. 29 with respect to t. Consequently, it has been shown that the trajectory of the wave front is always the characteristics curves, C^+ and C^-.

Integrating* Eqs. 75 and 76 yields the characteristic equations for C^+ and C^- respectively

$$C^+ \; : \; x = 3\sqrt{g\,y_{co}}\,t + \frac{1}{2}\,g\,S_o\,t^2 + A_1\,t^{1/3} \quad . \; . \; . \; . \; . \quad (79)$$

---

* Since Eqs. 75 and 76 are all the forms of the linear differential equation of first order, one can easily find out the integrating factors for Eqs. 75 and 76 respectively. For Eq. 75, the integrating factor is $t^{-1/3}$ and, for Eq. 76, t^{-1}.

$$C^- \; : \; x = B_1 t + \frac{1}{2} g S_o t^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

in which A_1 and B_1 are all parameters characterizing the families of the characteristic curves C^+ and C^-. If S_o is absent, Eqs. 79 and 80 simply become respectively

$$C^+ \; : \; x = 3\sqrt{g y_{co}}\; t + A_1 t^{1/3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

$$C^- \; : \; x = B_1 t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82)$$

These characteristics C^+ and C^- in the x, t-plane representing the paths of "water waves" have a basic property (Courant and Friedricks, 1948) of "centered simple wave": the characteristics of one kind, C^-, are straight lines in an x, t-plane and pass through the origin $0 : x = 0$, $t = 0$, see Fig. 7. At the origin 0, the quantities v and c as functions of x and t are discontinuous, however this discontinuity is immediately smoothed out and resolved into continuous flow in the subsequent motion.

The cross-characteristics C^+, Eq. 81, for large t, may have the asymptotic representation

$$x \sim 3\sqrt{g y_{co}}\; t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

When $A_1 = 0$ and $B_1 = 3\sqrt{g y_{co}}$, the two characteristics directions C^+ and C^- coincide along the trajectory of the wave front that thus possesses the double characteristics. Differentiating Eq. 81 with respect to t twice yields, for C^+,

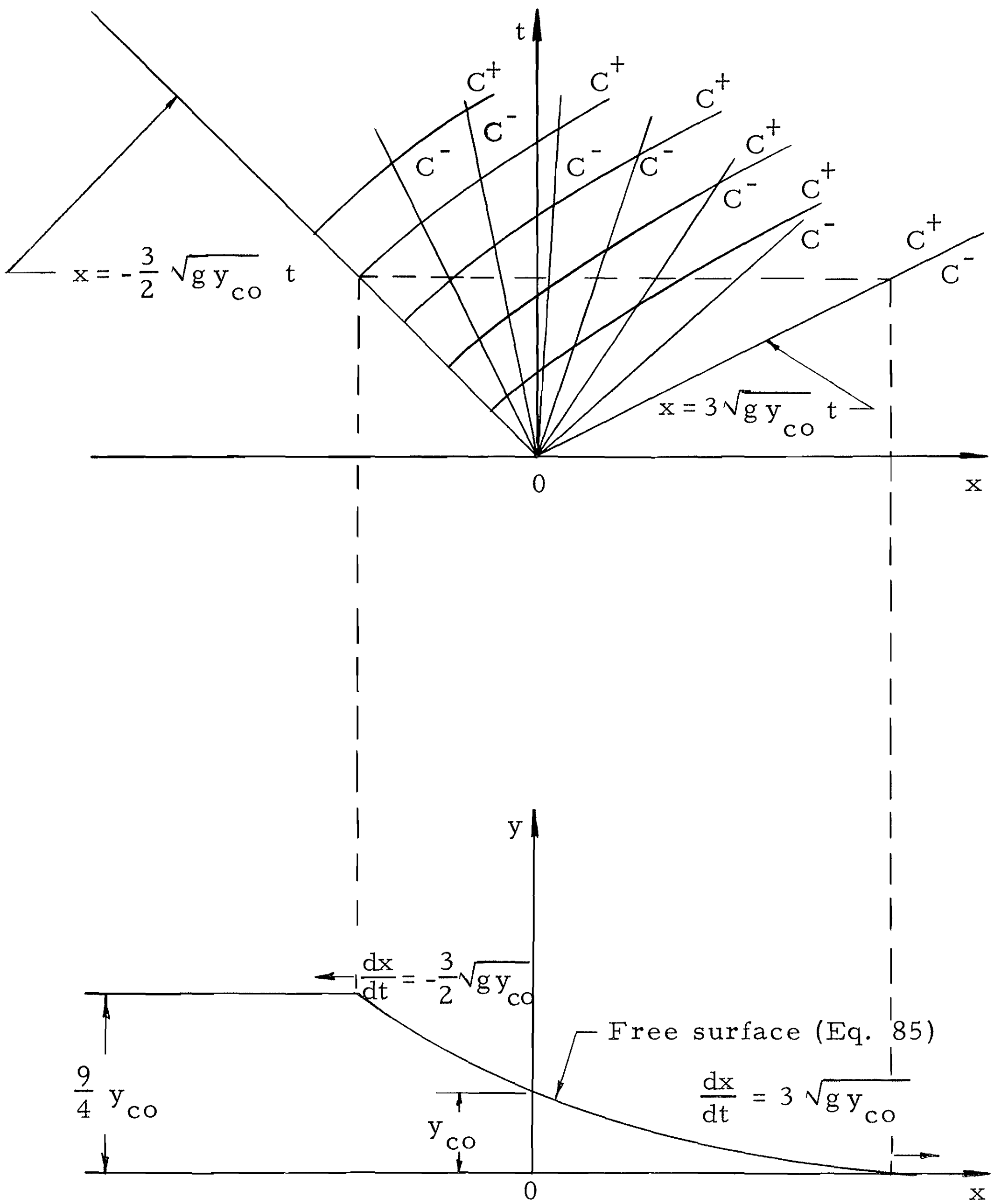


Fig. 7. Pattern of characteristics in a centered simple wave moving down the dry bed of a stream.

$$\frac{d^2x}{dt^2} = \frac{1}{9t} \left(- \frac{2x}{t} + 6\sqrt{g y_{co}} \right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84)$$

which is always positive for $x < 3\sqrt{g y_{co}}\, t$, therefore the cross-characteristics C^+ in the centered simple wave can be depicted as shown in Fig. 7. At a certain time t after the wave front forwarding a distance $x = 3\sqrt{g y_{co}}\, t$, the Ritter solutions, Eqs. 60 and 61, give the v and c distributions that are linear along the x-axis up to the tip of the wave front. The free surface profile becomes a parabolic curve, as indicated in Eq. 61.

$$y = \frac{1}{9g} \left(3\sqrt{g y_{co}} - \frac{x}{t} \right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

Those v, c, and y-distributions are plotted respectively in Figs. 7 and 8.

At $x = 0$, one observes from Eqs. 60 and 61 that v and c are both independent of t, and hence that the volume of water crossing this location $(x = 0)$ per unit time, q_o, is independent of time. In fact, at this point for all time t

$$v = \sqrt{g y_{co}}$$

and

$$y = y_{co}$$

whence

$$q_o = (g y_{co}^3)^{1/2} = \text{constant}$$

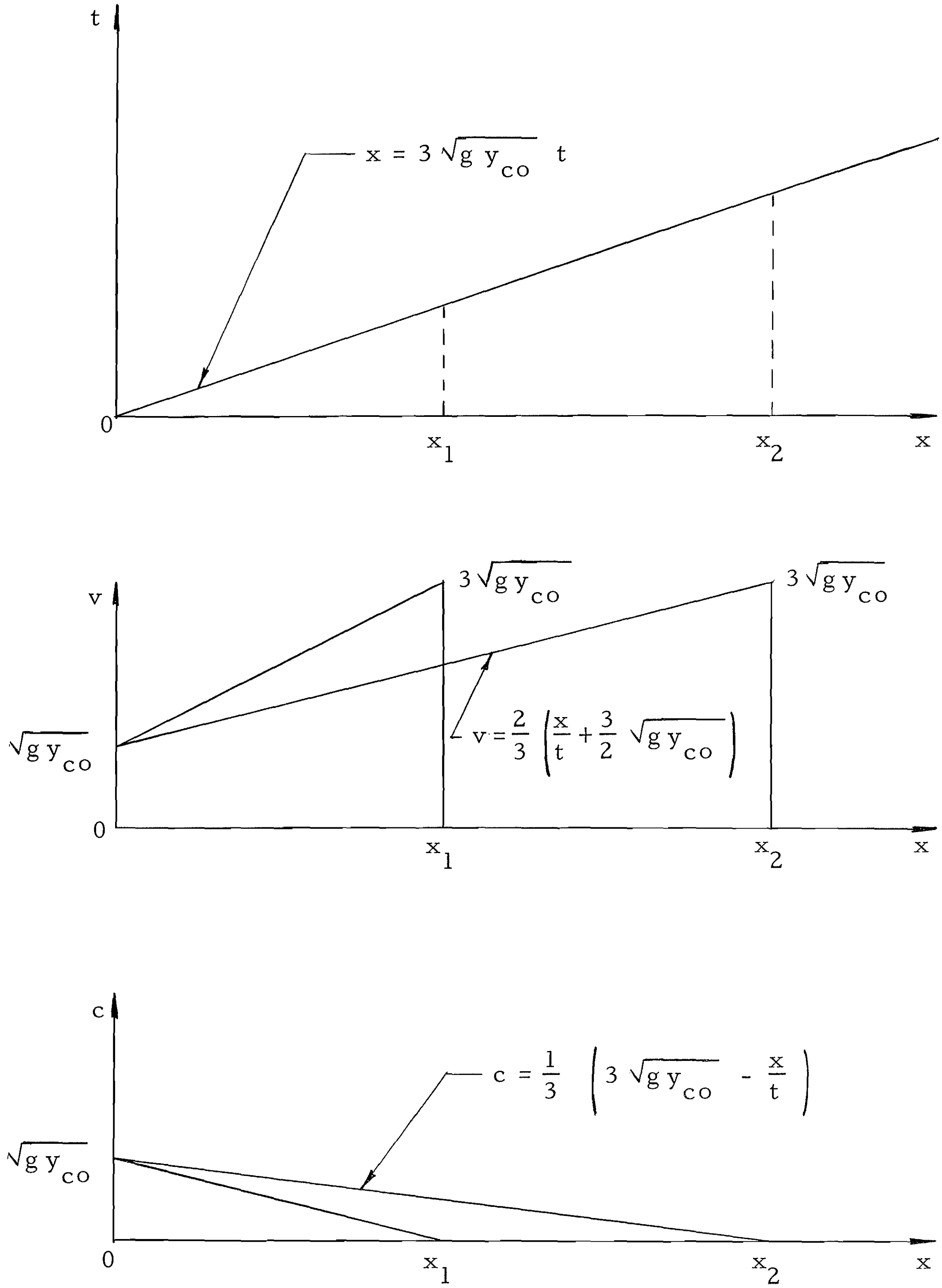


Fig. 8. The v and c distributions in a centered simple wave.

satisfying the boundary condition of the mathematical model formulated previously for the surface irrigation. This may be the only case in which the method of characteristics can yield the exact solutions satisfying both the equations of motion and the boundary condition. However, the foregoing results, representing only the simplified version of the general problem, is actually not complete in describing the whole physical picture of flow conditions because the channel slope, S_o, the frictional slope, S_f, and the infiltration rate, $\bar{i}$, are all not considered.

In addition to those factors considered in constructing the simplest model, if the channel slope, S_o, enters the equations of motion such as Eqs. 64 and 65 with the same boundary condition prescribed at $x = 0$, the method of characteristics for a transformed simple wave results in the solutions (Eqs. 73 and 74) that satisfy the equations of motion but not the boundary condition. Since v and c are no longer independent of t at $x = 0$ for all time t, (as indicated in Eqs. 73 and 74)

$$v = \frac{2}{3}\left(g S_o t + \frac{3}{2}\sqrt{g y_{co}}\right)$$

$$y = \frac{1}{9g}\left(\frac{1}{2} g S_o t + 3\sqrt{g y_{co}}\right)^2$$

at $x = 0$ and the volume of water crossing this point per unit time, q, is also dependent of time. That is

$$q = \frac{2}{27g}\left(g S_o t + \frac{3}{2}\sqrt{g y_{co}}\right)\left(\frac{1}{2} g S_o t + 3\sqrt{g y_{co}}\right)^2 \neq \text{const.}$$

for all $t > 0$ and $q = (g y_{co}^3)^{1/2} = q_o$ for $t = 0$.

Although the solutions obtained for another type of problem (i. e., a dam-breaking problem) in this particular case ($S_o \neq 0$) cannot be used to represent the solutions in the problem of surface irrigation where the different boundary condition is imposed at $x = 0$, it is still interesting and useful to make a schematic diagram of the characteristics pattern for this problem (see Fig. 9) and study the possible way to obtain the approximate solutions by the method of finite difference.

A rather complicated pattern of characteristics is expected when additional terms, S_f and $\bar{i}$, are considered. The trajectory of the forward wave front can no longer belong to both the C^+ and C^- families. Instead, it must be an envelope of C^+ and C^- characteristics because the reflected C^- curves caused by the effects of these two additional factors cannot intersect each other. This argument was given by Dressler (1952) for a different application. Through the perturbation method he obtained the approximate solutions for a dam-breaking problem and study the effect of the frictional resistance term on the flow characteristics on a dry horizontal bed having no lateral outflow.

The boundary condition in our problem which is different from that used by Dressler may lead to a complicated form of boundary condition at $x = 0$ for another system of partial differential equations in terms of correction functions through the perturbation procedure.

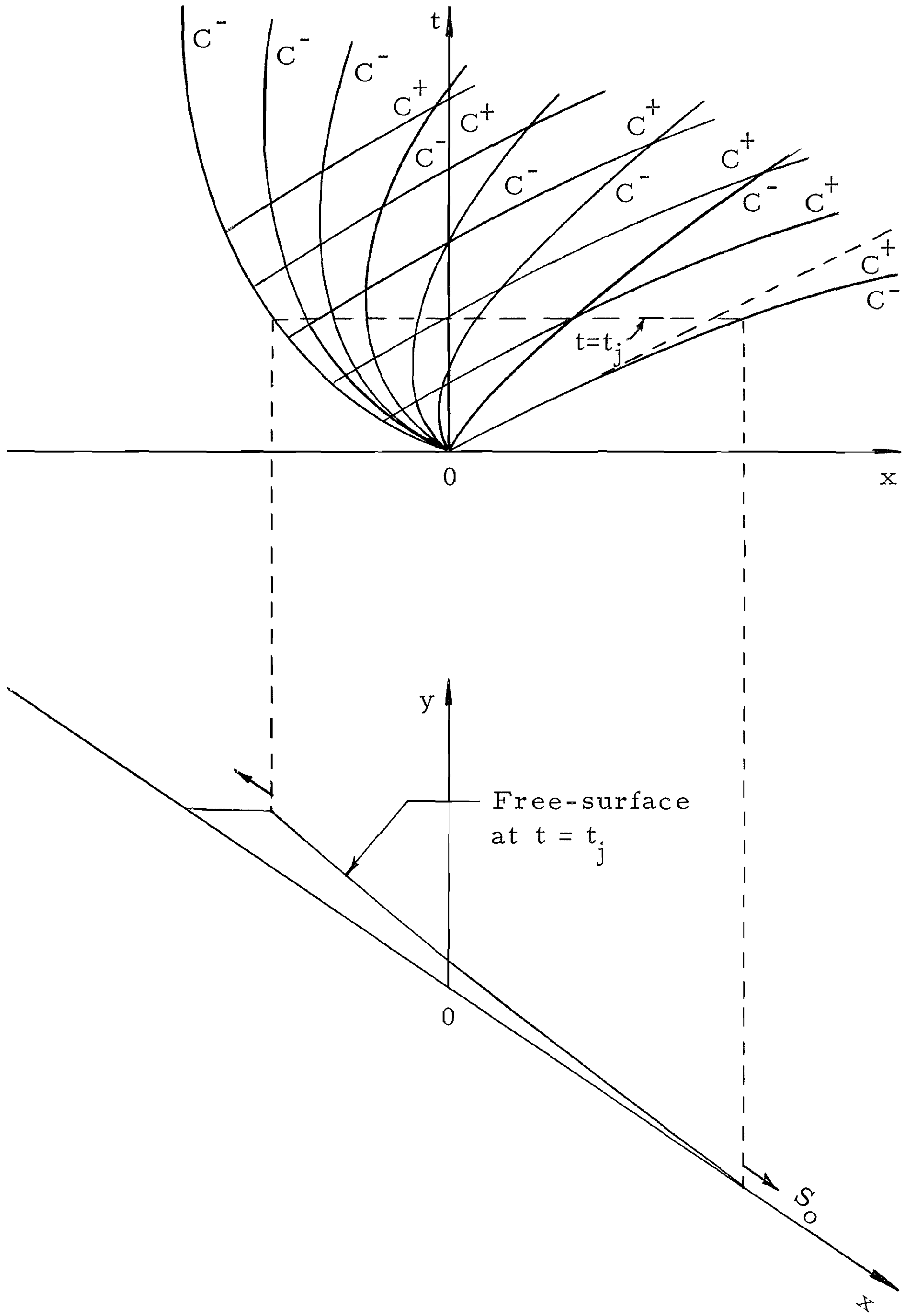


Fig. 9. Pattern of characteristics for $S_f = 0$ and $\bar{i} = 0$.
(The solutions obtained satisfy the equations of motion but not the boundary condition at $x = 0$.)

However, the similar initial conditions for these correction functions must be derived for the singularity at the origin. In general, as can easily be seen from the proceeding arguments, the slope, resistance, and infiltration effects on the main solution can be ignored at the initial instant. The necessary initial conditions for $v(x, t)$ and $c(x, t)$

$$\left.\begin{aligned} v(x, 0) &= v_o \\ c(x, 0) &= c_o \end{aligned}\right\} \text{ at } x = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (86)$$

in which $v_o = c_o = \sqrt{g y_{co}}$ must be prescribed for the singularity at the origin ($x = 0$) in order that one can successfully obtain numerical solutions through the finite-difference method.

In what follows, the method of finite differences based on the theory of characteristics that can be applied to all the cases possibly encountered in the problems of surface irrigation will be studied. Moreover, it can readily be verified from the finite-difference method that the solutions obtained from the prescribed initial condition at the singularity are compatible with the exact solutions obtained from the method of characteristics along the trajectory of the wave front for the simplest case of "reducible" hyperbolic flow equations discussed previously.

CHAPTER 5

FINITE-DIFFERENCE METHOD

As already understood from the previous discussions, a solution of the original dynamic equations, Eqs. 50 and 51, could be shown to be uniquely determined when appropriate initial conditions for $t = 0$ are prescribed. It follows that a solution of the new system of relations, Eqs. 56, 57, 58, and 59, is also uniquely determined when initial conditions are prescribed since the two systems of equations are equivalent.

The formulation of our problems in terms of the characteristics form is quite useful in studying properties of the solutions and also in studying the appropriateness of various boundary and initial conditions. In the problem of surface irrigation, the two families of characteristics determined by Eqs. 56 and 57 are not distinct along the trajectory of the wave front because of the fact that the water surface touches the bottom and hence $c = \sqrt{gy} = 0$ over there. The boundary condition prescribed at $x = 0$ (along the t-axis) is particularly unusual because neither v nor c is imposed on the boundary except that one can only formulate the relation

$$v c^2 = g q_o \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (87)$$

at the fixed points along the t-axis. In the case of subcritical flow, one boundary condition, either v or c, along the t-axis will in general suffice to lead to a unique determination of the other. This is not true for the case of supercritical flow, in which both v and c along the t-axis must be prescribed so that the values of v and c at any point within the flow region for $x > 0$ can be determined. The physical interpretation of the latter case is that the value of v is always greater than $c = \sqrt{gy}$ and this in turn implies that $v + c$ and $v - c$, which fix the slopes of the characteristics C^+ and C^-, are all positive in sign. Therefore, both v and c must be prescribed along the t-axis with the help of Eq. 87. This is the most difficult part of the technique of utilizing the finite-difference method to obtain the numerical solution for this typical problem.

In addition to those difficulties encountered in the formulation of finite-difference scheme in the problem of surface irrigation, it is already clear that the inherent property of a "centered simple wave" has a singularity at the origin ($x = 0$), where the quantities of v and c as functions of x and t are discontinuous, but this discontinuity is immediately smoothed out in the subsequent motion. To overcome this inappropriate singularity at the initial state, the proper ones such as Eq. 86 at $t = 0$ should be imposed at $x = 0$. These initial values (Eq. 86) are in fact the necessary ones for this typical finite-difference scheme being able to start with and compute the subsequent values of v and c at the proper grid-points in the x, t-plane. Probably, in this typical problem, the finite-difference scheme with the fixed-time intervals seems the only way to get rid of these difficulties. In the following, the pertinent method and technique are discussed very much in detail with the case of simple wave.

Let us begin by describing briefly a method of determining the characteristics and thus the solution of a simple-wave problem by a method of successive approximation. The characteristics and characteristic equations for this particular case ($S_o = 0$, $S_f = 0$, $\bar{i} = 0$) are:

$$C^+ : \frac{dx}{dt} = v + c \quad \ldots \ldots \ldots \ldots \quad (56)$$

$$C^- : \frac{dx}{dt} = v - c \quad \ldots \ldots \ldots \ldots \quad (57)$$

and

$$\text{along } C^+ : \frac{d}{dt}(v + 2c) = 0 \quad \ldots \ldots \ldots \quad (88)$$

$$\text{along } C^- : \frac{d}{dt}(v - 2c) = 0 \quad \ldots \ldots \ldots \quad (89)$$

and from Eqs. 88 and 89, one has the relations

$$\text{along } C^+ : v + 2c = k_1 = \text{const.} \quad \ldots \ldots \ldots \quad (90)$$

$$\text{along } C^- : v - 2c = k_2 = \text{const.} \quad \ldots \ldots \ldots \quad (91)$$

in which the constants k_1 and k_2 will be different on different curves in general.

Consider a series of points 0, 1, 3, 6, 10, ... on the t-axis (see Fig. 10) a small time, δt, apart. A sudden release of water at $x = 0$ require the necessary initial conditions (Eq. 86)

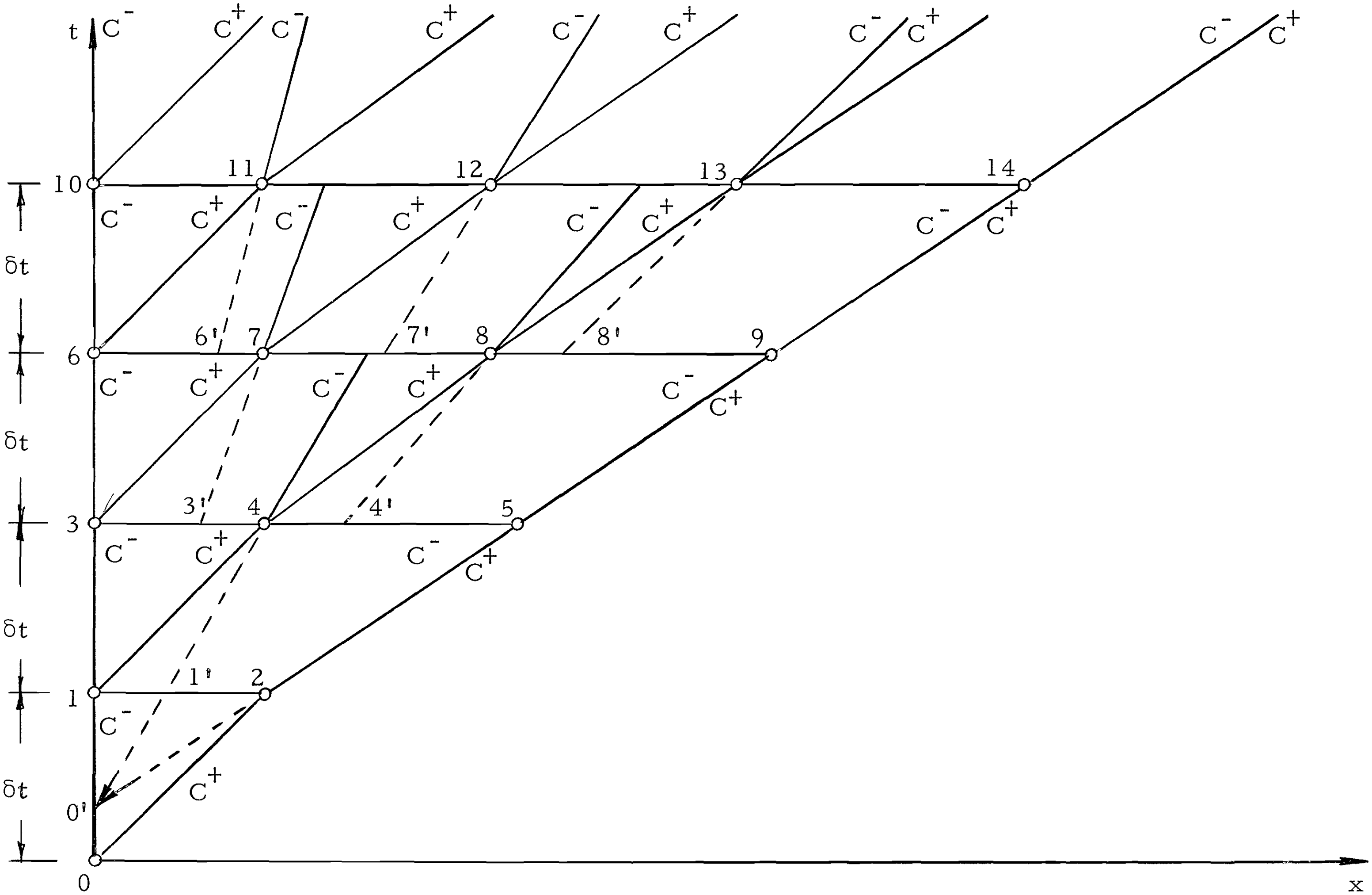


Fig. 10. Net points used in the finite-difference scheme.

$$v(x, 0) = v_o$$

$$c(x, 0) = c_o$$

which satisfy the boundary condition (Eq. 87) to yield

$$v_o c_o^2 = g q_o$$

in which $v_o = c_o = \sqrt{g y_{co}}$ and $q_o = \sqrt{g y_{co}^3}$ are all constants. At all of these points on the t-axis the values of v and c are unknown except that they satisfy Eq. 87. Consequently these values at points numbered as 1, 3, 6, 10, etc., are determined with the help of Eq. 87 in the course of calculation starting from the singularity (origin 0) and thus proceeding with the sequential increment of small time, δt. The value of c at those points on the trajectory of the wave front is zero since it touches the bottom of the channel bed. Thus the trajectory of the wave front belongs to both characteristics C^+ and C^- because Eqs. 56 and 57 both give the same slope on the curve; namely

$$\frac{dx}{dt} = v \quad \ldots\ldots\ldots\ldots \quad (92)$$

in which v is the velocity of the forward wave front that must also be determined by the characteristics equations, Eqs. 90 and 91.

The integration by finite differences can be proceeded as follows: the slopes of the characteristics curves C^+ and C^- at the singularity (origin 0) is

$$C^{+} \; : \; \frac{dx}{dt} = v_o + c_o = 2\,c_o$$

$$C^{-} \; : \; \frac{dx}{dt} = v_o - c_o = 0$$

because one already imposed $v_o = c_o = \sqrt{g\,y_{co}}$ at the singularity. From this point, straight line segments with these slopes are drawn until they intersect with a line distant δt along the t-axis from the x-axis, respectively, at points 1 and 2. Since the slope of the characteristics C^{-} is zero, the straight line segment $\overline{01}$ coincides with the t-axis. Along this segment, the characteristic equation, Eqs. 87 and 91, must be satisfied. Consequently, one has the relations

$$v_1 - 2\,c_1 = v_o - 2\,c_o = k_2 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (93)$$

$$v_1\,c_1^{\,2} = v_o\,c_o^{\,2} = g\,q_o \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (94)$$

for the velocity v_1 and celerity c_1 to be determined at the point 1. Since $v_o = c_o$, Eqs. 93 and 94 can be written as

$$v_1 - 2\,c_1 = -\,c_o \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (95)$$

$$v_1\,c_1^{\,2} = c_o^{\,3} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (96)$$

Substituting $v_1 = c_o^{\,3}/c_1^{\,2}$ from Eq. 96 into Eq. 95 yields

$$2\,c_1^{\,3} - c_o\,c_1^{\,2} - c_o^{\,3} = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (97)$$

Solving Eq. 97, one can easily obtain a possible solution (real root*) for c_1; that is exactly

$$c_1 = c_o$$

whence $v_1 = c_o$

The values of v and c at the point 2, specifically, will be calculated along a straight line segment issuing from the point 0. The line segment $\overline{02}$ is considered as the short tangent segment of the characteristic curve C^+ at the point. Therefore, the value of x at the point 2 for a given δt can be determined graphically, or by the relation (Eq. 56)

$$\delta x_2 = 2\, c_o\, \delta t$$

in which δx_2 is the distance of the segment between the point 1 and the point 2. Along the segment $\overline{02}$, the characteristic equation, Eq. 90, must be satisfied. Hence

$$v_2 + 2\, c_2 = v_o + 2\, c_o = k_1 \quad \dots \dots \dots \quad (98)$$

$$c_2 = 0 \quad \dots \dots \dots \dots \dots \quad (99)$$

from which one can readily see that

$$v_2 = 3\, c_o$$

---

* Equation 97 can be expressed as

$$2\left(\frac{c_1}{c_o}\right)^3 - \left(\frac{c_1}{c_o}\right)^2 - 1 = 0$$

That is $\left[\left(\frac{c_1}{c_o}\right) - 1\right] \left[2\left(\frac{c_1}{c_o}\right)^2 + \left(\frac{c_1}{c_o}\right) + 1\right] = 0$

The only positive root is $c_1/c_o = 1$

The values of v and c at points 1 and 2 agree with the Ritter solution (Eqs. 60 and 61) obtained from the method of characteristics. Once v and c are known at points 1 and 2, the slopes of the characteristics issuing from these points can be determined once more from Eqs. 56 and 57 and the entire process can be carried out again to yield the locations of the additional points 3, 4, and 5 in the x, t-plane for another time increment δt and hence the approximate values of v and c at these points.

The process to determine the values of v and c at the point 3 is exactly the same as that at the point 1. Nevertheless, special attentions must be focused on the techniques of determining the values of v and c at points 4 and 5 respectively. In what follows, these techniques will be discussed in detail since this is the vital point of the present method and that may be the only way accessible to numerical solutions in the problem of this typical flow condition. The analytical solutions obtained from the method of characteristics certainly confirm the validity of the following technique for the problem of simple wave.

Let us determine the values of v and c at the point 5 first because the technique being used will almost follow the similar process in determining the values of v and c at the point 2 except that the straight line segment issuing from the point 2 to the point 5 is now becoming both the characteristics C^+ and C^-. Consequently, the value of x at the point 5 for a given δt can be determined by Eq. 92.

$$\delta x_5 = v_2 \, \delta t = 3 \, c_o \, \delta t$$

in which δx_5 is the difference of x-coordinate between the point 5 and the point 2 in the x, t-plane. Furthermore, along the segment $\overline{25}$, both characteristics equations, Eqs. 90 and 91, are satisfied with the value of c being zero at these points; i.e., $c_2 = c_5 = 0$. Therefore, the relations

$$v_2 + 2\,c_2 = v_5 + 2\,c_5 = k_1$$

$$v_2 - 2\,c_2 = v_5 - 2\,c_5 = k_2$$

can be reduced to have a single result that

$$v_5 = v_2 = 3\,c_o$$

which agree again with the Ritter solution. Likewise, all the points 9, 14, ..., along the trajectory of the wave front have the same values of v and c:

$$v_2 = v_5 = v_9 = v_{14} = \ldots. = 3\,c_o$$

and

$$c_2 = c_5 = c_9 = c_{14} = \ldots. = 0$$

This argument proves the validity of the finite-difference method used for the case of simple wave.

The difficulty of determining the values of v and c at the point 4 is obvious from Fig. 10. The value of x at the point 4 is determined from the characteristic C^+ issuing from the point 1.

$$\delta x_4 = v_1 \delta t = 2 c_o \delta t$$

in which δx_4 is similarly the difference of x-coordinate between point 4 and point 1. Nevertheless, there is only one characteristic relation available along the straight line segment $\overline{14}$ which is the C^+ characteristic. From Eq. 90, the relation

$$v_4 + 2 c_4 = v_1 + 2 c_1 = k_1$$

can only be obtained to solve two unknowns, v_4 and c_4 since it is already known that $v_1 = c_1 = c_o$. The solutions of two unknowns from one equation is of course impossible unless another relation for these unknowns can be found. This can be done by the following way:

Since the segment $\overline{14}$ represents the short segment of the tangent of the characteristics C^+ at the point 4, another straight line segment for the characteristic C^- can be constructed. The inspection of Fig. 10 reveals that the slope of the characteristic C^- issuing from the point 4 stays right between those slopes at the point 2 and the point 5. The slopes of the characteristic C^- at these points are respectively those of the straight line segments $\overline{13}$ and $\overline{25}$. Extending these segments (a dotted line in the figure), a point of intersection 0' on the t-axis is reached. Connecting two points 0' and 4 (a dotted line in the figure) yields the interpolation slope of two C^- characteristics respectively at points 3 and 5. The segment $\overline{0'4}$ intersects with the segment $\overline{12}$ at the point 1'. The values of v and c at the point 1' can be obtained by using linear interpolation formulas* (Ralston and Wilf, 1964)

$$v_{1'} = v_1 + \frac{v_2 - v_1}{\delta x_2} (x_{1'} - x_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (100)$$

$$c_{1'} = c_1 + \frac{c_2 - c_1}{\delta x_2} (x_{1'} - x_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (101)$$

where $\delta x_2 = x_2 - x_1$ and x_1, $x_{1'}$, and x_2 are x-coordinates respectively at points 1, 1', and 2. The values of v_1, c_1, v_2, and c_2 are all known from the previous computations. The only one undetermined in Eqs. 100 and 101 is the value of $x_{1'}$ at the point 1'. However, $x_{1'}$ can be obtained from the following relations:

Since $\delta x_2 = x_2 - x_o = x_2 - x_1$

$$\delta x_5 = x_5 - x_2$$

are already known,

$$\delta x_2 + \delta x_5 = x_5 - x_o = x_5 - x_3$$

Moreover, since

$$\delta x_4 = x_4 - x_3$$

the following relation

---

* For the case of simple wave, the linear interpolation formulas may be sufficiently accurate; however, for other more complicated cases, in order to improve the accuracy, the quadratic interpolation formulas may be used.

$$x_{1'} = x_1 + \frac{x_2 - x_1}{x_5 - x_3}(x_4 - x_3)$$

or

$$x_{1'} = x_1 + \frac{(\delta x_2)(\delta x_4)}{\delta x_2 + \delta x_5} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (102)$$

exists to determine the value of x-coordinate at the interpolation point 1'. Consequently, substituting $x_{1'}$-value into Eqs. 100 and 101 yields the values of v and c at this point, or simply

$$v_{1'} = v_1 + (v_2 - v_1)\frac{\delta x_4}{\delta x_2 + \delta x_5}$$

$$c_{1'} = c_1 + (c_2 - c_1)\frac{\delta x_4}{\delta x_2 + \delta x_5}$$

If $v_1 = c_o$, $c_1 = c_o$, $v_2 = 3\,c_o$, $c_2 = 0$, $\delta x_2 = 2\,c_o\,\delta t$, $\delta x_4 = 2\,c_o\,\delta t$, and $\delta x_5 = 3\,c_o\,\delta t$ are all substituted into the foregoing equations, then $v_{1'}$ and $c_{1'}$ can be calculated; namely

$$v_{1'} = \frac{9}{5}\,c_o$$

$$c_{1'} = \frac{3}{5}\,c_o$$

By knowing the values of v and c at the point 1', the C^- characteristic relation, Eq. 91, can thus be formulated on the straight line segment $\overline{1'4}$.

$$v_4 - 2\,c_4 = v_{1'} - 2\,c_{1'} = k_2$$

The values of v_4 and c_4 can now be determined from the following two linear equations

$$v_4 + 2\,c_4 = 3\,c_o \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (103)$$

$$v_4 + 2\,c_4 = \frac{3}{5}c_o \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (104)$$

which simply yield $v_4 = \frac{9}{5}c_o$ and $c_4 = \frac{3}{5}c_o$. The proceeding entire process of calculation can be similarly carried out to determine the values of v and c at the interior points such as points 7, 8, 11, 12, 13, ... All the supplementary C^- characteristics segments drawn from those interior points are issuing from the point $0'$ as shown in Fig. 10. In the method of characteristics, as previously mentioned, the family of the C^- characteristics are the straight lines issued from the origin 0. Thus, the finite-difference method apparently does not give a correct approximate solution around the origin because of the property of singularity at the point. However, if δt is chosen sufficiently small, the position of the point $0'$ will be in good approximation to the position of the origin 0.

In order to improve the clumsy situation around the singularity, the starting point of the finite-difference scheme should be located so that the point $0'$ acting as a center of concurrent C^- characteristic curves in Fig. 10 will be the actual origin in the x, t-plane. If the initial point for $x = 0$ can be set at the point $0''$ (Fig. 11) $\frac{1}{2}\,\delta t$ downward from the origin 0, then the origin 0 becomes an actual center

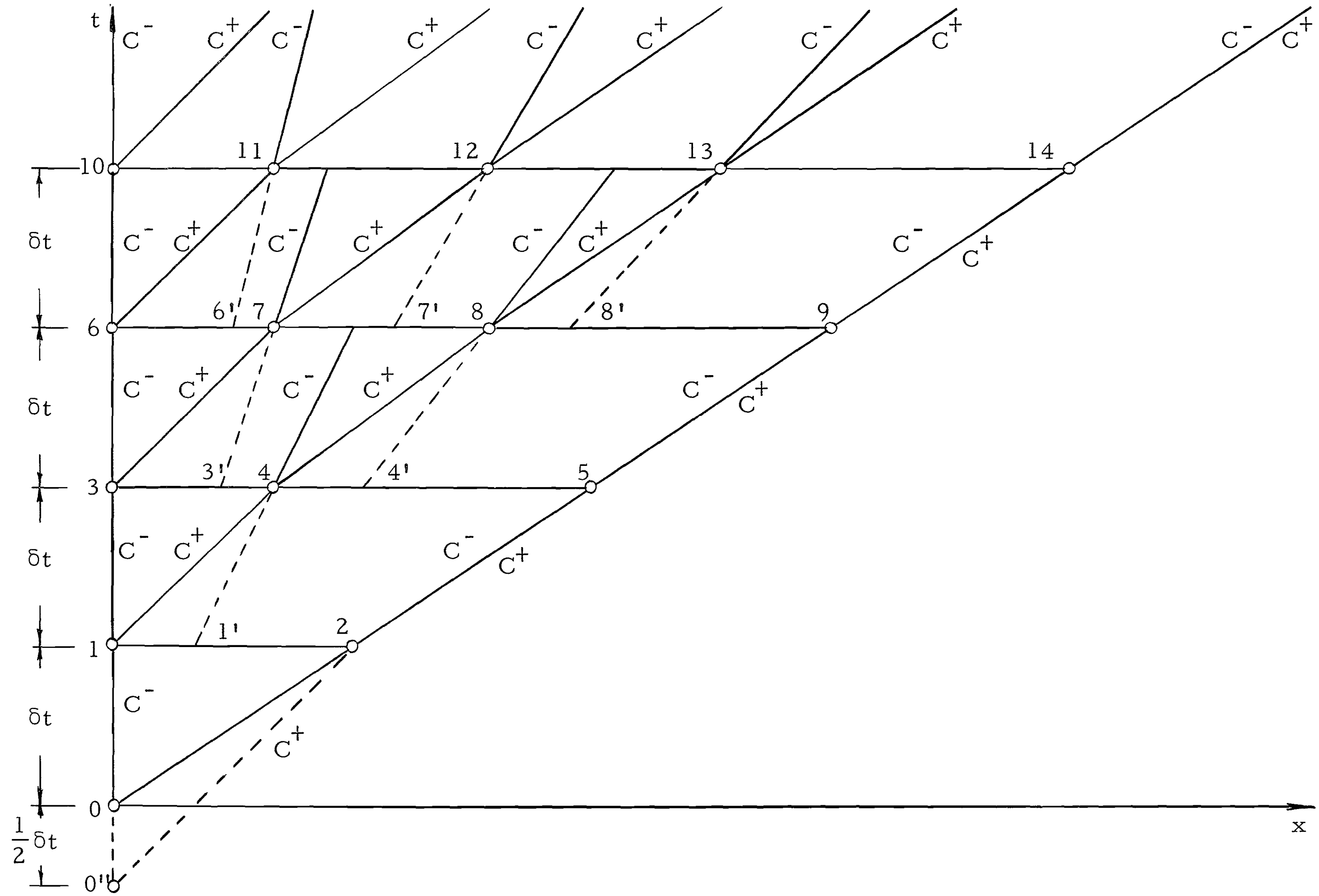


Fig. 11. Modified scheme used at the singularity in the finite-difference method.

of the concurrent C^- characteristic curves. The reason to take $\frac{1}{2}\delta t$ for the segment $\overline{00''}$ manifests itself in the previous arguments. The justifiable statement is given here rather than to recapitulate the whole discussion.

The initial conditions, $v_o'' = c_o'' = c_o$, are still prescribed at the point $0''$ so that the straight line segment $\overline{0''2}$ must be considered as the C^+ characteristic issued from the point $0''$. For this purpose,

$$
\begin{aligned}
\delta x_2 &= x_2 - x_{0''} \\
&= (\frac{1}{2}\delta t + \delta t)(v_{0''} + c_{0''}) \\
&= 3\, c_o\, \delta t
\end{aligned}
$$

The preceding result automatically proves that the origin 0 will be the center of the concurrent C^- characteristics. The rest of the procedures for computing the values of x, t, v, and c at every grid point in the finite-difference scheme will follow the same procedures as given before.

Thus far, we only confine ourselves to the method in solving the simplest version of nonlinear-waves problem, i. e., a "centered simple wave." The special technique for overcoming the initial discontinuity was introduced and the methods of constructing the finite-difference scheme and of evaluating the values of independent and dependent variables at every grid point with fixed-time intervals were presented. It is interesting to note that the technique and the method used can be extended in solving the more general form of the differential equations. In what follows, an additional term, the slope of channel, S_o, to those treated in the previous discussion will be considered.

With this additional term, the system of equations is no longer "reducible" but has the same characteristics C^+ and C^- (Eqs. 56 and 57). Nevertheless, the characteristic equations have the forms

$$\text{along } C^+ : \frac{d}{dt}(v + 2c) - gS_o = 0 \quad . \quad . \quad . \quad . \quad . \quad (105)$$

$$\text{along } C^- : \frac{d}{dt}(v - 2c) - gS_o = 0 \quad . \quad . \quad . \quad . \quad . \quad (106)$$

Integrating Eqs. 105 and 106 yield

$$\text{along } C^+ : v + 2c - gS_o t = k_1 = \text{const.} \quad . \quad . \quad . \quad (107)$$

$$\text{along } C^- : v - 2c - gS_o t = k_2 = \text{const.} \quad . \quad . \quad . \quad (108)$$

Using the analogous modified scheme of finite-difference method around the initial discontinuity, the initial conditions

$$v_{0''} = c_o$$

$$c_{0''} = c_o$$

are prescribed at the point $0''$ as shown in Fig. 12. The boundary condition at $x = 0$ is the same form as Eq. 87; hence

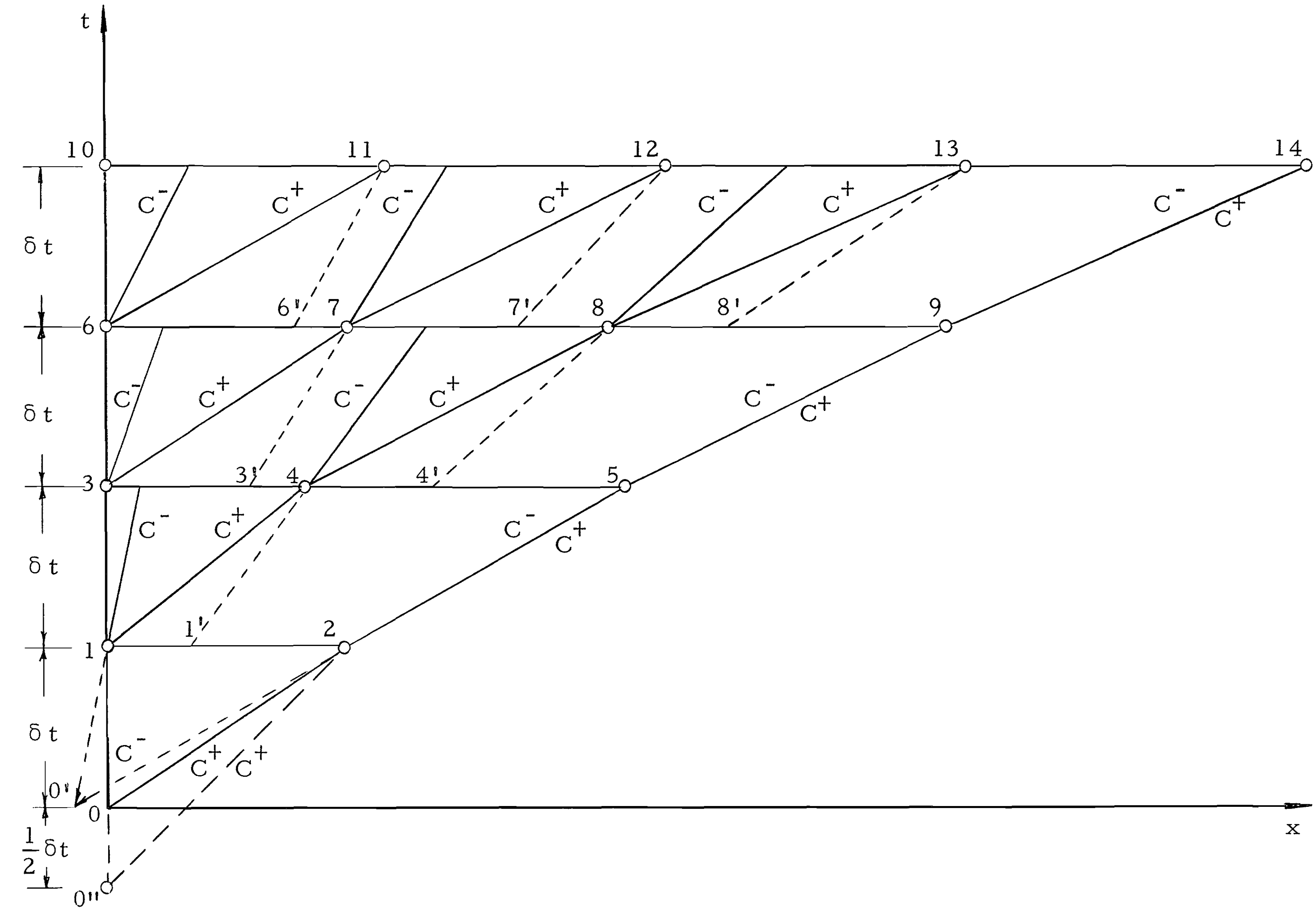


Fig. 12. Finite-difference scheme for the effect of channel slope S_o other than those from Fig. 11.

$$v_{0''} c_{0''}^{2} = g q_o$$

Following the same procedures as given in the previous example, one can readily determine the values of v and c at every propagated grid point where its x-coordinate with a known finite-time interval δt can be evaluated from Eq. 56 or 57. For instance, in the present case,

$$\delta x_2 = x_2 - x_{0''} = 3 c_o \delta t$$

Evidently, a straight line segment $\overline{0''2}$ is C^+ characteristic issuing from the point $0''$ and another segment $\overline{0''1}$ is C^- characteristic. Hence, along these segments, Eqs. 107 and 108 are satisfied respectively. Instead of the segments $\overline{0''2}$ and $\overline{0''1}$, the segments $\overline{02}$ and $\overline{01}$ are taken as the characteristics C^+ and C^-. Hence

$$v_2 + 2 c_2 - g S_o \delta t = v_o + 2 c_o \text{ along } \overline{02}$$

Remembering that $c_2 = 0$ since it is the trajectory of the wave front and assuming that $v_o = c_o = c_o$ at the point 0, the following is obtained,

$$v_2 = 3 c_o + g S_o \delta t$$

which is in agreement with the Ritter solution. Another relation, along $\overline{01}$, is

$$v_1 - 2 c_1 - g S_o \delta t = v_o - 2 c_o$$

$$= - c_o \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (109)$$

with the boundary condition at the point 1

$$v_1 c_1^2 = g q_o = c_o^3 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (110)$$

From Eq. 109, $v_1 = 2 c_1 + g S_o \delta t - c_o$ substituted into Eq. 110, one has

$$2 c_1^3 + (g S_o \delta t - c_o) c_1^2 - c_o^3 = 0 \quad . \; . \; . \; . \; . \; . \quad (111)$$

must be solved for c_1. Comparing Eq. 111 with Eq. 97 which has a solution $c_1 = c_o$, there exists a root for c_1 between 0 and c_o. Since $0 < c_1 < c_o$, from Eq. 110, there must be $v_1 > c_o$. Consequently, the t-axis will be no longer a C^- characteristic because

$$\frac{dx}{dt} = v_1 - c_1 > 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (112)$$

which issues a C^- characteristic segment from the point 1 to the right hand side of the t-axis. This is an interesting result which is not obtainable from the Ritter solution since it cannot satisfy the boundary condition at $x = 0$.

Since the C^- characteristics are no longer treated as the straighted lines issued from the origin as those mentioned in the case of a centered simple wave, the supplementary C^- characteristic (a dotted line in the figure) interpolated from two neighboring points at every interior grid point has a variable location of the point of intersection of the two C^- characteristics drawn at these neighboring points. For example, the point of intersection of the two C^-

characteristics drawn at the points 1 and 2 is the point 0', from which the interpolated C^- characteristic is drawn to the point 4. The straight line segment $\overline{1'4}$ becomes a supplementary C^- characteristic at the point 4 and the values of v and c at this point must be determined from the values of v and c at the points 1 and 1'. For this purpose, the values of v and c at the point 1' is again interpolated from the values of v and c at the points 1 and 2. The point of intersection of the two C^- characteristic segments is varying rather than fixed at the point 0' as in the previous case.

Attention must be paid computing values of v and c on the boundary at $x = 0$ by extrapolation from the values of v and c at those points to the right of the t-axis. The linear extrapolation formulas such as Eqs. 100 and 101 may be used; however, in order to improve the accuracy of approximation, the quadratic extrapolation formulas must be employed. Since one boundary condition (Eq. 87) is already prescribed at $x = 0$, simply one value of two dependent variables, either v or c, by extrapolation is enough to determine both values of v and c at $x = 0$. The rest of the entire procedures of computation will follow that of the former case.

In addition to the channel slope, S_o, if the frictional S_f and the lateral outflow $\bar{i}$ are added in the differential equations, the characteristic equations (Eqs. 58 and 59) obtained generally are not possible for integration. When the Chezy formula is used,

$S_f = \frac{g}{C^2} \left(\frac{v}{c}\right)^2$, thus Eqs. 58 and 59 can be rewritten as

$$\frac{d}{dt}(v + 2c) + \frac{g\bar{i}}{c}\left(1 - \frac{v}{c}\right) - gS_o + \left(\frac{g}{C}\right)^2 \left(\frac{v}{c}\right)^2 = 0 \quad . \quad . \quad (113)$$

along a characteristic curve C^+, $\frac{dx}{dt} = v + c$, and

$$\frac{d}{dt}(v - 2c) - \frac{g\bar{i}}{c}\left(1 + \frac{v}{c}\right) - gS_o + \left(\frac{g}{C}\right)^2 \left(\frac{v}{c}\right)^2 = 0 \quad . \quad . \quad (114)$$

along a characteristic curve C^-, $\frac{dx}{dt} = v - c$.

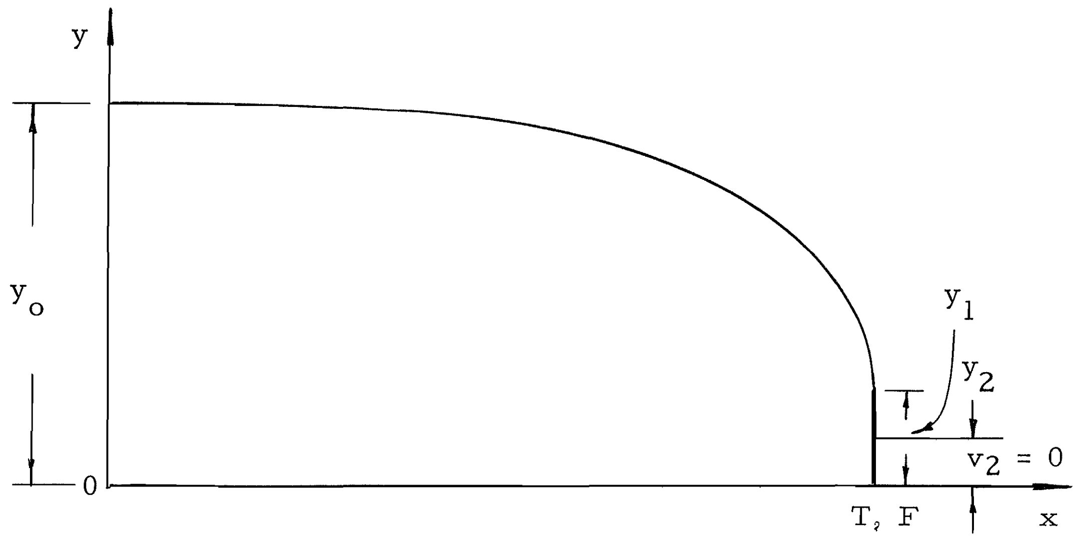
At the wave front, where $c = 0$, the inspection of Eqs. 113 and 114 reveals that the friction term and the infiltration term in the equations having c in the denominator are all becoming infinite values. Along the locus of the wave front, the velocity, v, evaluated must be equal to the reciprocal slope of the wave front curve, $\frac{dx}{dt}$, in the x, t-plane. Each of the Eqs. 56 and 57 is satisfied along the trajectory curve of the wave front because of $c = 0$ there and both characteristic directions C^+ and C^- coincide with the direction of the wave front curve. However, neither of relations, Eqs. 113 and 114, can hold along the wave front curve. Although the wave front cannot be a characteristic curve itself, it must be an envelope of C^+ and C^- curves for $C > 0$ and $\bar{i} > 0$.

We see from the experimental profiles obtained by Tinney and Bassett (1961) that the free-surface wave actually governed by resistance has a vertical slope at the tip of the wave front. To handle the tip region accurately, some type of boundary-layer technique would be necessary, as suggested by Dressler (1952). The tip region moves somewhat like a separate entity pushed along by the water behind it and in the tip, v should be changing rather slowly toward the front. The following approximate considerations to obtain some data about the wave front are necessary in order to overcome an inappropriate condition, $c = 0$, at the tip.

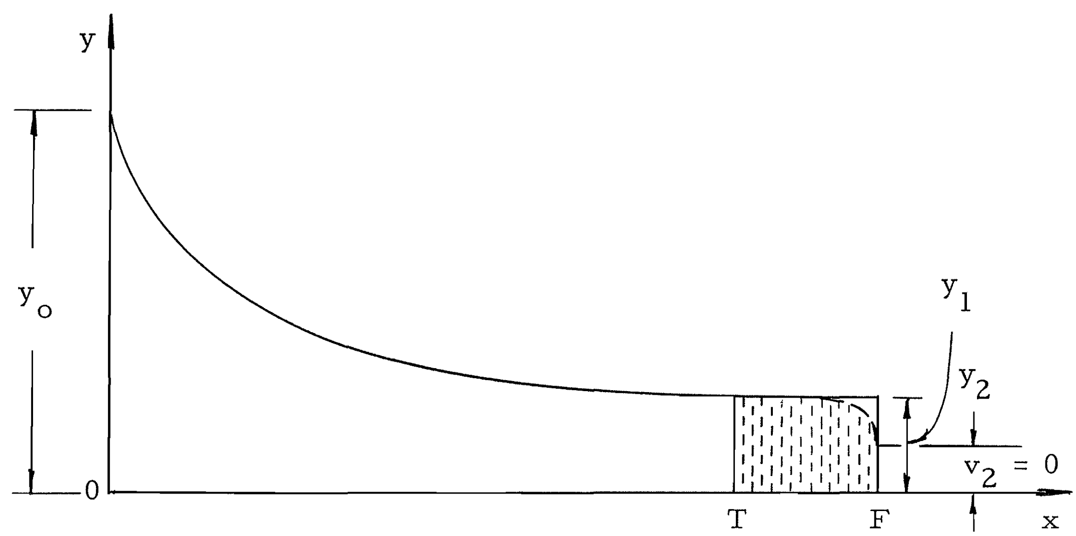
The simplest assumption to make would be considering the zone of quiet downstream which is terminated on the upstream side by a shockwave (or bore) and the zone of constant state in which the water is assumed to flow with uniform velocity, v. The latter zone may be called the transition region which connects the zone of quiet downstream and the zone of the actual wave behind. The question arises concerning the depth of water assumed in the zone of quiet downstream. In the absence of the S_f and $\bar{i}$ terms, there may exist a critical value of the ratio $y_2/y_o = 0.1384$, in which y_2 is the depth of water in the zone of quiet downstream and y_o is the depth of water at $x = 0$. If y_2/y_o is less than the critical value, the discharge rate at $x = 0$ will be independent of y_2 as well as independent of the time. This fact has been discussed briefly by Stoker (1957) on the dam-breaking problem. Furthermore, the height of the bore, y_1, caused by the assumed depth of water, y_2, in the zone of quiet downstream can be evaluated from the shock-wave equation if y_2/y_o is known. The assumed constant shock velocity can also be obtained from this equation. In this simple problem, it is implicit that if one can find or assume a judicious depth of flow in the front of the tip without causing any significant change on the flow characteristics behind the tip, the proposed method seems justifiable. However, even in this simplest case, since y_o, y_1, y_2, and the shock velocity are all interrelated to each other, there appears to be no feasible means to formulate a theoretical basis on the assumption of y_2 if the S_f and $\bar{i}$ terms are added in the equations.

The transition zone will be very narrow in the case of subcritical flow; contrarily, it will be comparatively lengthy in the case of supercritical flow, as may be expected from the characteristics of free-surface profiles. These two cases are schematically shown in Fig. 13. The appropriate selection of the depth of water, y_2, in the stationary zone for different conditions of flow is essentially a prerequisite of using this method.

Another interest in handling the tip problem is the assumptions of the values of v/c and $\bar{i}/c$ at the tip rather than having $c = 0$ there. From the general hydrodynamic equation of free-surface profile, Eq. 24, developed with the prescribed initial conditions, Chen and Hansen (1966) reasoned that either subcritical or supercritical flow has a free-surface profile (y) staying right between the normal-depth (y_n) curve and the critical-depth (y_c) curve. At the initial state $(t = 0)$, since $v = v_o = c_o$ and $c = c_o$,



(a) Subcritical flow case



(b) Supercritical flow case

Fig. 13. Transition zone caused by the stationary water at the tip.

it is justified from Eq. 24 that the slope of the free-surface profile at this stage is infinite. If there is no cutting-off of the inflow from the upstream end $(x = 0)$, the velocity v of flow will approach the normal velocity v_n when the time becomes infinite. As a matter of fact, this was already demonstrated in the derivation of Eq. 48 for the case of no infiltration.

If the infiltration function, $\bar{i}$, for the Kostiakov-Lewis (Philip and Farrel, 1964) type

$$\bar{i} = a t^b \qquad (-1 < b \leq 0) \quad . \quad . \quad . \quad . \quad . \quad (115)$$

is considered for simplicity in the present analysis, $\bar{i}$ becomes infinite at $t = 0$ because of $-1 < b < 0$. This is somewhat similar to the property of the wave front, where $c = 0$, $\frac{\partial c}{\partial x} = -\infty$, and $\frac{\partial c}{\partial t} = +\infty$. At the wave front, since $\bar{i}$ is maintained always at the initial state, the value of $\bar{i}/c$ is extremely large enough though c may be assumed as a constant at the tip. Thus the containment of the infiltration term in the equations is believed to be a major factor in the determination of the trajectory of the wave front. From this point of view, the "classical" approximate approach, a simple volume-balance method (i. e., using only the equation of continuity) in the surface irrigation should be justified. Nevertheless, as it is experimentally known, or can be seen from these equations, that a vertical wave front is maintained at the tip, one may logically assume a critical depth at the tip where the corresponding velocity is the critical velocity as well as the normal velocity. Consequently, at the tip, the value of v/c is assumed unity. This value is justified by the initial conditions prescribed at $x = 0$. When $v/c = 1$, the infiltration term in Eq. 113 always vanishes because $\left(1 - \frac{v}{c}\right) = 0$. Thus, with this assumption, the infiltration term is no longer a factor.

With the assumptions that $c = 0$ and $v/c = 1$ at the wave front, Eq. 113 yields

$$\frac{dv}{dt} - g S_o \left(1 - \frac{g}{C^2 S_o}\right) = 0$$

whence

$$v = \frac{dx}{dt} = g S_o \left(1 - \frac{g}{C^2 S_o}\right) t + A_1$$

$$x = \frac{1}{2} g S_o \left(1 - \frac{g}{C^2 S_o}\right) t^2 + A_1 t + B_1$$

The condition that $\frac{dx}{dt} = 3 c_o = 3 \sqrt{g y_{co}}$ and $x = 0$ as $t = 0$ gives $A_1 = 3 c_o$ and $B_1 = 0$, therefore the trajectory of the wave front is

$$x = 3 c_o t + \frac{1}{2} g S_o \left(1 - \frac{g}{C^2 S_o}\right) g^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (116)$$

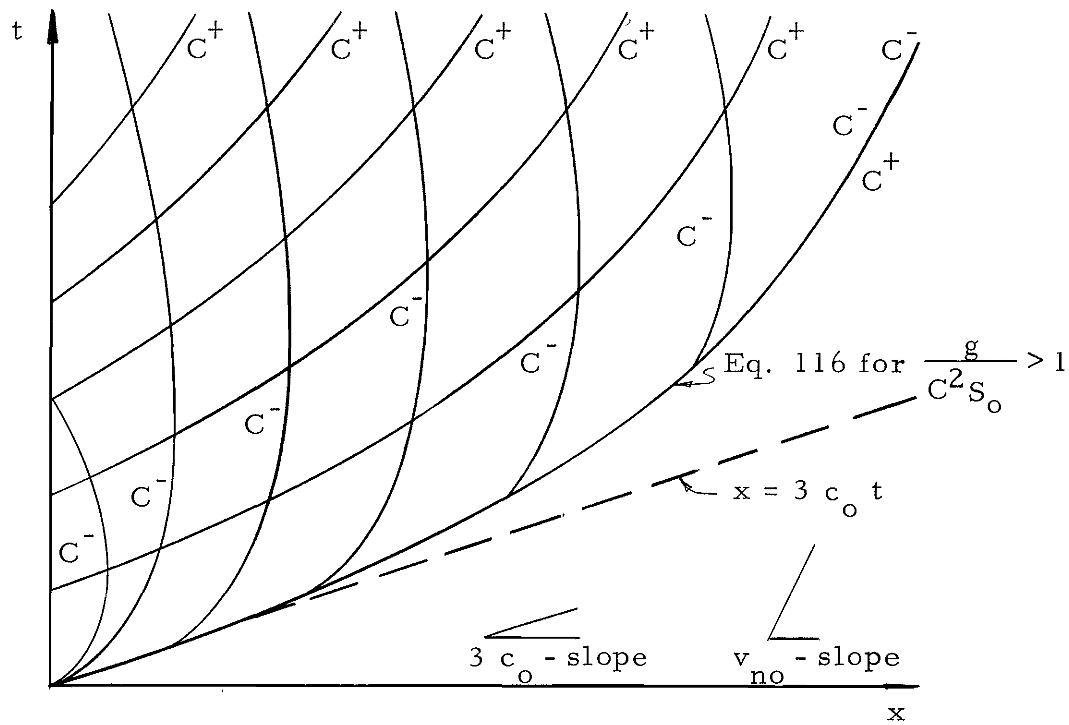
which is compatible with Eq. 29 except that an additional parameter, $g/C^2 S_o$, which is an alternative expression of the Froude number at the "normal" state, merges into the equation. The corresponding solutions for v and c analogous to Eqs. 73 and 74 are possible from Eq. 116. However, owing to the same reasons that Eqs. 73 and 74 do not staisfy the boundary conditions at $x = 0$, the finite-difference method must be employed for obtaining the numerical solutions.

The parameter, $g/C^2 S_o$, is a measure of the combined effects of the channel slope and friction on the flow characteristics. From the definitions of y_c and y_n given previously, the following relation can be formulated:

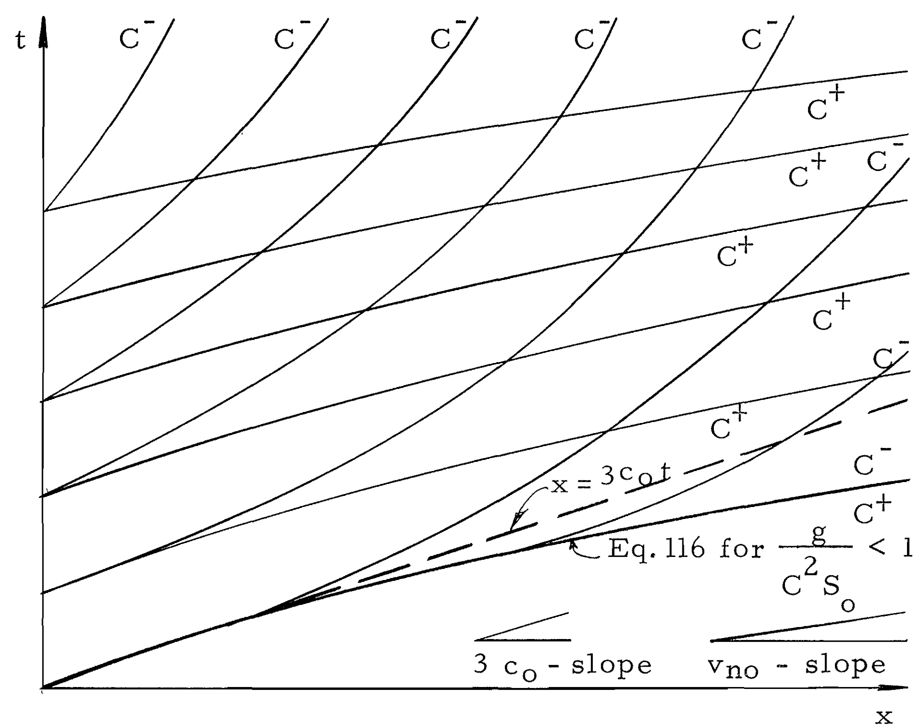
$$\frac{y_c}{y_n} = \left(\frac{C^2 S_o}{g}\right)^{1/3} = \mathbb{F}_n^{\,2/3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (117)$$

in which $\mathbb{F}_n = \dfrac{v_n}{\sqrt{g\, y_n}}$ = "normal" Froude number having the value being less than, equal to, or greater than unity respectively, corresponding to whether the flow is on a mild, critical, or steep slope. From Eq. 116, it can be readily seen that for a mild slope, the second term in the right hand side of the equation becomes a negative value, which has exactly a reverse trend of Eq. 29, where only the channel slope is considered. For a steep slope, the aforementioned term has a positive value which has on the contrary the same trend as Eq. 29 while for a critical slope, the term is zero and the trajectory is a straight line.

The characteristic patterns for flows over a mild slope and a steep slope are schematically shown in Fig. 14. For a flow over a



(a) In the case of mild slopes



(b) In the case of steep slopes

Fig. 14. Pattern of characteristics for flows on mild and steep slopes.

critical flow, it should look like the one for centered simple wave as shown in Fig. 7.

Looking into Fig. 14, one may question on the assumptions that $c = 0$ and $v/c = 1$ at the wave front. Because of the singularity at the origin ($t = 0$ and $x = 0$) where $v = c = c_o$ is imposed, evidently the assumption, $c = 0$, is not satisfied at that point. Furthermore, the characteristics C^+ and C^- basing on these assumptions are everywhere zero values along the wave front. This situation needs further investigation. As a matter of fact, at the initial state, $v/c = c_o/c_o = 1$ but not $c = 0$. After that, $c = 0$ but not $v/c = 1$. If $v/c \neq 1$ the infiltration term as well as the friction term in Eqs. 113 and 114 are important in determining the values of v and c along the trajectory of the wave front where $c = 0$. There seems no way to find out, with the present knowledge, what should be the values of v/c and i/c. Unless these relations can be established, the finite-difference method seemingly fails for this typical problem. The former approach to assume a certain amount of depth at the wave front is also ambigious, because with this assumption, the wave front is no longer an envelope of the characteristics, C^+ and C^-. It readily can be seen from Eqs. 56 and 57 that c must be zero in order to have C^+ and C^- characteristics along the wave front, that is, Eqs. 56 and 57 both are now equivalent to Eq. 92.

Another feature of finite-difference technique used by Kruger and Bassett (1965) must be mentioned here. Without considering the characteristics and formulating the characteristic equations, they developed the finite-difference recurrence equations from the basic differential equations and obtained the numerical solutions by operating them on a fixed rectangular grid in the x, t-plane. Although the fact that they assume the initial conditions at $x = 0$ a normal depth and velocity seems against the usual way, the solutions surprisingly converged in the limit to those obtained for a steady flow case. In addition, they also assumed a certain amount of depth at the wave front. Naturally, the question concerning the validity of the finite-difference method arises. Is it possible to obtain correct solutions without considering the characteristic relations and the difficult situations at the wave front? An answer for this question may be obtained by conducting the same problem by two different methods. The result may give a clue to the problem encountered in using the finite-difference method around the neighborhood of the wave front.

Remember that Eq. 116 is not the solution of the problem because of the inappropriate conditions imposed at the wave front, and that it is simply intended to express qualitatively the importance of the additional parameter merged intrinsically in the basic equations. The only exact solution that can be obtained from the simplest case of the problem are the Ritter solutions (i. e., Eqs. 60 and 61) because Eqs. 73 and 74 are not the writer's solutions either. The Ritter solutions may be perturbed through the perturbation method to explore the effects of the aforementioned parameter due to the free surface and the infiltration variable. In another type of the problems (dam-breaking problem), Dressler (1952) has successfully obtained the solutions due to the hydraulic resistance effect by the perturbation procedure.

CHAPTER 6

OTHER METHODS

There are many other methods used in solving simultaneously the system of equations, Eqs. 31 and 32, in the case of the two-dimensional flow. For the convenience of presenting the following discussion, these equations are recapitulated herein.

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + g\frac{\partial y}{\partial x} - \frac{v}{y}\,\bar{i} = g(S_o - S_f) \quad . \quad . \quad . \quad (31)$$

$$\frac{\partial y}{\partial t} + v\frac{\partial y}{\partial x} + y\frac{\partial v}{\partial x} = -\,\bar{i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

As $q = vy$ was previously defined, Eq. 32 can be written as

$$\frac{\partial y}{\partial t} + \frac{\partial q}{\partial x} = -\,\bar{i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (118)$$

If $\bar{i} = 0$ in Eq. 118, one can introduce a function $\psi(x, t)$ defined (Rouse, 1949) as

$$\frac{\partial \psi}{\partial x} = -\,y \qquad \text{and} \qquad \frac{\partial \psi}{\partial t} = q$$

which satisfy the equation

$$\frac{\partial y}{\partial t} + \frac{\partial q}{\partial x} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (119)$$

From these definitions, one may have the following relations:

$$\frac{\partial y}{\partial x} = -\,\frac{\partial^2 \psi}{\partial x^2}, \qquad \frac{\partial y}{\partial t} = -\,\frac{\partial^2 \psi}{\partial x \partial t}$$

$$\frac{\partial q}{\partial x} = \frac{\partial^2 \psi}{\partial x \partial t}, \qquad \frac{\partial q}{\partial t} = \frac{\partial^2 \psi}{\partial t^2}$$

$$v = \frac{q}{y} = - \frac{\dfrac{\partial \psi}{\partial t}}{\dfrac{\partial \psi}{\partial x}}$$

$$\frac{\partial v}{\partial x} = \frac{1}{y^2} \left(y \frac{\partial q}{\partial x} - q \frac{\partial y}{\partial x} \right) = \frac{1}{\left(\dfrac{\partial \psi}{\partial x}\right)^2} \left(- \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial x \partial t} + \frac{\partial \psi}{\partial t} \frac{\partial^2 \psi}{\partial x^2} \right)$$

$$\frac{\partial v}{\partial t} = \frac{1}{y^2} \left(y \frac{\partial q}{\partial t} - q \frac{\partial y}{\partial t} \right) = \frac{1}{\left(\dfrac{\partial \psi}{\partial x}\right)^2} \left(- \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial t^2} + \frac{\partial \psi}{\partial t} \frac{\partial^2 \psi}{\partial x \partial t} \right)$$

Substituting all of these into Eq. 31 for the case of $\bar{i} = 0$ yields

$$\left(\frac{\partial \psi}{\partial x}\right)^2 \frac{\partial^2 \psi}{\partial t^2} + \left(\frac{\partial \psi}{\partial t}\right)^2 \frac{\partial^2 \psi}{\partial x^2} - 2 \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial t} \frac{\partial^2 \psi}{\partial x \partial t}$$

$$+ g \left(\frac{\partial \psi}{\partial x}\right)^3 \frac{\partial^2 \psi}{\partial x^2} + g S_o \left(\frac{\partial \psi}{\partial x}\right)^3 + \frac{g}{C^2} \left(\frac{\partial \psi}{\partial t}\right)^2 = 0$$

. . . . (120)

The initial and boundary conditions for $\psi(x, t)$ are:

$$\frac{\partial \psi (x, 0)}{\partial x} = y_{co}, \qquad \frac{\partial \psi (x, \infty)}{\partial x} = - y_{no}$$

$$\frac{\partial \psi (0, t)}{\partial t} = q_o$$

in which

$$y_{no} = \left(\frac{q_o^2}{C^2 S_o}\right)^{1/3} \qquad y_{co} = \left(\frac{q_o^2}{g}\right)^{1/3}$$

Solving Eq. 120 for $\psi(x, t)$ is not an easy task; however, by this means, the possibility of applying an electrical analogy to the tracing of water waves in the surface irrigation was demonstrated. In the case that $\bar{i} \neq 0$, the function defined above is no longer valid because it fails to satisfy Eq. 118. A function similar to $\psi(x, t)$ must be redefined and used.

Another approximate solution obtained by the kinematic-wave method is based on the assumption (Henderson and Wooding, 1964) that the slope of the water surface relative to the slope of the channel bed can be neglected if the depth of water is sufficiently small. The assumption used is somewhat like the Dupit-Forchheimer theory used in the groundwater flow. The approximate equation of motion used has the form

$$q = K y^N \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (121)$$

in which K and N are all coefficients depending on which flow is considered. The following values of K and N are taken respectively for:

(1) laminar flow : $K = g S_o / 3\nu$, $N = 3$

(2) turbulent flow : $K = C S_o^{1/2}$, $N = 3/2$

in which ν = kinematic viscosity of the fluid (water). Knowing these values, one can obtain

$$q = \frac{g S_o}{3\nu} y^3 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (122)$$

for the laminar flow, and

$$q = C S_o^{1/2} \; y^{3/2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (123)$$

for the turbulent flow in the two-dimensional flow case. In fact the simplified equations of motion, Eqs. 122 and 123 (or in general, Eq. 121), are used to replace the equation of momentum (Eq. 31) for the convenience of manipulating equations. Therefore, Eqs. 118 and 121 constitute a kinematic-wave problem which can be solved by the method characteristics. This approach has been used extensively by many investigators in the field of overland-flow research. The assumption used may fail at the wave front, where the slope of the free surface is observed to be vertical to the bed; however, the fact that it fairly describes the rest of the profiles is justified.

The form of Eq. 121 means that Eq. 118 can be written as a total derivative; herein lies the essence of the method of characteristics. Differentiating 121 with respect to x yields

$$\frac{\partial q}{\partial x} = K N y^{N-1} \frac{\partial y}{\partial x} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (124)$$

Substituting Eq. 124 into Eq. 118, one has

$$\frac{\partial y}{\partial t} + K N y^{N-1} \frac{\partial y}{\partial x} = - \bar{i} \quad . \; . \; . \; . \; . \; . \; . \quad (125)$$

Let

$$\frac{dx}{dt} = K N y^{N-1} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (126)$$

be a characteristic*, then Eq. 125 becomes

$$\frac{\partial y}{\partial t} + \frac{dx}{dt}\frac{\partial y}{\partial x} = -\bar{i}$$

or

$$\frac{dy}{dt} = -\bar{i} \quad \ldots\ldots\ldots\ldots\ldots \quad (127)$$

As the same reasoning described in the general theory of characteristics, a system of equations, Eqs. 126 and 127, is equivalent to another system of equations, Eqs. 118 and 121. The solutions obtained from Eqs. 126 and 127 are thus uniquely determined. This means that along the characteristic (Eq. 126), the relation (Eq. 127) always holds. An alternative system of equations corresponding to Eqs. 118 and 121 can be obtained as follows:

Since from Eq. 121,

$$\frac{\partial q}{\partial t} = KNy^{N-1}\frac{\partial y}{\partial t}$$

$$\frac{\partial y}{\partial t} = \frac{y^{1-N}}{KN}\frac{\partial q}{\partial t}$$

substituted into Eq. 118 and rearranged

$$\frac{\partial q}{\partial t} + KNy^{N-1}\frac{\partial q}{\partial x} = -\bar{i}KNy^{N-1} \quad \ldots\ldots \quad (128)$$

---

* In this simplified method, there is only one characteristic, i.e., $\frac{dx}{dt} = KNy^{N-1}$. However, in the general method, two characteristics are defined, i.e., C^+ characteristic : $\frac{dx}{dt} = v + c$

C^- characteristic : $\frac{dx}{dt} = v - c$

If the same characteristic (Eq. 126) is defined, then Eq. 128 can be reduced to

$$\frac{dq}{dt} = -\bar{i}\,K N y^{N-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (129)$$

Hence, Eqs. 126 and 129 are equivalent to Eqs. 118 and 121. Correspondingly, one can easily deduce the following relations:

$$\frac{dy}{dx} = -\frac{\bar{i}}{KN}\, y^{1-N} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (130)$$

$$\frac{dq}{dx} = -\bar{i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (131)$$

A somewhat different initial condition at $x = 0$ must be imposed for this approximate method. Since only one characteristic is available, it is logical to assume that the flow starts immediately with the normal depth, y_{no}, and the normal velocity, v_{no}; that is

$$v = v_{no} = C\sqrt{S_o\, y_{no}}$$

$$y = y_{no} = \left(\frac{q_o^2}{C^2\, S_o}\right)^{1/3}$$

The boundary condition at $x = 0$ is essentially the same as

$$q = v\,y = q_o$$

The pattern of the characteristic (Eq. 126) with these initial and boundary conditions is schematically plotted in Fig. 15. The slope of the wave front characteristic at the origin is

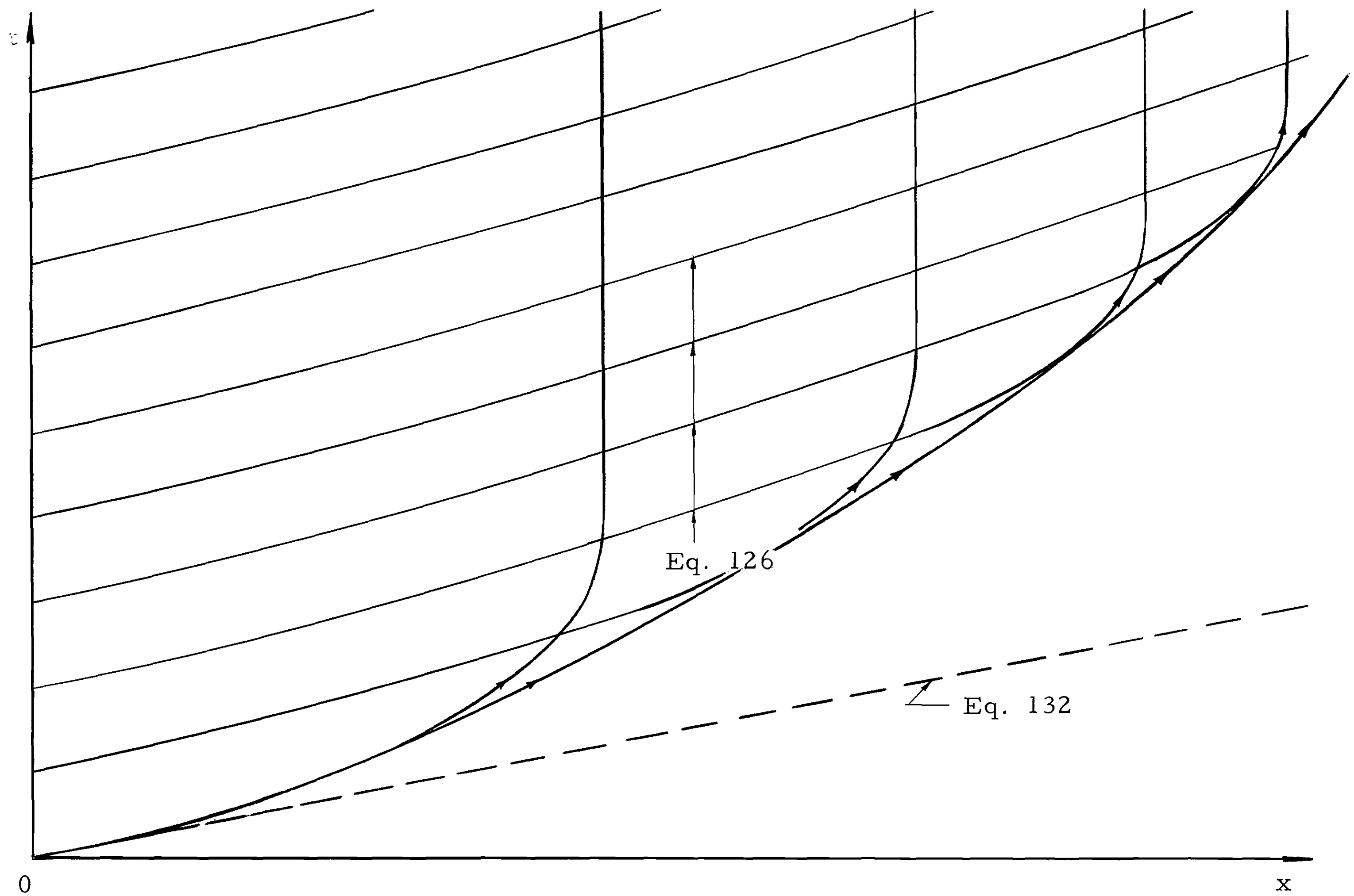


Fig. 15. Pattern of characteristic for $\bar{i} = at^b$ in the kinematic-wave method.

$$\frac{dx}{dt} = K N y_{no}^{N-1} \quad . . . . \quad (132)$$

The first increment of advance, Δx_1, is thus $\Delta x_1 = K N y_{no}^{N-1} \Delta t$ for small fixed time interval, Δt. Along the trajectory of the wave front, Eq. 127 is satisfied. If the infiltration function for the Kostiakov-Lewis type (Eq. 115) is considered, the finite-difference method for a small increment of time, Δt, gives the recurrence relation between the $(j + 1)$ point and the j point along the wave front.

$$y_{j+1} - y_j = - a(\Delta t)^b (\Delta t)$$

or

$$y_{j+1} = y_j - a(\Delta t)^{b+1} \quad . . . . \quad (133)$$

Specially, for $j = 0$, $y_o = y_{no}$, the next increment of advance, Δx_2, after $y_1 = y_o - a(\Delta t)^{b+1}$ is calculated from Eq. 133, namely

$$\Delta x_2 = K N y_1^{N-1} \Delta t$$

In general,

$$\Delta x_{j+1} = K N y_j^{N-1} \Delta t \quad . . . . \quad (134)$$

Thus, the sequential generation of the values of y and Δx along the trajectory is attained by both Eqs. 133 and 134.

The characteristic issued from the boundary $(x = 0)$ after elapsing Δt next to the wave front is somewhat modified if the same Kostiakov-Lewis infiltration function is used. The recurrence formula for y becomes

$$y_{j+1} = y_j - 2^b a (\Delta t)^{b+1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (135)$$

while the recurrence formula for Δx is still the same as Eq. 134. In general, for the k-th characteristic issued from the boundary after $(k - 1)\Delta t$ time has been elapsed, the recurrence formula for y is

$$y_{j+1} = y_j - k^b a (\Delta t)^{b+1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (136)$$

The calculation of the values of y and Δx for each characteristic must be stopped when they become zeroes at the same time. This time is called the time of equilibrium. For the Kostiakov-Lewis infiltration function, each characteristic apparently has a different time of equilibrium as shown in Fig. 15. The corresponding free-surface profiles for this type of infiltration function with each increment of the fixed-time interval, Δt, is schematically plotted in Fig. 16. Before Δt is reached, the depth of flow at the wave front is evidently not zero by this approximate method. There is also no maximum length of advance when the Kostiakov-Lewis infiltration function is adopted.

As a special case, however, if the infiltration function is constant (i.e., $\bar{i} = at^b$ for $b = 0$), the direct integration over the characteristic equation (Eq. 127) along the characteristic curve (Eq. 126) is possible because $\bar{i} = a =$ const. In this case, all of the characteristics issued from the boundary $(x = 0)$ at different times are all parallel to each other and have the same time of equilibrium, t_e. Hence, it can be readily seen that there must exist a maximum length of advance for this case. The trajectory of the wave front and the free-surface profiles can be expressed as follows:

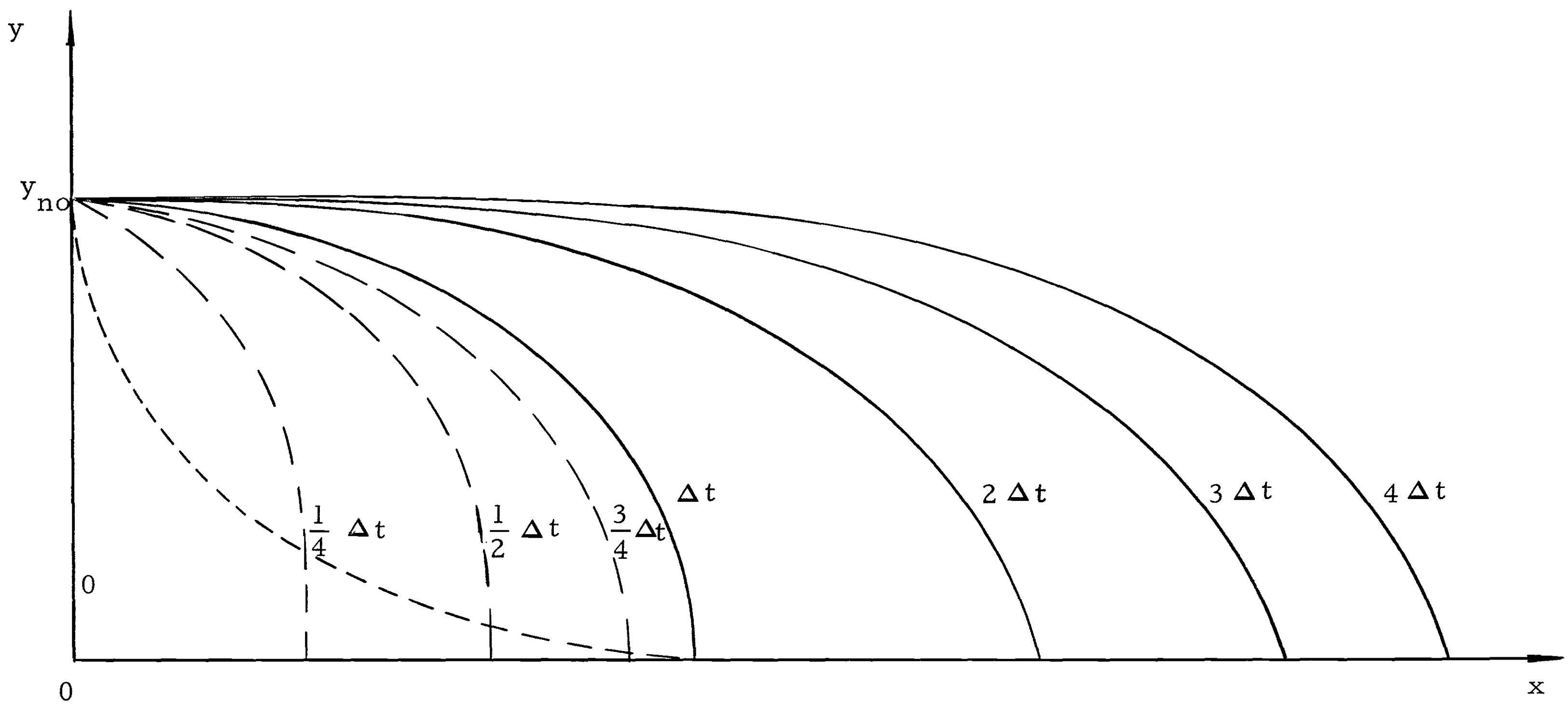


Fig. 16. Free-surface profiles for $\bar{i} = at^b$ in the kinematic-wave method.

From Eqs. 127 and 131, since $\bar{i} = a$, integrating with respect to t yields

$$y = -at + A_1$$

$$q = -ax + B_1$$

in which A_1 and B_1 are the constants of integration that must be determined from the initial and boundary conditions. The initial condition that $y = y_{no}$ as $t = 0$ gives $A_1 = y_{no}$ and the boundary condition that $q = q_o$ as $x = 0$ yields $B_1 = q_o$. Consequently, one has the forms of the flow depth at the wave front and discharge-distribution equations, respectively.

$$y = -at + y_{no} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (137)$$

$$q = -ax + q_o \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (138)$$

The maximum length of advance, L, can be obtained from Eq. 138 by setting $q = 0$, thus

$$L = \frac{q_o}{a} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (139)$$

An alternative form of Eq. 138 is

$$K y^N = K y_{no}^N - ax. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (140)$$

or

$$\left(\frac{y}{y_{no}}\right)^N = 1 - \frac{ax}{K y^N} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (141)$$

From Eq. 140,

$$y = \left(\frac{q_o}{K} - \frac{ax}{K}\right)^{\frac{1}{N}}$$

Substituting this y value into Eq. 126 and integrating yields

$$\frac{N}{a}(q_o - ax)^{\frac{1}{N}} = -K^{\frac{1}{N}} N t + A_1$$

in which A_1 is again a constant of integration. The condition that $x = 0$ as $t = 0$ gives

$$A_1 = \frac{N}{a} q_o^{\frac{1}{N}}$$

Consequently

$$q_o^{\frac{1}{N}} - (q_o - cx)^{\frac{1}{N}} = K^{\frac{1}{N}} a t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (142)$$

This is the equation of the trajectory of the wave front in the x, t-plane. The time of equilibrium, t_e, corresponding to the maximum length of advance, L, attained can be obtained from Eq. 142 by letting $t = t_e$ for $x = \frac{q_o}{a}$; namely

$$q_o^{\frac{1}{N}} = K^{\frac{1}{N}} a t_e$$

$$t_e = \frac{y_{no}}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (143)$$

which is also obtainable from Eq. 137. The pattern of characteristics and the free-surface profiles are plotted in Figs. 17 and 18 respectively for this special case.

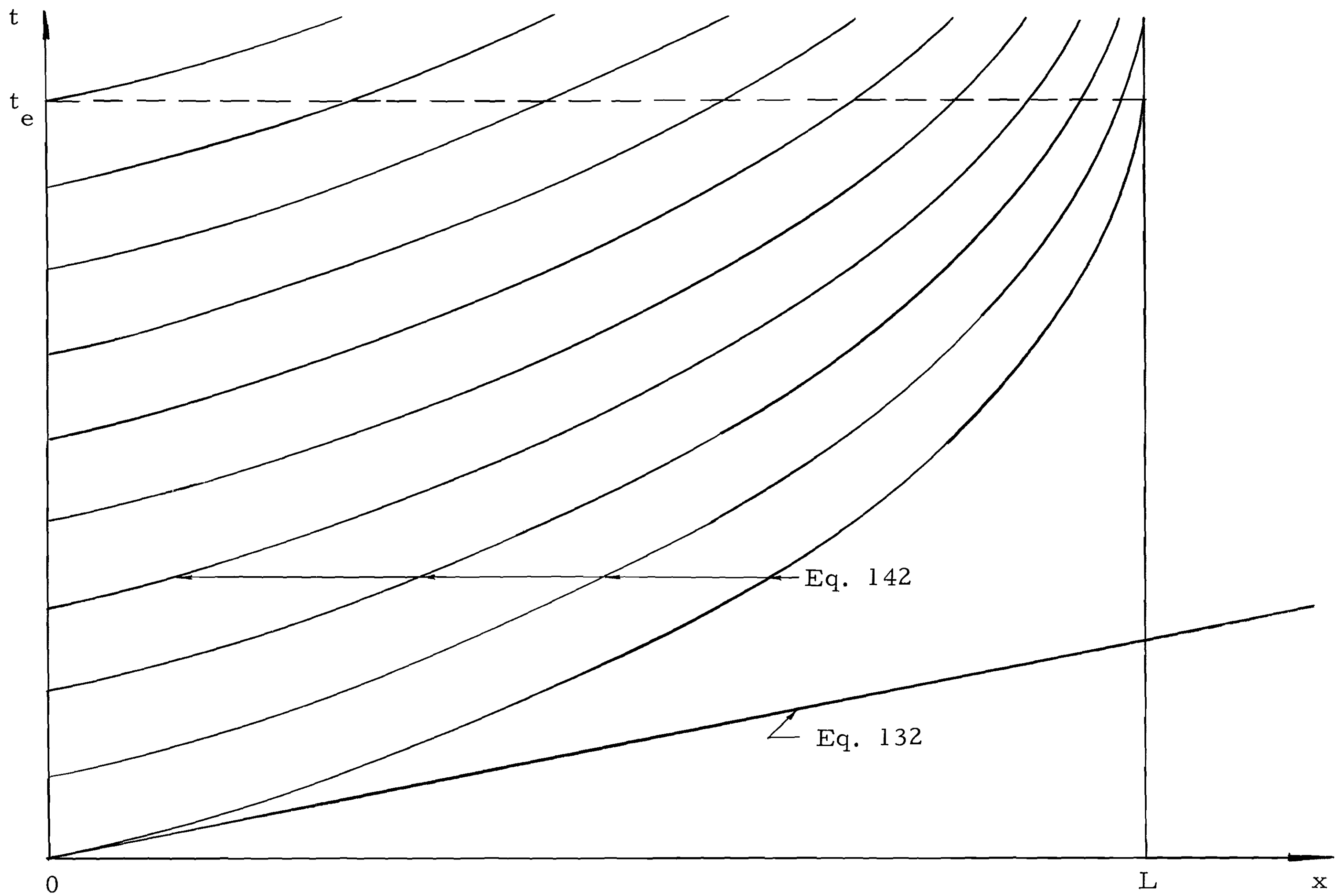


Fig. 17. Pattern of characteristic for $\bar{i} = a$ in the kinematic-wave method.

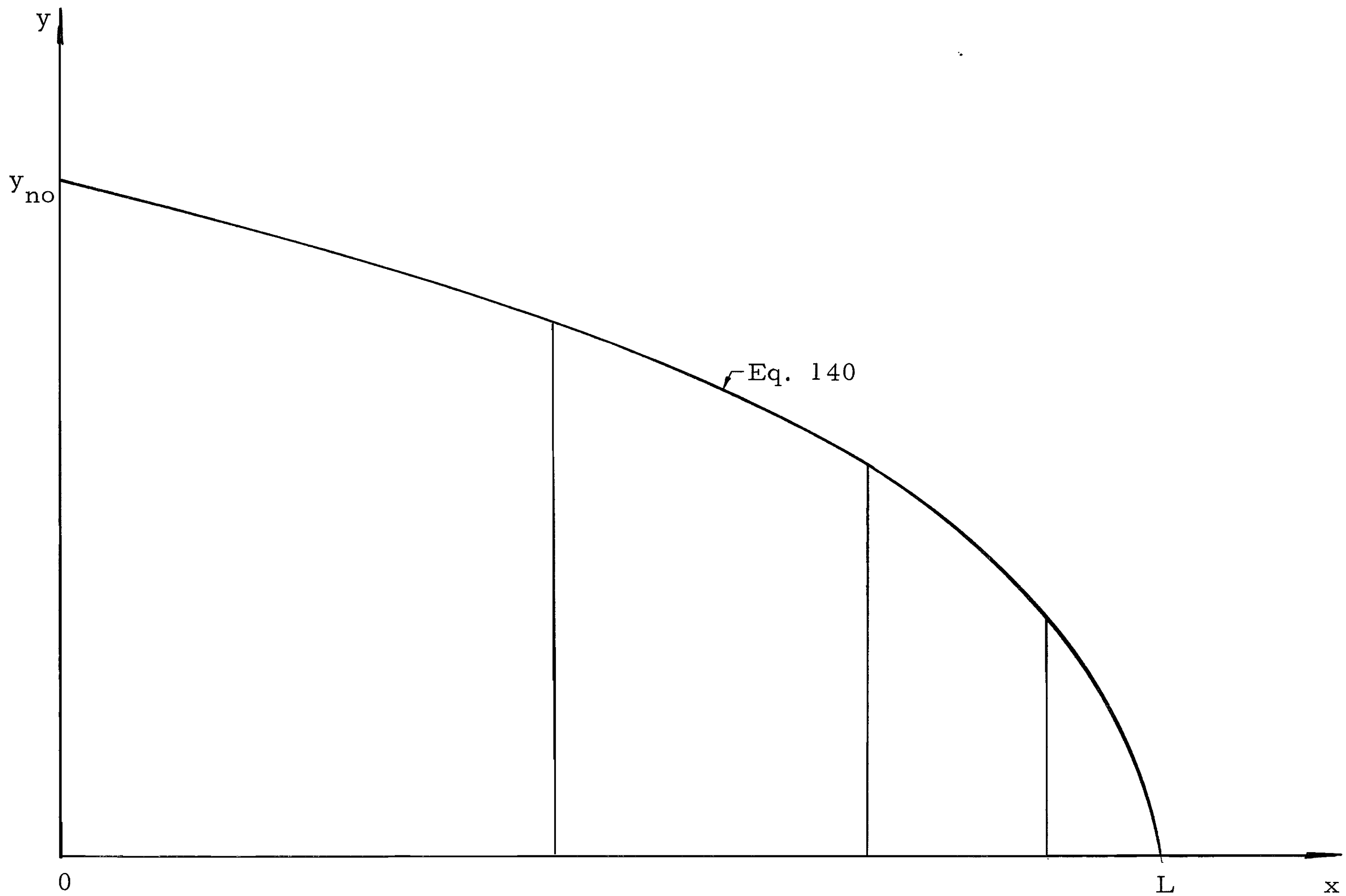


Fig. 18. Free-surface profiles for $\overline{i} = a$ in the kinematic-wave method.

The kinematic-wave method is the most simplified method so far as it can be deduced from the general equations without losing too much the generality of the fomation of the mathematical model for the flow of water-waves in the surface irrigation. The method still gives the equations of the free-surface profiles of forwarding waves. Further simplification of the method will result in the "classical" approach of the simple volume-balance concept, in which only the equation of continuity is considered. In the latter method, the writer's interest is only the solution of the trajectory of the wave front. Many investigations have been conducted in this area and among them, the studies of Chen (1965) and Philip and Farrell (1964) can be cited. Although the latter method is much simpler than the former method, no information about the free-surface profiles is obtainable from the latter method, so it is of no concern herein.

APPENDICES

REFERENCES

1. Chen, Cheng-lung. 1965. "Techniques of Border Irrigation by a Hydrologic Method of Routing." Technical Report PR-WR11-1, Utah Water Research Lab., Utah State Univ., Logan, Utah, 97 pp.

2. Chen, Cheng-lung. 1966. Discussion on the Article, "A Solution of the Irrigation Advance Problem," by Otto Wilke and Ernest T. Smerdon (Am. Soc. Civ. Engs., Proc. Paper 4471), J. of the Irrig. and Drain. Div., Am. Soc. Civ. Engs., Vol. 92, No. IR2, June, pp. 97-101.

3. Chen, Cheng-lung, and Vaughn E. Hansen. 1966. "Theory and Characteristics of Overland Flow." Trans., Am. Soc. Agr. Engs., St. Joseph, Michigan, Vol. 9, No. 1, pp. 20-26.

4. Chow, Ven Te. 1959. "Open-Channel Hydraulics." McGraw-Hill Book Company, Inc., New York, pp. 20-24, pp. 64-66, pp. 128-130.

5. Courant, R., and K. O. Friedricks. 1948. "Supersonic Flow and Shock Waves." Interscience Publishers, Inc., New York, pp. 92-115.

6. Dressler, R. F. 1949. "Mathematical Solution of the Problem of Rollwaves in Inclined Open Channels." Communications on Pure and Applied Mathematics, Vol. 2, pp. 149-194.

7. Dressler, R. F. 1952. "Hydraulic Resistance Effect upon the Dam-Breaking Functions." J. of Research of the National Bureau of Standards, Washington, D. C., Vol. 49, No. 3, September, Research Paper 2356, pp. 217-225.

8. Dressler, R. F. 1958. "Unsteady Non-linear Waves in Sloping Channels." Proc., Royal Soc. of London, 247A, pp. 186-198.

9. Henderson, F. M., and R. A. Wooding. 1964. "Overland Flow and Groundwater Flow from a Steady Rainfall of Finite Duration." J. of Geophysical Research, Vol. 69, No. 8, pp. 1531-1540.

10. Philip, J. R., and D. A. Farrell. 1964. "General Solution of the Infiltration-Advance Problem in Irrigation Hydraulics." J. Geophys. Res., Vol. 69, No. 4, pp. 621-631.

11. Kruger, W. E., and D. L. Bassett. 1965. "Unsteady Flow of Water Over a Porous Bed Having Constant Infiltration." Trans. Am. Soc. Agr. Engs., St. Joseph, Mich., Vol. 8, No. 1, pp. 60-62.

12. Ralston, Anthony, and Herbert S. Wilf. 1964. "Mathematical Methods for Digital Computers." John Wiley & Sons, Inc., New York, pp. 165-188.

13. Rouse, Hunter. 1949. "Engineering Hydraulics." Flood Routing Section by B. R. Gilcrest, John Wiley & Sons, Inc., pp. 635-710.

14. Stoker, J. J. 1957. "Water Waves." Interscience Publishers, Inc., New York.

15. Tinney, E. R., and D. I. Bassett. 1961. "Terminal Shape of Shallow Liquid Front." J. of the Hydraulics Div., Am. Soc. Civ. Engs., Vol. 87, No. HY5, Proc. Paper 2934, September, pp. 117-133.

NOTATIONS

The following symbols have been adopted for use in this study.

A = cross-sectional area of flow;

A_c = cross-sectional area corresponding to a critical depth;

A_n = cross-sectional area under consideration corresponding to a normal depth;

A_1 = integration constant;

a = infiltration constant;

B_1 = integration constant;

b = bottom width of the channel section; infiltration constant;

C = Chezy's roughness coefficient;

C^+ = characteristic curve;

C^- = characteristic curve;

c = celerity = $\sqrt{gy}$;

D = hydraulic depth;

D_c = hydraulic depth corresponding to a critical depth;

F = force on the element in the x-direction;

g = gravitational acceleration;

i = infiltration rate;

$\bar{i}$ = uniform infiltration rate across the top width;

K = conveyance of the channel section at an actual flow depth, coefficient;

K_n = conveyance of the channel section corresponding to a normal depth;

NOTATIONS (Continued)

k_1 = constant;

k_2 = constant;

L = maximum length of advance for the wavefront;

M = momentum flux of the element in the x-direction;

m = exponent;

N = coefficient;

n = exponent;

P = static pressure force on a normal section of the element;

Q = discharge of flow;

Q_o = constant discharge of inflow at the upstream end;

R_n = hydraulic radius of the channel section corresponding to a normal depth;

S_f = frictional slope of flow;

S_o = bed slope;

T = top width of the free surface at the section considered;

t = time;

t_e = time of equilibrium when the wavefront reaches the maximum length of advance;

U = uniform velocity of uniformly progressive wave flow;

V_w = uniform velocity of advancing wave

v_{co} = critical velocity at $x = 0$;

v_n = normal velocity;

v_{no} = normal velocity at $x = 0$;

NOTATIONS (Continued)

W = weight of the element per unit length;

x = distance in the direction of flow;

y = depth of flow;

$\bar{y}$ = depth of the centroid of the cross-sectional area from the free surface;

y_c = critical depth;

y_{co} = critical depth at $x = 0$;

y_n = normal depth;

y_o = depth of water at $x = 0$;

y_2 = depth of water in the zone of quiet downstream;

Z = Section factor for critical flow computation;

Z_c = critical section factor at a critical depth;

β = momentum coefficient;

θ = angle of inclination of the bed with respect to the horizontal surface;

υ = kinematic viscosity of the fluid;

ξ = coordinate variable = $x - Ut$;

ρ = density of the fluid;

τ_o = shearing stress on the boundary of the section considered;

ψ = a function used to define a flow depth and discharge for the case of no infiltration.