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The acceleration and dissolution of stars moving through the blackbody radiation of a collapsing universe.

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THE ACCELERATION AND DISSOLUTION OF STARS MOVING THROUGH THE BLACKBODY RADIATION OF A COLLAPSING UNIVERSE

A Dissertation presented

by

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THE ACCELERATION AND DISSOLUTION OF STARS
MOVING THROUGH THE BLACKBODY RADIATION OF A COLLAPSING UNIVERSE

September 1986

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This dissertation deals with the motion and ablation of stars in the collapse phase of a closed Friedmann universe. Stars are initially accelerated due to the collapse of space. Radiation drag becomes increasingly important, however, and in most of the cases considered leads to maximum speeds and rapid deceleration. The external blackbody radiation also leads to mass loss, which acts as an additional accelerating mechanism.

Three species of degenerate stars are considered: black dwarfs (BD), white dwarfs (WD), and neutron stars (NS). Each is assumed to have a non-degenerate, ionized atmosphere. In the star's rest frame the external blackbody radiation appears highly anisotropic, with most of the radiation entering the atmosphere through a narrow cone centered on the forward direction (opposite to the direction of motion). This radiation is Compton scattered. Atmospheric electrons (and hence ions) are accelerated azimuthally. After having travelled about one quarter of a circumference, they detach themselves from the star and

stream away. The atmosphere is constantly replenished by upwelling from the interior. Mass loss then is a result of mechanical forces and is not due to thermal boiling.

Four optical depths are considered for each species: 0, 1, 2, and 3. Maximum speeds characterized by $\gamma = 4.5\times10^3$, 1.7×10^4, 8.4×10^4, 1.2×10^5, 2.1×10^5, 8.1×10^7, 1.1×10^8, 1.3×10^8, and 1.5×10^8 are achieved at temperatures of 1.1×10^8 K, 1.2×10^8 K, 2.1×10^9 K, 2.1×10^9 K, 2.6×10^9 K, 2.0×10^{12} K, 2.3×10^{12} K, 2.8×10^{12} K, and 2.8×10^{12} K for BD0, BD1, WD0, WD1, WD2, NS0, NS1, NS2, and NS3 respectively, where $\gamma = (1 - v^2/c^2)^{-1/2}$. BD2, BD3, and WD3 never achieve maximum speeds because in these cases mass loss is too rapid. The model and points where it is expected to break down are described in detail.

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INTRODUCTION

Much study has been devoted to the formation of structure in the universe but comparatively little to its dissolution (Rees 1969). In closed Friedmann models dissolution mirrors formation only in the sense that the most tenuous structures tend to be the last created and the first destroyed.

In the collapse phase of the universe the average distance between groups of galaxies shrinks. One can single out a unique frame of reference, the comoving frame, in which the resulting galactic blue shifts appear isotropic. The blueshifts of individual groups of galaxies may deviate from the average blueshift at a given distance. This deviation reflects the group's peculiar velocity and is not due to the collapse of space.

Dissolution begins with the merging of clusters when the universal scale factor has shrunk by a factor of five and the temperature of the background radiation has risen to 15 K. Galaxies, which have peculiar velocities of order 300 km s^{-1}, begin to accelerate. Acceleration occurs because the gravitationally non-interacting galaxies always overtake comoving observers that move towards them. When the temperature of the radiation has reached 150 K and these velocities have increased to 3000 km s^{-1}, the average density of the universe becomes comparable to the mean density of galaxies ($\cong 10^{-24}$ g cm^{-3}). Galaxies release their constituent stars with peculiar velocities of order 3000 km s^{-1}. The stars accelerate and by 10^4 K become relativistic.

Acceleration continues, but, as the radiation field intensifies, they are slowly whittled away. Eventually, the universe contains only radiation, neutrinos, gas, and black holes. In this paper attention is focused mainly on the motion and ablation of stars in the radiation field of a collapsing universe.

We consider three species of stars: black dwarfs, white dwarfs, and neutron stars. In the first chapter these stars are taken to be structureless, indestructible spheres. Their motions are initially dominated by the acceleration due to universal collapse. Radiation drag becomes increasingly important, however, and eventually comparable to the acceleration. Maximum speeds are achieved, followed by rapid deceleration.

In the second chapter stellar ablation is taken into account. Stars now have an atmosphere, an interior, and a 'surface' separating the two. Radiation, which appears anisotropic in the star's rest frame, enters the atmosphere. Some is scattered by atmospheric electrons and some travels straight to the surface. The azimuthal component of scattered radiation accelerates atmospheric material. This material then leaves the star, causing it to accelerate to a higher velocity than would be attained if the collapse of space were the only means of acceleration. Radiation that reaches the surface either directly or indirectly via particle collisions acts as a decelerating force. When and if the drag force finally becomes comparable to the acceleration, deceleration occurs, but more rapidly than before.

In the third chapter miscellaneous topics are discussed. It was assumed in the calculation of chapters I and II that collisions are unimportant. In this chapter we show that such an assumption is justified. Additional contributions to deceleration (or acceleration) are also discussed. These include changes in the mass-radius relation, e^+e^- pair creation, and the presence of gas. Finally, we consider the possibility of supermassive black holes consuming large numbers of stars.

CHAPTER I

STELLAR MOTIONS ASSUMING NO MASS LOSS

The Stellar Model

As a first approximation, we assume that stars are structureless, indestructible spheres. We make no distinction between atmosphere and interior. It is only the total mass and radius that concern us here, and these remain constant throughout the period of interest. No distortion in shape is assumed to occur, even though the external radiation field appears anisotropic in the rest frame of the star.

We also assume that all stars are non-rotating. Observations indicate that most white dwarfs rotate slowly, if at all (Shapiro and Teukolsky 1983). Neutron stars (pulsars) are assumed to lose most of their rotational energy during the hundreds of millions of years that elapse between the dissolution of galaxies and the onset of relativistic motion in the stars. No new star formation is assumed to occur once the galaxies have dissolved.

In addition, stellar magnetic fields are ignored. Main sequence stars have typical surface magnetic fields of order 100 G (Shapiro and Teukolsky 1983). As they collapse to form the various types of degenerates, the strengths of the magnetic fields increase according to the inverse square of the radii. Hence, a neutron star whose radius is five orders of magnitude smaller than that of a typical main sequence star such as the Sun will have a magnetic field strength of order 10^{12} G.

These fields are thought to decay slowly. A homogeneous neutron star, for example, will lose its magnetic field in about 10^6 years. The decay time depends upon the detailed interactions between the electrons and other matter in the interior and can be much longer for more realistic models. White dwarfs, on the other hand, are for the most part non-magnetic. Only a few percent have significant magnetic fields and only a handful have magnetic fields as high as 10^8 G.

Drag Force Due to Background Radiation

In Cosmology there is a unique or absolute frame of reference, the comoving frame. This frame is distinguished from all others by the isotropy of matter and radiation. Properties, such as the radiation temperature, depend only upon how much time has elapsed since the onset of collapse. The star's rest frame, on the other hand, presents a different picture. It singles out a particular direction, the direction of motion. The observer still sees blackbody radiation wherever he or she looks, but its temperature is now angle dependent. Radiation coming in along the forward direction (opposite to the direction of motion) appears hottest and that coming in along the backward direction coolest. This radiation can be described by the anisotropic intensity distribution (Heer and Kohl 1968):

$$I(\bar{\theta})d\bar{\Omega} = \frac{\sigma/\pi \; T^4 \; d\bar{\Omega}}{\gamma_s{}^4(1 - \beta_s \cos\bar{\theta})^4} \cdot \qquad (1)$$

The angle $\bar{\theta}$ is measured from the forward direction (Fig. 1). T is the temperature of the blackbody radiation in the comoving frame, $v_s = \beta_s c$

the speed of the star through the comoving frame, where $\gamma_s = (1-\beta_s^2)^{-1/2}$, and σ the Stefan-Boltzmann constant.

The anisotropic intensity of equation (1) gives rise to a drag force:

$$\vec{f} = -\frac{1}{c} \int\int I(\bar{\theta}) \cos\theta' \cos\bar{\theta} d\bar{\Omega} d\Omega' R^2 \hat{z}. \tag{2}$$

The $\cos\theta'$ factor (Fig. 1) is always present when radiation does not enter normally and is due to the fact that an area viewed at an angle appears smaller than an area viewed face on. The $\cos\bar{\theta}$ factor gives the component of force in the forward ($-\hat{z}$) direction. R is the star's radius and c is the speed of light.

Substitution of equation (1) into equation (2) gives

$$\vec{f} = -\frac{1}{c}\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2\pi}\int_0^{\pi} \left[\frac{\sigma/\pi \; T^4}{\gamma_s^4(1 - \beta_s \cos\bar{\theta})^4}\right] \cos\bar{\theta}\sin\bar{\theta}d\bar{\theta}d\bar{\phi}$$
$$\cos\theta'\sin\theta'd\theta'd\phi' \; R^2 \hat{z}.$$

Integration is first performed over all inward normals ($-\hat{n}$) (keeping $\bar{\theta}$ and $\bar{\Phi}$ fixed) and then over all photon directions ($\hat{\Omega}$) for a given normal direction. We find

$$\vec{f} = -\frac{16\sigma T^4 \beta_s \gamma_s^2}{3c} \pi R^2 \hat{z}. \tag{3}$$

This is the relativistic drag force. Since it depends only upon the relative motion of the star and the radiation through which it moves, it can be calculated in either frame. We chose the star's rest frame because only in this frame does the star appear spherical. It is far easier to describe the anisotropic intensity distribution that occurs in the star's rest frame than to determine the star's shape in the comoving frame. A comoving observer sees more than a simple Lorentz contraction in the direction of motion.

Consider the case of a comoving observer whose line of sight to the star is perpendicular to the direction of motion (Fig. 2A). The observer is far enough away so that, to a high degree of accuracy, all rays appear to come in parallel. Divide the star up into infinitesimally thin rings with all points on any given ring being equidistant from the observer. Each ring appears Lorentz contracted in the direction of motion. In addition, light from rings closer to the observer must leave the star later than light from more distant rings. During these time intervals, however, the star has moved so that the Lorentz contracted rings are displaced by varying amounts in the direction of motion. What the observer sees, then, is a rotated hemisphere (Fig. 2B). The angle of rotation from the line of sight is given by

$$\theta_r = \cos^{-1}(1/\gamma_s),$$

where γ_s is the relativistic factor, $(1 - \beta_s{}^2)^{-1/2}$. This means that as $v_s \rightarrow c$ ($\gamma_s \rightarrow \infty$), $\theta_r \rightarrow \pi/2$ and the hemisphere is rotated into the direction of motion.

The apparent rotation described above is seen only by distant observers. This is because the distant observer is never able to see more than 50% of the star. The nearby observer, on the other hand, begins to see the 'back' of the star after a very short time. To such an observer, the rings of equidistant points no longer appear concentric. Distortion occurs. Because this distortion is difficult to describe in general, we do not attempt an analysis of the drag force in this frame of reference.

Stellar Speeds as a Function of Temperature

We now wish to find the speed of stars through the blackbody radiation of a collapsing universe as a fuction of time (or, equivalently, radiation temperature). If mass loss is not taken into account, acceleration obeys (Weinberg 1972)

$$\frac{d^2x^\mu}{d\tau^2} = -\Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} + \frac{f^\mu}{M}. \qquad (4)$$

The rate of acceleration (or deceleration) is determined by the relative strengths of the gravitational force per unit rest mass (first term on right) and non-gravitational force per unit rest mass (second term on right). The gravitational term can be computed from the Robertson-Walker line element, which describes spacetime intervals in a universe characterized by homogeneity and isotropy. The non-gravitational term is calculated from the drag force, equation (3).

Taking the time component of equation (4) and making use of the drag force of equation (3), we find

$$\gamma_s\frac{d\gamma_s}{dt_c} = \frac{1}{T}\frac{dT}{dt_c}(\gamma_s{}^2 - 1) - \frac{16\sigma T^4\beta_s{}^2\gamma_s{}^2}{3Mc^2}\pi R^2, \qquad (5)$$

where dt_c is the time interval measured in the comoving frame.

Before solving equation (5), we eliminate the time variable by finding a relation between blackbody temperature and time. Since we are concerned mostly with high temperatures, we can write this relation as (Weinberg 1972)

$$\frac{dT'}{dt_c} = \left(\frac{8\pi G\rho_t}{3}\right)^{1/2} T', \qquad (6)$$

which holds for temperatures higher than about 10^3 K. $T' \equiv T/T_o$, where

T_o = 150 K is the temperature of the radiation when galaxies merge. ρ_t is the total energy density with contributions from both matter and radiation. When T' reaches about 10^3, radiation dominates and we can write $\rho_t \cong \rho_\gamma \propto T'^4$. Substitution into equation (6) gives

$$dt_c = \frac{dT'}{C_1 T'^3}, \qquad (7)$$

with the constant

$$C_1 = (8\pi G\rho_{\gamma o}/3)^{1/2} = 4.9\text{x}10^{-17}\ s^{-1},$$

where $\rho_{\gamma o}$ is the energy density of radiation at 150 K.

Equation (5) can now be solved for $\gamma_s(T)$:

$$\gamma_s = [C_2 T^2 \exp(-C_3 T^2) + 1]^{1/2}. \qquad (8)$$

The constant C_2 is determined from the initial (at 150 K) stellar velocity, $v_{so} = 3\text{x}10^3$ km s^{-1}, and has the value

$$C_2 = 4.4\text{x}10^{-9}\ K^{-2}.$$

The constant C_3 has the values:

$$C_3 = \frac{16\pi R^2 \sigma T_o^2}{3MC_1c^2} = 8.0\text{x}10^{-17},\ 2.3\text{x}10^{-19},\ \text{and}\ 2.5\text{x}10^{-25}\ K^{-2}$$

for black dwarfs, white dwarfs, and neutron stars respectively.

In calculating the constant C_3, it was assumed that all black dwarfs have masses of 0.1 M⊙, all white dwarfs masses of 0.7 M⊙, and all neutron stars masses of 1.4 M⊙. The radii of the various species are then determined by employing the most appropriate mass-radius relation. Black dwarfs, for example, are well described by an n = 3/2 polytrope. This implies a radius of ≅ $6.0\text{x}10^9$ cm for stars composed entirely of hydrogen (Shapiro and Teukolsky 1983). White dwarfs can be described reasonably well by Chandrasekhar's simple degenerate model, which assumes a single species of non-interacting fermion. For stars containing no hydrogen, this model gives a radius of $\cong 8.1 \times 10^8$ cm. The radii of neutron stars are more uncertain due to the uncertainty in the equation of state for densities higher than that of nuclear matter. We take an average of the six different values predicted by the models described in Baym and Pethick (1979) and find the radius 1.2×10^6 cm.

The temperatures at which maximum speeds are achieved can be found be differentiating equation (8):

$$T_{max} = (1/C_3)^{1/2}. \tag{9}$$

These temperatures can then be used to find the maximum values of the relativistic factor, γ_s:

$$\gamma_{s,max} \cong 6.1 \times 10^{-1} (C_2/C_3)^{1/2}. \tag{10}$$

One can also obtain simple expressions for γ_s in the limits of low and high temperature. In the low temperature limit, $C_3 T^2 << 1$,

$$\beta_s \gamma_s \cong (C_2)^{1/2} T, \tag{11}$$

which simply says that, initially, stellar motions are dominated by the collapse of space. Later on, however, when $C_3 T^2 > 1$, radiation drag is dominant and

$$\gamma_s \cong (C_2)^{1/2} T \exp(-C_3 T^2/2). \tag{12}$$

Numerical Results

Figure (3) shows a plot of log γ_s versus log T for the three species of stars. The linear rise (see eq. [11]) reflects the collapse of space. Radiation drag does not become noticeable until the blackbody temperature has risen to about 5.0×10^7 K, 1.0×10^9 K, and 1.0×10^{12} K for black dwarfs, white dwarfs, and neutron stars respectively. By $T = 1.1\times10^8$ K, 2.1×10^9 K, and 2.0×10^{12} K, the two opposing forces have become equal (see eq. [9]). Maximum speeds are achieved, characterized by $\gamma_s = 4.5\times10^3$, 8.4×10^4, and 8.1×10^7 (see eq. [10]). Thereafter, radiation drag dominates.

CHAPTER II

STELLAR MOTIONS TAKING MASS LOSS INTO ACCOUNT

Brief Overview

We now give the star structure: an ionized atmosphere, an interior, and a distinct surface. Radiation enters the atmosphere and is Compton scattered by ambient electrons. Numerous collisions then ensure that energy is transferred to atmospheric ions, such that both species move off with comparable speeds. The azimuthal component of particle momentum leads to mass loss. The radial component is transferred to the surface and contributes to stellar deceleration. Material always upwells from the interior at a sufficient rate to replace what is lost.

In this chapter we derive the five equations needed to find stellar speeds as a function of blackbody temperature for stars that are losing mass. Many assumptions and simplifications are made along the way, which are best discussed as they arise. The five equations are then reduced to two sets of coupled first order differential equations: one set gives stellar acceleration and rate of mass loss in the limit of low blackbody temperature, and the other gives the same things in the limit of high blackbody temperature. Solution is by the Runge-Kutta method.

The Transfer of Radiation Through the Atmosphere

The Intensity

In the first chapter we showed that the external radiation field appears anisotropic to an observer in the star's rest frame if this observer moves with respect to a comoving observer. More radiation appears to come from the forward direction than from any other, and the faster the star moves, the more pronounced this effect becomes. Once speeds have become relativistic, almost all of the entering radiation comes through a narrow cone centered on the forward direction, which means that very little radiation strikes the back of the star. Because of this, we make the assumption that the front of the star is irradiated, while the back is not.

Radiation entering the atmosphere can be absorbed, scattered, or left unaffected by the presence of matter. Since ionization levels are high at the temperatures considered, we can ignore contributions to the absorption from bound-bound and bound-free transitions. Free-free transitions are also assumed to be unimportant because, by the time atmospheric densities have become high enough (which occurs late in the calculation), temperatures have become too high. We are therefore left with electron scattering as the dominant interaction. Multiple scatterings need not be considered because photons are expected to give up most of their energy upon scattering. This happens because incident photons are much more energetic than scattering electrons. Even though electrons initially take up most of the energy entering the atmosphere, they distribute it among neighboring ions (via collisions), such that both species develop comparable velocities within a short time. This, of course, leads to ion energies which are about m_i/m_e times greater than electron energies, where m_i and m_e are the masses of the ion and electron respectively. Photons that are not scattered are assumed to reach the surface directly.

Emission within the atmosphere is ignored because the photons produced are expected to be of much lower energy than the ones impinging upon the atmosphere.

So, ignoring absorption, multiple scatterings, and emission and assuming steady state conditions, we write the equation of transfer:

$$\hat{\Omega}\cdot\vec{\nabla}I(\nu,\hat{\Omega}) = -\int_0^\infty d\nu' \int_{4\pi} d\hat{\Omega}' \sigma_s(\nu\rightarrow\nu', \hat{\Omega}\rightarrow\hat{\Omega}')I(\nu,\hat{\Omega}). \tag{13}$$

$\hat{\Omega}$ and $\hat{\Omega}'$ are directions of photon travel, ν and ν' photon frequencies, and $\sigma_s(\nu\rightarrow\nu',\hat{\Omega}\rightarrow\hat{\Omega}')$ the scattering probability per unit length for radiation entering the atmosphere at ν and $\hat{\Omega}$ and being scattered into ν' and $\hat{\Omega}'$. The separate $\hat{\Omega}$ and $\hat{\Omega}'$ dependences, rather than a single scattering angle $\hat{\Omega}\cdot\hat{\Omega}'$, are due to the fact that the electrons are moving. There is a preferred direction and it is the direction of electron motion.

One should note that equation (13) is written for an inertial frame. Although the star's rest frame is not an inertial frame, we can envision it coinciding with such a frame at a given instant of time and over an arbitrarily small but finite interval of space.

We rewrite equation (13) as

$$\hat{\Omega}\cdot\vec{\nabla}I(\nu,\hat{\Omega}) = -\sigma(\nu,\hat{\Omega})I(\nu,\hat{\Omega})$$

with

$$\sigma(\nu,\hat{\Omega}) = \int_0^\infty d\nu' \int_{4\pi} d\hat{\Omega}' \sigma_s(\nu\to\nu',\hat{\Omega}\to\hat{\Omega}') \quad (14)$$

by bringing the intensity outside the integral.

We then solve:

$$\frac{I(\nu,\hat{\Omega})}{I_o(\nu,\hat{\Omega})} = \exp[-\sigma(\nu,\hat{\Omega})s]. \quad (15)$$

$I_o(\nu,\hat{\Omega})$ is the intensity of radiation entering the atmosphere and $I(\nu,\hat{\Omega})$ the intensity after having traversed a path length s. In the next two sections we find expressions for $\sigma(\nu,\hat{\Omega})$ and s.

The Scattering Probability

In order to calculate the scattering probability, $\sigma_s(\nu\to\nu',\hat{\Omega}\to\hat{\Omega}')$, we start with the Klein-Nishina formula for Compton scattering off stationary electrons. We then discuss how this scattering probability transforms when we go to frames in which the electrons are moving. One of these frames, the star's rest frame, is the frame in which we wish to integrate the scattering probability given by equation (14).

The Klein-Nishina scattering probability can be written:

$$\sigma_{se}(\nu_e\to\nu_e',\xi_e) = \frac{n_e r_o^2}{2\alpha_e \nu_e} [1 + \xi_e^2 + \alpha_e\alpha_e'(1-\xi_e)^2] \delta(\xi_e - 1 + \frac{1}{\alpha_e'} - \frac{1}{\alpha_e}), \quad (16)$$

where the 'e' subscripts denote the electron rest frame. The scattering angle $\xi_e \equiv \hat{\Omega}_e \cdot \hat{\Omega}_e'$ appears here because in this frame nothing is moving and there can be no preferred direction. n_e is the number density of electrons, r_o the classical electron radius, and $\alpha_e = h\nu_e/m_e c^2$ and

$\alpha_e' = h\nu_e'/m_e c^2$ dimensionless frequencies, where h is Planck's constant, m_e the electron's mass, and c the speed of light.

We now wish to find the scattering probability in the star's rest frame. It is assumed that all electrons move with the same speed because, although there is a thermal distribution, the speeds involved are expected to be much smaller than the bulk speed, which is uniform.

The transformation between the two frames (electron rest frame and stellar rest frame) is given by (Pomraning 1973)

$$\sigma_s(\nu\to\nu', \hat{\Omega}\to\hat{\Omega}') = \frac{D}{D'}\,\sigma_{se}(\nu_e\to\nu_e', \xi_e),$$

where $D = 1-\vec{\beta}\cdot\hat{\Omega}$, $D' = 1-\vec{\beta}\cdot\hat{\Omega}'$, and $\beta = v/c$. This transformation is, of course, only valid for inertial frames, but, as noted above, such frames can be constructed to coincide with the real frames at a given instant and over a small interval of space. One can now use the above transformation and transformations for the various quantities within $\sigma_{se}(\nu_e\to\nu_e', \xi_e)$ to show

$$\sigma_s(\nu\to\nu', \hat{\Omega}\to\hat{\Omega}') = \frac{nr_o^2}{2\alpha\gamma\nu}\{1 + [1-\frac{(1-\xi)}{\gamma^2 DD'}]^2 + \frac{\alpha\alpha'(1-\xi)^2}{\gamma^2 DD'}\}$$

$$\delta(\xi - 1 + \frac{\gamma D}{\alpha'} - \frac{\gamma D'}{\alpha}). \qquad (17)$$

The quantities γ and β have the usual definitions, $\gamma = (1-\beta^2)^{-1/2}$ and $\beta = v/c$. Other quantities are defined as in the Klein-Nishina formula. When $\gamma = 1$ equation (17) reduces to equation (16) as expected.

Before integrating the scattering probability over all exit frequencies and directions of photon travel, we make the following assumptions:

First, we choose the frequency at which the intensity, $I_o(\nu,\hat{\Omega})$, is a maximum to be the frequency of the incoming radiation. An observer in the star's rest frame sees the usual blackbody spectral distribution except for an angle dependent temperature:

$$I(\nu,\hat{\Omega})d\nu d\hat{\Omega} = \frac{2h\nu^3}{c^2}\{\exp[\frac{h\nu}{kT(\hat{\Omega})}] - 1\}^{-1}\, d\nu d\hat{\Omega}\ ,$$

where $T(\hat{\Omega}) = T(\bar{\theta}) = T/\gamma_s(1-\beta_s\cos\bar{\theta})$. $\bar{\theta}$ is measured from the forward direction, i.e., a ray coming in along the forward direction $(-\hat{z})$ would have $\bar{\theta} = 0$ (Fig. 1). T is the temperature of the blackbody radiation in the comoving frame and $v_s = \beta_s c$ is the velocity of the star. For $\bar{\theta} = 0$ and $\gamma_s >> 1$ we find the simplified blackbody temperature, $T(\bar{\theta}) \cong 2\gamma_s T$. The expression above can now be differentiated to give the frequency at which maximum intensity occurs. We write this in terms of α, the dimensionless frequency:

$$\alpha \cong 9.5\text{x}10^{-10}\ K^{-1}\ \gamma_s T. \tag{18}$$

Second, we note that most of the radiation comes in through a small solid angle centered on the forward direction. Hence, the forward direction can be taken to be the direction of travel for incoming photons, an assumption that has already been used in the derivation of equation (18).

Third, we assume a constant optical depth in θ, i.e.:

$$\sigma(\nu,\hat{\Omega})s = \text{constant in } \theta\ , \tag{19}$$

where θ is the angle between the forward direction $(-\hat{z})$ and the inward normal to the surface $(-\hat{n})$ (Fig. 1). This simplifies the calculation of the scattering probability, since we need not calculate it as a general function of θ.

With these assumptions in mind, we perform the integrations of equation (14) to find the total scattering probability. The first integral is simply a delta function in $\gamma D/\alpha'$ and can be solved to yield

$$\sigma_s(\nu, \hat{\Omega}\to\hat{\Omega}') = \frac{nr_o^2}{2} \frac{D}{[\gamma D'+\alpha(1-\xi)]^2}\{1 + [1-\frac{(1-\xi)}{\gamma^2 DD'}]^2$$

$$+ \frac{\alpha^2(1-\xi)^2}{\gamma D'[\gamma D'+\alpha(1-\xi)]}\}. \qquad (20)$$

The second integral is quite tedious but straightforward if we perform it at $\theta = \pi/2$. The assumption of constant optical depth allows us to calculate the scattering probability at any angle between $\theta = 0$ and $\theta = \pi/2$ (recall that radiation does not strike the back of the star in our model) and we choose $\theta = \pi/2$ because incoming photons, which come in along the forward direction, are moving parallel to the electrons. This untangles the complicated angular dependences and allows us to write

$$D = 1-\vec{\beta}\cdot\hat{\Omega} = 1-\beta, \quad D' = 1-\vec{\beta}\cdot\hat{\Omega}' = 1-\beta\cos\theta_s, \quad \text{and } \xi = \cos\theta_s.$$

The only angle remaining is the scattering angle, θ_s.

We, therefore, write

$$\sigma_s(\nu, \xi) = \frac{nr_o^2}{2} \frac{(1-\beta)}{[\gamma(1-\beta\cos\theta_s)+\alpha(1-\cos\theta_s)]^2}$$

$$\{1 + [1 - \frac{(1-\cos\theta_s)}{\gamma^2(1-\beta)(1-\beta\cos\theta_s)}]^2$$

$$+ \frac{\alpha^2(1-\cos\theta_s)^2}{\gamma(1-\beta\cos\theta_s)[\gamma(1-\beta\cos\theta_s)+\alpha(1-\cos\theta_s)]}\}.$$

This equation is to be integrated over all solid angle. Performing the ϕ_s integration and writing the equation in terms of the variable $x \equiv \cos\theta_s$, we find

$$\sigma_s(\nu, \theta=\pi/2) = \pi nr_o^2 \int_{-1}^{1} \frac{(1-\beta)}{[\gamma(1-\beta x)+\alpha(1-x)]^2}$$

$$\{ 1 + [1 - \frac{(1-x)}{\gamma^2(1-\beta)(1-\beta x)}]^2 + \frac{\alpha^2(1-x)^2}{\gamma(1-\beta x)[\gamma(1-\beta x)+\alpha(1-x)]}\} dx. \quad (21)$$

The above must be solved in pieces. The first term can be integrated easily:

$$\pi n r_o^2 \int_{-1}^{1} \frac{(1-\beta)\, dx}{[\gamma(1-\beta x)+\alpha(1-x)]^2} = \frac{\pi n r_o^2(1-\beta)}{\gamma^2(\beta + \alpha/\gamma)}[\frac{1}{1-\beta} - \frac{1}{(1+\beta) + 2\alpha/\gamma}]. \quad (22)$$

The second term, the one of the form $[\;]^2$, can be multiplied out and written in terms of the variable, $y \equiv \beta(1-x)/(1-\beta x)$:

$$\pi n r_o^2 \int_{2\beta/(1+\beta)}^{o} \frac{(1-\beta)}{(\gamma + \alpha y/\beta)^2}[1 - \frac{2y}{\beta\gamma^2(1-\beta)} + \frac{y^2}{\beta^2\gamma^4(1-\beta)^2}] \frac{dy}{\beta(\beta-1)}. \quad (23)$$

Integrating, we find

$$\frac{\pi n r_o^2}{\alpha\gamma}[1 - \frac{1}{1 + 2\alpha/(1+\beta)\gamma}] + \frac{2\pi n r_o^2}{\alpha^2\gamma^2(1-\beta)}\{\ln[\frac{1}{1 + 2\alpha/(1+\beta)\gamma}] + 1$$

$$- \frac{1}{1 + 2\alpha/(1+\beta)\gamma}\} - \frac{\pi n r_o^2}{\alpha^3\gamma^3(1-\beta)^2}\{2\ln[1 + \frac{2\alpha}{(1+\beta)\gamma}]$$

$$-[1 + \frac{2\alpha}{(1+\beta)\gamma} - \frac{1}{1 + 2\alpha/(1+\beta)\gamma}]\}. \quad (24)$$

The third term in equation (21) can also be written in terms of the variable y and integrated:

$$-\frac{\pi n r_o^2}{\gamma\alpha}\{\ln[\frac{1}{1 + 2\alpha/(1+\beta)\gamma}] + \frac{3}{2} - \frac{2}{1 + 2\alpha/(1+\beta)\gamma}$$

$$+ \frac{1}{2[1 + 2\alpha/(1+\beta)\gamma]^2}\}. \quad (25)$$

Adding the three terms (equations [22], [24], and [25]) together, assuming that the electron velocity is highly relativistic ($\gamma >> 1$), and simplifying, we find

$$\sigma_s(\nu, \theta{=}\pi/2) = \frac{\pi n r_o^2}{\gamma^2(1 + \alpha/\gamma)^2}[(\frac{\alpha}{2\gamma} + 9 + 16\frac{\gamma}{\alpha} + 8\frac{\gamma^2}{\alpha^2})$$

$$+ (\frac{\alpha}{\gamma} - 2 - 15\frac{\gamma}{\alpha} - 20\frac{\gamma^2}{\alpha^2} - 8\frac{\gamma^3}{\alpha^3})\ln(1 + \frac{\alpha}{\gamma})]. \quad (26)$$

Equation (26) can be written in two limits, a low temperature limit corresponding to $\alpha/\gamma \ll 1$ and a high temperature limit corresponding to $\alpha/\gamma \gg 1$. Rejecting terms that are less than 10% of the largest term (if $\alpha/\gamma < 0.1$ for the low temperature limit or if $\alpha/\gamma > 10$ for the high temperature limit), we find for the two limits

$$\sigma_s(\nu, \theta=\pi/2) \cong \frac{na_o}{2\gamma^2} \quad , \frac{\alpha}{\gamma} \ll 1 \tag{27}$$

and

$$\sigma_s(\nu, \theta=\pi/2) \cong \frac{3na_o}{8\alpha\gamma} \left[\frac{1}{2} + \ln(1+\frac{\alpha}{\gamma})\right], \ \frac{\alpha}{\gamma} \gg 1. \tag{28}$$

Equations (27) and (28) have been written in terms of the Thomson scattering cross section per electron, $a_o = 8\pi r_o{}^2/3$. These equations say that the scattering probability increases as the electron density increases and decreases as the electrons accelerate or the incoming photons become more energetic ($\alpha/\gamma \gg 1$ only).

The Photon Path Length

We now determine the photon path length, i.e., the distance that unscattered photons must travel (at $\theta = \pi/2$) in order to reach the star's surface. h is taken to be the height of the non-degenerate atmosphere and R the radius of the degenerate interior. Both h and R are assumed constant in θ for a given blackbody temperature. The two regions are divided by a distinct surface. We do not worry about partially degenerate transition layers, since they would be superfluous in the model. We need only an atmosphere to facilitate mass loss and an interior to replenish what is lost and to maintain equilibrium.

The photon path length can be shown to obey the relation (Fig. 4)

$$s \cong \left(\frac{2h}{R}\right)^{1/2} R \qquad (29)$$

for the condition $h \ll R$.

The optical depth, τ, is now simply the product of equations (27) and (29) for the low temperature limit and equations (28) and (29) for the high temperature limit. It has been found to depend upon electron density and speed, the height of the atmosphere, and the star's radius and speed through the blackbody radiation, all of which are ultimately functions of the radiation temperature. Our first constraint is therefore

$$\tau = \sigma(\nu, \hat{\Omega})s. \qquad (30)$$

τ is assigned a value, and the five variables above are constrained so as to satisfy equation (30).

The Acceleration of Atmospheric Particles

The Hydrodynamical Equations

In the last section we calculated the probability of photons scattering off electrons of a given density and moving at a given speed. In the present section we find how electron momentum changes as a result of interacting with these photons.

We start with the hydrodynamical equations for a relativistic, ideal fluid. In the Eulerian picture particle, momentum, and energy balances are performed on a differential volume element, which is fixed in space. Such a description is the natural choice for an observer in the star's rest frame because atmospheric quantities such as particle density or velocity are required at specific locations.

If $D(\vec{r},t)$ represents the density of the quantity under consideration, i.e., particle, momentum, or energy density, $\vec{F}(\vec{r},t)$ the flux of the quantity, and $S(\vec{r},t)$ any external source of the quantity per unit volume, we can write the general conservation equation (Pomraning 1973):

$$\frac{\partial D(\vec{r},t)}{\partial t} + \vec{\nabla}\cdot\vec{F}(\vec{r},t) = S(\vec{r},t).$$

Specifically:

$$\frac{\partial(\gamma\rho)}{\partial t} + \vec{\nabla}\cdot(\gamma\rho\vec{v}) = 0 \qquad (31)$$

for particle balance, where no external source of particles is assumed;

$$\frac{\partial}{\partial t}\left[\frac{\gamma^2}{c^2}(\rho c^2 + E_m + P_m)\vec{v} + \frac{F_r}{c^2}\right] + \vec{\nabla}P_m + \vec{\nabla}\cdot\left[\frac{\gamma^2}{c^2}(\rho c^2 + E_m + P_m)\vec{v}\vec{v} + P_r\right] = S_{mom}, \qquad (32)$$

for momentum balance, where S_{mom} is an external source of momentum; and

$$\frac{\partial}{\partial t}\left[\gamma^2(\rho c^2 + E_m + P_m) - P_m + E_r\right] + \vec{\nabla}\cdot\left[\gamma^2(\rho c^2 + E_m + P_m)\vec{v} + \vec{F}_r\right] = S_{en}, \qquad (33)$$

for energy balance, where S_{en} is an external energy source. ρ is the mass density of the fluid in its rest frame, E_m the energy density of the fluid in its rest frame in excess of the rest energy ($E_{total} - \rho c^2$), P_m the pressure of the fluid, and $\vec{v}$ the velocity of the fluid with $\gamma = (1-\beta^2)^{-1/2}$ and $\beta \equiv v/c$. Equations (31) through (33) include the presence of a radiation field with energy density E_r, flux $\vec{F}_r$, and pressure tensor P_r.

Radial Forces

The force on atmospheric material due to external photons has both an $\hat{r}$ and $\hat{\theta}$ component. Even though these two components of force are comparable, we do not expect the velocities in the two directions to be of similar magnitude. Azimuthal acceleration continues until atmospheric material detaches itself from the star at $\theta \cong \pi/2$, but radial acceleration only continues until the bottom of the atmosphere is reached. Since $R >> h$, we expect $v_\theta >> v_{r(ep)}$ at all angles except for $\theta \cong 0$, where $v_{r(ep)}$ is that component of the radial velocity due to external photons.

In addition to the inward force of external photons, we have an outward force due to upwelling material. Material from the interior must upwell at a sufficient rate to replace what is lost. We do not expect it to transfer much momentum to the atmospheric particles, however, because the area through which upwelling material may pass is much greater than the annulus through which material is lost. Assuming a constant density, this means that the velocity of upwelling material is much smaller than the atmosphere's azimuthal velocity (see equation [31]): $v_{r(um)} << v_\theta$, where 'um' stands for upwelling material. Since $v_{r(ep)}$ and $v_{r(um)}$ are separately much smaller than v_θ and since the two forces act in opposite directions, we expect $v_r << v_\theta$, where v_r is the vector sum of $v_{r(ep)}$ and $v_{r(um)}$.

Of course there are other radial forces such as gravity, pressure gradients, etc., but these are expected to be so small compared to either the force of external radiation or the force of upwelling material that they can be ignored.

The Momentum Balance Equation

We not only assume that $v_r << v_\theta$ but also that $v_\phi << v_\theta$. There is no external force in the $\hat{\phi}$ direction and thus no net movement of material. We can therefore write the total velocity as

$$v \cong v_\theta, \tag{34}$$

which means that the $\hat{\theta}$ momentum equation is now decoupled from the $\hat{r}$ and $\hat{\phi}$ momentum equations.

The equation for momentum balance in the $\hat{\theta}$ direction can be written out explicitly as

$$\frac{\partial}{\partial t}\left[\frac{\gamma^2}{c^2}(\rho c^2 + E_m + P_m)v_\theta + \frac{F_\theta}{c^2}\right] + \frac{1}{r}\frac{\partial P_m}{\partial \theta} + \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\gamma^2}{c^2}(\rho c^2 + E_m + P_m) v_r v_\theta\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\gamma^2}{c^2}(\rho c^2 + E_m + P_m)v_\theta v_\theta\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left[\frac{\gamma^2}{c^2}(\rho c^2 + E_m + P_m)v_\phi v_\theta\right] + \frac{1}{r}\frac{\gamma^2}{c^2}(\rho c^2 + E_m + P_m)(v_\theta v_r - \cot\theta v_\phi v_\phi) + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 P_{r\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta P_{\theta\theta}) + \frac{1}{r\sin\theta}\frac{\partial P_{\phi\theta}}{\partial\phi} + \frac{1}{r}(P_{\theta r} - \cot\theta P_{\phi\phi}) = S_{mom,\,\theta}. \tag{35}$$

Quantities are defined as in equations (31) through (33). The subscripts 'r', 'θ', and 'φ' refer to the components of the quantity under consideration in the $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ directions respectively. Double subscripts indicate tensor quantities. For example, $P_{r\theta}$ is the force due to radiation moving in the $\hat{\theta}$ direction impinging upon a unit area with $\hat{r}$ normal. The dyadic terms, i.e., those of the form $v_i v_j$, can be defined in a similar way. $\gamma^2/c^2\ (\rho c^2 + E_m + P_m)v_r v_\theta$, for example, is the momentum flux of material moving in the $\hat{\theta}$ direction and passing through a surface perpendicular to $\hat{r}$. The second through sixth terms are due to matter, the last four terms to radiation, and the first term to both matter and radiation.

Instead of finding the relativistic density, $D(\vec{r},t)$, and flux, $\vec{F}(\vec{r},t)$, and performing a balance of the quantity under consideration (as Pomraning does), one could have employed the more widely used tensor approach. This approach makes use of the energy-momentum tensor of a perfect fluid:

$$T^{\mu\nu} = Pg^{\mu\nu} + (P + \rho)u^{\mu}u^{\nu} \tag{36}$$

in order to solve the equation for conservation of energy-momentum:

$$T^{\mu\nu}{}_{;\nu} = 0. \tag{37}$$

u^{μ} is the local value of $dx^{\mu}/d\tau = \gamma v^{\mu}$ for a comoving fluid element, and P and ρ are the pressure and energy density measured by an observer in a locally inertial frame that moves with the fluid at the instant the measurement is made. $g_{\mu\nu}$, the metric tensor, is defined by

$$d\tau^2 \equiv -g_{\mu\nu}\,dx^{\mu}dx^{\nu} = dt^2 - (dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2).$$

Taking the $\hat{\theta}$ component, we arrive at equation (35) except for the radiation and external source terms. These too will emerge if we write equation (36) to include the radiation field and equation (37) to include an external source of energy-momentum. In writing out equation (37), one must remember that the components of the velocity v_r, v_θ, and v_ϕ used by Pomraning in equation (35) are neither the contravariant components, v^i, nor the covariant components, v_i, but functions of these components and the metric tensor.

We now return to equation (35) and discuss the various simplifications that can be made.

We start off by arguing that the thermal energy of particles is not only much smaller than the energy associated with bulk motion but is also non-relativistic. Most of the photons that enter the atmosphere are involved in scattering events, since absorption is not important at the high temperatures and not so high densities expected. These do not heat the gas. Scattered photons are available for heating, but their energies are much lower than they were when they entered the atmosphere. We therefore expect only a small percentage of the energy that enters the atmosphere to go into heating. Photons that are not removed from the original beam strike the surface and contribute to stellar deceleration.

One can therefore assume that the temperature of the atmosphere is on the order of T (blackbody temperature in comoving frame) and not many orders of magnitude higher. Now, since the condition

$$E_m + P_m \cong \frac{\rho kT}{m_H} << \rho c^2$$

holds for $T << 10^{13}$ K, thermal motions are non-relativistic for the entire period of interest. This is an important simplification because now the temperature of the atmosphere need not enter our discussion.

Steady state conditions can be assumed because, even though the blackbody temperature rises, its change is negligible over atmospheric timescales.

The third term and the $v_\theta v_r$ part of the sixth term can be ignored since $v_r << v_\theta$. The net momentum passing through the $\hat{r}$ surface is much less than the momentum passing through the $\hat{\theta}$ surface.

Terms involving v_ϕ are negligible.

One can also assume that no significant radiation field exists in the atmosphere. Scattered photons are of very low energy; photons, which have either been emitted or absorbed, are also of low energy; and photons from the original beam that have not been scattered or absorbed have no effect on particle momentum. We therefore ignore the last four terms on the left hand side of equation (35).

Finally, it can be assumed that the material pressure gradient in the $\hat{\theta}$ direction is zero. There is no a priori reason to expect a significant azimuthal variation, since the total incoming energy varies very little in θ (over the front face) when the star is moving very fast.

Taking the assumptions in the above paragraphs into account, we write equation (35) as

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}[\sin\theta\rho\gamma^2 v_\theta{}^2] = S_{mom,\theta}. \qquad (38)$$

The source term, $S_{mom,\theta}$, is the $\hat{\theta}$ component of the momentum of external photons scattered in a unit volume of atmosphere per second. In writing equation (38), we have assumed that scattering photons transfer all their momentum to electrons, which in turn transfer most of their momentum to ions such that both species end up moving with the same velocity. The density in equation (38) is therefore the total density and the velocity, v_θ, is the velocity of atmospheric particles, both electrons and ions, in the $\hat{\theta}$ direction.

Since we are not concerned with how quantities such as ρ and v_θ vary with depth, we write $S_{mom,\theta}$ as an average:

$$S_{mom,\theta} = \frac{P_{scat}}{h}. \tag{39}$$

A large number of photons enter the atmosphere through a unit area per second. Most of these are scattered by the time the original beam reaches the surface. P_{scat} is just the total $\hat{\theta}$ momentum of all these photons. Since steady state conditions are assumed to prevail, it is also the sum of the fluxes entering individual layers.

Equations (38) and (39) give

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}[\sin\theta\rho\gamma^2 v_\theta^2] = \frac{P_{scat}}{h}.$$

Since θ is the only variable over which quantities vary, we write

$$\frac{1}{R\sin\theta}\frac{d}{d\theta}[\sin\theta\rho\gamma^2 v_\theta^2] = \frac{P_{scat}}{h}. \tag{40}$$

R replaces r because, to the accuracy required, all atmospheric layers lie a distance R from the center.

Approximation for Incoming Intensity

Before calculating $P_{scat}(\theta)$, we make an approximation for the angular dependence of the intensity of incoming photons. Our approximate intensity is constant within a small solid angle centered on the forward direction and equal to the maximum value of the actual intensity, such that the integrated intensity within the small solid angle is equal to the total integrated intensity over all solid angle if no approximation is made. That is,

$$\int_0^{2\pi} \int_0^{\theta_0} I_{max} \sin\bar{\theta} d\bar{\theta} d\bar{\phi} = \int_0^{2\pi} \int_0^{\pi/2} I(\bar{\theta}) \sin\bar{\theta} d\bar{\theta} d\bar{\phi}, \tag{41}$$

where $\bar{\theta}$ is the angle between the incoming photons and the forward direction (Fig. 1). $I(\bar{\theta}) = \sigma T^4 / \pi \gamma_s{}^4 (1-\beta_s \cos\bar{\theta})^4$ is the exact intensity distribution, and $I_{max} = I(\bar{\theta}=0)$ is the intensity of radiation coming in along the forward direction. Carrying out the integrations of equation (41) and assuming that $\gamma_s >> 1$ and $\theta_0 \gtrsim 0$, we find

$$\theta_0 \cong \frac{1}{\sqrt{3}\gamma_s}$$

and hence the approximation for the intensity distribution of incoming photons

$$I_0 = \begin{cases} \dfrac{16\sigma T^4 \gamma_s{}^4}{\pi} & |\bar{\theta}| < \dfrac{1}{\sqrt{3}\gamma_s} \\ 0 & |\bar{\theta}| > \dfrac{1}{\sqrt{3}\gamma_s}. \end{cases} \tag{42}$$

The limits of integration in equation (41) are really only appropriate for $\theta = 0$. In general, the upper limit of $\bar{\theta}$ depends upon θ and ϕ. We do not worry about such complications, however, because at high stellar speeds the entering radiation is highly focused. One does not expect a noticeable variation in θ, since the same narrow cone through which most of the radiation passes impinges upon all normal surfaces except those very close to $\theta = \pi/2$.

The total momentum flux can be written

$$P_{scat} = \frac{1}{c} \int I_{scat} \cos\theta' \sin\theta' d\bar{\Omega} \tag{43}$$

I_{scat} is the intensity of photons scattered out of the original beam in traversing the atmosphere, and θ' is the angle between the incoming photons and the inward normal to the area under consideration (Fig. 1).

For simplicity, we assume that the optical depth of the atmosphere remains constant in time (or blackbody temperature). This allows us to write

$$\frac{I}{I_o} = \exp[-\sigma(\nu,\hat{\Omega})s] = \text{constant in } T \equiv C_4,$$

where I_o is given by equation (42). I_{scat} is now simply:

$$I_{scat} = I_o - I = I_o(1 - C_4). \tag{44}$$

Since most of the radiation enters the atmosphere through a narrow cone centered on the forward direction, we can write

$$\theta' \cong \theta.$$

Making use of equations (44) and (42), we perform the integration of equation (43) and find

$$P_{scat} \cong \frac{16\sigma T^4 \gamma_s{}^2(1 - C_4)}{3c} \cos\theta \sin\theta. \tag{45}$$

In a similar manner we find the energy flux:

$$F_{scat} = \int I_{scat} \cos\theta' d\bar{\Omega} \cong \frac{16\sigma T^4 \gamma_s{}^2(1 - C_4)}{3} \cos\theta. \tag{46}$$

Equation (40) can now be solved to give the speed of atmospheric particles. If the speed at $\theta = \pi/2$ is denoted by v_M, where $\gamma_M = (1 - v_M{}^2/c^2)^{-1/2}$ and the density by ρ_M, we can write

$$\gamma_M{}^2 \cong \frac{16\sigma T^4 \gamma_s{}^2(1 - C_4)}{9C_5 \rho_M c^3}. \tag{47}$$

C_5 is the ratio h/R, which, like C_4, is assumed constant in θ and time (or blackbody temperature).

The speed v_M will be taken to be the speed of all particles but really represents a maximum speed. Particles that start from rest at $\theta = 0$ will indeed achieve this speed by $\theta = \pi/2$, but particles which start from rest at other azimuthal positions (newly upwelled material) will not reach such high speeds. We do not worry about this hierarchy of speeds, however, and take v_M to be the speed of all particles.

Equation (47) tells us how fast atmospheric particles will be moving at $\theta = \pi/2$ if electrons Compton scatter and transfer most of their energy to nearby ions, such that both develop the same speed within a short time. Atmospheric material is expected to detach itself from the star very close to $\theta = \pi/2$ because the inward radial force due to external photons drops off very suddenly as this point is reached.

Mass Loss

The equation for energy balance in the atmosphere can be written in the general form:

$$\frac{\partial}{\partial t}[\gamma^2(\rho c^2 + E_m + P_m) - P_m + E_r] + \frac{1}{r^2}\frac{\partial}{\partial r}[r^2\gamma^2(\rho c^2 + E_m + P_m)v_r$$

$$+ r^2F_r] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}[\sin\theta\gamma^2(\rho c^2 + E_m + P_m)v_\theta + \sin\theta F_\theta]$$

$$+ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}[\gamma^2(\rho c^2 + E_m + P_m)v_\phi + F_\phi] = S_{en}. \tag{48}$$

Making the same assumptions that we did for equation (35), we find

$$\frac{1}{R\sin\theta}\frac{d}{d\theta}[\sin\theta\rho c^2\gamma^2 v_\theta] = \frac{F_{scat}}{h},$$

which can then be solved with the help of equation (46) to give

$$\rho_M c^2 \gamma_M{}^2 v_M = \frac{8\sigma T^4 \gamma_S{}^2 (1 - C_4)}{3C_5}. \tag{49}$$

To find the total energy lost per second as seen by an observer in the star's rest frame, we simply multiply the energy flux by the surface area of the annulus through which atmospheric material must pass, i.e., $(\rho_M c^2 \gamma_M{}^2 v_M)(2\pi Rh)$. This is then set equal to $-\dot{M}c^2\gamma_M$, where $-\dot{M} = -dM/dt$ is the rate at which the star itself loses mass. Solving for M, we find

$$\dot{M} = -\frac{8\sigma T^4 \gamma_S{}^2 (1 - C_4)}{3C_5} \frac{2\pi Rh}{c^2 \gamma_M}. \tag{50}$$

The Mass-Radius Relation

Every astronomical body that is massive enough to pull itself into a sphere has a unique equilibrium radius. Very low mass bodies, such as planets, do not have widely varying densities and so obey a mass-radius relation of the form

$$R \propto M^{1/3}.$$

In such bodies electrostatic solid state forces are much stronger than gravitational forces.

More massive bodies, such as small stars, become degenerate once nuclear reactions have ceased. In these stars equilibrium is maintained by the balance of degenerate electron or neutron pressure and gravity; electrostatic solid state forces are negligible. More massive degenerates are therefore smaller:

$$R \propto M^{-1/3}. \tag{51}$$

Equation (51) is really only valid for small degenerates, but will be assumed valid for all stars in this calculation.

As a white dwarf loses mass, its radius increases according to equation (51). The pressure due to degenerate electrons becomes less and less dominant and at some point becomes equal to the ordinary ionic pressure. The star may continue to lose mass, but we no longer call it a white dwarf. A maximum radius is achieved, followed by a gradual shrinking.

As a neutron star loses mass, its radius also increases, but once central densities drop below a few x 10^3 g cm^{-3}, it becomes unstable to β decay. Neutrons are transformed into protons, electrons, and anti-neutrinos. β decay was blocked at higher densities because there were no available energy levels for the emitted electrons to fill. Neutron stars don't gradually turn into white dwarfs; they become unstable.

It may happen that degenerates lose so much mass that equation (51) is no longer an adequate mass-radius relation. If this happens, however, it will be late in the collapse. Now, since the spatial collapse speeds up as the temperature of the radiation increases, the amount of time during which the star is no longer adequately described be equation (51) (if this occurs at all) is very small compared to the total integration time.

The Star's Motion through the Background Radiation

The Total Decelerating Force

Radiation enters the atmosphere and is Compton scattered. The $\hat{\theta}$ component of the scattering radiation accelerates atmospheric material azimuthally, while the $(-\hat{r})$ component accelerates it radially. In this section we consider the radial contribution to stellar deceleration. We mentioned earlier that the radial force on atmospheric particles has several components, the two major ones being due to external radiation and upwelling material. We ignore upwelling material here and assume that the radial force on particles is due entirely to scattering radiation. This force is transferred to the surface through collisions. The small number of photons from the original beam that do not scatter also contribute to stellar deceleration and are taken into account.

The decelerating force due to scattering radiation can be written:

$$\vec{f}_{scat} = \frac{-1}{c}\int\int I_{scat}\cos\theta'\cos\theta'\cos\theta d\bar{\Omega}d\Omega R^2\ \hat{z}. \tag{52}$$

Once again, θ' is the angle between the incoming radiation $(\hat{\Omega})$ and the inward normal to the surface $(-\hat{n})$, θ the angle between the forward direction $(-\hat{z})$ and the inward normal $(-\hat{n})$, and $\bar{\theta}$ the angle between the forward direction $(-\hat{z})$ and the incoming radiation $(\hat{\Omega})$ (Fig. 1). The $\cos\theta$ term arises because we are interested in the component of the radial force in the $(-\hat{z})$ direction. Making use of equations (44) and (42), we find

$$\vec{f}_{scat} = \frac{-8\sigma T^4\gamma_s{}^2(1 - C_4)}{3c}\pi R^2\ \hat{z}. \tag{53}$$

The decelerating force due to unscattered radiation can be written:

$$\vec{f}_{unscat} = \frac{-1}{c}\int\int I \cos\theta' \cos\bar{\theta} d\bar{\Omega} d\Omega R^2 \hat{z}. \qquad (54)$$

Making use of the fact that $I = I_0 C_4$, we find

$$\vec{f}_{unscat} = \frac{-16\sigma T^4 \gamma_s{}^2 C_4}{3c} \pi R^2 \hat{z}. \qquad (55)$$

In both of the integrations we have assumed that $\theta' \cong \theta$.

The total decelerating force, $\vec{f}$, is just the sum of the two components and is given by

$$\vec{f} = \frac{-8\sigma T^4 \gamma_s{}^2 (1 + C_4)}{3c} \pi R^2 \hat{z}. \qquad (56)$$

This is really more of an upper limit because the force exerted on atmospheric particles by upwelling material has not been taken into account.

The Equation of Motion

When mass loss is taken into account, the equation describing the star's motion through the blackbody radiation can be written:

$$\frac{d^2 x^\mu}{d\tau^2} = -\Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} + \frac{f^\mu}{M} - \frac{1}{M} \frac{dM}{d\tau} \frac{dx^\mu}{d\tau}. \qquad (57)$$

The first term on the right hand side is simply the gravitational force per unit rest mass and is the same as it was in Chapter I, the second term is the decelerating force per unit rest mass, and the third term is the additional accelerating force caused by the continuous ejection of stellar material.

Taking the time component of equation (57) and making use of equations (56) and (51), we find

$$\frac{d\gamma_s}{dT} = \frac{\gamma_s^2 - 1}{\gamma_s T} - \frac{C_6 \beta_s \gamma_s T}{M^{5/3}} - \frac{\gamma_s}{M}\frac{dM}{dT}. \qquad (58)$$

The constant C_6 is given by

$$C_6 = \frac{8\pi\sigma T_o^2(1 + C_4)K^2}{3C_1 c^2}, \qquad (59)$$

where K is the constant of proportionality in the mass-radius relation, equation (51).

Equation (58) can be simplified when stellar speeds are highly relativistic, i.e., when $\gamma_s >> 1$:

$$\frac{d\gamma_s}{dT} = \frac{\gamma_s}{T} - \frac{C_6 \gamma_s T}{M^{5/3}} - \frac{\gamma_s}{M}\frac{dM}{dT}. \qquad (60)$$

<u>Reduction to Two Coupled First Order Differential Equations</u>

We now have five equations: (27-28) together with (29), (47), (50), (51), and (60) for the five unknowns: ρ_M, γ_M, M, R, and γ_s. In this section we reduce them to two sets of coupled first order differential equations (one set for the low temperature limit and the other for the high temperature limit). Each set consists of a mass loss equation and an acceleration/deceleration equation.

We start with the scattering probability, equations (27-28). These equations have been written in terms of the electron number density, n, and not the mass density, ρ. To convert, we make use of the relation

$$n = \frac{\rho(1 + X)}{2m_H},$$

where X is the hydrogen mass fraction. We assume that black dwarfs are composed entirely of hydrogen (X = 1) and that white dwarfs and neutron stars are composed entirely of elements heavier than hydrogen (X = 0).

One can then use equations (27) and (29) or (28) and (29) to write the atmospheric mass density in the limits of low and high temperature:

$$\rho_M = \frac{4 m_H \gamma_M{}^2}{a_o(1+X)R} \frac{-\ln C_4}{(2C_5)^{1/2}}, \qquad \frac{\alpha}{\gamma_M} << 1$$

and

$$\rho_M = \frac{16 m_H \alpha \gamma_M}{3a_o(1+X)R} \frac{-\ln C_4}{(2C_5)^{1/2}} \left[\frac{1}{2} + \ln\left(1+\frac{\alpha}{\gamma_M}\right)\right]^{-1}, \qquad \frac{\alpha}{\gamma_M} >> 1,$$

where use has been made of the two assumptions, $I/I_o = \exp[-\sigma(\nu,\hat{\Omega})s] = C_4$ and $h/R = C_5$. The subscript 'M', as before, denotes the value the quantity takes on at $\theta = \pi/2$.

Next, we substitute these mass densities into equation (47) to find the atmospheric speed (written in terms of γ_M):

$$\gamma_M \cong \left[\frac{8\sigma a_o(1+X)}{9c^3 m_H} \left(\frac{1}{2C_5}\right)^{1/2} \left(\frac{1-C_4}{-\ln C_4}\right)\right]^{1/4} R^{1/4} T \gamma_s{}^{1/2}$$

and

$$\gamma_M \cong \left[\frac{2\sigma a_o(1+X)}{3c^3 m_H C_7} \left(\frac{1}{2C_5}\right)^{1/2} \left(\frac{1-C_4}{-\ln C_4}\right)\right]^{1/3} R^{1/3} T \gamma_s{}^{1/3}$$

$$\left[\frac{1}{2} + \ln\left(1 + \frac{C_7 \gamma_s T}{\gamma_M}\right)\right]^{1/3}$$

for $\alpha/\gamma_M << 1$ and $\alpha/\gamma_M >> 1$ respectively if atmospheric speeds are highly relativistic. C_7 is the constant of equation (18).

Finally, we write the mass loss equation, equation (50), as a function of γ_s, M, and T. Using the above two expressions to eliminate γ_M and making use of the mass-radius relation, equation (51), we find

$$\frac{dM}{dT} \cong \frac{-16\pi\alpha K^2 T_o^2}{3C_1c^2} \left[\frac{9c^3 m_H}{8\sigma a_o(1+X)K} (2C_5)^{1/2} (1-C_4)^3(-\ln C_4)\right]^{1/4} \frac{\gamma_S^{1/2}}{M^{7/12}}, \qquad \frac{\alpha}{\gamma_M} << 1 \tag{61}$$

and

$$\frac{dM}{dT} \cong \frac{-16\pi\alpha K^2 T_o^2}{3C_1c} \left[\frac{3m_H C_7}{2\sigma a_o(1+X)K} (2C_5)^{1/2} (1-C_4)^2(-\ln C_4)\right]^{1/3} \frac{\gamma_S^{2/3}}{M^{5/9}} \left[\frac{1}{2} + \ln(1+\frac{C_7\gamma_S T}{\gamma_M})\right]^{-1/3}, \qquad \frac{\alpha}{\gamma_M} >> 1. \tag{62}$$

The presence of γ_M in the ln( ) term of equation (62) will be discussed in the next section.

We now have the two sets of coupled first order differential equations, equations (61) and (60) for the low temperature limit and equations (62) and (60) for the high temperature limit.

In deriving equations (61) and (62), we had to convert time intervals in the star's rest frame into time intervals in the comoving frame and then into temperature intervals in the comoving frame. The first transformation is simply

$$dt_c = \gamma_S dt,$$

where dt indicates time intervals in the star's rest frame and dt_c time intervals in the comoving frame. This is identical to what one would expect from Special Relativity because the general gravitational field is homogeneous and changes very little in the time it takes a photon leaving the star to reach the comoving observer. The second transformation is the one used in Chapter I:

$$dt_c = \frac{T_o^2 dT}{C_1 T^3}.$$

We measure time in the comoving frame because we need a standard that is the same for all stars and for all stages in the acceleration/deceleration of a single star. Time intervals in the rest frame of a black dwarf, for example, will be different from those in the rest frame of a white dwarf, which moves at a different speed, or even from those in the rest frame of the black dwarf at a later time.

In writing equations (61) and (62), we did not take relativistic corrections to the time interval due to the star's own field into account. These corrections are negligible for black and white dwarfs but not for neutron stars. From the Schwarzschild metric one can show that the time interval in the weak field limit is given by

$$dt' = \left(1 - \frac{2GM}{Rc^2}\right)^{-1/2} dt,$$

where dt is the time interval in the rest frame of the star in the absence of gravitation (the time interval used in the hydrodynamical equations), and dt' the time interval taking the star's own gravitational field into account. dt' has the values 1.0dt, 1.0dt, and 1.3dt for black dwarfs, white dwarfs, and neutron stars respectively. Since we do not consider vertical atmospheric structure and hence do not take the star's gravitational field into account in the hydrodynamics, we also do not take it into account in calculating time intervals. This means that neutron stars lose mass slightly faster in our calculation than they would if the gravitational field were taken into account, i.e., $dM/dt = 1.3 dM/dt'$. Of course this rate will diminish in time, since $2GM/Rc^2 \propto M^{4/3}$ decreases in time.

Method of Solution

We can write the two equations, equations (61) or (62) and equation (60), in the form

$$\frac{dy}{dx} = f(x,y,z) \tag{63}$$

and

$$\frac{dz}{dx} = g(x,y,z) \tag{64}$$

by substitution of equation (61) or (62) into equation (60). Our new equations are then (61) or (62) and (60) with dM/dT written out explicitly.

Solution is by the Runge-Kutta method (Grove 1966):

$$y_{n+1} = y_n + 1/6(k_1 + 2k_2 + 2k_3 + k_4)$$

and

$$z_{n+1} = z_n + 1/6(m_1 + 2m_2 + 2m_3 + m_4).$$

The k's and m's have the following values and are to be calculated in the order in which they appear:

$$k_1 = hf(x_n, y_n, z_n),$$

$$m_1 = hg(x_n, y_n, z_n),$$

$$k_2 = hf(x_n + h/2, y_n + k_1/2, z_n + m_1/2),$$

$$m_2 = hg(x_n + h/2, y_n + k_1/2, z_n + m_1/2),$$

$$k_3 = hf(x_n + h/2, y_n + k_2/2, z_n + m_2/2),$$

$$m_3 = hg(x_n + h/2, y_n + k_2/2, z_n + m_2/2),$$

$$k_4 = hf(x_n + h,\ y_n + k_3,\ z_n + m_3),$$

$$m_4 = hg(x_n + h,\ y_n + k_3,\ z_n + m_3).$$

h is the size of the increment, i.e., $x_{n+1} - x_n$. In the above scheme one starts with the initial conditions, $y = y_o$ and $z = z_o$ at $x = x_o$ and, having chosen the size of the increment, calculates the new values y_1 and z_1. From these one then calculates y_2 and z_2, y_3 and z_3, etc.

In our analysis the incremental variable is the blackbody temperature, T. We start the calculation at $T = 8.6\text{x}10^5$ K, $1.4\text{x}10^6$ K, and $4.3\text{x}10^6$ K for black dwarfs, white dwarfs, and neutron stars respectively. These values were chosen in order to just barely satisfy the conditions $\gamma_M >> 1$ and $\gamma_s >> 1$. The increments are of equal size in logT:

$$\log(T_{n+1}) - \log(T_n) = 10^{-2} \qquad (65)$$

but not in T:

$$T_{n+1} - T_n = 2.3293\text{x}10^{-2}\ T_n.$$

Late in the calculation mass loss occurs very quickly and the increments must be made smaller in order to avoid negative masses. We therefore set the right hand side of equation (65) equal to 10^{-3}, 10^{-5}, or 10^{-7}, as needed.

The initial values of M and γ_s are taken from Chapter I. It is assumed that only a negligible amount of mass loss has occurred before the start of the integration. From these initial conditions, one can see that the values of α/γ_M satisfy the condition $\alpha/\gamma_M << 1$ for all three stellar types. We, therefore, start by solving equations (61) and (60). As soon as the inequality $\alpha/\gamma_M > 1$ is met, however, we switch over to the other set of equations, equations (62) and (60). The transition from one set of equations to the other is abrupt and is not smoothed out by some sort of averaging process.

We now discuss the presence of the variable γ_M in the ln( ) term of equation (62). This dependence arises because the equation for γ_M in the high temperature limit is a transcendental equation; we cannot find γ_M purely as a function of R, γ_S, and T. This does not present a problem in our analysis, however. The fact that the logarithm of a variable varies much more slowly than the variable itself allows us to use the old approximation for γ_M in the logarithmic term in order to find a new approximation. We also use the old approximations for γ_S and T (the values at the beginning of the interval) when they appear within the logarithm. Other than this, everything is computed according to the scheme outlined above.

Equations (61) and (62) still require that we assign values to C_4 and C_5. We consider three optical depths, $\tau = 1$, 2, and 3, and one atmospheric height to stellar radius ratio, $h/R = 10^{-3}$. An optical depth of zero is equivalent to no atmosphere and no mass loss and was treated in Chapter I.

The computer program used to solve the two sets of differential equations was written in Fortran 77 and is included as an Appendix. The constant A_1 is the constant that appears in the low temperature limit of γ_M, A_2 the constant in (61), AB the constant (59), B_1 the constant that appears in the high temperature limit of γ_M, and B_2 the constant in (62). The values of these constants in the nine cases considered are shown in Table 1.

TABLE 1

Constants of Differential Equations

Star	τ	A_1	B_1	AB	B_1	B_2
BD	1	2.9(-4)	1.3(41)	4.5(37)	1.8(-2)	2.2(39)
BD	2	2.7(-4)	2.1(41)	3.5(37)	1.6(-2)	3.3(39)
BD	3	2.5(-4)	2.5(41)	3.2(37)	1.4(-2)	4.2(39)
WD	1	1.8(-4)	1.5(40)	2.8(36)	9.5(-3)	2.8(38)
WD	2	1.6(-4)	2.3(40)	2.2(36)	8.4(-3)	4.2(38)
WD	3	1.5(-4)	2.7(40)	2.0(36)	7.5(-3)	5.3(38)
NS	1	3.6(-5)	2.5(35)	9.8(30)	1.1(-3)	8.1(33)
NS	2	3.3(-5)	3.9(35)	7.7(30)	9.9(-4)	1.2(34)
NS	3	3.1(-5)	4.6(35)	7.0(30)	8.8(-4)	1.5(34)

NOTE.- BD means black dwarf, WD means white dwarf, NS means neutron star. τ is optical depth. A_1, A_2, AB, B_1, and B_2 are defined in text. Numbers in parentheses are powers, e.g., 2.9(-4) means 2.9×10^{-4}.

Numerical Results

TABLE 2

Selected Results

Star	τ	$\gamma_{s,max}$	$T(\gamma_{s,max})$	$M/M_i(\gamma_{s,max})$	$M/M_i(T_f)$	T_f
BD	1	1.7(4)	1.2 (8)	3.3(-3)	6.3(-5)	1.2 (8)
BD	2	No max			5.8(-4)	8.0 (7)
BD	3	No max			6.7(-4)	7.1 (7)
WD	1	1.2(5)	2.1 (9)	7.1(-1)	2.7(-1)	4.9 (9)
WD	2	2.1(5)	2.6 (9)	1.8(-1)	1.3(-2)	2.8 (9)
WD	3	No Max			4.5(-4)	2.3 (9)
NS	1	1.1(8)	2.3(12)	8.5(-1)	6.2(-1)	1.0(13)
NS	2	1.3(8)	2.8(12)	6.6(-1)	3.6(-1)	6.6(12)
NS	3	1.5(8)	2.8(12)	5.3(-1)	1.9(-1)	5.3(12)

NOTE.- BD means black dwarf, WD means white dwarf, NS means neutron star. τ is optical depth. Other headings are identified in text. Numbers in parentheses indicate powers, e.g. 1.7(4) means 1.7×10^4. No max means that no maximum value occurs.

Table 2 gives the maximum value of the relativistic factor, $\gamma_{s,max}$; the temperature at which it occurs, $T(\gamma_{s,max})$; the fraction of the original mass at this temperature, $M/M_i(\gamma_{s,max})$; the final temperature, T_f; and the final fraction of the original mass, $M/M_i(T_f)$.

The most striking result is that in some cases (BD 2,3; WD3) a maximum speed is never achieved. This occurs for the larger optical depths because the higher the optical depth, the greater the mass loss (see eqs. [61] and [62]). Mass loss, in turn, leads to additional acceleration (in addition to that due to the collapse of space). In these cases, the ratio of this additional acceleration to the deceleration caused by radiation drag first decreases as it does in all the cases, but then increases (calculated in program, but not shown in Table 2). This increase guarantees that a maximum speed will never be reached no matter how long we continue the calculation and how small we take the temperature intervals. Black dwarfs are most susceptible to this because they lose the largest percentage of their mass. This presumably occurs because they have the deepest atmospheres, which means that the annulus through which mass loss occurs is the largest.

When a maximum speed is achieved, its value is higher and the temperature at which it occurs is higher than the same quantities in the case of no mass loss. This is simply due to the additional acceleration. Deceleration occurs more rapidly (Fig. 5) because the higher temperature and larger cross section give rise to a larger drag force.

The last column gives T_f. This is not necessarily the temperature at which stellar speeds drop below a certain value but simply the last value printed. It is a good approximation to the temperature at which stars come to rest (for those that actually do), however, because at this point mass loss is extremely rapid and the temperature intervals extremely small.

The second to last column is at best an upper limit to the mass that remains, since the calculation has been cut off at a point where mass loss is rapid.

The fact that the black dwarf values of $M/M_i(T_f)$ increase as optical depth increases seems to contradict what was said in the second paragraph above. This is spurious. The last two columns do not represent the same stage of acceleration/deceleration, etc., in each of the three cases. It can be seen from equations (61) and (62) that the rate of mass loss depends upon the mass present and the stellar speed and not just upon optical depth. If γ_s and M are held fixed, we would expect to see the behavior described in the second paragraph above.

CHAPTER III

MISCELLANEOUS TOPICS

Collisions

Implicit in the first two chapters is the assumption that stars do not collide. In this section it will be shown that collisions are rare until late in the collapse and can therefore be ignored.

The rates of loss of stars in the comoving volume R^3, where R is a universal scale factor, are given by

$$\frac{d(n_3R^3)}{dt_c} = -cn_3R^3(n_3\sigma_{33}), \qquad (66)$$

$$\frac{d(n_2R^3)}{dt_c} = -cn_2R^3(n_2\sigma_{22} + n_3\sigma_{32}), \qquad (67)$$

and

$$\frac{d(n_1R^3)}{dt_c} = -cn_1R^3(n_1\sigma_{11} + n_2\sigma_{21} + n_3\sigma_{31}). \qquad (68)$$

The number densities, n_1, n_2, and n_3, refer to black dwarfs, white dwarfs, and neutron stars respectively. The σ_{ij}'s are collisional cross sections for the species i and j. Equation (66) says that neutron stars are destroyed only by collisions with other neutron stars, equation (67) that white dwarfs are destroyed by collisions with either neutron stars or other white dwarfs, and equation (68) that black dwarfs are destroyed by collisions with neutron stars, white dwarfs, or other black dwarfs. This is a reasonable assumption because the densities of the three species are so disparate. In a neutron star/white dwarf collision, for example, the neutron star will pass right through the white dwarf without being disrupted or noticeably decelerated since the white dwarf is many orders of magnitude more tenuous than the neutron star. The white dwarf, on the other hand, will be destroyed. All species are assumed to be moving at approximately the speed of light.

Equation (66) can be solved to yield

$$N_3 = (C_{33}t_R + I_1)^{-1} \tag{69}$$

$N_3 \equiv n_3R^3/n_{3o}R_o{}^3$, where the 'o' subscripts refer to quantities at radiation temperatures of 150 K. This fraction is diminished only by collisions and is unaffected by the collapse of space. $C_{33} \equiv c\sigma_{33}n_{3o}$ and I_1 is a constant of integration, to be determined later. t_R is a temporary variable (it will disappear when simplifications are made) and is defined by $dt_R \equiv (R_o/R)^3 dt_c$.

Equation (67) can be solved with the help of equation (69):

$$N_2 = [I_2(C_{33}t_R + I_1)^{C_{32}/C_{33}} - \frac{C_{22}}{C_{32}-C_{33}}(C_{33}t_R + I_1)]^{-1}. \tag{70}$$

$N_2 \equiv n_2R^3/n_{2o}R_o{}^3$, $C_{22} \equiv c\sigma_{22}n_{2o}$, and $C_{32} \equiv c\sigma_{32}n_{3o}$. I_2 is another constant of integration.

Equations (69) and (70) can now be used to solve equation (68):

$$N_1 = \frac{X(C_{33}t_R + I_1)^{-C_{31}/C_{33}}}{I_3 + \int C_{11}X(C_{33}t_R + I_1)^{-C_{31}/C_{33}}\, dt_R}, \tag{71}$$

with

$$X = [I_2 - \frac{C_{22}}{C_{32}-C_{33}}(C_{33}t_R + I_1)^{-C_{32}/C_{33}} + 1]^{-C_{21}/C_{22}}.$$

$N_1 \equiv n_1R^3/n_{1o}R_o{}^3$, $C_{11} \equiv c\sigma_{11}n_{1o}$, $C_{21} \equiv c\sigma_{21}n_{2o}$, and $C_{31} \equiv c\sigma_{31}n_{3o}$. I_3 is yet another constant of integration. We do not worry about the integral in equation (71) until after simplifications are made.

In order to simplify equations (69), (70), and (71) we must determine the values of the constants C_{33}, C_{22}, C_{32}, C_{11}, C_{21}, and C_{31}. These in turn require that we find the values of the collisional cross sections and the number densities.

We start with the collisional cross sections. For stars of different species an actual collision need not occur for the more tenuous member to be destroyed. Disruption begins at

$$x_d = R_t (M_d/M_t)^{1/2},$$

where x_d is the distance from the center of the denser star to the surface of the more tenuous one, R_t the radius of the more tenuous star, and M_d and M_t the masses of the denser and more tenuous stars respectively. The collisional cross sections are then given by

$$\sigma_{dt} = \pi(x_d + R_t)^2. \tag{72}$$

These, of course, reduce to the geometrical cross sections when two stars of the same species interact and an actual collision occurs. For simplicity, we take the initial values of the masses and radii when computing actual cross sections.

The number densities of the three species can be found from the initial mass function:

$$\phi_m = Lm^{-(1+x)},$$

which gives the number of stars per unit mass per unit volume. The coefficient L and exponent x depend upon the mass range considered. We approximate ϕ_m by two straight lines (Miller and Scalo 1979) with

$$x = 1.5, \qquad 1 \leq M/M_\odot \leq 10,$$

$x = 0.4, \quad 0.1 < M/M_\odot < 1.$

For the purposes of this calculation, we assume that all matter has gone to stars, that the matter in black holes is negligible, and that 5% of the matter is to be found in black dwarfs. We must assume a black dwarf mass fraction, because black dwarfs have masses $< 0.1\ M_\odot$ (the cutoff for nuclear reactions) and observations of stars with such small masses are simply not adequate to determine a slope for ϕ_m.

To find the two values of L, we first calculate the mass density of the three species and set the sum equal to the total mass density, assumed to be equal to 2×10^{-29} g cm^{-3} at the present epoch (3K). We then find one L in terms of the other by requiring that the function ϕ_m be continuous at 1 $M_\odot$. The values of L are

$L_1 = 7.6\times10^{-50}\ g^{0.4}\ cm^{-3}$,

$L_2 = 3.2\times10^{-13}\ g^{1.5}\ cm^{-3}$,

where L_1 is the constant for $0.1 < M/M_\odot < 1$ and L_2 the constant for $1 < M/M_\odot < 10$. In calculating these two constants, we assumed that white dwarfs have initial masses between 0.1 $M_\odot$ and 4 $M_\odot$ and neutron stars initial masses between 4 $M_\odot$ and 10 $M_\odot$.

The constants C_{ij} are found to have the values:

$C_{33} = 1.6\times10^{-35}\ sec^{-1}$,

$C_{22} = 5.2\times10^{-28}\ sec^{-1}$,

$C_{32} = 9.6\times10^{-30}\ sec^{-1}$,

$C_{11} = 8.8\times10^{-27}\ sec^{-1}$,

$C_{21} = 9.5\times10^{-26}\ sec^{-1}$,

$C_{31} = 2.2\times10^{-27}\ sec^{-1}$.

Equation (69) can now be written in terms of the variable $T_i \equiv T_o/T$ and simplified by noting that $C_{33}/C_1 << 1$. The constant I_1 is determined from initial conditions. We find

$$N_3 \cong \frac{T_i}{C_{33}/C_1 + T_i}. \tag{73}$$

Equations (70) and (71) can be simplified in a similar fashion:

$$N_2 \cong \frac{T_i - C_{32}/C_1}{T_i + C_{22}/C_1} \tag{74}$$

and

$$N_1 \cong \frac{T_i - (C_{21}+C_{31})/C_1}{T_i + C_{11}/C_1}. \tag{75}$$

Collisions are assumed to become important when the original numbers have been depleted by 50% ($N_i = 0.5$). This occurs at blackbody temperatures of 3.6×10^{10} K, 1.4×10^{13} K, and 4.6×10^{20} K for black dwarfs, white dwarfs, and neutron stars respectively. Collisions are therefore expected to be of no importance for those stars that come to rest. Black dwarfs of optical depths 2 and 3 and white dwarfs of optical depth 3 never come to rest according to the calculations of Chapter II.

Possible Decelerating Mechanisms for Stars that do not Come to Rest

In chapter II we saw that black dwarfs of optical depths 2 and 3 and white dwarfs of optical depth 3 never come to rest. In this chapter we investigate several possible deceleration mechanisms.

We mentioned earlier that small degenerates obey a mass-radius relation of the form $R \propto M^{-1/3}$, while non-degenerates obey one of the form $R \propto M^{1/3}$. This means that stars that are losing mass will expand to a maximum radius and then contract. This maximum radius is characterized by a critical mass. For cold bodies composed entirely of H, He, or C, these critical masses are 6.4×10^{30} g, 2.2×10^{30} g, and 4.4×10^{30} g respectively (Zapolsky and Salpeter 1969). Such masses occur when blackbody temperatures have reached about 8.0×10^7 K, 7.1×10^7 K, and 2.3×10^9 K for black dwarfs of optical depth 2, black dwarfs of optical depth 3, and white dwarfs of optical depth 3. Black dwarfs are assumed to be composed of H and white dwarfs of either He or C (the temperature at which the critical mass is reached is not sensitive to composition in the case of white dwarfs since mass loss occurs very quickly at this point).

Now, if we take $R \propto M^{-1/3}$ before the critical mass is reached and $R \propto M^{1/3}$ after, we find

$$\frac{F_{n-d}}{F_d} = \left(\frac{M_{crit}}{M}\right)^{1/6}, \qquad \frac{\alpha}{\gamma M} << 1$$

and

$$\frac{F_{n-d}}{F_d} = \left(\frac{M_{crit}}{M}\right)^{2/9}, \qquad \frac{\alpha}{\gamma M} >> 1.$$

F_d is the absolute value of the ratio of the third term on the right hand side of equation (58) to the second term, i.e., it is the ratio of the acceleration due to mass loss to the deceleration due to radiation drag when a degenerate mass-radius relation is used. F_{n-d} is the same ratio using the non-degenerate mass-radius relation. One can see that for $M < M_{crit}$, F_{n-d} is higher than F_d, which means that the non-degenerate mass-radius relation causes even more acceleration than the degenerate mass-radius relation.

The critical mass and maximum radius discussed above occur for cold bodies in a vacuum. It is possible that this description is significantly altered by an intense external radiation field. Our analysis, for example, begins to predict (at high radiation temperatures) atmospheric densities that are higher than the average densities of the stars themselves. So, realistically, one would expect stars to begin to contract when such a situation arises and perhaps long before. To find out when and how quickly this occurs, one would have to incorporate the equations of stellar structure into the analysis. Account would also have to be taken of the fact that stellar interiors at some point become hot enough for nuclear reactions to occur. Any shrinkage at all, though, even one as slow as $R \propto M^{\varepsilon}$, where ε is very small, would lead to additional stellar acceleration. Deceleration could then only occur if these models were to lead to much thinner atmospheres than the ones assumed in the present analysis.

e^+e^- pair creation is expected to do little to halt the acceleration of black or white dwarfs. The density of e^+e^- pairs does not become comparable to the radiation density until the radiation has reached a temperature of about 5×10^9 K. Our calculations do not extend to such temperatures for the species and optical depths above, but by 8.0×10^7 K, 7.1×10^7 K, and 2.3×10^9 K (the final temperatures printed), black dwarfs of optical depths 2 and 3 and white dwarfs of optical depth 3 have F_d values of 31, 51, and 12 respectively. Mass loss is extremely rapid at this point. If drastic structural changes do not occur (leading to much thinner atmospheres and less mass loss), the stars will become unstable since structural readjustment cannot be expected to keep pace with mass loss indefinitely. e^+e^- pair creation will certainly contribute to stellar deceleration if stars are still around at 5×10^9 K, but its role will be minor.

Gas produced by mass loss is expected to contribute even less to stellar deceleration. We are far into the radiation era, which means that the density of matter is much less than the density of radiation.

Supermassive Black Holes

It has been suggested that supermassive black holes (10^6-10^8 M⊙) are the power sources of quasars and active galactic nuclei (Shapiro and Teukolsky 1983). If black holes of this size are abundant in the universe, they may be more effective at destroying stars than collisions.

As the universe collapses, supermassive black holes are accelerated to high speeds just as stars are, but instead of losing mass they gain it. We assume for simplicity that all matter is originally (at T_o = 150 K) contained in galaxies of 10^{11} M⊙. Each galaxy has at its center a black hole of 10^7 M⊙. This means that the average distance between supermassive black holes, d_o, is about 4.3×10^{22} cm at 150 K. Now, since distance scales as $d = d_o/T'$, where $T' = T/T_o$, we can find the temperature at which supermassive black holes begin to fill the entire universe. Written in terms of T', this is

$$T' = \frac{d_o c^2}{4GM} \qquad (76)$$

and will be taken to be the 'temperature' at which stars are destroyed by black holes.

Equation (76) requires that we find an expression for the mass, M, of the supermassive black hole as a function of blackbody temperature. To do this, we assume that the black hole sweeps up everything in the cylinder defined by its motion and that it moves at the speed of light throughout the period of interest. Its mass is found to depend on temperature according to

$$M \cong (L_3 - L_4T' - L_5T'^2)^{-1}.$$

The constants L_3, L_4, and L_5 have the values $1/M_o$, $4\pi G^2\rho_{mo}/C_1c^3$, and $2\pi G^2\rho_{ro}/C_1c^3$ respectively. M_o is the black holes's initial mass (at 150K) and ρ_{mo} and ρ_{ro} are the initial values of the average density of matter and radiation. Equation (76) can now be solved for T' or T. We find that supermassive black holes begin to fill the entire universe at about 3.5×10^9 K. Black holes, if they exist in such numbers, are therefore much more effective at destroying stars than collisions.

CONCLUSION

In the analysis of Chapters I and II stars were shown to accelerate to high speeds due to the collapse of space and the additional accelerating force caused by mass loss, and then in most cases to decelerate due to radiation drag. All stars obeyed a degenerate mass-radius relation of the form: $R \propto M^{-1/3}$ throughout the period of interest. In addition, they all had the same atmospheric height to stellar radius ratio, and this ratio remained constant in time. Four optical depths were considered (an optical depth of zero is really equivalent to no atmosphere and no mass loss). Stars with larger optical depths generally accelerated longer, but once maximum speeds were achieved, deceleration was more rapid. This was due to the fact that larger optical depths led to more mass loss and greater additional acceleration. This additional acceleration then caused the stars to reach maximum speeds at higher blackbody temperatures and to decelerate more quickly once these speeds were achieved because radiation drag was stronger at these higher temperatures.

There were three species of stars (black dwarfs of optical depths 2 and 3 and white dwarfs of optical depth 3) that never came to rest according to the analysis of Chapters I and II. The analysis, however, assumed that stellar interiors can always replace what is lost, no matter how rapid this loss becomes. This assumption will at some point break down. As time goes on, less and less of the presumed atmosphere will be replaced because the internal structure will simply be unable to readjust itself quickly enough to keep pace with mass loss. If the star is to remain stable then, the atmosphere will have to become thinner, which could lead to less mass loss and possible deceleration. On the other hand, the star will probably begin to contract, which would tend to increase mass loss and acceleration (see Chapter III). This would be required if one is to avoid a density inversion (atmospheric densities higher than the average densities of the stars themselves were obtained at high blackbody temperatures). The relative contributions of the two effects cannot be determined, however, unless the star's internal structure is considered in the analysis.

Other considerations are the possibility that stellar interiors at some point become hot enough for nuclear reactions to occur and, in the case of neutron stars, the creation of e^+e^- pairs at about 5×10^9 K. Both would lead to additional deceleration or perhaps instability.

It is also possible that surface stresses (that arise from that part of the azimuthal component of the external radiation that reaches the surface) will tear the stars apart.

Chapters I and II then give a first approximation as to what happens to stars moving through the blackbody radiation of a collapsing universe. Successive approximations could perhaps make use of some of the suggestions offered in the above paragraphs, particularly the suggestion about the inclusion of internal structure.

Stellar numbers are unlikely to be diminished while most of the action is taking place. Collisions were shown to be unimportant. Supermassive black holes begin to fill the entire universe while neutron stars are still accelerating, but this assumes that every galaxy has a supermassive black hole at its center and that all matter is contained in galaxies.

APPENDIX A

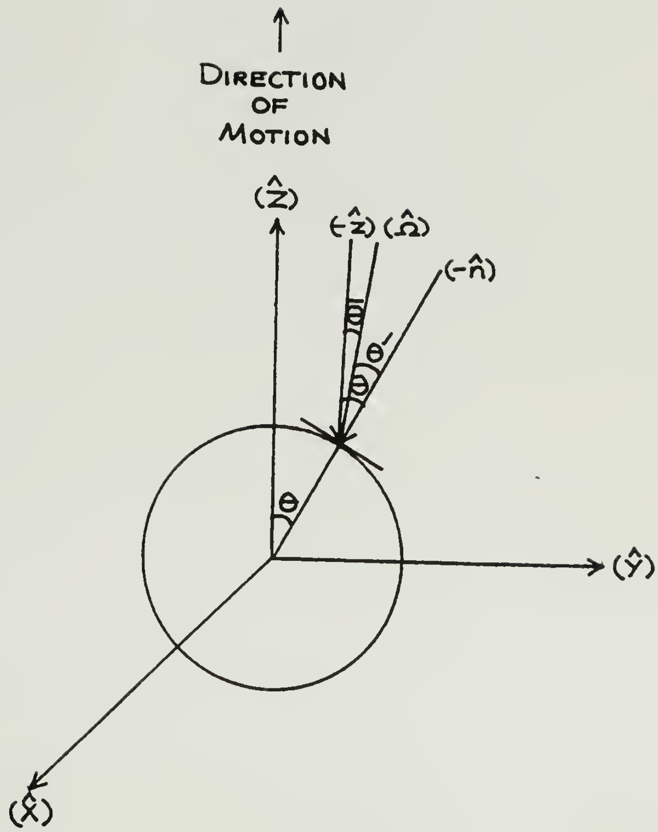


Figure 1

The various angles in the rest frame of the star. θ is the angle between the forward direction $(-\hat{z})$ and the inward normal $(-\hat{n})$, $\bar{\theta}$ the angle between the forward direction $(-\hat{z})$ and the direction of incoming photons $(\hat{\Omega})$, and θ' the angle between the direction of incoming photons $(\hat{\Omega})$ and the inward normal $(-\hat{n})$.

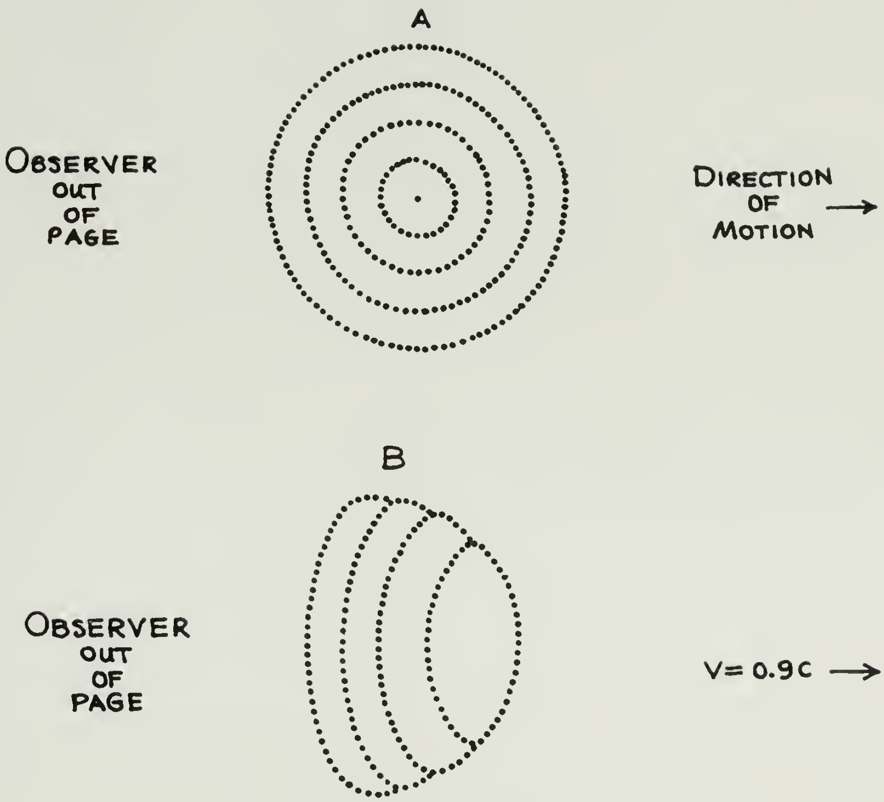
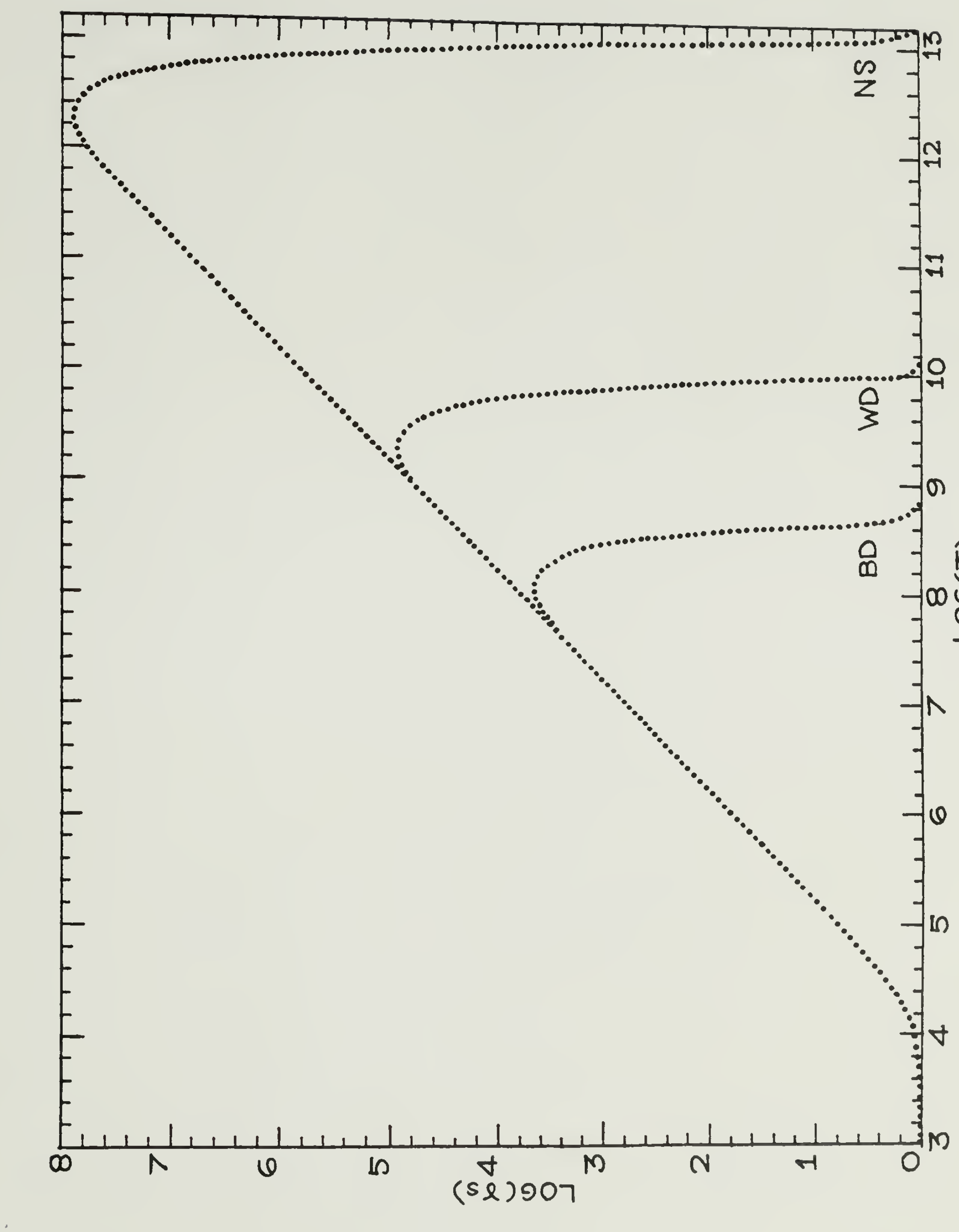


Figure 2

A distant observer views a star moving perpendicular to his or her line of sight. In (A) the star, which is at rest with respect to the observer, is divided up into concentric rings, such that all points on a given ring are equidistant from the observer. In (B) the star is moving at 0.9c and appears as a rotated hemisphere.

Figure 3

$\log(\gamma_s)$ versus $\log(T)$ for black dwarfs (BD), white dwarfs (WD), and neutron stars (NS) whose masses are constant. T is the temperature of the blackbody radiation in the comoving frame, and $\gamma_s = (1 - \beta_s^2)^{-1/2}$, where $v_s = \beta_s c$ is the velocity of the star in the comoving frame.


NS
WD
BD
LOG(T)

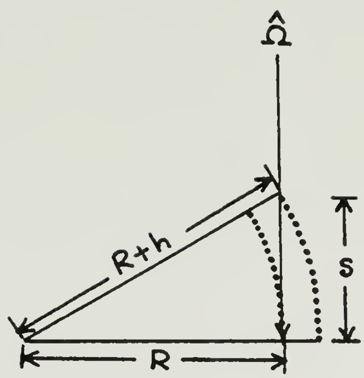


Figure 4

Photon path length, s, at $\theta = \pi/2$ (see Figure 1 for definition of θ and $\hat{\Omega}$). R is the star's radius and h the height of it's atmosphere.

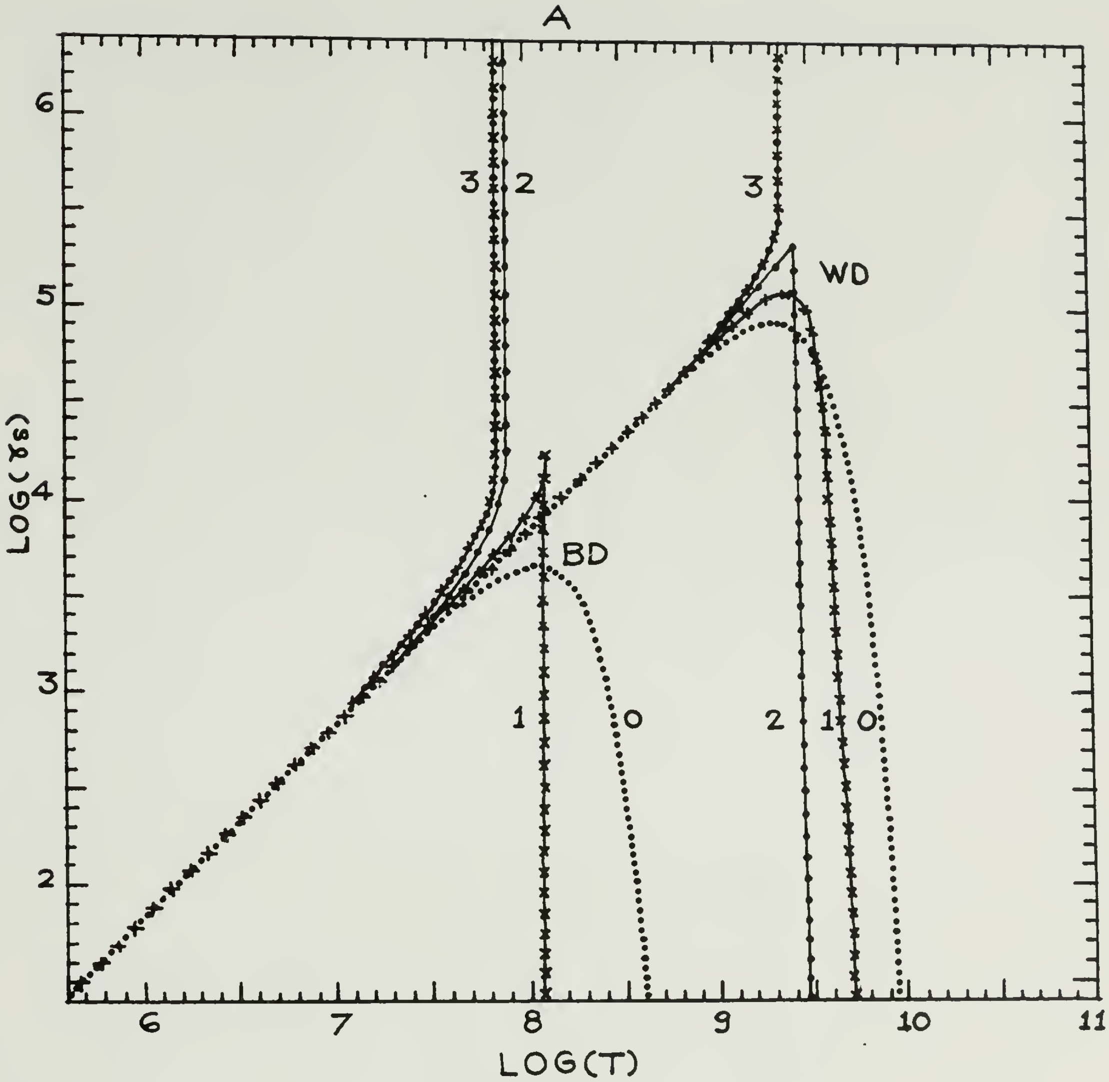


Figure 5

Same as for Figure 3 except stars are now losing mass. We consider three atmospheric optical depths for each species: 1, 2, and 3. An optical depth of 0 is equivalent to no atmosphere and no mass loss. In (A) black dwarfs and white dwarfs are considered, in (B) neutron stars.

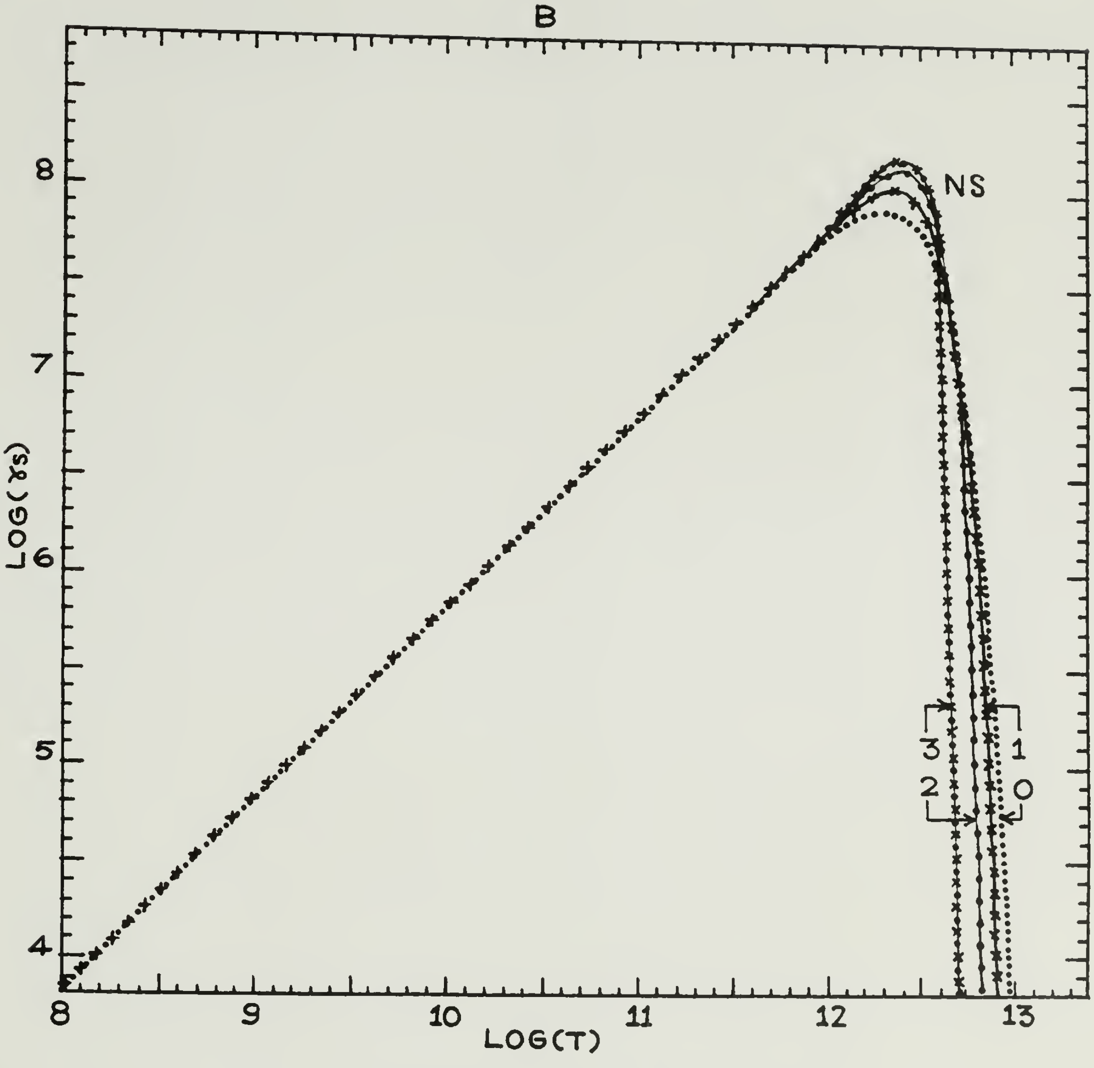


Figure 5 (continued)

APPENDIX B

```
      PROGRAM SOLVE (INPUT,OUTPUT)
* PROGRAM TO SOLVE TWO COUPLED FIRST ORDER DIFFERENTIAL EQUATIONS
* IN TWO LIMITS
* READ IN INITIAL CONDITIONS AND VALUES OF CONSTANTS
      READ 1, GS,T,EM,IRANGE
1     FORMAT (3E8.1,I3)
      READ 2, A1,A2,AB
2     FORMAT (3E8.1)
      READ 3, B1,B2
3     FORMAT (2E8.1)
      GM=A1*T*GS**0.5/EM**0.0833
* II TELLS HOW LARGE TEMP. INTERVAL SHOULD BE
      II=1
      DO 6, I=1,IRANGE
8     IF (II .EQ. 1) THEN
      JRANGE=10
      DT=2.3293E-2*T
      ELSE IF (II .EQ. 2) THEN
      JRANGE=100
      DT=2.3052E-3*T
      ELSE IF (II .EQ. 3) THEN
      JRANGE=100
      DT=2.3026E-5*T
      ELSE IF (II .EQ. 4) THEN
      JRANGE=100
      DT=2.3000E-7*T
      ELSE
      GO TO 7
      END IF
      DO 4, J=1,JRANGE
      TSTLG=LOG10(9.5E-10*GS*T/GM)
* TEST FOR LOW TEMP. LIMIT
* IF CONDITION MET DO FIRST BLOCK, OTHERWISE DO SECOND BLOCK
         IF (TSTLG .LE. 0.0) THEN
* SOLN. IN LOW TEMP. LIMIT
      X1=-A2*GS**0.5/EM**0.5833
      XK1=DT*X1
      XM1=DT*(-AB*GS*T/EM**1.6667-GS*X1/EM+GS/T)
      X2GS=GS+0.5*XM1
      X2EM=EM+0.5*XK1
      IF (X2EM .LE. 0.) THEN
      II=II+1
      GO TO 8
      ELSE
      END IF
      X2T=T+0.5*DT
```

```
      X2=-A2*X2GS**0.5/X2EM**0.5833
      XK2=DT*X2
      XM2=DT*(-AB*X2GS*X2T/X2EM**1.6667-X2GS*X2/X2EM+X2GS/X2T)
      X3GS=GS+0.5*XM2
      X3EM=EM+0.5*XK2
      IF (X3EM .LE. 0.) THEN
      II=II+1
      GO TO 8
      ELSE
      END IF
      X3T=T+0.5*DT
      X3=-A2*X3GS**0.5/X3EM**0.5833
      XK3=DT*X3
      XM3=DT*(-AB*X3GS*X3T/X3EM**1.6667-X3GS*X3/X3EM+X3GS/X3T)
      X4GS=GS+XM3
      X4EM=EM+XK3
      IF (X4EM .LE. 0.) THEN
      II=II+1
      GO TO 8
      ELSE
      END IF
      X4T=T+DT
      X4=-A2*X4GS**0.5/X4EM**0.5833
      XK4=DT*X4
      XM4=DT*(-AB*X4GS*X4T/X4EM**1.6667-X4GS*X4/X4EM+X4GS/X4T)
      DEM=1./6.*(XK1+2.*XK2+2.*XK3+XK4)
      IF (-DEM .GE. EM) THEN
      II=II+1
      GO TO 8
      ELSE
      END IF
      EM=EM+DEM
      GS=GS+1./6.*(XM1+2.*XM2+2.*XM3+XM4)
      T=T+DT
      FRLG=LOG10(-EM**0.6667*DEM/(AB*T*DT))
      GM=A1*T*GS**0.5/EM**0.0833
        ELSE
* SOLN. IN HIGH TEMP. LIMIT
* LOG TERM GIVEN VALUE AT BEGINNING OF INTERVAL
      QC=(0.5+LOG(1.+9.5E-10*GS*T/GM))**0.3333
      Y1=-B2*GS**0.6667/(QC*EM**0.5556)
      YK1=DT*Y1
      YM1=DT*(-AB*GS*T/EM**1.6667-GS*Y1/EM+GS/T)
      Y2GS=GS+0.5*YM1
      Y2EM=EM+0.5*YK1
* IF MASS LOSS TOO RAPID MAKE TEMP. INTERVAL SMALLER
      IF (Y2EM .LE. 0.) THEN
      II=II+1
      GO TO 8
      ELSE
```

```
      END IF
      Y2T=T+0.5*DT
      Y2=-B2*Y2GS**0.6667/(QC*Y2EM**0.5556)
      YK2=DT*Y2
      YM2=DT*(-AB*Y2GS*Y2T/Y2EM**1.6667-Y2GS*Y2/Y2EM+Y2GS/Y2T)
      Y3GS=GS+0.5*YM2
      Y3EM=EM+0.5*YK2
      IF (Y3EM .LE. 0.) THEN
      II=II+1
      GO TO 8
      ELSE
      END IF
      Y3T=T+0.5*DT
      Y3=-B2*Y3GS**0.6667/(QC*Y3EM**0.5556)
      YK3=DT*Y3
      YM3=DT*(-AB*Y3GS*Y3T/Y3EM**1.6667-Y3GS*Y3/Y3EM+Y3GS/Y3T)
      Y4GS=GS+YM3
      Y4EM=EM+YK3
      IF (Y4EM .LE. 0.) THEN
      II=II+1
      GO TO 8
      ELSE
      END IF
      Y4T=T+DT
      Y4=-B2*Y4GS**0.6667/(QC*Y4EM**0.5556)
      YK4=DT*Y4
      YM4=DT*(-AB*Y4GS*Y4T/Y4EM**1.6667-Y4GS*Y4/Y4EM+Y4GS/Y4T)
      DEM=1./6.*(YK1+2.*YK2+2.*YK3+YK4)
      IF (-DEM .GE. EM) THEN
      II=II+1
      GO TO 8
      ELSE
      END IF
      EM=EM+DEM
      GS=GS+1./6.*(YM1+2.*YM2+2.*YM3+YM4)
      T=T+DT
      FRLG=LOG10(-EM**0.6667*DEM/(AB*T*DT))
      GM=B1*QC*T*GS**0.3333/EM**0.1111
      TSTLG=LOG10(9.5E-10*GS*T/GM)
        END IF
* IF STELLAR SPEEDS FALL BELOW A CERTAIN VALUE, TERMINATE
      IF (GS .LE. 3.) THEN
      GO TO 7
      ELSE
      END IF
      GMLG=LOG10(GM)
      EMLG=LOG10(EM)
      GSLG=LOG10(GS)
      TLG=LOG10(T)
      IF (II .GE. 2) THEN
```

```
* EMLG IS LOG(MASS), GSLG IS LOG(GAMMA-S), TLG IS LOG(TEMP.),
* TSTLG IS LOG(TEST FOR LOW OR HIGH TEMP. LIMIT), GMLG IS
* LOG(GAMMA-M), AND FRLG IS LOG(ADDITIONAL ACCELERATION DUE
* TO MASS LOSS / DECELERATION DUE TO RADIATION DRAG)
      PRINT 9, II,EMLG,GSLG,TLG,TSTLG,GMLG,FRLG
9     FORMAT (I3,6F9.3)
      ELSE
      END IF
4     CONTINUE
      IF (II .EQ. 1) THEN
      PRINT 5, II,EMLG,GSLG,TLG,TSTLG,GMLG,FRLG
5     FORMAT (I3,6F9.3)
      ELSE
      END IF
6     CONTINUE
7     END
```

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