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Optimal Regional Potentiometric Surface Design: Least-Cost Water Supply/Sustained Groundwater Yield

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ABSTRACT

A distributed parameter groundwater management model utilizing quadratic programming to develop an optimal regional steady-state potentiometric surface and its sustained groundwater withdrawal strategy is presented. It minimizes the regional cost of attempting to satisfy the water needs of each finite-difference cell (a) from groundwater and diverted surface water or (b) from groundwater and reduction of water needs achieved by reducing production acreages. Groundwater elevations, withdrawal and recharge are constrained, satisfying legal and hydrologic constraints. The technique is applicable for assuring a regional sustained yield of groundwater in a conjunctive water management setting. It represents the first application of optimization in the "target objective" approach to regional groundwater management.

INTRODUCTION

In the Grand Prairie of eastern Arkansas (Fig. 1), most of the irrigation water needs for rice and soybeans have historically been provided by groundwater from an unconsolidated Quaternary alluvium. This extensive formation, the Mississippi alluvial aquifer, underlies much of eastern Arkansas as well as parts of neighboring states. An impermeable clay layer, however, prevents recharge of the aquifer in the Grand Prairie, except at some locations along the area's periphery where streams penetrate to the permeable material (Engler et al., 1945; Griffis, 1972; Peralta et al., 1985). As a result, recharge has not kept pace with groundwater pumping, and the potentiometric surface has been declining. Saturated thicknesses are decreasing and in some locations Quaternary groundwater cannot be obtained at useful discharge rates. This trend is projected to continue if current groundwater usage continues (Peralta et al., 1985). If stable groundwater levels are to be achieved and maintained, alternative sources of water will have to be developed to meet current water needs.

The Grand Prairie Water Supply Project was initiated to determine how best to physically and legally coordinate the uses of available water resources to meet long-term water needs. This requires developing the technical/institutional tools necessary to implement the resulting water management strategy, should that be desired. An overview of the subprojects, funding agencies and critical path approach to the effort is described by Peralta et al. (1984).

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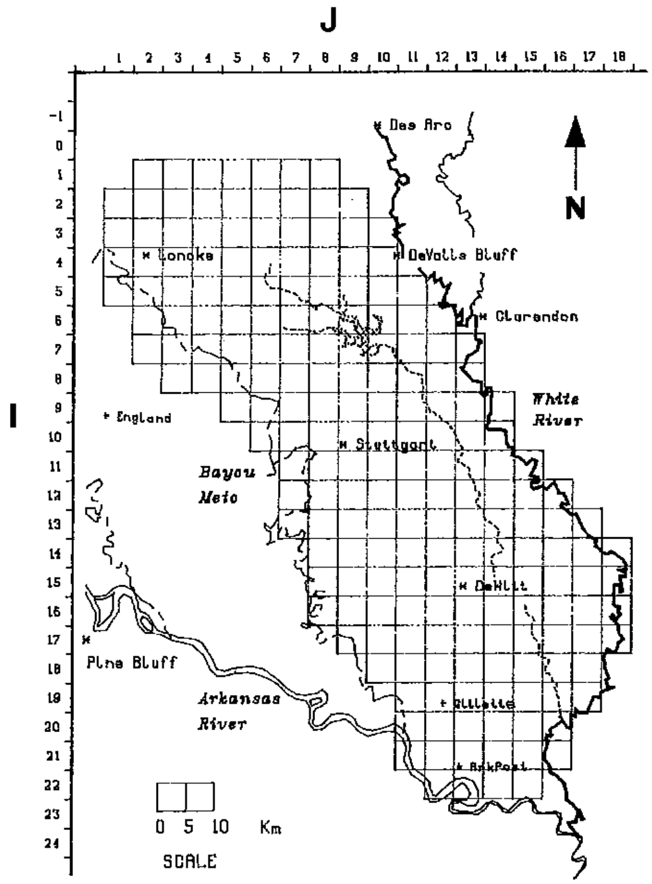

Fig. 1—The Grand Prairie study area.

One significant result of the project has been acknowledgement in the state water plan (Peralta and Peralta, 1984b) of the physical and legal feasibility of achieving a steady-state potentiometric surface. The approach, which requires the conjunctive use of groundwater and surface water, is considered only for areas with critical groundwater problems. In the approach, a "steady-state" potentiometric surface (i.e., target springtime groundwater levels) is maintained year after year. A finite difference form of the Boussinesq equation is used to determine the annual volume of groundwater which, if pumped from the aquifer in each cell, will maintain a particular set of target levels. As shown in Fig. 1 each cell is 5 km by 5 km in size.

Peralta and Peralta (1984a) presented an example, using dynamic simulation, in which spring target levels were maintained for ten years. In the example, simulated water needs and groundwater usage differed from month to month in accordance with climatologic influence on evapotranspiration and user operations. In addition, the sum of 12 consecutive monthly pumping values for a particular cell equalled that cell's annual withdrawal volume as calculated by the steady-state equation. The example demonstrated that maintaining a steady-state potentiometric surface over the long term can be achieved by limiting annual groundwater withdrawals to the volume calculated using the Boussinesq equation.

There are many possible sets of steady-state (target) groundwater levels for any area, each one corresponding to a particular strategy of sustained groundwater withdrawal. Depending on the management objective, one set of target levels is more desirable than the others. An important part of the Grand Prairie project is the development of a means of determining optimal target levels. The objective of this paper is to describe a model used to determine the regional potentiometric surface that results in the lowest annual expenditure for meeting water needs from ground and alternative water resources. The model is an optimizing computer program that incorporates a modified version of the quadratic programming subroutine written by Liefsson et al. (1981). Equations describing porous media flow are used as constraints to permit calculation of the groundwater withdrawals that will maintain the optimal surface. These spatially distributed groundwater withdrawals thus represent a sustained-yield pumping strategy.

METHODOLOGY

Governing Equation

Developing a regional steady-state set of target groundwater levels requires the use of a steady-state equation for each cell. The following has been developed for two-dimensional steady flow in a heterogeneous isotropic aquifer from both the linearized Boussinesq equation (Illangasekare and Morel-Seytoux, 1984) and the Darcy equation (Peralta and Peralta, 1984b):

$$q_{i,j} = - t_{i-1/2,j}\, s_{i-1,j} - t_{i+1/2,j}\, s_{i+1,j} + [t_{i-1/2,j} + t_{i+1/2,j} + t_{i,j-1/2} + t_{i,j+1/2}]\, s_{i,j} - t_{i,j-1/2}\, s_{i,j-1} - t_{i,j+1/2}\, s_{i,j+1} \qquad [1]$$

where

$q_{i,j}$ = the net vertical flux rate of groundwater moving into or out of the aquifer in cell (i,j). It is positive when flow is out of the aquifer, negative when flow is into the aquifer, L^3/T

$s_{i,j}$ = vertical distance between a horizontal datum located above the ground surface, and the potentiometric surface. In this paper $s_{i,j}$ is a steady state drawdown, L

$t_{i-1/2,j}$ = is the geometric average of the transmissivities of cells (i,j) and (i−1,j), L^2T

To express this equation in matrix form for a groundwater system, the row-column notation is replaced with single integer identification of each cell. Thus for a groundwater flow system of n cells:

$$(Q) = [T](S) \qquad [2]$$

where

(Q) = a n x 1 column vector of net steady-state flux values, L^3/T

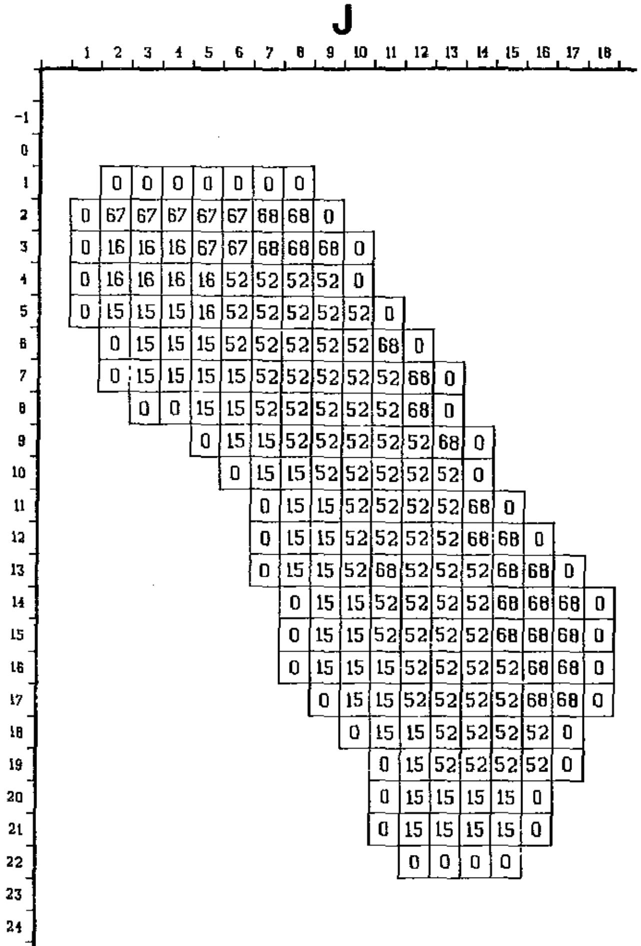

Fig. 2—Unit cost of "alternative" water ($/dam³).

[T] = a n x n symmetric diagonal matrix of finite difference transmissivities, L^2/T

(S) = a column vector of steady-state drawdowns, L

In applying this equation to the Grand Prairie, one considers the following. Transmissivities are based on a hydraulic conductivity of 82 m per day (Engler et al., 1945; Griffis, 1972; Peralta et al., 1985). Validation of an unsteady-state groundwater simulation model AQUISIM, developed by Verdin et al. (1981), demonstrated that the study area can be treated as a groundwater system surrounded by constant-head cells (Peralta et al., 1985). In the validation, the groundwater level in each constant-head cell equalled the average of ten years of observed springtime groundwater levels in that cell. Cells showing a 0 value in Fig. 2 were used as constant-head cells in the validation and in the management model presented in this paper.

The value in (Q) corresponding to a constant-head cell is the annual volume of water entering (−) or leaving (+) the aquifer at that cell. Since no groundwater withdrawal by wells is considered at constant-head cells, for those cells the value in (Q) represents the annual volume of water moving between the aquifer and either the surrounding aquifer system or a stream located within the cell. In this paper the term "pumping", p, is used to refer to groundwater withdrawal via wells.

Vertical recharge of the aquifer in the Grand Prairie is negligible for interior cells (non-constant-head cells). Therefore, the net annual vertical flux for each interior cell equals its groundwater pumping volume and the value in the (Q) vector corresponding to an interior cell is nonnegative.

Estimating Unit Costs of Supplied Water

In this paper, the term "water needs" refers to current groundwater usage. It is assumed that actual current needs being met by other means will continue to be met by those means. The problem the management model addresses is how best to replace current groundwater usage with a combination of new alternative water sources, use reduction, and groundwater. This section briefly describes the determination of unit costs of alternative water, reduction of water use and groundwater.

For purposes of the paper, new alternative sources of water include water diverted from the Arkansas River and water diverted from the White River. Diverted surface water is the new alternative source in all cells where surface water is assumed to be available. In all other cells, reduction in use (by reducing irrigated or aquacultural acreage) is the alternative. A state or local water management agency may consider other alternative water sources in using the model. Increased use of on-farm reservoirs or reduced water needs due to conservation measures can provide means of balancing supply and demand. Harper (1983) provides a preliminary assessment of the cost per unit volume and the potential quantitative availability of water from these on-farm practices for the Grand Prairie.

The cost per cubic dekameter of an appropriate alternative water source (c_a) or the opportunity cost of reduced pumping (also c_a) for each cell is shown in Fig. 2. Preliminary U.S. Army Corps of Engineers investigations indicate to which cells diverted Arkansas River water can be delivered. These cells show a price of \$15 or 16 per cubic dam (personal communication, Joe Clements) diverted from the Arkansas River. A price of 52 \$/dam^3 labels cells to which the Corps feels that water from the White River can probably be diverted (personal communication, Richard Coleman). Recent reconnaissance-level evaluation indicates that legally and physically available water from the Arkansas and White Rivers is adequate to meet water needs in the cells serviceable by those rivers, assuming average climatologic and hydrologic conditions (Dixon and Peralta, 1984).

Opportunity cost is adopted as the unit price of the loss in revenue due to reduced pumping in cells for which diverted surface water is unavailable. The opportunity cost associated with failure to satisfy water needs is assumed to be 67 \$/dam^3 for aquacultural production of fish or minnows and 68 \$/dam^3 for rice production. These values reflect the reduction in net benefits caused by having to replace aquacultural or rice production with an unirrigated crop. Conversion to fallow could also be considered.

The costs of delivering a unit volume of diverted surface water to a watercourse within a cell (c_a), that we have used, do not include the cost of delivering the water to a particular field within the cell. For this paper we assume that the total costs of conveying water from the watercourse to a field equal the fixed costs of a well and pump system for obtaining groundwater at the field. Therefore, in order to economically compare the use of diverted surface water with groundwater at a cell we consider c_a for that cell as well as the variable cost of obtaining a unit volume of groundwater at a field within the cell. We use two different unit costs together, c_e and c_m, to determine the variable cost of obtaining groundwater at the field. The first unit cost includes energy, repair and lubrication costs for the pumping power plant, and is associated with raising a unit volume one unit distance. The value used for c_e, \$0.48 per dam^3·m (cubic dekameter-meter), was developed assuming a representative mix of electric, natural gas and diesel power plants, current energy prices and a 500 gpm well pumping to meet irrigation needs. The second unit cost, a function of pump maintenance costs alone, is associated only with each unit volume of groundwater that is pumped. The value selected for c_m for this paper, \$1.34 per dam^3, is appropriate for a turbine pump. Using these unit costs, if the total dynamic head (h) of a sample well is 50 m, the total cost of obtaining one cubic decameter of groundwater at the ground surface is [(0.48 \$/dam^{3}m) (50 m) + 1.34 \$/dam^3] (1 dam^3) = \$25.34. This approach to determining groundwater costs is used in the subsequent section dealing with model formulation. It should be noted that the use of economic assumptions different than those described above simply requires the use of different unit cost values.

Formulation and Utilization of the Management Model

Assuming that the annual needed water volume w in a cell is met either by groundwater or alternative water, the volume of alternative water used equals (w − p), where p is the volume of groundwater used. Utilizing this assumption, a simple statement of the objective of minimizing the total cost, X, of satisfying annual water needs for a system of m internal cells is:

$$\min X = \sum_{k=1}^{m} [(c_{e_k} h_k + c_{m_k}) p_k + c_{a_k} (w_k - p_k)] \qquad [3]$$

where

X = the total cost of satisfying annual regional water needs from groundwater and alternative sources, \$/T

h_k = an estimate of the total dynamic head of a well pumping at the center of cell k, L

p_k = the annual groundwater pumping volume from cell k, L^3/T

$(c_{e_k} h_k + c_{m_k}) p_k$ = the total annual cost of groundwater pumped in cell k, \$/T

$c_{a_k} (w_k - p_k)$ = the total annual cost of alternative water or opportunity cost in cell k, \$/T

Equation [3] is an objective function containing the unknown variables h and p. There are two problems with the manner in which it is formulated. First, the stated goal is to design an optimal steady-state spring potentiometric surface. Using h as a variable is not as satisfactory for this purpose as the alternative of expressing h in terms of steady-state drawdown. Second, as discussed later, it is also best for p to be described in terms of steady-state drawdown. The next four paragraphs explain how the reformulation is accomplished. Needed definitions are presented first.

Static lift is the difference in elevation between a steady-state potentiometric surface and the ground surface. Recall that s is the distance between the potentiometric surface and a horizontal datum located above the ground surface. Defining s_e as the distance between the ground surface and the horizontal datum, static lift equals $(s - s_e)$. In this study area static lift is the major contributor to h.

The optimization procedure requires initially assumed values of steady-state drawdown for all cells. In the process of minimizing the value of X, these steady-state drawdowns change. A change in h caused by a change in the steady-state drawdown is approximately proportional to the change in static lift. Therefore, an estimate of h_* for a new (optimal) groundwater level for a cell is expressed as:

$$h_* = h_i\,(s_* - s_e)/(s_i - s_e) \qquad [4]$$

for

$s_i - s_e > 0$

where

h_* = the total dynamic head for the optimal steady-state drawdown for the cell, L

h_i = the initially estimated total dynamic head for the cell, L

s_* = the optimal steady-state drawdown for the cell, L

s_i = the initially assumed steady-state drawdown for the cell, L

Prior to optimization, h_i is estimated for a representative well and irrigation system in the center of each cell. It is developed using simulated pumping rates corresponding to irrigation scheduling, and the aquifer saturated thickness appropriate for the initially assumed groundwater level. It equals the average difference between the ground surface elevation and the groundwater level at the well during simulated pumping.

Since the h of equation [3] equals h_* at optimality, the right-hand side of equation [4] can be used to replace h in equation [3]. If in addition, f is defined as $c_e h_i/(s_i - s_e)$, ($/1^4$), equation [3] can be rewritten as:

$$\min X = \sum_{k=1}^{m} [f_k\, s_{*_k}\, p_k - (f_k\, s_{e_k} - c_{m_k} + c_{a_k})\, p_k + c_{a_k}\, w_k] \qquad [5]$$

As mentioned previously, the annual net vertical flux for each internal cell in the Grand Prairie equals the annual steady-state pumping volume for that cell. Thus p in equation [5] is equivalent to q in Equation 1 and can be expressed as a function of steady-state drawdowns. We next explain why this substitution is made.

The quadratic programming subroutine used in this model is based on the general differential algorithm (Wilde and Beightler, 1967) that was developed in considerable computational detail by Morel-Seytoux (1972). In order to insure that a local minimum is also a global minimum, the objective function must be convex. Convexity is assured if the symmetric n by n matrix (Hessian matrix) of second partial derivatives of the objective function with respect to the variables is positive definite. This is achieved by replacing each p in equation [5] with the right-hand side of equation [1]. Convexity is verified using the method of principal minors.

Equation [6] below is the resulting objective function for minimizing total regional cost, expressed in standard quadratic programming form. Equation [6] and its attendant constraint equations [7] and [8] represent the optimization management model. In these equations, $[T_i]$ is the m x m matrix of finite difference transmissivities (analogous to that of equation [2]) for the m interior cells of the system.

$$\text{Min } X = (1/2)\,(S_*)^t\,[T_{ia}]\,(S_*) - [T_{ib}]\,(S_*) + (Y) \qquad [6]$$

subject to

$$(L_q) \leqslant [T]\,(S) = (Q) \leqslant (U_q) \qquad [7]$$

$$(L_{s*}) < (S_*) \leqslant (U_{s*}) \qquad [8]$$

where

$(S_*)^t$ = the 1 x m transpose of the column vector of optimal drawdowns for the interior cells, L

$[T_{ia}]$ = the square symmetric matrix that results when each element of $[T_i]$, identified by its row k and column ℓ, is multiplied by a coefficient: $2f_k$ when k equals ℓ and $(f_k + f_\ell)$ when k does not equal ℓ, $\$/L^2 \cdot T$

$[T_{ib}]$ = the matrix that results when each column ℓ of [Ti] is multiplied by $((f_\ell)(s_{e_\ell}) - c_{m_\ell} + c_{a_\ell})$, $\$/L \cdot T$

(Y) = a m x 1 vector of constants. If the k^{th} cell is an internal cell, the value of the k element equals $(c_{a_k}\, w_k)$. If the k^{th} cell is a constant-head cell, the element also contains constants reflecting boundary conditions. (Y) is not included in the optimization, but is added to the output value to determine the total annual least cost, $/ T

(L_q) = actually a n x 1 column vector of the right-hand side of constraint equations. Since the equations equal net flux, however, it is referred to as the vector of lower bounds on net flux. The individual value is zero for all internal cells since no recharge occurs internally. For constant-head cells the lower bound on net flux is a negative number representing the estimated maximum physically feasible recharge at those cells, L^3/T

(Q) = a column vector representing the optimal steady-state flux values, i.e., optimal sustained-yield pumping values for all internal cells and the optimal volume fluxes for all constant-head cells, L^3/T

(U_q) = a column vector of upper bounds on the net flux in the cells. In this paper the upper bound for a particular internal cell is that cell's current annual groundwater pumping. The upper bound for constant-head cells equals a large positive number. This is our standard procedure since the total recharge for the entire system is limited to the total discharge. Limiting discharge from one system boundary may result in unnecessarily limiting recharge along a different boundary where recharge is needed, L^3/T

(L_{s*}) = a column vector of lower bounds on the optimal steady-state drawdown for internal cells, i.e., the ground surface elevation, L

(U_{s*}) = a column vector of upper bounds on the steady-state drawdown for internal cells, L

The upper bound on steady-state drawdown for each cell equals the drawdown that maintains 6 m of saturated thickness in the aquifer in that cell. Peralta et al. (1985) showed 6 m to be the spring saturated thickness at which representative wells pumping to meet the scheduled irrigation needs of rice in the Grand Prairie will begin to go dry during the irrigation season. In their simulations, they assumed one well per 50 acres of rice, no interference between wells, no hydraulic gradient other than that of the cone of depression, and average climatologic conditions. Site-specific studies can be conducted to more accurately determine the saturated thickness needed to satisfy certain conditions. For example, Dutram and Peralta (1984) show that 4 m is an adequate spring saturated thickness in cell (10,9), assuming droughty climatologic conditions, current acreages, existing wells, and the present hydraulic gradient.

The objective function of equation [6] uses transmissivities that are based upon the saturated thicknesses of initially assumed drawdowns. The optimization procedure changes the drawdowns. As drawdowns change, so should transmissivities. To accommodate this without introducing nonlinear constraints, sequential optimizations are performed: the second optimization uses as input the transmissivities corresponding to the optimal potentiometric surface of the first optimization, the third optimization uses the transmissivities appropriate for the results of the second optimization, etc, until the resulting optimal drawdowns are within an acceptable tolerance of the input drawdowns. In this example, after four successive optimizations, convergence to within 0.3 m was achieved for all cells. At the same time that the transmissivities are modified, values of f_k are changed to reflect total dynamic heads appropriate for the optimal drawdowns.

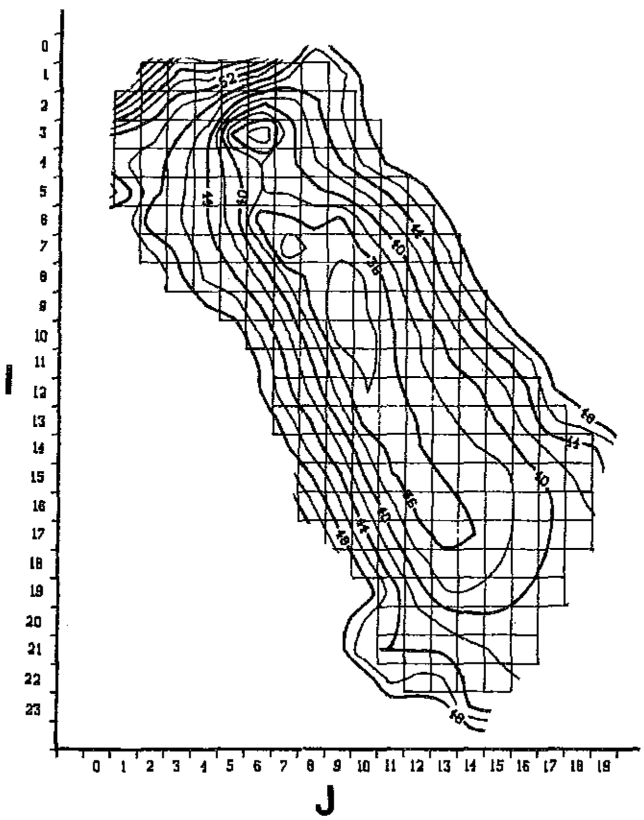

Fig. 3—Minimum-cost steady-state potentiometric surface (m above sea level).

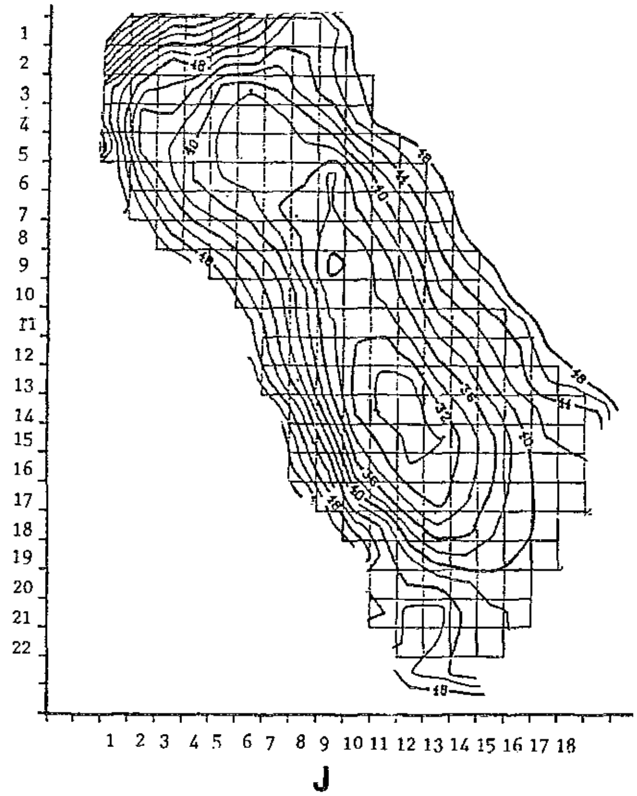

Fig. 4—Potentiometric surface in the springtime of 1982 (m above sea level).

RESULTS AND DISCUSSION

The optimal steady-state potentiometric surface based on the stated assumptions is shown in Fig. 3. Comparison of this surface with the potentiometric surface existing in 1982 (Fig. 4) indicates that in some locations, for example cell (14,11), the optimal surface is up to 8 m higher than the 1982 surface, while in other locations, for example cell (3,6), the optimal surface is down to 3 m lower than the 1982 surface.

The total cost per unit volume of groundwater in each cell, based on the optimal surface, is shown in Fig. 5. Study of Figs. 2 and 5 indicates that for the optimal potentiometric surface, groundwater is the less expensive source of water in most cells. Fig. 6 shows the percentage of the water needs of each cell met by groundwater in the optimal strategy. Comparing the cells serviced by surface water in Fig. 2 with the cells utilizing groundwater in Fig. 6, one realizes that under sustained-yield conditions, it is regionally less expensive to use surface water in most cells where it is available, despite the fact that groundwater is generally less expensive.

Fig. 7 shows cells in which water needs are not met by either groundwater or diverted surface water under the

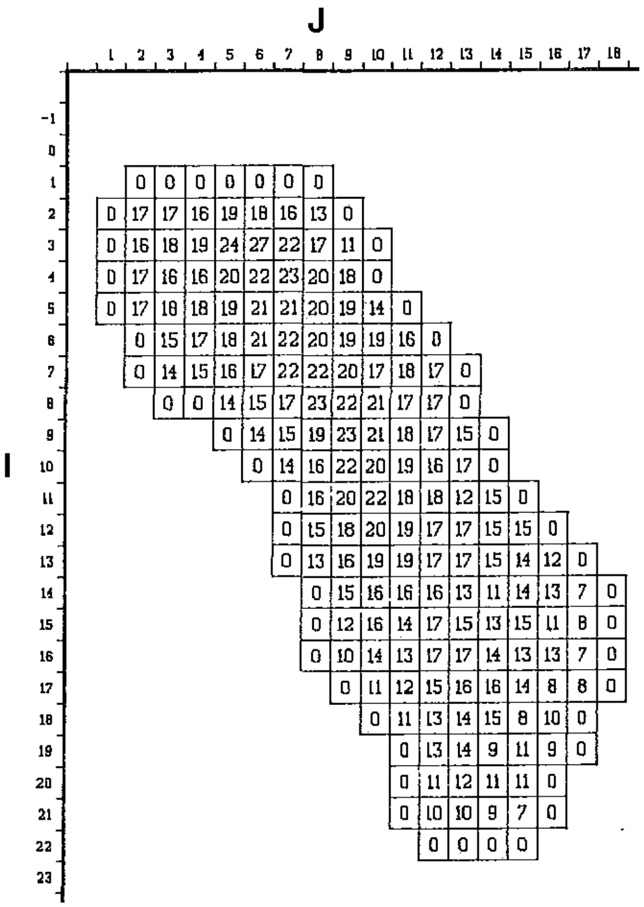

Fig. 5—Total unit cost of groundwater for the minimum-cost potentiometric surface ($/dam³).

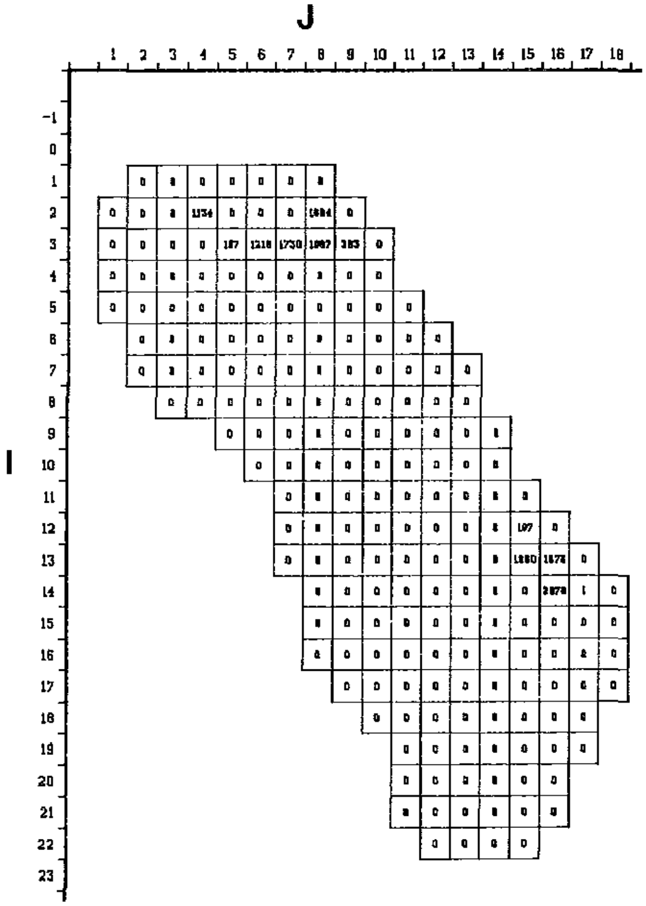

Fig. 7—Volume of unmet water needs under the minimum-cost strategy (dam³).

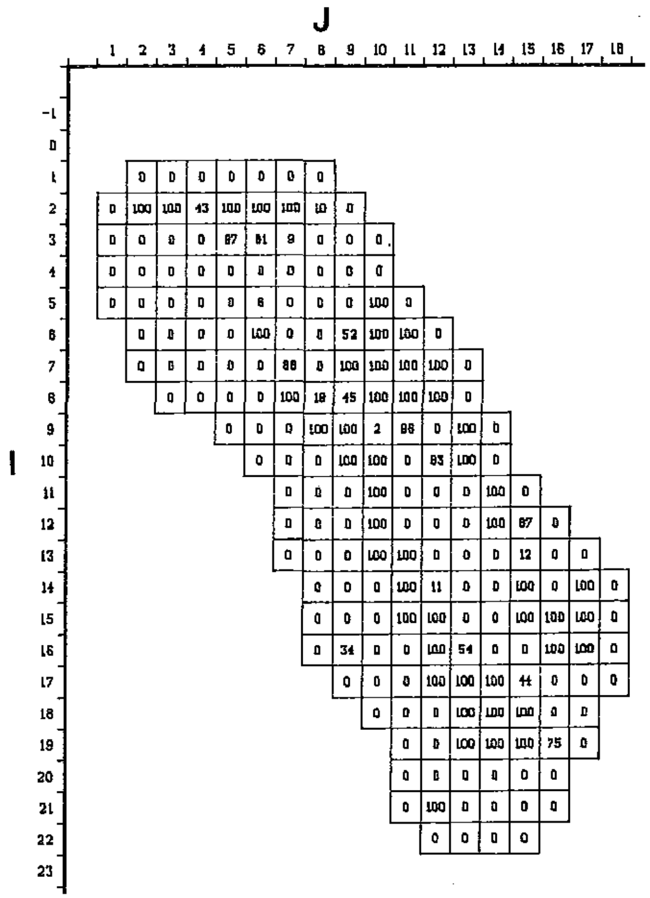

Fig. 6—Percent of water needs met by groundwater under the minimum-cost strategy.

adopted strategy. In these cells, the opportunity cost of lost aquacultural or rice production is used in the determination of the least-cost solution. Study of Figs. 2 and 7 reveals that water needs are unmet only in cells without diverted surface water. The volume of unmet water needs in these cells can be divided by 2.2 m or 0.6 m to estimate the acres of aquaculture or irrigated rice, respectively, that cannot be supported in these cells under the least-cost strategy.

The minimum value of the objective function, including the vector of constants (Y), is $9.1 million. Of this, $1 million is opportunity cost caused by conversion of aquacultural and rice acreage to nonirrigated corn acreage. The total volume of groundwater and diverted surface water provided is 341,000 dam³ at an average cost per cubic decameter of $24. As previously stated, preliminary evaluation indicates that the necessary volume of diverted surface water is physically and legally available from these rivers during climatologically "average" summers. A more detailed assessment of streamflow and demand is necessary before complete hydrologic feasibility can be determined. Of course, geohydrologic feasibility is assured by the use of bounds on recharge at peripheral cells and the inclusion of two-dimensional flow equations as constraints in the model.

The strategy described above uses specified unit costs for water. With time, these costs may change. One is interested in knowing how sensitive the optimal water-use strategy is to the assumed costs. In particular, one wants to know whether groundwater use and unmet water requirements will change.

Simple evaluation of the sensitivity can be performed by changing the cost per unit volume of alternative water (c_a) and the unit costs of groundwater, c_e and c_m. If all values of c_a, c_e and c_m are increased by either 50 or 100%, the total volumes of groundwater pumping and unmet water needs do not change. If groundwater unit costs are increased by 50% but all values of c_a remain the same, the changes are −4.5% and +2.6%, respectively. If the unit costs of groundwater are kept the same and the cost of alternative water increases 50%, the volume changes are 6.8% and 10.9%, respectively. Thus the optimal solution is relatively stable for minor changes in unit costs.

At the point of greatest difference between them, the spring 1982 potentiometric surface is about 8 m lower than the optimal potentiometric surface. At a different location, the optimal surface is about 18 m below a natural, unstressed potentiometric surface simulated for the area (using the same constant-head cell elevations). Thus, the optimal surface lies between the unstressed and the current surfaces.

As presented in this paper, the steady-state approach to determining an optimal solution does not include consideration of the time required for the optimal potentiometric surface to be attained. It is important to know how long it may take for groundwater levels to evolve to the optimal from specified initial conditions.

It has taken most of this century to dewater the aquifer to its present condition. Assuming that, beginning in 1982, the optimal pumping strategy presented in this paper were implemented, years would pass before actual levels approximated target levels. Dynamic simulations using AQUISIM (Verdin et al., 1981), validated for the Grand Prairie (Peralta et al., 1985), show that 95% of the cells would be within 4.5 m and 86 percent of the cells would be within 3 m of their target elevation within 10 years. After 30 years of pumping in accordance with the optimal strategy, 100% and 94% would be within 4.5 m and 3 m, respectively, of their target levels.

Had the optimal strategy been implemented at the initiation of aquifer development (i.e., beginning with unstressed conditions), after 10 years of pumping, 45% of the cells would have been within 4.5 m and 30% would have been within 3 m of their target elevations. After 30 years, 90% and 43% would have been within 6 and 3 m, respectively. Note that the water levels resulting from 30 years of implementation under this scenario are not as close to the optimal levels as are the levels resulting from 30 years of implementation beginning from 1982 levels. This results from the fact that the unstressed levels are much farther from the optimal levels than are the 1982 levels.

Assuming that the c_e unit cost of raising groundwater is $0.48/dam³·m, the 4.5-m difference between a simulated "actual" elevation and a target elevation represents a $2.16/dam³ difference between "actual" price and the price assumed in the development of the pumping strategy. Since optimal drawdowns are less than current drawdowns, the simulated total cost per unit volume of groundwater used for a particular cell is less than the cost that would actually be incurred were the strategy invoked at this time. However, the effect of this price difference on how much groundwater should be used is not greater than that resulting from increasing c_e and c_m by 50% without changing c_a, as described above. On the other hand, if an optimal pumping strategy is invoked at a time when actual drawdowns are less than the optimal drawdowns designed by the model, actual pumping costs during the evolutionary era are less than those predicted by the model.

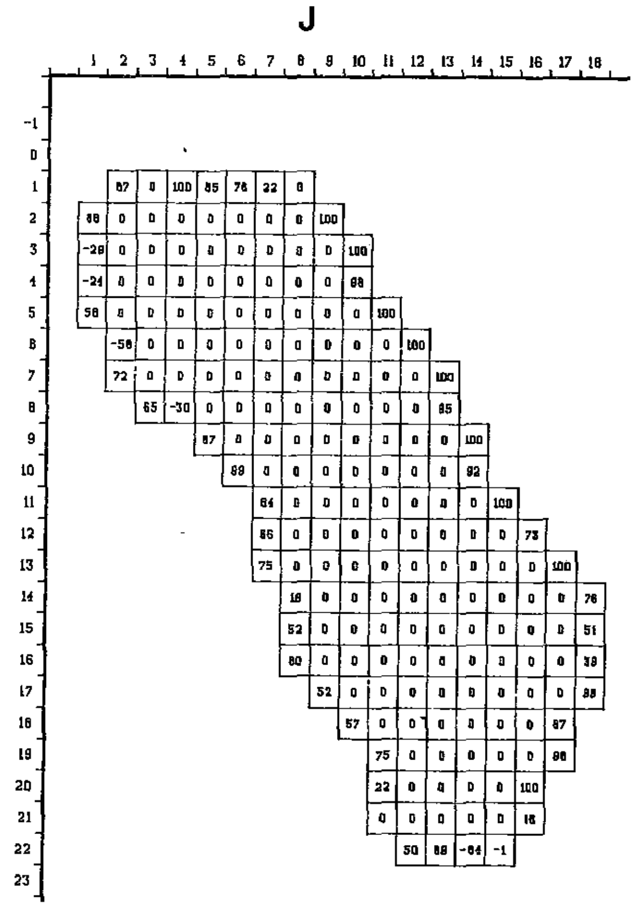

Fig. 8—Ratio of optimal recharge rate to maximum feasible recharge rate for constant-head cells under the minimum-cost strategy.

USING CONSTRAINED DERIVATIVES TO REFINE THE MANAGEMENT STRATEGY

Consider the situation simulated for the northern section of the Grand Prairie. In cell (1,4), 100% of the feasible recharge is utilized (Fig. 8). Fig. 7 shows that there is unsatisfied water demand in cell (2,4), directly south of cell (1,4). From the 67 $/dam³ cost of "alternative" water at cell (2,4) in Fig. 2, it is evident that no surface water is available in the cell. In Fig. 6 we see that only 43% of its water needs are met by groundwater.

Four subsequent examples illustrate how one may proceed to refine the optimal strategy to reduce unmet needs in cell (2,4). In the first two examples, approaches that will increase groundwater availability by relaxing or tightening existing constraints or bounds are explored. Of the four examples, the last three each result in increased regional expense. To aid the discussion, equation [6] is expressed as follows.

$$X = Z + (Y) \quad [9]$$

where

Z = the quadratic and linear portions of X which are optimized, $/T

(Y) = the vector of constant terms, $/T

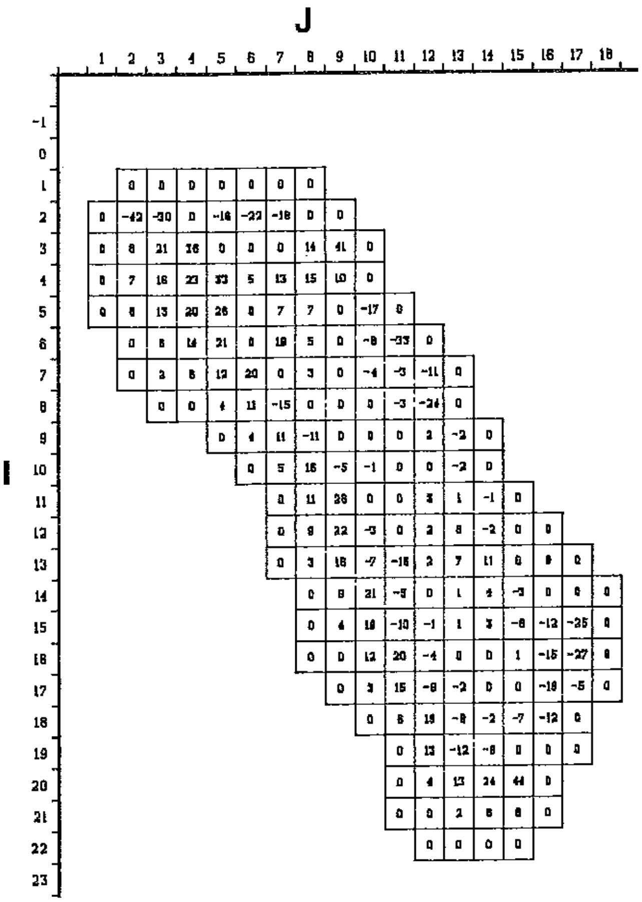

Fig. 9—Constrained derivatives ($\partial z/\partial p$) with respect to groundwater withdrawal under the minimum-cost strategy (\$/dam³).

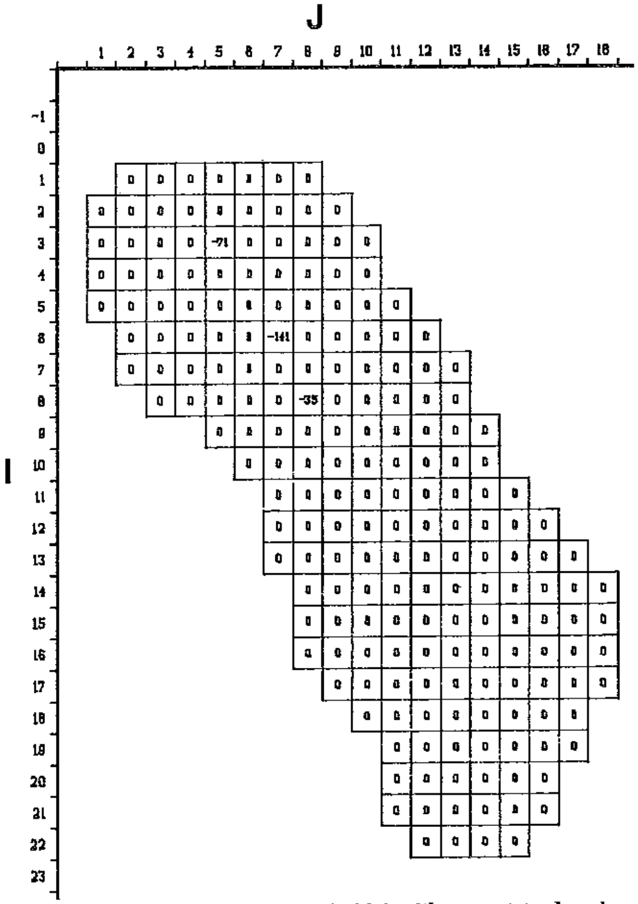

Fig. 11—Constrained derivatives ($\partial z/\partial s$) with respect to drawdown under the minimum-cost strategy (\$100/m).

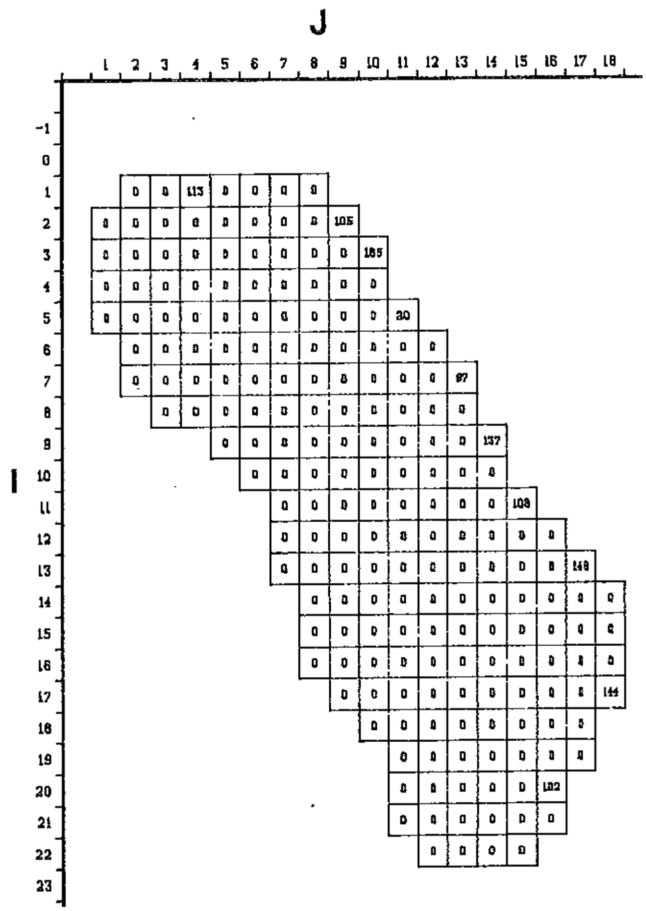

Fig. 10—Constrained derivatives ($\partial z/\partial q$) with respect to recharge under the minimum-cost strategy (\$/dam³).

The change in X resulting from any fine-tuning of the strategy is the sum of the changes in Z and (Y). From the description of (Y) following equations [6] to [8], we know that the value of (Y) changes if any product $c_{a_k}w_k$ changes. A change in the value of Z is estimated through the use of constrained derivatives (shadow prices).

Constrained derivatives are linear coefficients that estimate the effect on Z or on a variable caused by a unit change in another variable. Therefore, the change in Z approximately equals the product of the change in a variable and the appropriate constrained derivative. This procedure for determining the effect on Z of a change in a variable or variable bound can be applied as long as the change does not cause any other bounds or constraints to be exceeded. Because of the necessity of satisfying this criterion, the reductions of unmet water needs achieved in the four examples are not of comparable magnitude.

Figs. 9, 10 and 11 contain the constrained derivatives of Z with respect to pumping, recharge and drawdown respectively. A positive constrained derivative for Z indicates that the variable is at its lower bound, whereas a negative value shows that variable is tight against its upper bound.

Another type of constrained derivative is also utilized. This second type indicates the precise effect on variables caused by changes in other variables. They are not shown in the figures but are detailed where appropriate in the discussion. The sign of these constrained derivatives does not indicate tightness against a bound, but merely indicates the direction of resulting change.

Approach 1

Net flux at cell (1,4) is negative, meaning that the cell is a source of recharge. The constrained derivative showing the effect on Z caused by a change in flux at cell (1,4) is 113 $/dam³ (Fig. 10). Because this is a positive value, flux at cell (1,4) is at its lower bound. Relaxing the bound on recharge and decreasing flux (increasing recharge) at that cell by 40 dam³ will change Z by about (113$/dam³) (−40 dam³) or −$4520. Changing the volume of recharge does not change the value of (Y); therefore, the change in X is also −$4520.

The value of the constrained derivative that describes the effect on pumping at cell (2,4) caused by a unit change in flux at cell (1,4) is −2.71. (This constrained derivative is not shown in the figures.) Thus, by changing the recharge bound to permit a recharge increase of 40 dam³ at cell (1,4), one changes the pumping at (2,4) by (−2.71) (+40 dam³) or 108.4 dam³. This 2.71-unit increase in pumping for each unit of relaxed recharge constraint, coupled with the decrease in total regional cost, is attractive from a management perspective. In this situation the physical feasibility of relaxing the recharge constraint is seriously considered.

Approach 2

From the negative values in Fig. 11 we see that total regional cost can be reduced if the upper bound on drawdown at any of cells (3,5), (6,7) or (8,8) can be relaxed (increased). Reviewing the constrained derivatives (not shown) of the effect of drawdown at those cells on pumping at cell (2,4), however, indicates that in order to increase pumping the drawdown bound must be tightened rather than relaxed. In other words, one cannot both increase pumping at cell (2,4) and decrease total regional expense by altering any drawdown bounds.

The most desirable way of increasing pumping at cell (2,4) via changes in drawdown bounds is to force the model to provide at least 6.5 m of saturated thickness, instead of the original 6.0 m, in cell (3,5). The constrained derivative of drawdown at cell (3,5) on pumping in cell (2,4) is −163.5 dam³/m. The increase in pumping in cell (2,4) caused by a 0.5-m increase in saturated thickness (decrease in drawdown) in cell (3,5) is (−163.5 dam³/m) (−0.5 m) = 82 dam³. This is a 4 percent increase in the percent of water needs met by groundwater (from 43 to 47%) in cell (2,4).

Other consequences of this action, however, are to change the percent of water needs met by groundwater from 97 to 93 percent in cell (3,5) and from 9 to 10 percent in cell (3,7). This means that unmet water needs increase in cell (3,5) and decrease in cell (3,7).

The constrained derivative (Fig. 11) of drawdown at cell (3,5) on the total regional cost is −$7100/m. The increase in Z resulting from the 0.5-m increase in saturated thickness is approximately (−7100 $/m) (−0.5 m) = $3550. Since there is no change in (Y), the increase in X is also $3550. Whether the relatively slight increase in satisfied water needs in cells (2,4) and (3.7) is worth the increase in unmet water needs in cell (3,5) and the increase in total regional expense is a more difficult decision to make than that of the first example.

Approach 3

One hundred percent of the water demand of cell (2,3) is being met by groundwater (Fig. 6). To evaluate the effect of using some of that groundwater in cell (2.4), the constrained derivative of pumping in cell (2,3) on pumping at cell (2,4) is used. This value, −0.317, is not shown in the figures. In addition, the constrained derivative of pumping at cell (2,3) on Z (−30 $/dam³) is used. By reducing the pumping in cell (2,3) by 10 dam³, the pumping in cell (2,4) increases by (−0.317) (−10 dam³) or 3 dam³. This change in pumping causes a redistribution of some unmet water needs from cell (2,4) to (2,3) and a change in Z of approximately (−30 $/dam³) (−10 dam³) or $300. Since there is no change in the value of (Y), the total annual regional cost X increases by $300. The rate of exchange in this example is not favorable, but the approach may be socially desirable.

Approach 4

Assume that through improved on-farm water management and without additional personal expense, the water users of cell (2,3) can reduce water needs and consequently can reduce pumping by 10 dam³. The change in (Y) equals the product of the change in water needs and the opportunity cost at cell (2,3), (−10 dam³) (67 $/dam³) = −$670. The change in Z is estimated by the product of the change in pumping and the constrained derivative of pumping at cell (2,3) on Z, (−10 dam³) (−30 $/dam³) = $300. Using Equation 9, the resulting change in total regional cost X is $300 + (−$670) = −$370. As in the third approach, the pumping in cell (2,4) increases by 3 dam³.

As these examples illustrate, there is considerable flexibility in designing a sustained-yield groundwater water management strategy via a steady-state approach.

SUMMARY

This paper presents a procedure for minimizing, in steady-state, the cost of attempting to satisfy spatially distributed regional water needs currently supported by an aquifer system. Groundwater and either one alternative source of water or reduction in water needs (by reducing production acreages) can be considered in each cell (square) of the study area. The total cost for each cell is the sum of the costs of groundwater, and either alternative water or opportunity costs caused by reducing production in that cell. The cost per unit volume of the alternative water (i.e., diverted surface water) and the opportunity cost of reducing water needs by reducing production, are known as a priori.

It is assumed that the cost per unit volume of groundwater is a linear function of the distance between the ground surface and the potentiometric surface in each cell. The steady-state drawdowns (distance between a horizontal datum and the water table) comprising the potentiometric surface are the variables. The total cost of using groundwater is a function of both the volume of groundwater usage and the distance that water must be raised. The distance through which the water is raised in a particular cell is represented by the total dynamic head of a hypothetical well in the center of the cell. A finite difference form of the Boussinesq equation is used in lieu of the volume of groundwater withdrawal in the objective function and constraint equations. The result is an objective function that is quadratic in the drawdown variables.

The solution space of drawdowns is constrained by lower and upper bounds. The upper bounds serve to assure that sufficient saturated thickness exists to insure groundwater availability. For internal cells, functional equivalents of groundwater withdrawal are constrained to be nonnegative and to be less than water needs. Similarly, for constant-head cells (recharge sources), recharge is constrained to be less than a physically feasible upper limit. Thus, the constraints are used to imbed within the management model the necessary equations describing steady-state two-dimensional flow through a porous media. Because steady-state equations are used and recharge is limited to that which is assumed feasible, the groundwater withdrawal (pumping) strategy that will maintain the optimal steady-state potentiometric surface is a sustained-yield pumping strategy. Several approaches for refining the optimal strategy through the use of constrained derivatives are presented.

Depending on the difference between a current potentiometric surface and an optimal surface, it may take many years of pumping in accordance with an optimal sustained-yield pumping strategy before the optimal surface is attained. The optimization does not consider the period of evolving groundwater levels. It develops only the optimal steady-state levels. Simple analysis of the sensitivity of the results to the evolutionary era are presented. The approach's greatest potential lies in situations where a long-term guaranteed supply of groundwater is desired.

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