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Equivariant Intersection Cohomology of BXB Orbit Closures in the Wonderful Compactification of a Group

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EQUIVALENT INTERSECTION COHOMOLOGY OF $B \times B$ ORBIT CLOSURES IN THE WONDERFUL COMPACTIFICATION OF A GROUP

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by

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ABSTRACT

EQUIVARIANT INTERSECTION COHOMOLOGY OF $B \times B$ ORBIT CLOSURES IN THE WONDERFUL COMPACTIFICATION OF A GROUP

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This thesis studies the topology of a particularly nice compactification that exists for semisimple adjoint algebraic groups: the wonderful compactification. The compactification is equivariant, extending the left and right action of the group on itself, and we focus on the local and global topology of the closures of Borel orbits.

It is natural to study the topology of these orbit closures since the study of the topology of Borel orbit closures in the flag variety (that is, Schubert varieties) has proved to be interesting, linking geometry and representation theory since the local intersection cohomology Betti numbers turned out to be the coefficients of Kazhdan-Lusztig polynomials.

We compute equivariant intersection cohomology with respect to a torus action because such actions often have convenient localization properties enabling us to use data from the moment graph (roughly speaking the collection of 0 and 1-dimensional orbits) to compute the equivariant (intersection) cohomology of the whole space, an approach commonly referred to as GKM theory after Goresky, Kottowitz and MacPherson. Furthermore in the GKM setting we can recover ordinary intersection cohomology from the

equivariant intersection cohomology. Unfortunately the GKM theorems are not practical when computing *intersection* cohomology since for singular varieties we may not a priori know the local equivariant intersection cohomology at the torus fixed points. Braden and MacPherson address this problem, showing how to *algorithmically* apply GKM theory to compute the equivariant intersection cohomology for a large class of varieties that includes Schubert varieties.

Our setting is more complicated than that of Braden and MacPherson in that we must use some larger torus orbits than just the 0 and 1-dimensional orbits. Nonetheless we are able to extend the moment graph approach of Braden and MacPherson. We define a more general notion of moment graph and identify canonical sheaves on the generalized moment graph whose sections are the equivariant intersection cohomology of the Borel orbit closures of the wonderful compactification.

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CHAPTER 1

INTRODUCTION

1.1 Overview

The main object of study of this thesis is the wonderful compactification of a semisimple adjoint algebraic group over the complex numbers. It is a particular example of a more general construction first presented by De Concini and Procesi [14] in 1982. Their compactification, X, is defined for any symmetric variety G/H (so H is the set of fixed points of an involution on G) where G is a semisimple adjoint algebraic group over $\mathbb{C}$. It is smooth, equivariant (it carries a G-action that extends the left G-action on G/H), and most importantly the boundary $X \setminus (G/H)$ is a normal crossings divisor with finitely many G-stable divisors whose intersections give the G-orbit closures.

The work of De Concini and Procesi was in part motivated by some classical problems from enumerative geometry. Furthermore, compactifications are interesting from the point of view of algebraic geometry because they often shed some light on certain properties of the variety being compactified. Intuitively, the wonderful compactification is giving us information about G/H 'at infinity', and since it is complete it will have nicer properties, for instance its intersection theory will be more straightforward. The wonderful compactification is spherical, that is it has finitely many B-orbits where $B \subset G$ is a Borel subgroup, and has been used extensively in the study of spherical varieties. The papers [27], [26], [8], [10] are a few examples of the use of wonderful varieties to prove results (for example classification results) about spherical spaces.

An adjoint semisimple group G, acting on itself on the left and right (a $(G \times G)$-action), is easily expressed as a symmetric variety, $(G \times G/\mathrm{Diag}(G \times G))$, and applying the construction of De Concini and Procesi yields its wonderful compactification. It took some time before the construction of De Concini and Procesi was generalized to fields of arbitrary characteristic. Strickland [32] was able to construct the wonderful compactification of a semisimple adjoint *group* (but not an arbitrary symmetric space) over any algebraically closed field, with the general construction for *symmetric spaces* of any (semisimple adjoint) group over any algebraically closed field (of characteristic not equal to 2) achieved by De Concini and Springer [16].

More specifically we are interested in studying topological properties of the wonderful compactification of a group that are likely to give us useful information pertaining to the (geometric) representation theory of the group. We are motivated in part by the strong link between the representation theory of a reductive group G and the geometry of the associated flag variety G/B (here $B \subset G$ is a Borel subgroup). For example the local intersection cohomology Betti numbers of Schubert varieties give the coefficients of Kazhdan-Lusztig polynomials. Schubert varieties are the closures of Schubert cells (in the flag variety) and these correspond to $(B \times B)$-orbits in the group, so we study $(B \times B)$-orbit closures in the wonderful compactification. In some ways they are like the more familiar Schubert varieties, for example they are always normal and Cohen-Macaulay [11], and in other ways they are different, for instance they are more singular, essentially always being singular in codimension two [9] (this in distinction to the complicated situation for Schubert varieties [3]).

We are interested in the intersection cohomology of these $(B \times B)$-orbit closures, and furthermore we work equivariantly with the aim of utilizing the convenient localization properties that Goresky, Kottowitz and MacPherson [21] have shown often exist for equivariant (intersection) cohomology with respect to torus actions, and knowing that in their setting intersection cohomology can be recovered from equivariant intersection cohomology.

Actually, the Betti numbers for the intersection cohomology of these orbits has already been computed by Springer [31]. He shows how this intersection cohomology leads to "Kazhdan-Lusztig" type elements in a particular representation of the Hecke algebra associated to $G \times G$ that was constructed prior by Mars and Springer [28], and the existence of Kazhdan-Lusztig polynomials. In fact, these Kazhdan-Lusztig Polynomials are shown by Chen and Dyer [13] to be those of a Coxeter group that is non-canonically associated to the poset of $(B \times B)$-orbits in the wonderful compactification.

Our aim is to extend the work of Springer by giving a functorial description of the cohomology. One approach we hoped to emulate was that Strickland [33] used to give a description of the equivariant cohomology groups of the $G \times G$-orbits in the wonderful compactification. Her description is combinatorial, stated in the language of Stanley-Reisner systems.

A different approach, tailor made for computing equivariant *intersection* cohomology is that of Braden and MacPherson [5]. Their approach is applicable to a large class of varieties carrying Torus actions, which notably includes Schubert varieties. Their description of the equivariant intersection cohomology is combinatorial and expressed using 'sheaves' over the graph of 0 and 1-dimensional orbits, which they christen the moment graph. It is also algorithmic, showing how to systematically compute stalks of the intersection cohomology starting in the smooth part of the variety and into the lower (possibly singular) strata.

Fiebig has continued the use of moment graph techniques, showing how they may be used to conveniently translate representation theoretic problems into geometric ones. For instance he shows in [18] how the classical Kazhdan-Lusztig conjecture may be translated into a multiplicty conjecture on the stalks of Braden-MacPherson sheaves on the moment graph. He then furnishes a proof of the conjecture, which involves working with parity sheaves which were introduced by Juteau, Mautner and Williamson [25] working in positive characteristic.

A further interesting connection is the equivalence of the category of Braden MacPherson sheaves and the category of Soergel Bimodules, proved in [20]. Soergel Bimodules are used by Elias and Williamson to give a completely algebraic proof of the Kazhdan-Lusztig conjecture in the remarkable paper [17].

Our approach to computing equivariant intersection cohomology is to generalize the notion of moment graph and use similar algorithmic, combinatorial techniques to those of Braden and MacPherson.

1.2 Notation and Conventions

In this thesis we will work exclusively over the complex numbers. So unless otherwise specified all varieties, algebraic groups, sheaves and (co)homology groups should be understood to be defined over/have coefficients in $\mathbb{C}$. When we have an algebra A over $\mathbb{C}$ and an A-module M we will use $\overline{M}$ to denote the $\mathbb{C}$-vector space $M \otimes_A \mathbb{C}$. All algebraic groups considered are linear groups and we treat them as topological groups using the classical topology on the set of $\mathbb{C}$-points.

Given a topological group G, by a $\boldsymbol{G}$**-space** we will mean a space X with a continuous G-action. The stabilizer in G of a point x in a G-space X will be denoted $\boldsymbol{G_x}$. Given two G-spaces X and Y, the quotient of $X \times Y$ by the diagonal action of G will be denoted $\boldsymbol{X \times_G Y}$.

Given a G-space X, a H-space Y and a continuous homomorphism $\phi : G \to H$, a continuous map $f : X \to Y$ is $\boldsymbol{\phi}$**-equivariant** if $f \circ g = \phi(g) \circ f$ for all g in G. When $G = H$ and ϕ is the identity we simply say that f is **equivariant** or $\boldsymbol{G}$**-equivariant**.

When G is an algebraic group then we consider algebraic actions and homomorphisms and speak of $\boldsymbol{G}$**-varieties** rather than G-spaces. The center of G will be $\boldsymbol{Z(G)}$, $\mathrm{Diag}\boldsymbol{(G \times G)}$ will mean the diagonal in $G \times G$, and $\boldsymbol{R_u(G)}$ will denote the unipotent radical of G.

For reductive algebraic G we will always fix some choice $\boldsymbol{T} \subset \boldsymbol{B} \subset \boldsymbol{G}$ of maximal torus and Borel subgroup in G. Then the Weyl group $\boldsymbol{W}$ is the quotient of the normalizer of T by T. We use $\boldsymbol{\Phi}$, $\boldsymbol{\Phi^+}$, $\boldsymbol{Q}$ and $\boldsymbol{\Delta} = \{\alpha_1, ..., \alpha_r\} \subset \Phi^+$ to denote the corresponding root system, positive roots, root lattice and simple roots, respectively. The group of characters of T - denoted $\boldsymbol{X^*(T)}$ - will be written additively.

$\boldsymbol{P_I}$ will be the standard parabolic subgroup defined by $I \subset \Delta$, and $\boldsymbol{P_I^-}$ the opposite parabolic. Their intersection $\boldsymbol{L_I} := P_I \cap P_I^-$ is the Levi subgroup containing T and whose root system, denoted $\boldsymbol{\Phi_I}$, has basis I. Additionally, $\boldsymbol{W_I}$ will denote the parabolic subgroup of W generated by the simple reflections corresponding to the roots in I. We choose distinguished coset representatives $\boldsymbol{W^I} = \{x \in W \mid x(I) \subset \Phi^+\}$ of W/W_I and for each Weyl group element $w \in W$ we will use $\boldsymbol{w^I}$ to denote the unique element of W^I that is in the same coset as w. The longest element of W_I will be $\boldsymbol{w_I}$, with the length of an element $w \in W$ is denoted $\boldsymbol{\ell(w)}$. For each $w \in W$ we will use $\dot{\boldsymbol{w}}$ to denote some choice of coset representative; so $\dot{w}$ is in the normalizer of T and $w = \dot{w}T$.

In chapters 3 and 4, where we introduce the wonderful compactification and present our main theorems, we fix a semisimple adjoint algebraic group G and use X to refer exclusively to the wonderful compactification of G.

CHAPTER 2

EQUIVARIANT INTERSECTION COHOMOLOGY

In this chapter we give a quick overview of the theories of equivariant cohomology, intersection cohomology and equivariant intersection cohomology. Precise statements of the results that are most important for the main theorems of this thesis are to be found in Sections 2.4 and 2.5.

Some especially important notions in our study of group actions and the corresponding equivariant cohomology will be contracting actions and transverse slices to orbits. These are defined below.

Definition 2.1. *Consider an irreducible variety Y with an algebraic $\mathbb{C}^\times$ action and a subvariety $Y' \subset Y$ that is invariant under the action. We say the $\mathbb{C}^\times$ action* ***contracts*** *Y onto Y' if for all $y \in Y$ the limit $\lim_{z \to 0} z \cdot y$ exists and is in Y' (where $z \in \mathbb{C}^\times$), or equivalently if the action map $\mathbb{C}^\times \times Y \to Y$ extends to a morphism $\mathbb{C} \times Y \to Y$ sending $\{0\} \times Y$ to Y'.*

A useful lemma for showing the existence of a contracting $\mathbb{C}^\times$ is the following.

Lemma 2.2 ([7, Proposition A2])**.** *Given a torus T acting on a variety Y with fixed point y the following are equivalent.*

(1) The weights of T in the tangent space $T_y(Y)$ are contained in an open half space.

(2) There is an open subset $U \subset Y$ that is contracted onto y by some one dimensional subtorus of T.

If (2) holds and $\gamma : \mathbb{C}^\times \to T$ is a one parameter subgroup whose image contracts some neighbourhood U of y onto y, then

$$Y_y := \{x \in Y \mid \lim_{z \to 0} \gamma(z) \cdot x = y\}$$

is an open affine T-stable neighbourhood of y which admits a closed T-equivariant embedding into $T_y(Y)$.

Definition 2.3 ([28, 2.3.2])**.** *Given a connected linear algebraic group G acting on an irreducible variety Y, we say S is a* ***transverse slice*** *at $y \in Y$ to the orbit $G \cdot y$ if*

(1) S is a locally closed subset of Y containing y,

(2) the restriction of the G-action to S gives a smooth morphism $G \times S \to Y$,

(3) dim S = dim Y - dim $(G \cdot y)$.

The transverse slice S is ***attractive*** *if there is a one dimensional torus in G under whose action S is invariant and that contracts S onto y.*

Definition 2.4. *Given a connected linear algebraic group G acting on a smooth irreducible variety Y, a closed subgroup $H \subset G$, and $y \in Y$, we say N is a* ***normal space*** *to $G \cdot y$ along $H \cdot y$ if*

(1) N is a locally closed smooth subvariety of Y that is H-invariant

(2) $N \cap (G \cdot y) = H \cdot y$ (and in particular N contains $H \cdot y$)

(3) at all points of $H \cdot y$ the identity $T_xN + T_x(H \cdot y) = T_xY$ holds on tangent spaces.

The normal space N is ***attractive*** *if there is an algebraic $\mathbb{C}^\times$ action on N that fixes $H \cdot y$, contracts N onto $H \cdot y$, and commutes with the H-action on N.*

2.1 Equivariant Cohomology Basics

Given a topological group G acting continuously on a topological space X we may study the *equivariant cohomology* of X with respect to the action. This is a cohomology theory that is compatible with the group action; for example we will see that the equivariant cohomology of X with respect to a free G-action is equal to the regular cohomology of X/G. In the nicest situations we will be able to recover the regular cohomology $H^*(X)$ of a space X from the equivariant cohomology.

Of particular interest is the case where G is a torus. Then the computation of equivariant cohomology is greatly simplified by surprising localization results. We go over this in Section 2.5 and also touch on how to recover regular cohomology from equivariant cohomology.

To give the traditional definition of equivariant cohomology we need to show that for each topological group G there is a contractible space with a free action of G. Having fixed such a space, denoted EG, we see that $X \times EG$ is homotopic to X but is acted on freely by G. This makes the definition of equivarent cohomology given below somewhat intuitive.

Let $X \times_G EG := (X \times EG)/G$ denote the quotient of $X \times EG$ by the diagonal action of G. The following definition originated from ideas of Borel.

Definition 2.5 ([4]). *Given a G-space X the equivariant cohomology ring of X with respect to this action is*

$$H^*_G(X) := H^*(X \times_G EG),$$

graded by the equivariant cohomology groups $H^n_G(X) := H^n(X \times_G EG)$

One explicit construction of a space EG, due to Milnor in [29], is to take the topological join of infinitely many copies G with an appropriate topology. We will call the quotient $BG := EG/G$ the classifying space for G. The bundle $EG \to BG$ is a universal principle G-bundle and so the specific choice of EG is irrelevant: the spaces BG are homotopic for different choices of EG, and equivariant cohomology as presented above is well defined.

To avoid the complication of dealing with the (often) infinite dimensional space EG we consider a sequence of finite dimensional approximations $EG_0 \subset EG_1 \subset ... \subset EG_n \subset ...$ all of which are acted on freely by G, with EG_n being $(n-1)$-connected and $EG = \bigcup_n EG_n$. (In Milnor's construction, EG_n is the join of $n+1$ copies of G.) We then have finite dimensional approximations $X \times_G EG_0 \subset X \times_G EG_1 \subset ... \subset X \times_G EG_n \subset ...$ of $X \times_G EG$, and $H^*_G(X) = \varprojlim H^*(X \times_G EG_n)$.

It should not be surprising, given its definition, that equivariant cohomology posesses many of the same canonical properties as ordinary cohomology. For example:

1. There is a cup product on equivariant cohomology, giving the aforementioned ring structure.

2. Equivariant cohomology is *functorial* for equivariant maps. That is, given groups G and H acting on X and Y respectively, a homomorphism $\phi : G \to H$ and a ϕ-equivariant map $f : X \to Y$ we have a pullback $f^* : H^*_H(Y) \to H^*_G(X)$.

3. Applying functoriality to the map $X \to pt$ and using the ring structure we see that equivariant cohomology is always a module over $H^*_G(pt) = H^*(BG)$.

4. We have versions of excision, the Mayer-Vietoris sequence, the Künneth formula, the Leray spectral sequence, Chern classes, and (for orientable manifolds) Poincaré duality.

It is notable that $H^*_G(X)$ is a $H^*_G(pt)$-module because in most cases $H^*_G(pt)$ isn't simply the base ring. This is in contrast to regular (singular) cohomology where $H^*(pt)$ is just the base ring of coefficients for our cohomology.

Example 2.6. Consider $G = S^1$, the one dimensional compact torus. It acts freely on $\mathbb{C}^n \setminus \{0\}$ by scalar multiplication and the subset S^{2n-1} is G-stable and $(2n-2)$-connected.

Thus the odd dimensional spheres S^{2n-1} give us finite dimensional approximations of $EG := \bigcup_n S^{2n-1}$, and $BG = \bigcup_n S^{2n-1}/S^1 = \bigcup_n \mathbb{CP}^{n-1} = \mathbb{CP}^\infty$. So $H^*_{S^1}(pt)$ is the polynomial ring $\mathbb{C}[t]$ where t is an indeterminate of degree 2.

Example 2.7. More generally consider an algebraic torus T. Choose an isomorphism $T \simeq (\mathbb{C}^\times)^d$ and let $ET_n := (\mathbb{C}^n \setminus \{0\})^d$ carry the T-action given by termwise scalar multiplication. The action is free and ET_n is $(2n-2)$-connected. We have an inclusion $ET_n \subset ET_{n+1}$ where ET_n are the elements of ET_{n+1} that have all their $(n+1)$st coordinates all zero. The quotient $BT_n := ET_n/T \simeq (\mathbb{CP}^{n-1})^d$ is a finite dimensional approximation to the classifying space $BT = \bigcup_n B_n = (\mathbb{CP}^\infty)^d$. The cohomology ring $H^*(BT)$ isomorphic to $\mathbb{C}[t_1, ..., t_d]$, a polynomial ring on d indeterminates of degree 2.

Throughout this thesis we will identify this ring with $\boldsymbol{A} := \mathrm{Sym}(\mathfrak{t}^*)$ where $\mathfrak{t}$ is the Lie algebra of T. The identification is canonical. For an idea of how this identification works, consider a character $\chi : T \to \mathbb{C}^\times$. On the one hand it corresponds to its derivative $d\chi : \mathfrak{t} \to \mathbb{C}$ in $\mathfrak{t}^*$ and on the other hand to a complex line bundle $ET \times_T \mathbb{C}_\chi \to BT$ (where $\mathbb{C}_\chi$ is $\mathbb{C}$ acted on by T via the character χ and the map is induced by projection onto the first factor in $ET \times \mathbb{C}_\chi$) which corresponds to an element $c(\chi) \in H^2(BT)$, its first Chern class.

Some further properties of equivariant cohomology are:

1. If G acts trivially on X then $X \times_G EG = X \times BG$ and so $H^*_G(X) \simeq H^*_G(pt) \otimes H^*(X)$ by the Künneth formula. In particular $H^*_G(X)$ is free as a $H^*_G(pt)$-module.

2. If G acts on X almost freely - that is, with finite stabilizers - then $H^*_G(X) \simeq H^*(X/G)$.

3. A closed subgroup $H \subset G$ acts freely on EG and so $EG \to EG/H$ is a universal principal bundle for H. Given a H-space X we have an action of G on $G \times_H X$ (on the first factor) and $H^*_G(G \times_H X) \simeq H^*_H(X)$. So for example $H^*_G(G/H) \simeq H^*_H(G/G) = H^*_H(pt)$

One result that simplifies the calculation of the equivariant cohomology groups is that we may use the approximations to EG to calculate the groups $H^n_G(X)$. Specifically $H^n_G(X) = H^n(X \times_G EG_m)$ for $n \leq m$ and any compact topological G-manifold X of dimension less than or equal to m [24, Chapter IV, Theorem 13.1].

More significant for our purposes (and rather surprising) is the localization theorem, applicable to torus actions on certain spaces, that enables us to compute equivariant cohomology using only data from the fixed points and one-dimensional orbits of the action. We give the definition in its full generality in Section 2.5.

2.2 Intersection Homology and Derived Category Basics

Singular homology and cohomology are noticeably better behaved for smooth spaces than for singular. For instance Poincaré duality is not guaranteed for singular spaces. Because our objects of study are possibly singular, being algebraic varieties, we will work with intersection (co)homology.

Intersection *homology* was developed by Goresky and MacPherson [22] as a homology theory for singular spaces for which the Poincaré-Lefschetz theory of intersection of homology cycles would hold. The intersection homology groups share many other properties of ordinary homology but it is notable that they are not homotopy invariant (though they are invariant under homeomorphisms). On smooth spaces the intersection homology groups equal the singular homology groups.

The intersection homology groups are defined for pseudomanifolds. Roughly speaking, a pseudomanifold X is a space that admits a stratification

$$X = X_n \supset X_{n-2} \supset ... \supset X_{n-3} \supset ... \supset X_1 \supset X_0$$

where $X_n \setminus X_{n-2}$ is an oriented dense n manifold and $X_i \setminus X_{i-1}$ for $1 \leq i \leq n-2$ is an i manifold along which the normal structure of X is locally trivial.

The definition of the intersection homology groups involves functions called perversities. A *perversity* is a map $\overline{p} : \mathbb{Z}_{\geq 2} \to \mathbb{Z}_{\geq 0}$ such that both $\overline{p}(c)$ and $c - 2 - \overline{p}(c)$ are non-negative and non-decreasing functions of c. For each perversity $\overline{p}$ we have intersection homology groups $IH_i^{\overline{p}}(X)$. There is a distinguished middle perversity $\overline{m} := \frac{c-2}{2}$ (well defined at least for complex varietes, which have even dimensional strata), complementary to itself in the sense that $\overline{m} + \overline{m} = c - 2$ and when we don't specify a perversity the middle perversity should be assumed.

The groups $IH_i^{\overline{p}}(X)$ will be the homology groups of the subcomplex $IC_*^{\overline{p}}(X)$ of (the ordinary locally finite chains) $C_*(X)$ consisting of all i-dimensional chains that intersect each X_{n-k} in a set of dimension at most $p(k) + i - k$ for all $k \geq 2$ and whose boundaries intersect each X_{n-k} in a set of dimension at most $p(k) + i - k - 1$ for all $k \geq 2$.

So the perversity determines whether/to what extent chains intersect the singular part of X. The minimal perversity is $\overline{0}(c) = c - 1$ and the maximal perversity is $\overline{t}(c) = c - 2$. It turns out the intersection cohomology groups are independent of choice of stratification on X.

Among the results Goresky and MacPherson prove is a generalized Poincaré duality [22, Theorem 3.3]: for $i + j = \dim(X)$ and complementary perversities $\overline{p} + \overline{q} = \overline{t}$ there is a non-degenerate pairing

$$(IH_i^{\overline{p}}(X) \otimes \mathbb{Q}) \times (IH_j^{\overline{q}}(X) \otimes \mathbb{Q}) \to \mathbb{Q}$$

coming from intersection but augmented so as to count points with multiplicities.

It turns out to be beneficial to work sheaf theoretically, so in follow-up paper Goresky and MacPherson [23] express the intersection homology groups $IH_i^{\overline{p}}(X)$ as the cohomology groups of a complex of sheaves $\mathbf{IC}^\bullet_{\overline{p}}(X)$ that is well defined up to quasi-isomorphism. That is, $\mathbf{IC}^\bullet_{\overline{p}}(X)$ is an object in the bounded derived category $D^b(X)$, and intersection homology and the dual theory, intersection *cohomology*, are defined as hypercohomology of $\mathbf{IC}^\bullet_{\overline{p}}(X)$:

Definition 2.8 ([23, Section 2.3]). *The intersection **homology** groups are*

$$IH_i^{\overline{p}}(X) := \mathscr{H}^{-i}(\boldsymbol{IC}^\bullet_{\overline{p}}(X))$$

*and the intersection **cohomology** groups are*

$$IH^i_{\overline{p}}(X) := \mathscr{H}^{i-dim(X)}(\boldsymbol{IC}^\bullet_{\overline{p}}(X))$$

where $\mathscr{H}$ denotes hypercohomology.

See the next section for a description of the (bounded, constructible) derived category.

Using the powerful sheaf theoretic apparatus of the derived category Goresky and MacPherson show that $\mathbf{IC}^\bullet_{\overline{p}}(X)$ is constructible using the standard operations of sheaf theory and it is possible to give a characterization of $\mathbf{IC}^\bullet_{\overline{p}}(X)$ that is independent of any mention of a stratification.

Three different characterizations of $\mathbf{IC}^\bullet_{\overline{p}}(X)$ are given by Goresky and MacPherson. We recall here one of them that is originally due to Deligne. This construction is stratification dependent but the resulting sheaf is then shown to be stratification independent.

Definition 2.9 ([23, Section 3.1]). *Consider a psuedomanifold X and a choice of stratification $X = X_n \supset X_{n-2} \supset X_{n-3} \supset ... \supset X_1 \supset X_0$. We have a corresponding filtration by open sets $\varnothing = U_0 \subset U_2 \subset U_3 \subset ... \subset U_n \subset U_{n+1} = X$ where $U_i := X \setminus X_{n-i}$ for $i \leq n$ and $i_k : U_k \to U_{k+1}$ are the inclusions.*

For each stratification $\overline{p}$, we define complexes $\mathbb{P}^\bullet_k \in D^b(U_k)$ inductively by

$$\mathbb{P}^\bullet_2 := \mathbb{C}^\bullet_{U_2}[dim(X)]$$

$$\mathbb{P}^\bullet_{k+1} := \tau_{\leq p(k)-dim(X)} Ri_{k*}\mathbb{P}^\bullet_k \text{ for } k \geq 1$$

where $\tau_{\leq m}$ are truncation functors.

Set $\boldsymbol{IC}^\bullet_{\overline{p}}(X) := \mathbb{P}^\bullet_{n+1}$.

Remark 2.10. When a stratification is not specified the middle perversity $\overline{m}$ is assumed. So $\mathbf{IC}^\bullet(X) := \mathbf{IC}^\bullet_{\overline{m}}(X)$, $IH_i(X) = IH_i^{\overline{m}}(X) = \mathscr{H}^{-i}(\mathbf{IC}^\bullet(X))$ and $IH^i(X) = IH^i_{\overline{m}}(X)$

Example 2.11. Consider the flag variety G/B of an algebraic group G. The dimensions of the stalks of the intersection cohomolog sheaves of the strata of G/B are the coefficients of the Kazhdan-Lusztig polynomials. That is, from the Bruhat decomposition $G = \bigsqcup_{w \in W} BwB$, where W is the Weyl group of G/B, we have a stratification of G/B by Schubert cells $X_w := BwB/B$ and given $x \leq y$ in W we let $\mathbf{IC}^{\bullet}(\overline{X_y})_{X_x}$ denote the stalk of $\mathbf{IC}^{\bullet}(\overline{X_y})$ at any point of X_x. The Kazhdan-Lusztig polynomials $P_{x,y}$ are defined for any pair of elements of W and

$$P_{x,y} = \sum_i q^i \dim \mathbf{IC}^{\bullet}(\overline{X_y})_{X_x}$$

Another notable difference between intersection and regular cohomology (in addition to the lack of homotopy invariance on intersection cohomology) is that there isn't a cup product structure on $IH^*_{\overline{p}}(X)$.

When we pull back intersection cohomology by an inclusion $i : Y \hookrightarrow X$ (of a G-stable subset) we will use special notation,

$$IH^*(X)_Y := H^*(X; i^*\mathbf{IC}_X).$$

Lemma 2.12 ([23, Section 5.4])**.** *If $i : Y \to X$ is the inclusion of a subvariety then from the adjunction $\boldsymbol{IC}_X \to i_* i^* \boldsymbol{IC}_X$ we get a homomorphism $IH^*(X) \to IH^*(X)_Y$. If the inclusion is **normally nonsingular** then we have a canonical isomorphism $i^*\boldsymbol{IC}_X \simeq \boldsymbol{IC}_Y$, giving a homomorphism*

$$IH^*(X) \to IH^*(Y).$$

Remark 2.13. Open inclusions are normally nonsingular.

Definition 2.14. *Consider a G-space X that is a locally closed union of strata of an equivariant Whitney stratification of some smooth compact manifold M on which G acts smoothly. A complex of sheaves $\mathscr{F}_X$ on X is said to be **(cohomologically) constructible** with respect to the given stratification if its cohomology sheaves $\mathbb{H}^i(\mathscr{F}_X)$ are finite dimensional and are locally constant on each stratum of X.*

2.3 Equivariant Derived Category

Equivariant cohomology and equivariant intersection cohomology are more properly understood as the hypercohomology of objects in the equivariant derived category. Indeed many of the properties of these cohomology theories turn out to hold for all objects in the equivariant derived category. In particular this is true of the localization results.

The equivariant derived category was first defined in the topological context by Bernstein and Lunts [2]. We recall their definition following the expositions given in [21, Section 5.1] and [19, Section 2.1].

We fix a topological group G and universal principal bundle $EG \to BG$. Recall that for every G-space X we have the quotient $X \times_G EG := (X \times G)/G$ of $X \times EG$ by the diagonal G action, and a projection map (onto the first factor) p and a quotient map q

$$X \xleftarrow{p} X \times EG \xrightarrow{q} X \times_G EG$$

Definition 2.15 ([2, 2.1.3, 2.7.2])**.** *The* ***equivariant derived category*** *of sheaves on X with coefficients in $\mathbb{C}$, denoted* $\boldsymbol{D_G(X)}$*, has as objects triples $(\mathscr{F}_X, \overline{\mathscr{F}}, \beta)$ where $\mathscr{F}_X \in D(X)$, $\overline{\mathscr{F}} \in D(X \times_G EG)$ and $\beta : p^*\mathscr{F}_X \to q^*\overline{\mathscr{F}}$ is an isomorphism in $D(X \times_G EG)$. A morphism from $\eta : (\mathscr{F}_X, \overline{\mathscr{F}}, \beta) \to (\mathscr{G}_X, \overline{\mathscr{G}}, \gamma)$ is a pair $\eta = (\eta_x, \overline{\eta})$ where $\eta_X : \mathscr{F}_X \to \mathscr{G}_X$ and $\overline{\eta} : \overline{\mathscr{F}} \to \overline{\mathscr{G}}$ such that the diagram*

$$\begin{array}{ccc} p^*\mathscr{F}_X & \xrightarrow{\beta} & q^*\overline{\mathscr{F}} \\ \big\downarrow{\scriptstyle p^*\eta_X} & & \big\downarrow{\scriptstyle p^*\overline{\eta}} \\ p^*\mathscr{G}_X & \xrightarrow{\gamma} & q^*\overline{\mathscr{G}} \end{array} \tag{2.1}$$

commutes.

We have a **forgetful functor** $D_G(X) \to D(X)$ given by $(\mathscr{F}_X, \overline{\mathscr{F}}, \beta) \mapsto \mathscr{F}_X$. The object $(\mathscr{F}_X, \overline{\mathscr{F}}, \beta)$ is said to be an *equivariant lift* of the sheaf $\mathscr{F}_X \in D(X)$. The constant sheaf $\mathbb{C}_X$ has a canonical lift $\mathbb{C}_X^G := (\mathbb{C}_X, \mathbb{C}_{X \times_G EG}, I)$ to the equivariant derived category. Likewise, for any perversity $\overline{p}$ the sheaf $I^{\overline{p}}C_X$ of intersection cochains (with complex coefficients) has a canonical lift $I^{\overline{p}}C_X^G := (I^{\overline{p}}C_X, I^{\overline{p}}C_{X\times_G EG}, \beta)$ constructed in [2, Section 5.2]. When we omit the perversity, writing $\mathbf{IC}_X^G$, the middle perversity is to be assumed.

Definition 2.16. *The* ***bounded equivariant derived category*** $D_G^b(X)$ *is the full subcategory of* $D_G(X)$ *consisting of triples* $(\mathscr{F}_X, \overline{\mathscr{F}}, \beta)$ *with* $\mathscr{F}_X$ *in* $D^b(X)$*. The* ***constructible bounded equivariant derived category*** $D_{G,c}^b(X)$ *is the full subcategory of* $D_G^b(X)$ *consisting of triples* $(\mathscr{F}_X, \overline{\mathscr{F}}, \beta)$ *with* $\mathscr{F}_X$ *in* $D_c^b(X)$*.*

Note that $\mathbb{C}_X^G$ and $\mathbf{IC}_X^G$ are objects of $D_G^b(X)$. We will work exclusively in $D_G^b(X)$.

If we stipulate that X and Y be complex projective varieties with the metric topology, acted on continuously by a Lie group G then we are guaranteed the existence of the six Grothendieck operations (functors); (derived) pushforward, pullback, (derived) proper pushforward, proper pullback, tensor and $R\mathrm{Hom}$ (see [19, Section 2.2]). That is, given a G-map $f : X \to Y$, we have the induced maps $f \times \mathrm{id} : X \times EG \to Y \times EG$ and $f \times_G \mathrm{id} : X \times EG \to Y \times EG$, and the corresponding functors so that, for instance, given $\mathscr{F} = (\mathscr{F}_X, \overline{\mathscr{F}}, \beta) \in D_G^b(X)$ then $(Rf_*\mathscr{F}_X, R(f \times_G \mathrm{id})_*\overline{\mathscr{F}}, R(f \times \mathrm{id})_*(\beta))$ is an object in $D_G^b(Y)$, denoted $Rf_{G*}\mathscr{F}$. Likewise we have the functors $Rf_{G!}$, f_G^*, $f_G^!$, $\overset{L}{\otimes}$ and $R\mathrm{Hom}$. (The definitions of $Rf_!$ and $f!$ are complicated by the fact that $X \times_G EG$ and $Y \times_G EG$ are not locally compact in general but this is solved by considering $X \times_G EG$ as a direct limit of locally compact spaces [2].)

We will often (when it is unambiguous) abuse notation and just write Rf_*, $Rf_!$, f^* and $f^!$ in place of Rf_{G*}, $Rf_{G!}$, f_G^* and $f_G^!$ since all our 'sheaves' will be understood to be in the equivariant derived category.

The Cohomology Functors

Now, following Sections 5.4 and 5.5 of [21] we define the cohomology functors on $D_G^b(X)$. We will need the map to a point $c : X \to pt$. Given a sheaf $\mathscr{F} = (\mathscr{F}_X, \overline{\mathscr{F}}, \beta)$ in $D_G^b(X)$, the **equivariant cohomology** of X **with coefficients in** $\mathscr{F}$ (or the equivariant cohomology of $\mathscr{F}$) is

$$H_G^*(X; \mathscr{F}) := \mathbb{H}(\overline{\mathscr{F}}) = H^*(R(c \times_G \mathrm{id})_*\overline{\mathscr{F}})$$

and the **ordinary cohomology** of X **with coefficients in** $\mathscr{F}$ (or the ordinary cohomology of $\mathscr{F}$) is

$$H^*(X;\mathscr{F}) := \mathbb{H}(\mathscr{F}_X) = H^*(Rc_*\mathscr{F}_X)$$

As we've already been doing, when the sheaf isn't specified it is understood that by **equivariant cohomology** we mean that of the constant sheaf $\mathbb{C}_X^G$, denoted

$$H_G^*(X) := H_G^*(X;\mathbb{C}_X^G),$$

and by (ordinary) **cohomology** we mean that of the constant sheaf, $H^*(X) := H^*(X;\mathbb{C}_X^G)$.

By the **equivariant intersection cohomology** of X we mean the equivariant cohomology of the intersection cohomology sheaf $\mathbf{IC}_X^G$,

$$IH_G^*(X) := H_G^*(X;\mathbf{IC}_X^G)$$

and the **intersection cohomology** of X is the ordinary cohomology of the intersection cohomology sheaf, $IH^*(X) := H^*(X;\mathbf{IC}_X^G)$.

In summary, equivariant cohomology is a functor $D_G^b(X) \to H_G^*(pt)\text{-mod}^{\mathbb{Z}}$ where $H_G^*(pt)\text{-mod}^{\mathbb{Z}}$ denotes the category of $\mathbb{Z}$-graded modules over $H_G^*(pt)$. Likewise cohomology is a functor $D_G^b(X) \to H^*(pt)\text{-mod}^{\mathbb{Z}} = \mathbb{C}\text{-mod}^{\mathbb{Z}}$.

As we did in the non-equivariant case, when we pull back intersection cohomology by an inclusion $i : Y \hookrightarrow X$ (of a G-stable subset) we will use special notation,

$$IH_G^*(X)_Y := H_G^*(X;i^*\mathbf{IC}_X^G).$$

The equivariant counterpart to Lemma 2.12, which dealt with normally nonsingular inclusions is the following.

Lemma 2.17. *If $i : Y \to X$ is the inclusion of a subvariety then from the adjunction $\boldsymbol{IC}_X^G \to i_*i^*\boldsymbol{IC}_X^G$ we get a homomorphism $IH_G^*(X) \to IH_G^*(X)_Y$. If the inclusion is **normally nonsingular** then we have a canonical isomorphism $i^*\boldsymbol{IC}_X^G \simeq \boldsymbol{IC}_Y^G$, giving a homomorphism*

$$IH_G^*(X) \to IH_G^*(Y).$$

2.4 Equivariant Intersection Cohomology

In this section we highlight the properties specific to equivariant intersection cohomology that are of most interest to us. First we expand on what we've said about how almost free actions interact with equivariant intersection cohomology.

Theorem 2.18 ([2, Theorem 9.1])**.** *Let $\phi : H \to G$ be a surjective algebraic homomorphism of affine reductive (complex) algebraic groups with kernel $K = Ker(\phi)$. Let X and Y be complex algebraic H and G-varieties respectively, and $f : X \to Y$ be an algebraic ϕ-map. Assuming,*

1. *K acts on X with finite stabilizers, and*

2. *f is affine and is the geometric quotient map by the action of K,*

then

$$Rf_* \boldsymbol{IC}_X^H = \boldsymbol{IC}_Y^G[dim(K)].$$

Additionally the manner in which contracting actions interact with equivariant intersection cohomology will be of fundamental importance.

Lemma 2.19 ([5, Lemma 3.1])**.** *Suppose a T-space Y has a $\mathbb{C}^\times$ action that commutes with T and contracts a locally closed subvariety $Y_1 \subset Y$ onto another subvariety Y_2. Then the restriction $IH_T^*(Y)_{Y_1} \to IH_T^*(Y)_{Y_2}$ is an isomorphism.*

Theorem 2.20 ([5, Theorem 3.8])**.** *Suppose a torus T acts linearly on $\mathbb{C}^r$, there is a subtorus $\mathbb{C}^\times \subset T$ contracting $\mathbb{C}^r$ onto $\{0\}$, and $Y \subset \mathbb{C}^r$ is a T-invariant subvariety. Then the restriction homomorphism*

$$IH_T^*(Y) \to IH_T^*(Y \setminus \{0\})$$

makes $IH_T^*(Y)$ into a projective cover of $IH_T^*(Y \setminus \{0\})$ as $H_T^*(pt)$-modules.

Also, the kernel of the restriction map is isomorphic to the local equivariant intersection cohomology with compact supports, $IH_{T,c}^*(Y) \simeq IH_T^*(Y, Y \setminus \{0\})$. This is a free $H_T^*(pt)$-module, and $\overline{IH_{T,c}^*(Y)} = IH_c^*(Y)$. (Recall $\overline{IH_{T,c}^*(Y)} := IH_{T,c}^*(Y) \otimes_A \mathbb{C}$.)

2.5 Equivariant Localization

In this section we recall the approach of Goresky-Kottowitz-MacPherson [21] to computing torus equivariant cohomology using the graph of zero and one dimensional orbits. The first theorem we mention is their sheaf-theoretic presentation of a lemma of Chang and Skelbred [12].

We fix an algebraic torus T and a T-variety X. Let $F \subset X$ be the fixed locus of the T-action and

$$X_1 := \{x \in X \mid \operatorname{corank}(T_x) \leq 1\}$$

the union of all the 0 and 1-dimensional orbits. We consider an arbitrary element $\mathscr{F}$ of $D^b_T(X)$.

Theorem 2.21 ([21, Theorem 6.3])**.** *If $H^*_T(X;\mathscr{F})$ is a free module over $A \simeq H^*_T(pt)$ then the restriction homomorphism $H^*_T(X;\mathscr{F}) \to H^*_T(F;\mathscr{F})$ is an injection. In fact the sequence*

$$H^*_T(X;\mathscr{F}) \to H^*_T(F;\mathscr{F}) \xrightarrow{\delta} H^*_T(X_1, F;\mathscr{F})$$

is exact, where δ is the connecting homomorphism from the long exact sequence of the pair (X_1, F). Thus the image of $H^*_T(X;\mathscr{F})$ in $H^*_T(F;\mathscr{F})$ is identified with the kernel of δ.

Sheaves on X satisfying the assumption of the theorem have a special name:

Definition 2.22. *A sheaf $\mathscr{F} \in D^b_T(X)$ is **equivariantly formal** if the spectral sequence for its equivariant cohomology,*

$$E_2^{pq} = H^p_T(pt) \otimes H^q_T(X;\mathscr{F}) \Rightarrow H^{p+q}_T(X;\mathscr{F}),$$

degenerates at E^2.

If $\mathscr{F}$ is equivariantly formal then $H^*_T(X;\mathscr{F})$ is a free module over $H^*_T(pt)$ and so Theorem 2.21 is applicable.

The class of equivariantly formal spaces turns out to be quite large. Goresky-Kottowitz-MacPherson give nine examples in [21, Theorem 14.1] of large classes of varieties and equivariant sheaves on them that are equivariantly formal. For instance, given a T-variety X and $\mathscr{F} = (\mathscr{F}_X, \overline{\mathscr{F}}, \beta) \in D^b_T(X)$ we have equivariant formality in all the following instances:

- The ordinary cohomology $H^*(X; \mathscr{F})$ vanishes in odd degrees.
- $\mathscr{F} = \mathbb{C}^T_X$ is the constant sheaf and X has a cell decomposition by T-invariant subanalytic cells.
- $\mathscr{F} = \mathbb{C}^T_X$ is the constant sheaf and X is a nonsingular complex projective algebraic variety.
- $\mathscr{F} = \mathbf{IC}^G_X$ is the intersection cohomology sheaf with middle perversity and X is a complex projective algebraic variety.

2.6 The Weight Filtration on Intersection Cohomology

For one technical aspect of the proof of our main theorem we need to identify some special submodules of both the regular and equivariant intersection cohomology. These submodules come from a filtration on intersection cohomology given by Hodge theory. We will denote them HIH. For certain nice classes of varieties HIH will coincide with IH, and it possesses the useful property that the restriction $HIH^*(Y) \to HIH^*(U)$ is a surjection whenever $U \subset Y$ is an open subvariety.

To define the modules we use the weight filtration on the intersection cohomology of complex varieties that was constructed by Saito (see for example [30]) . We recall some essential features of this weight filtration following Section 3.4 of [5].

Given a complex variety Y and an open subvariety U, the weight filtration is an increasing filtration $W_i IH^*(Y)$ and $W_i IH^*(Y, U)$ on the intersection cohomology groups

$IH^*(Y)$ and $IH^*(Y,U)$. This filtration is strongly compatible with the homomorphisms in the long exact sequence for the pair (Y,U) in the sense that taking the associated graded Gr_k^W of all terms in the sequence gives another exact sequence.

There is also a weight filtration $W_i IH_T^*(Y)$ on the equivariant intersection cohomology $IH_T^*(Y)$, given an algebraic action of a torus T on Y. This is because the restriction homomorphism on IH induced by a normally nonsingular inclusion is strictly compatible with the weight filtration, and the inclusions $(Y \times ET_k)/T \to (Y \times ET_{k+1})/T$ are normally nonsingular.

Lemma 2.23 ([5, Theorem 3.3])**.** *$W_k IH^d(X,U) = 0$ if $k < d$.*

Definition 2.24.

$$HIH^d(Y) := Gr_d^W IH^d(Y) = W_d IH^d(Y),$$

using the above lemma, and

$$HIH^d(Y,U) := Gr_d^W IH^d(Y,U) = W_d IH^d(Y,U)$$

for the relative version. If X is a T-variety then

$$HIH_T^d(X) := W_d IH_T^d(X).$$

*We say Y is **pure** if $HIH^*(Y) = IH^*(Y)$.*

Proposition 2.25. *Projective varieties and quasiconical affine varieties are always pure.*

More generally, if Y has a $\mathbb{C}^\times$-action contracting Y onto $Y^{\mathbb{C}^\times}$ and $Y^{\mathbb{C}^\times}$ is proper then Y is pure.

Theorem 2.26 ([5, Theorem 3.5])**.** *If $U \subset Y$ is an open subvariety then the restriction $HIH^*(Y) \to HIH^*(U)$ is a surjection. If Y is a T-variety and U is T-invariant then $HIH_T^*(Y) \to HIH_T^*(U)$ is a surjection.*

CHAPTER 3

THE WONDERFUL COMPACTIFICATION

In 1982 De Concini and Procesi [14] introduced a particularly nice compactification of certain symmetric varieties that is now commonly referred to as the wonderful compactification. Recall that a symmetric variety is a homogeneous space G/H where G is a reductive algebraic group and H is the fixed point set of some involution (an automorphism of order two), and that G/H carries an action of G - multiplication on the left. De Concini and Procesi were working in the particular case where G is a semisimple simply connected algebraic group over the complex numbers. They define an equivariant compactification, X, of G/H; that is, X is a variety with a G action and containing G/H as a dense orbit.

Definition 3.1. *The* ***wonderful compactification*** *of a symmetric variety G/H is the minimal regular equivariant compactification of G/H, where by* ***regular*** *we mean that*

(a) X is smooth and

(b) $X\backslash(G/H)$ is a normal crossings divisor with finitely many G-stable divisors whose various intersections give the G orbit closures,

and by ***minimal*** *we mean that all the other regular compactifications of X are in most cases obtained from X by a series of blowups (as shown in De Concini and Procesi's follow-up paper [15]).*

The compactification is constructed by choosing a suitable G-module in which H is the stabilizer of a line l. Then G/H embeds into the projectivization of that module as the orbit $G \cdot l$ and we define X to be the closure of this orbit.

This thesis is concerned with the case where this construction is used to compactify a group. This construction yields a wonderful compactification of any semisimple *adjoint* group G via the identification $G \simeq (G \times G)/\mathrm{Diag}(G \times G)$ where $\mathrm{Diag}(G \times G)$ refers to the diagonal in $G \times G$ which is the fixed set of the involution $(x, y) \mapsto (y, x)$.

In fact, the construction is particularly easy to describe concretely in this case where we are compactifying a group. We choose a regular dominant weight λ of the simply connected cover of $\widetilde{G}$ of G and use V_λ to denote the corresponding irreducible representation of highest weight λ. As a $\widetilde{G} \times \widetilde{G}$ module, $\mathrm{End}(V_\lambda)$ is isomorphic to $V_\lambda \otimes V_\lambda^*$ and G is identified with the orbit of $[I]$ in $\mathbb{P}(\mathrm{End}(V_\lambda))$ where I is the identity in $\mathrm{End}(V_\lambda)$.

The wonderful compactification X is an irreducible, smooth, projective $G \times G$ variety containing G as an open $G \times G$ subvariety. The action of $G \times G$ on G is the two sided action $(g, h) \cdot x = gxh^{-1}$.

3.1 $G \times G$ Orbit Structure

We now recall some results on the structure of X that have been established in [16] and [9]. The orbit structure of X admits a nice description in terms of the of the root system structure of G. We make a choice of maximal torus and Borel subgroup $T \subset B \subset G$, and use Φ, Φ^+, Q, W and $\Delta = \{\alpha_1, ..., \alpha_r\} \subset \Phi^+$ to denote the corresponding root system, positive roots, root lattice, Weyl group and simple roots, respectively. Also, we write the group of characters of T - denoted $X^*(T)$ - additively.

We now examine the closure $\overline{T}$ of T in X. The negative fundamental Weyl chamber corresponding to Δ gives a T-stable chart Z of $\overline{T}$ that is canonically identified with r-dimensional affine space $\mathbb{A}^r$. The maximal torus T embeds into $\mathbb{A}^r$ via $t \rightarrow$

$(\alpha_1(t)^{-1}, ..., \alpha_r(t)^{-1})$, being identified with the points of $\mathbb{A}^r$ that have all coordinates non-zero. The sets

$$\mathbb{A}^r_I = \{(x_1, ..., x_r) \in \mathbb{A}^r \mid x_i = 0 \text{ for all } i \text{ such that } \alpha_i \in I\}, \quad I \subset \Delta$$

are the $T \times 1$-orbits of $\mathbb{A}^r$ and it is shown in [14] that the $(G \times G)$-orbits of X are all of the form $(G \times G) \cdot \mathbb{A}^r_I$. Thus the $(G \times G)$-orbits of X are in bijection with subsets of Δ and there are exactly 2^r orbits.

We will use X_I to denote the $G \times G$ orbit corresponding to $I \subset \Delta$. There is a unique closed orbit $X_\varnothing$, also $X_\Delta = G$, and the subset partial order on Δ gives the closure order on the orbits.

Each orbit X_I contains a unique basepoint h_I such that

(a) $(B \times B^-) \cdot h_I$ is dense in X_I, (3.1)

(b) there is a character λ of T with $\lim_{t \to 0} \lambda(t) = h_I$ ([9], Proposition A1).

As an example, h_Δ is the identity $[I] \in X_\Delta \simeq G$.

In addition to describing the number of orbits and the closure order on them we would like to be able to describe the $(G \times G)$-stabilizers and find attractive transverse slices at each distinguished basepoint.

We recall a description of a particularly nice transverse slice to X_I at h_I; it is isomorphic to affine space and $(T \times T)_{h_I}$-invariant.

Proposition 3.2 ([7]). *A $(T \times T)_{h_I}$-invariant attractive transverse slice to X_I at h_I is*

$$\Sigma_I := \left\{ x \in \overline{T} \;\middle|\; h_I \in \overline{(T \times T)_{h_I} \cdot x} \right\}.$$

The $\mathbb{C}^\times$ contracting Σ_I onto h_I is contained in $(T \times T)_{h_I}$.

Proof. Because $(T \times T)_{h_I} = (T \times 1)_{h_I} = (Z \times 1)_{h_I}$, the set Σ_I actually equals

$$\left\{ x \in \overline{T} \;\middle|\; h_I \in \overline{(Z \times 1)_{h_I} \cdot x} \right\}$$

which we know from [7, Section 3.1] is a transverse slice to $(T \times T) \cdot h_I$ at h_I in the smooth toric variety $\overline{T}$. It is isomorphic to affine space $\mathbb{A}^d$ where $d = \dim(\overline{T}) - \dim((T \times T)h_I) = |\Delta - I|$ and is contracted onto h_I by a $\mathbb{C}^\times$ contained in $(Z \times 1)_{h_I} \subset (T \times T)_{h_I}$.

It is clear from the definition that Σ_I is $(T \times T)_{h_I}$-invariant.

Since X is regular, $\dim(X) - \dim(X_I) = |\Delta - I| = d$ so Σ_I is also a transverse slice (in X) to X_I at h_I.

$\square$

Remark 3.3. It is clear from the definition that $\Sigma_I \cap X_I = \{h_I\}$.

To describe the $(G \times G)$-stabilizer of h_I we recall some notation. Given an algebraic group H, we use $R_u(H)$ to denote the unipotent radical of H and $Z(H)$ to denote the center of H. The diagonal of $H \times H$ is $\mathrm{Diag}(H \times H)$. Let P_I be the standard parabolic subgroup defined by $I \subset \Delta$, and P_I^- the opposite parabolic. Their intersection $L_I := P_I \cap P_I^-$ is the Levi subgroup containing T and whose root system, denoted Φ_I , has basis I.

Proposition 3.4 ([9, Proposition A1]). *The $(G \times G)$-stabilizer of h_I is a semi-direct product:*

$$(G \times G)_{h_I} = (R_u(P_I^-) \times R_u(P_I)) \rtimes (Z(L_I) \times \{1\})Diag(L_I \times L_I).$$

Note that $(Z(L_I) \times \{1\})Diag(L_I \times L_I) = \{(x, y) \in L_I \times L_I \mid xy^{-1} \in Z(L_I)\}$.

3.2 $B \times B$ Orbit Structure

The wonderful compactification has finitely many $B \times B$ orbits. In this section we use the Weyl group W to describe the orbit structure.

Given $I \subset \Delta$ let W_I denote the parabolic subgroup of W generated by the simple reflections corresponding to the roots in I. It is the Weyl group of Φ_I. Also, denote by

$W^I = \{x \in W \mid x(I) \subset \Phi^+\}$ the set of distinguished coset representatives of W/W_I. This selects the shortest element of a coset as a representative. For each $w \in W$ we will use w^I to denote the unique element of W^I that is in the same coset as w, that is $w \in w^I W_I$. Furthermore, let w_I denote the longest element of W_I. The Weyl group W (along with its subsets W_I and W^I) has the usual Bruhat order, and the length of an element $w \in W$ is denoted $\ell(w)$. For each $w \in W$ we will use $\dot{w}$ to denote some choice of coset representative; so $\dot{w}$ is in the normalizer of T and $w = \dot{w}T$. The $B \times B$ orbits are described in Lemma 1.3 of[31], parts of which we now recall.

Lemma 3.5.

1. *Each $(B \times B)$-orbit in X_I is of the form*

$$(B \times B) \cdot (x, w) \cdot h_I$$

   for unique elements $x \in W^I$ and $w \in W$. This orbit will be denoted $X_{[I,x,w]}$ and has a distinguished basepoint $h_{[I,x,w]} := (\dot{x}, \dot{w}) \cdot h_I$.

2. *Dim $\left(X_{[I,x,w]}\right) = |I| + \ell(w_D) - \ell(x) + \ell(w)$.*

Thus the $(B \times B)$-orbits in X_I are indexed by $W^I \times W$. An alternative, more 'symmetric', way to index these orbits would be by cosets $\overline{(x,w)} \in (W \times W)/\mathrm{Diag}(W_I \times W_I)$ instead of pairs $(x, w) \in W^I \times W$. This is via the isomorphism $W^I \times W \simeq (W \times W)/\mathrm{Diag}(W_I \times W_I)$ given by $(x, w) \to \overline{(x,w)}$.

We now recall two descriptions of the closure order $\leq$ on the $B \times B$ orbits. Possibly abusing terminology, we will refer to this order as a Bruhat order.

Proposition 3.6 ([31], 2.4)**.** *Let $x' \in W^I$, $x \in W^J$, $w, w' \in W$. Then $X_{[I,x',w']} \leq X_{[J,x,w]}$ if and only if $I \subset J$ and there exist $u \in W_I$, $v \in W_J \cap W^I$ with $xvu^{-1} \leq x'$, $w'u \leq wv$, and $\ell(wv) = \ell(w) + \ell(v)$. If this is so we have $x' \geq x$ and $-\ell(x') + \ell(w') \leq -\ell(x) + \ell(w)$.*

We may describe $\leq$ as being generated by two simpler order relations, leading to a more accessible description.

Proposition 3.7 ([31], 2.8). *Define* $X_{[I,x',w']} \leq_1 X_{[J,x,w]}$ *if* $I \subset J$ *and* $x' \geq x$, $w' \leq w$, *and* $X_{[I,x',w']} \leq_2 X_{[J,x,w]}$ *if* $I \subset J$ *and there exists* $z \in W_J$ *with* $x' \geq xz$ *and* $w' = wz$, $\ell(wz) = \ell(w) + \ell(z)$.

Then $\leq_1$ *and* $\leq_2$ *are both order relations and they generate* $\leq$.

Now we describe a $(T \times T)_{h_{[I,x,w]}}$-invariant attractive transverse slice to $X_{[I,x,w]}$ at $h_{[I,x,w]}$, for which we need the unipotent radicals $U^+ = R_u(B)$ and $U^- = R_u(B^-)$ of B and B^- respectively.

Proposition 3.8 ([31, Proposition 1.6]). *The morphism*

$$\phi : (U^- \cap xU^+x^{-1}) \times (U^- \cap wU^-w^{-1}) \times \Sigma_I \to X$$

that sends (g, h, z) *to* $(gx, hw) \cdot z$ *is a bijection onto its image and this image,*

$$\Sigma_{[I,x,w]} := Im(\phi)$$

is a $(T \times T)_{h_{[I,x,w]}}$*-invariant attractive transverse slice to* $X_{[I,x,w]}$ *at* $h_{[I,x,w]}$. *The* $\mathbb{C}^\times$ *that contracts* $\Sigma_{[I,x,w]}$ *onto* $h_{[I,x,w]}$ *is contained in* $(T \times T)_{h_{[I,x,w]}}$.

3.3 $T \times T$ Orbit Structure

For the purposes of calculating equivariant intersection cohomology, we are most interested in the lowest dimensional $(T \times T)$-orbits. This is in analogy with GKM Theory (see Section 2.5) where just the zero and one dimensional orbits of a torus action are used to compute equivariant cohomology.

Not every $(B \times B)$-orbit will contain $(T \times T)$-fixed points but each orbit $X_{[I,x,w]}$ does contain a unique smallest torus orbit as is established in the following lemma.

Lemma 3.9. *Given a* $(B \times B)$*-orbit* $X_{[I,x,w]}$

1. *The* $(T \times T)$*-stabilizer of every point in this orbit is contained in* $(T \times T)_{h_{[I,x,w]}}$, *the stabilizer of the basepoint.*

2. $(T\times T)_y = (T\times T)_{h_{[I,x,w]}}$ *only if* $y \in (T\times T)\cdot h_{[I,x,w]}$

Consequently $(T\times T)\cdot h_{[I,x,w]}$ *is the unique* $(T\times T)$*-orbit of* $X_{[I,x,w]}$ *of minimal dimension.* $(T\times T)\cdot h_{[I,x,w]}$ *has dimension* $|I|$

We will refer to $(T\times T)\cdot h_{[I,x,w]}$ as the **core** of $X_{[I,x,w]}$, and also as a **distinguished $(T\times T)$-orbit of type I**.

Proof. First we show that $(T\times T)\cdot h_{[I,x,w]}$ has dimension $|I|$. The $(T\times T)$-stabilizers of $h_{[I,x,w]}$ and h_I are conjugate: $(T\times T)_{h_{[I,x,w]}} = (\dot{x},\dot{w})(T\times T)_{h_I}(\dot{x}^{-1},\dot{w}^{-1})$. From the description of the $(G\times G)$-stabilizer of h_I in Proposition 3.4 we know that $(T\times T)_{h_I} = (Z(L_I)\times 1)\mathrm{Diag}(T\times T)$. The rank of $Z(L_I)$ is $|T|-|I|$ and so $(T\times T)_{h_I}$ is a subtorus of dimension $(\dim(T)-|I|)+\dim(T)$. It follows that

$$\dim((T\times T)\cdot h_{[I,x,w]}) = \dim((T\times T)h_I) = 2\dim(T) - ((\dim(T)-|I|)+\dim(T)) = |I|.$$

Statement 1 follows immediately from Corollary 3.14: the contracting $\mathbb{C}^\times$-action commutes with the $(T\times T)$-action.

For statement 2 we use the the description of $G\times G/(G\times G)_{h_I}$ as a fiber space,

$$p : G\times G/(G\times G)_{h_I} \to (G\times G)/P_I^-\times P_I,$$

from [31, Section 1.1] which is a $(G\times G)$-equivariant fiber bundle with fiber $L_I/Z(L_I)$. (Recall we have the identification $G\simeq G\times G/\mathrm{Diag}(G\times G)$ given by $g\mapsto [g,1]$ and $gh^{-1} \leftarrow\!\shortmid [g,h]$.) We use this fiber bundle to argue that points of $X_{[I,x,w]}\setminus (T\times T)\cdot h_{[I,x,w]}$ must have strictly smaller stabilizers than $(T\times T)_{h_{[I,x,w]}}$.

Keeping in mind that $(B\times B^-)h_{[I,x,w]}$ is open in X_I. We restrict the quotient map p to

$$B\times B^-/(B\times B^-)_{h_I} \to (B\times B^-)/((P_I^-\times P_I)\cap(B\times B^-)),$$

and

$$(B\times B^-)/((P_I^-\times P_I)\cap(B\times B^-)) \simeq (B\times B^-)/(B\cap P_I^-\times P_I\cap B^-) \simeq (B/(B\cap P_I^-))\times B^-/(P_I\cap B^-).$$

$B/(B\cap P_I^-)\times \times B^-/(P_I\cap B^-)$ is isomorphic to $U_{\Delta\setminus I}\times U_{\Delta\setminus I}^-$ and clearly the unique point in $U_{\Delta\setminus I}\times U_{\Delta\setminus I}^-$ with the smallest T-stabilizer will be the identity.

Since the $(T\times T)$-stabilizer of $[x,y]$ will be contained in the $(T\times T)$-stabilizer of $p([x,y])$, we only need to be concerned with elements of the domain that map (under the restriction of p) into $((P_I^-\times P_I)\cap(B\times B))$. Those will be the points $(BB^-\cap L_I)/(BB-\cap Z(L_I)) = (BB^-\cap L_I)/Z(L_I)$. Points with any non-zero 'unipotent coordinates' will have a torus stabilizer that is too big, so the points with the largest possible stabilizer are exactly the points $T/Z(L_I)$ which correspond exactly to the points of $(T\times T)h_I$. Since $(B\times B)(x,w)h_I\subset(B\times B^-)h_I$ we have the desired result.

$\square$

In the lemma above we have established that the smallest $(T\times T)$-orbits in the $(G\times G)$-orbit X_I all have dimension $|I|$ and are in one-to-one correspondence with the $(B\times B)$-orbits. These are our substitute for $(T\times T)$-fixed points.

The closure of each core is a toric variety whose boundary components are actually other cores, as shown below, and it will be important for us to understand the closure order on these core $(T\times T)$-orbits.

Lemma 3.10. *The $(T\times T)$-orbits in $\overline{(T\times T)\cdot h_{[I,x,w]}}$ are precisely the orbits $(T\times T)\cdot h_{[J,x',w']}$ satisfying $J\subset I$ and $\overline{(x',w')}=\overline{(x,w)}$ in $W\times_{W_I}W$.*

Proof. By consideration of dimension the $(T\times T)$-orbits in the closure of $(T\times T)\cdot h_{[I,x,w]}$ must not lie in $X_{[I,x,w]}$. But they must be in $\overline{X_{[I,x,w]}}$ and the lemma follows from the closure order on the $(B\times B)$-orbits. $\square$

It will be beneficial to us to identify subvarieties in X that are transverse to $(B\times B)$-orbits along their cores and contract onto those cores. This is in analogy with [5, Lemma 4.1] where at each fixed point there is a transverse slice to the stratum containing that fixed point and this slice is contained in an open neighbourhood of the fixed point that contracts onto the fixed point.

Proposition 3.11. *We can find a $(T \times T)$-invariant attractive normal space to each $(B \times B)$-orbit along its core. That is, given $X_{[I,x,w]}$ we can find N such that*

(1) N is a locally closed smooth $(T \times T)$-invariant subvariety

(2) $N \cap X_{[I,x,w]} = (T \times T) \cdot h_{[I,x,w]}$,

(3) at all points x of $(T \times T) \cdot h_{[I,x,w]}$ the identity $T_x N + T_x X_{[I,x,w]} = T_x X$ holds on tangent spaces.

(4) N contracts onto the core by a $\mathbb{C}^\times$ action that fixes each point of the core and commutes with the $(T \times T)$-action

Proof. Such a subvariety may be obtained by acting by $T \times T$ on a transverse slice to $X_{[I,x,w]}$ at $h_{[I,x,w]}$. In particular,

$$N_{[I,x,w]} := (T \times T) \cdot \Sigma_{[I,x,w]}$$

satisfies the properties. It is $(T \times T)$-invariant, locally closed and smooth by construction (since $\Sigma_{[I,x,w]}$ is locally closed and smooth). Any $\mathbb{C}^\times$ in $(T \times T)_{h_{[I,x,w]}}$ that contracts $\Sigma_{[I,x,w]}$ onto $h_{[I,x,w]}$ also contracts $N_{[I,x,w]}$ onto the core $(T \times T) \cdot h_{[I,x,w]}$ while fixing the core, and $N \cap X_{[I,x,w]} = (T \times T) \cdot h_{[I,x,w]}$ precisely because $\Sigma_{[I,x,w]} \cap X_{[I,x,w]} = h_{[I,x,w]}$.

Finally, the identity on tangent spaces holds at $h := h_{[I,x,w]}$ since

$$T_h N + T_h X_{[I,x,w]} \supset T_h \Sigma_{[I,x,w]} + T_h X_{[I,x,w]} = T_h X.$$

The identity must then hold at all points of $(T \times T) \cdot h_{[I,x,w]}$ since those points are $(T \times T)$-translates of h. $\square$

Proposition 3.12. *Any subtorus $(T \times T)^\perp$ that is complementary to $(T \times T)_{h_{[I,x,w]}}$ in $T \times T$ acts freely on $N_{[I,x,w]}$ and the quotient $N_{[I,x,w]}/(T \times T)^\perp$ is acted on by $(T \times T)_{h_{[I,x,w]}}$ in the obvious way and is isomorphic as a $(T \times T)_{h_{[I,x,w]}}$-variety to $\Sigma_{[I,x,w]}$.*

Proof. Fix a subtorus $(T \times T)^{\perp}$ such that $T \times T$ is the direct product of $(T \times T)^{\perp}$ and $(T \times T)_{h_{[I,x,w]}}$. We show that it acts freely on $N_{[I,x,w]}$ by showing that its intersection with the $(T \times T)$-stabilizer of any point of $N_{[I,x,w]}$ is trivial.

Since $T \times T$ is abelian the $(T \times T)$-stabilizers are the same at all points of $N_{[I,x,w]}$ in the same $(T \times T)$-orbit. Since $N_{[I,x,w]} = (T \times T)\Sigma_{[I,x,w]}$ we only need concern ourselves with the $(T \times T)$-stabilizers of points in $\Sigma_{[I,x,w]}$.

We know we can choose a one parameter subgroup $\gamma : \mathbb{C}^{\times} \to T \times T$ contracting $\Sigma_{[I,x,w]}$ onto $h_{[I,x,w]}$. Given $s \in \Sigma_{[I,x,w]}$ and (t_1, t_2) in $(T \times T)_s$ we have $(t_1, t_2)h_{[I,x,w]} = (t_1, t_2) \lim_{z \to 0} \gamma(z)s = \lim_{z \to 0}(t_1, t_2)\gamma(z)s = \lim_{z \to 0} \gamma(z)(t_1, t_2)s = \lim_{z \to 0} \gamma(z)s = h_{[I,x,w]}$, showing that $(T \times T)_s \subset (T \times T)_{h_{[I,x,w]}}$. This proves that $(T \times T)^{\perp}$ acts freely on $N_{[I,x,w]}$.

Since $T \times T$ is abelian it is trivially true that $(T \times T)_{h_{[I,x,w]}}$ normalizes $(T \times T)^{\perp}$ and thus its action on $N_{[I,x,w]}$ descends to an action on $N_{[I,x,w]}/(T \times T)^{\perp}$.

Finally, we show that the composition of the inclusion $\Sigma_{[I,x,w]} \hookrightarrow N_{[I,x,w]}$ and the quotient map $N_{[I,x,w]} \to N_{[I,x,w]}/(T \times T)^{\perp}$ is a $(T \times T)_{h_{[I,x,w]}}$-equivariant isomorphism. We will denote this composite map f. The $(T \times T)_{h_{[I,x,w]}}$-equivariance is clear: $f((t_1, t_2)s) = (T \times T)^{\perp}(t_1, t_2)s = (t_1, t_2)(T \times T)^{\perp}s = (t_1, t_2)f(s)$ for any $(t_1, t_2) \in (T \times T)_{h_{[I,x,w]}}$.

To define the inverse map we describe a $(T \times T)$-invariant map $N_{[I,x,w]} \to \Sigma_{[I,x,w]}$ that is constant on $(T \times T)^{\perp}$ orbits. Each element n of $N_{[I,x,w]}$ is by definition in the $(T \times T)$-orbit of some element of $\Sigma_{[I,x,w]}$. Furthermore, since $T \times T$ is the product of $(T \times T)^{\perp}$ and $(T \times T)_{h_{[I,x,w]}}$, and $\Sigma_{[I,x,w]}$ is $(T \times T)_{h_{[I,x,w]}}$-invariant, n is in the $(T \times T)^{\perp}$ orbit of some $s \in \Sigma_{[I,x,w]}$. We show now that this element s is unique; that is, the intersection $(T \times T)^{\perp}s \cap \Sigma_{[I,x,w]}$ consists only of s. If we have $s' \in (T \times T)^{\perp}s \cap \Sigma_{[I,x,w]}$ then $s' = (t_1, t_2)s$ for some $(t_1, t_2) \in (T \times T)^{\perp}$. Letting $\gamma : \mathbb{C}^{\times} \to T \times T$ denote a one parameter subgroup contracting $\Sigma_{[I,x,w]}$ onto $h_{[I,x,w]}$, we have $(t_1, t_2)h_{[I,x,w]} = (t_1, t_2) \lim_{z \to 0} \gamma(z)s = \lim_{z \to 0}(t_1, t_2)\gamma(z)s = \lim_{z \to 0} \gamma(z)(t_1, t_2)s = \lim_{z \to 0} \gamma(z)s' = h_{[I,x,w]}$, implying that $(t_1, t_2) \in (T \times T)_{h_{[I,x,w]}}$. Since (t_1, t_2) is also in $(T \times T)^{\perp}$, it must equal $(1, 1)$. Thus $s' = (1, 1)s = s$.

We now have a map $g : N_{[I,x,w]} \to \Sigma_{[I,x,w]}$ that sends n to the unique s in $(T \times T)^{\perp} \cap \Sigma_{[I,x,w]}$. Since the composition of the action map $(T \times T)^{\perp} \times \Sigma_{[I,x,w]} \to N_{[I,x,w]}$ and g gives projection onto the second factor in $(T \times T)^{\perp} \times \Sigma_{[I,x,w]}$, the map g must be regular.

Clearly g is constant on $(T \times T)^{\perp}$-orbits, resulting in an induced map $N_{[I,x,w]}/(T \times T)^{\perp} \to \Sigma_{[I,x,w]}$ that is $(T \times T)_{h_{[I,x,w]}}$-equivariant. It is immediate that this induced map is an inverse to f. $\square$

Lemma 3.13. *For each core $(T \times T) \cdot h_{[I,x,w]}$ and normal space N to $X_{[I,x,w]}$ along that core we can always find a $(T \times T)$-invariant open subvariety O that contains N and contracts onto it.*

Proof. For this proof let $h := h_{[I,x,w]}$. Choose a $(T \times T)$-invariant subspace $\mathfrak{u}'$ of $\mathfrak{u} \times \mathfrak{u} = \mathrm{Lie}(U^+ \times U^+)$ that is complementary to the Lie algebra of the $(U^+ \times U^+)$-stabilizer of the basepoint h. That is $\mathfrak{u}'$ is a sum of root spaces and is complementary to $\mathrm{Lie}\left((U^+ \times U^+)_h\right)$ in $\mathfrak{u} \times \mathfrak{u}$, the Lie algebra of $U^+ \times U^+$. There are only finitely many ways to make this choice.

Then $U' := \exp(\mathfrak{u}')$ is $(T \times T)$-invariant and isomorphic to affine space, and we now show that $O := U' \cdot N$ is a $(T \times T)$-invariant open subvariety of X that contracts onto N.

Let

$$f : U' \times N \to X,$$

be the restriction of the action map $(G \times G) \times X \to X$ to $U' \times N$. Then O is the image of f and we show that f is an injective open map.

Examining the differential of f at $((1,1),h)$ we see that the image $(df)(T_{((1,1),h)}((1,1) \times N)$ is $T_h N$ which is transverse to the image $(df)(T_{((1,1),h)}(U' \times h))$ by choice of U'. Thus the map f is étale at $((1,1),h)$. Since in addition $f^{-1}(h) = ((1,1),h)$, the map f is injective. It also follows that f is an open map and thus that O is open.

Now we argue that O contracts onto N. With the $(T \times T)$-action on $U' \times N$ given by

$$(t_1, t_2) \cdot ((u_1, u_2), n) := ((t_1 u_1 t_1^{-1}, t_2 u_2 t_2^{-1}), t_1 n t_2^{-1})$$

the action map f is $(T \times T)$-invariant.

A second $(T \times T)$-action on $U' \times N$ commuting with the first is given by

$$(t_1, t_2) * ((u_1, u_2), n) := ((t_1 u_1 t_1^{-1}, t_2 u_2 t_2^{-1}), n)$$

and under this action we can find a one parameter subgroup $\gamma : \mathbb{C}^\times \to T \times T$ contracting $U' \times N$ onto $(1,1) \times N$ while fixing $(1,1) \times N$. (Any $\mathbb{C}^\times$ contracting U' onto the identity will do, since this $(T \times T)$-action doesn't affect N.) $\square$

Corollary 3.14. $X_{[I,x,w]}$ *contracts onto* $(T \times T) \cdot h_{[I,x,w]}$.

Proof. It suffices to show that $X_{[I,x,w]}$ is contained in the open set O from Lemma 3.13 above. We do this by showing that the restriction of the map $f : U' \times N \to X$ from the proof above to $U' \times (T \times T) \cdot h_{[I,x,w]}$ has image $X_{[I,x,w]}$.

Note that because we are working in characteristic 0, the map $\exp : \mathfrak{u} \times \mathfrak{u} \to U^+ \times U^+$ is an isomorphism. Since $\mathrm{Lie}\left((U^+ \times U^+)_{h_{[I,x,w]}}\right) \oplus \mathrm{Lie}\,(U') = \mathfrak{u} \times \mathfrak{u}$, the restriction of the quotient $(U^+ \times U^+) \to (U^+ \times U^+)/(U^+ \times U^+)_{h_{[I,x,w]}}$ to U' is injective and surjective. So $U' \cdot h_{[I,x,w]} = (U^+ \times U^+) \cdot h_{[I,x,w]}$ and

$$\begin{aligned} f(U' \times (T \times T) \cdot h_{[I,x,w]}) = U'(T \times T) h_{[I,x,w]} &= (T \times T) U' h_{[I,x,w]} \\ &= (T \times T)(U^+ \times U^+) h_{[I,x,w]} \\ &= X_{[I,x,w]} \end{aligned}$$

since $T \times T$ normalizes U'. This completes the proof.

$\square$

Remark 3.15. Note that the contracting action of Lemma 3.13 preserves $(B \times B)$-orbits.

CHAPTER 4

EQUIVARIANT INTERSECTION COHOMOLOGY OF $B \times B$ ORBIT CLOSURES IN THE WONDERFUL COMPACTIFICATION

4.1 Generalized Moment Graphs

We provide a generalization of the notion of a moment graph given, for example, in [18]. Let $Y \simeq \mathbb{Z}^r$ be lattice of finite rank.

Definition 4.1. *A (directed)* ***generalized moment graph*** $\mathcal{G}$ *over* Y *is given by the following data:*

- *A finite graph* $(\mathcal{V}, \mathcal{E})$ *with vertices* $\mathcal{V}$ *and directed edges* $\mathcal{E}$ *such that there are no directed cycles and each pair of vertices is joined by at most one edge. To emphasize the direction of an edge* E *directed from a vertex* x *to a vertex* y *we will write* $E\colon x \to y$.

- *An assignment of a sublattice of* Y *to each edge and vertex,* $v : \mathcal{V} \sqcup \mathcal{E} \to \{\textit{sublattices of } Y\}$, *such that if vertices* x *and* y *are joined by an edge* E *then both* $v(x)$ *and* $v(y)$ *are contained in* $v(E)$.

There is a partial order $\leq$ on the vertices defined by $x \leq y$ if and only if $x = y$ or there is a directed path from x to y. We use the partial order to define a topology on $\mathcal{V}$: a subset $\mathcal{U} \subset \mathcal{V}$ is open if for any $x \in \mathcal{U}$ all vertices above x in the partial order are also in $\mathcal{U}$. That is, $\mathcal{U} = \cup_{x \in \mathcal{U}} \{y \in \mathcal{V} \mid y \geq x\}$.

We'll use the notation $\mathcal{V}_{\geq x}$ for the open set $\{y \in \mathcal{V} \mid y \geq x\}$; the notations $\mathcal{V}_{>x}$, $\mathcal{V}_{\leq x}$ and $\mathcal{V}_{<x}$ are used analogously.

4.2 Sheaves on Generalized Moment Graphs

Consider a generalized moment graph, $\mathcal{G} = (\mathcal{V}, \mathcal{E}, v)$, over Y. We denote by $A = \mathrm{Sym}(Y \otimes_{\mathbb{Z}} \mathbb{C})$ the symmetric algebra of Y over $\mathbb{C}$, which we consider as a $2\mathbb{Z}$-graded algebra with $Y \otimes_{\mathbb{Z}} \mathbb{C}$ sitting in degree 2. For each $w \in \mathcal{V} \sqcup \mathcal{E}$ we understand the sublattice $v(w)$ to be sitting inside A as the elements $v(w) \otimes_{\mathbb{Z}} 1 \subset Y \otimes_{\mathbb{Z}} \mathbb{C}$ and we use $\langle v(w) \rangle$ to denote the ideal in A generated by $v(w)$. The quotient rings $A/\langle v(w) \rangle A$ play a big role and will be denoted A_w. Each of the rings A_w is a polynomial ring since $A_w = \mathrm{Sym}(Y/v(w) \otimes_{\mathbb{Z}} \mathbb{C})$. All the modules we consider should be understood to be finitely generated graded modules.

Definition 4.2. *Given a generalized moment graph $\mathcal{G}$ over Y, a **sheaf** $\mathscr{F}$ is given by the data*

- *an A_w-module $\mathscr{F}^w$ for each vertex and edge $w \in \mathcal{V} \sqcup \mathcal{E}$,*
- *a homomorphism $\rho_{x,E} : \mathscr{F}^x \to \mathscr{F}^E$ of graded A_x-modules for every vertex x lying on an edge E.*

Remark 4.3. As defined our sheaves on $\mathcal{G}$ are not sheaves in the typical sense but we can make them so by using a different topology on $\mathcal{V} \sqcup \mathcal{E}$.

A morphim of $f : \mathscr{F} \to \mathscr{D}$ of sheaves on $\mathcal{G}$ is a collection of A-module homomorphisms $f_w : \mathscr{F}^w \to \mathscr{D}^w$ for all vertices and edges $w \in \mathcal{V} \sqcup \mathcal{E}$ that are compatible with the maps ρ, that is

$$\begin{array}{ccc} \mathscr{F}^x & \xrightarrow{f^x} & \mathscr{D}^x \\ {\scriptstyle \rho^{\mathscr{F}}_{x,E}}\downarrow & & \downarrow{\scriptstyle \rho^{\mathscr{D}}_{x,E}} \\ \mathscr{F}^E & \xrightarrow{f^E} & \mathscr{D}^E \end{array}$$

commutes whenever x is a vertex lying on an edge E.

Definition 4.4. *A sheaf $\mathscr{F}$ on $\mathcal{G}$ is* ***rigid*** *if the group of automorphisms of $\mathscr{F}$ consists only of multiplication by non-zero scalars.*

Example 4.5. One distinguished sheaf on $\mathcal{G}$ is the *structure sheaf* $\mathscr{A}$ defined by

- $\mathscr{A}^w := A/\langle v(w)\rangle A$ for all vertices and edges $w \in \mathcal{V} \sqcup \mathcal{E}$, and
- given a vertex x lying on an edge E the map $\rho_{x,E} : A/\langle v(x)\rangle A \to A/\langle v(E)\rangle A$ is the canonical quotient homomorphism (recall that $v(x) \subset v(E)$).

Each of the modules $\mathscr{A}^w$ is in fact a graded A-algebra and so $\mathscr{A}$ is a sheaf of rings.

For any subset $\mathcal{U} \subset \mathcal{V}$, we define the *space of sections* of $\mathscr{F}$ over $\mathcal{U}$ by

$$\Gamma(\mathcal{U}, \mathscr{F}) = \left\{ (m_x) \in \prod_{x\in\mathcal{V}} \mathscr{F}^x \;\middle|\; \rho_{y,E}(m_y) = \rho_{z,E}(m_z) \text{ for all } y, z \in \mathcal{U} \text{ joined by an edge } E \right\}.$$

The space of sections $\Gamma(\mathcal{U}, \mathscr{F})$ as defined is an A-module but is also a $\Gamma(\mathcal{U}, \mathscr{A})$-module since $\langle v(w)\rangle \mathscr{F}^w = 0$ for all $w \in \mathcal{V} \sqcup \mathcal{E}$.

For each subset $\mathcal{U} \subset \mathcal{V}$, the obvious restriction of a sheaf $\mathscr{F}$ to the subgraph of $\mathcal{G}$ with vertices $\mathcal{U}$ will be denoted $\mathscr{F}^{\mathcal{U}}$.

For any subsets $\mathcal{U}' \subset \mathcal{U} \subset \mathcal{V}$, the projection $\bigoplus_{x\in\mathcal{U}} \mathscr{F}^w \to \bigoplus_{x\in\mathcal{U}'} \mathscr{F}^w$ along the decomposition induces a restriction map $\Gamma(\mathcal{U}, \mathscr{F}) \to \Gamma(\mathcal{U}', \mathscr{F})$.

For an open subset $\mathcal{U}$ of $\mathcal{V}$ we call an element of $\Gamma(\mathcal{U}, \mathscr{F})$ a **section**. We call $\Gamma(\mathscr{F}) := \Gamma(\mathcal{V}, \mathscr{F})$ the **space of global sections**, and we will call $\Gamma(\mathscr{A})$ the *structure algebra* of $\mathcal{G}$. Each space of sections $\Gamma(\mathcal{U}, \mathscr{F})$ is a module over the structure algebra $\Gamma(\mathscr{A})$, via the restriction $\Gamma(\mathscr{A}) \to \Gamma(\mathcal{U}, \mathscr{A})$. In particular, $\Gamma(\mathscr{F})$ is a $\Gamma(\mathscr{A})$-module.

4.3 Construction of the Canonical sheaf $\mathscr{B}$ on a generalized moment graph $\mathcal{G}$

A natural question to ask is whether a given sheaf $\mathscr{F}$ is *flabby*, that is whether the restriction map $\Gamma(\mathscr{F}) \to \Gamma(\mathcal{U}, \mathscr{F})$ (restricting global sections) is surjective for all open sets $\mathcal{U}$. In the setting of [5], working with ordinary moment graphs, there is a class of sheaves that are universal with respect to the problem of extending local sections. We show that this is also true in this more generalized setting.

Definition 4.6. *A sheaf $\mathscr{B}$ on a generalized moment graph $\mathcal{G}$ is a* ***Braden-MacPherson sheaf*** *if it satisfies the following properties*

1. *For each $x \in \mathcal{V}$ the module $\mathscr{B}^x$ is a free $A/\langle v(x)\rangle A$-module.*
2. *Given an edge $E\colon x \to y$, the map $\rho_{y,E} : \mathscr{B}^y \to \mathscr{B}^E$ is surjective with kernel $\langle v(E)\rangle \mathscr{B}^y$.*
3. *For an open subset $\mathcal{U}$ of $\mathcal{V}$ the restriction map $\Gamma(\mathscr{B}) \to \Gamma(\mathcal{U}, \mathscr{B})$ is surjective.*
4. *The composition $\Gamma(\mathscr{B}) \subset \bigoplus_{z\in\mathcal{V}} \mathscr{B}^z \to \mathscr{B}^x$ is surjective for every x in $\mathcal{V}$.*

The following theorem gives a classification of Braden-MacPherson sheaves on a generalized moment graph.

Theorem 4.7. *Assume the generalized moment graph $\mathcal{G}$ to be such that for any vertex w the set $\mathcal{G}_{\leq w}$ is finite. Then*

(1) For any vertex w there is a unique (up to isomorphism) Braden-MacPherson Sheaf $\mathscr{B}(w)$ with the properties:

- *$\mathscr{B}(w)^w = A/\langle v(w)\rangle A$ and $\mathscr{B}(w)^x = 0$ unless $x \leq w$.*
- *$\mathscr{B}(w)$ is indecomposable in the category of sheaves on $\mathcal{G}$.*

(2) Let $\mathscr{B}$ be a Braden-MacPherson sheaf. Then there are $w_1, ..., w_n \in \mathcal{V}$ and $l_1, ..., l_n \in \mathbb{Z}$ such that

$$\mathscr{B} = \mathscr{B}(w_1)[l_1] \oplus ... \oplus \mathscr{B}(w_n)[l_n]$$

with the multiset $(w_1, l_1), ..., (w_n, l_n)$ being uniquely determined by $\mathscr{B}$.

Proof. This proof is virtually identical to that given in [19, Theorem 6.4] for regular moment graphs. We first prove the existence part of statement (1) by giving an inductive construction for $\mathscr{B}(w)$:

(1) Set $\mathscr{B}(w)^w := A/\langle v(w)\rangle A$ and $\mathscr{B}(w)^x = 0$ for $x \nleq w$.

(2) If $\mathscr{B}(w)^y$ has already been defined, then for each edge $E\colon x \to y$, set

$$\mathscr{B}(w)^E := \mathscr{B}(w)^y/\langle v(E)\rangle \mathscr{B}(w)^y$$

and let $\rho_{y,E} : \mathscr{B}(w)^y \to \mathscr{B}(w)^E$ be the canonical map.

(3) Suppose $\mathscr{B}(w)^y$ has already been defined for all y in an open set $\mathcal{U} \subset \mathcal{V}$ and $x \in \mathcal{V}$ is such that $\mathcal{U} \cup \{x\}$ is open as well. By step 2 above we have also defined $\mathscr{B}(w)^E$ and $\rho_{y,E}$ for each edge E directed upwards from x (to some vertex y) and we now need to define $\mathscr{B}(w)^x$ and the maps $\rho_{x,E}$. We can already calculate sections of $\mathscr{B}(w)$ on the open set $\mathcal{V}_{>x}$; the 'restrictions' of these sections to edges flowing up from x form a module $\mathscr{B}(w)^{\partial x}$ and $\mathscr{B}(w)^x$ is defined to be its projective cover. This construction is now described more explicitly.

First, we introduce notation for the "boundary vertices" of x in $\mathcal{V}$, by which we mean the vertices adjacent to and higher than x:

$$\mathcal{V}_{\partial x} := \{y \in \mathcal{V} \mid \text{there is an edge } E\colon x \to y\}.$$

Similarly we single out the edges that flow up from x:

$$\mathcal{E}_{\partial x} := \{E \in \mathcal{E} \mid E \text{ is directed upwards from } x\}.$$

.

The module $\mathscr{B}(w)^{\partial x}$ is defined as the image of the composition

$$\Gamma(\mathcal{V}_{>x}, \mathscr{B}(w)) \subset \bigoplus_{y>x} \mathscr{B}(w)^y \xrightarrow{p} \bigoplus_{y\in\mathcal{V}_{\partial x}} \mathscr{B}(w)^y \xrightarrow{\rho} \bigoplus_{E\in\mathcal{E}_{\partial x}} \mathscr{B}(w)^E$$

where p is projection along the direct sum decomposition and $\rho = \oplus_{y \in \mathcal{V}_{\partial x}} \rho_{y,E}$. Note that $\mathscr{B}(w)^{\partial x}$ is a module over $\oplus_{E \in \mathcal{E}_{\partial x}} A_E$ and thus also an A_x-module since we have the canonical homomorphisms $A_x \to A_E$ for all $E \in \mathcal{E}_{\partial x}$. Furthermore, $\mathscr{B}^{\partial x}$ is finitely generated, being a submodule of a Noetherian module. We define $d_x : \mathscr{B}(w)^x \to \mathscr{B}(w)^{\partial x}$ to be a projective cover in the category of graded A_x modules. It is free of finite rank, and the components of d_x (with respect to the inclusion $\mathscr{B}(w)^{\partial x} \subset \bigoplus_{E \in \mathcal{E}_{\partial x}}$) will be the maps $\rho_{x,E}$.

To confirm that this sheaf we have constructed is a Braden-MacPherson sheaf we must verify properties (1)-(4) of Definition 4.6. Property (1) holds because projective modules (our $\mathscr{B}(w)^x$) over polynomial rings (the A_x) are free and furthermore only finitely many of the stalks of $\mathscr{B}(w)$ are non-zero by assumption and each stalk is of finite rank by construction. Property (2) is built into the construction, and all that is left is to check (3) and (4), the surjectivity conditions.

From the construction of $\mathscr{B}(w)$ it is clear that for every x in $\mathcal{V}$ the restriction $\Gamma(\mathcal{V}_{\geq x}, \mathscr{B}(w)) \to \Gamma(\mathcal{V}_{>x}, \mathscr{B}(w))$ is a surjection (since lifting a section s over $\mathcal{V}_{>x}$ to $\mathcal{V}_{\geq x}$ amounts to choosing an element of $\mathscr{B}(w)^x$ that maps to the image of s in $\bigoplus_{E \in \mathcal{E}_{\partial x}} \mathscr{B}(w)^E$, which is an element of $\mathscr{B}(w)^{\partial x}$ onto which $\mathscr{B}(w)^x$ surjects).

More generally, given open $\mathcal{U} \subset \mathcal{V}$ and $x \notin \mathcal{U}$ such that $\{x\} \cup \mathcal{U}$ is open, $\Gamma(\mathcal{U} \cup \{x\}, \mathscr{B}(w)) \to \Gamma(\mathcal{U}, \mathscr{B}(w))$ is surjective because a section over $\mathcal{U}$ restricts to a section over $\mathcal{V}_{>x}$ which we can extend to a section over $\mathcal{V}_{\geq x}$ by the previous paragraph, thus determining an element of $\mathscr{B}(w)^x$ that extends the section over $\mathcal{U}$ to a section over $\{x\} \cup \mathcal{U}$.

Then, the procedure of the previous paragraph applied repeatedly enables us to extend a section over an open set $\mathcal{U}$ to a global section. (Keep in mind that only finitely many stalks of $\mathscr{B}(w)$ are nonzero.)

All that is left is to establish that $\Gamma(\mathscr{B}(w)) \to \mathscr{B}(w)^x$ is surjective for all x. Since we have shown that every local section extends to a global section, it is sufficient to prove that

$\Gamma(\mathcal{V}_{\geq x}, \mathscr{B}(w)) \to \mathscr{B}(w)^x$ is surjective for all x. This is immediate from the construction of the Braden-MacPherson sheaf: the image of $s_x \in \mathscr{B}(w)^x$ in $\bigoplus_{E \in \mathcal{E}_{\partial x}} \mathscr{B}(w)^E$ is in $\mathscr{B}(w)^{\partial x}$, that is, it is the 'restriction' of some section s on $\mathcal{V}_{>x}$ and s together with s_x determine a section on $\mathcal{V}_{\geq x}$ that 'restricts' to $s_x \in \mathscr{B}(w)^x$.

We are yet to prove the uniqueness part of statement (1) of the theorem, but it follows immediately from statement (2) which we now prove. Let $\mathscr{B}$ be an arbitrary Braden-MacPherson sheaf. We show it is of the form $\mathscr{B}(w_1)[l_1] \oplus ... \oplus \mathscr{B}(w_n)[l_n]$ using the same sort of inductive arguments we've been employing.

Assume that on an open set $\mathcal{U}$, the restriction of $\mathscr{B}$ decomposes as

$$\mathscr{B}^{\mathcal{U}} = \mathscr{B}(w_1)^{\mathcal{U}}[l_1] \oplus ... \oplus \mathscr{B}(w_n)^{\mathcal{U}}[l_n]$$

for some $w_1, ..., w_n \in \mathcal{U}$ and $l_1, ..., l_n \in \mathbb{Z}$, the multiset $(w_1, l_1), ..., (w_n, l_n)$ being unique. Furthermore assume $x \in \mathcal{V} \setminus \mathcal{U}$ is such that $\mathcal{U} \cup \{x\}$ is open.

The restriction $\Gamma(\mathscr{B}) \to \Gamma(\mathcal{V}_{>x}, \mathscr{B})$ is surjective by the definition of a Braden-MacPherson sheaf and thus so is $\bigoplus_{E \in \mathcal{E}_{\partial x}} \rho_{x,E} : \mathscr{B}^x \to \mathscr{B}^{\partial x}$ (by lifting a section over $\mathcal{V}_{>x}$ to a global section and then picking the component over x). The map $\bigoplus \mathscr{B}(w_i)^x[l_i] \to \bigoplus \mathscr{B}(w_i)^{\partial x}[l_i]$ is a projective cover by construction and thus we have a decomposition $\mathscr{B}^x = \bigoplus \mathscr{B}(w_i)^x[l_i] \oplus R$ for some graded free A_x-module R which lies in the kernel of d_x. Each isomorphism $R \simeq A_x[j_1] \oplus ... \oplus A_x[j_m]$ then yields an isomorphism

$$\mathscr{B}^{\mathcal{U} \cup \{x\}} \simeq \mathscr{B}(w_1)^{\mathcal{U} \cup \{x\}}[l_1] \oplus ... \oplus \mathscr{B}(w_n)^{\mathcal{U} \cup \{x\}}[l_n] \oplus \mathscr{B}(x)^{\mathcal{U} \cup \{x\}}[j_1] \oplus ... \oplus \mathscr{B}(x)^{\mathcal{U} \cup \{x\}}[j_m]$$

as desired. This construction also establishes the uniqueness of the multiset $(w_1, l_1), ..., (w_n, l_n)$.

□

4.4 Constructing a Moment Graph from X

From the wonderful compactification X we now construct a generalized moment graph $\mathcal{G}_X$ over $Q \times Q$, where Q is the root lattice. We already established that each $(B \times B)$-

orbit $X_{[I,x,w]}$ has a unique smallest $(T\times T)$-orbit $(T\times T)\cdot h_{[I,x,w]}$ which we will refer to as the core of the $X_{[I,x,w]}$. These cores are the vertices of $\mathcal{G}_X$. Thus there is a vertex corresponding to each $(B\times B)$-orbit. For convenience the vertex $(T\times T)\cdot h_{[I,x,w]}$ will be denoted $V[I,x,w]$. The sublattice assigned to the vertex $V[I,x,w]$ consists of all characters in $Q\times Q$ that annihilate the $(T\times T)$-stabilizer of $h_{[I,x,w]}$. We may describe this sublattice in terms of Q_I, the lattice generated by I. The action of W on Φ gives an action of $W\times W$ on $\Phi\times\Phi$ componentwise and

$$\begin{aligned} v\left(V[I,x,w]\right) &:= \Big\{(\beta,\gamma)\in Q\times Q \mid (T\times T)_{h_{[I,x,w]}}\subset \mathrm{Ker}(\beta,\gamma)\Big\} \\ &= (x^{-1},w^{-1})(\mathrm{Diag}(Q_I\times Q_I)). \end{aligned}$$

The edges directed upwards from a vertex $V[I,x,w]$ correspond to $(T\times T)$-orbits whose intersection with $\Sigma_{[I,x,w]}$ (the earlier defined transverse slice at $h_{[I,x,w]}$ to $X_{[I,x,w]}$) are closed irreducible $(T\times T)_{h_{[I,x,w]}}$-invariant curves. These curves have been described in [7, Theorem 3.1] and are of two types.

Firstly, consider the transverse slice to X_I at h_I, that is $\Sigma_I\simeq\mathbb{A}^d$ where $d=|\Delta|-|I|=r-|I|$. Because $(T\times T)_{h_I}$ acts on $\mathbb{A}^d\simeq\Sigma_I$ by d independent characters [7, Section 3.1], there are exactly d closed irreducible $(T\times T)_{h_I}$-stable curves, the coordinate lines. We denote these coordinate lines $C_1(h_I),\dots,C_d(h_I)$. The translates of these curves by $(\dot{x},\dot{w})$ give us closed irreducible curves in $\Sigma_{[I,x,w]}$,

$$C_i(h_{[I,x,w]}) := (\dot{x},\dot{w})\cdot C_i(h_I).$$

Secondly, consider $\beta\in\Phi$ with U_β denoting the corresponding unipotent subgroup. If $\beta\in\Phi^-\cap x(\Phi^+)$, then $U_\beta\times 1$ does not fix $h_{[I,x,w]}$ and thus

$$C^L(h_{[I,x,w]},\beta) := (U_\beta\times 1)\cdot h_{[I,x,w]}$$

is an irreducible closed curve in $\Sigma_{[I,x,w]}$ through $h_{[I,x,w]}$ and stable by $(T\times T)_{h_{[I,x,w]}}$. Similarly, given $\beta\in\Phi^-\cap w(\Phi^-)$,

$$C^R(h_{[I,x,w]},\beta) := (1\times U_\beta)\cdot h_{[I,x,w]}$$

is an irreducible closed curve of the desired type.

By [7, Theorem 3.1] these collections of curves comprise all the $(T\times T)_{h_{[I,x,w]}}$-invariant curves in $\Sigma_{[I,x,w]}$.

Thus the edges directed upwards from the vertex $(T\times T)\cdot h_{[I,x,w]}$ will be, firstly, the orbits $(T\times T)\cdot\big(C_i(h_{[I,x,w]})\setminus\{h_{[I,x,w]}\}\big)$, and secondly the orbits $(T\times T)\cdot\big(C^L(h_{[I,x,w]},\beta)\setminus\{h_{[I,x,w]}\}\big)$ and $(T\times T)\cdot\big(C^R(h_{[I,x,w]},\beta)\setminus\{h_{[I,x,w]}\}\big)$ for appropriate β (as defined above).

We will refer to the edges of the form $(T\times T)\cdot\big(C_i(h_{[I,x,w]})\setminus\{h_{[I,x,w]}\}\big)$ as type I edges. They are actually core $(T\times T)$-orbits: each coordinate line $C_i(h_I)$ corresponds to a particular $\alpha\in\Delta\setminus I$ and

$$(T\times T)\cdot\big(C_i(h_{[I,x,w]})\setminus\{h_{[I,x,w]}\}\big)=(T\times T)\cdot(x,w)\cdot h_{I\cup\{\alpha\}}=(T\times T)\cdot h_{[I,x,w]\uparrow\alpha}$$

where $[I,x,w]\uparrow\alpha := [I\cup\{\alpha\},x',wx^{-1}x']$, and $x' := x^{I\cup\{\alpha\}}$ is the distinguished coset representative of the coset $xW_{I\cup\{\alpha\}}$. We denote this edge, $(T\times T)\cdot h_{[I,x,w]\uparrow\alpha}$, by

$$E\colon V[I,x,w]\to V[I,x,w]{\uparrow}\alpha \quad\text{or}\quad E\colon V[I,x,w]\to V[I\cup\{\alpha\},x',wx^{-1}x'],$$

and the sublattice assigned to it consists of all characters in $Q\times Q$ that annihilate the $(T\times T)$-stabilizer of $h_{[I,x,w]\uparrow\alpha}$. That is,

$$v\left(E\colon V[I,x,w]\to V[I\cup\{\alpha\},x',wx^{-1}x']\right):=((x')^{-1},(x')^{-1}xw^{-1})(\mathrm{Diag}(Q_{I\cup\{\alpha\}}\times Q_{I\cup\{\alpha\}})).$$

Note that the edge is joining vertices in different $(G\times G)$-orbits. It will turn out that all edges joining vertices in different $(G\times G)$-orbits are of this type.

We've shown, then, that each core in addition to being a vertex in the graph also functions as multiple edges, from vertices (cores) that are in its closure and of codimension 1. That is, given cores C' and C such that $C'\subset\overline{C}$ and $\dim(C')+1=\dim(C)$ then C and C' are joined by an edge that corresponds to C. Applying Lemma 3.10 we see that the core $(T\times T)\cdot h_{[I,x,w]}$ corresponds to the edges

$$E\colon V[I\backslash\{\alpha\},(xz)^{I\backslash\{\alpha\}},wz((xz)^{I\backslash\{\alpha\}})^{-1}xz]\to V[I,x,w]\quad\text{for each }\ \alpha\in I\ \text{ and }\ z\in\ W^{I\backslash\{\alpha\}}\cap W_I.$$

The other edges, those of the form $(T \times T) \cdot \big(C^L(h_{[I,x,w]}, \beta) \setminus \{h_{[I,x,w]}\}\big)$ and $(T \times T) \cdot \big(C^R(h_{[I,x,w]}, \beta) \setminus \{h_{[I,x,w]}\}\big)$, will be called type II edges. They are not cores, rather $(T\times T)\cdot \big(C^L(h_{[I,x,w]}, \beta) \setminus \{h_{[I,x,w]}\}\big)$ is an orbit that contains $(T\times T)\cdot h_{[I,x,w]}$ in its closure and is contained in the $(B \times B)$-orbit $(B \times B) \cdot (r_\beta x, w) \cdot h_I$ where r_β is the reflection corresponding to the positive root $-\beta$. Thus $(T \times T) \cdot \big(C^L(h_{[I,x,w]}, \beta) \setminus \{h_{[I,x,w]}\}\big)$ is the edge

$$E\colon V[I,x,w] \to V[I,(r_\beta x)^I, w].$$

The sublattice assigned to this edge is the elements of $Q\times Q$ that annihilate the $(T\times T)$-stabilizer of any point in the orbit $(T \times T) \cdot \big(C^L(h_{[I,x,w]}, \beta) \setminus \{h_{[I,x,w]}\}\big)$. This sublattice is

$$v\left(E\colon V[I,x,w] \to V[I,(r_\beta x)^I, w]\right) := (x^{-1}, w^{-1})(\mathrm{Diag}(Q_I \times Q_I)) + \mathrm{Span}_{\mathbb{Z}}((\beta, 0)).$$

Similarly, $(T \times T) \cdot \big(C^R(h_{[I,x,w]}, \beta) \setminus \{h_{[I,x,w]}\}\big)$ is the edge

$$E\colon V[I,x,w] \to V[I,x,r_\beta w].$$

with associated sublattice

$$v\left(E\colon V[I,x,w] \to V[I,x,r_\beta w]\right) := (x^{-1}, w^{-1})(\mathrm{Diag}(Q_I \times Q_I)) + \mathrm{Span}_{\mathbb{Z}}((0, \beta)).$$

Clearly type II edges are always joining vertices in the same $(G \times G)$-orbit. Thus edges between vertices in different $(G\times G)$-orbits are precisely the type I edges and edges between vertices in the same $(G \times G)$-orbit are type II edges. We have now completed the description of the generalized moment graph $\mathcal{G}_X$.

Example 4.8. The generalized moment graph for the wonderful compactification of $\mathrm{PGL}_2(\mathbb{C})$ is given below. In this case $\Phi \simeq A_1$ and so Δ consists of only one element, which corresponds to the solitary simple reflection $s \in W$.

The subgraph $\mathcal{G}_{X_I}$ of $\mathcal{G}_X$ obtained by restricting our attention to the orbit X_I may be described as follows. Its vertices are in bijection with the set $(W \times W)/\mathrm{Diag}(W_I \times W_I)$,

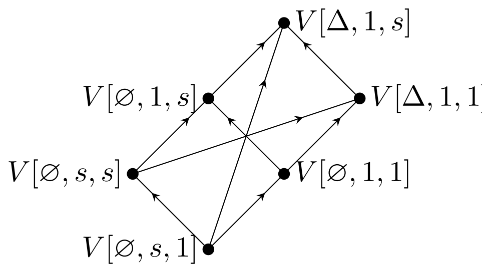

Figure 1. Generalized Moment graph for the wonderful compactification of $\mathbf{PGL}_2(\mathbb{C})$.

being in bijection with $(B \times B)$-orbits of X_I. Its edges correspond to the reflections in $(W \times W)/\mathrm{Diag}(W_I \times W_I)$, by which we mean all elements of the form $\overline{(r,1)}$ and $\overline{(1,r)}$ where $r \in W$ is a reflection. The set of reflections acts on $(W \times W)/\mathrm{Diag}(W_I \times W_I)$ on the left and vertices $\overline{(x_1, y_1)}$ and $\overline{(x_2, y_2)}$ are joined by an edge if and only if $\overline{(x_1, y_1)} = \overline{(r,1)}\,\overline{(x_2, y_2)}$ or $\overline{(x_1, y_1)} = \overline{(1,r)}\,\overline{(x_2, y_2)}$.

4.5 Statement of the Main Theorem

Our main result is that in the particular case where our generalized moment graph $\mathcal{G}_X$ is obtained from a wonderful compactification X, as described in Section 4.4, the indecomposable Braden-MacPherson sheaves on $\mathcal{G}_X$ actually encode the $(T \times T)$-equivariant intersection cohomology of the $(B \times B)$-orbit closures.

Definition 4.9. *For each vertex (core) $w \in \mathcal{V}$ there is an* ***intersection cohomology sheaf*** *$\mathscr{I}(w)$ on $\mathcal{G}_X$ defined as follows. For each vertex/edge u not in the subgraph $(\mathcal{G}_X)_{\leq w}$ (composed of vertices lower than or equal to w and the edges between them), $\mathscr{I}(w)^u := 0$. For each vertex/edge u in $(\mathcal{G}_X)_{\leq w}$,*

$$\mathscr{I}(w)^u := IH^*_{T \times T}\big(\overline{(B \times B)w}\big)_u$$

which is an A_u-module. (Recall that $A_u \simeq H^*_{T \times T}(u)$.)

For each edge $E\colon x \to y$ (within the subgraph $(\mathcal{G}_X)_{\leq w}$), the map $\rho_{x,E}$ is the composition

$$IH^*_{T\times T}\big(\overline{(B\times B)w}\big)_x \xleftarrow{\sim} IH^*_{T\times T}\big(\overline{(B\times B)w}\big)_N \to IH^*_{T\times T}\big(\overline{(B\times B)w}\big)_E$$

where the maps are the pullbacks by the inclusions of x and E into a normal space N of x as defined in Proposition 3.11. The first map is an isomorphism because N contracts onto x.

The map $\rho_{y,E} : \mathscr{I}^y \to \mathscr{I}^E$ is the composition

$$IH^*_{T\times T}\big(\overline{(B\times B)w}\big)_y \xleftarrow{\sim} IH^*_{T\times T}\big(\overline{(B\times B)w}\big)_{(B\times B)y} \to IH^*_{T\times T}\big(\overline{(B\times B)w}\big)_E$$

where the maps are the pullbacks by the inclusions of y and E into the orbit $(B\times B)y$. The first map is an isomorphism because $(B\times B)y$ contracts onto y. If E is a type I edge then $y = E$ and $\rho_{y,E}$ is the identity.

Proposition 4.10. *The intersection cohomology sheaves $\mathscr{I}(w)$ are rigid.*

Proof. This follows from the next proposition and the degree vanishing conditions for local intersection cohomology and compactly supported intersection cohomology. □

Proposition 4.11 ([6, Lemma 1.6],[1, Remark 1.8])**.** *A sheaf $\mathscr{F}$ on $\mathcal{G}$ is rigid if and only if for each $x \in \mathcal{V}$ there exists a number d so that $\mathscr{F}^x$ is generated in degrees $\leq d$ and $ker(\Gamma(\mathcal{V}_{\geq x}, \mathscr{F}) \to \Gamma(\mathcal{V}_{>x}, \mathscr{F}))$ -where the map is restriction- is generated in degrees $> d$.*

Our main result is that the indecomposable Braden-MacPherson sheaves $\mathscr{B}(w)$ compute intersection cohomology. This is stated more carefully in the two subsequent theorems. Given $w \in \mathcal{V}$ and an open subset $\mathcal{U} \subset \mathcal{V}$ we will use the notations $Z := \overline{(B\times B)w}$ and $Z_{\mathcal{U}} := \bigcup_{y\in\mathcal{U}\cap\mathcal{V}_{\leq w}} (B\times B)y$. Note that the union of strata $Z_{\mathcal{U}}$ is an open subset of Z.

Theorem 4.12. *The space of sections of $\mathscr{I}(w)$ over an open subset $\mathcal{U} \subset \mathcal{V}$ via the restriction maps $IH^*_{T\times T}(Z_{\mathcal{U}}) \to IH^*_{T\times T}(Z_{\mathcal{U}})_u$ is the intersection cohomology of the union of strata $Z_{\mathcal{U}}$:*

$$IH^*_{T\times T}(Z_{\mathcal{U}}) = \Gamma(\mathcal{U}, \mathscr{I}(w)).$$

This is analogous to GKM localization as described in Section 2.5.

Theorem 4.13. *The intersection cohomology sheaf $\mathscr{I}(w)$ is canonically isomorphic to the Braden-MacPherson sheaf $\mathscr{B}(w)$.*

The next section is devoted to the proofs of these theorems.

Corollary 4.14. *The $(T \times T)$-equivariant intersection cohomology of $(B \times B)$-orbit closures are given by global sections of the intersection cohomology sheaf,*

$$IH^*_{T\times T}\left(\overline{X_{[I,x,w]}}\right) = \Gamma(\mathscr{I}\left(V[I,x,w]\right)).$$

The $(T \times T)$-equivariant intersection cohomology of X is given by global sections of the intersection cohomology sheaf corresponding to the top most vertex of $\mathcal{G}_X$,

$$IH^*_{T\times T}(X) = \Gamma(\mathscr{I}\left(V[\Delta, 1, w_\Delta]\right)).$$

Since X is nonsingular this is also gives its $(T \times T)$-equivariant cohomology.

4.6 Proofs of the Main Theorems

4.6.1 Preliminaries: The Local Calculation

Given a core $(T \times T)$-orbit x and a $(B \times B)$-invariant locally closed subset $Y \subset X$ that contains x we want to establish the existence of transverse slices, normal spaces and contracting open sets to $(B \times B)x$ in Y. Let x_b denote the basepoint of x and Σ_x be the particular transverse slice to $(B \times B)x_b$ described in Proposition 3.8. This slice is $(T \times T)_{x_b}$-invariant and isomorphic to affine space. By Proposition 3.11 we obtain from this transverse slice a normal space $N_x := (T \times T) \cdot \Sigma_x$ to $(B \times B)x_b$ along x. Additionally, by Lemma 3.13 we know that acting on this normal space by an appropriate subset of $U^+ \times U^+$ we obtain an open $(T \times T)$-invariant subvariety O_x that contracts onto N_x.

Intersecting with Y gives us our desired subsets. $\Sigma := \Sigma_x \cap Y$ is a transverse slice in Y to $(B \times B)x$ at x_b. It is $(T \times T)_{x_b}$-invariant since both Σ_x and Y are, and it is attractive since the $\mathbb{C}^\times$ contracting Σ onto x_b is contained in $(T\times T)_{x_b}$ and thus preserves any $(T\times T)_{x_b}$-invariant set.

Likewise $N := N_x \cap Y = (T\times T)\Sigma$ is an attractive normal space (in Y) to $(B\times B)x$ along x and $O := O_x \cap Y$ is a $(T\times T)$-invariant open subset of Y that contracts onto N. (Recall that the contracting action on O_x preserves $(B\times B)$-orbits.)

Now we can present a version of the fundamental local calculation for intersection cohomology.

Lemma 4.15. *$IH^*_{T\times T}(N)$ is a projective cover of $IH^*_{T\times T}(N\setminus x)$.*

Proof. As in Proposition 3.12 we choose a subtorus $(T\times T)^\perp$ such that $T\times T$ is the direct product of $(T\times T)^\perp$ and $(T\times T)_{x_b}$. We take the quotient by the action of this subtorus to reduce the proof to a calculation in the transverse slice Σ.

We know by the proof of Proposition 3.12 that $(T\times T)^\perp$ acts freely on N_x. Consequently its action on the ($(T\times T)$-invariant) subsets $N := N_x\cap Y$ and $N\setminus x$ is also free. Thus we have well defined quotient varieties $N/(T\times T)^\perp$ and $(N\setminus x)/(T\times T)^\perp$ which, via the isomorphism $N_x/(T\times T)^\perp \simeq \Sigma_x$ from Proposition 3.12, are isomorphic to Σ and $\Sigma\setminus\{x_b\}$ respectively.

Let $f : N \to N/(T\times T)^\perp$ and $\phi : T\times T \to (T\times T)/(T\times T)^\perp$ be the quotient maps, keeping in mind that $(T\times T)/(T\times T)^\perp \simeq (T\times T)_{x_b}$. Then ϕ is a surjective homomorphism with kernel $(T\times T)^\perp$ and f is ϕ-equivariant. Since $(T\times T)^\perp$ acts freely on N, Theorem 2.18 is applicable and by it we obtain isomorphisms

$$\begin{aligned} IH^*_{T\times T}(N) &\simeq IH^*_{(T\times T)_{x_b}}(\Sigma)\ , \text{ and} \\ IH^*_{T\times T}(N\setminus x) &\simeq IH^*_{(T\times T)_{x_b}}(\Sigma\setminus x_b). \end{aligned} \tag{4.1}$$

Since there is a subtorus $\mathbb{C}^\times \subset (T\times T)_{x_b}$ contracting Σ to $\{x_b\}$ it is immediate from Theorem 2.20 that $IH^*_{(T\times T)_{x_b}}(\Sigma)$ is the projective cover of $IH^*_{(T\times T)_{x_b}}(\Sigma\setminus x_b)$. □

Another lemma that will be useful is the following.

Lemma 4.16. *The restriction*

$$IH^*_{T\times T}(N\setminus x)\to \bigoplus_{E\in\mathcal{E}_{\partial x},E\subset Y} IH^*_{T\times T}(N\setminus x)_E.$$

is an injection.

Proof. As in the preceeding proof we invoke Theorem 2.18 to reduce this to a calculation in the transverse slice Σ. Namely, since we have established the isomorphisms of Equation (4.1) we just need to show that $IH^*_{(T\times T)_{x_b}}(\Sigma\setminus x_b)\to \bigoplus_{E\in\mathcal{E}_{\partial x},E\subset Y} IH^*_{(T\times T)_{x_b}}(\Sigma\setminus x_b)_{E\cap\Sigma}$ is injective. The $\mathbb{C}^\times$ that contracts Σ onto x_b acts with finite stabilizers on both $\Sigma\setminus x_b$ and the $(T\times T)_{x_b}$-invariant curves it contains, which are the curves $E\cap\Sigma$ for each $E\in\mathcal{E}_{\partial x}$ that is contained in Y. Thus it suffices to show, again applying Theorem 2.18, that $IH^*_{(T\times T)_{x_b}/\mathbb{C}^\times}(\Sigma\setminus x_b)/\mathbb{C}^\times)\to \bigoplus_{E\in\mathcal{E}_{\partial x},E\subset Y} IH^*_{(T\times T)_{x_b}/\mathbb{C}^\times}(\Sigma\setminus x_b)/\mathbb{C}^\times)_{(E\cap\Sigma)/\mathbb{C}^\times}$ is injective.

Since the quotient $(\Sigma\setminus x_b)/\mathbb{C}^\times$ is projective, and the fixed locus of the $(T\times T)_{x_b}/\mathbb{C}^\times$ action is the points $(E\cap\Sigma)/\mathbb{C}^\times$, for $E\in\mathcal{E}_{\partial x}$ that are contained in Y, the restriction homomorphism is injective by the localization theorem.

□

For our two main theorems, recall that we are considering fixed $w\in\mathcal{V}$ and are concerned with relating $\mathscr{I}(w)^x$ to $\Gamma(\mathcal{V}_{>x},\mathscr{I}(w))$, where $x\le w$.

As before we define $Z:=\overline{(B\times B)w}$ and for an open subset $\mathcal{U}\subset\mathcal{V}$ of vertices,

$$Z_{\mathcal{U}}:=\bigcup_{y\in\mathcal{U}\cap\mathcal{V}_{\le w}}(B\times B)y$$

is an open subset of Z.

Of particular interest are open sets of the form $\mathcal{V}_{>x}$ and $\mathcal{V}_{\ge x}$. Corresponding to these we will have the open subsets

$$Z_{>x}:=Z_{\mathcal{V}_{>x}}=\bigcup_{w\ge y>x}(B\times B)y,\ \text{ and}$$

$$Z_{\ge x}:=Z_{\mathcal{V}_{\ge x}}=\bigcup_{w\ge y\ge x}(B\times B)y=Z_{>x}\cup(B\times B)x.$$

4.6.2 Proof of Theorem 4.12

The only relevant open sets are those that intersect $\mathcal{V}_{\leq w}$. The proof is inductive, starting at w and working down the partial order. Given an open set $\mathcal{U}$ whose intersection with $\mathcal{V}_{\leq w}$ is $\{w\}$,

$$IH^*_{T\times T}((B\times B)w) \simeq IH^*_{T\times T}(Z)_{(B\times B)w} \simeq IH^*_{T\times T}(Z)_w = \mathscr{I}(w)^w \simeq \Gamma(\mathcal{U}, \mathscr{I}(w))$$

because $(B\times B)w$ contracts onto w and $\mathscr{I}(w)^x = 0$ for all $x \notin \mathcal{V}_{\leq w}$.

Now suppose the property holds for an open set $\mathcal{U}$, and $x \in \mathcal{V}_{\leq w}$ is such that $\mathcal{U}\cup\{x\}$ is also open. We show the property also holds for $\mathcal{U}\cup\{x\}$.

Since the $(B\times B)$-invariant set $Z_{\mathcal{U}\cup\{x\}}$ is open we know from Section 4.6.1 that we have in $Z_{\mathcal{U}\cup\{x\}}$ an attractive normal space N to $(B\times B)x$ along x and a $(T\times T)$-invariant set O that is open in $Z_{\mathcal{U}\cup\{x\}}$ and contracts onto N. We establish that the property in Theorem 4.12 holds for $Z_{\mathcal{U}\cup\{x\}}$ by showing that in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & IH^*_{T\times T}(Z_{\mathcal{U}\cup\{x\}}) \arrow[r] \arrow[d] & IH^*_{T\times T}(Z_{\mathcal{U}}) \oplus IH^*_{T\times T}(O) \arrow[r] \arrow[d, "\wr"] & IH^*_{T\times T}(O\setminus x) \arrow[d] \\
0 \arrow[r] & \Gamma(\mathcal{U}\cup\{x\}, \mathscr{I}(w)) \arrow[r] & \Gamma(\mathcal{U}, \mathscr{I}(w)) \oplus \mathscr{I}(w)^x \arrow[r] & \bigoplus_{E\in\mathcal{E}_{\partial x}} \mathscr{I}(w)^E
\end{tikzcd}$$

the third vertical homomorphism is an isomorphism, the rightmost vertical homomorphism is injective, and the rows are exact.

The third vertical homomorphism is an isomorphism by our induction assumption and the fact that O contracts N which contracts onto x.

The rightmost vertical map is a composition of injective maps, namely

$$IH^*_{T\times T}(O\setminus x) \xrightarrow{\sim} IH^*_{T\times T}(N\setminus x) \to \bigoplus_{\substack{E\in\mathcal{E}_{\partial x}\\ E\subset Z_{\mathcal{U}\cup\{x\}}}} IH^*_{T\times T}(N\setminus x)_E$$

and

$$\bigoplus_{\substack{E\in\mathcal{E}_{\partial x}\\ E\subset Z_{\mathcal{U}\cup\{x\}}}} IH^*_{T\times T}(N\setminus x)_E \xleftarrow{\sim} \bigoplus_{\substack{E\in\mathcal{E}_{\partial x}\\ E\subset Z_{\mathcal{U}\cup\{x\}}}} IH^*_{T\times T}(Z)_E \subset \bigoplus_{E\in\mathcal{E}_{\partial x}} IH^*_{T\times T}(Z)_E = \bigoplus_{E\in\mathcal{E}_{\partial x}} \mathscr{I}(w)^E.$$

Their injectivity follows from Lemma 4.16 (for the second map) and the fact that the inclusions $N \setminus x \subset O \setminus x$ and $O \setminus x \subset Z$ are normally nonsingular and $O \setminus x$ contracts onto $N \setminus x$.

The top row is exact because it is part of the Mayer-Vietoris long exact sequence of the triple $(Z_{\mathcal{U}\cup\{x\}}, Z_{\mathcal{U}}, O)$.

As for the bottom row, the second map, which consists of a pair of restriction morphisms, is obviously injective. Exactness at the third module is also immediate: a pair (s,t), where $s = (s_y)_{y\in\mathcal{U}} \in \Gamma(\mathcal{U}, \mathscr{I}(w))$ and $t \in \mathscr{I}(w)^x$, maps to $0 \in \mathscr{I}(w)^{\partial x}$ if and only if $\rho_{y,E}(s_y) = \rho_{x,E}(s_)$ for all $E\colon x \to y$ in $\mathcal{E}_{\partial x}$, which exactly means that (s,t) is in the image of the leftmost map. (That is, the tuple $s' = (s'_y)_{y\in\mathcal{U}\cup\{x\}} \in \bigoplus_{y\in\mathcal{U}\cup\{x\}} \mathscr{I}(w)^y$, defined by $s'_y = s_y$ for $y \in \mathcal{U}$ and $s'_x = t_x$, is a section over $\mathcal{U} \cup \{x\}$ that clearly maps to (s,t).)

It follows from the four lemma that the second vertical map is surjective. It is easy to see that it is also injective. Thus the statement of the theorem holds for $\mathcal{U} \cup \{x\}$, completing the induction.

4.6.3 Proof of Theorem 4.13

We construct and isomorphism $\mathscr{I}(w) \simeq \mathscr{B}(w)$ and because $\mathscr{I}(w)$ is rigid this isomorphism must be canonical.

The proof will be inductive, working down the partial order. We will establish isomorphisms between the vertex (and edge) modules of $\mathscr{I}(w)$ and $\mathscr{B}(w)$ and it will be clear that these isomorphisms commute with the maps ρ on each sheaf.

For $u \in \mathcal{V}$ such that $u \not\leq w$ we have $\mathscr{I}(w)^u = 0 = \mathscr{B}(w)^u$. At w,

$$\mathscr{I}(w)^w = IH^*_{T\times T}(Z)_w \simeq H^*_{T\times T}(w) \simeq A_w = \mathscr{B}(w)^w$$

since the $(T \times T)$-orbit w is in the smooth part of $Z = \overline{(B \times B)w}$.

Now consider an edge $E\colon x \to y$ in $(\mathcal{G}_X)_{\leq w}$, with $\mathscr{I}(w)^y \simeq \mathscr{B}(w)^y$ already established.

The edge E is a $(T \times T)$-orbit contained within the single stratum $(B \times B)y$ so

$$IH^*_{T\times T}(Z)_E \simeq IH^*_{T\times T}(Z)_{(B\times B)y}/\langle v(E)\rangle IH^*_{T\times T}(Z)_{(B\times B)y},$$

since $\langle v(E)\rangle$ is the annihilator in A of the $(T \times T)$-stabilizer of E. (Recall that $A \simeq \mathrm{Sym}(\mathfrak{t}^* \times \mathfrak{t}^*)$.) Furthermore $IH^*_{T\times T}(Z)_{(B\times B)y} \simeq IH^*_{T\times T}(Z)_y$ since $(B \times B)y$ contracts onto y. Therefore,

$$\begin{aligned}\mathscr{I}(w)^E = IH^*_{T\times T}(Z)_E &\simeq IH^*_{T\times T}(Z)_{(B\times B)y}/\langle v(E)\rangle IH^*_{T\times T}(Z)_{(B\times B)y}\\ &\simeq IH^*_{T\times T}(Z)_y/\langle v(E)\rangle IH^*_{T\times T}(Z)_y\\ &\simeq \mathscr{I}(w)^y/\langle v(E)\rangle \mathscr{I}(w)^y\\ &\simeq \mathscr{B}(w)^y/\langle v(E)\rangle \mathscr{B}(w)^y\\ &\simeq \mathscr{B}(w)^E.\end{aligned}$$

Finally, we consider a vertex x such that for each edge $E\colon x \to y$ in $\mathcal{V}_{\partial x}$ we know $\mathscr{I}(w)^E \simeq \mathscr{B}(w)^E$ and furthermore for any vertex or edge z in the subgraph $(\mathcal{G}_X)_{>x}$ we have established $\mathscr{I}(w)^z \simeq \mathscr{B}(w)^z$, all isomorphism commuting with the relevant maps ρ. It follows that $\mathscr{I}(w)^{\partial x} \simeq \mathscr{B}(w)^{\partial x}$. We want to show that $\mathscr{I}(w)^x \simeq \mathscr{B}(w)^x$ and since $\mathscr{B}(w)^x$ is the projective cover of $\mathscr{B}(w)^{\partial x}$ it suffices to show that $\mathscr{I}(w)^x$ is the projective cover of $\mathscr{I}(w)^{\partial x}$.

Applying the discussion of Section 4.6.1 to the closed (and hence locally closed) set Z we see that we have a normal space, N, of x in Z. This normal space contracts onto x and its inclusion into Z is normally nonsingular. Thus $\mathscr{I}(w)^x = IH^*_{T\times T}(Z)_x \simeq IH^*_{T\times T}(Z)_N \simeq IH^*_{T\times T}(N)$. We know by Lemma 4.15 that $IH^*_{T\times T}(N)$ is the projective cover of $IH^*_{T\times T}(N \setminus x)$ so to complete the proof it suffices to show that $IH^*_{T\times T}(N \setminus x) \simeq \mathscr{I}(w)^{\partial x}$.

The overall strategy is the same as that used in [5] to prove their main theorem (Theorem 1.8); we show that in the following diagram

$$
\begin{tikzcd}
IH^*_{T\times T}(Z_{>x}) \arrow[r, "\alpha"] \arrow[d, "\gamma"] & \Gamma(\mathcal{V}_{>x}, \mathscr{I}(w)) \arrow[d, "\phi"] \\
IH^*_{T\times T}(N \setminus x) \arrow[r, "\beta"] & \bigoplus\limits_{\substack{E\in\mathcal{E}_{\partial x} \\ E\subset Z}} \mathscr{I}(w)^E
\end{tikzcd}
\tag{4.2}
$$

β is injective and α and γ are surjections. (Here β, α and γ are restriction homomorphisms and ϕ is the composition of the restriction $\Gamma(\mathcal{V}_{>x}, \mathscr{I}(w)) \to \Gamma(\mathcal{V}_{\partial x}, \mathscr{I}(w))$ and $\bigoplus\limits_{E} \rho_{y,E}$, the direct sum ranging over all edges $E\colon x \to y$ in $\mathcal{E}_{\partial x}$ that are contained in Z).

The map α is in fact an isomorphism by Theorem 4.12.

That β is injective is immediate from Lemma 4.16.

Showing γ is surjective is slightly more involved, requiring us to use results from Hodge theory.

Lemma 4.17. *The map γ in the commutative diagram 4.2 is surjective.*

Proof. We will prove this by showing that the restriction maps γ_2 and γ_1 in the diagram

$$
\begin{tikzcd}
IH^*_{T\times T}(Z) \arrow[r] \arrow[d, "\gamma_1"] & IH^*_{T\times T}(Z_{>x}) \arrow[d, "\gamma"] \\
IH^*_{T\times T}(N) \arrow[r, "\gamma_2"] & IH^*_{T\times T}(N \setminus x)
\end{tikzcd}
$$

are both surjective.

The surjectivity of γ_2 is part of Lemma 4.15: $IH^*_{T\times T}(N)$ is the projective cover of $IH^*_{T\times T}(N \setminus x)$.

Now recall from Section 4.6.1 that we may choose a $(T\times T)$-equivariant open set $O \subset Z$ that contracts onto N. We express γ_1 as the composition of restrictions $IH^*_{T\times T}(Z) \to IH^*_{T\times T}(O) \simeq IH^*_{T\times T}(N)$ and show that the first map is a composition of two identities and a surjection:

$$IH^*_{T\times T}(Z) = HIH^*_{T\times T}(Z) \twoheadrightarrow HIH^*_{T\times T}(O) = IH^*_{T\times T}(O).$$

Since Z is a projective variety, $IH^*_{T\times T}(Z) = HIH^*_{T\times T}(Z)$ by Proposition 2.25. Since O is an open $(T \times T)$-invariant subvariety of Z, we have the surjection $HIH^*_{T\times T}(Z) \twoheadrightarrow HIH^*_{T\times T}(O)$ by Theorem 2.26.

Finally, we want to show $HIH^*_{T\times T}(O) = IH^*_{T\times T}(O)$. Recall Equation (4.1), $IH^*_{T\times T}(N) \simeq IH^*_{(T\times T)_{x_b}}(\Sigma)$. Since Σ contracts onto a point, $IH^*_{(T\times T)_{x_b}}(\Sigma) = HIH^*_{(T\times T)_{x_b}}(\Sigma)$ by Proposition 2.25 and applying the isomorphism of Equation (4.1) we get $IH^*_{T\times T}(N) = HIH^*_{T\times T}(N)$. Applying the isomorphism $IH^*_{T\times T}(O) \simeq IH^*_{T\times T}(N)$ (from the contraction of O onto N) we deduce that $HIH^*_{T\times T}(O) = IH^*_{T\times T}(O)$.

□

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