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Research on Undergraduate Mathematics Education in Afghanistan:
How Students Understand and Learn the Concept of “Function”

A Thesis Presented By
Ahmad Khalid Mowahed

Submitted to the Graduate School of the
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of the requirements for the degree of

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ABSTRACT

The purpose of this study was to explore undergraduate students' conception of the concept of function in the education faculty of Balkh University, Afghanistan. In particular, the focus was to see what difficulties they had in learning the concept and to determine those factors that impeded/facilitated the learning of the concept in the curriculum. The research methodology was an exploratory approach where both qualitative and quantitative methods were used to analyze the data. The theoretical basis for the study was David Tall's theory which focuses on understanding one's knowledge and learning. David Tall's theory was used to analyze qualitative data and descriptive statistics, $t - test$ for two independent samples, and ANOVA were used to present the results of the qualitative data and to test the hypothesis developed after analyzing the qualitative data. Twenty fourth year undergraduate students worked on a questionnaire which was developed based on different aspects of the concept of function and were interviewed to elaborate their written work on the questionnaire. The participants completed the questionnaire and participated in in-depth interviews to reveal their understanding of the concept of function. The major finding was that those who had a good conceptual understanding of the concept could do better in various representations of the concept than those who did not. Also, the set – theoretic definition of the concept helped some of the students to be successful in dealing with various representations of the concept, but it challenged the majority of the students to comprehend this definition and apply it in certain situations. In addition, the concept was not taught with its theoretical aspects and through applied examples in Algebra and Calculus courses where the concept was introduced using two different approaches.

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INTRODUCTION TO THE STUDY

In Afghanistan, a post-conflict country, educational infrastructures have been destroyed by thirty years of war. There is a critical need to improve its education, especially teacher education. In order to improve mathematics teacher education in Afghanistan, we should be concerned about the curriculum and process of mathematics teacher education. I worked as a mathematics teacher for two years. During these two years, I felt that the students' mastery of mathematics did not meet the minimum standards. In order to improve student learning in mathematics, we need an understanding of students' difficulties in learning mathematics. This study will add to that understanding by exploring students' math concepts in a particular area—the concept of function—and their difficulties in understanding functions specifically.

The main purpose of this study was to understand pre-service and in-service fourth year students' conception of the concept of function and how they define and represent the concept of function. Having an understanding of their conception of function and their difficulties in understating it will form a basis for developing the mathematics curriculum. In addition, this study intended to contextualize the learning theories of mathematics education in the particular context of Balkh University located in the north of Afghanistan in order to improve the teaching of mathematics.

This Introduction will present a brief description of Afghanistan and the current situation of its undergraduate mathematics education, the history of function and claims about its importance, the statement of the problem, and the purpose of the study.

Afghanistan and Its Undergraduate Mathematics Education

Afghanistan is a Southern Asian country bordered by China, Iran, Pakistan, and the former soviet states of Turkmenistan, Uzbekistan, and Tajikistan. This geostrategic location has led Afghanistan to be in constant war for about 30 years after Soviet invasion in 1979, and the recent conflicts have had as negative consequences as the former invasion in 1979 on socio-economic, political, and educational realms. The constant conflicts over the course of 30 years, in turn, have led the country to be one of “the poorest” countries “in the world” (Sigsgaard, 2009, p. 15).

However, despite the unstable political situation in Afghanistan, its higher education has made some progress in rebuilding its educational infrastructures. The first and foremost achievement so far is that Ministry of Higher Education (MoHE) was able to reopen four-year in-service and pre-service undergraduate teacher training faculties in its 16 universities and 5 higher education institutions to prepare quality teachers for elementary, middle, and secondary schools. Following that, MoHE was able to develop the higher education strategic plan for the years 2005 to 2015 in which capacity building, quality assurance, curriculum development, and quality teachers and teaching are the most important components (Strategic Plan, 2004).

The strategic plan is also developed to address issues in undergraduate mathematics education that focuses on four dimensions such as (1) the content, (2) pedagogy, (3) technology, and (4) curriculum. The ultimate goal of mathematics education is to improve the teaching and learning of mathematical content through technology and sound pedagogical approaches.

Before 1979, some of these dimensions were developed with the coordination of foreign donor countries, including Germany, United States, and France. The content and curriculum of mathematics were primarily adopted and translated into the official languages, Pashto and Dari, by the professors who were sent to continue their education in donor countries. According to one of the professors, before 1979, during Daud Khan's administration, "the curriculum of mathematics was almost similar to the curricula that were followed in Germany, United States, and France" (a senior faculty). After the Soviet invasion in 1979, the curriculum was directed toward Russian mathematical school of thought in which every mathematical concept was presented with precise and sophisticated logical approaches.

However, the shift from one mathematical school of thought to the other, and then later in recent years the tendency of integrating both together, has had both positive and negative consequences. As a positive aspect, some aspects of the Russian mathematics education were more effective in presenting the concept of function in the textbooks published in 1986 than the textbooks published more recently because the participants who acquired the logical and set-theoretic representation of the concept introduced in the Russian curriculum were more successful than those who acquired the formal, $y = f(x)$, representation. On the other hand, most of the participants were not able to comprehend and apply these definitions in certain situations. In addition, professors who have had familiarity with the former Russian type of mathematics education, presented in Russian language, are still struggling to fit themselves in the current undergraduate mathematics education because the current curriculum requires English proficiency in order for them to be able to have access to 21st century mathematics education.

Statement of the Problem

Often, it is not possible to use sophisticated and formal definitions to introduce the concept of function in lower levels. However, experts in mathematics and mathematics education try to give simple and understandable definitions for functions using graphic, formulaic, tabular, mapping, and descriptive representations for functions. But introducing the concept of function through these approaches sometimes creates misconceptions for students in defining or determining whether a relation between two set of variables is a function or not. This results from using these approaches without integrating the core characteristic of the function.

Mathematics educators have studied how students construct the concept of function. Their research shows that grasping the function concept is not easy for students because understanding this concept depends on how students construct the concept (Spyrou, Elia & Gagatsis, 2004). So, when the mental construction for the function concept does not depend on its core characteristic, this creates problems in understanding the concept (Tall, 1992; Tall, McGowen, & DeMarois, 2000). The research also shows that using several approaches in presenting the concept of function can affect students' understanding positively. These different approaches also include using intervals, sets, graphs, machines and so forth.

Significance of the Study

Officials in the Ministry of Higher Education (MoHE) and mathematics teacher educators are searching for effective means to deepen in-service and prospective teachers' understanding of mathematical concepts through refining the existing

curriculum of mathematics teacher education as well as the teaching practices in educational institutions (MoHE's Strategic Plan, 2004). This study will respond to that call that asks for curriculum refinement and better preparation of secondary school mathematics teachers. The study will contribute to providing effective means for teachers and instructors to formulate their teaching around sound instructional strategies that facilitate long-term learning and for curriculum designers to design better the curriculum of mathematics teacher education congruent with the cognitive development of learners. The study will also help us to understand issues about the formal written curriculum of mathematics as well as about the taught curriculum of mathematics in that particular context. Finally, it will help us to find effective means to develop the mathematics curriculum in which the function concept is embedded so that students can learn it more easily from secondary school to higher education.

The Purpose of the Study

The main purpose of this study was to understand pre-service and in-service fourth year students' conception of the concept of function and how they define and represent the concept. This study addresses the following questions¹.

1. What is the function conception of fourth year pre-service and in-service students?
 - (a) How do students define the concept of function?
 - (b) What are students' mental images of functions?
 - (c) In what ways do students represent a function?

¹ Question (a) is adopted from Tall (1996); and questions (d) and (e) are suggested by Tall (1993, p. 5) to be explored.

2. What difficulties do they have in learning/understanding the concept of a function?
 - (d) What are the consequences of students' preference for procedural methods rather than conceptual understanding?
 - (e) What difficulties do students have in learning the concept of a function?
 - (f) What difficulties do students have in selecting and using appropriate representations?

The History of the Function Concept and Its Importance

Historically, the function concept dates back to the works of Persian mathematician called Sharaf al-Din al-Tusi who lived in the 12th century. He introduced the function concept while attempting to solve the third degree algebraic equation, $x^3 + d = ax^2$. In order to solve this equation, he transformed it to $x^2(a - x) = d$. He then proposed that “whether the equation has a solution depends on whether or not the function on the left side reaches the value d ” (Katz & Barton, 2007, p. 192). To determine whether or not the function on the left side reaches the value d , he found a maximum value for the function. This allowed him to state that if this value is less than d , there are no positive solutions; if it is equal to d , then there is one solution; and if it is greater than d , then there are two solutions.

As a mathematical term, in 1673, function was introduced by Gottfried Leibniz to describe a quantity related to a curve, such as a curve's slope at a point. The functions that Leibniz introduced are today called differentiable functions, which are at the heart of differential calculus (Thompson & Gardner, 1998). In 1718, Johann Bernoulli defined a

function as any expression made up of variables and some constants. Then during the mid-18th century, Leonard Euler used the word function to describe an expression or formula involving variables and constants; for example, $x^2 + 2x + 1$. The formal notion of a function, $y = f(x)$, was first introduced in approximately 1734 by Alexis Claude Clairaut and Euler (Eves, 1990).

It was during the end of the 19th century that mathematicians started to formalize all branches of mathematics by using set theory and tried to define every mathematical object as a set. Although there is a dispute about this claim, the credit for the formal definition of a function as a relation in which every first element has a unique second element was given to Dirichlet and Lobachevski (Eves, 1990).

The concept of function is used throughout secondary school to postsecondary school mathematics. The importance of function concept arises from its usage in every branch of mathematics such as calculus, algebra, geometry, trigonometry, number theory, probability, statistics, physics, engineering, economics, and daily life. Among them, algebra and calculus rely heavily on the concept of a function, and in fact most of the concepts such as limit, derivative, and integral are developed considering the properties of functions and the way they behave.

Definitions of Terms

Definition (Function): Tall (2002f) defines a function as:

Let A and B two sets other than empty set, and f is a relation defined from A to B . If

- i) $\forall x \in A, \exists y \in B$ s.t. $(x, y) \in f$
- ii) $(x, y_1) \in f$ and $(x, y_2) \in f \Rightarrow y_1 = y_2$

Then the relation f from A to B is called a *function* and denoted by $f : A \rightarrow B$ or $f \xrightarrow{f} B$. A is called the domain and B is called range (p. 3).

Descriptive form of the above formal definition:

The function or mapping, denoted by f , from a set A to a set B is a relation defined from A to B that satisfies the following conditions:

1. For every element x in A , there is an element y in B such that x is related to y , and
2. if x has relation with both y_1 and y_2 , then $y_1 = y_2$.

The set A is called the domain of the function and B is called co-domain. The relation given by f between x and y represented by the ordered pair $(x, f(x))$ is denoted as $f(x) = y$, and y is called the image of x under f . And the set of all images of the domain under f is called range of f .

The reason that this definition is preferred over other definitions that are given in some of the calculus textbooks is that they do not explicitly tell the reader the first condition (#1). Although the next three definitions give the exact definition of the function, but the former is easier to grasp and apply in certain situations because it is stated with explicit processes.

1. "A function is any process where numerical input is transformed into numerical output with the operating restriction that each unique input must lead to one and only one output" (Sparks, 2005, p. 25).
2. "A function f is a rule that assigns to each element x in set A exactly one element, called $f(x)$, in a set B " (Stewart, 2005, p. 12).
3. A function is a relation between two variables y and x , where "the variable y is said to be a function of variable x defined in interval $a < x < b$, if to every value of variable x in this interval corresponds only one value of variable y , independently from the form of correspondence" (Spyrou, Elia, and Gagatsis, 2004, p. 2).

REVIEW OF THE LITERATURE

This chapter presents the theoretical basis for the study, including its curricular and instructional implications, and reviews the literature concerning student's understanding of the function concept and their misconceptions and difficulties in learning/understanding it.

Theoretical Basis for the Study

The purpose of this section is to explain David Tall's theory which is an approach to the learning and understanding of mathematics. The section begins with an introduction of David Tall's theory which is concerned about knowing about an individual's knowledge and learning. This section also includes the curricular and instructional implications of the theory.

David Tall's theory

For several years David Tall and his colleagues² have been working on the ways in which “different kinds of mathematical concepts” are conceptualized. They were interested in finding the distinction between the “objects formed in geometry such as points, lines, and circles” and concepts such as “numbers, algebraic expressions, and limits” that are introduced in arithmetic, algebra, and calculus. So far, they have given two explanations for the distinction between the “objects formed in geometry” and concepts in arithmetic, algebra, and calculus (Tall, 2004a, p. 1). Tall (2004a) argue that

² The words in italic form in the sections below are taken from David Tall and his colleagues' works and cited throughout this paper without any changes.

the development of geometrical concepts followed a natural growth of sophistication in which objects were first perceived as whole gestalts [and] then roughly described with language growing more sophisticated (p. 1).

Then these descriptions turned into definitions appropriate for “deduction and proof”

(Tall, 2004a, p. 1). But, numbers and algebra started by

compressing the process of counting to the concept of number and grew in sophistication through the development of concepts where processes were first symbolized [and then] used as concepts such as sum, product, exponent, algebraic expression, and limit (Tall, 2004, p. 1).

Tall and his colleagues observed that there were not only “geometric, symbolic, and axiomatic” types of mathematical concepts, but also there were “three very different types of cognitive development which inhabited three distinct mathematical worlds”

(Tall, 2004, p. 2).

After collecting their findings, they developed a theoretical framework of “long-term human learning” that presents three ways in which “mathematical thinking develops” (Tall, 2008e, p. 1; Tall, 2007b, p. 1). The theoretical framework consists of three distinct mathematical worlds named as *conceptual embodiment*, *proceptual symbolism*, and *axiomatic formalism*. The other two important factors that facilitate long-term human learning are *set-before* in genes before birth and *met-before* that develops in development process (Tall, 2007b, p. 1). These three distinct but interrelated worlds of mathematics are closely related to mathematical thinking and concerned with the learning of mathematical concepts. These three worlds also “emphasize the construction of mental representations of concepts” (Juter, 2007, p. 53). Each of the worlds has its own hierarchical course of development of sophistication that focuses on the growth from the mathematics of “new-born baby to the mathematics of research mathematicians” (Tall, 2004, p. 285). The three worlds of mathematics are described below, as well as the

set-before and *met-before* which are crucial factors influencing *long-term human learning*. Tall (2005g) displayed the three worlds by the following diagram which shows the “Formal mathematics building on embodied and symbolic thought” (p. 1)

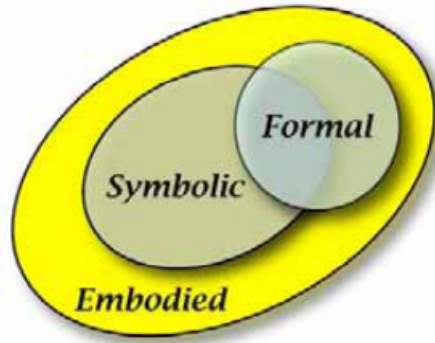


Figure 1: Three Words.

The *conceptual embodied world* grows from an individual’s perceptions of a mathematical concept and reflections on the properties of objects in real world through “mental experiments” (Juter, 2007, p. 54). The mental experiment can be students’ categorization of real-world objects such as an even number of items and later their intuitive conception of function. It also consists of one’s “thinking about things” that is perceived and sensed both in “the physical world” as well as in the “mental world of meaning” (Tall, 2004, p. 2). For example, individuals

conceptually embody a geometric figure such as a triangle consisting of three straight line-segments, [and then] imagine a triangle as such a figure and allow a specific prototype triangle to represent the whole class of triangles (Tall, 2007b, p. 2).

The *proceptual symbolic world* refers to the use of symbols that emerges from performing an action such as counting. When symbols are used for counting, it becomes a

thinkable concept such as number. For example, the symbols such as $4+5$ and $\int_1^3 x dx$

represent processes, addition and integration respectively, to be carried out. Also they are *thinkable concepts*, sum and integral respectively, which are produced by those processes. This combination of symbol, process, and concept constructed from the process is called an elementary *procept*; also a collection of these elementary *procepts* with the same output concept is called a *procept* (Tall, 2007, p. 2).

The *formal axiomatic world* refers to the formalism of Hilbert who stated that the formation of new concepts necessarily requires new signs that “reminds us of the phenomena which were the occasion for the formation of the new concepts” (Tall, 2007b, p. 3). The *formal axiomatic world* focuses on giving formal definitions to concepts and proving theorems by mathematical proof, which itself builds upon mathematicians’ embodied and symbolic knowledge. If we take the function concept as an example, first the definition for function is given by words that satisfy the conditions, and then the definition shifts from word form to a very sophisticated form given by Cartesian product and logic.

So far, we have used the terms *conceptual-embodiment*, *proceptual-symbolism*, and *axiomatic-formalism* for the three worlds of mathematics. However, as preferred by Tall and his colleagues, we will use *Embodied*, *Symbolic*, and *Formal* worlds instead of the former names.

As individuals move between the Embodied, Symbolic, and Formal worlds of mathematics, their needs and experiences change and their mental representations of concepts are formed and altered. Indeed, this forward and backward movement influences students’ conception and representation of a concept. We take the function

concept as an example. If the core definition of the function is given, even by word, and the students embody it correctly, then students have less difficulty in learning the definition of a function and determining whether a specific phenomenon is a function. However, when students reach university, the shift from definition from high school to a very formal definition through Cartesian product and logic challenges their ability to grasp the concept and apply it in certain situations.

One might argue that such complexity arises from the *met-befores* “where previously constructed cognitive connections are used to interpret new situations,” and students cannot connect it with the new form of the definition of the concept (Tall, 2007, p. 7). This is true and consistent with what Tall and his colleagues contend. They argue that sometimes a *met-before* is consistent with the new situation, but sometimes it is not. For example, the *met-before* adding two positive numbers gives more experience is true in whole numbers, integers, rationals, irrationals, real numbers, and complex numbers. But the *met-before*

taking away a subset leaves a smaller number of elements is consistent with finite sets, but is inconsistent in the context of infinite sets. For example, removing the even numbers from the counting numbers still leaves the odd numbers with the same cardinality (Tall, 2007, p. 7).

Curricular and Instructional Implications

In attempting to explore how long-term and sustainable learning can be achieved, we should take into account two factors that strongly influence students’ mathematical thinking and learning. The first one is the organization of mathematical concepts in the curricula and textbooks. We as teachers have to look at how the mathematical concepts are organized in the curriculum and textbooks. The development of logical sequence of

mathematical concepts in today's mathematics textbooks is not congruent with the cognitive development of most of the students (Reed, 2007). Often it is heard that learning mathematics is difficult; it is boring; it is not applicable; it has no relation with students' daily life and so forth. The reason that students find learning mathematical concepts difficult in such an organized fashion is because they have their own mental construction of a mathematical concept in the transition from the Embodied to the Symbolic and Formal worlds.

On the other hand, as we discussed earlier, the transition from Embodied world to other two worlds can be problematic for some of the students. The degree of difficulty depends on the extent to which they embodied the concept and how they encapsulated it into a *procept* to learn a sophisticated mathematical concept. So to address this issue we have to take into account the teaching of mathematical concepts as well.

The second factor is the teaching of mathematical concepts. Here some approaches are mentioned that enable students to learn a mathematical concept more effectively. The first and most important approach is that teachers use the Embodied approach to introduce a mathematical concept "before introducing any symbolism" (Tall, 2003a, p. 11). The teachers should not begin the concept with the Formal ideas of that concept but with the Embodied ideas of "graphical representations" of that concept (Tall, 2003a, p. 10). The Embodied approach can give initial meaning to the concept. In this approach, a student has an opportunity to reflect upon the concept using his/her own "mental experiment" (Juter, 2007, p. 54). For example, a teacher can show where a function is maximum or minimum through its graphical display before calculating them

numerically via the concept of derivative. Graphical representation is important because learners can easily study the properties of functions through their graphs.

Moreover, it should be understood that, although applied examples are important in improving one's conceptual understanding, the Embodied approach is not an approach that focuses entirely on "real-world applications". However, when a teacher wants to use the embodied approach to teach students, she/he should try to integrate the fundamental ideas of mathematics with the combination of "real-world applications" (Tall, 2003a, p. 11).

When this stage of presenting mathematical concepts has matured, then it is best to link the Embodied approach to the related worlds of Symbolism and later to the Formalism, because mathematical symbolism arises from embodied actions such as "counting, ordering," and "measuring" (Tall, 2007, p. 2). In order to have better transition from the first world to the other two, it is important to consider using computers in order to "represent" mathematical concepts "visually" through vivid examples (Tall, 2000g, p. 2; Tall, 2000h, p. 10).

However, a very productive approach is the student-centered method in which learners are the architects of their own learning by discovery. The student-centered teaching approach is "concerned not only with knowledge construction but also with the development of effective learning strategies" (Westwood, 2008, p. 26). The process of learning is more important than the actual acquisition of factual knowledge in a student-centered approach. Although the student-centered approach can be used with various types of methods such as inquiry and exploratory methods, the teacher is encouraged to

maintain the role of a facilitator and organizer of the resources from the beginning up to end of teaching and learning process.

The productivity of the teaching and learning process increases when the teacher uses the sophisticated *connectionist* classroom approach “in which the teacher as mentor orchestrates classroom activities”(Tall, 2007c, p. 10) and encourages learners “to make connections by focusing on essential ideas” (Tall, 2006e, p. 211; Tall, 2005e, p.15). This approach can be integrated with group work where students share ideas when challenged by problems.

Moreover, teachers should give students activities instead of giving just lectures which dominate today’s math classes. In addition to giving personal exercises and tasks, teachers should organize students in “cooperative groups” where group members investigate given examples (Reed, 2007, p. 12). The curriculum should not promote the use of the transmission approach where the teacher is at the center of activities and provides explanation for each single aspect of the topic. Nor should it teach to the tests, which is common in today’s schools.

To sum up, Tall’s theory proposes exploring learners’ mathematical thinking and asking them to explain how they connect their prior knowledge to the next stage of learning where the transition from their own perceived world or a concept to the other worlds, Symbolism and Formalism, takes place. This transition can be problematic for some and hinder their learning, depending on how they cope with the new form of a concept which was conceived in their own reflective world. It is important for educators to understand how students learn mathematics in college and university, because it has

specific implications both for the curricular organization of the current mathematics textbooks as well as for its teaching.

Studies about Learning the Concept of Function

The concept of function is one of the most complex concepts in mathematics. This complexity arises from the existence of graphic, formulaic, tabular, and descriptive representations for functions (Wilson, 1991) without introducing it through its core definition. Several studies have been done about students' understanding of the concept of function and their difficulties in learning it. These studies show that many students think that a "computational formula is necessary" to represent a function and insist that variables and formulas must exist in order to indicate input and output. Some students also think that there is a causal relationship between input and output variables (Reed, 2007, p. 45).

Tall and Bakar (1991) attempted to explore the hypothesis indicating that a student develop "prototype examples" such as $y=2x$ or $\sin(x)$ or $1/x$ of the function concept (p. 1). They argue that if the student is asked whether a graph is a function, he/she attempts to connect the graph with one of the model examples in his/her own mind, and then respond whether the given graph is a function or not. This indicates that the student does not have a clear understanding of the definition of the concept of function.

Tall and Bakar (1991) asked twenty eight students to explain in a sentence or two what they think a function is. Students' responses for the question consisted of definitions such as: (a) a function is an equation which has input and output (b) a function is ordered pairs which plots a curve or a straight line (c) a function is a term which will produce a

sequence of numbers when a random set of numbers are put into the term. The above definitions reveal that students expressed some idea of the *process* aspect of a function (p. 2). In addition to the former 28 students, the researchers also asked 109 university students to show whether the graphs of the parabolas that are concave toward *y-axis* and *x-axis*, respectively, are functions or not. All school students answered correctly that the first graph is a function, but 40% of them were able to recognize that the later graph is not a function. However, 97% of the university students were able to recognize that the first graph was a function, but only 20% of them were able to recognize that the later one is not a function. More interestingly, 29% of the school students were able to recognize that the graph of a circle is not a function, while 65% university students said that the graph of a circle is a function. Tall and Bakar (1991) concluded that teaching formal theory of function at the first stage seems to be unsuccessful, while emphasizing that practical experience can improve students' understanding of the concept of function (pp. 3-8).

Akkoc and Tall (2002) argue that the straightforward definition of the function that says that each element of a set A is mapped onto one and only one element of set B is a rather simpler and obvious definition that enables a mathematician to have access to further complex mathematical ideas. Some students can build upon this subtle combination of simplicity and complexity. For other students it is difficult to understand the concept because they construct their own understanding of the concept and its definition. However, this personal understanding of the concept of function can become a complicated collection of personal meanings that challenges student to learn the concept.

Akkoc and Tall (2002) attempted to explore the nature of this obvious mathematical combination of simplicity and complexity in the curriculum of the function concept in the upper-secondary school level in Turkey. The study was also concerned about the challenges that this combination poses to many students. A questionnaire consisted of questions asking whether the given graphs, equations, two set diagrams, and set of ordered pairs are functions was given to 100 upper-secondary school students. The results showed that none of the students responded yes to the question asking whether three similar representations, $y=4$, $y=4$ (for all values of x), and $y=4$ (for $x \geq 2$) are functions. The study also revealed that only two participants used the core function concept definition in an informal form to relate ideas across a range of different representational forms. The results also showed that the core function concept definition confused most of the students in the case of constant function. The researchers concluded that most of these misunderstandings stemmed from the isolated representations of functions in the Turkish curriculum, which is in fact similar to the depiction of the function concept in the secondary-school mathematics curriculum of Afghanistan.

DeMarois and Tall (1996) studied a number of community college students' *concept image* of function in terms of *facets* and *layers* by giving a questionnaire and doing interviews. The concept image simply means students' mental construction and understanding of the concept of function. Facets of the concept of function mean the "verbal (spoken), kinesthetic, written, geometric, numeric, symbolic, notation, and colloquial (informal) aspects" of the concept of function; and the layers refer to the "depth dimension" that starts with "pre-action" to "action", then to "process" and "object", and lastly to *procept* in the development of function concept by "cognitive

processes” (p. 2). It is believed that the more a student does better in both *facets* and *layers* of the concept of function, the more the student is able to perform well in different aspects and has profound understanding of the concept. As shown in Figure 1, doing better in both *facets* and *layers* means that the student is more toward the center of the circles, that is, the student knows the process as well as the concept (*precept*).

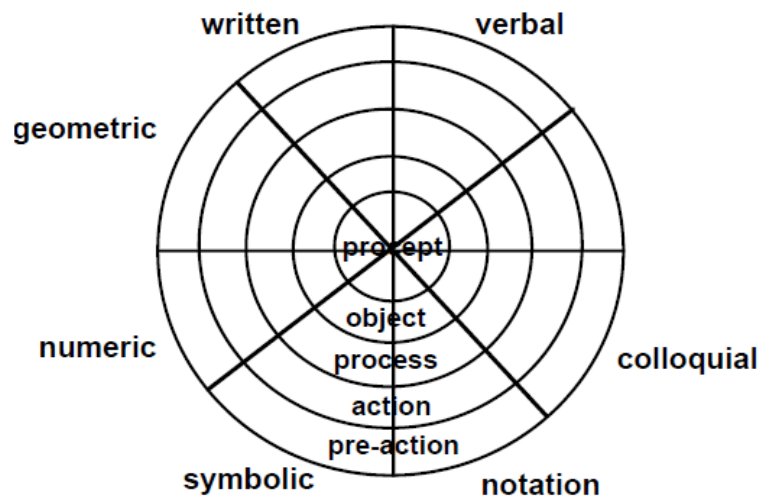


Figure 2: Facets and layers of the concept of function

The final analysis of one of the interviews indicated that the participant called an equation as a definition of the function concept, but the participant was more successful in symbolic facet than in the numeric facet, although the symbolic representation seems to be more sophisticated than numeric representation.

Reed (2007) reported in Sfard (1992) mentioned several misconceptions and difficulties of students in learning the function concept. He noted that students thought that the “function is its representation”. Also students were unable to connect “graphical, symbolic,” and “tabular” representations of the function concept. In addition, students gave different solutions to the “equivalent problems” when slight changes were brought

in the “notation.” The students also ignored the “domain and range” of the functions, which are very important in procedural performance and determining whether the relation between the elements of two arbitrary sets is a function. The students were also unable to do algebraic “operations on functions” (p. 46).

As reported in Reed (2007), Eisenberg (1992) noted that students tended to think of functions as algebraic expressions rather than graphic and had difficulty with connecting the graph of a function with its analytic representation. The research also indicated that even the students who had taken several courses in mathematics did not have a good understanding of the function concept.

These research findings show that the complexity in learning the function concept stems from the isolated multiple representations of it across the curriculum. Findings also show that emphasizing practical experience can improve students’ understanding of the concept.

METHODOLOGY

This section presents a brief introduction of the context of research, research design, sample, instrument, procedure, triangulation, categorization of data, and the pilot study.

The Context of Research

The study was carried out in the mathematics department of education faculty of Balkh University located in the north of Afghanistan. The education faculty is a four-year teacher training faculty that prepares both in-service and prospective teachers at the undergraduate level for secondary schools in various majors such as Islamic studies, mathematics, physics, chemistry, biology, history, geography, Pashto, Dari and English languages. A student has to complete a minimum of 145 credits in any of these majors to get a teaching licensure. The education faculty has two other branches one about 25km and the other 150km from the education faculty. Approximately, 2000 in-service and pre-service students study in the school of education and its two branches. Of those students, 830 are pre-service (312 female and 518 male) and more than 700 are in-service students studying in the school of education; and the other 400 students are studying in the two branches. These students are taught by more than 60 professional instructors.

Although some professional workshops were held during recent years to equip instructors with the new type of university curricula and teaching practices, there has been a little support in terms of providing remedial and enrichment programs for math instructors. This remedial and enrichment programs, if provided, could enable them to meet the contemporary expectations of mathematics education.

Design

The current study attempts to understand how fourth year undergraduate students construct the concept of function and what factors influence their learning. The collected evidence will be the basis of refining the curriculum for teaching functions and formulating sound teaching strategies in order to better teach students.

Both quantitative and qualitative methods were used in this study. The aim of using quantitative method was to organize, describe, and depict research findings so that one can have an overall understanding of the results of the study at a glance. Because the study aims to find about how students construct the concept of function and what influences their learning, descriptive statistics cannot give a satisfactory analysis of the students' experiences and the cognitive aspects of their learning. So the qualitative method is used to provide a better understanding of how they construct the concept of a function and what difficulties they have in learning it.

The emphasis on using both research methods to understand about students' learning will provide explanations for the following research questions:

1. What is the function conception of fourth year pre-service and in-service students?
 - (g) How do students define the function concept?
 - (h) What are students' mental images of functions?
 - (i) In what ways do students represent a function?
2. What difficulties do they have in learning/understanding the concept of a function?

- (j) What are the consequences of students' preference for procedural methods rather than conceptual understanding?
- (k) What difficulties do students have in learning the concept of a function?
- (l) What difficulties do students have in selecting and using appropriate representations?

In the current study, primarily two sources were used to collect the data. A 12-item questionnaire based on the concept of function was given to research participants to work on at the beginning of the study. After having done the initial analysis of the students' responses to the questionnaire, the researcher had one-to-one interviews with each of the participants. In these interviews, the same 12 questions that were in the questionnaire were asked from each participant. This time the aim was to collect more data about students' understanding of the concept of function and the process of their thinking rather than just interpreting their mathematical procedures and answers from the questionnaire.

In addition, the researcher collected and analyzed multiple curriculum documents to identify lessons taught about functions. This information played a significant role in interpreting students' responses as well as in proposing some constructive suggestions for curriculum designers, curriculum developers, and instructors.

Sample

During spring semester 2009, the study took place in the mathematics department of education faculty of Balkh University located in the north of Afghanistan. The study focused on fourth year undergraduate students' conception of the concept of function.

Two fourth year classes that were the only two senior classes of mathematics department were selected for this study. Senior students were purposefully chosen because the researcher intended to select those who already had completed most of the required courses taught in mathematics department. All participants had already taken several courses in different branches of mathematics. In particular, they had completed Algebra I, II, III, IV and Calculus I, II, III, IV courses focusing on the concept of function; and still they were studying the function concept in more than one dimension as well as in the complex plane.

The researcher first met with about 80 senior mathematics students, the overall population of in-service and pre-service senior students, and sought their cooperation to participate in the study. Participation in the study was voluntary. Sixteen pre-service and 10 in-service students agreed to participate in the study.

Of those sixteen pre-service students, 13 were male and 3 were female. Most of the pre-service participants were sort of average students in terms of their academic achievement except four of them who had higher GPAs compared to their classmates in the previous semester. Also, of those 10 in-service students, 6 were female and 4 were male. In-service students also are sort of average students in terms of their academic achievement except one of them who had the highest GPA compared to her classmates in the previous semester; she was absent during the administration of the questionnaire and was not interviewed. In addition, all participants aged between 22 and 32.

Table 1: Research participants

Participant ³	Gender	Pre-service	In-service	Teaches Math	Teaching experience (in years)
1	Male	√		in a private school	1
2	Male	√		in a private school	1
3	Male	√		in a private school	1
4	Male	√		NO	
5	Male	√		NO	
6	Female	√		NO	
7	Female	√		NO	
8	Male	√		NO	
9	Male	√		NO	
10	Male	√		in a private school	2
11	Male	√		NO	
12	Male	√		in a private school	3
13	Female	√		NO	
14	Male	√		NO	
15	Male	√		NO	
16	Male	√		in a private school	1
17	Female		√	in a public school	4
18	Female		√	in a public school	4
19	Female		√	in a public school	4
20	Female		√	in a public school	4
21	Female		√	in a public school	4
22	Female		√	in a public school	4
23	Male		√	in a public school	4
24	Male		√	in a public school	4
25	Male		√	in a public school	4
26	Male		√	in a public school	4

Two participants, one from in-service and the other from pre-service, were randomly chosen to participate in the pilot study, leaving a sample of 24. However, during the administration of the questionnaire, two participants were absent and two other students withdrew from the study. Then the remaining 20 participants worked on the

³ Similar numbers do not show the same person throughout this paper. They are used for the purpose of listing the participants.

questionnaire that consisted of 12 questions based on the concept of function. In addition, 19 of these students were interviewed individually, except one student who could not make the interview due to having final examinations.

There are several reasons for having such a small number of students in the sample. First, the researcher had a very short time and could interview a reasonable number of participants. In addition, it would have been difficult to analyze and interpret qualitatively the data if they were collected from a large number of participants. Furthermore, in-service students were teaching from morning up to noon in public schools and had little time to be interviewed. Finally, some female students probably did not have the permission from their parents to be interviewed although they made the majority of the students; they were about 50 to 55 per cent in the pre-service class and 80 to 85 per cent in the in-service class.

Instrument

The researcher developed a questionnaire for measuring the various aspects of the students' conception of functions. While developing the questionnaire, the researcher had the participants' mathematical knowledge in mind and tried to embed questions that were compatible with the existing curriculum used in that particular setting. All the questions were developed in a way that required high school level mathematics including some algebraic manipulative skills. However, only question #6 needed some knowledge of college level mathematics because this type of representation is not discussed in the high school curriculum.

Prior to conducting the research, the researcher consulted and piloted the questionnaire with one of his Afghan colleagues who used to teach math in one of the educational institutions in Afghanistan. After having his comments and suggestions, the questionnaire was developed. The questionnaire included the following questions. Question #1 was adapted from Tall (1996); #5 (a) from Tall (1991); #6 from Reed (2007); #7 and #8 from Stewart (2008); #4 and #9 from Tall (1996). One reason that these questions were adapted was that they could provide enough data for research questions and was already proved to be effective in collecting data about students' conception of the concept. The other reason was that they were compatible with the secondary school and university level mathematics curricula of Afghanistan. The other five were developed by the researcher in order to have better understanding of students' conception of the function and their difficulties in learning it.

Procedure

Before administering the questionnaire, the researcher piloted the 12-item questionnaire about students' conception of function. The questionnaire (see Appendix A) consisted of 12 questions asking students to respond to each question by writing a reasonable description of their responses. The questions were categorized as character strings, graphs, tables, equations, statements, and function machines. Each of these categories was developed to have an understanding of students' understanding of the concept of function and their ability to apply their own definitions of the concept of function in certain situations. As mentioned in details earlier in this chapter, the categories were focused on the *facets* and *layers*, and *concept image* aspects of students' understanding of the function concept.

Two of the participants who were randomly selected worked on the questionnaire. The aim of this piloting was to understand whether the questions are appropriate for students and to see if there was a need to refine the questionnaire. Also the aim was to see how much time would be needed to administer the questionnaire. The researcher piloted the questionnaire and did pilot interviews with two of the participants in late June, 2009. Both students completed the questionnaire in about 60 minutes. Based on this piloting, the questionnaire was revised. First, one question was removed completely because it did not provide solid evidence for research questions. Another question was transformed from character strings-ordered pairs to only character strings because the mixture was a little bit confusing to the students. Another question was split into two questions asking for conceptual understanding and procedural skills and the task of drawing its graph was removed (for these changes, see Appendix A, the initial questionnaire, and Appendix B, the refined one.)

The final questionnaire consisted of questions in the form of character strings, graphs, equations, function machines, statements, and tables, which are different representations of the function concept (see Appendix B for the full questionnaire). These types of questions are explained in the categorized framework for data analysis elaborated in this chapter.

After the first analysis of students' work on the questionnaire, 19 of the 20 participants were interviewed individually. Their responses on the questionnaire were the basis for the discussion during the interview. The participants were asked to explain their answers verbally in order to have a better understanding of their conception of the concept of function and their difficulties in learning/understanding it. Each interview was

taped and lasted between 30 to 60 minutes. Because the interviews were done in Dari language, the researcher directly translated each interview and transcribed it in English. These interviews and the completed questionnaires comprise the data for this study.

Triangulation

The questionnaire and interviews helped to provide consistent data about students' understanding of the concept of function. Students written responses on the questionnaires were further explored in the interviews. The one-to-one interviews provided significant data in a way that the questionnaire could not because some students were able to verbally elaborate their thinking and extend their responses in the interviews informally. In addition, the interviews provided an opportunity for participants to respond to the questions without considering mathematical and grammatical mistakes that usually occur in written responses.

The Concept of Function in the Curriculum

The researcher collected multiple curriculum documents that were available through the department and instructors in order to identify lessons taught about functions. After contacting with the head of the department and instructors, it was found that there was only one piece of curriculum called 'Plan Talimi' which can be translated as an educational plan. This document was a one page outline of courses, with their credit hours, that a student had to take them during the course of four years (see Appendix D).

Since there was not a detailed curriculum that would determine which aspects of the concept of function should be discussed during the course of four years, the researcher sought instructors and students help to find some documents about the

concept. The researcher asked a student to share some of his course materials in order to see what was taught about the concept. Since there were no syllabi available, the researcher could only get a copy of course handouts related to the calculus and algebra courses that both started at the second year and continued until the fourth year, each of them had 4 and 3 credits, respectively.

In the calculus course, the concept of function was reviewed very briefly because, according to a pre-service student, it was reasoned that students already studied the concept in the first year in the basic mathematics course. Furthermore, the handout showed that the calculus course was started by introducing the concept of limit of a function. It is important to note that starting calculus course without introducing the concept of function at the outset of the course can pose problems for students who are following the course. This is because almost all of the calculus is build upon the concept of function. Without having a good understanding of the concept, it is highly unlikely that one can do well not only in calculus course but also in other courses that come next.

Also the researcher could get only the handout for Algebra I course which was offered in the third semester, second year. This course also focused on the concept of function and most of its algebraic structures used the concept of function. In Algebra I course, the concept of function, types of functions, and its properties were presented in detail through set-theoretic approach. An important issue to be noted here is that the concept of function was presented first by introducing the set theory and then the Cartesian product of two or more stets that would made a special type of a relation called Mapping(the contemporary name of the concept of function).

Table 2: Topics taught about the concept of function in Calculus I – Calculus IV

Limit of a function: Definitions and properties
Limits of Trigonometric Functions
Limits of exponential and Logarithmic Functions
Discontinuity of a function at a point and in an interval
Continuous and Discontinuous Functions
Derivative of a function: Definitions and Properties
Derivative of an exponential function
Derivative Rules
Derivatives of Trigonometric Functions
Derivatives of Logarithmic Functions
Derivatives of Implicit Functions
Differentiation by Logarithm
Derivative of an Inverse Function
Derivative of the Inverse of a Trigonometric Function
The Derivative of a Parameterized Function
Hyperbolic Functions and Their Derivatives
Inverses of Hyperbolic Functions and Their Derivatives
Higher Order Derivatives of Functions
Functions with Several Variables and Partial Derivatives
Differential of a Function with Several Variables
Single and Multiple Integrals

Table 3: Topics taught about the concept of function in Algebra I

Set Theory
Mathematical Logic
Binary relation and Its Properties
Cartesian Product of Sets A and B
Composition of Relations
Equivalence Relation
Partitioning a set
Mapping (Function) of set A into set B
Surjective Mapping
Injective Mapping
Bijjective Mapping
Composition of Mappings
Inverse of a Mapping

In addition to these curriculum documents, one of the participants was asked to write a short report about things they studied in both courses. The report indicated that

the participant believed that function concept is at the core of topics in calculus and should be taught with its theoretical details. The following excerpt was written by the participants. It shows what topics were covered in those courses and indicates the student's thoughts which are almost similar to other participants' thoughts noted during informal conversations.

Over the course of three and half years, the function concept, composition of functions, one-to-one functions and etc. were discussed [in the Algebra I course]. After that, the concept of limit of a function, trigonometric functions, hyperbolic functions with its derivatives, definite and indefinite integrals were taught [in the calculus courses].

The shortcomings of abovementioned topics can be listed as the following. Since we know that higher mathematics is dependent on the concept of function, the function concept was not taught with its details and theoretical aspects. Therefore, students do not have a good understanding of functions and other topics taught over the course of several years. Thus, they cannot analyze the mathematical concepts taught before and unable to solve related problems. Since the function concept is related to the theory of sets and other topics not taught in details, we are unable to solve problems. We believe that if the function concept is taught in detail, students will not have any difficulty in understanding the function concept and other topics such as integrals, trigonometry, equations, analysis and so forth. As we know, the function concept is a very broad in scope and entails many other concepts in it. So far, all the students were deprived from its broader aspects (a pre-service student).

As discussed earlier in this section, it was found that two approaches were followed in the curriculum to introduce the concept. One approach was that the function concept was taught through set theory and Cartesian product in Algebra I course that followed the textbook published in 1986 based on Russian algebra curriculum. And the second approach was that the concept was briefly discussed through the formal representation, $y = f(x)$, and its limit using a recent textbook.

The Categorized Framework for Data Analysis

The theoretical basis for the study was David Tall's theory which was presented in the previous chapter in detail. Taking into account the fact that some of the questions are adapted from the studies that had already been carried out by Tall and his colleagues, the researcher wanted to develop a categorized framework for data analysis before doing data analysis. The data analysis framework is taken from several studies that described some of the factors that facilitated/ impeded the learning of the concept of function. Each category is presented next with its respective numbered question from the questionnaire at the beginning of each category.

The definition of a function

Question #1: What do you think a function is? Please, write it in a sentence or two. If you can, please give a definition.

This question was developed to measure how students define a function. This question is closely related to Tall's theory because different students define the concept differently. Their definitions are influenced by their own perceived world. That is, if a student is in Embodied world, he/she will define the concept in this world. Similarly, other students may define the concept in the other two worlds, Symbolic and Formal. The important issue here is that in which world a student should define the concept to be correct. There is no exact answer for this question because sometimes students define it correctly in their own perceived world which is neither Symbolic nor Formal; and some may define it in the other two worlds that may or may not be correct due to not satisfying the conditions, 1 and 2, given in the definition of the concept. Therefore, in addition to

those three worlds, we have to consider the following combinations of the three worlds, which are Embodied – Symbolic, Embodied – Formal, and Symbolic – Formal. The conditions presented in the Introduction are written here once again so that the reader knows what we mean by conditions 1 and 2 in the given table.

A function or mapping, denoted by f , from a set A to a set B is a relation defined from A to B that satisfies the following conditions:

1. For every element x in A , there is an element y in B such that x is related to y , and
2. if x has relation with both y_1 and y_2 , then $y_1 = y_2$.

Thus, below is a table that shows different combinations of those three worlds that determines where the correct definition could be. We will use this table to determine whether the participants define the concept correctly or not.

Table 4: Definitions counted to be correct or incorrect.

Worlds	Function is defined	To be correct
Embodied	by sentences that do not satisfy conditions 1 and 2.	NO
Symbolic	by mathematical symbols that do not satisfy conditions 1 and 2	NO
Formal	by sentences that satisfy conditions 1 and 2	Yes
Embodied – Symbolic	by sentences and symbols that do not satisfy conditions 1 and 2	NO
Embodied – Formal	by sentences that satisfy conditions 1 and 2	Yes
Symbolic – Formal	by sentences and symbols that satisfy conditions 1 and 2	Yes
	as : for every element in domain there is a unique element in co-domain ⁴	Yes

⁴ The responses may vary from one student to another, but the idea of having a unique output for each input would remain the same for all if they define it correctly.

Representations of a function

Question #2: Describe the ways a function can be represented.

This question was developed to understand in how many ways the participants could represent a function. It also determines a student's familiarity with various representations (such as graphic, tabular, analytic, diagram ...and ordered pairs) of the concept. The research shows that having various representations of functions can help students to learn the concept. In addition, this question was important because it aimed to determine whether these various representations were embedded in the existing curriculum of mathematics.

Question #3: What do you think when you see the notation $y = f(x)$? Or what do y , f , and x show? Please explain your answer.

The form $y = f(x)$ is an analytic representation of a function. This question was developed to see whether the participants can determine what each y , f , and x show. This is one of the most confusing representations that students reason to be the definition of the concept of function.

Function machine box

Question #4: Consider the following machine

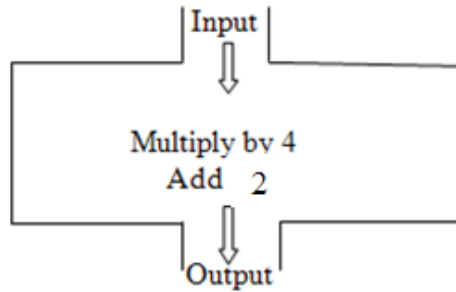


Figure 3: Function machine box.

What is the output if the input is $y(x) = x + 2$? Explain your answer.

The function machine box is a visual representation of a function, which is easily comprehended by the learner. It shows input-output *processes* in which there is a unique output for each specific input. This visual representation is fundamental and forms a basis for the formal representations of the concept of function (Tall, 2000e; 2000d). For a student to find an output, he/she must first recognize the input and then the *processes* that take place inside the machine. If these two factors are not recognized, the student will have difficulty in finding a specific output. However, if the student can find the output correctly, then it means that the student has a good *action* and *process* conception of the function concept.

Graphs

Question #5: Which one of the following graphs is a function? Please, explain your answer.

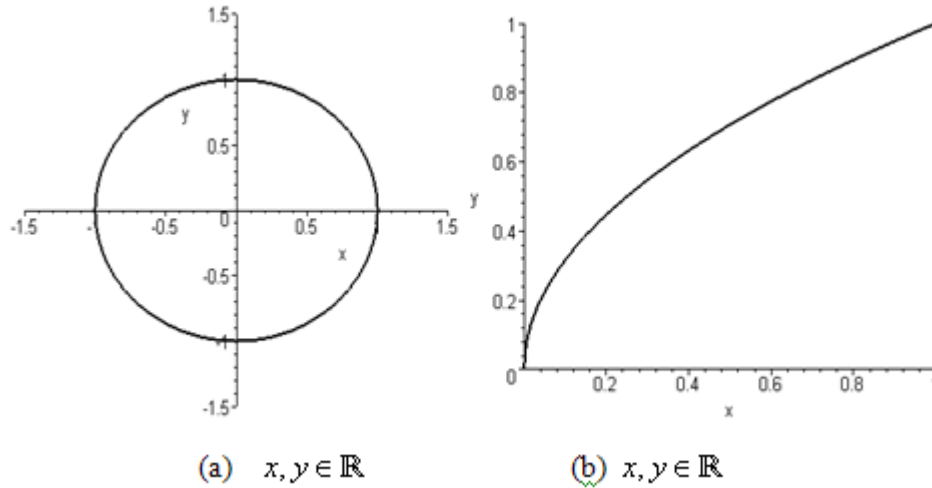


Figure 4: Graphical representation.

This question was developed to determine whether students can apply their definitions of the concept of function to determine whether a given graph is a function or not. Also it measures to what extent the student can do better in terms of *layers* which usually echoes ones' conceptual understanding.

The graphic representation of a function examines the process conception of students and is usually given to investigate students' *concept image* of the function concept (Tall & Bakar, 1991). If a learner has a profound concept image, he or she will be able to determine whether a given graph is a function or not.

However, sometimes it happens that students use the vertical line yardstick to determine whether a graph is a function or not. This vertical line yardstick overshadows the core characteristic of the function concept. Those students who solely rely on graphic representation of the function may have difficulty dealing with other representations and even in understanding the concept.

Character strings

Question #6: Can the following situation be called a function? Why or why not? Please, explain your answer.



Figure 5: Character string representation.

This question was developed to understand whether a student can apply his/her conceptual understanding in a situation which neither has an analytic form nor a curve or a graph.

This type of representation has “potential to indicate one’s ability to use a process conception of [a] function” (Reed, 2007, p. 82). On the other hand, it poses challenges to students to determine whether the given set of characters (discrete points on the xy -plane) can be called a function or not. This is because there are no inputs and processes that would give the outputs. This type of representation reveals the difficulties student face to understand the concept of function. One of the difficulties is that most of them say that the outputs must be continuous and form a graph.

Algebraic expressions as functions

Question #7: f is defined by $f(x) = \begin{cases} 1-x & x \leq 1 \\ x^2 & x > 1 \end{cases}$. Is f a function? Please explain

your answer.

Question #8: f is defined by $f(x) = \begin{cases} 1-x & x \leq 1 \\ x^2 & x > 1 \end{cases}$. Evaluate $f(0)$, $f(1)$, and $f(2)$.

Questions #7 and #9 were developed to understand how one's preference for procedural method influences his/her conceptual understanding when the domain and co-domain are not taken into consideration. That is, if the student does not consult his/her conceptual knowledge, what would be the results of his/her work and what these consequences would echo about his/her conceptual understanding.

This type of representation determines students' *action* (entering the inputs), *process* (manipulating the inputs under the acting f), and *concept image* (deciding whether the expression is a function or not) of the concept. The difficulty it causes, when the core characteristic is deemphasized, is that many students think that a given algebraic expression itself is the definition of the function, this shows that student's concept image of the concept is restricted. However, when the focus is more on a single algebraic expression as a function, students often tend not to consider an algebraic expression consisting of more than one expression as a function, and that indicates that students have a restricted image of the function concept.

Tables

Question #9: Consider the following tables:

Table 5: Composition

x	$f(x)$	x	$g(x)$
1	3	-2	3
2	-1	-1	1
3	1	0	5
4	0	1	2
5	-2	2	4

Evaluate $f(g(2))$.

Question #10: Consider $f(x) = 2x + 1$ and $g(x) = -2x - 1$. What is the value of $f(g(3))$?

These questions were developed to measure whether a student can find the image under a composition. Also, it aimed to show whether the student can distinguish the role of each letter in the $y = f(x)$ or $y = g(x)$ representation and extend it to the analytic form of $h = f(g(x))$.

This representation is similar to the ordered pairs. The only difference is that the inputs and outputs are situated in a table. It focuses on the process aspect of a student's concept image of the function concept as well as on the conceptual understanding.

Questions #9 and #10 are equivalent in terms of finding the composition of f and g at an arbitrary value, e.g. $x = 2$ in #9 and $x = 3$ in #10. Both questions determine a student's procedural knowledge and conceptual understanding. However, the representations such as tables and algebraic expressions challenge the student to find the output under composition if he/she does not have the core concept definition of the

concept of function. Students who are exposed to one of these may do well procedurally, but not conceptually.

Statements as functions

Question #11: Suppose 5 students who read about function concept got the following scores in a quiz: student1 read 5 hours and got 8 points; student2 read 4 hours and got 9.5 points; student3 read 3.5 hours and got 9.5 points; student4 read 3 hours and got 9.4 points; and student5 read 2 hours and got 9.3 points. Can this phenomenon be called a function? Why or why not?

Question #11 is the word form of a phenomenon. This statement can be represented by a table or diagram. However, the aim of this question was to see whether students can apply their conceptual understanding to determine whether the statement is a function or not and what factors influence her/him to decide it is or is not a function.

Statements, like question #11, describe a phenomenon that may or may not show a functional relationship depending on the type of relationship between quantities. These are the most open-ended type of questions and allow a student to construct any type of function. It not only shows the *action* and *process* aspects, but also shows the student's concept image of the function. The question #11 intends to determine the student's abstract conception of the concept and to understand if the student can apply the definition of the concept in daily life situations.

Question #1 is a conceptual question that includes all *layers* and *facets*. Its properties, the conditions mentioned earlier, cannot be derived from neither of the *facets*

presented in Table 6, but all *facets* share some characteristics in terms of *layers* and help a learner to better construct the concept. If we look at Figure 6, the *layers* are the most important stages that determine the learner’s conceptual knowledge through different representations and show how the learner constructs the concept.

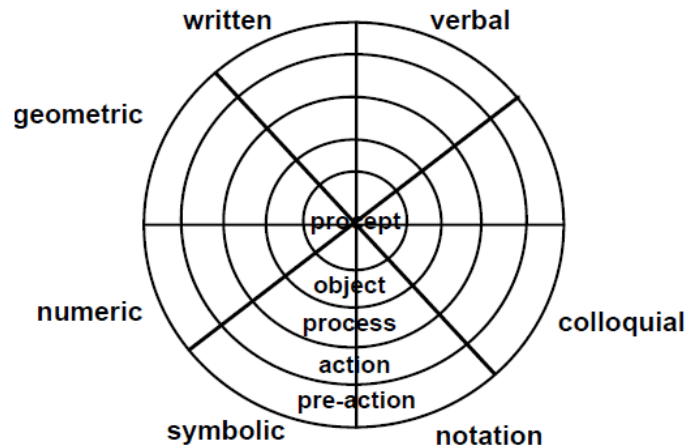


Figure 6: Facets and layers of the concept of function.

The hierarchical sophistication of *layers*, pre-action→action→process→object (concept), means that a student can do well in any of the representations, *facets*, if he/she has a good conceptual knowledge and successfully follow those *layers*. The pre-action and action (a broader term for pre-procedure and procedure in the case of having a specific examples like questions #4, #8, #9, and #10) indicate that the student has some manipulative skills, but it will not lead her/him to get the actual answer unless she/he follows the processes and finally decides whether it is a function (if asked to be determined).

Below is a table that lists all of the questions with their respective categories in terms of *facets* and *layers* that measures one’s knowledge and understanding in both dimensions.

Table 6: Questions in terms of facets and layers

Questions	<i>Facets</i>	<i>Layers</i>
#2	Written, verbal, visual(any type), geometric, numeric, Symbolic, notation, and colloquial	
#3	Analytic or notation and symbolic	Action and process
#4	Visual and colloquial	Pre-procedure, procedure, And process
#5	Geometric or graphic	Action, process, and concept
#6	Visual-character strings	Process and concept
#7	Analytic and Symbolic	Pre-action, action, process, and concept
#8	Notation or analytic	Pre-procedure, procedure, and process
#9	Numeric and analytic	Pre-procedure, procedure, process
#10	Analytic	Pre-procedure, procedure, process
#11	Written statements	Action, process, and concept

The Researcher

The researcher is a master candidate at University of Massachusetts, Amherst, and conducted this study as his master’s degree requirement. He did his undergraduate studies at the mathematics department of Education Faculty of Balkh University where the study was conducted. He worked in that university as a math instructor for about two years. The participants of the study were also taught by the researcher for about two semesters during 2005 and 2007 years. He knows most of those math students who were studying during these years, especially the participants. He had prior understanding of that context and intended to conduct this study to introduce new theories of learning about mathematics education that would identify the critical needs of the learners and teachers.

PRESENTATION OF DATA

This section presents the data collected through the questionnaire and one-to-one interviews with research participants during 2009 spring semester. The quantitative data presents a general description and depiction of students' written responses on the questionnaire on the basis of frequencies of correct and incorrect responses. The qualitative data presents a very detailed analysis of students' written responses on the questionnaire and their reflections in one-to-one interviews.

Quantitative Data

This section presents a general description and depiction of students' written responses on the questionnaire and their written work analysis on the basis of frequencies of correct and incorrect responses. It also presents an overall analysis of students' understanding of the function concept and their ability to apply it in certain situation.

There are two categories of students' written responses on the questionnaire. The first category presents how students defined the concept of function and in how many ways they could represent the function. And the second category presents students' ability to apply their definitions of the concept in certain situations.

How do students define the concept of function?

To measure how they define a function, all 20 participants were asked to define the function in their own words as well as mathematically (Question #1). In order to determine whether a student correctly/incorrectly defined the concept of function, Table 4, developed in the categorized framework in the previous chapter, was used. The aim was to show in which world a participant defined the concept. In addition to considering

those combined worlds, the least possible response to be counted as a correct answer was the statement which says that a relation between the elements of two sets, e.g. sets of objects, is called a function if for each input, e.g. x , in domain there is a unique output, e.g. y , in co-domain. If the student mentioned the first sentence without mentioning the complementary sentence in the above conditional statement (definition), then the response was counted to be incorrect because that the former statement echoes both condition 1 and condition 2 mentioned in the categorized framework.

Of these 20 students, 8 participants could give the correct definition of the concept. Of those 9 correct answers, 7 were set-theoretic definitions and 1 was a definition written by words. However, the incorrect definitions included 6 definitions related to the notion of the relation between two sets, two objects, or two things, 2 definitions related to the obedience of one variable from another variable, and the other 4 definitions included issues such as dependence of two things or objects, metaphors, and real life examples. These results are summarized in the following table.

Table 7: What do you think a function is?

Responses	Percentage	Frequency	Category of reasoning
Incorrect	60%	12	6 Relation
			2 Obedience
			4 Other
Correct	40%	8	7 Set-theoretic
			1 Other

In addition, Table 4 (see categorized framework in Methodology section) was used to see in which world a participant defined the concept and what type of reasoning he/she used in that particular world. As Table 8 shows, most of the participants correctly defined the concept in the Embodied-Formal and Symbolic – Formal worlds. This

indicated that these students were able to present their understanding of the concept in their own perceived world while preserving the Formal aspects of the concept. However, most of the students who could not give the complete definition of the concept were in the Embodied world although the majority used the set-theoretic approach. Although they could respond half of the definition, most of them missed the uniqueness condition. The uniqueness condition is a potential part of the definition of the concept. Once it is ignored, a student will call any situation to be a function.

Table 8: Reasoning type with its respective World

Participant	The World in which the concept was defined	Category and Type of Reasoning
1	Embodied – Formal	Set-theoretic
2	Embodied – Formal	A relation ⁵ between 2 variables
3	Embodied – Formal	A relation between 2 objects
4	Embodied – Formal	Set-theoretic
5	Symbolic – Formal	Set-theoretic
6	Symbolic – Formal	Set-theoretic
7	Embodied – Formal	Set-theoretic
8	Embodied – Formal	Set-theoretic
9	Embodied	Input-output
10	Embodied	Input-output
11	Embodied	Obedience
12	Embodied	A relation between 2 things or variables
13	Embodied	A relation between independent and dependent variables
14	Embodied	Obedience : x follows y as in $f(x)=x^2$
15	Embodied	A relation produced by Cartesian product of two sets
16	Embodied	A relation between tow objects
17	Embodied – Symbolic	A relation between two quantities as in $y = x$
18	Embodied	A relation between the elements of two sets
19	Embodied	Dependence of two variables
20	Embodied	Image of a person taken by a camera

⁵ The word comes from the notion of Cartesian product of two sets A and B defined as $A \times B = \{(a, b): a \text{ in } A \text{ and } b \text{ in } B\}$.

Looking once again at curriculum documents, it was found that the concept was introduced through Formal – Symbolic world. Although the Formal – Symbolic combination was a good approach to introduce the concept, it challenged most of the participants to construct the concept effectively.

It might have a potential implication for curriculum designers. That is the curriculum designers should embed this combination, Embodied-Formal, in the curriculum of the concept in addition to its Symbolic – Formal combination which is seen to be more appreciated in current textbooks.

What are students' mental images of functions?

To measure the participants' ability in applying their definitions in certain situations as well as their conceptual knowledge and procedural skills, a set of 9 questions in the form of analytic, $y=f(x)$, function machine, graphs, character strings, algebraic expressions, tables, equations and real life example were given (questions #3 through #11).

The question #3 was given to measure whether a participant can elaborate correctly what y , x , and f mean to him/her in $y=f(x)$ and what it shows. The correct answer for this question is to say y is a dependent variable, f is a rule or a relation that relates x to y , and x is independent variable. The question #4 was given as a function machine with an input, $y(x) = x+2$, and a process by first multiplying the input by 4 and then adding the result with 2; the students were asked to find the output for the given input. The correct answer for this question is $4x + 10$ because if we multiply $x + 2$ by 4, it gives $4x + 8$; and then adding 2 with $4x + 8$ gives $4x + 8 + 2 = 4x + 10$. The question #5 consisting two graphs, (a) a circle and (b) a graph of $y = +\sqrt{x}, 0 \leq x \leq \infty$, was given to

participants who were asked to determine which one shows a function. The correct answer for this question is to say that graph (b) is a function because for every x in its domain, x -axis, there is a unique image such as y in co-domain, y -axis, which satisfies the conditions mentioned in the definition of the concept of function (someone else also can draw a straight line parallel to y -axis and conclude that the graph is a function because the line cuts the graph at one and only one point, which is not a good reasoning from a conceptual point of view). In question #6, students were asked to determine whether the set of points (character strings) on the coordinate can be called a function. A reasonable correct answer for this question is to say that the situation is a function because each point on the plane has two components, one on horizontal axis and one on vertical axis, and there is an image (a point) for each element on the horizontal axis in the plane produced by both axes, and since neither of those different components on the horizontal axis has two images (uniqueness condition), then it is a function. In question #7, students were asked to show whether the algebraic expression $f(x) = \begin{cases} 1-x & x \leq 1 \\ x^2 & x > 1 \end{cases}$ is a function or not. Since f was defined in the set of Real Numbers, a correct answer for this question is to say that f is a function because there is an image in co-domain (set of real numbers) for every x in domain (the domain of $1-x$ is the set of real numbers less than or equal to 1 ; and the domain of x^2 is the set of real numbers greater than 1), and neither of these x_s has two images at the same time. Thus, it is a function. One can also draw a graph of the function or develop a table to show that f is a function; for example, those may look (values may differ) as

Table 9: Table for question #7

x	$1 - x$	X	x^2
\vdots	\vdots	1.01	1.02
.	.	2	4
-2	3	3	9
-1	2	4	16
0	1	\vdots	\vdots
1	0	.	.

and

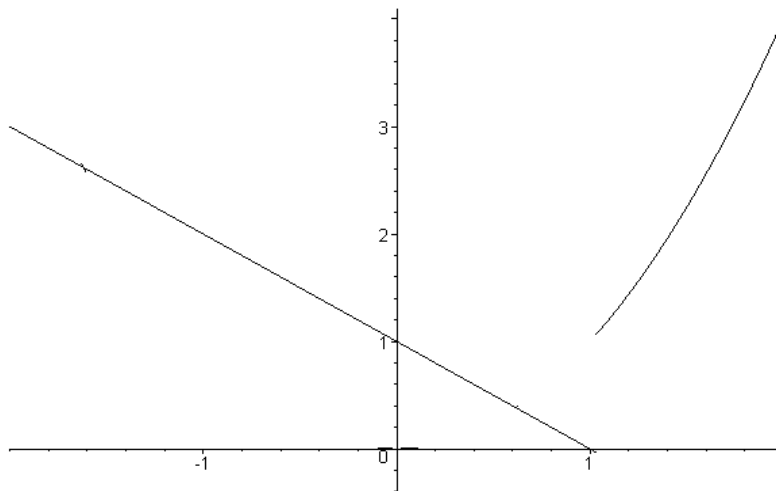


Figure 7: Graph of question #7

In question #8, students were asked to evaluate $f(0)$, $f(1)$, and $f(2)$ for the algebraic expression mentioned in question #7. The correct answer for this question is $f(0) = 1 - 0 = 1$, $f(1) = 1 - 1 = 0$, and $f(2) = (2)^2 = 4$. In question #9, students were asked to find $f(g(2))$ from Table 10.

Table 10: Composition, $f \circ g$.

x	$f(x)$	x	$g(x)$
1	3	-2	3
2	-1	-1	1
3	1	0	5
4	0	1	2
5	-2	2	4

A correct answer for this question is to first find the image of 2 under g , that is to find $g(2)$, and then find the image of $g(2)$ under f , that is to find $f(g(2))$, which is $f(g(2)) = f(4) = 0$. And the question #10 was asking students to find $f(g(3))$ where $f(x) = 2x+1$ and $g(x) = -2x-1$. A correct answer for this question is to either follow the process as $g(3) = -2(3) - 1 = -6-1 = -7$, then $f(-7) = 2(-7) + 1 = -14 + 1 = -13$, or $f(g(3)) = f(-2(3) - 1) = 2(-2(3)-1) + 1 = 2(-7)+1 = -14+1 = -13$. Finally, the question #11 was asking students whether a daily life phenomenon, Table 11, the relation between time spent on reading about the concept of function and the scores achieved in a test, can be called a function or not.

Table 11: Time spent on reading and achieved scores

Student	Time spent on reading(in hours)	Test scores (from 10 points)
1	5	8
2	4	9.5
3	3.5	9.5
4	3	9.4
5	2	9.3

Finally, a reasonable correct answer for this question would be to first apply the conditions mentioned in the definition of the concept of function and then say that it is a function because by those conditions there is a score for each specific reading hour and

each element in domain, set of reading hours, has a unique image in co-domain (set of test scores). This can simply be done by relating each input in the set of reading hours, by an arrow, with their respective scores.

All 20 Participants' written work on the questions showed the following results. The participants were able to do better in function machine, graphic, and analytic representations and least able to do in the algebraic expression and character string representations. This signals that students are mostly able to deal with function machine, graphic, and analytic representations of the concept of function.

Table 12: Responses for questions #3 – #11

Questions	Responses	Frequency	Percentage
#3: $y = f(x)$	Correct	16	80%
	Incorrect	4	20%
	Blank		
#4: Function Machine	Correct	14	70%
	Incorrect	6	30%
	Blank		
#5: Graphs	Correct	14	70%
	Incorrect	6	30%
	Blank		
#6: Character Strings	Correct	9	45%
	Incorrect	7	35%
	Blank	4	20%
#7: Algebraic Expression(concept)	Correct	13	65%
	Incorrect	6	30%
	Blank	1	5%
#8: Algebraic Expression (process)	Correct	8	40%
	Incorrect	9	45%
	Blank	3	15%
#9: Table (composition)	Correct	12	60%
	Incorrect	7	35%
	Blank	1	5%
#10: Analytic (composition)	Correct	17	85%
	Incorrect	2	10%
	Blank	1	5%
#11: Statement	Correct	14	70%
	Incorrect	5	25%
	Blank	1	5%

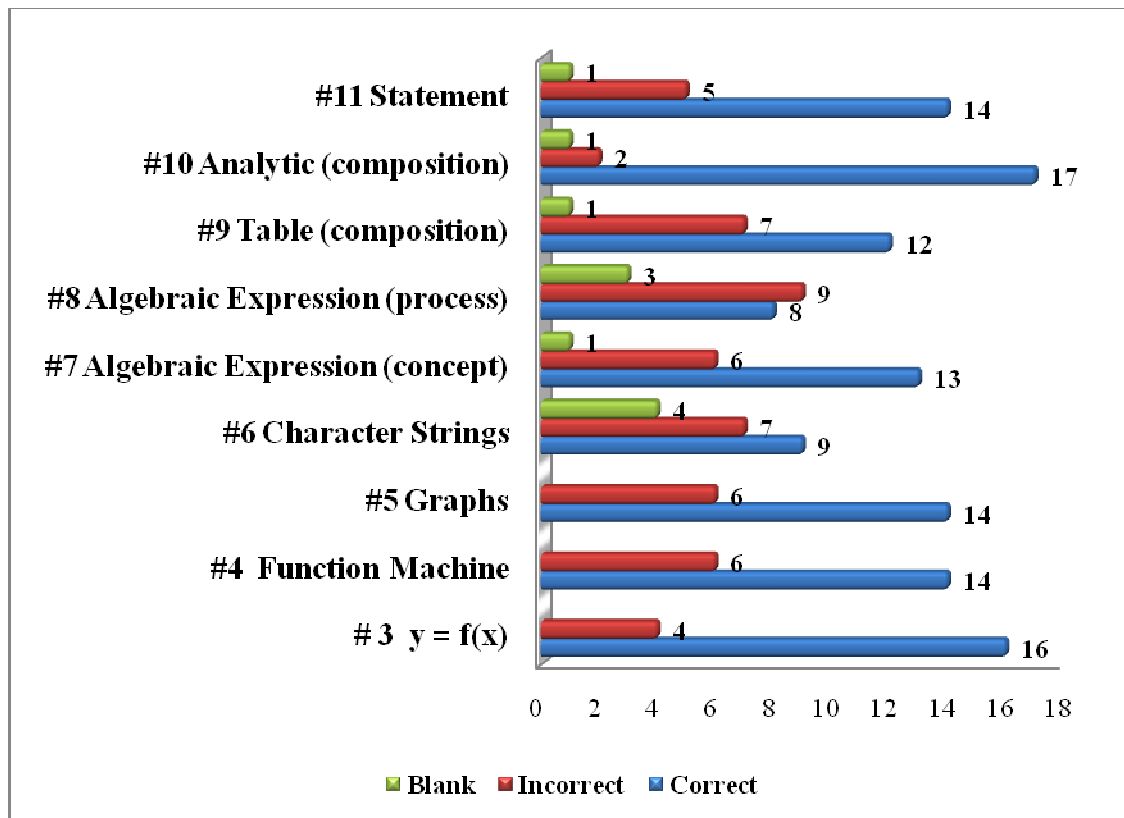


Figure 8: Answers for questions #3 through #11

Looking at Figure 8 and Table 7, two themes emerge from the quantitative data. First, how was it possible for the participants to answer most of the questions, shown in Figure 8, correctly despite the fact that only 8 of them were able to give a reasonable definition of the function? Does it mean that having the correct definition of the concept of function does not help students to deal with conceptual problems? Second, what caused the remaining 35-55% and 45-60% of students not be able to answer the Character Strings and in Algebraic Expression (process) questions, respectively?

The initial analysis of students' written responses showed that some students answered questions #5, #6, #7, and #11 only by writing short responses (such as Yes or No) without reasoning. In fact, these questions need more conceptual understanding to

explain why the answer is correct and needs more reasoning. Also, these questions are not from those kinds of questions that can only be answered by Yes or No responses because a student needs to think about the question and follow pre-action → action → process processes in mind and/or on paper to show whether the given questions show a function. That is why the reader can see the changes in the percentages (frequencies) of responses in both qualitative and quantitative sections. In addition, some students answered some of the questions through some tools (such as drawing a straight line parallel to the *y-axis*, straight line yardstick) other than the definition of the concept of function to answer them. These themes, along with others emerging from qualitative data, are further explored in the qualitative data section.

In what ways do students represent a function?

In order to understand in how many ways the participants can represent a function, they were asked to write in how many ways a function can be represented (Question #2). Taking into account the *facets*, presented in Methodology chapter, almost all of the participants answered this question correctly. Although their responses differed slightly, the responses included various representations depicted in Table 13.

All of these representations that facilitate learning are embedded in today's high school and university level mathematics curricula. It is believed that the more a student is exposed to different representation of the concept, the more he/she will deepen his/her understanding of the concept and deal with various situations effectively. The above table also echoes that such representations are embedded in the high school and university level mathematics curricula in Afghanistan.

Table 13: Responses for representations of a function

Participant	A function can be represented by
1	diagrams and graphs
2	graphs and equations
3	graphics and formal form ($y=f(x)$)
4	formal and sets
5	graphs, diagrams, tables, and ordered pairs
6	formal, graphs and real life examples
7	diagram, graphs, and tables
8	graphs, diagrams and sets
9	graphs, formal form, sets, and diagrams
10	graphs, sets, and formal form
11	diagrams, formal form, graphs, and ordered pairs
12	diagrams and equations
13	equations, graphs, sets, and diagrams
14	graphs, diagrams, sets, and formal form
15	diagrams, formal form, and graphs
16	graphs, sets, and formal form
17	graphs, sets, and tables
18	diagrams, formal form, and equations
19	a graph and a relation
20	sets, graphs, and formal form

However, what is important is to see to what extent these representations are pictured in textbooks in a way that does not impede learning. Looking at recent school textbooks printed in Afghanistan, it seems that the quality of the figures is not that much good and sometimes the figure lack some of the aspects of a mathematical concept. From an educational point of view, a misrepresented or low quality figure can pose challenges to the learner.

Qualitative Data

This section presents a more detailed analysis of students' written responses on the questionnaire and their reflections in one-to-one interviews by categorizing each participant's conception of the concept of function. Two categories of students'

conception of the concept are discussed in this section. The first one is the participants' conceptual understanding of the function concept. The second category is their procedural performance by applying their conceptual understanding. The focal point of this section is to provide answers for research questions by analyzing and presenting participants' written work and their reflections in the interviews. The researcher intended to present findings with their respective research questions in order to have a better understanding of the participants' conception of the concept and their difficulties in learning it.

It is important to mention that the researcher has purposefully selected those excerpts that are most relevant to the research questions. Only one interview is cited completely for the purpose of making a basis for an emerging theme called Teaching as Dialogue. This is because, during the one-to-one interviews, the researcher observed that teaching by dialogue was an effective approach for teaching mathematics.

Conceptual understanding: What are the consequences of students' preference for procedural methods rather than conceptual understanding?

The written work and reflections of the participants indicated that having a good conceptual understanding of the concept of function enables students to be successful in dealing with various situations, regardless of how they are represented. The sample was divided into two groups based on their responses for the conceptual question (question #1). The first group, Group⁶ 1, included 8 students who correctly defined the concept of function. And the second group, Group 2, included 12 students who could not give a

⁶ Two of those four pre-service students who had higher GPAs were not in this group!

correct definition of the concept. In order to find whether this conceptual difference between both groups influenced their procedural methods and their responses, conceptual questions #4 through #11 were used to explore this conceptual difference. It was found that students' conceptual understanding influenced their overall performance in various representations of the concept of function. For example, all students in Group 1 defined the concept correctly and were able to answer correctly almost all of the conceptual questions as well as those questions that needed conceptual understanding. They were able to elaborate their responses with good reasoning, except for the Character string question. Only 4 of them were able to answer this question. They had some misunderstandings about this question, which are discussed later in its respective section.

In contrast, one student in Group 2 could give a correct definition. Also students in Group 2 were able to do relatively better in Function Machine, Analytic (Composition), and Statement questions. Their responses to the questionnaire as well as their reflections in the interviews indicated that they were not able to elaborate their responses with good reasoning. Since they did not have a good definition, it could be one of the reasons that these students shortly answered some of the questions depicted in the quantitative data section. This analysis also provided answers to the issues that emerged in the qualitative data sections because these such responses were counted as correct responses (e.g. responses to questions #5 and #6).

The results, presented in Table 14, reflect a serious issue about the learning of the concept of function that should be addressed because 60% of the participants were not able to understand the concept and apply it in various situations. Therefore, it is important

both for those who design the curriculum as well as those who teach the concept. They should search for means to address this issue because written curriculum goes together with the taught curriculum which is usually developed by the teacher based on the expectations noted in the curriculum.

Table 14: Frequencies and percentages of the correct responses

Groups	Group 1(8 students):	Group 2 (12 students):
Correct Responses for	8 Student Defined Correctly	1 Student Defined Correctly
#4 Function Machine	8 (100%)	6(50%)
#5 Graphs	8(100%)	5(42%, approximately)
#6 Character Strings	4(50%)	1(8%, approximately)
# 7 Algebraic (Concept)	8(100%)	5(42%, approximately)
#8Algebraic (Process)	8(100%)	0(0%)
#9 Table (Composition)	8(100%)	4(33%, approximately)
# 10 Analytic (Composition)	8(100%)	9(75%)
#11 Statement	8(100%)	6(50%)

In order to explore more this difference, we underpin this argument by presenting and analyzing the qualitative data. We present some of the responses from which Table 14 was developed. We start off by presenting some of the responses of students from Group 1 followed by students' responses from Group 2. Students' responses, in Group1, showed that they were able to answer most of the questions correctly because, whenever they were asked a question, they first applied their own conceptual understanding of the function concept with its core characteristic to answer that question. Their definitions of the concept are cited bellow:

Group 1

1. *A function is a relation between two sets where every element of the first set is related to one and only one element in the second set.*

2. *A function is a relation between two variables, defined as y and x , where for every x there must be a unique y , not two y s, for one x .*
3. *A function is a relation between an object or a number with its image where the object cannot have two or more images, but the inverse is correct.*
4. *A function is a relation between [the elements of] two sets if and only if one element in set B is mapped for every element in set A .*
5. *A function is a relation between [the elements of] of two sets, such as A and B , where every element in A is related to only one element in set B . For example, if we have $A = \{a, b, c, d\}$, then the relation defined as multiply 2 with each element of set A is a function and is denoted as $R = \{(aR2a), (bR2b), (cR2c), (dR2d)\}$.*
6. *I think a function is a relation between set M and set N where every element in M is related to only one element in N .*
7. *The relation between the elements of two sets is a function if and only if every element of set A is mapped to only one element in set B .*
8. *A function is a relation where for every element of set A there is only and only one element in set B .*

Looking at their definitions, it seems that this group has a profound conceptual understanding of the concept. They presented the formal definition of the function in their own perceived world, the Embodied World. Their responses indicated that they defined the concept through Embodied – Formal world. Although their definitions varied in wording form, they had developed on their *met-before* conceptions of the concept from embodied world to symbolic and formal worlds without losing the core characteristic of the concept, which is the existence of a unique image in co-domain for every element in domain. This is similar to what Tall (2007) argued that sometimes a *met-before* can help some students to build upon their previous knowledge easily.

Further analysis of their written responses and reflections indicated that their substantial conceptual understanding helped them to perform better in various representations of the function as well as in procedural aspects. For example, when they were asked whether the phenomenon presented in question #11 is a function or not, they

first drew a diagram, and then related the number of hours to their respective scores with arrows.

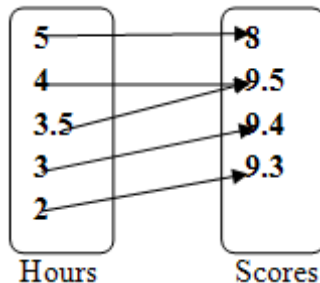


Figure 9: Diagram drawn by the participants

Then they applied their own definitions to see whether the phenomenon satisfied those conditions. When they saw that the phenomenon satisfied the conditions, they said that it is a function. In addition, in order to understand whether their conceptual knowledge had influence over their procedural performance, questions #7 and #8 were given to them.

Their answers are cited bellow.

Group 1

1. [the student first draw the graph of $f(x) = \begin{cases} 1-x & x \leq 1 \\ x^2 & x > 1 \end{cases}$, and then said that] *it is a function. [Because] for $x \leq 1$, it is $f(x) = 1-x$ in the first part. And in the second part, for $x > 1$, it is $f(x) = x^2$. [Therefore,] for ever point such as x , there exists only one point such $f(x)$. [Furthermore, he found that] $f(0) = 1, f(1) = 0, f(2) = 4$.*
2. *Yes, it is a function, with two rules, because when x varies, we get only one $f(x)$ for each unit of variation. [Furthermore, the student found that] $f(0) = 1, f(1) = 0, f(2) = 4$.*
3. *It is a function. We have the following ordered pairs*

$$y = \begin{cases} (1,0) & (0,-1) & (-1,2)\dots & x \leq 1 \\ (2,4) & (3,9) & (4,16)\dots & x > 1 \end{cases}$$

If the first two values are similar, then it is not a function. And if the first two values are not similar, then it is a function. Since all the first two values are not similar, then it is a function.

- 4. Yes, it is a function because for $x \leq 1$, it is $f(x) = 1-x$, which is the function of a straight line. And for $x > 1$, it is $f(x) = x^2$ which is a second degree function and its graph is a parabola. Therefore, it is function. Furthermore, $f(0)$ and $f(1)$ are defined for $f(x) = 1-x$ since $x \leq 1$; but $f(2)$ is not defined for this part function. And $f(2)$ is defined for $f(x) = x^2$ since $x > 1$; but $f(0)$ and $f(1)$ are not defined for this part of function.*

The other four also responded reasoning similarly. In addition, they were asked to write their responses for the Graph question. Their responses are cited below.

Group 1

- 1. Graph (b) is a function because for every x we have only one y . And the graph (a) is not a function because for a x there are two values. Therefore, by definition, it is not a function.*
- 2. The graph (b) is a function because for every x , y takes only one value; and it is one-to-one.*
- 3. The graph (b) is a function because if we draw a straight line parallel to y -axis, it cuts the graph in one point. Thus, by definition, it is a function. And the graph (a) is not a function because the straight line cuts the graph in two points, then, by definition, it is not a function.*
- 4. The graph (b) shows a function because if we draw a straight line parallel to y -axis, it cuts [the graph] only in one point. The graph (a) cannot be called a function because the straight line parallel to y -axis cuts the graph (a) in two points.*
- 5. Yes, the graph (b) is a function and is ascending. The graph (a) is not a function because if we draw a straight line parallel to y -axis, it cuts the graph (a) in two points.*
- 6. The graph (b) is a function because for every x we have only one y . Taking into account the definition of function, whenever x varies, the variable y also varies. In this case, when x increases, y also increases.*
- 7. The graph (b) is a function and the graph (a) is not because we know from the definition of the function that every element in the second set should be related to only one element in the first set. Also, if we draw a straight line parallel to y -axis, it should cut the graph in only one point. This [condition] is true for the graph (b), but it is not true for graph (a) because it cuts the graph in two points.*

8. *The graph (a) is not a function because, by definition, for every x we should have only one y . Also, if we draw a straight line parallel to y -axis, it cuts the graph in two points. And the graph (b) is a function because the straight line cuts the graph in only one point.*

They also found the correct answers for questions #4, #9, and #10, which are $4x + 10$, $f(g(2)) = 0$, and $f(g(3)) = -13$, respectively. However, although they had a good conceptual understanding of the concept, some of them had difficulty with question #6 further explored in its respective section called The essence of continuity and formula.

In contrast, Group 2 students' written work and reflections indicated that their restricted conceptual understanding and misconceived definitions negatively influenced their procedural methods and final outcomes. For example, they incorrectly answered some of the questions in their written work and were unable to develop a good reasoning strategy for most of the questions in the interviews. When they were asked to define a function, their responses included the following statements.

Group 2

1. *A function is a set where for every value a value is found.*
2. *Function means obedience; for example, obeying the law of an institution by its employees or obeying family rules and regulations.*
3. *The relation between two variables or two things or tow places is called function where one appears as an independent variable and the other as a dependent variable. For example, the second degree polynomial is a polynomial function defined for all real values. Also rational polynomials are functions for which the denominator must not be zero. And as a real-life example, a tree is a function of environment and water.*
4. *The function is a relation used for determining dependent and independent algebraic quantities.*

5. *I think a function means obeying the law of an institution. For example, I as a student of school of education should follow and obey its rules and regulations. Also a polynomial is a function because for every x we find a y , that is, y follows x . Another example is $f(x)=x^2$.*
6. *A function is a relation and is a direct relation between two sets. For example, if we consider sets like A and B , the set $A \times B$ [the Cartesian product of A and B] is a relation. Every function can be a relation, but every relation cannot be a function. For example, if we look at a mirror, we see one image on the mirror. If we see two images on the mirror, it cannot be called a function.*
7. *A function is a relation between two objects or two solid objects. And mathematically, it is a relation between two numbers such as a and b , where whatever value we give to a , we get a value for b .*
8. *A function is a relation between two quantities. And if we give a value for x , we get a value for y . For example, $y = f(x)$, $y = x$ and $y = \sin(x)$.*
9. *A function is a relation between two sets where set A is related to set B .*

In this group, one student's written work and his reflections indicated that he was not able to apply his definition of the concept although he defined it correctly. For example, he said that "a function is a relation between variables. The relation between two sets such as A and B is called a function if every element in A has only one element in set B ."

10. *A function is the relationship between two things; also we can say that the dependence of one variable on another variable is a function.*
11. *A function is made of three parts (1) Domain, (2) Range, and (3) Co-domain. For example, if we consider a camera by which a person's pictures can be taken. In this case camera is Domain and the person whose pictures are taken is Co-domain. Also, if a person lies two or three times, no one will trust him/her when he/she speaks the truth. Through these examples we can obtain the function concept; and in a function, we can have several Co-domains for a Domain.*

These definitions showed that they were able to state only half of the definition, the first condition, but they were unable to state the other half of the definition, the

uniqueness condition. Not mentioning the uniqueness condition makes their definitions to be incorrect although they were correct in stating half of the definition. Also these students were able to reason in the Embodied and Embodied – Symbolic worlds where none of their definitions were correct. These participants developed their *met-before* understanding of the concept by moving from Embodied world to the Symbolic and Formal worlds. But they could not preserve the core characteristic of the concept as they moved from the first world to the other two. Instead of relying on the conceptual knowledge, they merely relied on some tools, e.g. straight line yardstick, to measure whether the given phenomena were functions or not.

Further analysis of the responses of the 12 participants who defined the function less accurately showed that they had the following overall misconceptions. They believed that a function is a relation between two sets or two variables or two objects if

- for every input, an output is found, and even the outputs can be more than one
- one variable is independent and the other is dependent
- it is defined by a mathematical formula or expression, and
- one set is related to another set.

Looking once again at their overall misconceptions, it was found that only one of the participants, in Group 2, could define the concept correctly. The question is why only one student from a group of 12 could define the concept correctly and was not able to apply his definition in other situations. It has potential implications for the curricula that were followed in Calculus and Algebra courses. In fact, if we look at those definitions and misconceptions, they include two types of reasoning. First, most of them used the

notion of a relation between the elements of two sets or two objects, echoing the first half of the definition, which came from the Algebra curriculum. But they were unable to complete their definitions by stating the other half. This means that the way the concept was presented in the curriculum was not understandable for the majority of students (or at least in our sample). Second, some of them used the notion of dependence, obedience, and real life examples that probably came from the Calculus curriculum. Looking at these two types of reasoning, we can say that both courses followed two different approaches and introduced the concept in isolation. The literature about the learning of this concept indicated that introducing the concept in an isolated form challenged students to construct the concept. Therefore, what is important for curriculum designers and instructors in that context is to refine the existing curriculum of the concept by embedding similar definitions and representations in the curricula of both courses so that students learn it easily.

However, it does not mean that these 12 participants were very unsuccessful in dealing with questions asked in the questionnaire and in one-to-one interviews. They answered some of the questions correctly, but they could not develop a good reasoning strategy to elaborate how they came up with the responses. For the purpose of showing how their conceptual knowledge affected their procedural performance, only their written responses and reflections for questions #4 and #5 are cited below. The other responses are mentioned in their respective sections.

Responses for question #4: Except 4 students in Group 2, all other participants got the correct answer, which is $4(y(x)) = 4(x+2)+2 = 4x+8+2 = 4x + 10$, with good reasoning.

1	2	3	4
$y(4) = 4(x+2)$	$y(x) = x+2$	$y(x) = x+2$	<i>Has a direct relation because it is a function.</i>
$y(4) = 4x+8$	$y(4) = 4+2 = 6$	$y(4) = 4+2$	
$4x+8 = 0$		$y(4) = 6$	
$x = -2$			

The above responses indicated that these four students were unable to determine where to plug in the input. In addition, they misunderstood the order of the processes (input→process→output) to be carried out one after another. This indicates that they are in the *pre-procedural* and *pre-process layers* of the concept because none of them got the correct answer although they used their manipulative skills to find an answer.

Also students in Group 2 answered the question #5 as:

Group 2

- 1. The graph (b) shows a function because there are different values for different values. The graph (a) is also a function because for small values the area of the circle is small and for bigger values the area of the circle gets larger.*
- 2. The graph (b) is a function because for every value of the variable the function receives different values. The graph (a) is also a function because its points are ordered pairs of the form (x, y).*
- 3. Both graphs [(a) and (b)] are functions because there are points on y-axis for points on x-axis. Thus, a curve is obtained.*
- 4. Both graphs [(a) and (b)] are functions because there are numbers for y when the dependent variable x gets different values. Also, the graph (a) is a function because for every values of x and y, the sum of squares equals to 1, which is a constant.*

5. *Both graphs [(a) and (b)] are functions. The graph (a) is a trigonometric function used for determining angles. Also, the graph (b) is the graph of quadratic function.*
6. *The graph (b) is a parabolic function, and its relation is $y^2 = x$. If a value is put for x , two values are found. Thus, it cannot be called a function. Also, the graph (a) is a trigonometric function, and its relations are $y = \sin(x)$ and $y = \cos(x)$.*
7. *The graph (b) is a function because if we draw a straight line parallel to y -axis, it cuts the graph in one point. Also the graph (a) is not a function when consider it as having one variable, but if consider it as a function of two variables, it is a function.*
8. *The graph (b) is a function because it shows a curve and for every x , one y is found.*
9. *The graph (b) is a function because for every value, it has taken one value.*
10. *The graph (b) shows a function because there action line; and, by definition, the graph (a) is a function because for every x there is a y .*
11. *The graph (b) is a function because for every x a value is found for y . And I have not seen a graph such as (a), but there should be a mathematical relation in order to say whether it is a function or not.*
12. *The graph (b) is a quadratic function and its graph is a curve. Also, the graph (a) is not a function because the y -axis is cut in two points.*

Since question #4 was a conceptual question and needed conceptual understanding rather than determinative tools, e.g. straight line yardstick, the majority in this group responded incorrectly, except one who used straight line yardstick. In addition, most of the answers showed that they believed that both graphs were functions. This is a wrong answer because, as we explained earlier in this chapter, only graph (b) is a function because it satisfies those two conditions. Although some of them were right in answering a part of the question, they did not want to reason about the other one because they lacked the necessary conceptual knowledge.

These results can be linked to their definitions mentioned before in this section. Only one of them was able to define the concept correctly, but most of them were unable to define the concept and called every input-output process to be a function. The only

condition they had in mind was that there should be an element in the second set, co-domain, for every element in the first set, domain. Having this half of the definition caused them to answer question #5 incorrectly and also some other questions that are not cited here.

The overall conceptual knowledge and procedural skills of students in both groups and the effects of conceptual understanding on other aspects of the concept can be profiled in terms of *facets* (multiple representations) and *layers* (pre-action→ action→ process→ concept(object)) shown in the following figures. It is important to note that the more a group is close to the core of the diagram (colored toward the center), the more the group had good conceptual understanding and in turn it helped them to do better in other representations. Although there is an overlap among students' overall conceptual knowledge and procedural skill, because , in Group2, some of the students' work can be positioned in the process *layer*, both figures clearly show that having a good conceptual understanding enables a learner to deal with various aspects of the concept. It should be understood that these representations are the most influential factors that facilitate learning so they should not be underestimated.

However, if students prefer focusing on these representations rather than focusing more on understanding the concept, it will affect the outcome of the actions they do on a particular aspect of the concept because neither of the properties of the concept (such as the existence of a unique image for each element in domain) can be derived from these representations. The reason that Group1 seemed to be more toward the core, toward the concept, was that they developed their conceptual understanding through these multiple representations while preserving the core characteristic of the concept.

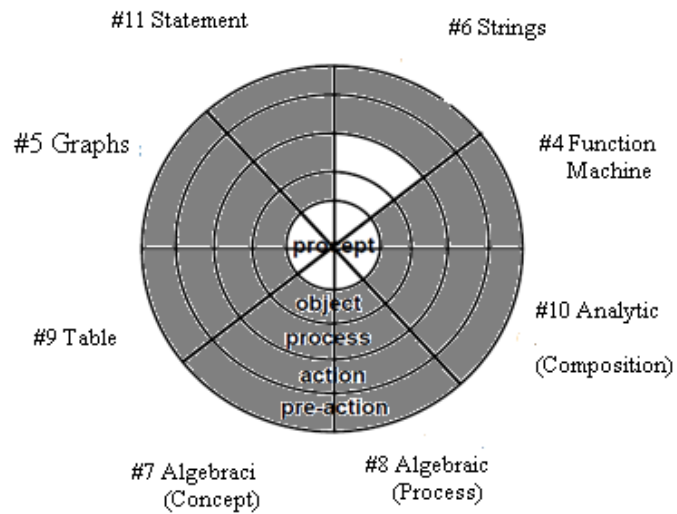


Figure 10: Group 1 in terms of facets and layers

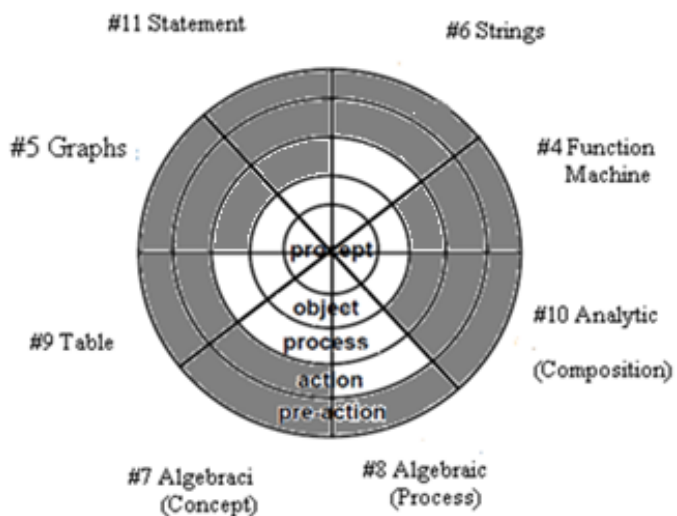


Figure 11: Group 2 in terms of facets and layers

Looking at the results of the qualitative and quantitative data, the following hypothesis can be developed.

Hypothesis: Having a good conceptual understanding of the concept of function enables students to be successful in dealing with various situations, regardless of how they are represented.

In order to test this hypothesis (only for this sample) statistically, ANOVA and t – test for independent samples (equality of variances was not assumed) on the difference between the means of the correct responses for both groups, Group1 and Group2, were used. The result was that the hypothesis was true for this sample. The students who correctly defined the concept, Group1, were able to respond more questions ($M = 7.50, SD = 1.41$) than the students who could not define the concept correctly, Group2, ($M = 4.50, SD = 2.88$). This difference was statistically significant at $\alpha = .05$ level, $t(10.19) = 2.65, p = .024$.

However, this result cannot be generalized for all fourth year undergraduate students because the sample was nonrandom. But we can argue that this result is true for the participants who voluntarily participated in the study. In order to generalize this significant difference for the overall population of fourth year undergraduate students, a random sample should be chosen. This is the second phase of this study that will be pursued next year.

What difficulties do students have in learning the concept of function?

During the interviews, the participants were asked to talk about their own difficulties in learning the function concept and reflect upon it. Their responses have resulted almost similar views mentioned in the literature by Tall and Bakar (2001), Akkoc and Tall (2002) DeMarios (1996), Reed(2007), and Eisenberge (1992) . Here, we cited some students' views about what difficulties they had in learning the concept. These views are important for those who are keen to explore further students' conception of function and their difficulties in learning it.

When the participants were asked what difficulties they had in learning the function concept, they mentioned the following issues. It is important to note that most of these views are written by students who could not define the concept completely. The majority of those who defined the concept correctly, they tended not to talk about their difficulties; the only thing they emphasized was to study more about theoretical aspects of the concept.

1. *I cannot show the composition of functions by its graph. Also, when I do the algebra of functions, I do not know what it means to [add, subtract, divide, or multiply] them.*
2. *I have difficulty dealing with trigonometric functions. And I do not know the applications of the function concept. I always memorize it, and then forget it.*
3. *I have difficulty in understanding the function concept itself. Also I do not know where the function is used in real life.*
4. *I have difficulty in applying the function concept in real-life situations. Whenever I study the function concept, I do not know which approach I should take in order to understand it better.*
5. *I have difficulty in understanding the function concept itself and have difficulty while doing the algebra of polynomial functions. Also, I have difficulty in determining the domain and co-domain of a function.*
6. *I have difficulty in understanding the function concept itself and have difficulty dealing with trigonometric functions, logarithmic functions, as well as in finding an inverse of a function.*
7. *I have difficulty in understanding the function concept itself and cannot determine the domain and co-domain of a function.*
8. *I do not understand the functions presented through mathematical formulas.*

These difficulties indicate that math major students, at least in our sample, are not able to comprehend the basic notions of the function concept. These difficulties are very serious and inform instructors and curriculum designers to think about what causes such

difficulties. These difficulties are potentially important to be addressed because these are the impeding factors that hinder one's understanding of the concept as well as other topics that requires a solid understanding of the concept of function. Both instructors and curriculum designers should look back at their teaching practices as well as at the curriculum and bring necessary changes in order address the abovementioned difficulties.

What difficulties do students have in selecting and using appropriate representations?

Participants' written responses and their reflections in the interviews indicated two other major issues that negatively affected their responses for Character String question (#6) and lead them to get the wrong answer for question #8(Algebraic (process)).

The essence of continuity and formula.

All participants' written work and reflections indicated that in order for a situation or a phenomenon represented by different representations to be a function, it is necessary for observations to be continuous and be related by a mathematical formula. To elaborate this issue, we cite some of the participants' responses for Character Strings question (#6 – the answer for this question was given earlier in this chapter and based on that it is a function. One also can say that since points are made of ordered pairs like {point1 = (a, b), point2 = (c, d), point3 = (e, d), point4 = (f, g), point5 = (h, i)}; an then say it is a function because there is an element in co-domain for each element in domain, and it is unique because none of elements appear twice as a first component. But the inverse is correct, e.g. point2 and point3 have the same output, because it does not violate the conditions). When they were asked whether the phenomenon presented in Figure 12

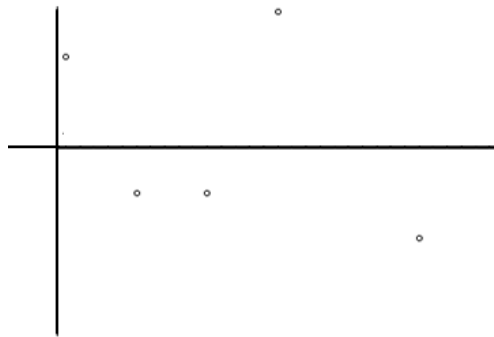


Figure 12: Character strings

is a function or not, their responses included three types of errors. First, four students thought a formula is necessary in order for each point to be determined. For example, these students responded as:

1. *I think it is not a function because here the x-axis and y-axis are not specified so that we can give values for x in order to get a value for y [the student meant that a formula is necessary to related x to y].*
2. *It is not a function because the points are situated in 2-D space and there is no mathematical relation for them.*
3. *It does not show a function because there is not mathematical rule or formula to relate horizontal and vertical axes.*
4. *It cannot be a function because a mathematical formula is not given to relate x and y.*
5. *It is not a function because it's the graph of $y^2 = x + 1$. And here we get two images for every value.*

These students were trying to first present the issue through a formula, in Symbolic world, and then decided whether it was a function or not. Since they could not find a formula for it, they concluded that it was not a function. This is similar to what Tall and Bakar (1991) stated that whenever students are given a graph, they try to find a “prototype example” as a formula in the form of $y = f(x)$ and then say whether the given

graph is a function or not (p. 1). In fact, this question does not need to be manipulated because there is nothing to be given as an input and there is nothing to show the process to be followed. The only thing that is given is the outputs. What it needs is to follow the *action→process→concept* processes mentally and perform mental actions such as thinking about each point's components on both axes and determining them. Once the components of each point are found in mind, then it is easy to apply the definition of the concept and decide that it is a function.

Other 4 students reasoned that the points must be continuous and they should be connected to each other so that it would make them to be a function.

6. *We cannot call it a function because these points neither are connected to each other and nor have the same dimension; and they do not have relation among themselves. They are produced by the intersection of x and y axes. If they were connected, they would make a function.*
7. *It is not a function because there is discontinuity [among points], also for each value of y , there are two value for x . Thus, it is not a function.*
8. *It is not a function because its graph does not exist.*
9. *It is not a function because there is a discontinuity among points; also we cannot find two x_s for a y .*

This is another type of misunderstanding of the concept because the only condition they put on the situation to be a function is that the points should be continuous. Having this notion negatively affects their understanding of the concept and causes them to make wrong decisions. It is possible that some of these students might have decided that graph (a) in question #5 was a function because all points were connected and there was an output for each input in the graph. One thought that can be given to explain this misunderstanding would be that their condition has nothing with determining whether the situation is a function or not. The continuity condition might have come from their previous

developed *met-before* called continuous functions in differential calculus. But whatsoever the result was, they applied it on this situation to have an answer for the question. They lacked that reasoning that could be derived from the definition of the concept to determine whether it was a function or not.

Some students reasoned that in order for the phenomenon to be a function, there must be an output for each input.

10. *It cannot be a function because there must be a value for every y, but in the above situation there is no value [for y].*

In his written work, he said that “*it is a function because the points are formed on coordinates.* But later in the interview, he said that “*it is not a function. Where are other points? We should have y_s for all x_s .*”

11. [Although he briefly wrote that it's a function, during the interview he said] *it is not a function because we should have all y_s for all x_s .*

These students were good in reasoning type because they wanted to apply the first condition, the existence condition, on the situation and then make a decision. Since they saw that it did not satisfy the first condition, they concluded that it was not a function. However, although their reasoning was strong, they misunderstood the situations where in one of them outputs (elements in co-domain) are not given and in the other outputs are given. In the former, the student has to check the existence condition with every single element in domain and see if its image is in co-domain. But in the later one, outputs are given. What the student can do is to find the components of each output and then apply the existence and uniqueness conditions.

Two other students said that this picture was new for them. They said the following.

12. *This picture [representation] is new for me, then I cannot say whether it is a function or not.*
13. *[his written work shows that he briefly said that it is a function but during the interview he said] I do not know whether it is a function or not.*

These students' responses have a point for this study, too. Every mathematical concept can be internalized by doing more practices and exposing to various representations of the concept. Since most of the student were not able to answer this question correctly, it is possible to say that this representation might have not been embedded in the curriculum, or at least it was deemphasized. It also has a potential implication for those who design the curriculum or teach students. They should try to embed this representation in the curriculum, if it has not been embedded yet, because as the recent discussions about the teaching and learning of functions show this type of representation can help students to improve their conceptual understanding through a series of reasoning processes.

To sum up, it seems that these participants consider a situation as a function if it has a continuous curve presented by a mathematical formula. This indicates that these participants do not have that much potential ability to use a "process conception of function" (Reed, 2007, p. 82) to answer questions consisting of character strings.

However, the written work of some of the participants indicated that they could answer the question correctly with good reasoning using the definition of function and the notion of ordered pairs. For example, 4 of the participants, who were in Group 1, after giving a discussion similar to the one mentioned before, said that "by definition of the function, it is a function because for every x there is only one value for y ." Also, one of

them said that “it s a function because each point is formed by an ordered pair such as (x, y) .”

Domain negligence.

Another important issue that some of the participants had difficulty with was the determination of the domain and co-domain of a function. Sometimes they even did not take them into account while answering the questions. In order to understand simultaneously both their conceptual understating and procedural performance, the question #8 was given to them to reflect on it. Some of the participants’ written work and reflections indicated that they tended to plug in an input in both of the expressions that made the function. This is because they thought that an input can be plugged into both expressions regardless of how the domain was defined. Some of their difficulties can be seen in the excerpts drawn from their written work and reflections in the interviews.

For example, 6 of the participants wrote: If $f(x) = \begin{cases} 1-x & x \leq 1 \\ x^2 & x > 1 \end{cases}$, then $f(0), f(1)$,

and $f(2)$ are:

$$\begin{aligned} f(0) &= 1 - 0 = 0 & f(0) &= 0^2 = 0 \\ f(1) &= 1 - 1 = 0 & f(1) &= 1^2 = 1 \\ f(2) &= 1 - 2 = -1 & f(2) &= 2^2 = 4. \end{aligned}$$

The above responses indicate that these students are in *pre-procedure layer*, meaning that they wanted to have an answer whether it is conceptually correct or not. In addition, it seems that they are in *pre-action* stage of the development of the concept, meaning that they just manipulated the expressions by plugging in various inputs without considering whether they were defined for that particular expression.

Teaching as Dialogue

One of the interesting issues that the researcher noticed during the interviews with participants was the fact that the oral question-and-answer discussions with the participants enabled them to discover more about the different aspects of the concept of function. The questions during the oral discussions were posed in a way that they did not contain answers. The informal question-and-answer discussions provided an opportunity for some of the participants to improve their conceptual understanding, especially in cases that they had not met them before frequently. For the purpose of explaining this issue, the researcher wanted to cite only one complete interview. Other interviews were also done in the same way.

Sample Interview

The interviewee was an in-service student. She was sort of average student comparing to her other classmates. She has been teaching math for four years in one of the public high schools in Balkh province, Afghanistan. Her written work indicated that her conceptual understanding was a little bit restricted, but she was good in some aspects of procedural performance. When she was asked to define a function, question #1, she wrote that “a function is a relation between two sets where set A is related to set B .” She put no condition on the relation. Putting no condition on this relation makes her definition colloquial and she is in *pre-procedural layer* (DeMarois, 1999, p. 3).

However, after having an informal dialogue, using question-and-answer approach during the interview, it was observed that she developed a good understanding of the

concept both in *facets* and *layers* and in procedural aspects. The following excerpt shows her conceptual knowledge and procedural skills.

Question #1: What you think a function is? Please write it in a sentence or more. If you can, please give a definition.

Interviewee: As I wrote, a function is a relation where set A is related to set B.

Interviewer: you mean that, without any condition, every relation between [the elements of] set A and set B can be called a function.

Interviewee: No, not every relation is a function.

Interviewer: Then, which relation can be called a function?

Interviewee: The relation where for each element of set A there is only one element in set B.

Here we can see that she strengthened her definition of function reasoning that there is a special relation that can be called a function. This shows that she developed her conception of function through a mental *process* putting a condition on the relation.

Question #2: Describe the ways a function can be represented.

Interviewee: [we can show a function] by graphs, diagrams, formal notation, and sets.

These four representations show that she can do well in these facets.

Question #3: What do you think when you see the notation $y = f(x)$? Or what do y , f , and x show? Please explain your answer.

Interviewee: f is a relation...I just know that f is a relation.

Interviewer: Then what are x and y ?

Interviewee: x is independent variable and y is dependent variable.

Question #6: Can the following situation be called a function? Why or why not?

Please, explain your answer.

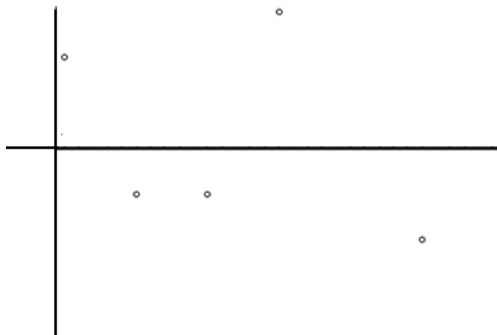


Figure 14: Character strings

Interviewer: You have written in your written work that the picture is new to you and you cannot tell whether it is a function or not. Can you explain why it is new for you?

Interviewee: As I have written, we have read [about functions] and just memorized it for the test, and after that we forgot what we have learned. We have not seen such a picture during our studies. As I said a function is a relation where each element of set A is related to one element in set B. But here...there are five points we cannot say whether it a function or not.

Interviewer: What if you apply the definition of function?

Interviewee: Well, here on x-axis [-y-axis], there are 3 points [5 points]. And for y-axis, there is one [two] point [s]. Therefore, it is not a function.

The reason she miscounted the points related to both axes was that she did not see the all points at the first glance. But she then found how many points were there. Again, here reasoning was not correct because she thought that these points were relevant to either y or $-y$, or x or $-x$ axes. She could not give a correct answer.

Question #7: f is defined by $f(x) = \begin{cases} 1-x & x \leq 1 \\ x^2 & x > 1 \end{cases}$. Is f a function? Please explain your answer.

Interviewee: f is a function because we give values for x [she is pointing at tables which she drew in her written work], once for $x \leq 1$ and then for $x > 1$.

Interviewer: If you apply the definition of function right now, is it a function?

Interviewee: Yes, because by definition for each element in set A [she points at the first table] there is one element in set B [she points at the second table].

Question #8: f is defined by $f(x) = \begin{cases} 1-x & x \leq 1 \\ x^2 & x > 1 \end{cases}$. Evaluate $f(0)$, $f(1)$, and $f(2)$.

Interviewer: You evaluated $f(0)$, $f(1)$, and $f(2)$ for both $1-x$ and x^2 . Does the domain of $1-x$ include 2?

Interviewee: Umm... domain of $1-x$.

Interviewer: If the domain of $1-x$ is $x \leq 1$, does this domain include 2?

Interviewee: No, it does not include.

Interviewer: let's look at x^2 with domain $x > 1$. You evaluated that $f(0) = 0$, $f(1) = 1$ and $f(2) = 4$. Can we include 0 and 1 in the domain defined as $x > 1$.

Interviewee: Umm... since $x > 1$, then 0 and 1 are not included in this domain.

Question #9: Consider the following tables:

Table 15: Table composition.

x	$f(x)$
1	3
2	-1
3	1
4	0
5	-2

x	$g(x)$
-2	3
-1	1
0	5
1	2
2	4

Evaluate $f(g(2))$.

Interviewer: You evaluated that $f(g(2)) = 0$. Can you please describe how you found this?

Interviewee: I have given four values and found the 0. It's a composite function.

Interviewer: Which values? Can you please explain how you found it? Please be specific.

Interviewee: Umm... [she was confused]

Interviewer: OK, what is the image of 2 under g ?

Interviewee: the image of 2 under g is -2...no it is 4.

Interviewer: Now, what is the image of 4 under f ?

Interviewee: It is 0.

Interviewer: then what is the value for $f(g(2))$?

Interviewee: It is 0.

Question⁸ #10: Consider $f(x) = 2x + 1$ and $g(x) = -2x - 1$. What is the value of $f(g(3))$?

Question #11: Suppose 5 students who read about function concept got the following scores in a quiz: student1 read 5 hours and got 8 points; student2 read 4 hours and got 9.5 points; student3 read 3.5 hours and got 9.5 points; student4 read 3 hours and got 9.4 points; and student5 read 2 hours and got 9.3 points. Can this phenomenon be called a function? Why or why not?

Interviewer: Can we call the abovementioned phenomenon a function?

Interviewee: Yes, because if we put number of hours in one set [she makes a set] and another for scores, and if we relate each number of hours with its respective score, then by the definition of function the given phenomena is a function.

Interviewer: And, what are the dependent and independent variables in this phenomenon?

Interviewee: Independent variable...[she is thinking to find it]....

Interviewer: if we look, students studied for different hours and got different scores. Now, is the number of hours independent variable or dependent variable?

Interviewee: it is dependent variable.

Interviewer: Can you explain why it is dependent variable?

⁸ This question was not asked during the interview.

Interviewee: Umm...the number of hours [pause] is independent variable and the scores are dependent variable because for different hours of reading they got different scores.

In addition, when another participant was asked to evaluate $f(0)$, $f(1)$, and $f(2)$ in

$f(x) = \begin{cases} 1-x & x \leq 1 \\ x^2 & x > 1 \end{cases}$, he said that “I cannot understand the functions defined as above

because I have not seen them frequently and no one has elaborated them clearly.”

However, when the interviewer, using the question-and-answer approach, discussed in detail what the function with each expression meant and what were their domains and co-domains, the interviewee got the correct answer with good reasoning and said:

Now I know how to solve such problems. If 100 similar questions are given to me, I can solve them. Before, I was plugging in the values in both expressions, but now I know where to put each value. I recommend that the function should be introduced through its domain and co-domain.

In fact, the research findings throughout the history of mathematics education have shown similar results saying that teaching as dialogue in the form of question-and-answer often has a prominent impact on students learning. In this method, students are interested in the questions because they have seen similar reasoning before and now would like to gather all the pieces of reasoning about a question to arrive at a final solution (Bagnato, 1973, p. 185).

Plato and Socrates believed that when people are questioned, if one puts the questions well, then they are able to answer the posed questions correctly. They believed that questioning uncover the ambiguities, creates desire, and “puts the listener in contact with the deep sources of his native intelligence” (Bagnato, 1973, p. 185). In addition, Polya (1973), the later developer of question-and-answer method, proposed

teachers use this approach to teach college and university mathematics. One of Polya's criticisms was that college professors lack such effective teaching method. He believed that a school teacher has an advantage over a college professor because in high school "one can use the dialogue form much more extensively than in the college" (Bagnato, 1973, p. 16).

However, whosoever uses the question – and – answer teaching method should be cautious of a very important rule, that is, the sub-questions posed professionally in this method to solve a problem or explain a concept should be consistent with one's knowledge and understanding and should not contain the ultimate answers.

How Mathematics Was Taught

Teaching practice is one of the most important factors that influences student learning. This is because whatever approaches a teacher uses to teach student, it affects their learning. It is of most importance if teachers know about teaching methods in mathematics and their students' learning styles. For the purpose of understanding how math, in particular the function concept, was taught in that particular institution, the researcher wanted to have participants' views about it in order to understand better how the teaching of the concept influenced their conceptual understanding and procedural performance. Only a few students talked about this issue. The majority of the views come from the group of students who could not define the concept correctly and had difficulty in learning the concept. It also includes a little bit of the views of the group who could define the concept correctly and did better in other aspects of the concept. Some participants reported the following issues about how they were taught.

Mathematical concepts were taught by transmission method where the teacher was only lecturing and students were memorizing the concepts.

While teaching the function concepts, the lecturers had not mentioned where it can be applied in real-life situations and had never used real-life examples. Also, the function concept was not explained clearly.

Lecturers use outdated teaching methods to teach mathematics. Also, they use those examples that are not understandable for students, and the examples are boring.

The function concept was not taught through its applications and real-life examples. It was taught only by some specific examples. This is why students forget the concept very quickly.

The function concept was introduced by various theoretical definitions that were not understandable and we forgot it quickly.

These views indicate that the transmission and lecturing teaching methods are still used in that institution. Also, it shows that the concept of function was not taught through real – life examples. What this indicates is that teachers should use other teaching methods and present the mathematical concepts through applied examples.

However, as we discussed in Methodology section, there was not provided any remedial or enrichment program during the course of four years so that math instructors could benefit from the new developments in mathematics education. Providing remedial and enrichment programs for mathematics teachers can be one of the ways to improve the existing teaching practices. Also, there might be some other issues such as teaching load and lack of academic incentives that prevents teachers to acquire and practice contemporary developments that are brought into the teaching and learning of mathematics. These issues are important to be explored further so that one can give a reasonable explanation to why teachers still use only those traditional teaching methods.

CONCLUSIONS

Findings

This study explored undergraduate students' conception of the concept of function and their difficulties in learning and understanding it. Although it was the first time that David Tall's theory was applied in that particular context of Afghanistan, the theory helped us to have a better understanding of students' knowledge and learning as well as their difficulties in learning the concept of function. The theory also indicated how exploring one's knowledge and understanding can propose effective means for improving the learning and teaching of mathematics. After having done an extensive analysis of the collected data through the questionnaire, one-to-one interviews, and curriculum documents, in summary, the following issues were found in this study.

First, the set-theoretic and Cartesian product definitions of the concept of function in the curriculum adapted from the Russian mathematics curriculum helped some of the students to deal with various representations, while most of the students had difficulty in comprehending the concept and applying it in certain situations. This difficulty not only stemmed from the negligence of the condition such as uniqueness of an image in the co-domain set for each element in the domain set, but also from the isolated introduction of the concept in both Calculus and Algebra curricula. This isolated introduction of the concept in both subjects might have been influenced by the evolution of the concept of function during the course of nine centuries, especially during 19th through 21st centuries, from its simplest representation to a very sophisticated form of logic and set representations.

Second, it was found that the function concept was presented in calculus and algebra courses in isolation, which can be counted as another factor that made the learning of the concept difficult for students. The calculus course used the analytic representation, $y = f(x)$, while the algebra course used the set-theoretic and Cartesian product representations, adopted from the Russian mathematics curriculum, of the concept of function. The calculus course was started with a brief informal introduction of the concept followed by other traditional concepts such as limit, derivative, and integral. This brief introduction of the concept in that course might be another factor that the majority of the students could not understand the concept.

Moreover, the study also showed that having a good conceptual understanding enables students to perform better in various representations of the concept of function and improves their reasoning skills while working on various types of mathematical problems. More importantly, it was also observed that teaching as dialogue facilitated conceptual understanding and seemed to be an effective approach for teaching the concept of function. This is because the participants were able to consider their understanding and improve it during the process of explaining their responses in one-to-one interviews.

In addition, the study showed that the participants from that particular institution had almost similar difficulties mentioned in the research literature collected across the world about the learning of the concept of function. For instance, in addition to other difficulties mentioned throughout the paper and in the literature, the participants thought that in order for a phenomenon to be a function, there should exist a mathematical formula and each pairs of observations, outputs, must be connected by a curve and be

continuous. Also, they tended to neglect the domain of a given function made up of more than one rule or expression. It was found that this type of function was deemphasized in the formal and taught curricula. This could be another reason to explain why some of the students faced problems in understanding such functions. Deemphasizing such functions would hinder learning the concept such as limit, derivative, and integral.

Finally, according to students' responses, it was found that transmission and lecturing teaching methods were used when teaching mathematical concepts. The concept of function was not taught with its theoretical aspects, nor was it presented with its applications in real-life situations.

Recommendations

Curriculum designers as well as math instructors should take into consideration certain measures to address students' difficulties discussed throughout this paper. They should embed the concept of function in the curriculum in a way that most of the students learn the concept easily. One of the ways that may help students to learn mathematical concepts easily (e.g. the concept of function) is to organize these concepts in the curriculum in a way that is consistent across various subjects that share the same concepts and utilize effective teaching methods integrated with technology to teach the concepts to students so that they understand and learn them through their own perceived world while preserving the Formal aspects of those concepts.

In addition, it is important to consider the teaching as dialogue, question – and – answer, teaching method to teach students. This can be an effective approach to address students' difficulties in learning mathematical concepts, especially the concept of

function. This is because the oral question-and- answer discussions during the interviews with participants enabled them to discover more about the different aspects of the function concept. The informal question-and-answer discussions provided an opportunity for some of the participants to improve their conceptual understanding, especially in instances where they had not met them before frequently.

Moreover, curriculum designers should embed the Embodied-Formal combination in the curriculum of mathematical concepts in addition to their Symbolic – Formal combination because this study showed that most of the student can comprehend the concept through this combined world.

Finally, the officials in the Ministry of Higher Education as well as in the Education Faculty of Balkh University should provide some remedial and enrichment programs for math instructors so that they will have access to 21st century developments in mathematics education in terms of content, curriculum, technology, and sound pedagogical approaches. Providing such programs will enable teachers to move from the existing teaching practices to very effective teaching practices, integrated with technology, that help learners to learn mathematical concepts easily and become life – long learners.

Future Studies

The first and foremost issue that should be explored in the future is the hypothesis developed after analyzing the collected data. It was hypothesized that having a good conceptual understanding of the concept of function enables students to be successful in dealing with various situations, regardless of how they are represented, was true for the

nonrandom sample that was selected for this study. As a future study, the researcher wants to test the hypothesis for a large random sample that will be selected from a population of undergraduate mathematics students.

In addition, one can find several issues to be explored from this paper while reading it. The first and foremost research topics for an Afghan researcher or other researchers might be the issues outlined under a topic called students' difficulties in learning the concept of function. These issues are very important to be explored because these are students' views about their difficulties in learning the concept of function. In addition, a researcher can explore the following questions. How can computer-generated graphic representations help students to learn the composition of two or more than two functions? Will teaching the concept of function through real-life examples improve students' conceptual understanding?

Limitations

Doing research in a particular context on measuring certain variables for having an understanding of that context is no longer free of limitations. This study also has some limitations that might have influenced the collection and interpretation of the data collected from that particular context. One important limitation is that the findings cannot be generalized to the whole population of fourth year undergraduate students because the sample was a nonrandom sample. Another limitation is that the researcher was not able to observe some classes because during that time the final examinations started. These observations could provide more explanations about how the function

concept was taught by teachers and learned by the students. This data would have affected the interpretation of the data if collected through observing classes.

In addition, fourth year undergraduate students might have not been able to call upon their knowledge of functions when working on the questionnaire because sometimes it takes a little bit of time to call upon what they have learned during the course of four years. Also, the level of anxiety created by the questions could have affected their responses negatively. Finally, some students might have misunderstood some of the questions when the questionnaire was administered.

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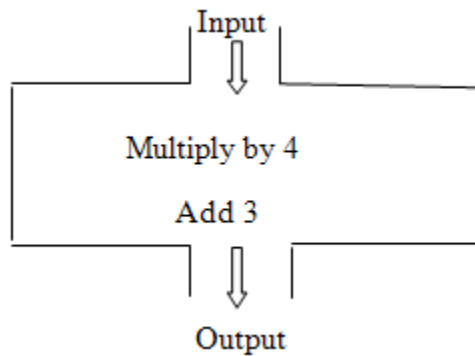
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Appendices

A. Initial Questionnaire.

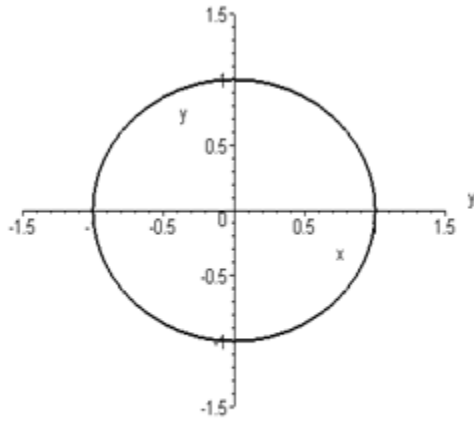
Functions

1. Explain in a sentence or so what you think a function is. If you can give a definition for a function then do so.
2. Describe the ways a function can be represented.
3. What do you think when you see the notation $y(x)$?
4. Consider the following machine:

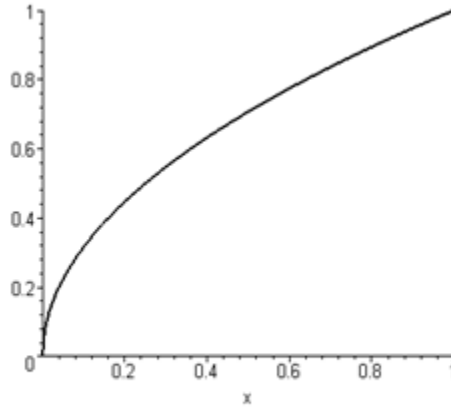


What is the output if the input is $y(x) = x + 2$?

5. What are the dependent and independent variables in $A = \pi r^2$, the area of a circle with radius r ?
6. Can you determine whether the followings graphs are functions? Why or why not? Explain your answer.

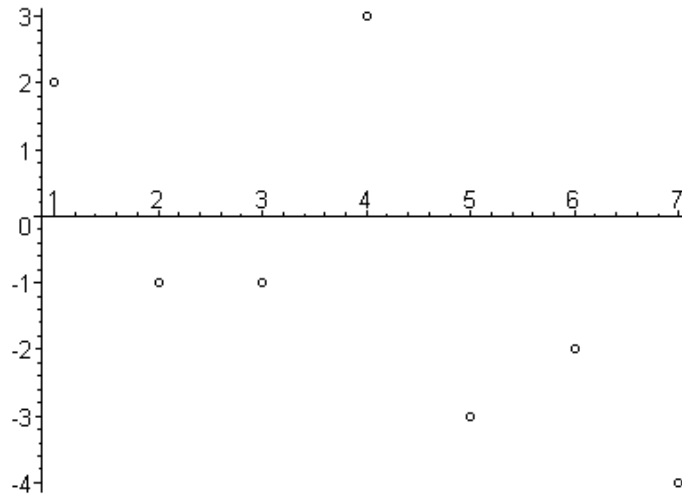


(a) $x, y \in \mathbb{R}$



(b) $x, y \in \mathbb{R}$

7. Is the following relation a function? Why or why not? Explain your answer.



8. A function f is defined by

$$f(x) = \begin{cases} 1-x & x \leq 1 \\ x^2 & x > 1 \end{cases}$$

Evaluate $f(0)$, $f(1)$, and $f(2)$ and sketch the graph.

9. The following tables are given:

x	$f(x)$
1	3
2	-1
3	1
4	0
5	-2

x	$g(x)$
-2	3
-1	1
0	5
1	2
2	4

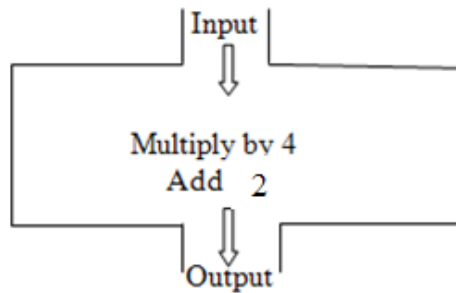
What is the value of $f(g(2))$?

10. Consider the functions f and g defined as $f(x)=2x+1$ and $g(x)=-2x-1$. What is $f(g(3))$? Describe what you did.
11. Suppose 5 students who read about function concept got the following scores in a quiz: student1 read 5 hours and got 8 points; student2 read 4 hours and got 9.5 points; student3 read 3.5 hours and got 9.5 points; student4 read 3 hours and got 9.4 points; and student5 read 2 hours and got 9.3 points. Can this phenomenon be called a function? Why or why not?
12. Personally, what you find difficult in learning/understanding function concept?

B. Second Questionnaire.

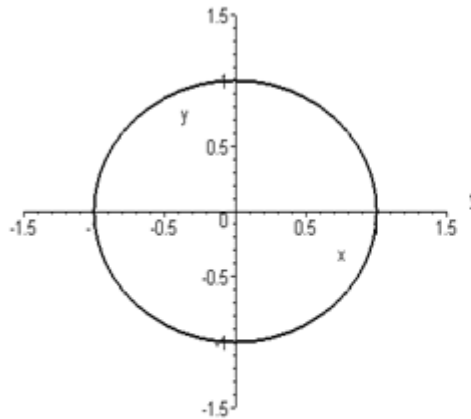
Function

1. Explain in a sentence or so what you think a function is. If you can give a definition for a function then do so.
2. Describe the ways a function can be represented.
3. What do you think when you see the notation $y(x)$?
4. Consider the following machine:

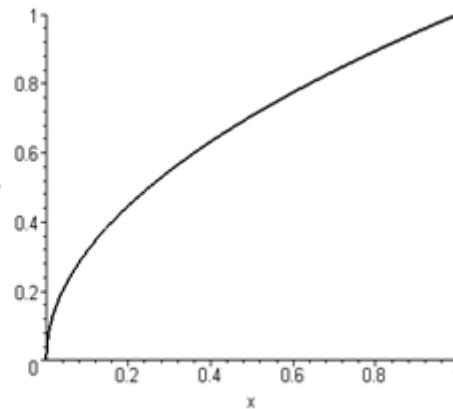


What is the output if the input is $y(x) = x + 2$?

5. Can you determine whether the followings graphs are functions? Why or why not? Explain your answer.

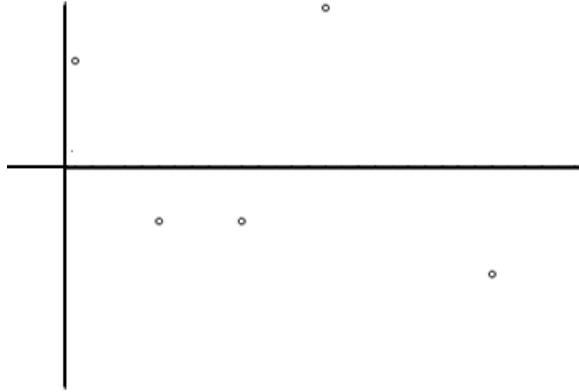


(a) $x, y \in \mathbb{R}$



(b) $x, y \in \mathbb{R}$

6. Can the following situation be called a function? Why or why not? Please, explain your answer.



7. f is defined by $f(x) = \begin{cases} 1-x & x \leq 1 \\ x^2 & x > 1 \end{cases}$. Is f a function? Please explain your answer.

8. f is defined by

$$f(x) = \begin{cases} 1-x & x \leq 1 \\ x^2 & x > 1 \end{cases}$$

Evaluate $f(0)$, $f(1)$, and $f(2)$.

9. The following tables are given:

x	$f(x)$
1	3
2	-1
3	1
4	0
5	-2

x	$g(x)$
-2	3
-1	1
0	5
1	2
2	4

What is the value of $f(g(2))$?

10. Consider the functions f and g defined as $f(x)=2x+1$ and $g(x)=-2x-1$. What is $f(g(3))$? Describe what you did.

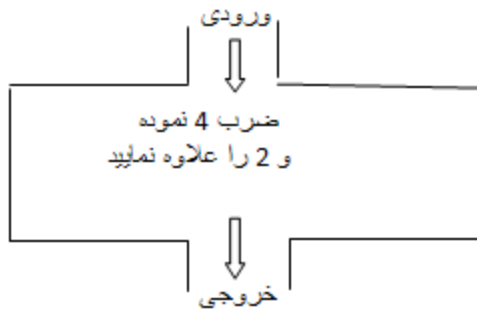
11. Suppose 5 students who read about function concept got the following scores in a quiz: student1 read 5 hours and got 8 points; student2 read 4 hours and got 9.5 points; student3 read 3.5 hours and got 9.5 points; student4 read 3 hours and got 9.4 points; and student5 read 2 hours and got 9.3 points. Can this phenomenon be called a function? Why or why not?

12. Personally, what do you find difficult in learning/understanding function concept?

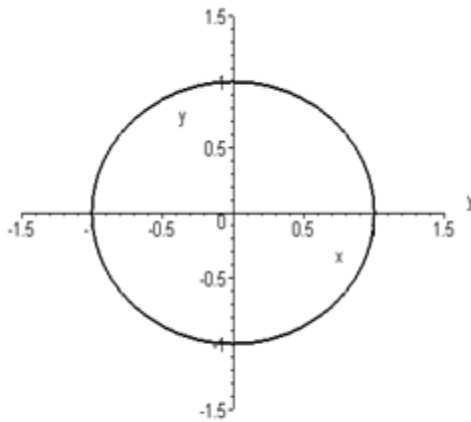
C. Second Questionnaire (Dari version).

پرسشنامه : تابع

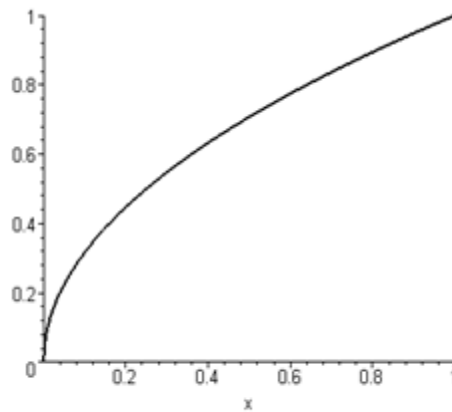
1. از نظر شما، تابع چه است؟ لطفاً در یک و یا چند جمله واضح نمایید. هرگاه امکان داشته باشد، یک تعریف ی از آن نیز بدهید.
2. به چند روش می توان تابع را نمایش داد؟ لطفاً پاسخ خود را توضیح دهید.
3. وقتی $y=f(x)$ را می بینید، در مورد آن چه فکر می کنید؟ و یا به عباره ای دیگر $y=f(x)$ چه را نشان می دهد؟ لطفاً پاسخ خود را توضیح دهید.
4. ماشین زیر را در نظر بگیرید:



5. هرگاه ورودی $y(x) = x + 2$ باشد، آنگاه خروجی را دریافت کنید. لطفاً پاسخ خود را توضیح دهید.
- کدام یک از گراف های زیر یک تابع را نشان می دهد؟ لطفاً پاسخ خود را توضیح دهید.

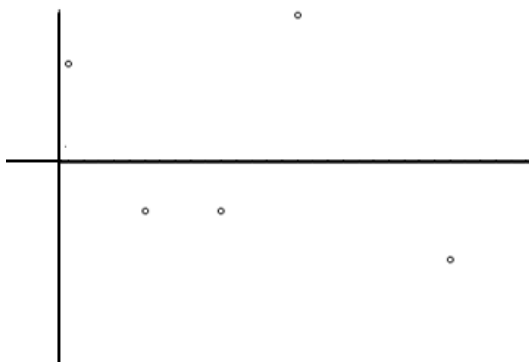


(a) $x, y \in \mathbb{R}$



(b) $x, y \in \mathbb{R}$

6. آیا می توان ست نقاط زیر را یک تابع نامید؟ لطفاً پاسخ خود را توضیح دهید.



7. f به شکل زیر تعریف شده است:

$$f(x) = \begin{cases} 1-x & x \leq 1 \\ x^2 & x > 1 \end{cases}$$

آیا f تابع است؟ لطفاً پاسخ خود را توضیح دهید.

8. f به شکل زیر تعریف شده است:

$$f(x) = \begin{cases} 1-x & x \leq 1 \\ x^2 & x > 1 \end{cases}$$

$f(0)$ ، $f(1)$ و $f(2)$ را دریافت نمایید.

9. جداول زیر داده شده اند:

x	$f(x)$
1	3
2	-1
3	1
4	0
5	-2

x	$g(x)$
-2	3
-1	1
0	5
1	2
2	4

قیمت $f(g(2))$ را دریافت کنید.

10. توابع $f(x)=2x+1$ و $g(x)=-2x-1$ را در نظر بگیرید. قیمت $f(g(3))$ را دریافت کنید.

11. فرض کنید 5 شاگرد که در مورد مفهوم تابع مطالعه نمودند نمرات زیر را در یک امتحان صنفی اخذ نمودند:

شاگرد اول مدت 5 ساعت مطالعه نمود و 8 نمره را اخذ کرد؛ شاگرد دوم 4 ساعت مطالعه نمود و 9.5 نمره را اخذ کرد؛ شاگرد سوم 3.5 ساعت مطالعه نمود و 9.5 نمره را اخذ کرد؛ شاگرد چهارم 3 ساعت مطالعه نمود و 9.4 نمره را اخذ کرد؛ شاگرد پنجم 2 ساعت مطالعه نمود و 9.3 نمره را اخذ کرد. آیا می توان رابطه بین مدت مطالعه و نمرات امتحان صنفی شاگردان را تابع نامید؟ چرا و چرا نه؟ متحول های مستقل و غیر مستقل در این موضوع کدام ها اند؟ لطفاً پاسخ خود را توضیح دهید.

12. شخص خود شما چه چیز را در فراگیری و فهمیدن مفهوم تابع مشکل می یابید؟

D. Curriculum Documents

Educational Plan.

پوښتون تعليم و تربيه کابل

پلان تعليمي ديارتمنت (رياضي) پوهنځي علوم طبيعي به سيستم کريدت

سال ۱۳۸۵

ردیف	موضوع	کد	دوره	دوام	دوام	دوام	دوام	دوام	صنف اول				صنف دوم		صنف سوم		صنف چهارم	
									۱	۲	۳	۴	۵	۶	۷	۸		
۱	رياضي عمومي I	Sc-111	5	80	60	20												
۲	رياضي عمومي II	Sc-113	4	64	50	14				5								
۳	اصول هئمه عمومي	Sc-226	3	48	43	8				4								
۴	فيزيک عمومي	Sc-112	4	64	50	14					3							
۵	ثقافت I	Sc-103	1	16	13	3				4								
۶	ثقافت II	Sc-104	1	16	13	3				1								
۷	اصول I	Sc-102	2	32	28	12					1							
۸	اصول II	Sc-104	2	32	28	12				2								
۹	تاريخچه رياضيات	Sc-227	2	32	28	4					2							
۱۰	مربوط کورس و فلسفي	Sc-436	3	48	40	8					2							
۱۱	اصول رياضتي	Sc-439	4	64	50	14												3
۱۲	مربوط رياضتي در فزيک	Sc-438	4	64	50	14												4
۱۳	تاريخ مفسر	Sc-203	2	32	30	2												4
۱۴	خدايه تعليمي در منطق	Sc-114	4	64	50	14					2							
۱۵	هئمه تربيه	Sc-326	3	48	30	18				4								
۱۶	الجر I	Sc-228	3	48	40	8								3				
۱۷	الجر II	Sc-230	3	48	40	8								3				
۱۸	الجر III	Sc-327	3	48	40	8								3				
۱۹	الجر IV	Sc-230	3	48	40	8								3				
۲۰	الجزر I	Sc-229	4	64	50	14									3			
۲۱	الجزر II	Sc-231	4	64	50	14					4							
۲۲	الجزر III	Sc-328	4	64	50	14								4				
۲۳	الجزر IV	Sc-331	4	64	50	14								4				
۲۴	ويگور انليز	Sc-332	3	48	40	8									4			
۲۵	نوري اجزالت	Sc-421	4	64	50	14												4
۲۶	انليز منطلي	Sc-428	4	64	50	14												4
۲۷	رياضي محاسبي	Sc-431	4	64	50	14												4
۲۸	حل مسائل رياضتي	Sc-432	2	32	2	30												2
۲۹	روشنامي عمومي	Sc-181	4	64	44	20												4
۳۰	روشنامي انگليسي	Sc-102	4	64	44	20				4								2
۳۱	روشنامي عربي	Sc-201	4	64	44	20					4							4
۳۲	اساسيات علوم و تربيه	Sc-202	4	64	44	20					4							4
۳۳	مربوط عمومي کورس و پلان تعليم و تربيه	Sc-301	4	64	44	20						4						4
۳۴	تفصيلي فلسفي و تربيه	Sc-303	4	64	44	20												4
۳۵	کورس آموزي	Sc-461	4	64	30	34												4
۳۶	خدايه تعليمي در اخلاص	Sc-232	4	64	50	14								4				
۳۷	خدايه تعليمي	Sc-333	4	64	50	14									4			
۳۸	فيزيک مودرن	Sc-115	3	48	30	18												4
۳۹	فيزيک برق	Sc-325	3	48	30	18				3								4
۴۰	تعليمات صحت	Sc-204	1	16	10	6												3
۴۱	مخيمت و سلامت	Sc-205	1	16	10	6												3
۴۲	تفصيل و و بزرگ	Sc-303	1	16	6	10												1
۴۳	تربيت بدني	Sc-286	1	16	6	10												1
۴۴	گيتور	Sc-101	2	32	8	24												2
۴۵	مولوگراف و سيمپلر		2															2
مجموع ساعت ها در يك سمنتر			144	2192	1590	442												3 4
تعداد مشاغل در يك سمنتر										18	18	18	18	18	18	18	18	18
										6	6	6	6	6	6	6	6	6

دو سرشع مشاغل پلان بدو سرشع در سمنتر

دو سرشع مشاغل پلان بدو سرشع در سمنتر

دو سرشع مشاغل پلان بدو سرشع در سمنتر