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Optimizing Consumer-Centric Assortment Planning under Cross-Selling Effects

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OPTIMIZING CONSUMER-CENTRIC ASSORTMENT PLANNING UNDER CROSS-SELLING EFFECTS

A Dissertation Outline Presented

by

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To the memory of my father, Aly, and my brother Khaled..

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I praise God for walking me through the process of writing this dissertation.

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ABSTRACT

OPTIMIZING CONSUMER-CENTRIC ASSORTMENT PLANNING UNDER CROSS-SELLING EFFECTS

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Central to modern-time, consumer-focused retailing is the ability to provide attractive and reasonably-priced product assortments for different customer profiles. To this end, retailers can benefit from the use of data analytics in order to identify distinct customer segments, each characterized by their buying power, shopping behavior, and preferences. Further, retailers can also benefit from a careful examination of alternative procurement options and cost levers associated with products that are considered for inclusion in the assortment. Issues of assortment planning lie at the interface of operations and marketing. Profitable planning trade-offs can be identified using an optimization methodology and are simultaneously driven by consumer preferences and supply cost considerations. This dissertation proposes and investigates novel, integrated optimization models for assortment planning with the

following overarching objectives: (i) To reveal insights into assortment decisions under product substitutability or complementarity and multiple customer segments; (ii) to improve the computational tractability of (nonlinear discrete) optimization models that arise in such contexts and to demonstrate their efficacy for large-scale data instances.

In the first essay, we investigate the joint optimization of assortment and pricing decisions for complementary retail categories with relatively popular products having high and stable sales volumes, such as fast-moving consumer goods. Each category comprises substitutable items (e.g., different coffee brands) and the categories are related by cross-selling considerations that are empirically observed in marketing studies to be asymmetric in nature. That is, a subset of customers who purchase a product from a primary category (e.g., coffee) can typically opt to also buy from one or several complementary categories (e.g., sugar and/or coffee creamer). We propose a mixed-integer nonlinear program that maximizes the retailer's profit by jointly optimizing assortment and pricing decisions for multiple categories using a deterministic maximum-surplus consumer choice model. A linear mixed-integer reformulation is developed, which effectively enables an exact solution to large, industry-sized problem instances using commercial optimization solvers. Our computational study indicates that overlooking cross-selling between retail categories can result in substantial profit losses, suboptimal (narrower) assortments, and inadequate prices. The demonstrated tractability of the proposed model paves the way for "store-wide" optimization of categories that exhibit significant complementarity, which retailers can infer from market basket analysis.

The second essay addresses an assortment packing problem where a decision-maker optimizes the assortment and release times of products that belong to different categories over a multi-period planning horizon. Products in a same category are substitutable, whereas products across categories may exhibit complementarity

relationships. All products have a longevity over which their attractiveness gradually decays (e.g., electronics or fashion products), while being positively or negatively impacted by the specific mix of substitutable or complementary products that the retailer has introduced. Our proposed 0-1 fractional program employs an attraction demand model and subsumes recent assortment packing models in the literature. We highlight the effect of overlooking cross-selling and cannibalization on the profit using an illustrative example. We develop linearized reformulation that afford exact solutions to small-sized problem instances. Furthermore, a linear programming-based heuristic approach is devised and is demonstrated to yield near-optimal solutions for large-scale computationally challenging problem instances in manageable times. Model extensions are discussed, especially in the context of the movie industry where exhibitors have to decide on the assortment of movies to display and their optimal display times.

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CHAPTER 1

INTRODUCTION AND MOTIVATION

Retail is a complex and ever-changing business environment that has contributed to the economic growth of many nations. In the US, the retail industry is the highest employer compared to other industries[1]. Figure ?? shows a growth chart of the levels of retail sales in the US over the last 5 years[2]. Due to its importance, retail operations management gained a lot of research attention recently.

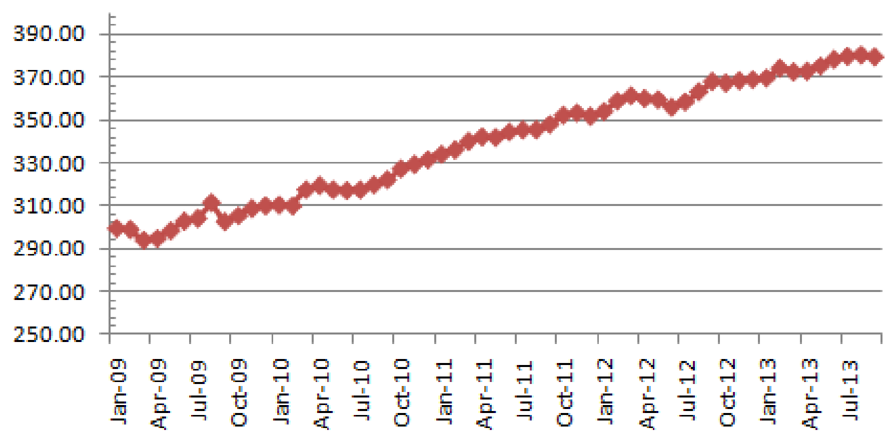

Figure 1.1: US Retail Sales Data in Billion Dollars (Jan 2009 - Sep 2013)

One of the key revenue management challenges for a retailer is to determine the set of products to carry in each store over time. The choice of a set that meets consumers' desires and preferences has a direct positive impact on sales and profits.

[1] "CPI Detailed Report Data for February 2013," Bureau of Labor Statistics, US Department of Labor, last retrieved from http://www.bls.gov/cpi/cpid1302.pdf on 11/1/13

[2] Source: Advance Monthly Retail Trade Report, United States Census Bureau, last retrieved from http://www.census.gov/retail/marts/www/download/text/adv44000.txt on 11/1/13

Thus, the goal of assortment planning is to find a set of products that maximizes a retailer's total profit.

Successful firms do appreciate the importance of assortment planning in today's retail marketplace. According to Aberdeen Group's *Precision Merchandising* study[3], the two most important key drivers for the increased focus on assortment planning solutions were the need to maximize margin (56%) and the need to maximize inventory returns (48%). This study also suggests that 71% of the best-in-class companies are able to create tailored specific assortments.

The challenge arises from the relationship between the number of products chosen and the total profit. A narrower assortment may result in losing store traffic as customers who seek more variety and competition may choose to shop elsewhere. On the other hand, a broader assortment implies more fixed and handling costs, slow inventory and poor product availability.

1.1. Background on Retail Management and Analytics

Retail management started thousands of years ago from the Mediterranean regions and spread to Egypt and Babylonia. It flourished in Rome, then after the destruction of the Roman Empire it spread across the globe. With the sophistication of modern life, the increase in population, rise of big data and the evolution of technology, retail management became more challenging. Focusing on the behavioral aspects of consumers and using the right tools and techniques in analytics, retailers can manage their business in a competitive market so as to attract the maximum number of customers and thus maximizing their profits.

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[3] "A Roadmap to Integrated Assortment Planning," last retrieved from http://www.popai.com/store/downloads/WhitePaper-Roadmap-To-Intergrated-Assortment-Planning-2010.pdf on 11/1/2013

Key decisions that are associated with retail management are vast. Variety, or the assortment planning decision, is concerned with finding the best set of products to display from a set of all candidate products. Replenishment and inventory planning decisions are concerned with the optimal amounts of products to order at a certain time period. The pricing decision defines how prices should be set in a way that enables maximum profits while preserving customer satisfaction and loyalty. Shelf-space allocation is another key decision that determines the amount of space that should be allocated to each product. Forecasting, promotions and discounts management and loss prevention are among many other decisions in retail management.

Analytics is the discovery and communication of meaningful patterns in data. It is of special importance in fields that are rich in data such as retail business and banking. Analytics relies on the application of statistical analysis, computer programming and operations research to quantify performance. What makes it a broader term than analysis in a way that it extends the descriptive and predictive modeling of the data analysis phase, into a prescriptive modeling phase that recommends an action and guides the decision making process (see Cooper 2012, and van Harmelen and Workman 2012 for more detailed information about analytics).

The advancement in technology plays an important rule in retail management. For many years, transactional data was the only resource that retailers had. After the rise of analytical data, retailers have been empowered with the ability to understand their business better and make decisions once they had access to the point-of-sale (POS) data. This type of historic customer transactions data opened new analytical insights that has not been discovered before. Retail business became able to analyze customer behavior and obtain useful patterns and trends that affect sales and profits (see Davenport et al. 2010 for more discussion about analytics at work). Three examples of such useful patterns that are used in this thesis are: (i) Cross-selling effect between complementary categories and how it can significantly affect the retailer's assortment and pricing decisions; (ii) cannibalization effects between substitutable products; and (iii) attractiveness of a product that decays over time specially with products like apparel or movies.

1.2. Consumer Choice Models

Consumer choice models, or demand models, serve as the fundamental base for assortment planning optimization, and can be divided into two categories: (i) Utility-based models, which assume that customers associate a utility value with each product, and (ii) exogenous demand models that specify exactly the demand for each product as well as the substitution behavior.

1.2.1 Utility-Based Models

Utility is the measure of satisfaction obtained from consuming a good or service. Utility-based consumer choice models assume that each customer segment i has a certain utility for each product j denoted by U_{ij}. In addition, each customer segment i has a utility for the no-purchase option as well that is denoted by U_{i0}. Anything that makes the customer better off is assumed to raise her utility. Thus, given a certain assortment of products, each customer makes a decision that is based on her highest utility.

Several utility-based models were used in the literature. The deterministic maximum utility model used by Dobson and Kalish (1988), is one of the simplest models which assumes that the customer chooses a product, from the given assortment, that yields the highest positive utility. Utility here is sometimes called *surplus* and it is calculated as the difference between the price that the customer is willing to pay or what is called *reservation price*, and the actual price set by retailer. If such product does not exist, or in other words, if all products yield a negative surplus, the customer chooses not to purchase. In the treatment of the joint assortment and pricing problem in Chapter 2, a deterministic utility-based choice model is utilized.

Deterministic utility models imply that a customer would make the same choice over time when faced with the same set of alternatives. In practice, however, this is not the case. Variations in choice were observed with similar customers similar alternatives. Expected utility or probabilistic choice models take into account that variation in behavior.

The Multinomial Logit (MNL) model is the most popular probabilistic utility-based model in the literature (Ben-Akiva and Lerman 1985, Anderson et al. 1992). Under the MNL, the utility for customer segment i for product j at time t is: $U_{ijt} = u_{ijt} + \epsilon_{ijt}$, where u_{ijt} is the deterministic expected utility for item j and the utility of the no-purchase option is: $U_{i0t} = u_{i0t} + \epsilon_{i0t}$, where u_{i0t} is the expected no-purchase utility and ϵ_{ijt} represent the random component of the utility and are independent and identically distributed (i.i.d.) Gumbel random variables. A customer chooses the item with the highest utility among the set of available choices. Thus, the probability that a customer chooses item j is:

$$\Pi_{ijt} = Pr\{U_{ijt} = max_{k=0}^{m}(U_{ikt})\}$$

This probability together with the closed maximization property of Gumbel distribution lead to the closed form expressions for purchase probabilities, given by:

$$\Pi_{ijt} = \frac{e^{u_{ijt}/\mu}}{\sum_{k=0}^{n} e^{u_{ikt}/\mu}}, \quad \forall i, j, t,$$

where Π_{i0t}, is the probability that the customer purchases nothing, $\forall i, t$.

The main drawback of the MNL model comes from a property that is referred to as the Independence of Irrelevant Alternatives (IIA). This property holds if the ratio of choice probabilities of two alternatives is independent of the other alternatives in the choice process. IIA property would not hold in cases where there are subgroups of products in the choice set such that the products within the subgroup are more similar with each other than across subgroups. The Nested Logit Model, introduced by Ben-Akiva and Lerman (1985), deals with the IIA property, in which a two-stage nested process is used to model choice.

The locational choice model is yet another utility-based model that found its roots from the study of pricing and location decisions by Hotelling (1929). Lancaster (1966) extended the work and proposed a locational model of consumer choice. Each product is assumed to be located as a vector on the characteristics space based on its attributes. Each consumer is located in a point X in the same space, this point represents the consumers most preferred set of product attributes. The utility of product j to the consumer is: $U_j = U - g(|X - b_j|)$, where U represents the utility of a product at the ideal location and g(.) is a function representing the disutility associated with deviation from the ideal location, where $|X - b_j|$ is the euclidean or rectilinear distance between the product and the consumers ideal location.

1.2.2 Exogenous Demand Models

In the exogenous demand model, every customer chooses a favorite product from the set of all products Ω. Let p_j is the probability a customer chooses product j, where $\sum_{j\in\Omega+\{0\}} p_j = 1$. If the customers favorite product is not available, she chooses her second favorite with probability δ or chooses not to purchase with probability $1 - \delta$, the same procedure repeats if the product is not available by choosing her third favorite product and so forth. The probability of substituting product j with another product k is α_{kj}, which is defined by a substitution matrix.

The exogenous demand model is the most commonly used consumer choice model in the literature on inventory management for substitutable products (see Yücel et al. 2009). It is more flexible than utility-based models, and can easily accommodate different substitution structures, but is sometimes difficult to estimate all its parameters in practice.

1.3. Assortment Planning

A category in retail business is a group of related products that share similar attributes. For example, in grocery stores, coffee could be a category that includes several subcategories like regular coffee and decaffeinated coffee.

Substitution is a term that is commonly related to a customer's choice decision. It can be defined as the willingness of customers to purchase another similar product if their favorite product is not available at the time of purchase. Suppose there is a customer with a particular product in mind (it could be a certain brand), and on looking for that product in store she can not find it. If there is a probability that she will pick another brand from the same category, then substitution occurs. Two types of substitution exist, the first one is called dynamic (or stockout-based) substitution and it occurs when the retailer originally carries this product in the assortment but it stocked out at the time of purchase. The second is called static (or assortment-based) substitution and occurs when the customer's favorite product is not part of the retailer's assortment.

Two products are complementary if they relate to and complement each other. Take for instance coffee and creamer or torch and batteries. With a negative cross-price elasticity of demand between complementary products we know that the increase in price of one of them would decrease the demand of the other and vice versa.

Assortment optimization research can be viewed under two main modeling styles: (i) Stylized models, aimed at providing insights into structural properties of the optimal solution, and (ii) optimization models, aimed at providing decision makers in the retail business with techniques and insights that help the decision making process.

1.3.1 Stylized Models

In assortment planning, the work by van Ryzin and Mahajan (1999) is one of seminal work in this stream. van Ryzin and Mahajan (1999) were the first to study assortment planning and inventory decisions under the MNL model and static substitution. They provide insights into the structural properties of the optimal solution, the main insight is that the optimal assortment consists of the most popular (or highest utility) products from the finite set of potential products to offer. The work in Mahajan and van Ryzin (2001) studies the same problem under dynamic substitution.

The van Ryzin and Mahajan model acts as a basic model in assortment planning under MNL and thus was extended in various ways. For example, Maddah and Bish (2007) extend the van Ryzin Mahajan model by considering the pricing decisions as well. The work by Cachon et al. (2005) study the same problem under the effect of consumer search. Aydin and Porteus (2008), Maddah and Bish (2007), and Maddah et al. (2013), who endogenize pricing. Caro and Gallien (2007) develop a stylized model that formulates the dynamic assortment planning problem faced by apparel retailers.

Related works on assortment planning within a newsvendor-type supply setting are those adopting choice models other than logit such as locational choice by Gaur and Honhon (2006) and exogenous choice by Smith and Agrawal (2000).

1.3.2 Optimization Models

Optimization models in the assortment planning literature provide the retailer with either exact or approximation methods that help in solving the problem. That stream began with the product line design problem by Green and Krieger (1985). They formulated a problem where a firm chooses k products out of the set of all potential products as to maximize both the consumers welfare and the firms profits.

Dobson and Kalish (1988, 1993) and Kohli and Sukumar (1990) extend the work on product line design and suggest more heuristics. Other works include Kök and Fisher (2007) who develop a heuristic for the joint assortment and inventory problem and Anupindi et al. (2009) who formulate an integer programming problem for the variety and stocking decisions in retail category management, they added a penalty if the consumers preferred item is not offered in the assortment, which affects the long-term profits. Subramanian and Sherali (2010), propose MIP formulations to model certain optimal pricing problems that arise within real-life analytic applications implemented at Oracle Corporation, and Ghoniem and Maddah (2013) integrate three key decisions by jointly optimizing assortment, pricing and inventory in a single category setting.

Other works include practical considerations such as bundling (Hanson and Martin 1990), generalized choice models (Hanson and Martin 1996), shelf space elasticity (Corstjens and Doyle 1981, Irion et al. 2011 and Martín-Herrán et al. 2006).

Our work is this dissertation can be classified under the optimization models stream. We use mathematical programming to design the problem and solve it using exact methods highlighting managerial insights that guides the decision maker.

1.4. Integrated Decision Making

Recently, more complex retail problems started to emerge that are based on jointly optimizing either one or several retail decisions under specific consumer behavior observation that became available through analytics. Such models aim at capturing a more realistic aspect of the problem. For instance, Cachon et al. (2005) study the van Ryzin and Mahajan (1999) model of assortment planning in the presence of consumer search. The paper by Maddah and Bish (2009) on "locational tying" does consider some aspects of integrated decisions under cross-selling within a stylized two-product, single-period, newsvendor-like framework.

Our work in this thesis is very related to that stream of literature as we investigate the effect of cross-selling in a multi-category assortment planning and pricing problem in the first essay. While on the second essay we investigate the effects of cross-selling and cannibalization, and integrate aspects of product attractiveness with the assortment planning decision as well.

1.5. Organization of the Dissertation

The remainder of this document is organized as follows: Chapter 2 investigates the joint optimization of assortment and pricing decisions for complementary retail categories, where each category comprises substitutable items and the categories are related by asymmetric cross-selling considerations. Chapter 3 examines a multi-category, multi-period assortment packing problem where a firm seeks to determine the optimal assortment and release times of products that decay over time. Finally, Chapter 4 concludes the document by the findings from Chapter 2 and Chapter 3, and discusses directions for future research.

CHAPTER 2

OPTIMIZING ASSORTMENT AND PRICING OF MULTIPLE RETAIL CATEGORIES WITH CROSS-SELLING

This chapter investigates the joint optimization of assortment and pricing decisions for complementary retail categories. Each category comprises substitutable items (e.g., different coffee brands) and the categories are related by cross-selling considerations that are empirically observed in marketing studies to be asymmetric in nature. That is, a subset of customers who purchase a product from a primary category (e.g. coffee) can opt to also buy from one or several complementary categories (e.g., sugar and/or coffee creamer). We propose a mixed-integer nonlinear program that maximizes the retailer's profit by jointly optimizing assortment and pricing decisions for multiple categories under a classical deterministic maximum surplus consumer choice model. A linear mixed-integer reformulation is developed which effectively enables an exact solution to relatively large problem instances using commercial optimization solvers. This is encouraging, because simpler product line optimization problems in the literature have posed significant computational challenges over the last decades and have been mostly tackled via heuristics. Moreover, our computational study indicates that overlooking cross-selling between retail categories can result in substantial profit losses, suboptimal (narrower) assortments, and inadequate prices.

2.1. Introduction and Motivation

Retailers face the challenging problem of selecting a subset of products to carry in stores or online and setting prices over time in a manner that appeals to a variety of consumers and maximizes profit. This has motivated a rich literature on the so-called *product line optimization problem*; a difficult combinatorial optimization problem that seeks to determine product selection and pricing strategies under anticipated consumer behavior and choice rationales (Dobson and Kalish 1993). With the advent of retail analytics, the focus of practitioners gradually shifted from single-product, "brand management" to multi-product "category management" (CM), e.g. Basuroy et al. (2001), Hall et al. (2010), and Zenor (1994). With growing interest in category management and market basket analysis (e.g., Mild and Reutterer 2003 and Russell and Petersen 2000), it has become apparent, however, that certain retail categories are interdependent and, therefore, should not be planned in isolation. Despite this industry trend, the effect of *cross-selling* on the optimization of interdependent retail categories remains understudied in the academic literature (see Maddah et al. 2011 for a review of works on category optimization in retailing).

Marketing studies provide strong empirical support for the notion of "cross-category" shopping or cross-selling (e.g., Mulhern and Leone 1991; Walters 1988, 1991). For example, a price discount on a product category (e.g., spaghetti, cake mix, fabrics) can substantially stimulate sales for a complementary product category (e.g., pasta sauce, cake frosting, sewing tools). In addition, an *asymmetric* cross-selling effect is often observed, whereby the sales of one "primary" category (e.g., cake mix) drive the demand of another "secondary" complementary category (e.g., cake frosting), with the opposite effect (secondary product driving the primary product demand) being rather negligible. For example, Walters (1991) argue that price promotions of spaghetti resulted in a significant increase in the sales of the spaghetti sauce, whereas the reverse phenomenon did not occur with price promotions on spaghetti sauce. Mulhern and Leone (1991) also report similar results indicating an asymmetric cross-selling effect of cake mix over cake frosting (see also Manchanda et al. 1999 and Shankar and Kannan 2014 for further discussion).

Our research is prompted by the emerging topic of managing multiple retail categories using mathematical programming and makes the following conceptual and computational contributions. First, as far as the marketing literature is concerned, this chapter contributes to the notion of multi-category management under cross-selling by proposing a mixed-integer nonlinear programing model. The developed model jointly optimizes assortment and pricing decisions for multiple complementary retail categories under a classical maximum-surplus consumer choice model. All retail categories are composed of substitutable products that reflect the same need for the consumer but differ in some minor attributes (e.g., different brands of coffee or sugar of the same size). A fraction of shoppers of the primary category consider buying from the secondary categories in a way that reflects asymmetric cross-selling effects. The second contribution of this chapter is computational in nature. Specifically, the proposed mixed-integer linear reformulation of the model is demonstrated to enable exact solutions to large-scale instances, which suggests that integrated optimization of multiple dependent categories is now computationally tractable. This is encouraging, because much of the extant literature on product line optimization examined more simplified settings (e.g. Dobson and Kalish 1988, 1993) and, recognizing the difficulty of such mixed-integer (nonlinear) programs, resorted to using constructive and greedy heuristics.

The remainder of the chapter is organized as follows. Section 2.2 briefly reviews the literature related to product line optimization and cross-selling. In Section 2.3, the problem is formally stated along with our notation, the proposed mixed-integer nonlinear programming model, and its linear reformulation. In Section 2.4, we present an illustrative example, followed by a computational study that involves relatively large problem instances. Section 2.5 concludes the chapter with a discussion of our findings and insights.

2.2. Literature Review

This chapter relates to the literature on retail category decision analysis and optimization. In particular, it involves the so-called *product line optimization problem* (simultaneous product *selection* and *pricing*) and cross-selling – a phenomenon that is identified by market basket analysis. These two aspects of the literature are discussed in the remainder of this section with greater focus on data-driven optimization-based approaches, as opposed to stylized models that investigate analytical results under simplified assumptions. The more inquisitive reader is referred to Maddah et al. (2011).

2.2.1 Product Line Optimization

The product line optimization problem, which integrates product line selection and pricing decisions, lies at heart of our work. Specifically, product line selection is concerned with optimizing the assortment or variety of products or services offered in a product line by a seller. In the context of our work, the seller is a retailer, the buyer is a shopper, and product lines of interest are retail categories. Early discussions of product line pricing with substitutable and complementary products date back to Dean (1950). Seminal conceptualizations of product line optimization problems appeared later and continue to motivate a great deal of research due to their practical relevance. A key element of these studies is the adoption of an adequate consumer choice model which captures the behavior of consumers and their anticipated purchase decisions in reaction to assortments and prices set by a seller. In particular, Zufryden (1977) and Green Krieger (1985) consider the "single-choice, deterministic, behavior" whereby a consumer chooses a single product that yields a greatest nonnegative *surplus* (if available). Furthermore, a consumer is assumed to refrain from buying if all products yield negative surpluses. Here, the consumer surplus for a product is defined as the difference between the consumer *reservation price* (or the maximum monetary value he/she is willing to pay for this product) and the price set by the seller. To incorporate this consumer choice model in a mathematical program for planning purposes, it is necessary to estimate reservation prices for different products across distinct customer segments. This can be achieved using different techniques, including conjoint analysis (see, for example, Zufryden 1977; Green and Srinivasan 1978, 1990; Dobson and Kalish 1988, 1993; Hanson and Martin 1990; Shioda et al. 2011; Smith et al. 2009, and references therein).

As noted in Kraus and Yano (2003), the deterministic single-choice consumer model is employed in "most articles on product line optimization." Zufryden (1977) discusses justification for this choice model and refers to earlier studies that provide empirical evidence in support of this consumer behavior (Best 1976; Braun and Srinivasan 1975; Pessemier et al. 1971). In particular, Johnson (1976) supports this behavioral assumption for applications with a high degree of sensitivity to the surplus; in this case, a customer commits to a product that yields a maximum surplus and would consider switching only if a new product is introduced with a better surplus. For other applications where the consumer choice may be less driven by surplus considerations, and more by brand image or quality etc., then a stochastic model may be more adequate. Ghoniem and Maddah (2013) also provide empirical support for this deterministic consumer choice model based on Tuna data from a grocery store in the Northeast US. The data spans a year and a half of transactions for a light Tuna product line, whereby the demand distribution across substitutable products is largely due to price discounts introduced by the retailer and is well-approximated by the deterministic single-choice consumer model.

In Zufryden (1982), a 0-1 integer program is formulated in order to tackle the joint product line selection and design problem with the objective of maximizing the weighted sum of consumers choosing a product line, under the single-choice deterministic rule. Beyond the modeling contribution, no solution methodology is delineated.

Green and Krieger (1985) examine two product line selection problem variants: The first is a *buyer-welfare* problem that optimizes assortment decisions in a fashion that maximizes the total consumer surplus, whereas the second is a *seller-welfare* problem that maximizes the seller's profit. Both variants are examined with the assumption of a single-choice deterministic consumer model and are solved using heuristics, with a dismissal of optimization-based approaches. McBride and Zufryden (1988), however, re-examine the seller-welfare product line selection problem using integer programming and argue that optimal solutions are attainable for their simulated instances on mainframe and personal computers.

Dobson and Kalish (1988, 1993) consider the more challenging product line optimization problem with simultaneous product selection (assortment) and pricing decisions under the single-choice deterministic consumer model. The joint optimization of assortment and pricing decisions prompts a 0-1 mixed-integer *nonlinear* programming formulation that is shown to be NP-Complete. Due to the perceived intractability of this formulation, the authors resort to using constructive heuristics (Dobson and Kalish, 1993). Shioda et al. (2011) revisit the product line optimization problem in Dobson and Kalish (1993) with the goal of enhancing its tractability via valid inequalities and refined heuristics. They refer to the single-choice deterministic consumer model as "the maximum utility or the envy-free pricing model." The latter designation is borrowed from the microeconomics literature (e.g., Walras 1954).

Describing the single-choice deterministic consumer model as a "max-surplus choice rule," Burkart et al. (2012) investigate product line pricing for services over a selling horizon with capacitated offerings. As consumers commit to products, these are depleted and may become unavailable, in which case consumers dynamically substitute and choose an available product that yields a maximum, nonnegative surplus. Ghoniem and Maddah (2013) also examine an extension of the nonlinear MIP formulation of the product line optimization problem in Dobson and Kalish (1993) whereby inventory considerations are integrated with assortment and pricing decisions over a multi-period horizon. The authors propose an effective linear reformulation of the model and develop several managerial and computational insights.

Although the deterministic consumer choice model has been widely used in the literature (Kraus and Yano 2003), several studies consider product line pricing problems under probabilistic consumer choice models. For example, Chen and Hausman (2000) examine the product line optimization problem under the logit choice model, discrete price values, and lower and upper bounds on the size of the assortment. Kraus and Yano (2003) investigate a similar problem under a so-called *share-of-surplus* choice model. Here, the fraction of a customer segment that buys a positive-surplus product is determined as the ratio of the surplus of this product over the total surplus across products having a positive surplus for that segment. This ratio determines the relative probabilities of customers buying products and involves positive-surplus products only. This contrasts with the multinomial logit choice model, where a customer can buy a negative-surplus product with a positive (albeit small) probability. Subramanian and Sherali (2010) also model category pricing under logit demand and propose a reformulation of a nonlinear fractional program using an effective linearization scheme. They also take into account several common industry practices related to targets on volume and sales levels, discrete prices exhibiting a ladder structure, and relative pricing rules for store vs. national brands. In a recent paper, Keller et al. (2014) investigate product line pricing problems under attraction demand models. The authors identify conditions under which non-convexities that arise in the formulation can be circumvented by recasting the model as a convex optimization problem, thereby significantly enhancing the problem tractability.

2.2.2 Cross-Selling Considerations

Although studies on using cross-selling to optimize product line planning is still scarce, two research streams are emerging: The first considers two-product settings, whereas the second addresses category optimization under cross-selling. Few works consider asymmetric cross-selling within a two-product context whereby demand for a primary product drives that for a secondary product, as in our present work. Aydin and Ziya (2008) analyze an *up-selling* practice, where upon the purchase of a *regular* product, whose price is exogenous, the buyer is offered to buy a *promotional* product, possibly at a discount. They focus on utilizing dynamic pricing for clearing the inventory of the promotional product. Beyond certain similarities in the consumer choice model with Aydin and Ziya (2008), our work has a distinctive focus on static pricing and assortment optimization for multiple complementary categories. Zhang et al. (2011) consider the effect of cross-selling on inventory decisions within a joint replenishment model of two products, a *major* and a *minor* one, with a common ordering cycle. The authors capture the effect of reduction in the demand of the minor product as a result of the major product planned stock-out (in a backordering setting), with the classic economic order quantity (EOQ) setting.

Maddah and Bish (2009) also investigate a stylized model for the notion of *locational tying* of two retail products, a *primary* and a *secondary* one, where the secondary product is offered in two distinct locations in a store, its own department and the primary product's department. This leads to two demand streams for the secondary product, an indirect one (which depends on the primary product price) due to cross-selling at the primary product location, and a direct one at its appropriate department. The demand model in this chapter also considers two demand streams for secondary products, even though secondary products are displayed in their own department only.

The second stream of literature, which relates to this chapter, is on category optimization under cross-selling. Agrawal and Smith (2003) consider joint assortment and inventory optimization under exogenous choice for substitutable sets. Each exogenous set is a combination of complementary products. This, however, introduces the hurdle of explicitly enumerating all possible combinations of complementary products for which consumer reservation prices need to be estimated. In contrast, we model complementarity across categories and fewer parameters that reflect the cross-selling potential of a customer segment need to be estimated (as detailed next in Section 2.3). Moreover, our focus is different than Agrawal and Smith (2003) as we consider pricing and assortment decisions.

Also of interest is the work by Cachon and Kök (2007) where the assortments of two categories offered simultaneously by two retailers are optimized under a competitive duopoly setting. According to a nested logit choice, customers choose a store first, and then choose to buy from one or both categories in the store. Three customer segments are considered pertaining to the two categories and to the "basket" composed of products from both categories. The distinctive feature of our model, with respect to Cachon and Kök (2007), is that we consider multiple customer segments for each category with customer purchases being endogenously deduced from asymmetric cross-selling effects. In addition, while Cachon and Kök (2007) consider assortment decisions only with a cost which is convex in the assortment size (akin to a newsvendor-type supply setting), we consider assortment and pricing decisions with a variable linear cost and a fixed cost for offering a product in the assortment.

Rodríguez and Aydin (2011) consider assortment and pricing decisions for two complementary categories, involving a *required* and an *optional* product, respectively, in a newsvendor-type supply setting and under logit demand. The authors study a stylized model with a single customer segment with two purchase scenarios: (i) Purchase with a combined utility for both products or (ii) a sequential purchase approach, where a customer first buys a required product and then considers buying an optional product. The demand model in the sequential setting bears certain similarities with our setting, with the difference that we also consider a direct demand stream for the secondary (optional) category. Our work focuses on developing an optimization model with a maximum-surplus choice model, multiple customer segments, and possibly more than two categories.

2.3. Problem Statement and Formulation

This section provides a formal problem statement for multiple category optimization with cross-selling and introduces our notation along with our proposed mixed-integer nonlinear formulation. The model is then recast as a mixed-integer linear reformulation and can, therefore, be solved using standard commercial optimization solvers such as CPLEX.

2.3.1 Mixed-Integer Nonlinear Formulation

We examine the setting where a retailer seeks to jointly optimize assortment and pricing decisions for multiple retail categories under asymmetric cross-selling. We adopt the classical assumption that any customer buys at most one product from a given category of substitutable products, as is common under the maximum-surplus choice rule. Specifically, we consider $\mathcal{L}$ distinct categories, where the first category is referred to as the primary category, whereas the remaining $|\mathcal{L}| - 1$ categories are secondary in that each complements the primary category. The chosen assortment for any category, denoted by $\mathcal{P}_\ell$, shall comprise substitutable products that are selected from a broader set of candidate products Ω_ℓ, with $\mathcal{P}_\ell \subseteq \Omega_\ell$, $\forall \ell \in \mathcal{L}$, and $\Omega_{\ell_1} \cap \Omega_{\ell_2} = \emptyset$, $\forall \ell_1, \ell_2 \in \mathcal{L}, \ell_1 \neq \ell_2$. For clarity in the notation, we shall designate by j^ℓ the j^{th} product in Ω_ℓ. For example, products 1^1 and 3^2 respectively refer to the first candidate product of the primary category and the third candidate of a secondary category. For any category ℓ, let $\mathcal{C}_\ell$ be the set of customer segments or *direct customers* interested in buying from category ℓ. It is assumed that customer segments in the same category and accross different categories are disjoint. For example, Figure 1 represents a setting with two categories (laptops and printers), where the first category has three distinct customer segments and the second has two customer segments. For clarity in the notation, let i^ℓ be the i^{th} customer segment of category ℓ. For example, customer segments 2^1 and 3^2 respectively refer to the second customer segment of the primary category and the third customer segment of the secondary category. Furthermore, the retailer can estimate from experience and historical data (or anticipates based on surveys and market analysis) that a fraction γ_i^k of customer segment $i \in \mathcal{C}_1$, upon purchasing a product from the primary category ($\ell = 1$), would also consider purchasing a product from a secondary category $k \in \mathcal{L} \setminus \{1\}$. Such customers will be referred to as *cross-selling customers*. For example, in Figure 1, only a fraction of the three customer segments of the primary category would consider cross-selling.

We denote by α_{ij}^ℓ the reservation price (or valuation) of customer segment $i \in \mathcal{C}_\ell$ for product $j \in \Omega_\ell$. Likewise, let β_{ij}^k be the reservation price of a cross-selling customer $i \in \mathcal{C}_1$ for a secondary product $j \in \Omega_k$, $\forall k \in \mathcal{L} \setminus \{1\}$. Reservation prices are assumed to be known to the retailer and can be estimated using such techniques as those discussed in Section 2. The adopted maximum-surplus consumer choice model stipulates that a direct or a cross-selling customer would only buy a product that yields a maximum, nonnegative surplus. The latter is measured as the difference between the exogenous customer reservation prices and the endogenous prices set by the retailer. If prices are set so that all surplus values turn out to be negative for a given customer segment, the customer is priced out of the market and will opt not to buy from this retailer in this planning horizon. The cost structure we consider involves variable wholesale costs, as well as an additional fixed cost for offering a product in the assortment at the beginning of the selling season, which is also typical in the literature (e.g. Dobson and

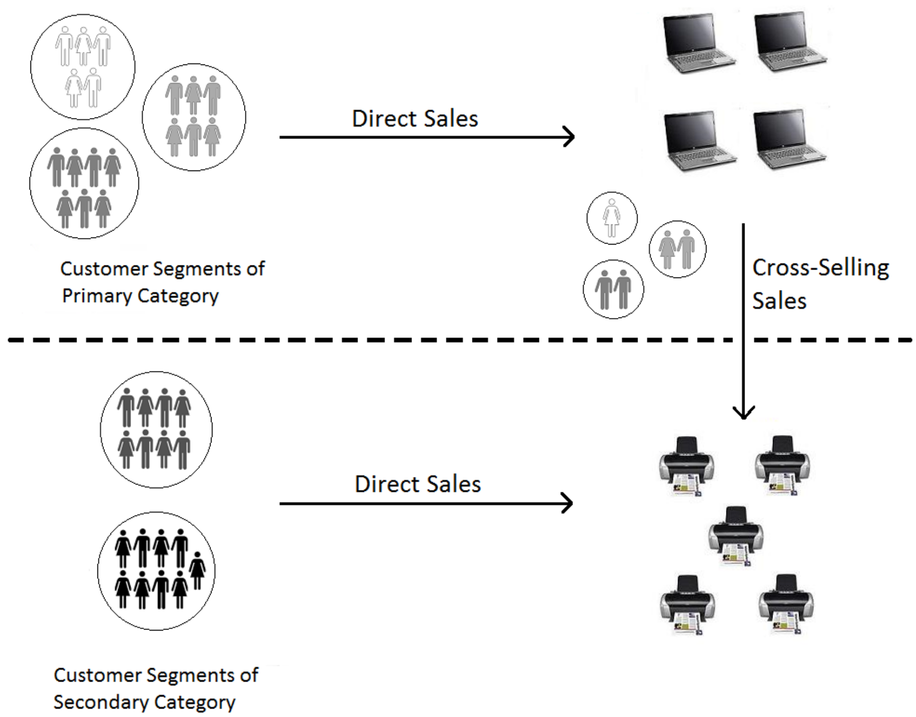

Figure 2.1: Example of a Store Grid Layout

Kalish 1988, 1993; Agrawal and Smith 2003; Anupindi at al. 2009). The objective is to maximize the retailer's profit by selecting an optimal assortment for each category, along with optimal pricing decisions.

Consider the following notation:

Input Parameters

- $\mathcal{L}$: Set of distinct product categories.
- Ω_ℓ : Set of all potential substitutable products in category ℓ, $\forall \ell \in \mathcal{L}$.
- $\mathcal{C}_\ell$: Set of customer segments that are interested in purchasing from category ℓ, $\forall \ell \in \mathcal{L}$. In Figure 1, $\mathcal{C}_1$ and $\mathcal{C}_2$ comprise 3 and 2 distinct customer segments, respectively.

- s_i^ℓ : Number of customers in segment i for category ℓ, $\forall i \in \mathcal{C}_\ell, \ell \in \mathcal{L}$. For example, s_1^1 refers to the number of customers in segment 1 for category 1, whereas s_1^2 designates the number of customers in segment 1 for category 2. In Figure 1, $s_1^1 = 5$ and $s_1^2 = 8$.

- α_{ij}^ℓ : Reservation price of customer segment i for product j in category ℓ, $\forall i \in \mathcal{C}_\ell, j \in \Omega_\ell, \ell \in \mathcal{L}$. For example, α_{13}^2 is the reservation price of customer segment 1, *direct customer of category 2*, for product 3.

- β_{ij}^k : Reservation price of customer segment i of the *primary* category for a *secondary* product j in category k, $\forall i \in \mathcal{C}_1, j \in \Omega_k, k \in \mathcal{L} \setminus \{1\}$. For example, β_{13}^2 is the reservation price of customer segment 1, *direct customer of the primary category*, for product 3 in category 2. This relates to a customer (of the primary category) who buys from a secondary category by cross-selling.

- γ_i^k : Fraction of customer segment i of the primary category who, upon purchasing a primary product from Ω_1, considers purchasing a complementary product from the secondary category set Ω_k, $\forall i \in \mathcal{C}_1, k \in \mathcal{L} \setminus \{1\}$. For example, referring to the first customer segment of the primary category in Figure 1, only 1 out of 5 customers would consider cross-selling and, hence, $\gamma_1^2 = 0.2$.

- f_j^ℓ : Fixed cost for the inclusion of product j into $\mathcal{P}_\ell$, the chosen assortment from category ℓ, $\forall j \in \Omega_\ell, \ell \in \mathcal{L}$.

- c_j^ℓ : Unit ordering cost for product j in category ℓ, $\forall j \in \Omega_\ell, \ell \in \mathcal{L}$.

- u_j^1: Upper bound on the price of product j in the primary category; $u_j^1 \equiv \max_{i \in \mathcal{C}_1}\{\alpha_{ij}^1\}$, $\forall j \in \Omega_1$.

- u_j^k: Upper bound on the price of product j in a secondary category k; $u_j^k \equiv \max\{\max_{i \in \mathcal{C}_k}\{\alpha_{ij}^k\}, \max_{r \in \mathcal{C}_1}\{\beta_{rj}^k\}\}$, $k \in \mathcal{L} \setminus \{1\}, j \in \Omega_k$.

Decision Variables

- $z_j^\ell \in \{0,1\}$: $z_j^\ell = 1 \Leftrightarrow$ product $j \in \Omega_\ell$ is offered in the assortment $\mathcal{P}_\ell$, $\forall j \in \Omega_\ell, \ell \in \mathcal{L}$.

- $x_{ij}^\ell \in \{0,1\}$: $x_{ij}^\ell = 1 \Leftrightarrow$ customer segment i of category ℓ purchases product $j \in \Omega_\ell$, $\forall i \in \mathcal{C}_\ell, j \in \Omega_\ell, \ell \in \mathcal{L}$. The x-variables are used to account for direct purchases.

- $y_{ij}^k \in \{0,1\}$: $y_{ij}^k = 1 \Leftrightarrow$ A fraction γ_i^k of customer segment i of the primary category purchases product j in a secondary category k, $\forall i \in \mathcal{C}_1, j \in \Omega_k, k \in \mathcal{L} \setminus \{1\}$. The y-variables are used to represent cross-selling.

- p_j^ℓ : Price of product j on category ℓ, $\forall j \in \Omega_\ell, \ell \in \mathcal{L}$.

- d_j^ℓ : Demand for product j in category ℓ, $\forall j \in \Omega_\ell, \ell \in \mathcal{L}$, calculated endogenously as a function of purchase decisions and the size of customer segments.

The multi-category cross-selling (MCCS) problem can be stated as the following mixed-integer nonlinear program. The objective function (2.1a) maximizes the retailer's profit over the selling horizon, that is, the difference between the retailer's revenue and variable ordering costs and fixed costs for including products in the assortment. As is clear from the remainder of the model constraints, $d_j^\ell p_j^\ell$ is a nonlinear term that involves an endogenously predicted demand and retail prices.

$$\text{Maximize} \sum_{\ell \in \mathcal{L}} \sum_{j \in \Omega_\ell} \left(d_j^\ell p_j^\ell - c_j^\ell d_j^\ell - f_j^\ell z_j^\ell \right). \tag{2.1a}$$

Constraints (2.1b) ensure that a customer segment would choose, from amongst offered products, one that maximizes her surplus, provided that it yields a nonnegative surplus as enforced by Constraints (2.1c).

$$\sum_{k\in\Omega_\ell}(\alpha_{ik}^\ell - p_k^\ell)x_{ik}^\ell \geq (\alpha_{ij}^\ell - p_j^\ell)z_j^\ell, \quad \forall \ell \in \mathcal{L}, i \in \mathcal{C}_\ell, j \in \Omega_\ell \tag{2.1b}$$

$$\sum_{j\in\Omega_\ell}(\alpha_{ij}^\ell - p_j^\ell)x_{ij}^\ell \geq 0, \quad \forall \ell \in \mathcal{L}, i \in \mathcal{C}_\ell. \tag{2.1c}$$

Note that Constraints (2.1b) are equivalent to the following (more aggregate) constraints where the right-hand-side considers the max-surplus choice: $\sum_{k\in\Omega_\ell}(\alpha_{ik}^\ell - p_k^\ell)x_{ik}^\ell \geq \max_{j\in\Omega_\ell}\{(\alpha_{ij}^\ell - p_j^\ell)z_j^\ell\}, \quad \forall \ell \in \mathcal{L}, i \in \mathcal{C}_\ell$.

Likewise, Constraints (2.1d)–(2.1e) stipulate that a customer segment $i \in \mathcal{C}_1$ would buy a complementary product $j \in \Omega_k$ provided that j yields a maximum, nonnegative surplus among all complementary products included in the assortment of this secondary category. The value for M in Constraints (2.1d) is set to $u_j^k \equiv \max\{\max_{i\in\mathcal{C}_k}\{\alpha_{ij}^k\}, \max_{r\in\mathcal{C}_1}\{\beta_{rj}^k\}\}, \forall k \in \mathcal{L}\setminus\{1\}, j \in \Omega_k$.

$$\sum_{h\in\Omega_k}(\beta_{ih}^k - p_h^k)y_{ih}^k \geq (\beta_{ij}^k - p_j^k)z_j^k - M(1 - \sum_{r\in\Omega_1} x_{ir}^1), \quad \forall k \in \mathcal{L}\setminus\{1\}, i \in \mathcal{C}_1, j \in \Omega_k \tag{2.1d}$$

$$\sum_{j\in\Omega_k}(\beta_{ij}^k - p_j^k)y_{ij}^k \geq 0, \quad \forall k \in \mathcal{L}\setminus\{1\}, i \in \mathcal{C}_1. \tag{2.1e}$$

Constraints (2.1f) ensure that, for a certain category, any customer segment will purchase at most one product from amongst the substitutable products offered in the assortment. Constraints (2.1g) guarantee that a customer segment $i \in \mathcal{C}_1$ would buy some secondary product only if she is also purchasing a primary product. Constraints (2.1h)–(2.1i) ensure that any product cannot be purchased by a customer or cross-sold, unless it is included in the assortment.

$$\sum_{j \in \Omega_\ell} x_{ij}^\ell \leq 1, \quad \forall \ell \in \mathcal{L}, i \in \mathcal{C}_\ell \tag{2.1f}$$

$$\sum_{h \in \Omega_k} y_{ih}^k \leq \sum_{j \in \Omega_1} x_{ij}^1, \quad \forall k \in \mathcal{L} \setminus \{1\}, i \in \mathcal{C}_1 \tag{2.1g}$$

$$x_{ij}^\ell \leq z_j^\ell, \quad \forall \ell \in \mathcal{L}, i \in \mathcal{C}_\ell, j \in \Omega_\ell \tag{2.1h}$$

$$y_{ij}^k \leq z_j^k, \quad \forall k \in \mathcal{L} \setminus \{1\}, i \in \mathcal{C}_1, j \in \Omega_k. \tag{2.1i}$$

Constraints (2.1j) aggregate the demand for any product in the primary category based on customer direct purchases and the size of the different customer segments. Likewise, Constraints (2.1k) express the demand of any product in the secondary category by aggregating direct sales and sales due to cross-selling. Note that demand is price-sensitive in that it depends on the consumer choice variables (i.e., x- and y-variables) which, in turn, depend on the assortment and pricing decisions and are governed by the maximum-surplus consumer choice model.

$$d_j^1 = \sum_{i \in \mathcal{C}_1} s_i^1 x_{ij}^1, \quad \forall j \in \Omega_1 \tag{2.1j}$$

$$d_j^k = \sum_{i \in \mathcal{C}_k} s_i^k x_{ij}^k + \sum_{r \in C_1} \lfloor \gamma_r^k s_r^1 \rfloor y_{rj}^k, \quad \forall k \in \mathcal{L} \setminus \{1\}, j \in \Omega_k. \tag{2.1k}$$

Constraints (2.1l) enforce upper bounds on prices based on the greatest reservation prices across customer segments, with $u_j^\ell \equiv \max\{\max_{i \in \mathcal{C}_\ell}\{\alpha_{ij}^\ell\}, \max_{r \in \mathcal{C}_1}\{\beta_{rj}^\ell\}\}$, $\forall \ell \in \mathcal{L} \setminus \{1\}, j \in \Omega_\ell$, and logically relates the pricing and the assortment variables. Constraints (2.1m) introduce logical binary and non-negativity restrictions on decision variables.

$$p_j^\ell \leq u_j^\ell z_j^\ell, \quad \forall \ell \in \mathcal{L}, j \in \Omega_\ell \tag{2.1l}$$

$$\mathbf{x}, \mathbf{y}, \mathbf{z} \in \{0, 1\}, \mathbf{p}, \mathbf{d} \geq 0. \tag{2.1m}$$

Model MCCS, which comprises (2.1a)-(2.1m), optimizes the retailer's assortment and pricing decisions, while predicting consumer decisions (maximizing their surplus) and the associated expected demand. To illustrate the rationale in the consumer choice model, we give a numerical example. Consider the primary category ($\ell = 1$). Suppose the model would like to introduce products 3 and 4 ($\in \Omega_1$), i.e., $z_3^1 = 1$, $z_4^1 = 1$, and $z_k^1 = 0$, for any product $k \in \Omega_1 \setminus \{3, 4\}$. Further, suppose the retailer would like to set the prices to $p_3^1 = 10$ and $p_4^1 = 25$. Consider now customer segment 1, with $\alpha_{13}^1 = 10$ and $\alpha_{14}^1 = 20$. Noting that $x_{1k}^1 = 0, \forall k \in \Omega_1 \setminus \{3, 4\}$ because of Constraints (2.1h), then Constraints (2.1b) reduce to:

$$(\alpha_{13}^1 - p_3^1)x_{13}^1 + (\alpha_{14}^1 - p_4^1)x_{14}^1 \geq \alpha_{13}^1 - p_3^1 \ (\equiv 0)$$
$$(\alpha_{13}^1 - p_3^1)x_{13}^1 + (\alpha_{14}^1 - p_4^1)x_{14}^1 \geq \alpha_{14}^1 - p_4^1 \ (\equiv -5)$$

Therefore, customer segment 1 is expected to buy product 3, i.e., $x_{13}^1 = 1$. Likewise, let customer segment 2 have $\alpha_{23}^1 = 12$ and $\alpha_{24}^1 = 30$, then Constraints (1b) for this segment enforce:

$$(\alpha_{23}^1 - p_3^1)x_{23}^1 + (\alpha_{24}^1 - p_4^1)x_{24}^1 \geq \alpha_{23}^1 - p_3^1 \ (\equiv 2)$$
$$(\alpha_{23}^1 - p_3^1)x_{23}^1 + (\alpha_{24}^1 - p_4^1)x_{24}^1 \geq \alpha_{24}^1 - p_4^1 \ (\equiv 5)$$

Therefore, customer segment 2 is expected to buy product 4, i.e. $x_{24}^1 = 1$. At last, suppose that customer segment 3 had reservation prices $\alpha_{33}^1 = 8$ and $\alpha_{34}^1 = 20$ with implied surpluses $\alpha_{33}^1 - p_3^1$ ($\equiv -2$) and $\alpha_{34}^1 - p_4^1$ ($\equiv -5$). In this case, segment 3 is simply priced out of the market, and does not buy anything, i.e., $x_{3j}^1 = 0, \forall j \in \Omega_1$.

Nonlinear solvers such as KNITRO and LINGO did not show success in finding a near-optimal solution to the MCCS problem even for very small instances. This motivated the investigation of linearized reformulation that can be tackled using linear solvers such as CPLEX as discussed in the next subsection.

2.3.2 Mixed-Integer Linear Reformulation

Model MCCS is a mixed-integer nonlinear formulation that jointly optimizes assortment and pricing decisions with cross-selling considerations. The simpler product line optimization problem under a maximum-surplus choice model is a special case of our problem and is shown to be NP-Complete in Dobson and Kalish (1993). This is indicative of the difficulty of our problem which poses computational challenges due to the discreteness of key decision variables (e.g., assortment and customer purchase decisions) and nonlinearities that arise in the expression of the revenue (with price-sensitive demand) and in the customer choice and cross-selling constraints. The computational intractability of MCCS can, however, be greatly alleviated by developing an equivalent mixed-integer linear reformulation. To this end, the following proposition shows that $p_j^\ell z_j^\ell = p_j^\ell$:

Proposition 1. It is valid to substitute $p_j^\ell z_j^\ell \equiv p_j^\ell$ in Model MCCS.

Proof.

- If $z_j^\ell = 0$, then $p_j^\ell z_j^\ell = 0$ and $p_j^\ell = 0$ by Constraint (2.1l), and thus $p_j^\ell z_j^\ell \equiv p_j^\ell$.
- If $z_j^\ell = 1$, then $p_j^\ell z_j^\ell = p_j^\ell$. $\square$

The result established in Proposition 1 is intuitive in that a product j that is not selected in the assortment will not be priced by the retailer. Note, however, that a similar result does not necessarily hold for $p_j x_{ij}$, i.e., $x_{ij} = 0$ does not necessarily imply that $p_j = 0$. In fact, a product that is not selected by one customer segment i could indeed be purchased by another segment and ought to be priced by the retailer.

We first linearize the objective function (2.1a). To this end, we introduce the following auxiliary nonnegative continuous variables (2.2a-2.2b) in lieu of nonlinear terms in the objective function, as in (2.3a), along with the linearizing constraints (2.3b)-(2.3g):

$$g_{ij}^{\ell} \equiv p_j^{\ell} x_{ij}^{\ell}, \quad \forall \ell \in \mathcal{L}, i \in \mathcal{C}_{\ell}, j \in \Omega_{\ell} \tag{2.2a}$$

$$q_{ij}^{k} \equiv p_j^{k} y_{ij}^{k}, \quad \forall k \in \mathcal{L} \setminus \{1\}, i \in \mathcal{C}_1, j \in \Omega_k. \tag{2.2b}$$

$$\text{Maximize} \sum_{k \in \mathcal{L} \setminus \{1\}} \sum_{r \in C_1} \sum_{h \in \Omega_k} \lfloor \gamma_r^k s_r^1 \rfloor q_{rh}^k + \sum_{\ell \in \mathcal{L}} \sum_{i \in \mathcal{C}_{\ell}} \sum_{j \in \Omega_{\ell}} s_i^{\ell} g_{ij}^{\ell} - \sum_{\ell \in \mathcal{L}} \sum_{j \in \Omega_{\ell}} \left(c_j^{\ell} d_j^{\ell} + f_j^{\ell} z_j^{\ell} \right) \tag{2.3a}$$

$$g_{ij}^{\ell} \leq u_j^{\ell} x_{ij}^{\ell} \;, \quad \forall \ell \in \mathcal{L}, i \in \mathcal{C}_{\ell}, j \in \Omega_{\ell} \tag{2.3b}$$

$$g_{ij}^{\ell} \geq p_j^{\ell} - u_j^{\ell}(1 - x_{ij}^{\ell}) \;, \quad \forall \ell \in \mathcal{L}, i \in \mathcal{C}_{\ell}, j \in \Omega_{\ell} \tag{2.3c}$$

$$g_{ij}^{\ell} \leq p_j^{\ell} \;, \quad \forall \ell \in \mathcal{L}, i \in \mathcal{C}_{\ell}, j \in \Omega_{\ell} \tag{2.3d}$$

$$q_{ij}^{k} \leq u_j^{k} y_{ij}^{k} \;, \quad \forall k \in \mathcal{L} \setminus \{1\}, i \in \mathcal{C}_1, j \in \Omega_k \tag{2.3e}$$

$$q_{ij}^{k} \geq p_j^{k} - u_j^{k}(1 - y_{ij}^{k}), \quad \forall k \in \mathcal{L} \setminus \{1\}, i \in \mathcal{C}_1, j \in \Omega_k \tag{2.3f}$$

$$q_{ij}^{k} \leq p_j^{k}, \quad \forall k \in \mathcal{L} \setminus \{1\}, i \in \mathcal{C}_1, j \in \Omega_k. \tag{2.3g}$$

The model linearization is completed by substituting constraints (2.3h)-(2.3k) in lieu of constraints (2.1b)-(2.1e) as follows:

$$\sum_{k \in \Omega_{\ell}} (\alpha_{ik}^{\ell} x_{ik}^{\ell} - g_{ik}^{\ell}) \geq \alpha_{ij}^{\ell} z_j^{\ell} - p_j^{\ell}, \quad \forall \ell \in \mathcal{L}, i \in \mathcal{C}_{\ell}, j \in \Omega_{\ell} \tag{2.3h}$$

$$\sum_{j \in \Omega_{\ell}} (\alpha_{ij}^{\ell} x_{ij}^{\ell} - g_{ij}^{\ell}) \geq 0, \quad \forall \ell \in \mathcal{L}, i \in \mathcal{C}_{\ell} \tag{2.3i}$$

$$\sum_{j \in \Omega_k} (\beta_{ij}^{k} y_{ij}^{k} - q_{ij}^{k}) \geq 0, \quad \forall k \in \mathcal{L} \setminus \{1\}, i \in \mathcal{C}_1 \tag{2.3j}$$

$$\sum_{h \in \Omega_k} (\beta_{ih} y_{ih}^{k} - q_{ih}^{k}) \geq \beta_{ij}^{k} z_j^{k} - p_j^{k} - M(1 - \sum_{r \in \Omega_1} x_{ir}^{1}), \quad \forall k \in \mathcal{L} \setminus \{1\}, i \in \mathcal{C}_1, j \in \Omega_k. \tag{2.3k}$$

Note that in constraint (2.3h), the nonlinear term $p_j^\ell z_j^\ell$ is replaced by p_j^ℓ, as a result of Proposition 1.

Model L-MCCS can be stated as follows:

L-MCCS: {Maximize (2.3a): (2.3b)-(2.3k), (2.1f)-(2.1l), and $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \{0, 1\}, \mathbf{p}, \mathbf{d}, \mathbf{g}, \mathbf{q} \geq 0$}.

For completeness, the following proposition establishes the validity of L-MCCS:

Proposition 2. Model L-MCCS is a valid reformulation of Model MCCS.

Proof.

Consider the substitution relationships $g_{ij}^\ell \equiv p_j^\ell x_{ij}^\ell, \forall \ell, i, j$, and note that:

- If $x_{ij}^\ell = 0$, then $p_j^\ell x_{ij}^\ell = 0$, and we need to verify that $g_{ij}^\ell = 0$. By (2.3b) and the non-negativity restriction on the g-variables, we have that $g_{ij}^\ell = 0$. Under this condition, constraints (2.3c) and (2.3d) hold true, and $g_{ij}^\ell = p_j^\ell x_{ij}^\ell$.

- If $x_{ij}^\ell = 1$, we need to verify that $g_{ij}^\ell = p_j^\ell$, which is jointly enforced by constraints (2.3c) and (2.3d).

Consider the substitution relationships $q_{ij}^k \equiv p_j^k y_{ij}^k, \forall k, i, j$, and note that:

- If $y_{ij}^k = 0$, then $p_j^k y_{ij}^k = 0$, and we need to verify that $q_{ij}^k = 0$. By (2.3e) and the non-negativity restriction on the q-variables, we have that $q_{ij}^k = 0$. Under this condition, constraints (2.3f) and (2.3g) hold true, and $q_{ij}^k = p_j^k y_{ij}^k$.

- If $y_{ij}^k = 1$, we need to verify that $q_{ij}^k = p_j^k$, which is jointly enforced by constraints (2.3f) and (2.3g). $\square$

As a consequence of linearization, the linear model L-MCCS contains an six additional sets of constraints (2.3b) - (2.3g) and two additional variables, namely g_{ij}^ℓ and q_{ij}^k.

2.4. Computational Study

In this section, we present an illustrative example followed by our computational results for large-scale instances. The illustrative example discusses the planning of a primary category and a single secondary category. The computational study demonstrates the tractability of the proposed model reformulation and the usefulness of adopting an integrated approach that incorporates cross-selling considerations. The larger instances considered are scaled with respect to different parameters, namely, the number of candidate products in each category, $|\Omega_\ell|, \forall \ell \in \mathcal{L}$, and the number of direct customer segments for each category, $|\mathcal{C}_\ell|, \forall \ell \in \mathcal{L}$. All runs were performed with AMPL/CPLEX 12.4.0.0 on Microsoft Windows 7 Professional with an Intel Core i7-2600, 3.40 GHz processor and 12 GB RAM.

2.4.1 Illustrative Example: A Single Secondary Category

This illustrative example involves optimizing assortment and pricing decisions for a primary category and a secondary category. For each of the two categories, the retailer may select from among three substitutable products, i.e., $|\Omega_1| = 3$ and $|\Omega_2| = 3$. Further, the retailer has identified two direct customer segments for each category, that is, $|\mathcal{C}_1| = 2$ and $|\mathcal{C}_2| = 2$. Table 2.1 summarizes other input parameter values pertaining to customer segment sizes, customer reservation prices (or valuations), cross-selling parameters, and fixed and variable costs for the different products. Table 2.2 reports the solution obtained under two policies: (i) Our proposed integrated approach that optimizes both categories under cross-selling as in Model MCCS and (ii) a disjoint approach where each category is planned in isolation, thereby ignoring cross-selling effects by setting all γ values to zero.

The results demonstrate the importance and usefulness of the proposed integrated model. Under the integrated approach, the optimal assortments for the primary and secondary categories, respectively, are $\mathcal{P}_1 = \{1^1, 3^1\}$ and $\mathcal{P}_2 = \{2^2, 3^2\}$ (where the

Table 2.1: Data for Illustrative Example with a Single Secondary Category

Primary category						Secondary category									
		α^1_{ij}						α^2_{ij}				β_{ij}			
$i \in \mathcal{C}_1$	s^1_i	$j=1$	$j=2$	$j=3$	γ	$i \in \mathcal{C}_2$	s^2_i	$j=1$	$j=2$	$j=3$	$i \in \mathcal{C}_1$	$j=1$	$j=2$	$j=3$
$i=1$	880	95	97	90	0.2	$i=1$	600	115	120	125	$i=1$	115	120	125
$i=2$	1020	80	83	85	0.4	$i=2$	800	110	115	120	$i=2$	110	115	115
f^1_j		620	1082	1000		f^2_j		1035	748	633				
c^1_j		79	82	83		c^2_j		96	100	98				

Table 2.2: Solution for Illustrative Example with a Single Secondary Category

Integrated Solution with Cross-Selling Effects											
Primary category				Secondary category							
$i \in \mathcal{C}_1$	x^1_{ij}			$i \in \mathcal{C}_2$	x^2_{ij}			$i \in \mathcal{C}_1$	y^2_{ij}		
$j \in \Omega_1$	$j=1$	$j=2$	$j=3$	$j \in \Omega_2$	$j=1$	$j=2$	$j=3$	$j \in \Omega_2$	$j=1$	$j=2$	$j=3$
$i=1$	1	0	0	$i=1$	0	0	1	$i=1$	0	0	1
$i=2$	0	0	1	$i=2$	0	0	1	$i=2$	0	1	0
z^1_j	1	0	1	z^2_j	0	1	1	Total Profit = 48,234.6			
d^1_j	880	0	1020	d^2_j	0	408	1576				
p^1_j	90	0	85	p^2_j	0	115	120				
Disjoint Solution without Cross-Selling Effects											
Primary category				Secondary category							
$i \in \mathcal{C}_1$	x^1_{ij}			$i \in \mathcal{C}_2$	x^2_{ij}			$i \in \mathcal{C}_1$	y^2_{ij}		
$j \in \Omega_1$	$j=1$	$j=2$	$j=3$	$j \in \Omega_2$	$j=1$	$j=2$	$j=3$	$j \in \Omega_2$	$j=1$	$j=2$	$j=3$
$i=1$	1	0	0	$i=1$	0	0	1	$i=1$	0	0	1
$i=2$	0	0	0	$i=2$	0	0	1	$i=2$	0	0	0
z^1_j	1	0	0	z^2_j	0	0	1	Total Profit = 42,872.6			
d^1_j	880	0	0	d^2_j	0	0	1400				
p^1_j	95	0	0	p^2_j	0	0	120				

superscript of a product identifies its category). In particular, product 3 in the primary category was introduced at an affordable price for customer segment 2 ($\in \mathcal{C}_1$), which resulted in profitable cross-selling transactions and the inclusion of product 2 in the secondary category. When cross-selling was overlooked, the retailer did not perceive benefit in including product 3 in primary category and product 2 in the secondary category, thereby yielding suboptimal, narrower assortments denoted by $\tilde{\mathcal{P}}_1 = \{1^1\}$ and $\tilde{\mathcal{P}}_2 = \{3^2\}$. Such suboptimal assortment and/or pricing decisions are, of course, accompanied by a significant profit loss of around 13%. Further, it results in a reduced business volume whereby 3,884 transactions are anticipated under the integrated approach as opposed 2,280 transactions under the disjoint approach. This can have two damaging consequences for the retailer. The first is the risk of underestimating demand and, therefore, having to lose or backorder certain transactions. The second, as a result of narrower assortments, can cause an overall reduction of customer footprint (Hess and Gerstner 1987; DeGraba 2003) – a major concern to retailers.

2.4.2 Results for Larger Instances

In this section, we report in Table 2.3 results for large-scale instances that we randomly generated using the data generation scheme in the appendix. Central to our computational study is a comparison between our proposed integrated approach which accounts for cross-selling and a disjoint approach that overlooks cross-selling and optimizes each category in isolation, as explained in Section 2.4.1. Each of the 18 instances reported in Table 2.3 is identified by its number and is characterized by the number of candidate, substitutable products in each category. All instances in our computational study involve one primary category and two secondary categories (i.e., $|\mathcal{L}| = 3$). For the integrated approach, Table 2.3 reports $|\mathcal{P}_1|$, $|\mathcal{P}_2|$, and $|\mathcal{P}_3|$ – the size of the optimal assortments for the primary category and the two secondary categories. It also reports the profit as a percentage of the total revenue and the CPU time (seconds) to solve the instance to optimality. For the disjoint approach, we also report the size of selected assortments, $|\tilde{\mathcal{P}}_1|$, $|\tilde{\mathcal{P}}_2|$, and $|\tilde{\mathcal{P}}_3|$. For each category, we also report the number of products under the disjoint approach that are common to the optimal assortment, i.e., $|\mathcal{P}_\ell \cap \tilde{\mathcal{P}}_\ell|, \forall \ell \in \mathcal{L}$. The last two columns report the profit loss and the CPU time (secs) under the disjoint approach.

From a computational viewpoint, it is worthwhile to note that the linear MIP reformulation, L-MCCS, solved to optimality all instances, with up to 5 customer segments for each category and over 75 substitutable products in each category. For most instances in our test-bed, the solution effort required less than one CPU minute. For the larger and more difficult instances, the CPU time ranged between 2 and 11 CPU minutes. This empirically observed computational tractability of the proposed MIP reformulation is encouraging and bears the potential of benefiting retailers for large-scale, industry-sized problem instances. The disjoint approach confirms that optimizing single-category decisions, when pertinent, is computationally very manageable with the available computing power.

Table 2.3: Computational Analysis of MCCS for Single-Period Instances

				Integrated Approach					Disjoint Approach							
Inst.	$\|\Omega_1\|$	$\|\Omega_2\|$	$\|\Omega_3\|$	$\|\mathcal{P}_1\|$	$\|\mathcal{P}_2\|$	$\|\mathcal{P}_3\|$	Profit	CPU	$\|\tilde{\mathcal{P}}_1\|$	$\|\mathcal{P}_1 \cap \tilde{\mathcal{P}}_1\|$	$\|\tilde{\mathcal{P}}_2\|$	$\|\mathcal{P}_2 \cap \tilde{\mathcal{P}}_2\|$	$\|\tilde{\mathcal{P}}_3\|$	$\|\mathcal{P}_3 \cap \tilde{\mathcal{P}}_3\|$	Prof. Loss	CPU
Instances with $\|\mathcal{C}_1\|=\|\mathcal{C}_2\|=\|\mathcal{C}_3\| = 3$																
1	25	25	25	3	2	3	5.1%	1.0	3	3	2	2	2	2	65.4%	0.6
2	25	50	75	2	2	2	2.0%	3.9	1	1	2	2	2	1	11.8%	0.7
3	50	50	50	2	3	3	2.8%	3.4	1	1	2	2	2	2	50.8%	0.8
4	50	75	100	1	2	2	2.4%	3.3	1	1	1	1	2	0	11.0%	1.0
5	75	50	25	1	1	1	2.3%	1.9	1	1	1	1	1	1	0.0%	0.7
6	75	100	150	2	2	1	2.2%	10.4	1	0	2	1	1	1	7.9%	2.1
7	100	75	50	2	1	2	2.2%	2.1	2	2	1	0	2	2	19.4%	0.8
Instances with $\|\mathcal{C}_1\|=\|\mathcal{C}_2\|=\|\mathcal{C}_3\| = 4$																
8	25	50	75	2	2	2	2.5%	6.4	2	2	2	2	1	1	5.59%	0.9
9	50	75	100	2	3	2	2.2%	28.6	2	2	3	1	2	1	17.2%	1.4
10	75	50	25	2	2	3	2.0%	4.8	2	2	2	0	3	2	31.7%	1.2
11	75	75	75	3	2	4	4.2%	37.1	3	3	2	0	3	3	57.9%	1.5
12	75	100	150	2	3	2	2.4%	113.3	2	2	2	1	2	1	6.5%	2.1
13	100	75	50	1	3	2	2.2%	51.3	1	1	2	1	2	1	9.9%	1.4
Instances with $\|\mathcal{C}_1\|=\|\mathcal{C}_2\|=\|\mathcal{C}_3\| = 5$																
14	25	50	75	2	2	2	2.3%	15.5	2	2	2	2	2	1	5.6%	1.2
15	50	75	100	3	3	4	2.3%	155.3	3	3	2	1	3	0	20.9%	2.3
16	75	50	25	2	3	2	2.2%	7.6	1	1	2	2	2	2	22.0%	1.1
17	75	100	150	2	3	3	2.2%	652.9	2	2	2	0	3	1	18.1%	10.2
18	100	75	50	2	3	4	2.1%	333.1	1	0	3	0	4	4	8.9%	3.0

From a managerial point of view, the following observations and insights are in order.

1. *Profit reduction.* Over the 18 instances in our test-bed, the disjoint approach coincidentally yielded an optimal solution for only one instance (Instance 5). This atypical situation arises when direct customers are more profitable than cross-selling customers and it is optimal for the retailer to plan assortment and pricing decisions without consideration for cross-selling. For all the other instances, the profit reduction caused by the disjoint approach ranged from 5.6% to 65.4%.

2. *Suboptimal, narrower assortments.* One recurrent disadvantage of the disjoint approach is that it tends to yield suboptimal (and often narrower) assortments. For Instance 18, the primary category comprises two products under the integrated approach, whereas only a single product forms the primary category under the disjoint approach. Further, the latter product is not part of the pair of products chosen in the integrated approach. Likewise, the three products selected in the first secondary category ($\ell = 2$) do not overlap at all with the three products selected under the integrated approach. The larger assortments observed under the integrated approach are often due to the introduction of a primary product as an incentive for attractive cross-selling customers. This, in turn, may result in the inclusion of additional secondary products that can secure profitable cross-selling transactions. A more aggressive version of this phenomenon relates to the concept of "loss-leaders' (Hess and Gerstner 1987)' whereby a retailer would sell a product at loss with the anticipation that customers who purchase it would also buy secondary products that are more lucrative.

3. *Suboptimal pricing.* The disjoint approach is also observed to yield suboptimal prices. Of particular interest are the prices of products that are selected under both the integrated and disjoint approaches. This comparison is pertinent when the entire assortment of a category is common to both approaches.

   - *Over-pricing products.* When the product valuations by cross-selling customers are relatively lower than those by direct customers for secondary categories, there is a risk of over-pricing under the disjoint approach. In fact, here, the retailer overlooks cross-selling and chooses higher prices based solely on direct customers. When cross-selling customers visit the store, they may find the assortment relatively interesting, but would perceive the prices of secondary products as expensive (yielding negative utility). This would result in lost sales opportunities for the retailer.
   - *Under-pricing products.* When, on the contrary, cross-selling customers have relatively high product valuations for secondary categories, they would perceive the prices set by the retailer under the disjoint approach as quite attractive. This will generate a substantial stream of cross-selling purchases which will accelerate the depletion of the secondary products ordered by the retailer and are likely to cause stock-outs.

2.5. Conclusion

We have examined the multi-category cross-selling (MCCS) problem, where a retailer seeks to jointly optimize assortment and pricing decisions for a primary category and several related secondary categories – each of which is composed of substitutable products. We developed a novel mixed-integer nonlinear formulation that maximizes the retailer profit under a maximum utility consumer choice model. We highlight that the nonlinearity of this model can be circumvented by introducing auxiliary variables and accompanying linearization constraints. The linear MIP reformulation is empirically observed to afford exact solutions to large-scale, industry-sized problem instances in manageable times (ranging from a few CPU seconds to a few CPU minutes). We have demonstrated the importance of jointly planning retail categories that are related by cross-selling. In fact, failing to do so results in substantial profit losses (ranging from 5% to 65% in our computational experience), suboptimal (and often narrower) assortments, and inadequate prices. When such retail categories are planned in isolation, price inadequacy is evidenced by over-pricing certain secondary products, which can cannibalize cross-selling transactions, or under-pricing which stimulates cross-selling purchases that were unaccounted for to extent of causing stock-outs.

The approach articulated in the chapter can help overcome computational difficulties noted in the literature, e.g. in the work by Dobson and Kalish (1993). In the latter, only heuristics approaches were devised for a single product selection and pricing problem under a maximum surplus consumer choice model. Our work can also serve as a cornerstone for future research on the integration of additional decisions related to inventory holding and shelf space allocation. Another direction that we recommend for future research is to analyze the effect of promotional campaigns. Finally, we recommend examining product line optimization problems with probabilistic consumer choice models, especially to address applications for which the deterministic consumer choice model may not be adequate.

CHAPTER 3

A MULTI-CATEGORY ASSORTMENT PACKING PROBLEM UNDER CROSS-SELLING AND CANNIBALIZATION EFFECTS

Central to the management of product variety in retail is the issue of dynamically "refreshing" product assortments. In doing so, retailers seek to project an attractive image of their business in a competitive market by meeting customers' expectations and by sparking their interest for new(er) products. In this chapter, we examine an assortment packing problem where a decision-maker optimizes the assortment and the market entry time of products that belong to multiple interdependent categories over a multi-period planning horizon. It is assumed that products in the same category are substitutable, whereas products across categories may exhibit asymmetric complementarity relationships. Products are also assumed to have a limited longevity over which their attractiveness gradually decays (e.g., electronics or fashion products). Upon its introduction, the decaying attractiveness of a product can be further positively or negatively impacted by the specific mix of substitutable or complementary products that the retailer introduces. We propose a 0-1 fractional optimization model that employs an attraction demand model and subsumes recent assortment packing models in the literature. We develop a linearized reformulation that affords exact solutions to small-sized problem instances. Furthermore, a linear programming-based heuristic approach is devised and demonstrated to yield near-optimal solutions for large-scale computationally challenging problem instances in manageable times. A model extension in the context of the movie industry is discussed, where exhibitors decide on the assortment of movies to display and their optimal display times.

3.1. Introduction and Motivation

To maintain a competitive edge, firms need to carefully manage the life cycles of their products which encompass four main stages, spanning their introduction, growth, maturity, and decline. The latter ends with the product being withdrawn from the market or its production discontinued, as newer versions of it or substitutable options become available. The management of product variety over time is grounded in a firm's ability to strategically plan the introduction and withdrawal of products. Such planning issues arise in a broad spectrum of applications, including the management of movie theaters, DVD rentals, automotive industry, high fashion, and electronic devices to name a few. In such applications, products have a limited life cycle that may be measured in weeks (e.g., movies) or a few years (e.g., cars). The dynamic planning of product assortments is further complicated by two key factors. The first is the inherent decline of a product attractiveness over time, as it "ages." This phenomenon, henceforth referred to as a product decay, is further complicated by the additional impact of other products in the assortment. The introduction of more attractive substitutable items (e.g., new version of a smart-phone) can be detrimental to and further accelerate the decay of a (cannibalized) product. In contrast, the decaying attractiveness of a product can be invigorated, to some extent, by the introduction of certain complementary items that increase the utility or the appeal of the former and create the opportunity of *cross-selling*. This requires a strategic planning of the specific mix of products that are made available for customers in order to meet, but also spark, their interest in new(er) products. Given the combinatorial nature of such assortment decisions, it is judicious to develop optimization models that capture the relative market share of decaying products as different subset of products get introduced into the assortment over time.

Firms, especially in the apparel business, are particularly interested in dynamically refreshing their stores with new products over different selling seasons. However, as indicated in Caro et al. (2014), they usually employ a manual, *ad hoc* strategy based on their own experience and subjective judgment. Such approaches typically lack integration and can benefit from business analytics and optimization. In this regard, Caro et al. (2014) suggest that even simple constructive or greedy heuristics can produce assortments that outperform those constructed by retailers using manual approaches.

We consider the general case of multiple interdependent categories with product substitutability in a given category and complementarity between products across categories. We also consider a multi-period horizon whereby subsets of products are dynamically introduced over time and their relative market shares are represented using an attraction demand model (Bell et al. 1975). Each product attractiveness decays over time, until its complete decline and withdrawal at the end of its longevity. After its introduction, an item decay may be strategically accelerated or slowed down due to cannibalization or cross-selling effects. This model extends the extant literature on assortment packing. It subsumes the special case of a single-category, single-period model with an attraction demand model (Talluri and van Ryzin 2004; Kök et al. 2008). It also extends the recent work by Caro et al. (2014) which examines a single-category, multi-period problem with decaying attractiveness but without any cannibalization or cross-selling effects. The special case addressed in Caro et al. (2014) was shown to be NP-hard and the intractability of their proposed (nonlinear) 0-1 fractional program motivated the use of several constructive heuristics.

As in Caro et al. (2014), we model demand in the form of market shares following the attraction model by Bell et al. (1975). In its original form, a demand attraction model defines a competitive market share for n sellers following the relationship (us)/(us+them). It expresses the individual market share, $m(s_i)$, of a given seller, $s_i \in S$, as a function of the attraction value for each seller $a(s_i)$ in the form: $m(s_i) = a(s_i)/\sum_{j=1}^{n} a(s_j)$. Our model uses a similar relationship to express the market share of the retailer as a function of the different attraction values for each product that decays over time.

The remainder of the chapter is organized as follows. Section 3.2 briefly reviews the related literature. In Section 3.3 the problem is formally stated along with notation and our 0-1 fractional program is introduced. We also discuss some features and assumptions of the model and highlight parameter estimation methods. Section 3.4 presents two solution approaches. In Section 3.5, we present an illustrative example, followed by a computational study. Section 3.6 provides an extension of the model in the movie industry. Section 3.7 concludes the chapter with a discussion of our findings.

3.2. Literature Review

The work presented in this chapter is generally related to the assortment planning literature, where the retailer decides on the set of products to be carried in store. Kök et al. (2008) present an extensive review of the literature in this area which can be classified into two main streams: i) *Static* assortment planning, and ii) *dynamic* assortment planning.

Static assortment planning problems seek to determine a fixed assortment selection for the entire season. The stylized model by Talluri and van Ryzin (2004) provides an analysis of the single-period revenue management problem of deciding which subset of fare products to offer. Their analysis provides a characterization of optimal policies under a general choice model of demand and shows that the optimal assortment is a set comprising of highest-margin products.

Dobson and Kalish (1988, 1993) proposed early studies of static assortment and pricing problems. The authors proposed several modeling contributions but no exact solution methodology was presented. Instead, the authors opted to use heuristics to cope with the intractability of their models. Van Ryzin and Mahajan (1999) and Smith and Agrawal (2000) also consider the static assortment problem with a stochastic demand model and static product substitution. In contrast, Mahajan and van Ryzin (2001) capture the dynamic substitution due to stockouts in the assortment planning problem. Kök and Fisher (2007) estimated demand under substitution where their assortment decision is followed by an inventory decision. Honhon et al. (2010) find the optimal stocking levels under random demand, and Rodríguez and Aydin (2011) solve for assortment and pricing decisions for configurable products under uncertain demand. The more complex problem of jointly solving for pricing and inventory decisions for an assortment is studied by Maddah and Bish (2007), Aydin and Porteus (2008), and Ghoniem and Maddah (2013).

In the aforementioned studies, even when the problem is in a multi-period setting, the assortment in all periods tends to be static. Dynamic assortment planning addresses the need to revise or change the assortment selection over the time. Fashion and apparel retailers would benefit from such ability to revise their assortment specially after the reduction that some companies have recently made in their supply chain response time. Traditionally, the design-to-shelf lead time for the apparel supply chain is 6-9 months. However, innovative retailers redesigned their supply chain architecture, and reduced the lead time to 2-5 weeks. Raman et al. (2001) describe the organizational changes in the supply chain that allowed the Japanese apparel company "World Co." to achieve much shorter lead times. The work by Caro and Gallien (2007) develop a stylized model that formulates the dynamic assortment planning problem faced by fashion retailers. Bernstein et al. (2013) dynamically customizes the assortment over time, depending on customer's preferences and inventory levels.

Research in assortment planning has primarily focused on single category problem decisions. Single category models overlooks the dependency and relationships across multi-category items that can affect the key retail decisions (see Russell et al. 1997 for a review). In contrast, our work is related to the stream of research that examines multi-category effects on retail decisions. Work on this stream includes the paper by Manchanda et al. (1999) that introduced co-incidence and heterogeneity along with complementarity as factors affecting items dependency in the shopping basket. They argue that not accounting for these three factors simultaneously could lead to erroneous inferences of the problem. Van den Poel et al. (2004) measure the complementary effects of retail promotions for a large number of product pairs using the market basket analysis. Akçura and Srinivasan (2005) study the role of customer information on cross-selling and risks accompanied by obtaining these information. They show that by a firm's commitment on a cross-selling level, it can obtain customer intimacy and benefit from detailed customer information, resulting in higher profits and lower prices.

The work by Caro et al. (2014) is the first to tackle the assortment packing problem, deciding on the optimal introduction timing of products to the assortment. Their model follows the attraction model by Bell et al. (1975). Their formulation was deemed intractable and heuristic approaches were presented. Our work extends the problem by Caro et al. (2014) and proposes a multi-category assortment packing model where cross-selling and cannibalization effects between products are observed. In contrast to Caro et al. (2014), our approach to solve the problem is based on exact methods. As such, our proposed model is, to the best of our knowledge, the first approach to solve the multi-category assortment packing problem to optimality and reveal important managerial insights.

3.3. Mathematical Programming Formulation

This section provides a formal problem statement for the multi-period multi-category assortment packing problem under cross-selling and cannibalization effects. We introduce the notation along with a 0-1 fractional program.

3.3.1 Binary Fractional Model

Consider the multi-period multi-category assortment packing problem where a firm seeks to optimize the release times of new products into the market, in a way that maximizes its market share over the entire planning horizon. Products belong to a set of categories, where two types of relations exist between pairs of products: i) Cross-selling relations between a pair of complementary products from two distinct categories, ii) cannibalization relations between a pair of substitutable products within the same category.

We consider a set of n candidate products of multiple categories. The assortment planning is dynamic, in the sense that there is no one fixed assortment that is adopted over the whole time horizon $\mathcal{T}$. Each product i is characterized by its attractiveness v_i when it is first introduced into the market, its gross margin r_i, and a decay factor k_{id} that determines the longevity ℓ_i of the product in the market.

The cross-selling effect is the increase in market share of one product as a result of introducing a complementary product, while cannibalization is the reduction in the market share of one product as a result of introducing a substitutable product. Thus, these two phenomenon can be modeled in the assortment packing problem as either an increase or reduction in the decay function's intensity of the existing product. More specifically, the matrix γ_{ij} reflects the change fraction that should be applied to the decay function. The matrix γ_{ij} is defined in a way that captures the effects between each pair of products whether they belong to the same category or not. That allows us to capture cross-category effects (such as cross-selling) as well as inter-category effects (such as cannibalization). The difference between cross-selling and cannibalization effects in the matrix is that cross-selling values range from 0 to 1, while cannibalization values range from -1 to 0. We further assume, for computational simplicity, that products are introduced to the market once and are not withdrawn from the market before they decay. This assumption will be further relaxed in the model extension in Section 3.6 where product removal is allowed.

Consider the following notation:

Parameters

- $\mathcal{I} = \{1, \ldots, n\}$: Set of all n candidate products in all the categories that the firm can introduce over the planning horizon.
- $\mathcal{T} = \{1, \ldots, T\}$: Set of all T periods in the planning horizon.
- α_t: Discount (or seasonality) factor at period t, ranging between 0 and 1, $\forall t \in \mathcal{T}$.
- r_i: Unit gross margin of product i, $\forall i \in \mathcal{I}$.
- v_i: Weight, or attractiveness, of product i when it is first introduced into the market, $\forall i \in \mathcal{I}$.
- v_0: Weight, or attractiveness, of the outside option due to competition.
- ℓ_i: Longevity of product i (in periods), $\forall i \in \mathcal{I}$.
- k_{id}: Decay factor for product i after d periods of its initial introduction, ranging between 0 and 1. As such, $k_{i0} = 1$, since the product did not start decaying yet, and $k_{id} = 0, \forall i \in \mathcal{I}, d \geq \ell_i$.
- γ_{ij} : Fraction of the increase (decrease) in the decay value of product i if product j is introduced within its life cycle as a result of cross-selling (cannibalization) effects, with the assumption that $\sum\limits_{j:\gamma_{ij}<0} \gamma_{ij} \geq -1, \forall i \in \mathcal{I}$.
- $h = \max\{1, t - \ell_i + 1\}$, a pointer to whether product i is active (did not decay) at period t.

Decision Variables

- $x_{it} \in \{0,1\}$: $x_{it} = 1$ if and only if product i is introduced in period t, $\forall i \in \mathcal{I}, t \in \mathcal{T}$.

The multi-period multi-category Assortment Packing Problem with Cross-Selling and Cannibalization effects, denoted by **APPCS**, can be stated as the following 0-1 nonlinear fractional model:

$$\textbf{APPCS}\text{: Maximize: } \sum_{t=1}^{T} \alpha_t \sum_{i=1}^{n} r_i \left(\frac{v_i \sum\limits_{u=1}^{t} [k_{i,t-u} x_{iu} + \sum\limits_{j \neq i} \sum\limits_{h}^{t} k_{i,t-u} \gamma_{ij} x_{jh} x_{iu}]}{v_0 + \sum\limits_{s=1}^{n} v_s \sum\limits_{u=1}^{t} [k_{s,t-u} x_{su} + \sum\limits_{j \neq s} \sum\limits_{h}^{t} k_{s,t-u} \gamma_{sj} x_{jh} x_{su}]} \right) \tag{3.1a}$$

$$\text{subject to: } \sum_{t=1}^{T} x_{it} \leq 1, \quad \forall i \in \mathcal{I} \tag{3.1b}$$

$$\mathbf{x} \text{ binary.} \tag{3.1c}$$

The objective function in (3.1a) is the sum of the discounted (gross) profits over all products in all categories for all periods. The expression between parentheses is the market share of product i at period t. Constraint (3.1b) ensures that each product is introduced at most once. Constraint (3.1c) imposes binary restrictions on the decision variables.

Building on the single category model (Caro et al. 2014), our model is also based on the attraction demand model (Bell et al. 1975), where the contribution of each product's market share is proportional to its attractiveness in each period. Let $z_{it} \equiv \sum\limits_{u=1}^{t} [k_{i,t-u} x_{iu} + \sum\limits_{j \neq i} \sum\limits_{h}^{t} k_{i,t-u} \gamma_{ij} x_{jh} x_{iu}]$ denote the contribution of product i to the attractiveness of the assortment in period t. The first term in z_{it}, $k_{i,t-u} x_{iu}$, refers to the decay effect of the normal life cycle of the product, while the second term, $\sum\limits_{j \neq i} \sum\limits_{h}^{t} k_{i,t-u} \gamma_{ij} x_{jh} x_{iu}$, refers to the delayed or accelerated decay effect due to cross-selling or cannibalization respectively. Then, let $z_t \equiv \sum_{i=1}^{n} v_i z_{it}$ denote the total attractiveness load of all products in period t, and $\phi_t \equiv \frac{z_t}{v_0+z_t}$ represents the firm's total market share at period t.

3.3.2 Model Discussion and Parameter Estimation

In this section we discuss our model, its key features and assumptions, and parameter estimation. The model formulated above is a generic model, however, it can be customized to fit specific applications by adding more constraints and/or customizing its variables and parameters as illustrated in Section 3.6. The model is assumed to run once, at the beginning of the season, and thus all release dates are specified in advance and cannot be adjusted later in the season. Motivation to this assumption is derived from applications with long production lead times, such as electronics and fashion products. On the other hand, for applications with short production lead times, one can adjust the model to include a learning phase. This can be done by running the model at the beginning of the season, implementing its optimal decision for the first few periods, get feedback from those periods and use them to adjust parameters (learning phase), and then rerun the model for the rest of the season. This learning process can be repeated again during the season when needed.

Another assumption in the model is that each product is introduced into the assortment at most once, and remains in the assortment until the end of the season. Given that it is not usual in retail settings to introduce a product and then remove it from the assortment and then re-introduce it again (due to associated costs like logistics, handling, and merchandising costs), this assumption in the generic model is realistic. However, in some applications, like movie scheduling, a product (movie) can be introduced to the assortment of "now displaying" movies, then discontinued for some reason, and re-introduced again for a few more weeks. This assumption can therefore be relaxed according to the application as shown in the model extension in Section 3.6 which allows for product removal.

An important characteristic of the model is the adoption of the attraction demand model by Bell et al. (1975). Following the same adoption made by Caro et al. (2014) in the single category assortment packing problem, we utilize the concept that marketing model builders frequently use. Where relationships of the form (us)/(us + them) are used to express the effects of "us" variables on purchase probability and market share. This form is a commonly used demand model in the marketing literature that captures assortment-based substitution (see Kök et al. 2008).

For the single category assortment packing problem, Caro et al. (2014) prove that the problem is NP-hard even for two periods only. Thus, building over the complexity of the single category problem by expanding it to a multiple category setting, and consider inter-related product effects, makes our optimization problem an NP-hard problem.

The data used in this chapter is a realistic randomly generated data. However, we have developed an understanding of how the different parameters of the model APPCS are estimated. Estimating the initial attractiveness v_i and the decay parameter $k_{i,d}$ of a product is done by identifying a matching product from the database of previously displayed products. From which an estimate of the attractiveness can be made based on the number of units sold in the first period after the product's introduction, and an estimate of the decay parameter can be made based on the change in the number of units sold in subsequent periods. Sawhney and Eliashberg (1996), Eliashberg et al. (2001) and Ainslie et al. (2005) illustrates the process with application to movie release dates. The way we realistically estimate the decay parameter $k_{i,d}$ in our computations is that we follow an exponential decay pattern illustrated as follows: randomly assign $k_{i,d}$ a fraction that is uniformly distributed between 0 and 1. For each period $t \in 1..T$ we raise that fraction to the power of $d = t - 1$. For example for a three period problem, $d = 0, 1$ and 2, and $k_{i,d} = 0.8$, then $k_{1,0} = 0.8^0 = 1$, $k_{1,0} = 0.8^1 = 0.8$ and $k_{1,0} = 0.8^2 = 0.64$.

Estimating cross-selling and cannibalization matrix γ_{ij} is the process of measuring the positive/negative effect seen when a product is introduced at the same period where a complementary/substitutable product is active. That type of parameters needs analysis of previous situations in previous years. Amount of sales in overlapping periods are compared to non-overlapping ones so as to measure the percent of increase in decay if any. Krider and Weinberg (1998) discuss the rationale of avoiding the competition in the general context of product introduction timing.

3.4. Exact and Heuristic Solution Approaches

This section describes and examines different solution approaches to the multi-category assortment packing problem. We first reformulate the problem as a linear mixed integer program that is capable of solving small-size instances and discuss the implications of using traditional integer program solver to computationally solve the problem. We then describe a heuristic solution approach that is based on the continuous relaxation solution of the APPCS problem.

3.4.1 Linearization Scheme

Model APPCS is a 0-1 nonlinear formulation that optimizes assortment and entry timing decisions over the planning season with cross-selling and cannibalization considerations. The problem is NP-hard and poses computational challenges due to the discreteness of key decision variables and nonlinearities that arise in the objective function in the fractional expression of the market share. The computational intractability can, however, be largely alleviated by developing a linear reformulation of APPCS.

The objective function of APPCS has two types of nonlinearities: i) Existence of a quadratic term as a result of multiplying two x variables ($x_{iu}x_{jh}$), and ii) fractional expression of the market share. To overcome the nonlinearity of the quadratic term we first introduce the following auxiliary binary variable in lieu of the nonlinear term in the objective function:

$$q_{ijuh} \equiv x_{iu}x_{jh}, \quad \forall i \in \Omega, j \in \Omega \setminus \{i\}, u, h \in \mathcal{T}. \tag{3.2a}$$

The variable q_{ijuh} is set to 1 if and only if both variables x_{iu} and x_{jh} equal to 1. The following linearizing constraints are added for all $i \in \Omega$, $j \in \Omega \setminus \{i\}$, $u, h \in \mathcal{T}$:

$$q_{ijuh} \leq x_{iu}, \tag{3.3a}$$

$$q_{ijuh} \leq x_{jh}, \tag{3.3b}$$

$$q_{ijuh} \geq x_{iu} + x_{jh} - 1. \tag{3.3c}$$

As a result, APPCS can be reformulated as follows:

$$\text{Maximize} \quad \sum_{t=1}^{T} \alpha_t \sum_{i=1}^{n} r_i \left(\frac{v_i \sum_{u=1}^{t} [k_{i,t-u}x_{iu} + \sum_{j \neq i} \sum_{h=\max\{1,t-\ell_j+1\}}^{t} k_{i,t-u}\gamma_{ij}q_{ijuh}]}{v_0 + \sum_{s=1}^{n} v_s \sum_{u=1}^{t} [k_{s,t-u}x_{su} + \sum_{j \neq s} \sum_{h=\max\{1,t-\ell_j+1\}}^{t} k_{s,t-u}\gamma_{sj}q_{sjuh}]} \right) \tag{3.4a}$$

$$\text{subject to:} \quad \sum_{t=1}^{T} x_{it} \leq 1, \quad \forall i \in \Omega \tag{3.4b}$$

$$q_{ijuh} \leq x_{iu}, \quad \forall i \in \Omega, j \in \Omega \setminus \{i\}, u, h \in \mathcal{T} \tag{3.4c}$$

$$q_{ijuh} \leq x_{jh}, \quad \forall i \in \Omega, j \in \Omega \setminus \{i\}, u, h \in \mathcal{T} \tag{3.4d}$$

$$q_{ijuh} \geq x_{iu} + x_{jh} - 1, \quad \forall i \in \Omega, j \in \Omega \setminus \{i\}, u, h \in \mathcal{T} \tag{3.4e}$$

$$x \text{ binary}, q \geq 0. \tag{3.4f}$$

To overcome nonlinearity due to fractional expression, we first apply the Charnes and Cooper (1962) transformation technique to the objective function in (3.4a), then we introduce new variables to substitute for the resulting nonlinear terms. An explanation of the transformation method is discussed next. We denote by

$$w_t \equiv \frac{1}{v_0 + \sum_{s=1}^{n} v_s \sum_{u=1}^{t} [k_{s,t-u} x_{su} + \sum_{j \neq s} \sum_{h=\max\{1,t-\ell_j+1\}}^{t} k_{s,t-u} \gamma_{sj} q_{sjuh}]}, \quad \forall t \in \mathcal{T} \tag{3.5a}$$

where v_0 is assumed to be greater than or equal to 1 and where w_t is a continuous variable between 0 and 1, $\forall t \in \mathcal{T}$. We also let

$$y_{it} \equiv \sum_{u=1}^{t} [k_{i,t-u} x_{iu} + \sum_{j \neq i} \sum_{h=\max\{1,t-\ell_j+1\}}^{t} k_{i,t-u} \gamma_{ij} q_{ijuh}] w_t, \quad \forall i \in \Omega, t \in \mathcal{T} \tag{3.5b}$$

The nonlinear terms in the objective function are linearized using substitutions from (3.5a) and (3.5b) as follows:

$$\text{Maximize} \quad \sum_{t=1}^{T} \alpha_t \sum_{i=1}^{n} r_i v_i y_{it} \tag{3.5c}$$

We also append the following linear constraints:

$$v_0 w_t + \sum_{i=1}^{n} v_i y_{it} = 1, \quad \forall t \in \mathcal{T} \tag{3.5d}$$

$$y_{it} \leq \sum_{u=1}^{t} [k_{i,t-u} x_{iu} + \sum_{j \neq i} \sum_{h=\max\{1,t-\ell_j+1\}}^{t} k_{i,t-u} \gamma_{ij} q_{ijuh}], \quad \forall i \in \Omega, t \in \mathcal{T} \tag{3.5e}$$

$$y_{it} \leq k_{i,t-u} w_t + \sum_{j \neq i} \sum_{h=\max\{1,t-\ell_j+1\}}^{t} k_{i,t-u} \gamma_{ij} x_{jh} w_t + 1 - x_{iu},$$
$$\forall i \in \Omega, t \in \mathcal{T}, u \leq t \tag{3.5f}$$

Where constraint (3.5d) represents and introduces the relationship in (3.5a) into the model in a linear format, and constraints (3.5e) and (3.5f) are linearizing constraints to ensure that y_{it} carries the correct value of the market share in each iteration. The last step in this procedure is to substitute the nonlinear term in (3.5f) with the following auxiliary variable:

$$p_{jht} \equiv x_{jh} w_t, \quad \forall j \in \Omega, h \in \mathcal{T}, t \in \mathcal{T}. \tag{3.6a}$$

Because the x-variables are binary and the w-variables are continuous between 0 and 1, the following constraints are added to complete this linearization scheme for all $j \in \Omega$, $h, t \in \mathcal{T}$:

$$p_{jht} \leq x_{jh}, \tag{3.6b}$$

$$p_{jht} \leq w_t, \tag{3.6c}$$

$$p_{jht} \geq x_{jh} + w_t - 1. \tag{3.6d}$$

Where constraint (3.6b) forces p_{jht} to be equal to zero if $x_{jh} = 0$, while constraints (3.6c) and (3.6d) ensures that if $x_{jh} = 1$, then the value of p_{jht} will be set to w_t and thus the substitution $p_{jht} \equiv x_{jh} w_t$ is valid.

The complete linear reformulation of APPCS is denoted by L-APPCS and is stated as follows:

$$\textbf{L-APPCS}\text{: Maximize: } \sum_{t=1}^{T} \alpha_t \sum_{i=1}^{n} r_i v_i y_{it} \tag{3.7a}$$

$$\text{subject to: } \sum_{t=1}^{T} x_{it} \leq 1, \quad \forall i \in \Omega \tag{3.7b}$$

$$q_{ijuh} \leq x_{iu}, \quad \forall i \in \Omega, j \in \Omega \setminus \{i\}, u, h \in \mathcal{T} \tag{3.7c}$$

$$q_{ijuh} \leq x_{jh}, \quad \forall i \in \Omega, j \in \Omega \setminus \{i\}, u, h \in \mathcal{T} \tag{3.7d}$$

$$q_{ijuh} \geq x_{iu} + x_{jh} - 1,$$
$$\forall i \in \Omega, j \in \Omega \setminus \{i\}, u, h \in \mathcal{T} \tag{3.7e}$$
$$v_0 w_t + \sum_{i=1}^{n} v_i y_{it} = 1, \quad \forall t \in \mathcal{T} \tag{3.7f}$$
$$y_{it} \leq \sum_{u=1}^{t} [k_{i,t-u} x_{iu} + \sum_{j \neq i} \sum_{h=\max\{1,t-\ell_j+1\}}^{t} k_{i,t-u} \gamma_{ij} q_{ijuh}],$$
$$\forall i \in \Omega, t \in \mathcal{T} \tag{3.7g}$$
$$y_{it} \leq k_{i,t-u} w_t + \sum_{j \neq i} \sum_{h=\max\{1,t-\ell_j+1\}}^{t} k_{i,t-u} \gamma_{ij} p_{jht} + 1 - x_{iu},$$
$$\forall i \in \Omega, t \in \mathcal{T}, u \leq t \tag{3.7h}$$
$$p_{jht} \leq x_{jh}, \quad \forall j \in \Omega, h, t \in \mathcal{T} \tag{3.7i}$$
$$p_{jht} \leq w_t, \quad \forall j \in \Omega, h, t \in \mathcal{T} \tag{3.7j}$$
$$p_{jht} \geq x_{jh} + w_t - 1, \quad \forall j \in \Omega, h, t \in \mathcal{T} \tag{3.7k}$$
$$\mathbf{x} \text{ binary}, \mathbf{q}, \mathbf{w}, \mathbf{y}, \mathbf{p} \geq 0. \tag{3.7l}$$

L-APPCS is a linear mixed-integer program that enables optimal solutions using standard commercial optimization solvers such as CPLEX. The model is capable of solving small-size instances to optimality. However, a downside of this linearized model is the increased number of variables and constraints compared to APPCS, thus solving large store-wide instances is still intractable. In the following section, we propose a heuristic that solves large instances in convenient time with a very small optimality gap.

3.4.2 RelaxMax Heuristic

We introduce an approximation algorithm to solve the APPCS problem. The algorithm, which we refer to as the *RelaxMax* heuristic, is based on the continuous relaxation solution of the problem. The procedure is as follows: solve the continuous relaxation, then introduce each product at the period with the highest fraction. The heuristic is formally introduced in Algorithm 1. In the following section, results of the heuristic will be compared against other solution approaches discussed in this chapter.

Algorithm 1 RelaxMax Heuristic

```
solve continuous relaxation
for all products i ∈ 1..I do
    x_max ⇐ max_{t=1}^{T} x[i, t]
    if x_max ≠ 0 then
        for all periods t ∈ 1..T do
            if x[i, t] = x_max then
                fix x[i.t] ← 1
                break
            end if
        end for
    end if
end for
```

3.5. Computational Study

In this section, we use simulated data to illustrate the proposed multi-category assortment packing problem and then test the tractability of the different solution approaches. All mathematical programs are coded in AMPL on a Dell XPS 8300 workstation having Intel Core(TM) i7-2600 CPU 3.40 GHz processor and 12 GB of RAM.

3.5.1 Illustrative Examples

We analyze an illustrative example to obtain insights regarding the effects of complementarity and substitution on the optimal assortment selection and the expected profit. The instance consists of five products introduced over the range of ten periods. Products 1 and 2 belong to the same category while products 3, 4 and 5 belong to a different category. Cannibalization effect is observed between products 1 and 2, due to substitutability, with $\gamma_{2,1} = -0.2$, meaning that decay of product 2 is accelerated by 20% faster if product 1 is introduced within its life cycle. Moreover, cross-selling effect is observed between products 2 and 4, due to complementarity, with $\gamma_{2,4} = 0.3$, meaning that current decay of product 2 is delayed by 30% if product 4 is introduced within its life cycle. All five products have equal margins ($r_i = 1$, $\forall i \in 1, ..., 5$) and product attractiveness $v_1 = 10, v_2 = 50, v_3 = 20, v_4 = 30$ and $v_5 = 10$, while the attractiveness of the outside option $v_0 = 40$. An exponential decay function is used with parameters: $k_1 = 0.9, k_2 = 0.8, k_3 = 0.3, k_4 = 0.2$ and $k_5 = 0.7$.

To show the consequences of ignoring substitution and complementarity effects, we first solve this instance without accounting for the substitution and complementarity effects by setting $\gamma_{ij} = 0, \forall i, j$. We refer to this case as the *disjoint approach*. This approach is enabled by deactivating the cross-selling and cannibalization matrix γ_{ij}, thus setting $\gamma_{ij} = 0 \quad \forall i, j$. Table 3.1 shows the results where the two substitutable products 1 and 2 are introduced close to each other in the first two periods, while the two complementary products 2 and 4 are introduced far away from each other at periods 1 and 9 respectively. The optimal objective value calculated in this case is \$83.44.

Table 3.1: Solution of the Disjoint Approach

	Solution x_{it}									
i, t	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$	$t = 6$	$t = 7$	$t = 8$	$t = 9$	$t = 10$
$i = 1$	0	**1**	0	0	0	0	0	0	0	0
$i = 2$	**1**	0	0	0	0	0	0	0	0	0
$i = 3$	0	0	0	0	0	0	**1**	0	0	0
$i = 4$	0	0	0	0	0	0	0	0	**1**	0
$i = 5$	0	0	0	0	**1**	0	0	0	0	0

By activating the γ_{ij} values of the instance (i.e., set $\gamma_{2,1} = -0.2$ and $\gamma_{2,4} = 0.3$), the solution to the *integrated approach* is given in Table 3.2 with an objective value of \$82.73.

Table 3.2: Solution of Integrated Approach

	Solution x_{it}									
i,t	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t=6$	$t=7$	$t=8$	$t=9$	$t=10$
$i=1$	0	0	0	0	0	**1**	0	0	0	0
$i=2$	0	**1**	0	0	0	0	0	0	0	0
$i=3$	0	0	0	0	0	0	0	0	**1**	0
$i=4$	**1**	0	0	0	0	0	0	0	0	0
$i=5$	0	0	0	0	0	**1**	0	0	0	0

The following insights are in order:

- Knowledge of the cannibalization effect between products 1 and 2 encouraged the model to increase the gap between their introductions to the market to minimize the negative impact of cannibalization on sales of product 2.
- Knowledge of the cross-selling effect between products 2 and 4 encouraged the model to decrease the gap between their introductions to the market to maximize the positive impact of cross-selling on sales of product 2.

Although having less gross profits with the integrated model might seem counter intuitive, however we highlight here that the objective value obtained from the disjoint model is not a correct interpretation of the real situation as it ignored the effects of cross-selling and cannibalization which will occur in reality regardless of the fact that the firm is aware of its existence or not. To calculate the actual gross profits in the disjoint case and demonstrate the effects of overlooking substitution and/or complementarity effects, we solve the instance again using the integrated model, with the active values of γ_{ij} while forcing the model to fix the myopic solutions of the disjoint case in Table 3.1. The optimal objective value obtained in this case is \$79.67. This means that when cannibalization and cross-selling effects are overlooked, the actual objective value of \$79.67 is less than the (falsely) expected objective of \$83.44 due to the suboptimal timing of products. Which is, in turn, 3.7% less than the optimal integrated objective value (\$82.73) that is obtained when planning considers cannibalization and cross-selling effects, which in considered a valuable increase in terms of gross profits in retail business.

3.5.2 Tractability Analysis

Table 3.3 shows results for 12 instances of size up to 100 products within 12 periods. Solutions for the nonlinear MIP formulation (APPCS) are obtained using the solver KNITRO, while the linear reformulation (L-APPCS) results are obtained using CPLEX and the RelaxMax heuristic results are obtained using MINOS. Optimality gaps for the nonlinear formulation are reported by the solver, while gaps for the heuristic are computed relative to solutions and gaps of KNITRO, except for instance 12 and due to the absence of a KNITRO solution, the gap is calculated relative to the continuous relaxation solution using MINOS.

Table 3.3: Computational Analysis of L-APPCS, APPCS and RelaxMax

		Linear Formulation		Nonlinear Formulation		RelaxMax	
Instance	n	Objective	Time (sec)	Gap	Time (sec)	Gap	Time (sec)
1 Period ($T = 1$)							
1	10	12.19	0.20	0%	0.03	0%	0.03
2	50	11.37	0.37	0%	0.05	0%	0.05
3	100	11.48	2.33	0%	0.06	0%	0.09
3 Periods ($T = 3$)							
4	10	33.38	2.09	0%	0.03	0%	0.06
5	50	32.28	*	0.003%	0.19	0%	0.13
6	100	32.10	*	0.001%	1.28	0%	0.51
6 Periods ($T = 6$)							
7	10	60.07	737.54	0%	0.19	0%	0.03
8	50	61.08	*	0%	8.94	0.03%	0.75
9	100	59.02	*	0%	67.92	0%	3.80
12 Periods ($T = 12$)							
10	10	105.75	*	0%	1.34	0%	0.09
11	50	112.69	*	0.001%	1611.27	0.003%	3.87
12	100	106.11	*	*	*	0.87%	32.98

* Exceeded the 1 hour limit imposed on the solver.

As shown in Table 3.3, solving the problem to optimality using the linear formulation L-APPCS was enabled for small-size instances, however for large instances CPLEX exceeded the one hour limitation to find the solution. With the exception of instance 12, the nonlinear solver KNITRO could obtain near-optimal solutions (optimal in 8 out of 12 instances) to the problem APPCS. The optimality gap is reported for each instance which ranged from 0% to 0.003%, while the CPU time taken to find a solution ranged between 0.03 seconds and about 27 minutes. The RelaxMax heuristic is shown to achieve very close to optimal results (optimal in 9 out of 12 instances). In terms of CPU time, it ranged from 0.03 seconds to a maximum of around 33 seconds. This empirically observed computational tractability of the proposed heuristic, with very close to optimal solutions, is encouraging and bears the potential of benefiting retailers for large-scale, industry-sized problem instances.

3.6. Extension

In this section we develop an extension of the general model APPCS in the movie industry. A formal statement of the multi-period multi-genres assortment packing problem of movies under cannibalization effects is provided and an example is solved to illustrate the model behavior.

3.6.1 Model Formulation

Movie scheduling in the motion picture industry can be considered one of the most important applications to the assortment packing problem (see Eliashberg et al. 2006 for a review on movie industry problems). Each movie theater (exhibitor) decides on a weekly basis on the set of movies (assortment) to start showing in the coming week. The set of candidate movies to choose from $\mathcal{I}$, is the set of all movies released, or will be released, within the planning horizon $\mathcal{T}$. Usually the number of candidate movies exceeds the number of screens in the theater and thus the exhibitor must choose an optimal set for each week, within its capacity of screens H, that maximizes the gross profit. Each movie i is characterized by a release date (rd_i) and an optional due date (dd_i), together they act as a time-window for each movie to be displayed within. Distributors often mandates an obligation period (ob_i) which is the minimum number of weeks the exhibitor is required to display the movie (see Swami et al. 1999 and Eliashberg et al. 2001 for more information about the movie scheduling problem).

The fact that customers' interest in a movie decays over time, makes the movie a product with a limited life-cycle. This creates more challenge on theaters deciding on the best release time and removal time of a movie. A successful decision support system needs to incorporate different decay functions into its structure. A distinction between two different movie types in terms of their decay function is made by Sawhney and Eliashberg (1996). They classified movies as either a *blockbuster* or a *sleeper* movie, where sales of blockbusters start at its peek then decay exponentially over time, while sales of sleepers build up gradually and usually peek in 3 to 6 weeks after its release then it decays over time. This decay function characterizes the movie's life cycle, and thus a movie is discontinued when it either totally decays (that is when customers lose interest in watching the movie) or when it is more profitable to discontinue this movie and introduce another. Our formulation is capable of incorporating both decay patterns.

The effects of cannibalization between each pair of movies (i, j) of the same genre is captured using the matrix γ_{ij}. This effect is represented by a fraction ranging from 0 to 1 indicating the intensity of the accelerated decay that will result from introducing a same-genre movie while another is still displaying. Cross-selling effects are not typically detected between movies and therefore are not considered in this extension.

Consider the following notation of the optimal entry timing problem in the movie industry:

Parameters

- $\mathcal{I} = \{1, \ldots, n\}$: Set of candidate movies in all genres that the exhibitor can show over the planning horizon, listed consecutively from 1 to n.
- $\mathcal{T} = \{1, \ldots, T\}$: Set of all T weeks in the planning horizon.
- H: Number of screens in the movie theater.
- α_t: Seasonality factor in period t, used to reflect peeks due to holiday weeks, $\forall t \in \mathcal{T}$.
- r: The average unit gross margin of a sold ticket.
- v_i: Weight (attractiveness) of movie i on the first week, $\forall i \in \mathcal{I}$.
- v_0: Weight (attractiveness) of the outside option due to competition.
- rd_i: Release date of movie i, $\forall i \in \mathcal{I}$.
- dd_i: Due date of movie i, $\forall i \in \mathcal{I}$.
- ob_i: Obligation period in weeks of movie i, $\forall i \in \mathcal{I}$.
- $m_i = max\{ob_i, dd_i - rd_i + 1\}$: Maximum possible number of weeks movie i can be displayed, $\forall i \in \mathcal{I}$.
- k_{id}: Decay factor for movie i after d weeks of its initial introduction, ranging between 0 and 1. As such, $k_{i0} = 1$, since the product did not start decaying yet at the first period.
- γ_{ij}: Fraction of the decrease in the decay value of movie i if movie j is introduced within its life cycle as a result of cannibalization effects, with the assumption that $\sum_{j:\gamma_{ij}<0} \gamma_{ij} \geq -1, \forall i \in \mathcal{I}$.

Decision Variables

- $x_{itw} \in \{0,1\}$: $x_{itw} = 1$ if and only if movie i is introduced in week t for w weeks, $\forall i \in \mathcal{I}, t \in \mathcal{T}, w \in \mathcal{T}$.

The multi-period multi-genres optimal assortment packing problem with cannibalization effects that is oriented towards the movie industry, denoted by **APPMV**, can be stated as the following 0-1 nonlinear (fractional) program:

APPMV: Maximize:

$$\sum_{t=1}^{T} \alpha_t r \sum_{i=1}^{n} \left(\frac{v_i \sum\limits_{u=1}^{t} \sum\limits_{w=ob_i}^{m_i} x_{iuw} [k_{i,t-u} + \sum\limits_{j \neq i} \sum\limits_{h=u}^{\min\{dd_i, u+w-1\}} k_{i,t-u} \gamma_{ij} \sum\limits_{y=ob_j}^{m_j} x_{jhy}]}{v_0 + \sum\limits_{s=1}^{n} v_s \sum\limits_{u=1}^{t} \sum\limits_{w=ob_s}^{m_s} x_{suw} [k_{s,t-u} + \sum\limits_{j \neq s} \sum\limits_{h=u}^{\min\{dd_i, u+w-1\}} k_{s,t-u} \gamma_{sj} \sum\limits_{y=ob_j}^{m_j} x_{jhy}]} \right) \tag{3.8a}$$

$$\text{subject to:} \quad \sum_{t=1}^{T} \sum_{w=ob_i}^{m_i} x_{itw} \leq 1, \quad \forall i \in \mathcal{I} \tag{3.8b}$$

$$\sum_{i=1}^{n} \sum_{w=ob_i}^{m_i} \sum_{h=\max\{1, t-w+1\}}^{t} x_{ihw} \leq H, \quad \forall t \in \mathcal{T} \tag{3.8c}$$

$$\sum_{w=1}^{T} x_{itw} = 0, \quad \forall i \in \mathcal{I}, t \in \mathcal{T} : t < rd_i \text{ or } t > dd_i \tag{3.8d}$$

$$x \text{ binary.} \tag{3.8e}$$

The objective function in (3.8a) is the sum of the seasonal (gross) profits over all movies in all genres for all weeks in the planning horizon. The expression between parentheses is the market share of movie i in week t. Constraint (3.8b) ensures that each movie is introduced at most once. Constraint (3.8c) ensures that the number of selected movies in each week does not exceed the number of screens in the theater. Constraint (3.8d) prevents the introduction of a movie before its release date or after its due date. Constraint (3.8e) imposes binary restrictions on the decision variable x.

3.6.2 Illustrative Example

Consider a movie theater with a capacity of four screens ($H = 4$), having to choose from a pool of twenty released movies ($n = 20$). We examine a case with a planning horizon of 4 weeks ($T = 4$), assuming that all movies are available during the planning horizon, and thus $rd_i = 1, \forall i \in \mathcal{I}$ and $dd_i = 4, \forall i \in \mathcal{I}$. Furthermore, we assume no holidays and thus no seasonal factor ($\alpha_t = 1, \forall t \in \mathcal{T}$), and a cannibalization effect between movies 6 and 10 of the same genre, that is $\gamma_{6,10} = -0.2$, and $\gamma_{ij} = 0$ otherwise. The outside factor $v_0 = 160$ and the unit gross margin of a sold ticket $r = \$1$. The rest of the parameters' data is listed in Table 3.4.

Table 3.4: Data for Illustrative Example

Parameter	Movies									
	1	2	3	4	5	6	7	8	9	10
v_i	7	35	41	22	20	35	42	36	33	44
k_{i1}	0.3	0.4	0.1	0.4	0.5	1	0.3	0.5	0.3	0.2
	11	12	13	14	15	16	17	18	19	20
v_i	19	42	28	16	4	9	13	17	18	19
k_{i1}	1	0.4	0.9	0.3	0.4	0.7	0.7	0.1	0.6	0.6

The nonlinear solver KNITRO took only 2.4 seconds to find an optimal solution to the problem (optimality gap = 0%). The recommendations are as follows: for week 1, movies 6, 8, 11 and 13 are introduced, the schedule remains the same until week 3 where movies 6, 8 and 11 are discontinued and movies 2, 7 and 12 are introduced instead, the movie assortment becomes (2, 7, 12 and 13). In week 4, movie 13 is discontinued and replaced by movie 10 with a final assortment of movies (2, 7, 10 and 12) for the last week in the time horizon with a total gross revenue of \$1.87. Table 3.5 summarizes the recommended schedule with newly introduced movies highlighted in bold. Note that movies that were introduced within the time horizon are assumed to continue displaying beyond that horizon if necessary. For instance, movie 10 is introduced in week 4, however it will continue to be displayed until week 6 which is beyond the timing horizon of the instance.

Table 3.5: Recommended Schedule for the Illustrative Example

	Screens			
Week	i	ii	iii	iv
1	**6**	**8**	**11**	**13**
2	6	8	11	13
3	**2**	**7**	**12**	13
4	2	7	**10**	12

Model APPMV is, to the best of our knowledge, the first movie scheduling model that considers competition (using the outside option) and accelerated decay due to cannibalization effects between movies of the same genre.

3.6.3 Tractability Analysis

Table 3.6 shows results for 9 instances of size up to 100 movies within periods of 3, 6 and 12 weeks. In all instances, six screens are assumed to be available in the theater ($H = 6$). Solutions for the nonlinear model APPMV are obtained using KNITRO solver and optimality gaps are reported by the solver. We also use the RelaxMax heuristic introduced in Section 3.4.2 to solve the problem. Optimality gaps are reported for each instance. For instances 1 to 6, RelaxMax gaps are calculated based on the performance of KNITRO and the optimality gap given by the solver. However, for instances 7 to 9, and due to the absence of a solution by KNITRO, gaps are calculated by comparing the solution obtained by RelaxMax to the continuous relaxation solution of the problem obtained by the solver MINOS.

As shown in Table 3.6, the nonlinear solver KNITRO is capable of solving instances of up to 100 movies and 6 periods, the time it takes to solve these instances ranged from 0.03 seconds to around 8 minutes. For instances with 12 periods or higher, the solution time exceeded one hour. The RelaxMax heuristics reports optimal and very close to optimal results for all instances within a few minutes. The solution time of RelaxMax ranged from 0.06 seconds to around 5 minutes.

Table 3.6: Computational Analysis of APPMV

		Nonlinear Solver		RelaxMax	
Instance	n	Gap	Time (sec)	Gap	Time (sec)
3 Periods ($T = 3$)					
1	10	0%	0.031	0%	0.06
2	50	0%	0.16	0%	0.28
3	100	0%	0.55	0%	1.65
6 Periods ($T = 6$)					
4	10	0%	6.63	0.14%	1.72
5	50	0%	90.34	0%	4.18
6	100	0%	412.53	0%	22.15
12 Periods ($T = 12$)					
7	10	*	*	0.05%	23.01
8	50	*	*	0.12%	56.63
9	100	*	*	0%	306.80

* Exceeded the 1 hour limit imposed on the solver.

3.7. Conclusion

We have examined the multi-category assortment packing problem under the effect of cross-selling and cannibalization (APPCS), where a retailer seeks to optimize assortment and entry-timing of products into the market. A novel 0-1 nonlinear fractional model is developed that maximizes the retailer's gross profit over the planning horizon. We solve the nonlinear model using the nonlinear commercial solver KNITRO, results shows that near-optimal solutions with very small gaps (0% in many cases) can be obtained in a reasonable time. We highlight two more solution approaches, the first is formulating a linear model by circumvented the nonlinearity by introducing auxiliary variables and accompanying linearization constraints. The linear MIP reformulation is empirically observed to optimally afford exact solutions to small-size instances in manageable times. The second solution approach is a heuristic that is based on the continuous relaxation solution of the problem, and it is shown that it affords near-optimal solutions for large-scale instances.

We have demonstrated the importance of the integrated planning of multiple retail categories that are inter-related by cross-selling or intra-related through cannibalization. Our illustrative example shows that disjoint optimization of isolated categories results in sub-optimal solutions and therefore profit loss occurs.

We designed an extension of the general assortment packing problem model for a specific application in movie industry, that is to design a decision support system that helps the decision maker to choose the optimal (near-optimal) assortment of movies to display and to decide about the entry timing of each movie to the market given that certain movies can cannibalize each other. The model is based on the attraction model of demand and thus involves competition effect both internally, due to cannibalization, and externally when customers decide to watch a movie elsewhere. An illustrative example that uses an average-size instance of 20 movies over a period of 4 weeks is solved using KNITRO in 2.4 seconds with a 0% optimality gap, and a computational analysis is carried to show tractability of the solution approaches.

CHAPTER 4

CONCLUSIONS AND DIRECTIONS FOR FUTURE RESEARCH

This dissertation contributes to the existing literature by proposing new optimization models and more effective solution approaches for the challenge of designing optimal retail product lines. The work is based on the integration of mathematical modeling to optimally solve decision problems and the useful data patterns that can be extracted by the market-basket analysis techniques. Such data patterns that are used in this dissertation are: (i) Cross-selling effects between complementary products, (ii) cannibalization effects between substitutable products and (iii) product decay pattern over time.

4.1. Summary of Findings

The first essay models the assortment and pricing optimization problem under cross-selling effects. The work is grounded in mathematical optimization and linearization techniques where the problem is first introduced as a nonlinear program and then reformulated into a linear MIP program that can afford exact solutions to large-size problem instances. We show the consequences of overlooking cross-selling data in terms of profit reductions, sub-optimal assortments and inadequate prices. This essay makes the following contributions: (i) It introduces a novel assortment and pricing optimization problem, (ii) it enables an exact solution approach to solve the problem, and (iii) it provides insights on the consequences of overlooking cross-selling information.

The second essay models the multi-category multi-period assortment packing problem, where a retailer seeks to optimize the assortment and the introduction time of products to the market. To integrate some of the useful patterns in data with the decision support system, our model considers the effects of both cross-selling and cannibalization on the problem decisions. We show the effects of overlooking cross-selling and cannibalization on the optimal solution using an illustrative example. This essay makes the following contributions: (i) It introduces a novel assortment and introduction timing optimization problem, (ii) it enables an exact solution approach for small-size instances, and (iii) it enables a very close to optimal solution for large-size instances through implementing a linear programming-based heuristic.

4.2. Directions for Future Research

For the first essay, we recommend for future research the integration of additional retail decisions related to inventory holding and shelf space allocation. It can be also worthwhile to analyze the effect of promotional campaigns (see Su and Geunes 2012, 2013). Finally, we recommend examining product line optimization problems with probabilistic consumer choice models, especially to address applications for which the deterministic consumer choice model may not be adequate.

For the second essay, we believe different applications can benefit from the proposed model and thus what we recommend for future research is to apply the model into mobile phone introduction and fashion products. Exploiting the product introduction problem within the context of on-line shopping and e-commerce is also a promising direction for future research.

For both essays, extensions including supply chain considerations shall be continued. We foresee interest in investigating the impacts of inventory shortage policies on transportation and truck capacities, and implications of economies of scale on the single/multiple category management problem with multiple suppliers.

BIBLIOGRAPHY

Agrawal, N. and Smith, S. A. (2003). Optimal retail assortments for substitutable items purchased in sets. *Naval Research Logistics*, 50(7): 793-822.

Ainslie, A., Drèze, X. and Zufryden, F. (2005). Modeling movie life cycles and market share. *Marketing Science*, 24(3): 508-517.

Akçura, M. T. and Srinivasan, K. (2005). Research note: Customer intimacy and cross-selling strategy. *Management Science*, 51: 1007-1012.

Anderson, S. P., de Palma, A. and Thisse, J.-F. (1992). *Discrete choice theory of product differentiation*, Cambridge, MA: MIT Press.

Anupindi, R., Gupta, S., and Venkataramanan, M. (2009). Managing variety on the retail shelf: Using household scanner panel data to rationalize assortments. In *Retail Supply Chain Management*, Springer US, 122: 155-182.

Aydin, G. and Porteus, E. L. (2008). Joint inventory and pricing decisions for an assortment. *Operations Research*, 56(5): 1247-1255.

Aydin, G. and Ziya, S. (2008). Pricing promotional products under upselling. *Manufacturing & Service Operations Management*, 10(3): 360-376.

Basuroy, S., Mantrala, M. and Walters, R. (2001). The impact of category management on retail prices and performnance: Theory and evidence. *Journal of Marketing*, 65: 16-32.

Bell, D. E., R. L. K. and Little, J. D. C. (1975). A market share theorem. *Journal of Marketing Research*, 12(2): 136-141.

Ben-Akiva, M. E. and Lerman, S. R. (1985). *Discrete choice analysis : theory and application to travel demand*, MIT Press.

Bernstein, F., Kök, A. G. and Xie, L. (2013). Dynamic assortment customization with limited inventories. *Working paper*, Duke University.

Best, R. J. (1976). The predictive aspects of a joint-space theory of stochastic choice. *Journal of Marketing Research*, 13: 198-206.

Braun, M. A. and Srinivasan, V. (1975). Amount of information as a determinant of consumer behavior toward new products. *Proceedings of the American Marketing Association*, 373-378.

Burkart, W. R., Klein, R. and Mayer, S. (2012). Product line pricing for services with capacity constraints and dynamic substitution. *European Journal of Operational Research*, 219: 347-359.

Cachon, G. P. and Kök, A. G. (2007). Category management and coordination in retail assortment planning in the presence of basket shopping consumers. *Management Science*, 53(6): 934-951.

Cachon, G. P., Terwiesch, C. and Xu, Y. (2005). Retail assortment planning in the presence of consumer search. *Manufacturing & Service Operations Management*, 7: 330-346.

Caro, F. and Gallien J. (2007). Dynamic assortment with demand learning for seasonal consumer goods. *Management Science*, 53(2): 276-292.

Caro, F., Martinez-de Albeniz, V. and Rusmevichientong, P. (2014). The assortment packing problem: Multiperiod assortment planning for short-lived products. Forthcoming in *Management Science*.

Charnes, A., and Cooper, W. W. (1962). Programming with linear fractional functionals. *Naval Research Logistics Quarterly*, 9: 181-185.

Chen, K., and Hausman, W. H. (2000). Mathematical properties of the optimal product line selection problem using choice-based conjoint analysis. *Management Science*, 46(2): 327-332.

Cooper, A. (2012). A Brief History of Analytics. *CETIS Analytics Series*, 1: 9.

Corstjens, M. and Doyle, P. (1981). A model for optimizing retail space allocations. *Management Science*, 27(7): 822-833.

Davenport, T. H., Harris, J. G. and Morison, R. (2010). *Analytics at Work: Smarter Decisions, Better Results*. Harvard Business Press.

Dean, J. (1950). Pricing policies for new products. *Harvard Business Review*, 28: 45-53.

Dobson, G. and Kalish, S. (1988). Positioning and pricing of a product-line: Formulation and heuristics. *Marketing Science*, 7: 107-125.

Dobson, G. and Kalish, S. (1993). Heuristics for Pricing and Positioning a Product-Line Using Conjoint and Cost Data. *Management Science*, 39(2): 160-175.

Eliashberg, J., Elberse, A. and Leenders, M. (2006). The motion picture industry: Critical issues in practice, current research, and new research directions. *Marketing Science*, 25(6): 638-661.

Eliashberg, J., Swami, S., Weinberg, C. B. and Wierenga, B. (2001). Implementing and evaluating SilverScreener: A marketing management support system for movie exhibitors. *Interfaces*, 31(3): S108-S127.

Gaur, V. and Honhon, D. (2006). Assortment planning and inventory decisions under a locational choice model. *Management Science*, 52(10): 1528-1543.

Ghoniem, A. and Maddah, B. (2013). Integrated retail decisions with multiple selling periods and customer segments: Optimization and insights. *Working paper*, University of Massachusetts, Amherst.

Green, P. E. and Krieger, A. M. (1985). Models and Heuristics for Product Line Selection. *Marketing Science*, 4(1): 1-19.

Green, P. E. and Srinivasan, V. (1978). Conjoint analysis in consumer research: Issues and outlook. *Journal of Consumer Research*, 5: 103-23.

Green, P. E. and Srinivasan, V. (1990). Conjoint analysis in marketing: New developments with implications for research and practice. *Journal of Marketing*, 54(4): 3-19.

Hall, J. M., Kopalle, P. K. and Krishna, A. (2010). Retailer dynamic pricing and ordering decisions: Category management versus brand-by-brand approaches. *Journal of Retailing*, 86(2): 172-183.

Hanson, W. and Martin, R. K. (1990). Optimal bundle pricing. *Management Science*, 36: 155-174.

Hanson, W. and Martin, R. K. (1996). Optimizing multinomial logit profit functions. *Management Science*, 42: 992-1003.

Hess, J. D. and Gerstner, D. (1987). Loss leader pricing and rain check policy. *Marketing Science*, 6: 4, 358-374.

Honhon, D., Gaur, V. and Seshadri, S. (2010). Assortment planning and inventory decisions under stockout-based substitution. *Operations Research*, 58(5): 1364-1379.

Hotelling, H. (1929). Stability in Competition. *The Economic Journal*. 39(153): 41-57.

Irion, J., Lu, J-C., Al-Khayyal, F. A. and Tsao, Y-C. (2011). A hierarchical decomposition approach to retail shelf space management and assortment decisions. *Journal of the Operational Research Society*, 62: 1861-1870.

Johnson, R. M. (1976). Trade-off analysis of consumer values. *Journal of Marketing Research*, 11: 121-127.

Keller, P. W., Levi, R. and Perakis, G. (2014). Efficient formulations for pricing under attraction demand models. *Mathematical Programming*, 145: 1-2, 223-261.

Kohli, R. and Sukumar, R. (1990). Heuristics for Product-Line Design Using Conjoint Analysis. *Management Science*, 36(12): 1464-1478

Kök, A. G. and Fisher, M. (2007). Demand estimation and assortment optimization under substitution: Methodology and application. *Operations Research*, 55(6): 1001-1021.

Kök, A. G., Fisher, M. and Vaidyanathan, R. (2008). Assortment planning: Review of literature and industry practice. *Retail Supply Chain Management*, volume 122 of International Series in Operations Research & Management Science, Springer New York, 99-154.

Kraus, U. G. and Yano, C. A. (2003). Product line selection and pricing under a share-of-surplus choice model. *European Journal of Operational Research*, 150: 653-671.

Krider, R. E. and Weinberg, C. B. (1998). Competitive dynamics and the introduction of new products: The motion picture timing game. *Journal of Marketing Research*, 35(1): 1-15.

Lancaster, K. 1966. A new approach to consumer theory. *Journal of Political Economy*, 74: 132-57.

Maddah, B. and Bish, E. K. (2007). Joint pricing, assortment, and inventory decisions for a retailer's product line. *Naval Research Logistics*, 54: 315-330.

Maddah, B. and Bish, E. K. (2009). Locational tying of complementary retail items. *Naval Research Logistics*, 56: 421-438.

Maddah, B., Bish, E. K. and Munreo, B. (2011). Pricing, variety, and inventory decisions for categories of substitutable items. *Planning Production and Inventories in the Extended Enterprise: A State of the Art Handbook*, Eds. K. Kemph, P. Keskinocak, and R. Uzsoy. Kluwer Academic Publishers.

Maddah, B., Yassine, A., Salameh, M. K. and Ghaddar, L. (2013). Joint replenishment with lost sales. *Working paper*, American University of Beirut.

Mahajan, S. and van Ryzin. G. (2001). Stocking retail assortments under dynamic consumer substitution. *Operations Research*, 49(3): 334-351.

Manchanda, P., Ansari, A. and Gupta, S. (1999). The shopping basket: A model for multicategory purchase incidence decisions. *Marketing Science*, 18(2): 95-114.

Martín-Herrán, G. Taboubi, S. and Zaccour, G. (2006). The impact of manufacturers' wholesale prices on a retailer's shelf space and pricing decisions. *Decision Sciences*, 37(1): 71-90.

McBride, R. D. and Zufryden, F. S. (1988). An integer programming approach to the optimal product line selection problem. *Marketing Science*, 7(2):126-140.

Mild, A. and Reutterer T. (2003). An improved collaborative filtering approach for predicting cross-category purchase based on binary market basket data. *Journal of Retail Consumer Services*, 10: 123-133.

Mulhern, F. J. and Leone, R. P. (1991). Implicit price bundling of retail products: A multiproduct approach to maximizing store profitability. *Journal of Marketing*, 55(4): 63-76.

Pessemier, E. A., Burger, P., Teach, R. and Tigert, D. (1971). Using laboratory brand preference scales to predict consumer brand purchases. *Management Science*, 17: 371-385.

Raman, A., Fisher, M. L. and McClelland, A. S. (2001). Supply chain management at World Co., Ltd. *Harvard Business School Case*, 601-072.

Rodríguez, B. and Aydin, G. (2011). Assortment selection and pricing for configurable products under demand uncertainty. *European Journal of Operational Research*, 210(3): 635-646.

Russell, G. J., Bell, D.R. et al. (1997). Perspectives on multiple category choice. *Marketing Letters*, 8(3): 297-305.

Russell, G. J. and Petersen, A. (2000). Analysis of cross category dependence in market basket selection. *Journal of Retailing*, 76: 367-392.

Sawhney, M. S. and Eliashberg, J. (1996). A parsimonious model for forecasting gross box-office revenues of motion pictures. *Marketing Science*, 15(2): 113-131.

Shankar, V. and Kannan, P. K. (2014). An across-store analysis of intrinsic and extrinsic cross-category effects. *Customer Needs and Solutions Journal*, 1(2): 143-153.

Shioda, R., Tuncel, L. and Myklebust, T .G. J. (2011). Maximum utility product pricing models and algorithms based on reservation price. *Computational Optimization and Applications*, 48: 157-198.

Smith, J.C., Lim, C., and Alptekinoğlu, A. (2009). New product introduction against a predator: A bilevel mixed-integer programming approach. *Naval Research Logistics*, 56: 714-729.

Smith, S.A. and Agrawal, N. (2000). Management of multi-item retail inventory system with demand substitution. *Operations Research*, 48: 50-64.

Subramanian, S. and Sherali, H. (2010). A fractional programming approach for retail category price optimization. *Journal of Global Optimization*, 48(2): 263-277.

Su, Y. and Geunes, J. (2012). Price promotions, operations cost, and promotions in a two-stage supply chain. *Omega*, 40(6): 891-905.

Su, Y. and Geunes, J. (2013). Multi-period price promotions in a single-supplier, multi-retailer supply chain under asymmetric demand information. Forthcoming in *Annals of Operations Research*.

Swami, S., Eliashberg, J. and Weinberg, C. B. (1999). SilverScreener: A modeling approach to movie screens management. *Marketing Science*, 18(3): 352-372.

Talluri, K. and van Ryzin, G. J. (2004). Revenue management under a general discrete choice model of consumer behavior. *Management Science*, 50(1): 15-33.

Van den Poel, D., Schamphelaere, J. D. and Wets, G. (2004). Direct and indirect effects of retail promotions on sales and profits in the do-it-yourself market. *Expert Systems with Applications*, 27(1): 53-62.

van Harmelen, M. and Workman, D. (2012). Analytics for Understanding Research. *CETIS Analytics Series*, 1: 4.

van Ryzin, G. and Mahajan, S. (1999). On the relationship between inventory costs and variety benefits in retail assortments. *Management Science*, 45(11): 1496-1509.

Walras, L. (1954). Elements of pure economics or the theory of social wealth. Allen and Unwin.

Walters, R. G. (1988). Retail promotions and retail store performance: A test of some key hypotheses. *Journal of Retailing*, 64: 153-180.

Walters, R. G. (1991). Assessing the impact of retail price promotions on product substitution, complementary purchase, and interstore sales displacement. *Journal of Marketing*, 55: 16-28.

Yücel, E., Karaesmen, F., Salman, F. S. and Türkay, M. (2009). Optimizing product assortment under customer-driven demand substitution. *European Journal of Operational Research*, 199(3): 759-768

Zenor, M. J. (1994). The profit benefits of category management. *Journal of Marketing Research*, 31: 202-213.

Zhang, R., Kaku, I. and Xiao, Y. (2011). Deterministic {EOQ} with partial backordering and correlated demand caused by cross-selling. *European Journal of Operational Research*, 210(3): 537-551.

Zufryden, F. S. (1977). A conjoint measurement-based approach for optimal new product design and market segmentation. *Analytic approaches to product and marketing planning*, Ed. Shocker, A. D., Cambridge, MA, 100-114.

Zufryden, F. S. (1982). Product line optimization by integer programming. *Proceedings of the Annual Meeting of ORSA/TIMS*, San Diego, CA.