

2000

Young children's reasoning about the inverse relation between the number and sizes of parts : early fraction understanding and the role of material type.

Rachel E. Wing

University of Massachusetts Amherst

Follow this and additional works at: <https://scholarworks.umass.edu/theses>

Wing, Rachel E., "Young children's reasoning about the inverse relation between the number and sizes of parts : early fraction understanding and the role of material type." (2000). *Masters Theses 1911 - February 2014*. 2371.

Retrieved from <https://scholarworks.umass.edu/theses/2371>

This thesis is brought to you for free and open access by ScholarWorks@UMass Amherst. It has been accepted for inclusion in Masters Theses 1911 - February 2014 by an authorized administrator of ScholarWorks@UMass Amherst. For more information, please contact scholarworks@library.umass.edu.



312066 0275 8132 2

YOUNG CHILDREN'S REASONING ABOUT THE INVERSE RELATION
BETWEEN THE NUMBER AND SIZES OF PARTS: EARLY FRACTION
UNDERSTANDING AND THE ROLE OF MATERIAL TYPE

A Thesis Presented

by

RACHEL E. WING

Submitted to the Graduate School of the
University of Massachusetts Amherst in partial fulfillment
of the requirements for the degree of

MASTER OF SCIENCE

September 2000

Psychology

© Copyright by Rachel E. Wing 2000

All Rights Reserved

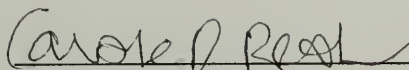
YOUNG CHILDREN'S REASONING ABOUT THE INVERSE RELATION
BETWEEN THE NUMBER AND SIZES OF PARTS: EARLY FRACTION
UNDERSTANDING AND THE ROLE OF MATERIAL TYPE

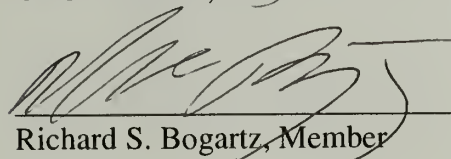
A Thesis Presented

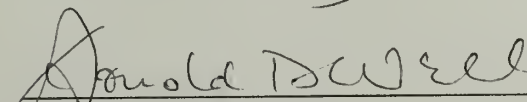
by

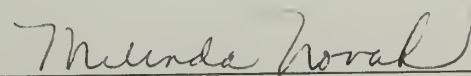
RACHEL E. WING

Approved as to style and content by:


Carole R. Beal, Chair


Richard S. Bogartz, Member


Arnold D. Well, Member


Melinda Novak, Department Head
Psychology Department

DEDICATION

Always intrigued by your words and actions, you kids are the reason for my interest in childhood cognition. I lovingly dedicate this thesis to you:

Patrick, Kevin, Anthony, Zachary, Katherine and Daniel.

ACKNOWLEDGEMENTS

I am grateful to the members of my committee, Arnold D. Well and Richard S. Bogartz, for their suggestions and statistical expertise throughout the development and completion of this thesis. In addition I would like to give special thanks to Carole R. Beal, my advisor and the chair of my committee. She has guided me through this process in a way that has really allowed it to be a tremendous learning experience.

Additional thanks go to all the parents, children and teachers from Deerfield Elementary and Avon Village School who contributed to this project. Their enthusiasm and willingness to participate made this study possible.

Finally, I want to express gratitude toward my fellow graduate students and my parents. When working out new ideas, both were often the sounding boards I needed. Their support and enthusiasm helped to make my thesis a less daunting task.

ABSTRACT

YOUNG CHILDREN'S REASONING ABOUT THE INVERSE RELATION BETWEEN THE NUMBER AND SIZES OF PARTS: EARLY FRACTION UNDERSTANDING AND THE ROLE OF MATERIAL TYPE

SEPTEMBER 2000

RACHEL E. WING, B.A., WASHINGTON UNIVERSITY

M.S., UNIVERSITY OF MASSACHUSETTS AMHERST

Directed by: Professor Carole R. Beal

Kindergartners' and first graders' inverse reasoning abilities with regard to fractional quantities were assessed. The effects different types of to-be-fractionated materials have on fraction task performance were also examined. In Experiment 1, unlike kindergartners, first graders demonstrated inverse reasoning by choosing the optimal number of recipients in a sharing task. Their performance was affected differentially by the 3 material types (continuous, discontinuous and blended). Performance on blended materials exceeded that with continuous. In Experiment 2, each child was only exposed to 1 of the 3 material types and this allowed kindergartners' emerging fraction knowledge to surface: children in the blended materials condition exceeded chance performance.

TABLE OF CONTENTS

	Page
ACKNOWLEDGEMENTS	v
ABSTRACT	vi
LIST OF TABLES	x
LIST OF FIGURES	xi
CHAPTER	
1. INTRODUCTION	1
Aspect of Fraction Knowledge	2
Fractional Parts are Equal in Size	2
A Proportional Relation	7
Exhaustive Division of Whole/Set	11
A Quantity Between Whole Numbers	13
A Part is Smaller than the Whole	16
Inverse Relation Between Number of Parts and their Size	23
Materials	26
Current Study	32
Predictions and Analyses	35
2. EXPERIMENT 1	39
Method	39
Participants	39
Design	39
Materials	42
Photographs	42
Desirable Materials	42
Clay	42

Finger-paint.....	43
Donuts.....	43
Cupcakes.....	43
Undesirable Materials.....	44
Brown Play-Doh.....	44
Worms.....	44
Peas.....	44
Pickles.....	44
Props.....	45
Recording Equipment.....	45
Procedure.....	45
Results.....	47
Scoring.....	47
Preliminary Analyses.....	48
Main Analyses.....	48
Age Analyses.....	52
Individuals' Patterns of Performance.....	54
Discussion.....	55
Effect of Material Type.....	56
Understanding in Kindergartners.....	57
3. EXPERIMENT 2.....	59
Method.....	59
Participants.....	59
Design.....	59
Materials.....	61
Procedure.....	61
Results.....	61
Scoring.....	61
Preliminary Analyses.....	62
Main Analyses.....	62
Kindergartners from Experiment 1 and 2.....	63

4. GENERAL DISCUSSION.....	65
Early Emergence of Inverse Reasoning	66
Material Types.....	67
Question Type	69
Limitations of this Study	70
Conclusions and Educational Implications	71
APPENDICES	
A. DONUT PHOTOGRAPHS	73
B. SET-UP.....	74
BIBLIOGRAPHY	75

LIST OF TABLES

Table	Page
1. Number of trials presented with each material and question type	35
2. Mean number of correct responses to most amount and least amount questions in the presence of desirable and undesirable continuous, blended and discontinuous materials.....	49
3. Mean number of correct responses for each material type separated by grade.....	51
4. Mean number of correct responses with each material type separated by question type	51
5. Number and percent of children classified as demonstrating correct, chance and incorrect patterns of performance at each grade level	54
6. Number and percent of children classified as demonstrating correct, chance and incorrect patterns of performance at each age	55
7. Mean number of correct responses with each material type.....	64

LIST OF FIGURES

Figure	Page
1. Material-question type pairings for the two MQP sets	41
2. First four descriptions in Trial Order 1 and the materials they described from MQP Sets 1 and 2	42
3. Mean number of correct responses for kindergartners and first graders.....	50
4. Mean number of correct responses at each age level.....	53
5. Material-question type pairs for the two MQP sets	60
6. First four descriptions in Trial Order 1 and the materials they described from MQP Sets 1 and 2.....	60
7. Mean number of correct responses for kindergartners in each material type condition.....	63

CHAPTER 1

INTRODUCTION

Fractions are a core component of the elementary mathematics curriculum. When fractions are formally introduced in fourth and fifth grade, students often have trouble working with and understanding them. For example, ordering fractions according to size is often a difficult task. Commonly, children will judge a fraction like $\frac{1}{6}$ to be greater than $\frac{1}{3}$, attending only to the denominator and viewing it as a whole number (Gelman & Gallistel, 1978). Another difficult task is performing fraction algorithms. To add two fractions together, children will frequently add the numerators and then the denominators. Appreciating the part-whole relation that exists between the numerator and denominator is a challenge for these children in both the ordering and adding problems.

The extent to which most children struggle with this area of mathematics suggests that, unlike the whole number counting procedure, fraction reasoning may draw on new concepts rather than flow from informal arithmetic understanding (Gelman & Gallistel, 1978). Fractions require reasoning about proportions; that is, the capacity to think about relations rather than just concrete objects. Stating that someone has eaten one third of a cookie implies the individual ate one of three equal parts of the cookie. Since children are extremely concrete, and relations like these are rather abstract in nature, this is a type of reasoning that would not necessarily be expected from young children. Therefore, it may be the case that an understanding of fractions can only be obtained through explicit instruction.

In spite of the complex reasoning abilities presumed necessary for fraction knowledge, recent work suggests early forms of fraction understanding emerge prior to formal instruction, during the preschool period. Many different aspects of fraction reasoning

come together in the conceptualization of a fraction. Research (Hunting & Sharpley, 1988; Frydman & Bryant, 1988; Goswami, 1989; Mix, Levine & Huttenlocher, 1999; Winer, 1980; Markman, 1978; Sophian, Garyantes & Chang, 1997) indicates that an understanding of the following components essential to fraction knowledge may originate during early childhood: the fractional parts of a whole/set are equal in size/number; a fraction is a proportion, a part is always in reference to a whole; division into fractional pieces is an exhaustive process; a fraction is a quantity falling in between whole numbers; fractional parts are smaller than the original whole; the size of the parts depends on the number of parts into which the whole is divided. In the following sections, I will review the research that has been conducted with young children for these individual aspects of fraction knowledge.

Aspects of Fraction Knowledge

Fractional Parts are Equal in Size

An understanding that the fractional parts of a whole/set are equal in size/number is a very important aspect of fraction reasoning. This knowledge has most often been explored with discrete sets, or discontinuous materials, and with a focus on the one-to-one distribution process called dealing. Dealing is a very common partitioning strategy among young children in which singular items from a set are given to each recipient in turn. Studies are generally designed such that no items remain after an equal distribution. In a study by Hunting and Sharpley (1988), more than 60% of preschoolers used the dealing method during a distribution task. While children were prone to using this method, it is not clear whether or not they were fully aware of its numerical significance (i.e. that the shares would be equal). It is possible that the preschoolers were simply repeating a procedure that they had seen performed several times before either with the sharing of a snack or their toys.

They may have had a very limited understanding of what they were doing. Frydman and Bryant (1988) conducted several studies that more thoroughly explore the depth of children's understanding.

In the first study, children were asked in a rather explicit manner whether they could infer the numerosity of one share from another after an equal subdivision. If they used the dealing strategy and truly understood its purpose, children should be able to infer the number of items in one shared set when they knew the number in another.

In part one of this first experiment, Frydman and Bryant (1988) presented 4-year-olds with red plastic blocks and instructed the children to think of them as sweets. Twelve or 24 "sweets" were placed in front of a child at which time she was told to share the "sweets" among two, three or four dolls. Emphasis was placed on the fact that each doll must be given the same number of "sweets". These trials were conducted simply to confirm Hunting and Sharpley's (1988) previous finding that one-to-one dealing was the most common strategy employed by young children in this type of task.

During part two of this study, children were given either 15 or 18 "sweets" and instructed to distribute them among three dolls. Again, they were informed that each doll was to receive an equal number of items. For these trials, children were corrected if they erred during the distribution process. Once all the blocks were dealt, the experimenter pointed to a single doll and asked the child how many sweets that doll had. The child was then questioned with regard to the second and third dolls. While the children were allowed to count in the case of the first doll, counting was prohibited for subsequent dolls. This portion of the study examined the extent to which children understood the quantitative significance involved in their process of distribution.

Results from the first part of this study confirmed Hunting and Sharpley's (1988) finding and once again revealed dealing in a one-to-one manner to be a popular distribution strategy among young children. Seventy-six percent of the 4-year-olds in Frydman and Bryant's study dealt the "sweets" out to the dolls (69% dealt correctly and 7% erred while dealing). As far as whether or not they fully understood the numerical significance of their actions, the second portion of the study revealed this not to be the case. Very few of the children realized that through dealing, equivalence was ensured. Every child resorted to counting when they were questioned about the second and third dolls. However, it should be noted that when counting was prevented, ten of the 24 children were able to produce the correct answer. This may be an indication that the 4-year-olds were just beginning to understand dealing as a way of attaining equal divisions. Perhaps, it was not until the tried and true method of counting was prohibited that they turned to this new emerging knowledge about dealing and divisions.

In order to determine the extent to which dealing is just a mechanical action devoid of meaning, Frydman and Bryant (1988) conducted two additional studies. These subsequent studies looked at recognizing the need to adjust the typical one-to-one manner of dealing when the portions were no longer equal.

Unlike the first study, the second one had children distribute different size quantities to different dolls. Rather than each doll receiving one "sweet" at a time, some dolls received doubles (two red blocks stuck together) and others received triples (three red blocks stuck together). The 4- and 5-year-olds in this study were always required to share a set of "sweets" between two dolls. In some trials, one doll received only singles while the other received only doubles. In other trials, one doll received singles while the other received

triples. It was explained to the children that despite the difference in portion sizes, both dolls should ultimately be given the same total quantity. Children who understood the numerical significance of their typical dealing strategy should have been able to adjust their behavior taking into account the new portion sizes.

Results indicated that 4-year-olds continued to deal in a one-to-one manner regardless of the different portion sizes. It appeared that for them it was the number of actions rather than the amount dealt to each doll that was important. It was only by 5 years of age that children fully understood the end result of dealing. Unlike the 4-year-olds, the 5-year-olds were able to set aside the one-to-one strategy in place of one that took into account the different portion sizes. However, the exact strategy employed by these preschoolers was not addressed here. It seems unclear whether the children were compensating for a difference in the two portions' masses or they were taking block numerosity into account. The next study provides more insight into the children's reasoning.

A third study conducted by Frydman and Bryant (1988) attempted to make the numerosity of the doubles and triples more salient in the hope that 4-year-olds would be more likely to take it into account. The 4-year-olds were separated into experimental and control conditions. Each child was given a pretest and posttest containing distribution tasks identical to those in the previous study (doubles and triples were again composed of red blocks). Those in the experimental condition participated in an intermediate training session in which doubles and triples were comprised of different colored blocks. For example, a double had one yellow and one blue block joined together. The goal was to allow the children to focus on the individual blocks and the different numerosities of each doll's portions. In this training session, the task was identical to that in the pretest and posttest;

only the color of the blocks differed. The children in the control condition went through an intermediate training session as well. During their session, however, doubles and triples were made up of either all yellow or all blue blocks. No added emphasis was placed on the individual blocks.

The experimental group performed better both during the training session and on the posttest. Not only was a child's attention drawn to the numerosity of each portion while viewing the different colored blocks, but the numerosity apparently remained significant when the blocks were all one color again as well. The fact that added salience to the numerosity of the portions was so beneficial implies that it may be numerosity rather than mass, a more spatial aspect, that guided these preschoolers to an equal distribution of blocks. If it was a spatial characteristic like area or mass that helped the children equate the portions, altering the colors of the blocks would not have made any difference in the children's performance.

Frydman and Bryant's (1988) studies reveal that 5-year-olds have a reasonable understanding of the quantitative rationale behind dealing. This means they understand how to attain equal divisions of a set, a crucial element in fraction reasoning. Four-year-olds do not spontaneously relate sharing or dealing to numerical equivalence. However, with some extra emphasis on numerosity, 4-year-olds were able to adjust their behavior when the typical dealing procedure failed in achieving numerical equivalence. It seems that at this age an understanding of equal divisions is somewhat incomplete.

In these particular studies, children were explicitly asked to ensure equal divisions. However, partitioning a material into equal portions is not the only option when dividing. It

may be that young children believe equivalent divisions to be “more correct” than other possible divisions, however, the field of fraction research lacks studies addressing this possible bias. In the future, this may be an exciting avenue to explore.

A Proportional Relation

Another important aspect of fraction reasoning is understanding that a fraction is a proportion, a relationship between a part and its whole. To grasp what a fraction is children must understand that a fraction does not refer to an absolute amount of material. Rather, it refers to a portion of something always in reference to a whole amount. As the whole changes in size, the fractional amount may contain more “stuff” but the proportion remains unchanged. Research with young children indicates that they may just be starting to understand this aspect of fractions.

Goswami (1989) looked at two tasks involving proportional reasoning: finding equivalent proportions, and solving analogies which include proportions. In the first task, 4- and 6-year-olds were required to match the equivalent proportions of different shapes. Five shapes were used throughout the experiment. The appropriate segments of the shapes were colored bright yellow in order to depict the proportions. Three cards were laid out in front of the child and upon each card there was an outline of a different shape (circle, square, rectangle, triangle or diamond). Within each outlined shape, the identical proportion was shaded bright yellow. A distance away from the three cards displaying identical proportions, four additional cards showing alternatives for completing the pattern were set out. Each child was instructed, “Pick the card which finishes the pattern.” It was made clear that no shapes were to be repeated in the pattern. The four alternatives were of the following type:

correct shape (not already in the pattern) and correct proportion, correct shape and wrong proportion, wrong shape and correct proportion, and wrong shape and wrong proportion.

In the second task, children dealt with an analogical relation between proportions and shapes. The children completed pictorial analogies involving proportions. Again, three cards with outlined shapes (circles, squares, rectangles, triangles or diamonds) and shaded proportions were laid out in front of the child. The shaded area was either a quarter, half or three quarters of the shape. While the first two cards differed in shape, their shaded proportions were identical. The third card, which was a short distance from the first two, showed the same shape as the first card but a different proportion was shaded yellow. After all these cards were placed in view, the child was told, "We are going to play a game about matching patterns. I am going to put down three cards, cards 1 and 2 on this side, and card 3 over here, and you have to pick the card to finish the pattern." The following is an example of an analogy problem that a child might see: $\frac{1}{2}$ circle: $\frac{1}{2}$ rectangle :: $\frac{1}{4}$ circle: ?. Five possible alternatives were provided for the fourth position in the analogy: correct shape and correct proportion, correct shape and wrong proportion, wrong shape and correct proportion, card identical to card 2, and card identical to card 3.

Results from this study indicated that very few 4-year-olds were capable of reasoning about proportions. Only 53% of them were able to successfully complete the equivalent proportion task at a rate above chance. Their performance dropped dramatically, with only 27% scoring above, when they were required to use proportions in the analogical task. However, by the age of 6 years, children appeared more capable of understanding

proportions. Not only did the 6-year-olds perform well on the equivalent proportion task in which 94% scored above chance, but they also did quite well on the analogies where 88% performed above chance.

One source of difficulty for the 4-year-olds in both the equivalent proportion task and the proportional analogies may have been the number of alternative solutions that were presented. Perhaps selecting from among four or five partly shaded shapes was distracting or even confusing. The preschoolers' performance may have improved if there were fewer options from which to choose.

In the previous study, Goswami (1989) varied the shape of the whole when she was examining children's knowledge about the proportional relationship of fractions. Subsequently, Spinillo and Bryant (1991) manipulated the size of the whole instead. The rectangular shape was used throughout their study. In varying the size of the rectangle, Spinillo and Bryant wanted to test whether young children understood that absolute size was irrelevant in a ratio or proportional relation.

On each trial, 4-, 5-, 6- and 7-year-olds were shown three different rectangular boxes with a certain fraction ($1/8$, $3/8$, $5/8$ or $7/8$) colored blue and the remaining fraction colored white. The first box was smaller than the other two and was shown in picture form. In addition to this small picture, two actual rectangular boxes were shown to the children. These boxes contained both blue and white bricks, were identical to each other in size, but differed in their blue to white brick ratio. The proportion of blue to white in one of the boxes matched that of the pictured box. The children were asked to identify the box that was represented by the picture.

To determine if the children were relying on shape to solve this problem, the shaded portion of the picture (not the actual boxes) took on a vertical orientation for half of the trials and a horizontal orientation for the remaining half. In doing this, it was not always the case that the shape of the shaded region was the same in both the picture and the matching box. In order to select the correct alternative, children had to rely on their knowledge about proportions.

The results of this study showed that 4-year-olds performed quite poorly. While 5-year-olds did succeed more often, less than half of them scored above chance on the task. It was not until 6 years of age that more than 50% of the children scored above chance. All age groups were more successful on trials in which the orientation of the proportions were the same. This indicates that young children could be relying on a spatial representation to a great extent. However, the 6-year-olds may have progressed to actual proportional reasoning since they did significantly better than the younger children on the different orientation trials. They may truly be focusing on the part's size with respect to the whole rather than some spatial configuration.

The studies performed by Goswami (1989) and Spinillo and Bryant(1991) indicate that by the ages of 6 and 7, children understand proportional relations. Young children were able to perform comparisons between proportions as well as complete proportional analogies. Four and 5-year-olds showed hints of this type of reasoning, especially when the task and materials were structured to support inferences about fractions. Therefore, it may be that the origins of proportional reasoning are present as early as preschool.

Exhaustive Division of Whole/Set

The exhaustive nature of fractions, the understanding that the sum of the parts must equal the original whole, is yet another aspect of fraction knowledge that has been investigated. While asking children to divide both continuous and discontinuous types of materials, experimenters have exploited children's prior understanding of sharing. One drawback to the sharing scenario is that while preschoolers realize that sharing requires a division of the materials, they do not always conceive of it as an exhaustive process.

In order to determine if preschoolers possess any informal knowledge with regard to the exhaustive distribution of continuous and discontinuous types of quantities, Hunting and Sharpley (1988) conducted a sharing study with 3-, 4- and 5-year-olds. In their study, they presented preschoolers with both continuous and discontinuous types of materials and asked them to share the materials among several dolls. For the continuous type tasks, the material used was skipping rope. Children were told to divide the rope between two and among three dolls. First, children were instructed to cut the rope "so that it is all used up and each doll gets an even share." Then they were allowed to cut the rope themselves or show the interviewer where they would like it to be cut. For the discontinuous type tasks, the preschoolers were told to share 12 crackers among both three and four dolls. Again, the children were instructed to use all of the materials and to ensure that each doll received an equal amount.

The preschoolers often did not distribute the materials exhaustively. The situation in which skipping rope was divided between two dolls was the only exception to the phenomenon of non-exhaustive distribution. For this task, 89% of the preschoolers made only one cut, using up all of the original rope. However, when they were asked to divide the

skipping rope among three dolls, only 40% made the appropriate number of cuts. Numerous children erred by trimming excess rope in an effort to equalize their pieces. Others cut their rope into several pieces and then simply selected the three closest in length and distributed those to the dolls. Non-exhaustive distribution was also observed with the discontinuous type of material. Prior to additional prompting, over half of the preschoolers failed to share all 12 crackers.

While preschoolers were able to exhaustively divide a continuous material into two parts, this was not the case when the number of parts increased or when the materials were discontinuous rather than continuous. Therefore, it appears that for the most part preschoolers either lack or have a very weak understanding of the fact that the sum of the fractional parts is equal to the original whole. This study indicates that the exhaustive nature of fractions may be an aspect of fraction reasoning that has not yet fully developed in preschoolers. However, it should not necessarily be said that absolutely no knowledge of this aspect of fraction reasoning is present in preschoolers.

Both the rope and cracker tasks required children to produce equal shares. A task involving comprehension rather than production may show that preschoolers can recognize the equal division of materials. While they may be unable to produce the shares themselves, perhaps they would be able to choose the correct alternative from among three possible distributions.

Additionally, throughout this study the term “sharing” was used in an attempt to motivate equal divisions and this may be another point of contention. It is possible that the sharing term was misleading and that it may not be the best way to examine children’s knowledge of fractions. While the children were explicitly instructed to give “even” shares

to the dolls, the word “share” may have a very strong connotation for children and it may be all they focused on. The problem with most of their attention being placed on the instruction to share is that sharing may mean different things to different children. “Share” could be interpreted as giving each doll “some” or just the amount that they think the doll could consume, given its size. Sharing may not necessarily imply an equal division for these young children. While the concept of sharing was used in an attempt to assist children’s understanding of fractions and division, it may have impeded that very type of reasoning and knowledge. Therefore, preschoolers may be capable of exhaustive distribution. They may just need to be asked about it in a different manner.

A Quantity Between Whole Numbers

Another type of reasoning that may provide some insight into a child’s fraction knowledge is the ability to reason about fractional quantities. To date, researchers have come to different conclusions regarding preschoolers’ ability to reason about fractions. Gelman (1991) believes that children have the ability to acquire the counting principles very early and that that inhibits their capacity to grasp quantities outside of whole numbers. In contrast, Mix, Levine and Huttenlocher (1999) argue that a major source of difficulty is simply interpreting the numeric symbols. Mix et al. (1999) tested whether or not young children could reason about fractional quantities without the use of written or verbal fraction labels. They assessed reasoning about fractional quantities by way of fraction calculations.

Mix et al. (1999) investigated simple fraction and mixed number calculation skills of 3- through 7-year-olds. Their first experiment addressed children’s ability to make simple fractions calculations, and the second addressed reasoning with mixed numbers.

The first experiment involved 3-, 4- and 5-year-olds and required the preschoolers to perform two different kinds of tasks. The first task entailed simple fraction calculations using quarter pieces of a circular sponge. The child was first presented with $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$ or 1 entire sponge circle. Presentations were all created from quarter sponge pieces that had been velcroed together. After allowing the child to view the sponge presentation for a few seconds, a screen was put in place to block the child's view. Next, sponge pieces ($\frac{1}{4}$, $\frac{1}{2}$ or $\frac{3}{4}$) were either added to or subtracted from the area behind the screen. When something was added or subtracted it was done so as a single unit (quarter pieces velcroed together). An example of a problem a child could receive is $\frac{1}{4} + \frac{3}{4}$. After the problem had been presented, the child was shown a response book that pictured $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$ and 1 whole circle. The child was then asked to choose the correct resulting representation.

The second part of this first experiment dealt with whole-number calculations. The procedure was very similar to that in the first part, however, circular black disks were used instead of quarter pieces of a circular sponge. One, two, three or four disks were shown to the child. After a brief period of time, the disks were covered. Staying within the limit of sums and minuends of four, disks were added or removed from the covered area. Adding one disk to three already behind the screen is an example of this type of problem. After being presented with the problem, the child was shown another response book from which she was to select the correct image of the final presentation. The choices in the response book pictured one, two, three or four black disks.

In the first experiment, 3-, 4- and 5-year-olds all performed above chance on whole-number problems. Only the 4- and 5-year-olds scored greater than chance on the simple fraction problems. While 4- and 5-year-olds were more proficient in the case of whole

numbers, performance on the fraction tasks was still quite impressive. Their performance suggests that they indeed realize that numerical values exist in between whole numbers.

In Experiment 2, the issue of mixed numbers was addressed with 4- through 7-year-olds. The procedure was identical to the simple fraction task in experiment one except that there were sums and minuends of up to three. Also, this time the amounts being added to or subtracted from the sponges behind the screen could range from $\frac{1}{4}$ to $2\frac{3}{4}$ in one-quarter increments. For example, a problem may be $2\frac{1}{4} - \frac{3}{4}$. Similar to Experiment 1, the response book contained four choices: the correct answer and three foils.

Reasoning about mixed numbers proved to be a more challenging task than reasoning about simple fractions. Six and 7-year-olds were able to score above chance in this second experiment, while 4- and 5-year-olds were not. It appears that it is only later in development that more complex fraction calculations are possible.

In addressing how young children perform these manipulations, several possibilities should be considered. Although older school children familiar with the verbal labels for fractions could keep track of the changing array through the terminology, these young children were not yet aware of fraction terminology. Perhaps the children simply counted the individual quarter pieces. Frydman and Bryant's (1988) third "sweets" study mentioned previously indicated that preschoolers attended to numerosity rather than attributes that were spatial in nature. However, in this study, counting seems to be an unlikely strategy since fraction problems failed to order for difficulty according to the maximum number of pieces. Maybe in this case children did do something spatial and constructed a mental model of the situation. It may be that while the sponge pieces were being added to or removed from the original presentation, the child's mental image changed accordingly and the resulting array

provided the answer to the problem (Huttenlocher, Jordan & Levine, 1994). Although the use of a mental model would in fact express impressive spatial abilities, the exact implications of this with respect to fraction reasoning are unclear.

A Part is Smaller than the Whole

Another component of fraction knowledge is the understanding that fractional parts are smaller than the whole out of which they were created. This part-whole relationship has many facets. One that has been extensively studied is the ability to reason about the inclusion relation; that is, that a part is included within the whole. This ability has primarily been assessed through Piagetian class-inclusion tasks.

A typical example of a class-inclusion problem entails showing a child a bouquet containing seven roses and three daisies and then asking her to determine whether there are more roses or flowers. Children rarely solve such problems prior to age 7 or 8. Eight-year-olds succeed on only half of the class-inclusion problems presented to them (Winer, 1980). It is not until the age of 10 or later that children's performance rises above chance to a success rate of 75%.

A major source of difficulty stems from the fact that children tend to treat category terms as mutually exclusive, preferring only one label per object (Markman, 1994). For example, words like "car" and "vehicle" violate the mutual exclusivity assumption. "Vehicle" is the superordinate of "car", however, young children do not consider this relation. Their assumption of mutual exclusivity leads them to assign only one label per object. Since in class-inclusion problems one category is included within another, both

category names can refer to the same object (flower and rose both refer to the rose).

Conceiving of a subordinate category as an entity itself as well as part of a superordinate category conflicts with a child's natural assumption of mutual exclusivity.

Another difficulty children experience with class-inclusion tasks is maintaining the asymmetry of the hierarchical relationship (Markman, 1989). When the superordinate class is divided into subclasses, the whole or superordinate class becomes less salient.

Consequently, part-part comparisons are often made between the subclasses rather than part-whole comparisons between the superordinate class and the appropriate subclass. The problems of conceptualizing asymmetrical relationships between categories and overcoming the assumption of mutual exclusivity are major obstacles when attempting to reason about class-inclusion.

Even by the sixth grade, children fail to fully understand the hierarchical structure involved in class-inclusion (Markman, 1978). Markman conducted a study with second through sixth graders and it was only by way of empirical evidence (direct observation of the materials) that the children were able to solve the class-inclusion problems. By the age of 12, children were still unaware that the hierarchical relation between a part and its whole was based on logical necessity rather than empirical fact.

Many of the studies have shown that emphasis on the whole is a key factor in conceptualizing the hierarchical relationship in the inclusion problem. Markman (1973, 1979) reasons that the typical class-inclusion problems tend to weaken the salience of the whole. Whether it is that the class term weakens the whole or simply that it does not have

the strength of cohesion to overpower the two concrete subordinate classes, the result is that the subclasses are more salient. This encourages a part-part comparison rather than a part-whole comparison.

In order to strengthen the salience of the whole in the part-whole comparison, Markman replaced class terms with collection terms. Unlike class terms, collection terms possess a psychological coherence and, therefore, promote the conceptualization of the whole. Terms like “forest”, “army” and “team” tend to facilitate the part-whole comparison. A tree is definitely a part of the forest, a soldier is clearly a part of the army, and a player is obviously a part of the team. In addition to adding to the cohesiveness of the whole, collection terms do not violate the mutual exclusivity assumption that many children make. An individual tree cannot be labeled a tree and a forest; it is only a tree. An object is not given two labels. When considering the class terms that parallel the previously mentioned collection terms, “trees”, “soldiers” and “players”, the part-whole relationship is not as clear. A particular tree being part of the trees violates the assumption of mutual exclusivity.

In placing more emphasis on the whole, collection terms promote the hierarchy or asymmetrical relationship that exists between a part and its whole. The hierarchical levels present in class terms are not as distinct which often leads children to a symmetrical analysis in which the inclusion relationship is lost.

Collection terms have been found to be useful in conceptualizing the whole with children as young as 4- and 5-years olds. Markman (1979) has found that collection terms assist in making the aggregate more salient during a conservation task. The collection term allows a child to interpret the original set in terms of a whole having a definite and constant number of parts. This helps her to overlook misleading transformations.

Markman (1979) presented 4- and 5-year-olds with a conservation task using four different sets of plastic toys: soldiers, football players, animals (bears, elephants, a hippopotamus and a giraffe), and pigs. The children were divided into two different conditions, the class condition and the collection condition. When referencing the plastic toys, those in the class conditions heard the following labels: soldiers, football players, animals and pigs. The children in the collection condition heard: army, football team, animal party, and family.

Each child received four conservation problems, one with each set of toys. For each problem, two rows of toys were set up in one-to-one correspondence, the experimenter's and the child's. In the case of the toy soldiers, a child in the class condition was instructed, "These are your soldiers and these are my soldiers," while the experimenter pointed to the appropriate sets. Then the child was asked, "What's more, my soldiers, your soldiers, or are they both the same?" After spreading out the toys in one of the sets, the experimenter asked the same question again.

The collection condition was very similar except while pointing to the two rows the experimenter stated, "This is my army and this is your army." The child was then asked, "What's more, my army, your army, or are they both the same?" As in the class condition, one row was then spread out and the experimenter asked the question once again. For both the class and collection conditions this procedure was carried out for all four sets of toys.

Children in the collection condition performed significantly better than those in the class condition. While those who heard collection terms solved an average of 3.18 problems correctly, those presented with class terms solved only 1.46 correctly. When asked to justify their answers, there were three general justifications given for correct responses. The

children either counted explicitly, stated that nothing had been added or taken away, or referred to the irrelevance of the transformation. While children in both conditions provided the first two justifications, reference to the irrelevant transformation was only made by those in the collection condition. This suggests that the collection terms were able to make the aggregate more salient to children than the class terms. The whole, the aggregate, was more clearly perceived and able to withstand an irrelevant transformation. This, however, was not the case in the class condition where the attention was placed on the parts rather than the whole. The only two justifications provided by children in this condition made reference to individual toys rather than a quality possessed by the whole set. Through the use of collection terms this study demonstrates that the ability to simultaneously comprehend parts (soldiers) as a whole (army) may be present in preschoolers.

Additional support for the role of collection terms in facilitating part-whole reasoning is found in Markman (1973). In this study Markman used collection terms in class-inclusion problems. She first presented 6-, 7- and 8-year-olds with two class-inclusion problems. The children were shown a picture of five white daisies and two yellow daisies. Then they were asked whether there were more white daisies or more daisies. The same procedure was followed with blue and green balls. Both Piagetian class-inclusion questions were answered incorrectly by every child.

Next, the experimental group was shown a picture containing six dogs that were identical except for their size. Two of the dogs were larger and it was explained that they were the mother and father dogs. The children were told that these six dogs were a family of

dogs and then asked, "Now who would have more pets, someone who owned the baby dogs or someone who owned the family?" The identical procedure was performed for rabbits, parrots and frogs.

The control group was shown the same pictures; however, the phrasing of the questions was different. Rather than using the term family, the class terms dogs, rabbits, parrots and frogs were used. For example, the question referring to the dogs was, "Who would have more pets, someone who owned the baby dogs or someone who owned the dogs?"

The results of the study indicated that the collection term "family" was of great assistance in these part-whole comparison problems. Fifty-five percent of the children in the experimental condition were able to solve all four problems while none of the children in the control group were able to solve even one problem. Therefore, it is apparent that 6-, 7- and 8-year-olds can reason about inclusion relations when the whole is made more salient by way of psychologically cohesive collection terms.

The effect collection terms have had on part-whole reasoning implies that other ways of emphasizing the whole may also facilitate part-whole reasoning. Perhaps a purely verbal presentation of the problem would be more beneficial than one involving pictures of the subclasses. Pictures may undermine the whole by diverting attention away from the more abstract whole and toward the now very concrete subclasses.

Wohlwill (1968) conducted a study in which 4- and 5-year-olds participated in both a pictorial and a verbal condition. Pictures of the following were presented: four owls and two pigeons, four butterflies and two eagles, six roses and two violets, five strawberries and two apples, and seven dogs and three horses. In the pictorial condition the picture was shown and

the class-inclusion question was asked. For the case of the owls and pigeons, the children were asked, "Are there more birds or more owls?" Questions of similar form were asked regarding the other four pictures.

In the verbal condition, no pictures were shown. Children were simply asked the class-inclusion question. In the case of the roses and violets, the children were asked, "If I had six roses and two violets, would I have more flowers or more roses?"

With a total of five problems, the mean number of correct responses given in the pictorial condition was .87. For the verbal condition, the mean was 2.27. While the score in the verbal condition is near chance, there is evidence to argue against children actually responding randomly. After each trial the children were asked to justify their answers. In the verbal condition, the majority of the correct responses were followed up by correct explanations.

Without pictures, subclasses and part-part comparisons were weakened. Consequently, more focus was directed on the part-whole comparison and performance on the class-inclusion tasks improved. Not only does this study demonstrate the power of presenting a class-inclusion problem only in verbal a format, but it also shows once again that 4- and 5-year-olds are able to reason about the part-whole relation.

Subsequent studies on the role of pictures in class inclusion tasks have not always found consistent results. Winer (1980) suggested that it may be the presence of the extra verbal and numerical cues instead of the absence of pictures that assisted children in Wohlwill's all verbal condition. Regardless, Wohlwill's results, along with Markman's studies, suggest that 4- and 5-year-olds may have some ability to reason about the part-whole relation.

Inverse Relation Between Number of Parts and their Size

Another element of fraction knowledge is the understanding that the size of the parts depends on the number into which the whole is divided. It is an inverse relationship that exists between the number of parts a material is divided into and the size of each individual part. This type of relation may seem confusing to young children. Their previous knowledge regarding whole numbers and counting informs them that as the number of things increase, the total amount also increases but the size of each thing remains constant. With fractions, as the number of parts increases the whole amount remains constant and the individual parts decrease in size. When first attempting to reason about fractions, children may overgeneralize their knowledge about whole numbers and be led astray.

Middle school children commonly err by judging a part's size by the size of the denominator (Gelman, 1991). Referencing the concept of sharing has been effective in combating this misconception. The effectiveness of a sharing scenario has been demonstrated with continuous types of materials such as pizza (Empson, 1995). When choosing between six and eight divisions of a pizza first graders were able to understand the inverse relationship that existed between the number of friends the pizza would be shared among and the size of each individual's slice. In an attempt to target the origins of this inverse reasoning, Sophian, Garyantes and Chang (1997) turned to a younger age group.

Children ages 5 and 7 were presented with a scenario in which they were asked to help the "Pizza Monster" choose between two ways of sharing his pizza. Quasicontinuous materials, orange lentils within a small clear box, were used to represent the Pizza Monster's small pizzas. The lentils functioned as neither a continuous nor discontinuous type of material since they were discrete but could not be counted when placed in front of the child.

Within the study, the children were questioned about three different comparisons. They were asked to compare one and two, two and three, and two and five recipients of the Pizza Monster's pizza. To help set up the representation of the two options being compared, two separate boxes filled with "pizza" were placed before the child. Then, in the case of the comparison between two and three recipients, two clear bowls were set in front of one of the boxes, while three bowls were placed in front of the other. The bowls were used to represent the recipients of the pizzas. It was explained that for each alternative, one of the bowls of pizza would be given to the Pizza Monster. The child was then asked, "Will the Pizza Monster get more to eat if we pour these pizzas (pointing to one box) into these bowls (pointing to the associated bowls) and give him one bowl or if we pour these pizzas (pointing to the other box) into these bowls (pointing to the associated plates) and give him one bowl?" After the child responded, the material was divided. However, it was only divided for the chosen alternative so that no comparisons could be made between the two alternatives. The identical procedure was carried out in the one to two and two to five comparisons.

Surprisingly, no effect of the different recipient ratios was found. However, age was a significant factor. Five-year-olds only chose the correct response one third of the time while 7-year-olds did so close to 80% of the time. It appears that 7-year-olds had a strong grasp of the inverse relation. Although the 5-year-olds did not score well, their pattern of results revealed that it was not due to strong misconceptions regarding the inverse relation. They were not necessarily being led astray by their knowledge of whole numbers. In other words, there was no bias towards selecting the larger number of recipients. Rather, it appeared that the children were simply unaware of the inverse relationship and uncertain about how to choose the correct answer. Even with the 1:2 ratio in which the children were

choosing between the monster retaining all of the pizzas for himself and having to share them, there was little indication that 5-year-olds could reason about the inverse relation.

There was some indication that 5-year-olds might have had a rudimentary notion of the inverse principle that could be elicited through training. Sophian et al. (1997) conducted a study with a "Pizza Monster" scenario very similar to that of the previous study, but in this study recipient comparisons were made using the following ratios: 2:3, 2:4, 2:5, 3:4 and 3:5. The study also differed in that it included pretest, training and posttest sessions.

During the training session, the Pizza Monster question was similar to that in the previous study. However, in this case, feedback was given and explanations were provided. After a child's first response, the "pizzas" were divided among the bowls in both of the hypothesized cases so that the child was able to compare the two alternatives. Then the child was asked the question again. If they answered incorrectly, the interviewer explained why they were incorrect.

No feedback was provided during the pretest or posttest sessions. Both involved novel comparison ratios so that the efficacy of the training and the generalizability of any learning could be assessed. The results of the pretests and posttests revealed that training did in fact help the 5-year-olds. They were able to perform better on the novel comparisons at the posttest than a control group of 5-year-olds that had received no training. Therefore 5-year-olds can become cognitively capable of solving this task.

Results from these studies (Sophian et al., 1997) indicate that both 5- and 7-year-olds can reason about the inverse relation. While 7-year-olds grasp the concept, 5-year-olds need training in order to demonstrate their emerging ability. Frydman and Bryant's (1988) "sweets" study suggests that counting may be of some assistance in fraction reasoning.

Therefore, while 5-year-olds need training in order to succeed with a quasicontinuous material like a bowl of lentils, a stronger ability to reason about the inverse relation may be unveiled through the use of discontinuous types of materials. Perhaps a change in material type could also reveal inverse reasoning in even younger children.

A limitation found in many of the studies discussed thus far is that either continuous or discontinuous types of materials were used, rarely both. In the next section, I review research showing that the type of material used may influence performance.

Materials

Whether preschoolers were asked to perform mental manipulations with quarter slices from a circular sponge or subdivide twelve “sweets” among three dolls (Mix et al., 1999; Frydman & Bryant, 1988), the studies discussed in the previous sections had a common goal. They all attempt to add to the existing knowledge regarding young children’s ability to reason about fractional quantities. Within this common goal, however, each individual task makes a unique contribution to the understanding of early fraction knowledge. Some tasks may lead experimenters to conclude that preschoolers understand the proportional aspect of fractions while others may shed light on when children first start to understand fractions’ exhaustive nature. Performance on the many different fraction tasks does not point to one specific age at which fraction reasoning is suddenly acquired. Rather, it is something that emerges gradually with separate components of fraction reasoning developing at different times. Even studies addressing specific aspects of fraction knowledge may find varying results due to differences in task and material type.

Just as success on different fraction tasks can reveal the acquisition of different aspects of fraction knowledge, the choice of materials used within these tasks may also have

implications regarding fraction reasoning. Specifically, continuous and discontinuous types of materials have been used throughout early fraction research and may in and of themselves have an effect on children's performance. Continuous types are those in which the whole being referenced is a single object. An example is the circular sponge used in Mix et al.'s (1999) calculation study. Discontinuous types, on the other hand, are those in which the whole refers to a discrete set of objects. Hunting and Sharpley (1988) used a set of 12 crackers to represent this type of material. The type of materials involved in a fraction task, their impact on performance and any reasons surrounding such effects on performance, are topics in need of more attention.

In research conducted with older children, the simple manipulation of material type (continuous or discontinuous) has been shown to have a significant effect. It is not only in the quest for fraction knowledge that researchers have looked at these material types and their effects. Studies addressing middle-schoolers' reasoning about percentages have also attended to this issue.

Gay and Aichele (1997) presented seventh and eighth graders with two kinds of questions. The first kind referenced a single rectangle within which a portion was shaded dark. The children were asked to choose which of the following statements described the shaded portion of the rectangle: it is greater than (given percentage), it is less than (given percentage), it is equal to (given percentage), can't tell or I don't know. While this first kind of question involved a continuous object, the second dealt with discontinuous objects, or discrete sets. A set of circles were shown in which a certain number of the circles were shaded dark. The children were then asked the same type of question that was asked in regard to the continuous rectangle.

Both seventh and eighth graders scored higher on the first kind of question. This result may suggest that children find it less difficult to conceptualize a continuous object, rather than a discontinuous set, as a whole or 100%.

A study by Behr, Wachsmuth and Post (1988) will help to redirect this issue toward fraction reasoning. In their study, Behr et al. divided fourth and fifth graders into an experimental and control group. During the instruction periods, the experimental group worked with manipulatives while the control group focused on algorithmic manipulations presented in their textbooks. The study included two instruction sessions. During the first session, those children in the experimental group only worked with fraction manipulatives of a continuous nature. It was not until the second session that discontinuous manipulatives were introduced. Throughout both sessions the control group continued to do algorithmic work from their textbooks.

After each session, both the experimental and the control group were presented with continuous and discontinuous fraction problems. While both types of problems used an egg carton, the task attempted to shift the children's focus to very different aspects of the carton in each case. In the case of the continuous fraction problems, the design of the problem was aimed at directing the children's focus onto the carton as a whole. During the presentation of the problem, a segment from another egg carton was placed over the main carton covering a portion of it. The child was then asked, "What fraction of the egg carton is covered?" The children were to respond verbally. In the discontinuous problems, the aim was to draw the child's attention to the individual segments in the carton. The children were instructed, "Put eggs into the carton so that $(\frac{3}{4}, \frac{2}{6}$ or $\frac{1}{3})$ of the holes are filled." Unlike the continuous case, this was a production task.

The results indicated that while the experimental group was capable of solving the continuous problems after the first session, so were those in the control group. Even with no exposure to continuous manipulatives, only algorithmic procedures, the control group children were able to solve the continuous fraction problems. This was not the case, however, with the discontinuous problems where the control group was unable to solve a single problem. These results suggest that reasoning about fractions in reference to a continuous representation may be more intuitive than that with discontinuous representations.

Looking at the infant research, findings support the intuitiveness of continuous objects being perceived as coherent wholes. Infants perceive objects as wholes due to their basic physical properties. In the instance of discontinuous sets, which have to be constructed or assembled to be thought of as a whole, the conceptualization of the whole is much less stable (Huttenlocher & Gao, 1999).

Another interesting finding after the first session was the performance of the experimental group on the discontinuous problems. While none of the children in the control group were able to solve the problems, over half of the experimental children were successful. This was only after the first session and, therefore, before the experimental group had ever been exposed to discontinuous manipulatives. It appears that the exposure to continuous manipulatives somehow facilitated working with discontinuous types of materials. The inverse situation in which exposure to discontinuous manipulatives would precede the testing of continuous types of materials was never tested. Therefore, whether or not working with a discontinuous type of material somehow facilitates working with continuous objects remains unknown.

The item used in portraying both material types may have affected the results of this experiment. While the use of the same material substance in both continuous and discontinuous problems is an asset to this type of study, an egg carton may not be the best example. Rather than being viewed as either a continuous or discontinuous material in the respective problems, it may have been viewed as a combination, or blended form, of the two. While the two different tasks attempt to emphasize either the carton as a whole or the individual holes within the carton, at all times both aspects of the carton were still present and possibly influential. Surely an example like the egg carton is not useless in the analysis of continuous and discontinuous representations. The possible representations that young children can map their early fraction concepts onto do not have to exist in the strict continuous/discontinuous dichotomy. Perhaps there is a way of representing a fraction that falls somewhere between a continuous and a discontinuous type of representation; maybe a combination or blending of the two.

While these studies with older children tend to support the idea that continuous representations of fractions are more intuitive, research with younger children is less clear. McDermit (1983) conducted a study with first graders in which they were to subdivide both continuous and discontinuous types of materials. The tasks were presented in a story context as well as a direct form. Regardless of presentation form, performance was superior when the children worked with discontinuous objects. This research suggests that the ability to subdivide discontinuous materials precedes that of continuous.

Hunting and Sharpley (1988) incorporated both continuous and discontinuous material types into their study as well. However, by using skipping rope and crackers as the continuous and discontinuous materials, the two distribution tasks differed in several ways in

addition to material type. In the case of the rope, scissors were brought into the task, and may have affected performance, as many preschoolers are not yet proficient with their use. Another problem was the lack of consistency between the items in the two material types. Conclusions could not be drawn solely on the basis of the continuous or discontinuous nature of the material. Children could have distributed the skipping rope and crackers differently simply because one was food and one was a toy. Perhaps preschoolers perceive the sharing of food differently than they do the sharing of a toy. The substance of the material should be identical if the goal is to isolate the effect of altering the material type, and only the type. In this study, no clear conclusion could be drawn regarding whether continuous or discontinuous representations were more intuitive.

Although the overarching question seems to be whether it is continuous or discontinuous representations that are more intuitive, perhaps it depends upon the type of question being asked. During a production task, discontinuous material types may be easier to distribute because from an early age, children have a procedure for this (Hunting & Sharpley, 1988; Frydman & Bryant, 1988). However, with continuous objects no procedure is available. The children must actually divide the material and in doing so must always keep the whole in mind. In this case keeping the whole and parts in mind simultaneously could be cause for difficulty. When dealing discontinuous objects, the whole never needs to be referenced.

When the task involved is a comprehension task rather than a production task, the benefits of procedural dealing are lost. Now continuous representations may be more intuitive than discontinuous. As previously mentioned, there is research demonstrating infants' inclination to perceive objects as cohesive (Huttenlocher et al., 1999). In their

knowledge about whole objects, infants may already have some intuitions about the division of a whole. With a discontinuous set, the whole is not evident. The individual objects must be assembled in order to perceive them as a cohesive whole. This seems like a difficult task for young children. Therefore, continuous objects may be easier to reason about during a comprehension task when children are only able to think about the division.

Due to contradictions found in the few studies that compare continuous and discontinuous materials, more research is needed to explore this issue. Additionally, research needs to be conducted in a more systematic manner such that differences between the continuous and discontinuous material substances as well as their respective tasks are at a minimum. If a study could be conducted in which the materials in the continuous and the discontinuous tasks only differed in numerosity and the tasks involved were identical, a clearer picture of the early fraction reasoning may emerge.

Current Study

The present study concerns a single component of fraction knowledge, the inverse relation. Even into middle school, children have difficulty understanding the inverse relation that exists between the number of parts in a whole and their sizes (Gelman & Gallistel, 1978). It is common for children to judge a fraction like $1/6$ to be greater than $1/3$. Eventually, children do relate the number of segments a whole is divided into to the resulting sizes of those pieces and this study is aimed at exploring the very rudimentary aspects of this reasoning.

Another rationale behind choosing this particular aspect of fraction knowledge was that the inverse relation embodies several of the other components involved in fraction reasoning. For example, in order to understand the inverse relation children must already be

able to reason about equal divisions and the exhaustive nature of fractions. Without these concepts already in place, reasoning about the different numbers and sizes of parts would be impossible.

The final incentive for addressing the inverse relation in particular was that Sophian et al. (1997) provided an excellent starting point for this research. Her study's design in which children were asked to choose between two possible ways of sharing pizza, directly targets the relationship between the number of divisions and their corresponding sizes. While this study was modeled after Sophian et al.'s Pizza Monster task, new goals required several modifications.

Sophian et al. (1997) tested 5- and 7-year-olds and found the 7-year-olds to be far more capable of reasoning about the inverse relation. Five-year-olds were only successful in the pizza sharing task after participating in a training period. Since 6-year-olds were not looked at in that study, the current study included this age group and attempted to determine their abilities with regard to the inverse relation. Five-year-olds were included once again so their abilities could be tested more thoroughly. In Sophian et al.'s study, 5-year-olds were tested with and without training and it was only through training that they were able to succeed. This study attempted to elicit correct responses without training.

Rather than using a quasicontinuous type of material like lentils, three different material types were employed. In addition to the typical continuous and discontinuous types, a third material type referred to as blended was included. The third material type contained attributes from both the continuous and discontinuous types. Somewhat reminiscent of Markman's collection terms, the blended materials facilitated a conceptualization of the whole while the individual parts were still very concrete. To facilitate a direct comparison

among the types of materials, it was essential that all three types, continuous, blended and discontinuous, were represented with the identical material substance. Therefore, a quasicontinuous type was not included since any material expressed in the quasicontinuous format rarely lent itself to representation in the three other formats (continuous, discontinuous and blended). By using continuous, blended and discontinuous material types and performing the identical task with each one, performance shed light on the most intuitive forms of representation.

Another modification that was made to the original study was that the children were allowed to place the doll in the sharing situation they believe to be correct. Rather than simply telling the experimenter which situation was ideal, the children personally put the doll in the situation that they thought was most desired. This assisted in maintaining the children's attention throughout the entire study.

Finally, unlike Sophian et al.'s study in which the goal was always to select the situation in which the largest portions were attained, in this study sometimes large portions were sought after while other times it was small portions. If children were able to succeed on both kinds of trials, it demonstrated a thorough understanding of the inverse relation.

Children with a thorough understanding of the inverse relation must be able to represent and reason about the two possible divisions suggested. Not only must they represent the whole, but they must represent two different division possibilities for the whole. These divisions must then be translated into piece sizes. This translation from the number of pieces to the actual size of the pieces is the very essence of the inverse relation. It was at this stage that the type of material (continuous, blended or discontinuous) could have significantly affected performance. In a production task, dealing would be of great assistance

with the discontinuous material type. After dealing, the only comparison would be that between two and three items. However, in a comprehension task like this, the benefits of dealing were absent. It may be continuous material types that produce better performance since in assessing piece size the whole must be referenced. For continuous material types the whole is concrete while a discontinuous material's whole must be constructed and, therefore, is quite abstract.

Once the piece sizes are represented, a comparison is the necessary final step to solving the problem. The comparison is where the question type plays a role. Some questions required children to choose the alternative that provided them with the largest pieces (most amount questions), while others asked for the smallest (least amount questions). Therefore, the children's understanding of the question guided them to the basis for their final decision. Operating with their understanding of the question the children chose what they believed to be the optimal piece size and, consequently, the correct number of divisions.

Predictions and Analyses

Each child participated in a total of 24 trials. For each trial, children received a score of one when they were correct and a score of zero when they were incorrect. Table 1 depicts the number of trials that fell within each material and question type.

Table 1. Number of trials presented with each material and question type

Continuous Materials		Blended Materials		Discontinuous Materials	
Most	Least	Most	Least	Most	Least
4	4	4	4	4	4

An effect of age was expected since the reasoning abilities of children are continually developing. Sophian's success with training 5-year-olds suggests that kindergartners may understand the relation between the number of parts and their sizes. However, this understanding may be a bit shaky and the kindergartners may not yet be capable of consistently coordinating the number of parts with the size of those parts. Due to this lack of coordination, their responses to this task could mimic a random pattern.

An effect of material type was also expected for two reasons. Not only is it that children's early understanding of the physical world assists them in conceptualizing the breaking apart of a continuous whole (Huttenlocher et al., 1999), but Gelman has also suggested that being able to count the individual objects within a discontinuous set may actually interfere with fraction reasoning. Therefore, with the extra intuitive nature of the continuous materials and the possible distractions involved in the discontinuous sets, we anticipated that children would be most successful on the continuous trials. Since the blended type of material contains attributes of both the continuous and discontinuous forms, any interference from the individual objects could be offset by the cohesive nature of the whole. Therefore, children should have been more successful on these trials than on the discontinuous, but less so than on the continuous.

Finally, an effect of question type was anticipated. Since children have generally been presented with the sharing scenario in instances where the goal is to retain the most amount possible, the least amount trials in which they attempted to retain the smallest pieces should have been relatively difficult.

In addition to the three main effects, three interactions were expected from the 2(grade level: kindergartner, first grade) x 3(material type: continuous, blended,

discontinuous) x 2(question type: most, least) x 2(desirability: desirable, undesirable) mixed design ANOVA as well. In particular, we predicted a Grade Level x Material Type interaction. It was expected that as children get older they should be more capable of conceptualizing the discontinuous sets as wholes, therefore, performing better on the discontinuous trials than the younger children. Another thought was that kindergartners would consistently do poorly with the discontinuous material type because they would resort to counting the number of recipients and let that take precedence over the size of the parts.

Older children's more extensive experience with sharing led to another predicted interaction. The first graders should be more capable of understanding an instance in which one would want to share with more recipients in an attempt to decrease the shares.

It was reasoned that young children may best understand the goals of attaining the most and least amounts when the materials involved are desirable and undesirable, respectively. Therefore, an interaction between question type and desirability was expected.

Children's performance was assessed to determine whether or not it differed significantly from chance. Due to the eventual success of the 5-year-olds in Sophian et al.'s study, it was predicted that kindergartners would score above chance on continuous and possibly even the blended materials. Six year-olds were not addressed in Sophian et al.'s experiment. However, since 7-year-olds performed very well and 5-year-olds were able to succeed after some training, these first graders were expected to perform above chance with the continuous, blended and possibly the discontinuous materials.

Preliminary ANOVA's were performed to investigate the effects of group (item-question combination sets to be outlined in the next section), gender, order, and classroom. No effects of MQP set, gender or trial order were expected. An ANOVA on the 2(grade

level: kindergarten, first grade) x 2(MQP set: 1, 2) x 2(gender: male, female) x 2(trial order: 1, 2) mixed design should demonstrate this. Additionally, no effect of classroom was expected. An ANOVA was conducted in order to determine whether or not different classrooms performed significantly different from one another.

CHAPTER 2

EXPERIMENT 1

Method

Participants

A total of 77 children participated in this task. Five were eliminated, 3 because English was their second language and 2 due to missing second session. Of the remaining 72 children, 33 were kindergartners (16 girls and 17 boys; ages 5 years 3 months to 6 years 5 months, \underline{M} = 5 years 7 months) and 39 were first graders (21 girls and 18 boys; ages 5 years 11 months to 7 years 2 months, \underline{M} = 6 years 8 months). When the two grade levels were combined, there were fourteen 5-year-olds (7 girls and 7 boys; ages 5 years 3 months to 5 years 5 months, \underline{M} = 5 years 4 months), fifteen 5½-year-olds (7 girls and 8 boys; ages 5 years 6 months to 5 years 11 months, \underline{M} = 5 years 8 months), sixteen 6-year-olds (6 girls and 10 boys; ages 6 years to 6 years 5 months, \underline{M} = 6 years 3 months), sixteen 6½-year-olds (13 girls and 3 boys; ages 6 years 6 months to 6 years 11 months, \underline{M} = 6 years 9 months) and eleven 7-year-olds (4 girls and 7 boys; ages 7 years to 7 years 2 months, \underline{M} = 7 years 1 month).

Children were drawn from three kindergarten and three first grade classrooms at Deerfield Elementary in South Deerfield, Massachusetts. Written parental consent was obtained for each child. Children participated in two 12-trial sessions; each session consisting of a one-on-one interview situation between a single child and the experimenter. Upon completion of a session, children received a sheet of Winnie the Pooh stickers.

Design

The task used photographs involving eight different kinds of materials. Half of the pictured materials were desirable and half were undesirable. The four desirable materials

were clay, finger-paint, donuts and cupcakes. The undesirable ones were brown play-doh, worms, peas and pickles. Notice for both the desirable and the undesirable materials, there were two food and two nonfood items. This ensured that the results would be generalizable and due to nothing uniquely true of foods or nonfoods. Finally, each of the eight materials was presented as three different material types: continuous, blended (a combination of continuous and discontinuous) and discontinuous. Children viewed 24 photographs in all. See APPENDIX A for an example of the different material types.

Two kinds of questions were asked during the experiment. At the start of every trial, children were told that the material shown in the photograph could either be divided between two or among three dolls. Then, on half the trials, children were asked to select the situation in which a single doll received the most amount of the material. These trials were labeled most amount trials. On the other 12 trials, children were asked to select the situation in which a single doll received the least amount of material. These were called the least amount trials. All three types (continuous, blended and discontinuous) of a given material were paired with either most amount or least amount questions.

In order to avoid confounding the materials with the two question types, two unique sets of material-question type pairings (MQPs) were composed. The pairings for MQP Set 1 and MQP Set 2 are shown in Figure 1. While every child viewed all eight materials, each in its three forms (continuous, blended and discontinuous), the type of question paired with a particular material varied. As seen in Figure 1, those children who received MQP Set 1 were asked questions of the most amount type regarding finger-paint, cupcakes, worms and pickles. The same children were asked least amount questions about clay, donuts, brown play-doh and peas. Those children who received MQP Set 2 experienced the converse. They

were asked most amount questions about clay, donuts, brown play-doh and peas. Least amount questions were asked with regard to finger-paint, cupcakes, worms and pickles.

	<u>MQP Set 1</u>	<u>MQP Set 2</u>
Most Amount	finger-paint cupcakes worms pickles	clay donuts brown play-doh peas
Least Amount	clay donuts brown play-doh peas	finger-paint cupcakes worms pickles

Figure 1: Material-question type pairings for the two MQP sets.

Two random trial orders were generated for the presentation of the material-question type pairs. The orders were constructed in such a way that even though the material-questions type pairs were not the same in MQP Sets 1 and 2, both sets were able to employ the same trial orderings. A given trial was described with respect to the following attributes: desirability, material type, food or nonfood status and question type. No reference was made to a specific material's substance (clay, donut, etc.). Rather, it was descriptions including everything but the actual materials that were randomized. The only stipulation in composing the trial orders was that consecutive trials differed by more than just material type. This ensured adjacent trials dealt with unique materials. See Figure 2 for how Trial Order 1 began.

<u>Trial Order 1</u>	<u>MQP Set 1</u>	<u>MQP Set 2</u>
most amount – desirable – continuous – food	cupcake	donut
most amount – undesirable – discontinuous – nonfood	worms	brown play-doh
most amount – desirable – blended – nonfood	finger-paint	clay
least amount – desirable – discontinuous – nonfood	clay	finger-paint

Figure 2: First four descriptions in Trial Order 1 and the materials they described from MQP Sets 1 and 2.

To avoid confounding trial order and MQP set, half of the children who received MQP Set 1 and half of the children who received MQP Set 2 were given Trial Order 1. The remaining children received Trial Order 2.

Finally, the 24 trials were divided equally over two sessions. Children simply received the first 12 trials in their order during the first session and the second 12 during the second session.

Materials

Photographs

An Olympus C-2000 digital camera was used to photograph all three different forms (continuous, blended and discontinuous) for each of the eight materials. The photos were taken from overhead and at a slight angle. All photos were enlarged to 8.5x11 inches and in order to ensure clarity of all objects, some images were altered using the computer program Adobe PhotoDeluxe. Finally, each photo was laminated to increase its durability.

Desirable Materials

Clay. The continuous clay photograph was of a single block of blue modeling clay. The blended photo showed a yellow cardboard box containing six blocks of the same blue clay. The discontinuous picture showed the six blocks of clay in a random arrangement. The

arrangement was such that all six blocks of clay were still easily visible. The size of the clay blocks remained constant across all three photographs.

Finger-Paint. The photograph of finger-paint in its continuous type showed a single cup of red paint. The blended type showed six cups contained within a cardboard carton. The carton was yellow in color and had separate circular cutouts for each finger-paint cup. The discontinuous picture showed the six tubs of red finger-paint in a random arrangement. The sizes of the finger-paint cups remained constant across all three photographs.

Donuts. The photo presenting a donut as continuous contained a single donut with white frosting and colored sprinkles. The blended type photo showed six identical sprinkled donuts in a purple cardboard box. The discontinuous picture showed six sprinkled donuts in a random arrangement. The arrangement was such that all six donuts were still easily visible. The size of the donuts was kept constant across all three pictures.

Cupcakes. The photograph displaying the continuous type of material showed a single cupcake with chocolate frosting. The photo of the blended type showed six cupcakes with chocolate frosting in a red metal baking tin. Finally, the photograph presenting cupcakes as discontinuous showed six cupcakes in a random arrangement. Visibility of all six cupcakes was ensured in the arrangement. The size of the cupcakes and their appearance was identical across all three photos.

Undesirable Materials

Brown Play-Doh. The picture demonstrating the continuous material type showed a single ball of brown play-doh. The photo depicting the blended type pictured six balls of brown play-doh inside a red container similar to an egg carton. A discontinuous set was presented by showing the six balls in a random arrangement. The size of the play-doh balls were kept constant for all three pictures.

Worms. The continuous version showed a single toy worm. The blended type pictured six toy worms in a green tackle box. The tackle box was divided into six compartments of equal size. A discontinuous set was shown by way of six worms in a random arrangement. The arrangement was such that all six were easily visible. The size of the worms was kept constant across all three photographs.

Peas. The continuous material type was shown as a picture of a single pea. The blended type was shown as six peas within their pod. Finally, the discontinuous material type was represented by six peas in a random arrangement. The size of the peas was kept constant across all three photos.

Pickles. The continuous form pictured one pickle slice. The blended form was presented as six pickle slices inside a red jar. The discontinuous material type showed six pickle slices in a random arrangement. The arrangement was such that all six pickle slices were still easily visible. The size of the pickle slices was kept constant for all three photographs.

Props

Four stuffed characters from the children's Winnie the Pooh stories were used to engage children in the task. The sharing scenario was acted out with Rabbit, Tigger, Piglet and Pooh dolls that were approximately 12 inches tall. Two cardboard shoeboxes covered in gray contact paper were used as placeholders for the dolls. In one shoebox, two cardboard dividers were inserted to create three compartments of equal size. In a second shoebox, two-thirds the size of the first, another divider was used to evenly partition the box into two compartments. As a result, the second shoebox contained two compartments that were not only equal in size to one another, but also equal in size to those of the first shoebox. Finally, a small wooden easel was used to display and suspend the photos approximately 12 inches from the ground. See APPENDIX B for an exact set-up of the props.

Recording Equipment

A portable audiocassette recorder (General Electric model number 3-5385A) was used to record both the child and the experimenter during the two testing sessions.

Procedure

Children were tested in a quiet space just outside their classroom. Each child participated in a total of 24 trials. The trials were split between two sessions, with each session containing 12 trials. Before the first session began, Pooh's desire to share things equally among his friends was demonstrated with a collection of pennies. A pile of pennies was placed in front of Pooh and the experimenter stated Pooh could either share his pennies with one or two friends. Then the experimenter demonstrated what would happen if Pooh decided to share his pennies with one friend. The pennies were evenly divided between Pooh and the one friend (Piglet, Rabbit or Tigger). The exhaustive distribution and equity of the

piles were stressed. Next, the experimenter demonstrated how Pooh would share with two friends and the pennies were equally divided among three dolls. Again, the exhaustive and equal distribution was emphasized. This was purely a demonstration and the child was not required to participate in any way. After the demonstration, the first trial of the experiment began. The penny demonstration was only given before the first session. At the time of the second session, the experimenter went immediately into the sharing trials.

On a given trial, one photograph was placed on the wooden easel in view of the child. The Pooh doll was held above the easel, centered over the photo. Placed on one side of the easel was the shoebox with two subdivisions. On the other side was the shoebox containing three subdivisions. One of Pooh's friends (Rabbit, Tigger or Piglet) was placed in the shoebox with two subdivisions, while the other two were put into the other shoebox (the characters rotated positions from one trial to the next). At this point there was a single vacancy in each of the two shoeboxes.

For the most amount trials, after a photograph was set on the easel the child was told Pooh loved the pictured material. Then it was explained that Pooh could either share this material with one friend or two. Next, the experimenter asked the child to put Pooh where he would receive the largest amount of the material. The child then placed Pooh either in the opening next to one or two friends. At this point, the experimenter verbalized the action taken by the child ("Okay, Pooh will share with Piglet." or "Okay, Pooh will share with Tigger and Rabbit.") and an audiocassette recorder recorded the information so that it could be coded later. (Halfway through testing the recorder malfunctioned so some responses were logged with paper and pencil.) No feedback was given regarding the child's correctness.

The scenario was very similar for the least amount trials. When a photo was on the easel, the child was told that Pooh did *not* like the pictured material. Then the child was instructed to help Pooh make a choice. As in the most amount trials, the choice was between sharing with one or two friends. However, in these trials the child was asked to place Pooh where he would receive the least amount of the material. The child responded by placing Pooh either in the opening next to one friend or two. Then the child's action was verbalized by the experimenter in order to be recorded by the audiocassette recorder. Once again, no feedback was provided as far as the correctness of the child's action.

For each session, 12 trials (a combination of the most and least amount type) were carried out in this fashion. After placing Pooh with either one or two friends on the last trial of a session, the experimenter asked the child why s/he decided to put Pooh there as opposed to with the other friend(s). The child's response was recorded (either by the audiocassette recorder or with paper and pencil). At the conclusion of each session, children were given a sheet of Winnie the Pooh stickers and thanked for their help.

Results

Scoring

On each trial, a child's response was recorded as either correct or incorrect. In the few instances in which a child changed his/her answer, the first answer was always the one recorded. Children who failed to place Pooh in one of the two openings after the original prompting were once again encouraged to put Pooh where they thought he should go.

Each child received a total score ranging from 0 to 24 that corresponded to the number of questions answered correctly. These scores were subdivided for specific analyses. In the instance of materials types, the 24 questions were divided into the three different kinds

of materials they addressed. Children received scores ranging from zero to eight for each of the three materials: continuous, blended and discontinuous. Scores for the two question types and two levels of material desirability were handled in the same manner. For question type scores, the 24 trials were separated into two sets of 12, one containing questions aimed at attaining the most amount for Pooh and one containing questions aimed at getting the least amount. Here, scores ranged from 0 to 12 for each level of question type. Finally, the trials were separated into those involving desirable and undesirable materials. Scores ranged from 0 to 12 for each desirability level.

Preliminary Analyses

Preliminary analyses were conducted to assess effects of gender, session, MQP set, trial order and classroom. The ANOVA performed on the 2(gender: male, female) x 2(session: 1, 2) x 2(MQP set: 1, 2) x 2(trial order: 1, 2) mixed design revealed no effects of gender, session, MQP set or trial order. In addition, one-way ANOVA's performed at both grade levels revealed no effect of classroom for either the kindergartners or first graders. Due to the lack of significant findings, data were collapsed across these variables in subsequent analyses.

Main Analyses

An ANOVA was carried out on the 2(grade: kindergarten, first grade) x 3(material type: continuous, blended, discontinuous) x 2(question type: most amount, least amount) x 2(desirability: desirable, undesirable) mixed design, with number of correct responses as the dependent variable. Responses from all 24 trials were used since, as stated in the preliminary analyses, session had no effect. The data are presented in Table 2.

Table 2: Mean number of correct responses to most amount and least amount questions in the presence of desirable and undesirable continuous, blended and discontinuous materials.

Material Type	Kindergartners	First Graders
<u>Continuous</u>		
Most amount questions		
Desirable materials	.97 (.88)	1.59 (.75)
Undesirable materials	1.00 (.90)	1.56 (.82)
Least amount questions		
Desirable materials	.88 (.96)	1.39 (.82)
Undesirable materials	.73 (.84)	1.23 (.87)
<u>Blended</u>		
Most amount questions		
Desirable materials	.70 (.85)	1.46 (.82)
Undesirable materials	.76 (.90)	1.62 (.78)
Least amount questions		
Desirable materials	.88 (.93)	1.49 (.82)
Undesirable materials	.91 (.91)	1.59 (.75)
<u>Discontinuous</u>		
Most amount questions		
Desirable materials	.97 (.92)	1.56 (.79)
Undesirable materials	.94 (.83)	1.49 (.76)
Least amount questions		
Desirable materials	.73 (.88)	1.49 (.79)
Undesirable materials	.85 (.80)	1.46 (.76)

Note. Scores ranged from zero to two. Standard deviations in parentheses.

The analysis revealed first graders significantly outperformed kindergartners, $F(1,70)=13.55$, $p<.05$. This difference is depicted in Figure 3. First graders gave correct responses on an average of 17.92 questions ($SD=8.29$), while kindergartners did so on an average of only 10.30 questions ($SD=9.28$). Additional analyses using t-tests indicated only first graders exceeded chance performance, $t(38)=4.46$, $p<.05$.

Returning to the results of the main four-way ANOVA, material type failed to produce an overall effect on performance. However, an interaction between material type and grade was found, $F(2,140)=4.59$, $p<.05$. Table 3 shows the mean number of correct responses for each of the three material types separated by grade level. To investigate the material type by grade interaction and detect possible simple effects of material type, one-

way ANOVA's were performed at both levels of grade. The analyses revealed changing material type had no significant effect on kindergartners' performance, but had a marginal effect on the performance of first graders, $F(2,76)=3.05$, $p<.10$. Looking closer at the first graders, t-tests revealed it was only their performance on blended and continuous materials that differed significantly from one another, $t(38)=3.21$, $p<.05$ (with Bonferroni adjustment). First graders were more successful on the sharing task when they worked with blended materials as opposed to continuous. Additional t-tests revealed that, while performance with one material type may have been better than another, first graders scored above chance with all three types; continuous, $t(38)=3.94$, $p<.05$; blended, $t(38)=4.72$, $p<.05$; discontinuous, $t(38)=4.46$, $p<.05$.

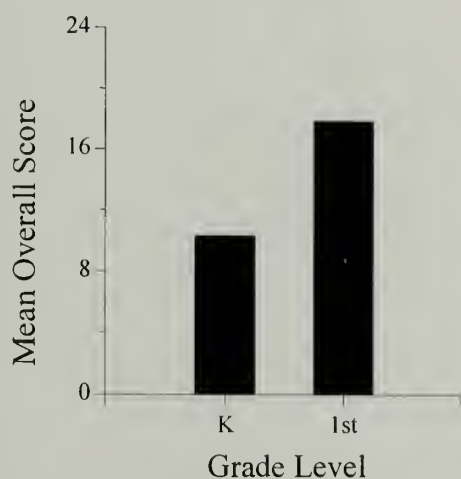


Figure 3: Mean number of correct responses for kindergartners and first graders. K = kindergartners; 1st = first graders.

Table 3: Mean number of correct responses for each material type separated by grade.

Material Type	Kindergartners	First Graders
Continuous	3.58 (3.25)	5.77 (2.81)
Blended	3.24 (3.19)	6.15 (2.85)
Discontinuous	3.49 (3.00)	6.00 (2.80)

Note. Scores ranged from zero to eight. Standard deviations in parentheses.

Question type, the next variable of interest from the initial ANOVA, significantly affected children's performance, $F(1,70)=13.35, p<.05$. Children were more successful on most amount questions ($M=7.47, SD=4.81$) than least amount ($M=6.96, SD=4.76$). In addition to this main effect, analyses also revealed an interaction between question and material type, $F(2,140)=5.66, p<.05$. Table 4 shows the mean number of correct responses with each material type separated by question type.

Table 4: Mean number of correct responses with each material type separated by question type.

Question Type	Material Type		
	Continuous	Blended	Discontinuous
Most Amount	2.61 (1.70)	2.33 (1.72)	2.53 (1.61)
Least Amount	2.15 (1.69)	2.49 (1.74)	2.32 (1.64)

Note. Scores ranged from zero to four. Standard deviations in parentheses.

In order to explore possible simple effects of material type, one-way ANOVA's were performed at both levels of question type. Results indicated that material type significantly affected both the most amount, $F(2,142)=3.96, p<.05$, and least amount questions,

$F(2,142)=3.91, p<.05$. These simple effects were investigated further using t-tests to compare performance with the different material types. Analyses revealed it was only children's performance with the continuous and blended materials that differed significantly from one another for both the most amount, $t(71)=2.47, p<.05$ (with Bonferroni adjustment), and least amount questions, $t(71)=-2.81, p<.05$ (with Bonferroni adjustment). As the means presented in Table 4 indicate, the direction of the difference, however, was different for the two question types. On the most amount questions, performance with continuous materials was better than with blended, but the least amount question, performance with blended materials was better than with continuous.

Finally, the original four-way ANOVA revealed no effects of a material's desirability. Children showed no difference on desirable versus undesirable materials.

Age Analyses

The original four-way ANOVA was rerun with age replacing grade. The new design was a 5(age: 5, 5½, 6, 6½, 7) x 3(material type: continuous, blended, discontinuous) x 2(question type: most amount, least amount) x 2(desirability: desirable, undesirable) mixed design, with number of correct responses as the dependent variable. Once again, responses from all 24 trials were used.

A main effect of age was revealed, $F(4,67)=5.15, p<.05$. Figure 4 shows the mean number of correct responses at each age level. A trend analysis indicated that the increase in average scores was linearly related to age, $F(1,67)=14.33, p<.05$. Additionally, t-tests were used to compare average scores of the different age groups to chance performance. Five-

year-olds' performance was shown to be significantly below chance, $t(13)=-2.47, p<.05$, while that of 6½- and 7-year-olds was above, $t(15)=5.21, p<.05$ and $t(10)=2.17, p<.10$, respectively. The means of both 5½- and 6-year-olds were at chance.

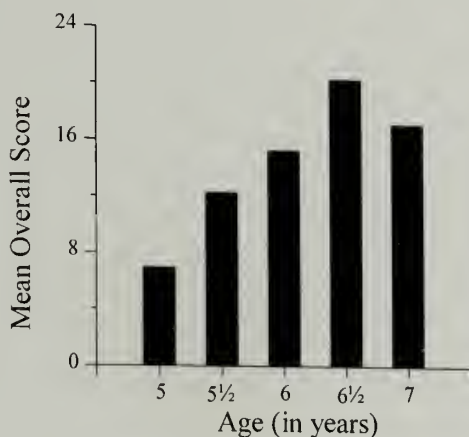


Figure 4: Mean number of correct responses at each age level.

Referring back to the age ANOVA, the four-way analysis replicated numerous findings from the grade level ANOVA. Once again, the main effect of question type was detected, $F(1,67)=15.32, p<.05$, indicating children's greater success with most amount questions ($M=7.47, SD=4.81$) compared to least amount questions ($M=6.96, SD=4.76$). Additionally, the age analysis showed the question type by material type interaction again, $F(2,134)=5.25, p<.05$.

The age ANOVA also resulted in some findings that differed from those in the grade level ANOVA. When grade was used as a factor in the main ANOVA, an interaction was found between material type and grade level. However, in this analysis where age was used in place of grade, no material type by age interaction was found. On the other hand, there

was a new result found: an interaction between material type and desirability, $F(2,134)=3.08$, $p<.05$. Two one-way ANOVA's, one at each level of desirability, were conducted to further investigate this new interaction. The analyses revealed no simple effects of material type at either level of desirability.

Individuals' Patterns of Performance

So far all analyses have looked at group patterns rather than response patterns of individuals. The following results address the performance of individuals. Each individual was labeled as having demonstrated one of three response patterns: correct, chance or incorrect. The binomial table of probabilities indicated that in order to be classified as having a correct response pattern a child must have correctly answered at least 17 of the 24 trials. Similarly, the binomial table required a score of seven or below in order for a child to be classified as having an incorrect response pattern. Finally, those identified as demonstrating a pattern of chance performance must have scored in the 8 to 16 range (binomial $p=.04$, two-tailed).

Table 5 shows the number and percent of children from both grade levels that demonstrated each of the three response pattern categories.

Table 5: Number and percent of children classified as demonstrating correct, chance and incorrect patterns of performance at each grade level.

Grade Level	n	Response Pattern		
		Correct	Chance	Incorrect
Kindergarten	33	10 (30%)	6 (18%)	17 (52%)
First Grade	39	28 (72%)	4 (10%)	7 (18%)

Note. Percentage for the grade level in parentheses.

A chi-square test was conducted to determine whether there were more first graders than kindergartners displaying a correct pattern. After condensing Table 5 into a 2x2 table containing the categories “correct” and “not correct” (combined chance and incorrect patterns), chi-square analyses revealed that indeed a significantly greater number of first graders demonstrated a pattern of correct performance, $\chi^2(1, N=72)=12.29, p<.05$.

Table 6 shows the number of children at the different ages that displayed each one of the three patterns. No analyses were performed on these frequencies.

Table 6: Number and percent of children classified as demonstrating correct, chance and incorrect patterns of performance at each age.

Age	n	Response Pattern		
		Correct	Chance	Incorrect
5 years	14	2 (14%)	3 (22%)	9 (64%)
5½ years	15	6 (40%)	2 (13%)	7 (47%)
6 years	16	9 (56%)	2 (13%)	5 (31%)
6½ years	16	14 (88%)	1 (6%)	1 (6%)
7 years	11	7 (64%)	2 (18%)	2 (18%)

Note. Percentage for the age in parentheses.

The free responses children gave upon being asked why they decided to place Pooh where they did (on the 12th trial of each session) were generally uninformative. A few that were of interest will be referred to in the general discussion.

Discussion

Although many fourth, fifth and sixth grade students have difficulty ordering fraction sequences, it appears that as early as first grade children have a solid understanding of the

inverse relation. In the Winnie the Pooh sharing task, first graders showed proficiency through both their group and individual performance. As a group, first graders were successful on an average of 18 of the 24 trials. Looking at individual patterns of performance, however, yields an even stronger impression of their competence. Seventy-two percent of the first graders demonstrated a pattern of correct performance. This is in stark contrast to the kindergartners: only 30% of whom displayed similar patterns of success. As a group, kindergartners performed well within the bounds of chance performance, averaging correct answers on only 10 of 24 questions.

Effects of Material Type

While first graders scored above chance with all three material types, their success rate with blended materials was significantly better than that with continuous. It may be that children benefited from the combined attributes in the blended form of materials. In the domain of language and categorization, Markman's collection terms were able to facilitate the conceptualization of unified groups while continuing to convey the existence of individual units. Here, in the realm of fraction knowledge, blended materials may similarly succeed in facilitating the conceptualization of a whole while simultaneously portraying the individual parts. As a result, children may have benefited from a combination of the division strategies typically used with sets and continuous wholes.

In contrast to first graders, kindergartners showed no effect of material type. Considering 52% displayed a consistently incorrect pattern of performance, the absence of a material type effect was not surprising. Gelman (1991) has suggested that the young children's whole number reasoning may interfere with early attempts at fraction reasoning. Perhaps children overgeneralized this familiar knowledge about whole numbers and were led

astray when they attempted to reason about the fractional parts. Kindergartners may have used their knowledge about whole numbers and compared two versus three dolls rather than comparing the relative portion sizes those dolls would have received. No reference to any material is necessary in this case so a material type effect would be unlikely. This strategy would also lead to incorrect answers and overall poor performance.

Understanding in Kindergartners

Although a great number of kindergartners consistently demonstrated incorrect response patterns, there were still 48% of them who did not wholeheartedly use a faulty strategy. Eighteen percent performed at chance, possibly wavering back and forth between correct and incorrect strategies. Another 30 % reliably chose the correct answer. In addition to kindergartners' performance here, the success Sophian et al. (1997) had in training 5-year-olds on a similar task suggests that there may be some understanding of the inverse relation present at this age.

Kindergartners' understanding of the inverse relation may be in the early stages and may still be very fragile. As Hunting and Sharpley (1988) and Frydman and Bryant (1988) have shown, children are just starting to understand the significance of one-to-one dealing around the age of 5. Perhaps this dealing strategy was helpful in partitioning discontinuous and blended materials. In the case of continuous materials, on the other hand, kindergartners' strategies may have entailed the visualization of materials breaking apart and then a comparison between the piece sizes for the two and three recipient cases. Regardless of whether or not dealing and breaking apart imagery were the strategies of choice for these children, throughout the changes in material type these kindergartners may have been trying to switch back and forth between two or possibly even more strategies. Making a choice

among strategies that were very new to them, the children may have felt overwhelmed. At times, they may have simply reverted back to more familiar whole number reasoning. Cutting down on the possible number of strategies may improve these children's performance.

In Experiment 2, attempts were made to decrease the overwhelming nature of the children's task. Material type was changed from a within-subject factor to a between-subject factor. The goal here was to narrow down the number of strategies the children would have to be using. By reducing complexity and only presenting each kindergartner with a single material type, these children may be better able to utilize their emerging knowledge regarding the inverse relation.

CHAPTER 3

EXPERIMENT 2

Method

Participants

A total of 45 kindergartners participated in Experiment 2. There were twelve 5-year-olds (8 girls and 4 boys; ages 5 years 2 months to 5 years 5 months, $M = 5$ years 3 months), twenty-six 5½-year-olds (13 girls and 13 boys; ages 5 years 6 months to 5 years 11 months, $M = 5$ years 9 months), six 6-year-olds (2 girls and 4 boys; ages 6 years to 6 years 4 months, $M = 6$ years 1 month) and one 6½-year-old (1 boy; age 6 years 7 months). Children were drawn from two kindergarten classes at Avon Village School in Avon, Ohio. Written parental permission was attained for each participant. Upon completion of the task, children were given a sheet of Winnie the Pooh stickers.

Design

Material type was treated as a between-subjects factor. Therefore, while all 24 photos from Experiment 1 were used, each child only viewed 8, requiring only one testing session.

Again, both most amount and least amount questions were asked. Two sets of material-question type pairs (MQP) were created to avoid confounding these question types with particular materials. As shown in Figure 5, those kindergartners who received MQP Set 1 were asked most amount questions regarding finger-paint, brown play-doh, donuts and peas. The same children were asked least amount questions in reference to pickles, cupcakes, clay and worms. The opposite material-question type pairs existed for those children receiving MQP Set 2.

	<u>MQP Set 1</u>	<u>MQP Set 2</u>
Most Amount	finger-paint donuts peas brown play-doh	worms pickles cupcakes clay
Least Amount	pickles cupcakes clay worms	finger-paint donuts peas brown play-doh

Figure 5: Material-question type pairs for the two MQP sets.

Two random trial orders of the material-question type pairs were generated. These trial orders were created in a similar way to the orders in Experiment 1. The only difference being that in the trial descriptions that were randomized, no reference was made to material type this time. In this study, a given trial was only described with respect to a material's level of desirability, food or nonfood status and the question type. See Figure 6 for the first few trial descriptions of Trial Order 1. An effort was made to avoid confounding trial order and MQP set. Within each condition of material type, half of the children who received MQP Set 1 and half who received MQP Set 2 were assigned Trial Order 1. The remaining children in each condition were assigned Trial Order 2.

<u>Trial Order 1</u>	<u>MQP Set 1</u>	<u>MQP Set 2</u>
least amount – undesirable – food	pickles	peas
least amount – desirable – food	cupcakes	donuts
most amount – undesirable – nonfood	brown play-doh	worms
most amount – desirable – food	donuts	cupcakes

Figure 6: First four descriptions in Trial Order 1 and the materials they described from MQP Sets 1 and 2.

Materials

Materials in this experiment were identical to those in Experiment 1. The same photographs, easel and Winnie the Pooh dolls were used.

Procedure

Children participated in a single session with eight trials. At the start of the session, the penny demonstration was given to show the kindergartners how Pooh always shared his materials equally and exhaustively among recipients. Then each child took part in four most amount and four least amount trials in one of two predetermined randomized orders. After completing all eight trials and responding to the experimenter's question about their reason for placing Pooh where they did for the final trial, participants received a sheet of Winnie the Pooh stickers.

Results

Scoring

For each trial, children's responses were recorded as either correct or incorrect. As in Experiment 1, only first responses were recorded and children who failed to respond after the initial question were prompted once again.

Since there were three conditions with each child viewing a single material type, children participated in only eight trials and overall scores ranged from zero to eight. These scores were subdivided for specific analyses. Half of the eight trials were phrased as most amount questions, while the other half were phrased as least amount questions. Therefore, each child received a score from zero to four for each of the two levels of question type.

Similarly, half of the trials involved desirable materials and half involved undesirable materials. Children were given scores ranging from zero to four for each of the two levels of material desirability.

Preliminary Analyses

An ANOVA performed on the 2(gender: male, female) x 2(MQP set: 1, 2) x 2(trial order: 1, 2) x 2(classrooms) between subject design revealed no effects of gender, MQP set, trial order or classroom. Thus, data were collapsed across these factors in subsequent analyses.

Main Analyses

A three-way analysis of variance analyzed the 3(material type: continuous, blended, discontinuous) x 2(question type: most amount, least amount) x 2(desirability: undesirable, desirable) mixed design. Material type was found to have a marginally significant effect on kindergartners' performance, $F(2,42)=2.96$, $p<.10$. In Figure 7, the mean number of correct responses for each material type condition are shown. Similar to the first graders' results in Experiment 1, it was only performances on continuous ($M=3.06$, $SD=3.37$) and blended materials ($M=5.64$, $SD=2.74$) that significantly differed from one another. A t-test revealed that performance with blended materials marginally exceeded that with continuous, $t(27)=-2.25$, $p<.10$ (with Bonferroni adjustment). Additional t-tests indicated that it was only blended scores that exceeded chance, $t(13)=2.25$, $p<.05$. Lastly, referring back to the main three-way ANOVA, no effects of question type or desirability were found.

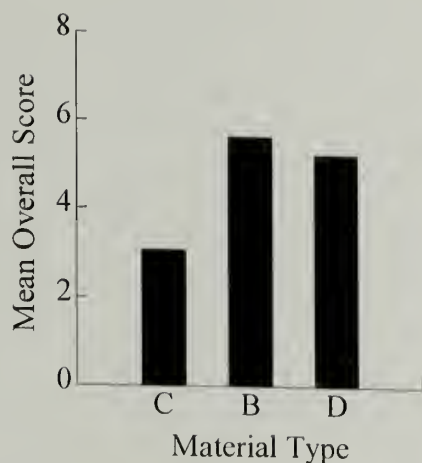


Figure 7: Mean number of correct responses for kindergartners in each material type condition. C = continuous; B = blended; D = discontinuous.

Kindergartners from Experiment 1 and 2

Table 7 shows kindergartners' average scores with each material type. For purposes of comparison, the separate means are given for each experiment. Independent groups t-tests were conducted to determine whether exposure to *only* that particular material type (Experiment 2) significantly improved performance with any of the materials. Performance with both blended and discontinuous materials was found to improve from Experiment 1 to Experiment 2. With blended materials, performance significantly improved, $t(45)=-2.45$, $p<.05$. In the case of discontinuous materials, scores increased a marginal amount, $t(47)=-1.92$, $p<.10$.

Table 7: Mean number of correct responses with each material type.

Material Type	Experiment 1 Kindergartners	Experiment 2 Kindergartners
Continuous	3.58 (3.25)	3.07 (3.37)
Blended	3.24 (3.19)	5.64 (2.74)
Discontinuous	3.49 (3.00)	5.25 (3.04)

Note. Scores ranged from zero to eight. Standard deviations in parentheses.

CHAPTER 4

GENERAL DISCUSSION

Middle-school children struggle with fractions when they are first presented in school. Fractions pose a challenge not only because they require the simultaneous consideration of two dimensions, but also because in many respects fraction reasoning conflicts with the logic of the whole number system (Gelman & Gallistel, 1978). Up until the presentation of fractions, students understand an increase in number to indicate an increase in size. While a fraction's numerator is in line with this notion, the denominator is in direct conflict with it. Often students are not fully cognizant of these differential aspects of the numerator and denominator.

Despite older students' difficulty with the formal introduction of fractions, first graders seemed to have a pretty solid understanding of one of the basic concepts involved in fractions: the inverse relationship existing between the number and size of fractional parts. First graders in Experiment 1 not only performed well in a general sense, but they exceeded chance performance with each of the three material types (when randomly sequenced) as well. In contrast, kindergartners' performance in Experiment 1 was quite poor. One year seems to have had a profound effect on children's performance.

While children first possess a solid understanding of the inverse relation during year-one of elementary school, this is not to say that kindergartners failed to express any fraction knowledge. Their success in Experiment 2, although limited to only the blended and discontinuous materials, does suggest that an understanding of the inverse relation is starting to emerge just prior to elementary school.

Early Emergence of Inverse Reasoning

An understanding of the inverse relation appears to develop gradually. In Experiment 1, both mean scores and individual patterns of performance at each age support this assertion. The average scores for each age linearly progress from below chance (5-year-olds), through chance (5½- and 6-year-olds) and finally to above chance (6½- and 7-year-olds). Individual response patterns for the above and below chance age groups simply reinforce that 5-year-olds do not seem to possess a solid understanding of the inverse relation, while 6½- and 7-year-olds do. However, by looking at the patterns displayed by the age groups yielding mean scores close to chance, new information is gained. It seems that 5½- and 6-year-old children do not perform randomly as their group averages imply. Instead, these are the age groups in which the majority of children are split between correct and incorrect response patterns. This suggests that perhaps an understanding of the inverse relation is an all or none situation. When performances of Experiment 1 and Experiment 2 kindergartners (mostly 5- and 5½-year-olds) are compared, however, it becomes clear that this is not the case. When the sharing task is altered to deal with only one material type, kindergartners express a competence with blended materials that they did not show previously. When all three material types were presented, kindergartners performed at chance with each one. However, when some of the complexity of the task is decreased, they are successful with the blended type. This implies that their understanding may be fragile and only expressed in certain situations. These results, along with Sophian et al.'s (1997) success in training 5-year-olds at their task, suggest that knowledge about the inverse relation may be something that develops gradually.

In addition to suggesting a gradual emergence of fraction knowledge, this study has also provided evidence in support of a much earlier emergence of fraction reasoning than had previously been thought. Together with Sophian's study, these two experiments portray children as young as 5 years old to have the beginnings of inverse reasoning.

Material Types

Some materials seem to facilitate newly developing fraction knowledge, while others do not (Behr et al., 1988; Gay & Aichele, 1997; McDermitt, 1983). Both the kindergartners who only viewed a single material type and the first graders who saw all three, performed better with blended materials in comparison to continuous. While discontinuous scores were not significantly better than continuous, for Experiment 2 kindergartners and Experiment 1 first graders, discontinuous scores were close to those of blended. These findings suggest that blended, and possibly discontinuous, materials may assist children in their attempts at fraction reasoning.

Frydman and Bryant (1988) found that children understand the numerical significance of dealing as early as 5 years of age. Perhaps the reason children scored higher with the blended and discontinuous materials lies in their newly learned strategy of dealing. When children in this task were questioned about their placement choice for Pooh, several mentioned how they mentally divided out the materials in a one-to-one manner and then drew a comparison between dolls' portions. This ability to deal with both blended and discontinuous materials may have been the facilitating factor in the use of inverse reasoning.

Although performances with blended and discontinuous materials were not significantly different from one another in either experiment, in both cases, children (first graders and Experiment 2 kindergartners) performed better with the blended materials. This

could mean there is some added benefit to having sets encased within a cohesive whole. Perhaps children did actually attend to the additional aspects of these materials. However, since no children made mention of the blended materials' continuous aspect (in their free responses) and the difference between blended and discontinuous was not of statistical significance, any influence may have been very subtle.

In contrast to the blended and discontinuous materials, performance with continuous materials was rather poor. First graders from Experiment 1 did score above chance with them, but their continuous scores were significantly lower than those attained with blended materials. Unlike the first graders, kindergartners from both experiments failed to even score above chance with these materials. With this type of material, no procedure like dealing is available to help in making up even shares. Children need to imagine the breaking apart of the whole. Not only that, but they must do it in two different ways and then assess the relative piece sizes. When children were questioned regarding their placement choice for Pooh in continuous cases, few insightful comments were given. Statements like, "because Pooh wants the most amount he can get," were very common. On the few occasions where children did actually refer to making divisions, they could not speak very clearly about what happened in the instance of three friends. One girl stated, "here (2 recipients), each guy would get half of the donut and over here (3 recipients)...they would...I don't know."

The difficulty young children seem to have with continuous materials could be the instigating factor behind the overall poor performance of Experiment 1 kindergartners. Their overall mean score was near chance and significantly worse than their first grade counterparts. Their performance was equally poor with all three materials showing no effect of material type. While it may be true that kindergartners can utilize a dealing strategy with

blended and discontinuous materials, it is probably a very new idea for them. Any strategies they have for continuous materials are likely to be less procedural and less developed. Therefore, these kindergartners from Experiment 1 may have been overwhelmed by the prospect of navigating back and forth between two or more very novel strategies. They may have, at times, reverted back to more familiar whole number reasoning to compare the number of dolls rather than the portion sizes those dolls received. If enough children behaved this way (52% of kindergartners consistently chose incorrect responses) in which referencing the material was not necessary, material type's lack of effect would not be surprising.

Finally, it seems that young children have not yet developed a general rule about the inverse relation that is independent of the materials being divided. This fact, along with previous research by Behr et al. (1988), Gay and Aichele (1997) and McDermit (1983) that suggests material type can affect performance in many different numerical tasks, suggests results attained with only one type of material should not be used to definitively exert the presence of any fraction concept. Numerical reasoning should be demonstrated with multiple materials.

Question Type

Regardless of grade, performance in Experiment 1 was better on the most amount questions than least amount. While the difference was significant, it was not large. Children's scores on most amount questions only exceeded those of least amount by about half a question. The reason for this small difference may be due to the fact that children have more experience in sharing situations in which the goal is to retain a lot of some material. Rarely do they have to share something in which the goal is to retain as little as possible.

An interaction between question type and material type was also found in Experiment 1. It seems that in the case of most amount questions, scores with blended materials were significantly lower than those with continuous. With least amount questions, the opposite was true. As far as the least amount questions, the increased performance that came along with the blended materials may be indicative of their added helpfulness in more difficult scenarios. In reference to the lower blended scores with the most amount questions, the reason is unclear.

The fact that neither the main effect nor the interaction were found in Experiment 2 suggests that they may have just been artifacts of the first experiment. With this as a possibility, it is hard to draw any clear conclusions from these findings in Experiment 1.

Limitations of this Study

The main limitation of this study is the use of the sharing scenario. Just as Hunting and Sharpley (1988) found nonexhaustive distribution of crackers and jumping rope in their study, children's comments during this study indicated that they, too, did not always perceive sharing as an exhaustive process. Hunting and Sharpley (1988) tried to stress the importance of equal and exhaustive shares and I demonstrated how Pooh always shared his things equally with nothing left over. Neither procedure had the desired effect. Several children in this study mentioned that they would give each of Pooh's friends one of the objects and then Pooh would get whatever was left over. "One for Piglet, one for Tigger and one for Pooh," was also a comment given by numerous children. Sharing does not necessarily seem to imply an equal and exhaustive division for these young children. While the concept of sharing has been used in an attempt to assist children's understanding of fractions and

division, it may be masking that very reasoning. It may be that preschoolers are capable of exhaustive distribution and that they just need to be asked about it in a different manner.

Another possible constraint of the experimental design was the use of pictures rather than actual materials. It may be the case that children are better able to use their dealing and visualization strategies when they have concrete materials before them.

Finally, there was the use of the Winnie the Pooh characters. While familiar dolls were intentionally chosen in order to engage the children's attention, the familiarity could have also caused some distraction. If children had a particular affinity for one of the characters over another, they may have made Pooh share with that doll more often than with the other two. There was one instance in which upon being asked why she chose to put Pooh where she did, a kindergartner replied, "because Pooh really likes Tigger."

Conclusions and Educational Implications

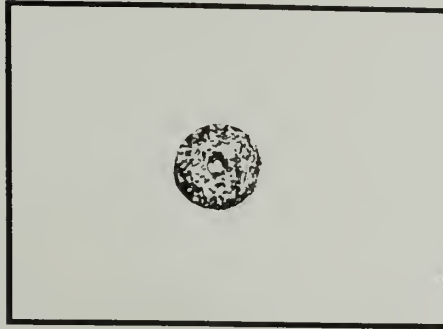
This study adds to previous research (Frydman & Bryant, 1988; Hunting & Davis, 1991; Hunting & Sharpley, 1988; Mix et al., 1999; Sophian et al., 1997; Spinillo & Bryant, 1991) calling attention to the fact that very young children can reason about fractions. In this task, as early as kindergarten and first grade, children were able to make some sense of the inverse relationship that exists between the number and size of parts. Since children possess this knowledge at such an early age, perhaps the difficulties middle-school students have with sequencing fractions lie more with understanding the symbolic representation of fractions rather than understanding their relative sizes. Research conducted by Mix et al. (1999) has shown that 4- and 5-year-olds can perform fraction calculations with quarter pieces of circular sponges. Again, with this type of reasoning in place so early, it is surprising that fourth, fifth and sixth graders struggle so much with these concepts when they

are introduced in school. A greater effort needs to be made in explicating the symbolic structure of fractions. A fraction can be discussed as representing several different states. The representation $1/2$ can be indicative of one from a set of two, half of a continuous whole or the ratio one-to-two. These distinctions need to be made clear and children need to be made aware of what the numerator and denominator refer to in each case.

This study clearly demonstrated a material types' effect on performance. Prior to this study, no direct evaluation of the different material types had been done. Some textbooks have chosen to introduce fractions with continuous objects, while others have used discrete sets. There seems to be nothing systematic about the choices. Since with some materials, children are better able to reason about fractions than with others, this haphazard manner of choosing the introductory material is cause for concern. In the case of the inverse relation, this study has shown that blended and discontinuous materials provide the most benefit. The possible facilitative nature of the both blended and discontinuous material should be translated into the school curriculums. Perhaps children would benefit most from being introduced to fractions with discontinuous materials because of most children's previous experience with the dealing procedure. Then blended materials could be used in transitioning students from reasoning about sets to reasoning about continuous wholes. Regardless of exactly how the schools use this information about the facilitative nature of different materials, it should be used in some manner to inform the fraction curriculum.

APPENDIX A
DONUT PHOTOGRAPHS

Continuous Type



Blended Type

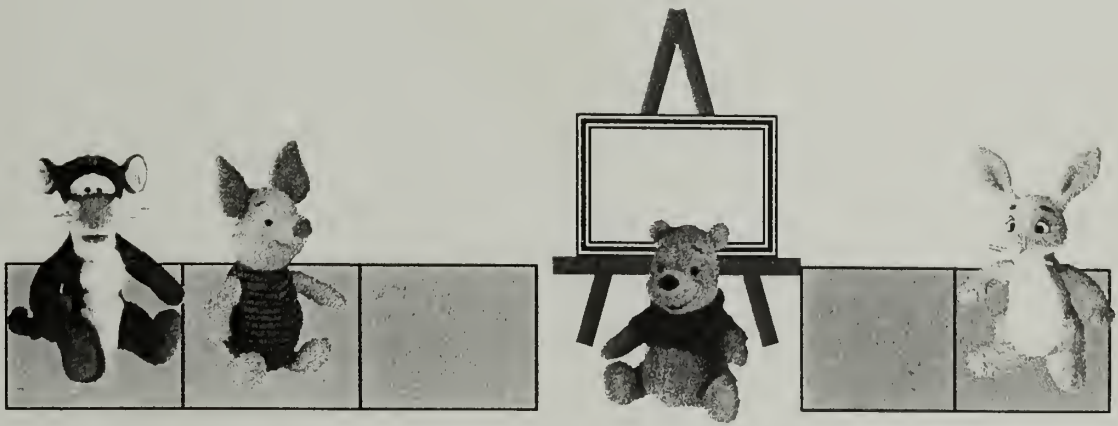


Discontinuous Type



APPENDIX B

SET-UP



BIBLIOGRAPHY

- Behr, M.J., Wachsmuth, I., & Post, T. (1988). Rational number learning aids: Transfer from continuous models to discrete models. Focus on Learning Problems in Mathematics, 10, 1-18.
- Empson, S.B. (1995). Equal sharing and shared meaning: The development of fraction concepts in a first-grade classroom. Paper presented at the meeting of the American Educational Research Association, San Francisco.
- Frydman, O., & Bryant, P. (1988). Sharing and the understanding of number equivalence by young children. Cognitive Development, 3, 323-339.
- Gay, S.A., & Aichele, D.B. (1997). Middle school students' understanding of number sense related to percent. School Science and Mathematics, 97, 27-36.
- Gelman, R. (1991). Epigenetic foundations of knowledge structures: Initial and transcendent constructions. In S. Carey & R. Gelman (Eds.), Epigenesis of mind: Essays on biology and cognition (pp. 293-322). Hillsdale, NJ: Erlbaum.
- Gelman, R. & Gallistel, C.R. (1978). The child's understanding of number. Cambridge: Harvard University Press.
- Goswami, U. (1989). Relational complexity and the development of analogical reasoning. Cognitive Development, 4, 251-268.
- Hunting, R.P. & Davis, G. (1991). Early fraction learning. New York: Springer-Verlag.
- Hunting, R.P., & Sharpley, C.F. (1988). Preschoolers' cognitions of fractional units. British Journal of Educational Psychology, 58, 172-183.
- Huttenlocher, J., & Gao, F. (1999, April). Coding of continuous quantity by infants. Paper presented at the biennial meeting of the Society for Research in Child Development, Albuquerque, NM.
- Huttenlocher, J., Jordan, N.C., & Levine, S.C. (1994). A mental model for early arithmetic. Journal of Experimental Psychology: General, 123, 284-296.
- Markman, E.M. (1973). The facilitation of part-whole comparisons by use of the collective noun "family". Child Development, 44, 837-840.
- Markman, E.M. (1978). Empirical versus logical solutions to part-whole comparison problems concerning classes and collections. Child Development, 49, 168-177.
- Markman, E.M. (1979). Classes and collections: conceptual organization and numerical abilities. Cognitive Psychology, 11, 395-411.

- Markman, E.M. (1989). Categorization and naming in children: Problems of induction. Cambridge: The MIT Press.
- Markman, E.M. (1994). Constraints on word meaning in early language acquisition. Lingua, 92, 199-227.
- McDermit, S.A. (1983). First-graders' performances on subdivision tasks involving discrete and continuous materials. Dissertation Abstracts, 44, 1013.
- Miller, S.A. (1986). Certainty and necessity in the understanding of Piagetian concepts. Developmental Psychology, 22, 3-18.
- Mix, K.S., Levine, S.C., & Huttenlocher, J. (1999). Early fraction calculation ability. Developmental Psychology, 35, 164-174.
- Sophian, C., Garyantes, D., & Chang, C. (1997). When three is less than two: Early developments in children's understanding of fractional quantities. Developmental Psychology, 33, 731-744.
- Spinillo, A.G., & Bryant, P. (1991). Children's proportional judgements: The importance of "half". Child Development, 62, 427-440.
- Winer, G.A. (1980). Class-inclusion reasoning in children: A review of the empirical literature. Child Development, 51, 309-328.
- Wohlwill, J.F. (1968). Responses to class-inclusion questions for verbally and pictorially presented items. Child Development, 39, 449-465.

