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Hypersurfaces in hyperbolic space with support function

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ABSTRACT

Based on [19], we develop a global correspondence between immersed hypersurfaces $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$ satisfying an exterior horosphere condition, also called here horospherically concave hypersurfaces, and complete conformal metrics $e^{2\rho} g_{\mathbb{S}^n}$ on domains Ω in the boundary $\mathbb{S}^n$ at infinity of $\mathbb{H}^{n+1}$, where ρ is the horospherical support function, $\partial_\infty \phi(\mathrm{M}^n) = \partial\Omega$, and Ω is the image of the Gauss map $G : \mathrm{M}^n \to \mathbb{S}^n$. To do so we first establish results on when the Gauss map $G : \mathrm{M}^n \to \mathbb{S}^n$ is injective. We also discuss when an immersed horospherically concave hypersurface can be unfolded along the normal flow into an embedded one. These results allow us to establish general Alexandrov reflection principles for elliptic problems of both immersed hypersurfaces in $\mathbb{H}^{n+1}$ and conformal metrics on domains in $\mathbb{S}^n$. Consequently, we are able to obtain, for instance, a strong Bernstein theorem for a

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complete, immersed, horospherically concave hypersurface in $\mathbb{H}^{n+1}$ of constant mean curvature.

1. Introduction

In a recent paper [19], the authors observed a very interesting fact that the principal curvatures of a hypersurface satisfying an exterior horosphere condition, also called here a horospherically concave hypersurface (see Definition 2.2 and Remark 2.1 for possible confusion), $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$ ($n \geq 3$) can be calculated in terms of the eigenvalues of the Schouten tensor of the horospherical metric $\hat{g} = e^{2\rho} g_{\mathbb{S}^n}$ via its horospherical support function ρ (see also [17,16]). In that paper [19], the authors called such exterior horosphere condition as *horospherically convex*. We adopt here *horospherically concave* since it is geometrically more natural.

This observation creates a correspondence that opens a window for more interactions between the study of elliptic problems of Weingarten surfaces in hyperbolic spaces and the study of elliptic problems of conformal metrics. We will assume throughout the paper that the dimension $n \geq 3$ or as stated otherwise.

Later it was pointed out in [5] that such correspondence can be seen as the association of a conformal metric at infinity with level surfaces of the geodesic defining functions of the conformal metric. In fact, the level surfaces of the geodesic defining function form the regular part of the normal flow (cf. [16]) of the horospherically concave hypersurfaces both in the hyperbolic metric and the conformally compactified metric. We refer to the part of the normal flow where each leaf is embedded as the regular part.

At first the horospherical support function $\tilde{\rho}$ is defined on the parameter space M^n of an immersed horospherically concave hypersurface $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$. Hence the so-called horospherical metric $g_h = e^{2\tilde{\rho}} G^* g_{\mathbb{S}^n}$ is originally defined on M^n too. It is much more useful if the horospherical support function $\tilde{\rho}$ as well as the horospherical metric g_h can be pushed on a domain in $\mathbb{S}^n$ through the Gauss map $G : \mathrm{M}^n \to \mathbb{S}^n$. Indeed, when the Gauss map is injective, we may view the hypersurface as a "graph" of the horospherical support function $\rho = \tilde{\rho} \cdot G^{-1}$ over the domain $G(\mathrm{M}^n)$ in $\mathbb{S}^n$. Though the Gauss map of a compact horospherically concave hypersurface is always injective, the Gauss map of an immersed, complete, horospherically concave hypersurface in general may not be injective.

We notice that the Gauss map of a horospherically concave hypersurface is naturally a development map. Hence, as a consequence of the celebrated injectivity result of Schoen and Yau [42,43], we obtain the following:

Theorem 3.3: *Suppose that $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$ is an immersed, complete, horospherically concave hypersurface and suppose that*

$$\sum_{i=1}^{n} \frac{2}{1+\kappa_i} \leq n,$$

where κ_i are principal curvatures of ϕ. Then its Gauss map is injective.

In general, to avoid wild behavior of the end of a horospherically concave hypersurface, we require that the Gauss map is regular at infinity (cf. Definition 3.4). An immediate consequence of such regularity is the following:

Lemma 3.4: *Suppose that $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$ is a properly immersed, complete, horospherically concave hypersurface with the Gauss map G regular at infinity. Then*

$$\partial G(\mathrm{M}^n) \subseteq \partial_\infty \phi(\mathrm{M}^n).$$

Using the uniformly horospherical concavity (cf. Definition 3.1) to ensure the completeness of the horospherical metric, we then establish the following injectivity theorem:

Theorem 3.2: *Suppose that $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$ is a properly immersed, complete, uniformly horospherically concave hypersurface with the Gauss map regular at infinity. And suppose that the boundary at infinity $\partial_\infty \phi(\mathrm{M}^n)$ is small in the sense that its Hausdorff dimension is less than $n-2$. Then the Gauss map $G : \mathrm{M}^n \to \mathbb{S}^n$ is injective.*

One of the most important issues in hypersurface theory is about when an immersed hypersurface is embedded. In contrast to the Hadamard type theorem established in [13] (cf. [26,45]), it is pointed out in [19] that even a horospherical ovaloid does not have to be embedded. But we observe the following:

Proposition 3.4: *Suppose that $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$, $n \geq 2$, is a connected, compact, immersed, horospherically concave hypersurface. Then the leaves ϕ_t in (2.10) in the normal flow are embedded spheres when t is large enough.*

Our approach here is to use the connection between normal flows, geodesic defining functions, and conformal metrics at the infinity for the hyperbolic metric $g_{\mathbb{H}^{n+1}}$ observed in [5]. In this paper we extend the correspondence shown in [19] and establish the following correspondence between uniformly horospherically concave hypersurfaces and complete conformal metrics with bounded curvature. Based on the Hadamard type theorem established in [13] (cf. [10,20]) we are able to obtain one of our main results:

Main Theorem A (Theorem 3.6): *Suppose that $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$ is an immersed, complete, uniformly horospherically concave hypersurface with injective Gauss map $G : \mathrm{M}^n \to \mathbb{S}^n$. Then it induces a complete conformal metric $e^{2\rho} g_{\mathbb{S}^n}$ on $G(\mathrm{M}^n) \subset \mathbb{S}^n$ with bounded curvature, where ρ is the horospherical support function and*

$$\partial_\infty \phi(\mathrm{M}^n) = \partial G(\mathrm{M}^n).$$

In addition, if we assume that the boundary at infinity $\partial_\infty \phi(\mathrm{M}^n)$ is a disjoint union of smooth compact embedded submanifolds with no boundary in $\mathbb{S}^n$, then ϕ can be unfolded into an embedded hypersurface along its normal flow eventually.
Equivalently, suppose that $e^{2\rho} g_{\mathbb{S}^n}$ is a complete conformal metric on a domain Ω in $\mathbb{S}^n$ with bounded curvature. Then it induces properly immersed, complete, uniformly horospherically concave hypersurfaces

$$\phi_t = \frac{e^{\rho+t}}{2}\left(1 + e^{-2\rho-2t}\left(1 + |\nabla\rho|^2\right)\right)(1, x) + e^{-\rho-t}(0, -x + \nabla\rho) : \Omega \longrightarrow \mathbb{H}^{n+1}$$

and

$$\partial_\infty \phi_t(\Omega) = \partial\Omega$$

for t large enough. In addition, we assume that the boundary $\partial\Omega$ is a disjoint union of smooth compact embedded submanifolds with no boundary in $\mathbb{S}^n$, then the hypersurface

$$\phi_t = \frac{e^{\rho+t}}{2}\left(1 + e^{-2\rho-2t}\left(1 + |\nabla\rho|^2\right)\right)(1, x) + e^{-\rho-t}(0, -x + \nabla\rho) : \Omega \longrightarrow \mathbb{H}^{n+1}$$

is embedded when t is large enough.

It is interesting in the hypersurface side to note that one also gets to know the end structure in the proof of the above theorem (cf. Remark 3.1). The embedding conclusions in the above theorem is particularly useful when combining with injectivity theorems in this paper and therefore gives us opportunities to apply the Alexandrov refection principle in dealing with immersed hypersurfaces in hyperbolic spaces. Based on a slight extension of the Alexandrov–Bernstein theorem in [12] we obtain the following:

Theorem 4.3 *Suppose that $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$ is an immersed, complete, horospherically concave hypersurface with constant mean curvature $H = \sum_{i=1}^n \kappa_i$ and*

$$\sum_{i=1}^{n} \frac{2}{1+\kappa_i} \leq n. \tag{4.19}$$

Then it is a horosphere if its boundary at infinity is a single point in $\mathbb{S}^n$.

This is a strong Bernstein theorem for immersed hypersurfaces in hyperbolic space. The condition (4.19) is used to apply Theorem 3.3 and also implies that $H \geq n$. Similarly, we establish a general Alexandrov reflection principle for immersed, complete, horospherically concave hypersurfaces satisfying general elliptic Weingarten equations.

Elliptic Weingarten equations for hypersurfaces and fully nonlinear elliptic Yamabe type equations for conformal metrics have been extensively studied. Both subjects have a long history and both are very important subjects in the fields of differential geometry and partial differential equations. Although they are mostly treated separately, there is a clear indication that these two subjects should be intimately related in terms of the types of problems and the tools that have been used.

For the sake of clarity, we will explain here what we mean for *elliptic Weingarten equations* for hypersurfaces and *elliptic Yamabe equation* for conformal metrics (see Section 4 for details). First, we introduce the conformally invariant elliptic PDE in the context of our discussions. Since we focus on realizable conformal metrics, we denote

$$\mathcal{C} := \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_i < 1/2, i = 1, \cdots, n\}$$

and

$$\Gamma_n := \{(x_1, \cdots, x_n) : x_i > 0, i = 1, 2, \cdots, n\}.$$

Consider a symmetric function $f(x_1, \cdots, x_n)$ of n-variables with $f(\lambda_0, \lambda_0, \cdots, \lambda_0) = 0$ for some number $\lambda_0 < \frac{1}{2}$ and

$$\Gamma = \text{ an open connected component of } \{(x_1, \cdots, x_n) : f(x_1, \cdots, x_n) > 0\}$$

satisfying

(4.1) $(\lambda, \lambda, \cdots, \lambda) \in \Gamma \cap \mathcal{C}, \forall \ \lambda \in (\lambda_0, \frac{1}{2})$,

(4.2) $\forall \ (x_1, \cdots, x_n) \in \Gamma \cap \mathcal{C}$, $\forall \ (y_1, \cdots, y_n) \in \Gamma \cap \mathcal{C} \cap ((x_1, \cdots, x_n) + \Gamma_n)$, $\exists$ a curve γ connecting $(x_1, \cdots, x_n)$ to $(y_1, \cdots, y_n)$ inside $\Gamma \cap \mathcal{C}$ such that $\gamma' \in \Gamma_n$ along γ,

(4.3) $f \in C^1(\Gamma)$ and $\frac{\partial f}{\partial x_i} > 0$ in Γ.

Suppose $g = e^{2\rho} g_{\mathbb{S}^n}$ is a conformal metric on a domain Ω of $\mathbb{S}^n$ satisfying

$$f(\lambda(\mathrm{Sch}_g)) = C \text{ and } \lambda(\mathrm{Sch}_g) \in \Gamma \cap \mathcal{C} \text{ in } \Omega, \tag{4.4}$$

for some nonnegative constant C, where $\lambda(\mathrm{Sch}_g)$ is the set of eigenvalues of the Schouten curvature tensor of the metric g. We refer to Eq. (4.4) as the conformally invariant elliptic problem of the conformal metrics on the domain Ω.

On the other hand, we now introduce the elliptic problems of Weingarten hypersurfaces in our context. Again, our focus is on admissible hypersurfaces with the canonical orientation. Let

$$\mathcal{K} := \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_i > -1, i = 1, \cdots, n\}.$$

Consider a symmetric function $\mathcal{W}(x_1, \cdots, x_n)$ of n-variables with $\mathcal{W}(\kappa_0, \kappa_0, \cdots, \kappa_0) = 0$ for some number $\kappa_0 > -1$ and

$$\Gamma^* = \text{ an open connected component of } \{(x_1, \cdots, x_n) : \mathcal{W}(x_1, \cdots, x_n) > 0\}$$

satisfying

(4.5) $(\kappa, \kappa, \cdots, \kappa) \in \Gamma^* \cap \mathcal{K}, \forall\ \kappa \in (\kappa_0, \infty)$,

(4.6) $\forall\ (x_1, \cdots, x_n) \in \Gamma^* \cap \mathcal{K}$, $\forall\ (y_1, \cdots, y_n) \in \Gamma^* \cap \mathcal{K} \cap ((x_1, \cdots, x_n) + \Gamma_n)$, $\exists$ a curve γ connecting $(x_1, \cdots, x_n)$ to $(y_1, \cdots, y_n)$ inside $\Gamma^* \cap \mathcal{K}$ such that $\gamma' \in \Gamma_n$ along γ,

(4.7) $\mathcal{W} \in C^1(\Gamma^*)$ and $\frac{\partial \mathcal{W}}{\partial x_i} > 0$ in Γ^*.

Suppose $\phi : \mathrm{M} \to \mathbb{H}^{n+1}$ is a hypersurface satisfying

$$\mathcal{W}(\kappa_1, \cdots, \kappa_n) = K \text{ and } (\kappa_1, \cdots, \kappa_n) \in \Gamma^* \cap \mathcal{K} \text{ on } \phi, \tag{4.8}$$

for some nonnegative constant K, where $(\kappa_1, \cdots, \kappa_n)$ is the set of principal curvatures of the hypersurface ϕ. We refer to Eq. (4.8) as the elliptic problem of Weingarten hypersurfaces.

So, by means of Theorem 3.6, the next step is to relate elliptic Weingarten equations for hypersurfaces and elliptic Yamabe equations for hypersurfaces:

Theorem 4.1: *There is an one-to-one correspondence between elliptic Weingarten equations for admissible horospherically concave hypersurfaces in hyperbolic space $\mathbb{H}^{n+1}$ and elliptic Yamabe problems for realizable conformal metrics on $\mathbb{S}^n$.*

We would like to remark that it is not just desirable but imperative for us to consider general fully nonlinear elliptic problems (4.4) and (4.8) other than, for example, just the mean curvature equation for hypersurfaces. Because, in order to gain the embeddedness and apply the Alexandrov reflection principle, we need to unfold a given hypersurface along the normal flow, in which the curvature equation usually does not remain the same. This is seen, for instance, in the proof of Theorem 4.2 in Section 4. We also like to make a remark that our conditions (4.2) and (4.6) in the definitions of ellipticity are different from those in the past. We first want to point out that conditions (4.2) and (4.6) are equivalent under the curvature relation (2.7). We then refer readers to the proof of Theorem 4.4 to check how conditions (4.2) and (4.6) work as good as before.

The correspondence established in Theorem 3.6 and Theorem 4.1 identifies the problem of finding a properly immersed and complete hypersurface $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$ that satisfies certain geometric equation (4.8) with a prescribed boundary at infinity $\partial_\infty \phi(\mathrm{M}^n)$ in $\mathbb{S}^n$ [44,25] with the problem of finding a complete conformal metric $e^{2\rho} g_{\mathbb{S}^n}$ that satisfies the corresponding geometric equation (4.4) on the domain $\Omega \subset \mathbb{S}^n$ whose boundary $\partial\Omega$ is the same as the prescribed boundary at infinity $\partial_\infty \phi(\mathrm{M}^n)$ [8,36]. For instance, the method of Alexandrov reflection for embedded hypersurfaces in hyperbolic space $\mathbb{H}^{n+1}$ in [3,27,12] and the method of moving planes (or spheres) in [24,7] are seen to be the same under the correspondence.

As a consequence of our general Alexandrov reflection principle for horospherically concave hypersurfaces satisfying elliptic Weingarten equations (4.8), we also establish a general Alexandrov reflection principle for conformal metrics satisfying fully nonlinear elliptic equations (4.4). To state our second main result, we need introduce some more notation. Let E be the equator in $\mathbb{S}^n$ and P be the totally geodesic hyperplane whose boundary is E. Let $R : \mathbb{H}^{n+1} \to \mathbb{H}^{n+1}$ stand for the reflection in $\mathbb{H}^{n+1}$ with respect to the hyperplane P, and $\Phi : \mathbb{S}^n \to \mathbb{S}^n$ the unique conformal transformation induced by R. Hence,

Main Theorem B (Theorem 4.6): *Suppose that $(\mathcal{W}, \Gamma^*)$ satisfies (4.5)–(4.7). Let $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$ be an admissible hypersurface with the canonical orientation satisfying (4.8), whose boundary $\partial_\infty \phi(\mathrm{M}^n)$ at the infinity is a disjoint union of smooth compact submanifolds with no boundary in E. Then $\partial_\infty \phi(\mathrm{M}^n)$ cannot be E and the surface ϕ is R-invariant.*
Equivalently, suppose that (f, Γ) satisfies (4.1)–(4.3). Let g be a realizable metric satisfying (4.4) on Ω such that $\partial\Omega \subset E$ is a disjoint union of smooth compact submanifolds with no boundary. Then $\partial\Omega$ cannot be E and g is Φ-invariant.

From this general Alexandrov reflection principle, we derive, for example, the following Delaunay type theorem:

Corollary 4.2: *Suppose that $(\mathcal{W}, \Gamma^*)$ satisfies (4.5)–(4.7). Let $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$ be an admissible hypersurface with the canonical orientation satisfying (4.8), whose boundary $\partial_\infty \phi(\mathrm{M}^n)$ at the infinity consists of exactly two points. Then the surface ϕ is rotationally symmetric with respect to the geodesic joining the two points at the infinity of ϕ.*
Equivalently, suppose that (f, Γ) satisfies (4.1)–(4.3). Let g be a realizable metric satisfying (4.4) on $\Omega = \mathbb{S}^n \setminus \{p, q\}$. Then g is cylindric with respect to the geodesic joining the two points in $\partial\Omega$.

At this point, as pointed out by the referee, we must do a couple of remarks. This Delaunay type theorem should be compared with those in [32–34], in particular with [32, Theorem 1.2]. In [32, Theorem 1.2], the author assumes the scalar curvature is nonnegative, while ours assumes the Schouten tensor is bounded. Also, in [32, Theorem 1.2], the metric is not assumed to be complete. We do this assumption since, in this way, the result looks cleaner. Assuming

$$e^{2\rho(x)} + |\nabla\rho|^2(x) \to +\infty \text{ as } x \to \{p, q\}$$

it is enough by means of Corollary 3.1.

The paper is organized as follows: In Section 2 we recapture the works in [19] and [5] and clarify the relation of geodesic defining functions and normal flows. In Section 3 we develop the global correspondence between admissible hypersurfaces $\mathbb{H}^{n+1}$ and realizable metrics on domains in $\mathbb{S}^n$. We also prove that an admissible hypersurface can be unfolded into an embedded one along the normal flow when the boundary at infinity is a disjoint union of smooth compact submanifolds with no boundary in $\mathbb{S}^n$. In Section 4 we establish the full correspondence between elliptic problems from the two sides. In particular, we compare Alexandrov theorems with Obata theorems, Bernstein theorems with Liouville theorems and even Delaunay type theorems. In fact we extend a general symmetry result in [31] for both admissible hypersurfaces and realizable metrics based on our embedding theorem.

2. Local theory

In this section we will recapture the works in [19,5] and set the stage to develop a global theory of the correspondence between hypersurfaces in hyperbolic space $\mathbb{H}^{n+1}$ and conformal metrics on domains of the conformal infinity $\mathbb{S}^n$ of hyperbolic space $\mathbb{H}^{n+1}$. In [19], Espinar, Gálvez and Mira discovered that (see also [17,16]), given a piece of horospherically concave hypersurface $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$, there is a locally conformally flat metric g_h on M^n, whose curvature is explicitly related to the extrinsic curvature of the hypersurface in $\mathbb{H}^{n+1}$. Conversely, one may construct an immersed, horospherically concave hypersurface in hyperbolic space $\mathbb{H}^{n+1}$ from a conformal metric on a domain in the infinity $\mathbb{S}^n$. It was later observed in [5] that such correspondence can be seen as the association of conformal metrics on domains of $\mathbb{S}^n$, geodesic defining functions, and level surfaces of geodesic defining functions (see also [37]).

2.1. Horospherical concavity and horospherical metrics

We will briefly introduce the construction developed in [19]. Let us denote by $\mathbb{R}^{1,n+1}$ the Minkowski spacetime, that is, the vector space $\mathbb{R}^{n+2}$ endowed with the Minkowski spacetime metric $\langle , \rangle$ given by

$$\langle \bar{x}, \bar{x} \rangle = -x_0^2 + \sum_{i=1}^{n+1} x_i^2,$$

where $\bar{x} \equiv (x_0, x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+2}$. Then hyperbolic space, the de Sitter spacetime, and the positive null cone are given, respectively, by the hyperquadrics

$$\begin{aligned}
\mathbb{H}^{n+1} &= \left\{ \bar{x} \in \mathbb{L}^{n+2} : \langle \bar{x}, \bar{x} \rangle = -1,\ x_0 > 0 \right\} \\
\mathbb{S}_1^{n+1} &= \left\{ \bar{x} \in \mathbb{L}^{n+2} : \langle \bar{x}, \bar{x} \rangle = 1 \right\} \\
\mathbb{N}_+^{n+1} &= \left\{ \bar{x} \in \mathbb{L}^{n+2} : \langle \bar{x}, \bar{x} \rangle = 0,\ x_0 > 0 \right\}.
\end{aligned}$$

The ideal boundary at infinity of hyperbolic space $\mathbb{H}^{n+1}$ will be denoted by $\mathbb{S}^n$.

An immersed hypersurface in hyperbolic space $\mathbb{H}^{n+1}$ is given by a parametrization

$$\phi : \mathrm{M}^n \longrightarrow \mathbb{H}^{n+1}.$$

On the hypersurface ϕ, an orientation assigns a unit normal vector field

$$\eta : \mathrm{M}^n \longrightarrow \mathbb{S}_1^{n+1}.$$

Hence, associated to ϕ, one may consider the map

$$\psi = \phi - \eta : \mathrm{M}^n \longrightarrow \mathbb{N}_+^{n+1}, \tag{2.1}$$

which is called the associated light cone map of ϕ. We will use horospheres to define the Gauss map of an oriented, immersed hypersurface in hyperbolic space $\mathbb{H}^{n+1}$. In the above hyperboloid model, horospheres in $\mathbb{H}^{n+1}$ are the intersections of affine null hyperplanes of $\mathbb{R}^{1,n+1}$ with $\mathbb{H}^{n+1}$.

Definition 2.1. (See [15,17,6].) Let $\phi : \mathrm{M}^n \longrightarrow \mathbb{H}^{n+1}$ be an immersed, oriented hypersurface in $\mathbb{H}^{n+1}$ with the orientation $\eta : \mathrm{M}^n \to \mathbb{S}_1^{n+1}$. The *Gauss map*

$$G : \mathrm{M}^n \longrightarrow \mathbb{S}^n$$

of ϕ is defined as follows: for every $p \in \mathrm{M}^n$, $G(p) \in \mathbb{S}^n$ is the point at infinity of the unique horosphere $\mathcal{H}_p$ in $\mathbb{H}^{n+1}$ passing through $\phi(p)$ and with the inner unit normal the same as $-\eta(p)$ at $\phi(p)$.

The associated light cone map ψ is strongly related to the Gauss map G of ϕ. Indeed, the ideal boundary $\mathbb{S}^n$ of $\mathbb{H}^{n+1}$ can be identified with the projective quotient space $\mathbb{N}_+^{n+1}/\mathbb{R}_+$ in such a way that we have

$$\psi = e^{\tilde{\rho}}(1, G), \tag{2.2}$$

where $\tilde{\rho}$ is the so-called horospherical support function for the hypersurface ϕ. Note that horospheres are the unique hypersurfaces such that, with outward orientation, the associated light cone map, as defined by (2.1), as well as the Gauss map are constant. Moreover, if we write $\psi = e^{\tilde{\rho}}(1, x)$ for a given horosphere, then $x \in \mathbb{S}^n$ is the point at infinity of the horosphere and $\tilde{\rho}$ is the signed hyperbolic distance of the horosphere to the point $\mathcal{O} = (1, 0, \ldots, 0) \in \mathbb{H}^{n+1} \subseteq \mathbb{R}^{1,n+1}$. The intrinsic geometry of a horosphere is Euclidean. Therefore one may introduce a notion of convexity–concavity based on horospheres. Namely,

Definition 2.2. (See [41].) Let $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$ be an immersed, oriented hypersurface and let $\mathcal{H}_p$ denote the horosphere in $\mathbb{H}^{n+1}$ that is tangent to the hypersurface at $\phi(p)$

and whose inward unit normal at $\phi(p)$ agrees with unit normal $\eta(p)$ to the hypersurface ϕ at $\phi(p)$. We will say that $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$ is horospherically concave at p if there exists a neighborhood $V \subset \mathrm{M}^n$ of p so that $\phi(V \setminus \{p\})$ does not intersect with $\mathcal{H}_p$. Moreover, the distance function of the hypersurface $\phi : V \to \mathbb{H}^{n+1}$ to the horosphere $\mathcal{H}_p$ does not vanish up to the second order at $\phi(p)$ in any direction.

We have the following characterization of horospherically concave hypersurfaces:

Lemma 2.1. *(See [19].) Let $\phi : \mathrm{M}^n \longrightarrow \mathbb{H}^{n+1}$ be an immersed, oriented hypersurface. Then ϕ is horospherically concave at p if and only if all principal curvatures of ϕ at p are simultaneously < 1 or > 1. In particular, dG is invertible at p if ϕ is horospherically concave at p.*

To see the second statement, if $\{e_1, \cdots, e_n\}$ denotes an orthonormal basis of principal curvature directions of ϕ at p and $\kappa_1, \cdots, \kappa_n$ are the principal curvatures respectively, i.e.

$$\begin{aligned} d\phi(e_i) &= e_i \\ d\eta(e_i) &= -\kappa_i e_i, \end{aligned} \tag{2.3}$$

it is then immediate [19] that

$$\langle (d\psi)_p(e_i), (d\psi)_p(e_j) \rangle = (1+\kappa_i)^2 \delta_{ij} = e^{2\tilde{\rho}} \langle (dG)_p(e_i), (dG)_p(e_j) \rangle_{\mathbb{S}^n}. \tag{2.4}$$

From now on, unless stated otherwise, we will take the orientation on a horospherically concave hypersurface so that all principal curvatures satisfy $\kappa_i > -1$.

Remark 2.1. We like to remark here that horospherical concavity (cf. Definition 2.2) is a weaker notion of convexity for oriented immersed hypersurfaces in hyperbolic space. To clarify, if we choose the orientation so that $\kappa_i > -1$, the hypersurface lies locally at the concave side of its tangent horosphere at each point, according to Definition 2.2. Such orientation will be called the canonical orientation.

Note that for a compact embedded horospherically concave hypersurface the canonical orientation is the inward orientation. For a totally umbilical sphere, which is horospherically concave, the canonical orientation is the one which makes all its principal curvatures positive.

Now we are ready to introduce the horospherical metric on an immersed horospherically concave hypersurface as follows:

Definition 2.3. Let $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$ be an immersed horospherically concave hypersurface. Then the Gauss map $G : \mathrm{M}^n \to \mathbb{S}^n$ is a local diffeomorphism. We consider the locally conformally flat metric

$$g_h = \psi^* \langle , \rangle = e^{2\tilde{\rho}} G^* g_{\mathbb{S}^n} \tag{2.5}$$

on M^n and call it the horospherical metric of the horospherically concave hypersurface ϕ.

It is clear that g_h is the induced metric on the immersed hypersurface $\psi : \mathrm{M}^n \to \mathbb{N}^{n+1}_+$, when ψ is spacelike. Considering $\psi : \mathrm{M}^n \to \mathbb{N}^{n+1}_+ \subset \mathbb{R}^{1,n+1}$ as a surface of co-dimension 2 in the Minkowski spacetime $\mathbb{R}^{1,n+1}$, we know that $\phi(p)$ and $\eta(p)$ are two unit normal vectors at $\psi(p)$ and the second fundamental form is

$$II_\psi(e_i, e_j) = (\frac{1}{1+\kappa_i}\phi - \frac{\kappa_i}{1+\kappa_i}\eta) g_h(e_i, e_j).$$

Hence, the sectional curvature of the metric g_h is

$$\mathrm{K}_{g_h}(\frac{e_i}{1+\kappa_i}, \frac{e_j}{1+\kappa_j}) = 1 - \frac{1}{1+\kappa_i} - \frac{1}{1+\kappa_j}$$

and Schouten tensor is

$$\mathrm{Sch}_{g_h}(e_i, e_j) = (\frac{1}{2} - \frac{1}{1+\kappa_i}) g_h(e_i, e_j).$$

When the Gauss map $G : \mathrm{M}^n \to \mathbb{S}^n$ of a horospherically concave hypersurface $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$ is a diffeomorphism, one may push the horospherical metric g_h onto the image $\Omega = G(\mathrm{M}^n) \subset \mathbb{S}^n$ and consider the conformal metric

$$\hat{g} = (G^{-1})^* g_h = e^{2\rho} g_{\mathbb{S}^n},$$

where $\rho = \tilde{\rho} \circ G^{-1}$. For simplicity, we also refer to this conformal metric $\hat{g}$ as the horospherical metric. On the other hand, given a conformal metric $\hat{g} = e^{2\rho} g_{\mathbb{S}^n}$ on a domain Ω in $\mathbb{S}^n$, one immediately recovers the light cone map $\psi(x) = e^\rho(1, x) : \Omega \to \mathbb{N}^{n+1}_+$. It turns out that one can solve for the map $\phi : \Omega \to \mathbb{H}^{n+1}$ and the unit normal vector $\eta : \Omega \to \mathbb{S}^{n+1}_1$ such that $\phi - \eta = \psi$.

Theorem 2.1. *(See [19].) Let $\phi : \Omega \subseteq \mathbb{S}^n \longrightarrow \mathbb{H}^{n+1}$ be a piece of horospherically concave hypersurface with Gauss map $G(x) = x$. Then $\psi = e^\rho(1, x)$ and it holds*

$$\phi = \frac{e^\rho}{2}\left(1 + e^{-2\rho}\left(1 + |\nabla\rho|^2\right)\right)(1, x) + e^{-\rho}(0, -x + \nabla\rho). \tag{2.6}$$

Moreover, the eigenvalues λ_i of the Schouten tensor of the horospherical metric $\hat{g} = e^{2\rho} g_{\mathbb{S}^n}$ and the principal curvatures κ_i of ϕ are related by

$$\lambda_i = \frac{1}{2} - \frac{1}{1+\kappa_i}. \tag{2.7}$$

Conversely, given a conformal metric $\hat{g} = e^{2\rho} g_{\mathbb{S}^n}$ defined on a domain of the sphere $\Omega \subseteq \mathbb{S}^n$ such that the eigenvalues of its Schouten tensor are all less than $1/2$, the map ϕ given by (2.6) defines an immersed, horospherically concave hypersurface in $\mathbb{H}^{n+1}$ whose Gauss map is $G(x) = x$ for $x \in \Omega$ and whose horospherical metric is the given metric $\hat{g}$.

To end this subsection, for the convenience of readers, we recall that on a Riemannian manifold (M^n, g), $n \geq 3$, the Riemann curvature tensor can be decomposed as

$$\mathrm{Riem}_g = W_g + \mathrm{Sch}_g \odot g,$$

where W_g is the Weyl tensor, $\odot$ is the Kulkarni–Nomizu product, and

$$\mathrm{Sch}_g := \frac{1}{n-2}\left(\mathrm{Ric}_g - \frac{S_g}{2(n-1)} g\right)$$

is the Schouten tensor, where Ric_g and S_g stand for the Ricci curvature and scalar curvature of g respectively. The eigenvalues of Sch_g are defined as the eigenvalues of the endomorphism $g^{-1}\mathrm{Sch}_g$.

Remark 2.2. To avoid confusion we remind readers that in our convention, for instance, the principal curvatures of a geodesic sphere in hyperbolic space $\mathbb{H}^{n+1}$ with respect to the inward orientation are bigger than $+1$. As we have said above, the inward orientation corresponds to the canonical orientation as defined in this paper.

Finally we want to remark that we have changed the orientation with respect to the one used in [19] and [5]. This is because it is more natural according to our notion of horospherical concavity.

2.2. Geodesic defining functions and normal flows

In this section we briefly summarize the work in [5]. We will take a viewpoint that is more reflective of conformal geometry and reinterpret the correspondence, Theorem 2.1, as the association of conformal metrics and geodesic defining functions. Here one can think of geodesic defining functions as ways of describing foliations of hypersurfaces, or level set representations of normal flows.

A defining function for a part of the infinity $\Omega \subset \mathbb{S}^n$ of hyperbolic space $\mathbb{H}^{n+1}$ is a smooth function satisfying

1. $r > 0$ in $\Omega \times (0, \epsilon_0) \subset \mathbb{H}^{n+1}$;
2. $r = 0$ on $\Omega \times \{0\} \subset \mathbb{S}^n$; and
3. $dr \neq 0$ on $\Omega \times \{0\} \subset \mathbb{S}^n$.

The hyperbolic space $(\mathbb{H}^{n+1}, g_{\mathbb{H}^{n+1}})$ is conformally compact in the sense that $r^2 g_{\mathbb{H}^{n+1}}$ extends to the infinity for any defining function r when considering $\Omega = \mathbb{S}^n$. The metrics

$r^2 g_{\mathbb{H}^{n+1}}|_{r=0}$ recover the standard conformal class of metrics on $\mathbb{S}^n$ when the defining functions vary.

Definition 2.4. A defining function r is said to be geodesic defining function if

$$|dr|_{r^2 g_{\mathbb{H}^{n+1}}} = 1, \tag{2.8}$$

at least in a neighborhood of the infinity (i.e. $\Omega \times [0, \epsilon_0)$ for some positive number ϵ_0). With geodesic defining function r we have

$$g_{\mathbb{H}^{n+1}} = r^{-2}(dr^2 + g_r),$$

where g_r is a family of metrics on $\Omega \subset \mathbb{S}^n$. It is easily seen that there is a canonical association between the choice of conformal metric $r^2 g_{\mathbb{H}^{n+1}}|_{r=0}$ and the geodesic defining function r.

The advantage of using geodesic defining functions is evident from the following lemma of Fefferman and Graham [21].

Lemma 2.2. *(See [21].) Suppose that g is a metric conformal to the standard round metric $g_{\mathbb{S}^n}$ on a domain $\Omega \subset \mathbb{S}^n$ and that r is the geodesic defining function associated with g. Then*

$$g_{\mathbb{H}^{n+1}} = r^{-2}(dr^2 + g_r)$$

where

$$g_r = g - r^2 \mathrm{Sch}_g + \frac{r^4}{4} Q_g \tag{2.9}$$

and

$$(Q_g)_{ij} = g^{kl}(\mathrm{Sch}_g)_{ik}(\mathrm{Sch}_g)_{jl}.$$

By the definition of geodesic defining functions above, it is useful to realize that the level surfaces of a geodesic defining function r are the normal flow of the boundary into $\mathbb{H}^{n+1}$ in the conformally compactified metric $r^2 g_{\mathbb{H}^{n+1}}$ as well as the normal flow of a horospherically concave hypersurface toward the infinity in $\mathbb{H}^{n+1}$ in the hyperbolic metric $g_{\mathbb{H}^{n+1}}$, which was called parallel flows in [16]. After identifying the level surfaces of a geodesic defining function as horospherically concave hypersurfaces in $\mathbb{H}^{n+1}$, the relation (2.7) in Theorem 2.1 is a direct consequence of the expansion (2.9) as observed in [5].

For the convenience of readers we calculate the expansion (2.9) using Ricatti equations for principal curvatures in hyperbolic space of the normal flow. Let $\Omega \subset \mathbb{S}^n$ be a domain in the sphere and $\phi : \Omega \to \mathbb{H}^{n+1}$ be an oriented horospherically concave hypersurface so that $G(x) = x$ for all $x \in \Omega \subset \mathbb{S}^n$. Let $\{\phi_t\}_{t\in\mathbb{R}}$ denote the (past) normal flow of ϕ in hyperbolic space $\mathbb{H}^{n+1}$, that is,

$$\phi_t(x) := \exp_{\phi(x)}(-t\eta(x)) = \phi(x)\cosh t - \eta(x)\sinh t : \Omega \longrightarrow \mathbb{H}^{n+1} \subset \mathbb{R}^{1,n+!}, \tag{2.10}$$

where exp denotes the exponential map for the hyperbolic metric $g_{\mathbb{H}^{n+1}}$. Due to the Ricatti equations, the principal curvatures κ_i^t of ϕ_t are given by

$$\kappa_i^t(p) = \frac{\kappa_i(p) + \tanh(t)}{1 + \kappa_i(p)\tanh(t)}, \tag{2.11}$$

and the first fundamental form of ϕ_t is given by

$$I_t(e_i, e_j) = (\cosh(t) + \kappa_i \sinh(t))^2 \delta_{ij}, \tag{2.12}$$

where $\{e_1, \cdots, e_n\}$ is an orthonormal basis of principal curvature directions of ϕ. From here one can easily check that the Gauss maps G_t remain invariant under this flow and the horospherical metric of ϕ_t is $g_t := e^{2t} g_h$, where g_h is the horospherical metric of ϕ. Moreover, the change of variable $r = 2e^{-t}$ shows that (2.12) is equivalent to (2.9).

Conversely, given a conformal metric $\hat{g} := e^{2\rho} g_{\mathbb{S}^n}$ on $\Omega \subset \mathbb{S}^n$ with Schouten tensor bounded from above, one considers a family of rescaled metric $\hat{g}_t = e^{2t}\hat{g}$. Choosing t_0 large so that $e^{-2t_0}\mathrm{Sch}_{\hat{g}} \leq \frac{1}{2}$, it follows from Theorem 2.1 that the foliation of hypersurfaces

$$\phi_t = \frac{e^{\rho+t}}{2}\left(1 + e^{-2\rho-2t}\left(1 + |\nabla\rho|^2\right)\right)(1, x) + e^{-\rho-t}(0, -x + \nabla\rho) : \Omega \longrightarrow \mathbb{H}^{n+1} \tag{2.13}$$

for $t > t_0$ consists of immersed, horospherically concave hypersurfaces with Gauss map $G_t(x) = x : \Omega \to \mathbb{S}^n$ the identity.

3. Global theory

From the previous section, we know that, for a piece of horospherically concave hypersurface in hyperbolic space $\mathbb{H}^{n+1}$, the Gauss map induces a canonical conformal metric on the infinity $\mathbb{S}^n$ locally. Conversely, given a conformal metric on a domain of $\mathbb{S}^n$, there is an immersed, horospherically concave hypersurface in hyperbolic space $\mathbb{H}^{n+1}$ whose horospherical metric is the given metric up to a rescale. In this section we establish a global correspondence between properly immersed, complete, horospherically concave hypersurfaces and complete conformal metrics on domains of $\mathbb{S}^n$. Given a complete, properly immersed, horospherically concave hypersurface $\phi : \mathrm{M}^n \longrightarrow \mathbb{H}^{n+1}$, the issues that concern us are the following:

- When is the horospherical metric g_h complete?
- When is its Gauss map injective?
- When does the boundary at infinity of the hypersurface coincide with the boundary of the Gauss map image?

In the other direction, given a complete conformal metric $\hat{g}$ on a domain Ω of the infinity $\mathbb{S}^n$ with Schouten tensor bounded from above, we are concerned with the following issues:

- When does it correspond to a complete, immersed, horospherically concave hypersurface?
- When is the corresponding hypersurface proper?
- When does the boundary of the domain coincide with the boundary at infinity of the hypersurface?

A final, yet most important question is: when are the leaves of the normal flow given in (2.10) or (2.13) eventually embedded? Equivalently, one may ask when there is a geodesic defining function associated with a given complete conformal metric on a domain $\Omega \subset \mathbb{S}^n$ defined for a positive distance uniformly in the domain Ω.

3.1. Uniform concavity vs bounded curvature

We are able to make a satisfactory correspondence if we restrict ourselves to the cases where hypersurfaces are uniformly horospherically concave or equivalently the conformal metrics are of bounded curvature. Let us start with the definition of uniformly horospherically concave.

Definition 3.1. Let $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$ be an immersed, oriented hypersurface. We say that ϕ is uniformly horospherically concave if there is a number $\kappa_0 > -1$ such that all the principal curvatures κ_i at all points in M^n are greater than or equal to κ_0.

Hence, in the light of (2.7), one can easily see that, for a conformal metric $\hat{g} = e^{2\rho} g_{\mathbb{S}^n}$ on a domain $\Omega \subset \mathbb{S}^n$ with Schouten tensor bounded from above, the corresponding hypersurface ϕ_t given in (2.13) is an immersed, uniformly horospherically concave hypersurface for t large enough if and only if the Schouten tensor of $\hat{g}$ is also bounded from below. On the other hand, when the conformal metric is of bounded curvature, the corresponding hypersurfaces ϕ_t given in (2.13) are immersed and uniformly horospherically concave with bounded principal curvatures for t large enough. Based on the above observation we make the following definition.

Definition 3.2. An oriented hypersurface $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$ is said to be admissible if it is properly immersed, complete, uniformly horospherically concave with injective Gauss map $G : \mathrm{M}^n \to \mathbb{S}^n$. Meanwhile, a complete conformal metric $\hat{g} = e^{2\rho} g_{\mathbb{S}^n}$ on a domain $\Omega \subset \mathbb{S}^n$ is said to be realizable if it is of bounded curvature.

When we start with a properly immersed, complete, horospherically concave hypersurface $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$ with injective Gauss map $G : \mathrm{M}^n \to \mathbb{S}^n$, from Theorem 2.1, we know ϕ induces a conformal metric $\hat{g}$ on the image of the Gauss map $\Omega = G(\mathrm{M}^n) \subset \mathbb{S}^n$ with Schouten tensor bounded from above by one half. Then, the question to ask is if the conformal metric $\hat{g}$ is complete? One can easily construct an example to show that the answer in general is negative. We will present a properly immersed, complete, horospherically concave hypersurface whose horospherical metric is not complete at the end of this subsection. On the other hand, when the hypersurface is uniformly horospherically concave, the completeness of the horospherical metric is a simple consequence of (2.4).

Lemma 3.1. *Suppose that $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$ is a complete, immersed, uniformly horospherically concave hypersurface. Then the horospherical metric g_h is complete on M^n.*

When we start with a complete conformal metric $\hat{g} = e^{2\rho} g_{\mathbb{S}^n}$ on a domain $\Omega \subset \mathbb{S}^n$ with Schouten tensor bounded from above, from Theorem 2.1, we know that for t large enough the hypersurface ϕ_t given by (2.13) is immersed and horospherically concave. Then a natural question to ask is if the hypersurface ϕ_t is complete and proper. One again easily observes

Lemma 3.2. *Suppose that $\hat{g} = e^{2\rho} g_{\mathbb{S}^n}$ is a complete conformal metric on a domain $\Omega \subset \mathbb{S}^n$ with Schouten tensor bounded from above and that $\phi_t : \Omega \to \mathbb{H}^{n+1}$ given by (2.13) is immersed. In addition we assume that*

$$\beta(x) := e^{2\rho(x)} + |\nabla \rho|^2(x) \longrightarrow +\infty \quad \text{as } x \to \partial\Omega.$$

Then ϕ_t is a properly immersed, complete, horospherically concave hypersurface for t large enough.

Proof. Here we shall use the Poincaré ball model of $\mathbb{H}^{n+1}$. We like to use stereographic projection in Minkowski spacetime to realize the coordinate change between the two models. Namely,

$$\begin{array}{ccc} \mathbb{H}^{n+1} \subset \mathbb{R}^{1,n+1} & \longrightarrow & \mathbb{B}^{n+1} \subset \mathbb{R}^{n+1} = \{\bar{x} \in \mathbb{R}^{1,n+1} : x_0 = 0\} \\ & \tau & \\ (x_0, x_1, \cdots, x_{n+1}) & \longrightarrow & \dfrac{1}{1+x_0}(x_1, \cdots, x_{n+1}) \end{array}.$$

Hence, omitting the variable t for simplicity, we have

$$\tau \circ \phi = \frac{e^{2\rho} + |\nabla \rho|^2 - 1}{e^{2\rho} + 2e^{\rho} + |\nabla \rho|^2 + 1} (x + Y(x)) : \Omega \longrightarrow \mathbb{B}^{n+1}$$

with

$$Y(x) = \frac{2}{e^{2\rho} + |\nabla\rho|^2 - 1}\nabla\rho.$$

Now it is easily seen that if $\beta(x) \to +\infty$, then

$$\left(\frac{e^{2\rho} + |\nabla\rho|^2 - 1}{e^{2\rho} + 2e^{\rho} + |\nabla\rho|^2 + 1}\right)(x) \longrightarrow 1$$

and

$$\left(\frac{2}{e^{2\rho} + |\nabla\rho|^2 - 1}\nabla\rho\right)(x) \longrightarrow \mathbf{0}.$$

Therefore, if $\beta(x) \to +\infty$ as $x \to x_0 \in \partial\Omega$, it then follows that

$$\tau \circ \phi(x) \to x_0 \tag{3.1}$$

as desired. $\square$

One important side product of the proof of Lemma 3.2 is the following:

Corollary 3.1. *Suppose that $\hat{g} = e^{2\rho} g_{\mathbb{S}^n}$ is a complete conformal metric on a domain $\Omega \subset \mathbb{S}^n$ with Schouten tensor bounded from above and that $\phi_t : \Omega \to \mathbb{H}^{n+1}$ is given in (2.13). In addition we assume that*

$$\beta(x) := e^{2\rho(x)} + |\nabla\rho|^2(x) \longrightarrow +\infty \quad \text{as } x \to \partial\Omega.$$

Then ϕ_t is a properly immersed, complete, horospherically concave hypersurface, and

$$\partial_\infty \phi_t(\mathrm{M}^n) = \partial\Omega,$$

for t large enough.

One may refer to Definition 3.3 for the boundary at infinity $\partial_\infty\phi(\mathrm{M}^n)$ of a hypersurface ϕ in hyperbolic space $\mathbb{H}^{n+1}$. It seems to us that it is a rather subtle issue to determine when $\beta(x) \to +\infty$ as $x \to \partial\Omega$ if one only assumes the metric $\hat{g}$ to be compete and the Schouten tensor to be bounded from above. We settle the issue by using Proposition 8.1 in [8], where it is shown that the conformal factor $\rho \to +\infty$ as $x \to \partial\Omega$ if the scalar curvature is bounded from below. Notice that in our context, since we always assume the Schouten tensor is bounded from above, the fact that the scalar curvature is bounded from below implies the curvature of the conformal metric $\hat{g}$ is bounded.

Proposition 3.1. *Suppose that $\hat{g} = e^{2\rho} g_{\mathbb{S}^n}$ is a complete conformal metric on a domain $\Omega \subset \mathbb{S}^n$ with Schouten tensor bounded and that $\phi_t : \Omega \to \mathbb{H}^{n+1}$ is given in (2.13). Then ϕ_t*

is a properly immersed, complete, uniformly horospherically concave hypersurface with uniformly bounded principal curvature, and

$$\partial_\infty \phi_t(\mathrm{M}^n) = \partial\Omega,$$

for t large enough.

Again, one may refer to Definition 3.3 for the boundary at infinity $\partial_\infty\phi(\mathrm{M}^n)$ of a hypersurface ϕ in hyperbolic space $\mathbb{H}^{n+1}$. To summarize we have the following main result in this subsection for the global correspondence.

Theorem 3.1. *Suppose that $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$ is an admissible hypersurface with the hyperbolic Gauss map $G : \mathrm{M}^n \to \Omega \subset \mathbb{S}^n$. Then it induces a realizable metric on the domain Ω. Moreover $\partial_\infty\phi(\mathrm{M}^n) = \partial\Omega$.*

On the other hand, suppose that $e^{2\rho}g_{\mathbb{S}^n}$ is a realizable metric on a domain $\Omega \subset \mathbb{S}^n$. Then ϕ_t given in (2.13) is an admissible hypersurface with bounded principal curvature and $\partial_\infty\phi_t(\mathrm{M}^n) = \partial\Omega$, for t large enough.

The above result provides a back-and-forth relationship between complete conformal metrics on domains of the sphere and horospherically concave hypersurfaces in $\mathbb{H}^{n+1}$ with prescribed boundary at infinity. This allows to relate the results of [35] and [36] for singular solutions for conformal metrics on the sphere with those of, among others, [44,25] for hypersurfaces in $\mathbb{H}^{n+1}$ with prescribed boundary at infinity.

Before we end this subsection we would like to present an easy example to show that one in general does not get the completeness of horospherical metric. Let us consider $\Omega = \mathbb{S}^{n-1}\times(-1,1) \subset \mathbb{S}^{n-1}\times(-\frac{\pi}{2},\frac{\pi}{2}) = \mathbb{S}^n\setminus\{\mathbf{S},\mathbf{N}\} \subset \mathbb{S}^n$. In this parameterization, the standard round metric is given as

$$g_{\mathbb{S}^n} = ds^2 + \cos^2 s g_{\mathbb{S}^{n-1}}$$

and the Christoffel symbols are

$$\Gamma^s_{ss} = \Gamma^s_{si} = 0 \text{ and } \Gamma^s_{ij} = \tan s (g_{\mathbb{S}^n})_{ij}$$

for $i,j = 2,3,\cdots,n$. Let

$$\rho(\theta,s) = \rho(s) = -\frac{1}{2}\log(1-s^2)$$

and $\hat{g} = e^{2\rho}g_{\mathbb{S}^n}$ be the conformal metric on Ω. If we consider the meridian $\gamma : (0,1) \to \Omega$ given by $\gamma(s) = (\theta_0, s)$ where $\theta_0 \in \mathbb{S}^{n-1}$ is fixed, then we easily see that

$$\int_\gamma e^\rho \, dv_{g_{\mathbb{S}^n}} = \int_0^1 \frac{1}{\sqrt{1-s^2}} ds < +\infty,$$

which implies that $\hat{g}$ is not complete in Ω. On the other hand, we recall

$$\mathrm{Sch}[\hat{g}]_{ik} = \mathrm{Sch}[g_{\mathbb{S}^n}]_{ik} - \rho_{i,k} + \rho_i \rho_k - \frac{1}{2}|\nabla \rho|^2 (g_{\mathbb{S}^n})_{ik}$$

and calculate

$$\rho_s = \frac{s}{1-s^2}$$

and the only nonzero terms for the Hessian are

$$\rho_{s,s} = \frac{1+s^2}{(1-s^2)^2}, \quad \rho_{i,j} = -\frac{s \tan s}{1-s^2}(g_{\mathbb{S}^n})_{ij}.$$

Hence we notice

$$-\rho_{s,s} + \rho_s^2 - \frac{1}{2}\rho_s^2 = -\frac{1+s^2}{(1-s^2)^2} + \frac{1}{2}\frac{s^2}{(1-s^2)^2} = -\frac{1+\frac{1}{2}s^2}{(1-s^2)^2} < 0$$

and

$$-\rho_{i,j} + \rho_i \rho_j - \frac{1}{2}|\nabla \rho|^2 (g_{\mathbb{S}^n})_{ij} = (\frac{s \tan s}{1-s^2} - \frac{s^2}{2(1-s^2)^2})(g_{\mathbb{S}^n})_{ij} \leq C(g_{\mathbb{S}^n})_{ij}$$

for some $C > 0$, $i, j = 2, 3, \cdots, n$, and $s \in (-1, 1)$. Therefore we consider the immersed, horospherically concave hypersurface ϕ_t given by (2.13) corresponding to $(\Omega, \hat{g})$ for t sufficiently large. Since $\rho \to +\infty$ as s approaches 1, from Lemma 3.2, we know that ϕ_t is proper and complete. We remark here that in fact $\mathrm{Sch}[\hat{g}]$ is not bounded from below in this example, which implies that the hypersurface ϕ_t is not uniformly horospherically concave.

3.2. Injectivity of hyperbolic Gauss maps

We next describe an explicit example to show that indeed the Gauss map of a noncompact, complete, properly immersed oriented horospherically concave hypersurface may not be injective. The essential reasons are that one can have a convex, self-intersecting, closed curve in $\mathbb{H}^2$ and that higher dimensional hyperbolic space $\mathbb{H}^{n+1}$ is a foliation of totally geodesic $\mathbb{H}^2$ via translation isometries.

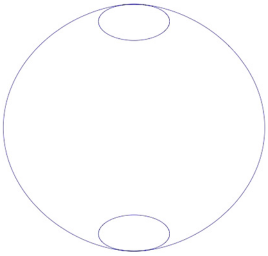
Let $r, R : \mathbb{R} \to \mathbb{R}$ be smooth 4π-periodic functions defined by

$$r(u) := \sin(\frac{u}{2})\cos(u), \quad R(u) := \cos(\frac{u}{2}) - \frac{1}{3}\cos(\frac{3u}{2}),$$

and let $\alpha(u) : \mathbb{R} \to \mathbb{H}^2 \subset \mathbb{R}^{1,2}$ be given by

$$\alpha(u) = (\cosh(r(u))\cosh(R(u)), \sinh(r(u))\cosh(R(u)), \sinh(R(u))).$$

Then α is non-embedded and has nonnegative curvature. Actually, in the geodesic coordinate, its profile is as depicted:



$$\alpha(u) := (\sin(\tfrac{u}{2})\cos(u), \cos(\tfrac{u}{2}) - \tfrac{1}{3}\cos(\tfrac{3u}{2}))$$

So the desired hypersurface is generated from the above immersed convex closed curve in a totally geodesic surface $\mathbb{H}^2$ by $(n-1)$-families of translation isometries along geodesics orthogonal to the totally geodesic surface $\mathbb{H}^2$ in $\mathbb{H}^{n+1}$. The resulting hypersurface is a properly immersed convex hypersurface $\phi : \mathbb{R}^{n-1} \times \mathbb{S}^1 \to \mathbb{H}^{n+1}$ where by construction the principal curvatures of the hypersurface are all zero except one is positive. Hence, the scalar curvature of the horospherical metric of such immersed convex hypersurface is strictly negative when $n \geq 3$. Also, the boundary at infinity of this hypersurface is an $\mathbb{S}^{n-2}$. In other words, the image of the Gauss map is $\mathbb{S}^n \setminus \mathbb{S}^{n-2} \simeq \mathbb{R}^{n-1} \times \mathbb{S}^1$, which is not simply connected. Considering the normal vector along the profile curve one sees that the Gauss map is a three-sheet covering map.

In general, it is a rather difficult issue to determine when the Gauss map is injective. On the other hand, the Gauss map of an immersed, horospherically concave hypersurface in hyperbolic space is a development map from the parameter space M^n equipped with the horospherical metric into the sphere. Hence, due to Kulkarni and Pinkall [30], we have the following:

Lemma 3.3. *Suppose that $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$ is an immersed, horospherically concave hypersurface and that the horospherical metric g_h is complete on M^n. Then the Gauss map is a covering onto its image in the sphere. Hence, the Gauss map is injective if its image in the sphere is simply connected.*

In this subsection we will show that the Gauss map of a properly immersed, complete, horospherically concave hypersurface is injective when it is regular at infinity (cf. Definition 3.4) and the boundary at infinity (cf. Definition 3.3) is small, which will be made precise below. We will also show that the injectivity of the Gauss map follows under certain curvature conditions on the hypersurface, which is a straightforward consequence of the celebrated injectivity of development maps of Schoen and Yau [42,43].

In the light of Lemma 3.3 the Gauss map is injective when its image in the sphere is simply connected. A good way to study images of Gauss maps is to consider the boundaries at infinity of hypersurfaces. Let us first define the boundary at infinity of a noncompact hypersurface in $\mathbb{H}^{n+1}$.

Definition 3.3. Suppose that $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$ is a properly immersed hypersurface. We define the boundary at infinity $\partial_\infty \phi(\mathrm{M}^n)$ to be the collection of points $x \in \mathbb{S}^n$ such that there is a sequence x_n on the hypersurface in the Poincaré ball $\mathbb{B}^{n+1}$ model of hyperbolic space that converges to x in $\overline{\mathbb{B}^{n+1}}$ in Euclidean topology.

In general, the end behaviors of properly immersed, complete, horospherically concave hypersurfaces may be very wild. The following regularity of Gauss maps at infinity seems to be a very efficient and geometric way to restrict the behavior of the end and in many ways excludes the persistent sharp turns of a surface approaching the boundary at infinity.

Definition 3.4. Suppose that $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$ is a properly immersed hypersurface. The Gauss map is said to be regular at infinity if, for each $p \in \partial_\infty \phi(\mathrm{M}^n) \subset \mathbb{S}^n$,

$$\lim_{i\to\infty} G(q_i) = p$$

for $q_i \in \mathrm{M}^n$, $\phi(q_i) \to p$.

As a consequence of the regularity of the Gauss map at infinity, we have the following:

Lemma 3.4. *Suppose that $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$ is a properly immersed, complete, horospherically concave hypersurface and that the Gauss map $G : \mathrm{M}^n \to \mathbb{S}^n$ is regular at infinity. Then*

$$\partial G(\mathrm{M}^n) \subset \partial_\infty \phi(\mathrm{M}^n). \tag{3.2}$$

Proof. Let $p \notin \partial_\infty \phi(\mathrm{M}^n)$. We would like to show that $p \notin \partial G(\mathrm{M}^n)$. Otherwise, $p \in \partial G(\mathrm{M}^n)$, which means that $p \notin G(\mathrm{M}^n)$ and there is a sequence $p_i \in G(\mathrm{M}^n)$ such that $p_i \to p$ in $\mathbb{S}^n$. Let $q_i \in \mathrm{M}^n$ such that $G(q_i) = p_i$. At least for a subsequence, we may assume $\phi(q_i)$ converges to $x \in \bar{\mathbb{B}}^{n+1}$. By the completeness of the hypersurface, if $x \in \mathbb{B}^{n+1}$, then $p = G(q)$ for some $q \in \mathrm{M}^n$ and $\phi(q) = x$, which contradicts the fact that $p \notin G(\mathrm{M}^n)$. On the other hand, if $x \in \partial\mathbb{B}^{n+1}$, one may conclude that $x \in \partial_\infty \phi(\mathrm{M}^n) \subset \mathbb{S}^n$ by the regularity of the Gauss map, which contradicts the fact that $p \notin \partial_\infty \phi(\mathrm{M}^n)$. □

We also observe the following:

Proposition 3.2. *Suppose that $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$ is a properly immersed, complete, horospherically concave hypersurface with complete horospherical metric and that the Gauss map is regular at infinity. Then either the Gauss map is a finite covering or*

$$G(\mathrm{M}^n) \subset \partial_\infty \phi(\mathrm{M}^n).$$

Proof. For any $p \in G(\mathrm{M}^n)$, we consider the preimage $G^{-1}(p) \subset \mathrm{M}^n$ and the set $P = \{\phi(q) : q \in G^{-1}(p)\} \subset \mathbb{B}^{n+1}$ of points on the surface. First we show that no limit point of P is inside $\mathbb{B}^{n+1}$. Otherwise, suppose that $x \in \mathbb{B}^{n+1}$ is a limit point of P. Then $x \in \phi(\mathrm{M}^n)$ due to the completeness of the surface. By the properness of the immersion ϕ we may conclude that $G^{-1}(p)$ has a limit point in M^n, which contradicts the fact that the Gauss map is a local diffeomorphism.

On the other hand, since ϕ is proper, when $G^{-1}(p)$ is infinite so is the set P. In this case P can only have limit points in the boundary at infinity $\partial_\infty \phi(\mathrm{M}^n)$. Therefore, $p \in \partial_\infty \phi(\mathrm{M}^n)$ due to the regularity of the Gauss map at infinity. The conclusion of this proposition then follows from Lemma 3.3. $\square$

Proposition 3.2 tells us that the Gauss map is a finite covering when the Gauss map is regular at infinity and the boundary at infinity $\partial_\infty \phi(\mathrm{M}^n)$ of the surface has no interior points. We know that a subset in $\mathbb{S}^n$ has no interior point if, for example, it is of Hausdorff dimension less than n. In fact, when the boundary at infinity $\partial_\infty \phi(\mathrm{M}^n)$ of a surface is of Hausdorff dimension less than $n-2$, it turns out that the Gauss map has to be injective.

Theorem 3.2. *Suppose that $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$ is a properly immersed, complete, horospherically concave hypersurface with complete horospherical metric and that its Gauss map is regular at infinity. Then the Gauss map is injective and*

$$\partial G(\mathrm{M}^n) = \partial_\infty \phi(\mathrm{M}^n),$$

provided that $\partial_\infty \phi(\mathrm{M}^n) \subset \mathbb{S}^n$ is small in the sense that its Hausdorff dimension is less than $n-2$.

Proof. By the above Lemma 3.4, we know that

$$\partial G(\mathrm{M}^n) \subset \partial_\infty \phi(\mathrm{M}^n)$$

is small in the sense that its Hausdorff dimension is less than $n-2$. Then $G(\mathrm{M}^n)$ is connected and simply connected in $\mathbb{S}^n$. Because, any loop in $G(\mathrm{M}^n)$ can be deformed into a point in $\mathbb{S}^n$ without leaving $G(\mathrm{M}^n)$, when $\mathbb{S}^n \setminus G(\mathrm{M}^n)$ is of codimension bigger than 2 in $\mathbb{S}^n$. Notice that $\mathbb{S}^n \setminus G(\mathrm{M}^n) = \partial G(\mathrm{M}^n)$ when $\partial G(\mathrm{M}^n)$ is of Hausdorff dimension less than $n-1$.

Thus, in the light of Lemma 3.3, the Gauss map is injective. This implies that no point in $\partial_\infty \phi(M^n)$ can be in the image $G(\mathrm{M}^n)$ of the Gauss map, which implies

$$\partial G(M^n) = \partial_\infty \phi(M^n). \qquad \square$$

As noted earlier, the Gauss map is a development map from a locally conformally flat manifold (M^n, g_h) into $\mathbb{S}^n$. Therefore, we may apply the celebrated result on the injectivity of the developing map in [42,43].

Theorem 3.3. *Suppose that $\phi : M^n \to \mathbb{H}^{n+1}$ is an immersed, complete, horospherically concave hypersurface and suppose that*

$$\sum_{i=1}^{n} \frac{2}{1+\kappa_i} \leq n, \tag{3.3}$$

where κ_i are the principal curvatures of ϕ in $\mathbb{H}^{n+1}$. Then the Gauss map is injective. Hence, the hypersurface ϕ is admissible and

$$\partial G(\mathrm{M}^n) = \partial_\infty \phi(\mathrm{M}^n).$$

Proof. This turns out to be a rather straightforward consequence of Theorem 3.5 on page 262 in [43]. First, the assumption (3.3) implies that the hypersurface is in fact uniformly horospherically concave. Hence, the horospherical metric g_h is complete in the light of (2.4). Secondly, due to the explicit relation (2.7) in Theorem 2.1, the assumption (3.3) implies that the scalar curvature of the horospherical metric g_h is nonnegative. Thus, by Theorem 3.5 on page 262 in the book [43], the Gauss map G of ϕ as a development map is injective. The remaining claim then follows from the fact that the surface ϕ is now known to be admissible. $\square$

For the convenience of readers we provide Theorem 3.5 on page 262 of [43] in the following:

Theorem 3.4. *(See [43].) Let (M^n, g) be a complete Riemannian manifold with nonnegative scalar curvature. Suppose that $\Phi : M^n \to \mathbb{S}^n$ is a conformal map. Then Φ is injective and $\partial\Phi(M^n) \subset \mathbb{S}^n$ has zero Newton capacity.*

3.3. Embeddedness

An important issue in the theory of hypersurfaces is to know when an immersed hypersurface is in fact embedded. To use the convexity to gain embeddedness is a classic idea traced back to Hadamard [26,45]. Here we will combine the ideas from [10,20] and the connection between normal flows and geodesic defining functions observed in [5] to obtain some embedding theorems, based on the embedding theorem in [13]. First we state some extremal cases where the hypersurfaces ϕ_t in (2.13) are embedded from known results in [2,10].

Proposition 3.3. *Let $\hat{g} = e^{2\rho} g_{\mathbb{S}^n}$ be a realizable metric on a domain $\Omega \subset \mathbb{S}^n$.*

a) *If the Schouten tensor of the conformal metric g is nonnegative, then Ω is either $\mathbb{S}^n$ or $\mathbb{S}^n \setminus \{point\}$. In the first case the corresponding hypersurface ϕ_t given in (2.13) is an embedded ovaloid when t is large. In the later case the corresponding hypersurface ϕ_t is a horosphere with the inward orientation for each t.*

b) *On the other hand, if the Schouten tensor is nonpositive, then Ω is homeomorphic to $\mathbb{R}^n$ and the corresponding horospherically concave hypersurface ϕ_t given in (2.13) is properly embedded for all t.*

Proof. a) By Theorem 3.1, we know that ϕ_t as in (2.13) is a properly immersed, complete, uniformly horospherically concave hypersurface when t is large enough. Moreover, the nonnegativity of the Schouten tensor implies that all principal curvatures of the surface ϕ_t are bigger than or equal to $+1$. Thus, from [10], the surface is either an embedded n-sphere or a horosphere.

b) As in the above, we construct properly immersed, complete, uniformly horospherically concave hypersurfaces ϕ_t via Theorem 3.1. This time the nonpositivity of the Schouten tensor of a realizable metric implies that all principal curvatures of the surface ϕ_t are between -1 and 1. Thus, from [2], the surface is properly embedded and homeomorphic to $\mathbb{R}^n$. $\square$

In general, one simply cannot expect all admissible hypersurfaces are embedded, as it was pointed out in [19], even a horospherical ovaloid may not be embedded. But what we can hope is that every admissible hypersurface can be unfolded along the (past) normal flow into an embedded one, which is shown to be the case for a horospherical ovaloid in Corollary 3.4. We recall that the geodesic defining function r and its level surfaces give rise to both the normal flow in the hyperbolic metric $g_{\mathbb{H}^{n+1}}$ (called parallel flows in [16]) and the normal flow in the compactified metric $r^2 g_{\mathbb{H}^{n+1}}$. It is worth mentioning that the geodesic defining function is not well defined when the surfaces are no longer embedded, while the normal flow of immersed surfaces is still well defined. Therefore, the embeddedness of a hypersurface is equivalent to the existence of a geodesic defining function, which is equivalent to solving the noncharacteristic first order partial differential equation (2.8). Interestingly, solving (2.8) by the characteristic method is equivalent to solving the normal flow in the compactified metric. It then becomes a standard geometric question of how far one can push a totally geodesic hypersurface along the normal flow in a Riemannian manifold without any focal points, to which the standard Riemannian comparison theorems apply (cf. [9]). This reconfirms that given a conformal metric with Schouten tensor bounded from above, the hypersurface ϕ_t given by (2.13) is an immersion when t is large enough (cf. [19]).

Theorem 3.5. *Suppose that $\hat{g} = e^{2\rho} g_{\mathbb{S}^n}$ is a conformal metric on $\Omega \subset \mathbb{S}^n$ with Schouten tensor bounded from above. Then ϕ_t given in (2.13) is an immersion for t large enough.*

Proof. Given a point $x \in \Omega$, there exists an open neighborhood U of x inside Ω on which the geodesic defining function associated with the given metric $\hat{g} = e^{2\rho} g_{\mathbb{S}^n}$ on Ω is defined and reaches out for a positive number ϵ. The key point is to show that $\epsilon \geq \epsilon_0$ for some positive number ϵ_0 which is independent of where x is in Ω. In the compactified metric $r^2 g_{\mathbb{H}^{n+1}}$ the infinity U becomes a totally geodesic boundary and how far the geodesic defining function can be defined is determined by how far the boundary can be pushed in along the normal flow without encountering any focal points. In other words, in the metric $r^2 g_{\mathbb{H}^{n+1}}$, we would like to know how far the geodesics from points in U in the directions normal to the boundary U can be extended without collisions locally.

Let us calculate the sectional curvatures for the metric $r^2 g_{\mathbb{H}^{n+1}}$ in the plane containing the direction normal to U. To do so, we set a normal coordinate x with respect to the metric $\hat{g}$ at a point x_0 in U. We may assume that the Schouten tensor of the conformal metric $\hat{g}$ is in diagonal form under the chosen coordinates at x_0. Hence, in the coordinate (r, x) for $\bar{g} = r^2 g_{\mathbb{H}^{n+1}}$,

$$\bar{R}_{irir} = \frac{1}{2}(-\partial_r\partial_r\bar{g}_{ii} - \partial_i\partial_i\bar{g}_{rr} + \partial_i\partial_r\bar{g}_{ir} + \partial_r\partial_i\bar{g}_{ir}) - \bar{g}^{\alpha\beta}(\overline{[ii,\alpha]}\,\overline{[rr,\beta]} - \overline{[ir,\alpha]}\,\overline{[ir,\beta]}),$$

where the Christoffel symbols of second kind are given by

$$\overline{[\alpha\beta,\gamma]} = \frac{1}{2}(\partial_\beta\bar{g}_{\alpha\gamma} + \partial_\alpha\bar{g}_{\beta\gamma} - \partial_\gamma\bar{g}_{\alpha\beta})$$

What is good here is that we only need to take second order derivatives for $\bar{g}_{rr}$ with respect to x variables. It is helpful at this point to recall from Lemma 2.2 that

$$\bar{g} = r^2 g_{\mathbb{H}^{n+1}} = dr^2 + \hat{g} - r^2\mathrm{Sch}_{\hat{g}} + \frac{r^4}{4}Q_{\hat{g}}.$$

Hence,

$$\begin{aligned}\bar{R}_{irir} &= -\frac{1}{2}\partial_r\partial_r\bar{g}_{ii} + \bar{g}^{ii}[ir,i][ir,i]\\ &= \lambda_i - \frac{3}{2}r^2\lambda_i^2 + (1 - r^2\lambda_i + \frac{1}{4}r^4\lambda_i^2)^{-1}(r\lambda_i - \frac{1}{2}r^3\lambda_i^2)^2\\ &= \lambda_i - \frac{1}{2}r^2\lambda_i^2. \end{aligned} \tag{3.4}$$

Now one may apply the second Rauch comparison theorem of Berger, Theorem 1.29 on page 30 of the book [9] to conclude that, for any given neighborhood V of x_0 such that $\bar{V} \subset U$, there is a positive number ϵ_0 such that the geodesic defining function reaches beyond ϵ_0 from any point in V, since $\mathrm{Sch}_{\hat{g}}$ is assumed to bounded from above. $\square$

Consequently, even though a horospherical ovaloid may not be embedded (cf. [19]), it can be expanded along the normal flow into an embedded one.

Proposition 3.4. *Suppose that $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$, $n \geq 2$, is a connected, compact, immersed, horospherically concave hypersurface. Then the leaves ϕ_t in (2.10) in the normal flow are embedded spheres when t is large enough.*

This should be compared with the Hadamard type theorem established by Do Carmo and Warner in Section 5 in [13] (cf. [26,45]). For the convenience of readers we state their result as follows:

Do Carmo–Warner theorem [13]: *Suppose that $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$, $n \geq 2$, is a connected, compact, immersed hypersurface with all principal curvature nonnegative. Then ϕ is an embedded ovaloid.*

It turns out that Do Carmo–Warner theorem [13] is one of the important key ingredients in our approach to establish the embeddedness of leaves in the normal flow from a noncompact admissible hypersurface. Another key ingredient is also a consequence of Theorem 3.5.

Lemma 3.5. *Suppose that $\phi : \Omega \to \mathbb{H}^{n+1}$, $n \geq 2$, is an immersed, horospherically concave hypersurface with Gauss map $G(x) = x : \Omega \to \mathbb{S}^n$. Then, for any compact subset $K \subset \Omega$, the hypersurfaces $\phi_t : K \to \mathbb{H}^{n+1}$ given in (2.10) are embedded when t is sufficiently large.*

In order to apply Do Carmo–Warner theorem [13] we use the following:

Lemma 3.6. *Suppose that $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$ is an admissible hypersurface. Then*

$$\kappa_i^t = \frac{\kappa_i + \tanh t}{1 + \kappa_i \tanh t} \longrightarrow +1 \text{ as } t \to \infty, \tag{3.5}$$

where κ_i^t are the principal curvatures for the hypersurface ϕ_t given by (2.13).

Proof. This is simply because the principal curvatures $\kappa_i \geq \kappa_0$ for some $\kappa_0 > -1$ from the uniformly horospherical convexity. □

Now, we are ready to prove:

Theorem 3.6 *(Main Theorem A). Suppose that $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$ is an immersed, complete, uniformly horospherically concave hypersurface with injective Gauss map $G : \mathrm{M}^n \to \mathbb{S}^n$. Then it induces a complete conformal metric $e^{2\rho} g_{\mathbb{S}^n}$ on $G(\mathrm{M}^n) \subset \mathbb{S}^n$ with bounded curvature, where ρ is the horospherical support function and*

$$\partial_\infty \phi(\mathrm{M}^n) = \partial G(\mathrm{M}^n).$$

In addition, if we assume that the boundary at infinity $\partial_\infty \phi(\mathrm{M}^n)$ is a disjoint union of smooth compact embedded submanifolds with no boundary in $\mathbb{S}^n$, then ϕ can be unfolded into an embedded hypersurface along its normal flow eventually.

Equivalently, suppose that $e^{2\rho}g_{\mathbb{S}^n}$ is a complete conformal metric on a domain Ω in $\mathbb{S}^n$ with bounded curvature. Then it induces properly immersed, complete, uniformly horospherically concave hypersurfaces

$$\phi_t = \frac{e^{\rho+t}}{2}\left(1+e^{-2\rho-2t}\left(1+|\nabla\rho|^2\right)\right)(1,x)+e^{-\rho-t}(0,-x+\nabla\rho):\Omega\longrightarrow\mathbb{H}^{n+1}$$

and

$$\partial_\infty\phi_t(\Omega)=\partial\Omega$$

for t large enough. In addition, we assume that the boundary $\partial\Omega$ is a disjoint union of smooth compact embedded submanifolds with no boundary in $\mathbb{S}^n$, then the hypersurface

$$\phi_t = \frac{e^{\rho+t}}{2}\left(1+e^{-2\rho-2t}\left(1+|\nabla\rho|^2\right)\right)(1,x)+e^{-\rho-t}(0,-x+\nabla\rho):\Omega\longrightarrow\mathbb{H}^{n+1}$$

is embedded when t is large enough.

Proof. We only need to focus on the proof of the embeddedness. The previous follows from Lemma 3.1, Proposition 3.1 and Theorem 3.5.

First, in the light of Lemma 3.5, one only needs to focus on each end. This is because, for the conformal metric $e^{2\rho}g_{\mathbb{S}^n}$ corresponding to the given admissible hypersurface ϕ, we know $\rho\to\infty$ when approaching $\partial\Omega$. Hence (3.1) holds, which implies the ends of hypersurface are well separated near the infinity.

Consider one of the connected components $\mathrm{E}^k\subset\partial\Omega$, which is a smooth compact submanifold with no boundary in $\mathbb{S}^n$. Let U be an open neighborhood of E^k in $\mathbb{S}^n$ whose closure $\bar{U}$ is a compact subset in $\mathbb{S}^n$ intersecting no other component of $\partial\Omega$. We denote the tubular neighborhood of E^k inside U with size λ_0 in $\mathbb{S}^n$ as

$$\mathrm{N}_{\lambda_0}(\mathrm{E}^k)=\mathrm{E}^k\times B^{n-k}_{\lambda_0}\subset\mathbb{S}^n,$$

where B^{n-k}_λ is geodesic ball in $\mathbb{S}^{n-k}$ for $n-k\geq 1$ and λ_0 is some small positive number. Let $\mathbb{S}^{n-k-1}_\lambda\subset B^{n-k}_{\lambda_0}$ denote the family of round spheres with $\lambda<\lambda_0$ and centered at the center of $B^{n-k}_{\lambda_0}$. Finally, for a point $p\in\mathrm{E}^k$, let $D^{n-k}_\lambda(p)\subset\mathbb{H}^{n+1}$ denote the totally geodesic hyperbolic subspace that has the boundary $\{p\}\times\mathbb{S}^{n-k-1}_\lambda\subset\mathrm{N}_{\lambda_0}(\mathrm{E}^k)$ at infinity.

From Lemma 3.5 we know that there exists t_0 large enough so that the hypersurface

$$\phi_t:\Omega\bigcap(\bar{U}\setminus\mathrm{N}_{\frac{1}{2}\lambda_0}(\mathrm{E}^k))\longrightarrow\mathbb{H}^{n+1}$$

is embedded and the hypersurface

$$\phi_t:\Omega\bigcap(\mathrm{N}_{\frac{1}{2}\lambda_0}(\mathrm{E}^k)\setminus\mathrm{E}^k)\longrightarrow\mathbb{H}^{n+1}$$

lies inside $\bigcup_{p\in \mathrm{E}^k}\bigcup_{\lambda<\frac{2}{3}\lambda_0} D^{n-k}_\lambda(p)$ for each $t\geq t_0$. Now let us consider the intersection $I^t_{p,\lambda}=D^{n-k}_\lambda(p)\cap\phi_t(\Omega)$ for each $p\in\mathrm{E}^k$ and $\lambda\leq\lambda_0$. First of all, one sees that each $I^t_{p,\lambda}$ is non-empty for $\lambda<\lambda_0$. This is a consequence of the fact that E^k is linked with each $\mathbb{S}^{n-k-1}_\lambda$ in $\mathbb{S}^n$ when λ is appropriately small, so E^k is still linked with D^{n-k}_λ in the ball $\mathbb{B}^{n+1}$. It is then clear that $I^t_{p,\lambda}$ is a connected, embedded convex ovaloid ($n-k\geq 3$), or a simple closed convex curve ($n-k=2$), or a single point ($n-k=1$), in a totally geodesic hyperbolic subspace $D^{n-k}_\lambda(p)$ when $\lambda\in(\frac{2}{3}\lambda_0,\lambda_0)$ and $t\geq t_0$. Another simple observation is the fact that each intersection $I^t_{p,\lambda}$ is compact, since the boundary at infinity $\mathbb{S}^{n-k-1}$ of $D^{n-k}_\lambda(p)$ does not intersect with the boundary at infinity $\partial\Omega$ of the hypersurface ϕ_t. Our theorem is true if, for any given $t\geq t_0$, we are able to show that each $I^t_{p,\lambda}$ is a connected, embedded, convex ovaloid ($n-k\geq 3$), or a simple closed convex curve ($n-k=2$), or a single point ($n-k=1$) for all $p\in\mathrm{E}^k$, $\lambda<\lambda_0$.

Let us first establish the cases $k=0$, i.e. E^k is a point $p\in\mathbb{S}^n$. We observe that the intersection of a complete, strictly convex, immersed, hypersurface and a totally geodesic hyperbolic hyperplane can only be a union of connected, convex, immersed hypersurfaces and possibly finitely many other points in the hyperbolic hyperplane. This is because a strictly convex hypersurface is either transversal to a totally geodesic hyperbolic hyperplane or locally stays strictly on one side of the totally geodesic hyperbolic hyperplane at the intersection point.

We claim that for any $\lambda<\lambda_0$ and $t\geq t_0$ each intersection $I^t_{p,\lambda}$ is a connected, immersed, compact, convex hypersurface ($n\geq 3$) or closed convex curve ($n=2$) in the totally geodesic hyperbolic hyperplane $D^n_\lambda(p)$. Hence our theorem, when $k=0$, follows from this claim and Do Carmo–Warner theorem [13]. Note that, in case $n=2$, we instead use the fact that a connected, immersed, convex, closed curve is embedded if it is a limit of connected, embedded, convex, closed curves.

It is clear that $I^t_{p,\lambda}$ is a connected, compact, embedded, convex ovaloid when λ is close to λ_0. This convex ovaloid stays as an embedded convex ovaloid before some points emerge in $I^t_{p,\lambda}$ as λ decreases from λ_0 in the light of Do Carmo–Warner theorem [13]. To show that no point ever emerges in $I^t_{p,\lambda}$ we may assume otherwise I^t_{p,λ_1} contains a point q for the first time as λ decreases from λ_0. One sees that the hyperbolic hyperplane $D^n_{\lambda_1}(p)$ is a support hyperplane at q for the hypersurface ϕ_t. It is clear that near q, the hypersurface ϕ_t lies locally on the side of the hyperplane $D^n_{\lambda_1}(p)$ that contains p and the normal to the hypersurface ϕ_t at q points to the same side due to the regularity of the Gauss map at infinity. But that would contradict (3.5). Therefore our theorem is proven when $k=0$.

The other extremal case is $k=n-1$. In this case $D^1_\lambda(p)$ is a geodesic with ends $(p,-\lambda)$ and (p,λ) in $\mathrm{N}_{\lambda_0}(\mathrm{E})\subset\mathbb{S}^n$. Instead of using hyperbolic hyperplanes we consider the ruled hypersurface $\Sigma_\lambda=\bigcup_{p\in\mathrm{E}} D^1_\lambda(p)$ and the intersection $I^t_\lambda=\bigcup_{p\in\mathrm{E}} I^t_{p,\lambda}$. Assume otherwise, that for the first time, for some λ_1, among all $p\in\mathrm{E}$ and λ decreasing from λ_0, the intersection I^t_{p,λ_1} contains more than just a single point. Then the hypersurface ϕ_t at the touch point, which has just emerged in I^t_{p,λ_1}, is tangent to the ruled hypersurface

Σ_{λ_1} and would possess some principal curvature nonnegative, which contradicts (3.5), similar to the situation dealt in the case $k = 0$.

For general k between 0 and $k-1$, we are going to use the combination of the above two special cases. We claim that each $I^t_{p,\lambda}$ is an embedded, convex ovaloid or a simple closed convex curve in the totally geodesic hyperbolic subspace $D^{n-k}_\lambda(p)$. Again we will use the ruled hypersurface

$$\Sigma^k_\lambda = \bigcup_{p\in \mathrm{E}^k} D^{n-k}_\lambda(p),$$

instead of hyperbolic hyperplanes. Assume otherwise, that for the first time, for some λ_1, among all $p \in \mathrm{E}^k$ and λ decreasing from λ_0, some points emerge in the intersection I^t_{p,λ_1} other than the connected immersed convex surface ($n - k \geq 3$) or the connected convex closed curve ($n - k = 2$). Then, similar to the case $k = n - 1$ above, the hypersurface ϕ_t at the touch point, which has just emerged in I^t_{p,λ_1}, is tangent to the ruled hypersurface $\Sigma^k_{\lambda_1}$ so would have some principal curvature nonnegative, which contradicts (3.5). This completes the proof. □

Remark 3.1. It is worth mentioning that the argument above is local in the sense that each component E^k is investigated independently. In other words, one may conclude that for t large enough the hypersurface ϕ_t is embedded near those ends which are of manifold structure. It is also worth mentioning that, in fact, with the above argument we have shown that each end has the structure

$$\mathrm{E}^k \times \mathbb{S}^{n-k-1} \times (0, \infty),$$

where $\mathbb{S}^{n-k-1}$ stands for a single point for $k = n - 1$.

4. Elliptic problems

In this section we compare the elliptic problems associated with Weingarten hypersurfaces in hyperbolic space $\mathbb{H}^{n+1}$ to those of conformal metrics on domains of the conformal infinity $\mathbb{S}^n$. Both subjects have a long history and have been extensively studied. Although they are mostly treated separately, there is a clear indication that these two subjects should be intimately related in terms of the types of problems and the tools that have been used to study them. Our work here is an attempt to give a unified framework for the two subjects with a hope that in doing so, it will shed light on further investigation and research. For instance, comparing Obata type theorems and Alexandrov type theorems, we derive a new Alexandrov type theorem, which does not assume the hypersurface to be embedded. Similarly, comparing Liouville type theorems and Bernstein type theorems, we also obtain some new results.

4.1. Corresponding elliptic problems

For a comprehensive introduction of conformally invariant elliptic PDE we refer readers to the papers [32,25,44,33,34] and references therein. We will briefly introduce the conformally invariant elliptic PDE in the context of our discussions. Since we focus on realizable conformal metrics, we denote

$$\mathcal{C} := \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_i < 1/2, i = 1, \cdots, n\}$$

and

$$\Gamma_n := \{(x_1, \cdots, x_n) : x_i > 0, i = 1, 2, \cdots, n\}.$$

Consider a symmetric function $f(x_1, \cdots, x_n)$ of n-variables with $f(\lambda_0, \lambda_0, \cdots, \lambda_0) = 0$ for some number $\lambda_0 < \frac{1}{2}$ and

$$\Gamma = \text{ an open connected component of } \{(x_1, \cdots, x_n) : f(x_1, \cdots, x_n) > 0\}$$

satisfying

$$(\lambda, \lambda, \cdots, \lambda) \in \Gamma \cap \mathcal{C}, \forall\ \lambda \in (\lambda_0, \frac{1}{2}), \tag{4.1}$$

$$\begin{aligned} &\forall\ (x_1, \cdots, x_n) \in \Gamma \cap \mathcal{C},\ \forall\ (y_1, \cdots, y_n) \in \Gamma \cap \mathcal{C} \cap ((x_1, \cdots, x_n) + \Gamma_n),\ \exists \text{ a curve } \gamma \\ &\text{connecting } (x_1, \cdots, x_n) \text{ to } (y_1, \cdots, y_n) \text{ inside } \Gamma \cap \mathcal{C} \text{ such that } \gamma' \in \Gamma_n \text{ along } \gamma, \end{aligned} \tag{4.2}$$

and

$$f \in C^1(\Gamma) \text{ and } \frac{\partial f}{\partial x_i} > 0 \text{ in } \Gamma. \tag{4.3}$$

Suppose $g = e^{2\rho} g_{\mathbb{S}^n}$ is a conformal metric on a domain Ω of $\mathbb{S}^n$ satisfying

$$f(\lambda(\mathrm{Sch}_g)) = C \text{ and } \lambda(\mathrm{Sch}_g) \in \Gamma \cap \mathcal{C} \text{ in } \Omega, \tag{4.4}$$

for some nonnegative constant C, where $\lambda(\mathrm{Sch}_g)$ is the set of eigenvalues of the Schouten curvature tensor of the metric g. We refer to Eq. (4.4) as the conformally invariant elliptic problem of the conformal metrics on the domain Ω.

Definition 4.1. In (4.4), a positive constant C is admissible for a given curvature function f if $f(\bar{\lambda}_0, \bar{\lambda}_0, \cdots, \bar{\lambda}_0) = C$, $\frac{\partial f}{\partial x_i}(\bar{\lambda}_0, \bar{\lambda}_0, \cdots, \bar{\lambda}_0) > 0$, and $\bar{\lambda}_0 > \lambda_0$.

On the other hand, we have the following elliptic problems of Weingarten hypersurfaces. For a comprehensive introduction of Weingarten hypersurfaces we refer to the papers [14,23,22,29] and references therein. We will briefly introduce the elliptic problems of Weingarten hypersurfaces in our context. Again, our focus is on admissible hypersurfaces with the canonical orientation. Let

$$\mathcal{K} := \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_i > -1, i = 1, \cdots, n\}.$$

Consider a symmetric function $\mathcal{W}(x_1, \cdots, x_n)$ of n-variables with $\mathcal{W}(\kappa_0, \kappa_0, \cdots, \kappa_0) = 0$ for some number $\kappa_0 > -1$ and

$$\Gamma^* = \text{ an open connected component of } \{(x_1, \cdots, x_n) : \mathcal{W}(x_1, \cdots, x_n) > 0\}$$

satisfying

$$(\kappa, \kappa, \cdots, \kappa) \in \Gamma^* \cap \mathcal{K}, \forall\ \kappa \in (\kappa_0, \infty), \tag{4.5}$$

$$\begin{aligned} &\forall\ (x_1, \cdots, x_n) \in \Gamma^* \cap \mathcal{K},\ \forall\ (y_1, \cdots, y_n) \in \Gamma^* \cap \mathcal{K} \cap ((x_1, \cdots, x_n) + \Gamma_n),\ \exists \text{ a curve} \\ &\gamma \text{ connecting } (x_1, \cdots, x_n) \text{ to } (y_1, \cdots, y_n) \text{ inside } \Gamma^* \cap \mathcal{K} \text{ such that } \gamma' \in \Gamma_n \text{ along } \gamma, \end{aligned} \tag{4.6}$$

and

$$\mathcal{W} \in C^1(\Gamma^*) \text{ and } \frac{\partial \mathcal{W}}{\partial x_i} > 0 \text{ in } \Gamma^*. \tag{4.7}$$

Suppose $\phi : \mathrm{M} \to \mathbb{H}^{n+1}$ is a hypersurface satisfying

$$\mathcal{W}(\kappa_1, \cdots, \kappa_n) = K \text{ and } (\kappa_1, \cdots, \kappa_n) \in \Gamma^* \cap \mathcal{K} \text{ on } \phi, \tag{4.8}$$

for some nonnegative constant K, where $(\kappa_1, \cdots, \kappa_n)$ is the set of principal curvatures of the hypersurface ϕ. We refer to Eq. (4.8) as the elliptic problem of Weingarten hypersurfaces.

Definition 4.2. In (4.8), a positive number K is admissible for a given curvature function $\mathcal{W}$ if $\mathcal{W}(\bar{\kappa}_0, \bar{\kappa}_0, \cdots, \bar{\kappa}_0) = K$, $\frac{\partial \mathcal{W}}{\partial x_i}(\bar{\kappa}_0, \bar{\kappa}_0, \cdots, \bar{\kappa}_0) > 0$, and $\bar{\kappa}_0 > \kappa_0$.

Remark 4.1. For the motivation of (4.2) and (4.6), please see the proof of Theorem 4.4, where (4.6) is shown to be sufficient to apply the Alexandrov reflection method. On the other hand, it is more appropriate to use curves instead of rays in (4.2) and (4.6), since the curvature relation is non-linear.

To relate these two elliptic problems, in the light of Theorem 2.1, we consider

$$\mathcal{T}(x_1, \cdots, x_n) = \left(\frac{1}{2} - \frac{1}{1 + x_1}, \cdots, \frac{1}{2} - \frac{1}{1 + x_n}\right) : \mathcal{K} \to \mathcal{C}. \tag{4.9}$$

Let us discuss the correspondence between conformally invariant elliptic problems of realizable metrics and elliptic problems of admissible Weingarten hypersurfaces. By our definitions, only $\Gamma \cap \mathcal{C}$ is relevant for a realizable metric and only $\Gamma^* \cap \mathcal{K}$ is relevant for an admissible hypersurface. Below we list some fundamental relations and facts for the correspondence between the elliptic problems of conformal metrics and Weingarten hypersurfaces.

Symmetric functions:

$$\mathcal{W}_f = f \circ \mathcal{T} \text{ and } f_{\mathcal{W}} = \mathcal{W} \circ \mathcal{T}^{-1}. \tag{4.10}$$

Domains:

$$\mathcal{T}(\Gamma^* \cap \mathcal{K}) = \Gamma \cap \mathcal{C}. \tag{4.11}$$

It is clear that

$$\mathcal{T}((\kappa_0, \kappa_0, \cdots, \kappa_0) + \Gamma_n) = ((\lambda_0, \lambda_0, \cdots, \lambda_0) + \Gamma_n) \cap \mathcal{C}, \tag{4.12}$$

where $\lambda_0 = \frac{1}{2} - \frac{1}{1+\kappa_0}$. In fact,

$$\mathcal{T}((x_1, \cdots, x_n) + \Gamma_n) = (\mathcal{T}(x_1, \cdots, x_n) + \Gamma_n) \cap \mathcal{C} \tag{4.13}$$

for all $(x_1, \cdots, x_n) \in \mathcal{K}$. Therefore (4.1) holds for $\Gamma \cap \mathcal{C}$ if and only if (4.5) holds for $\Gamma^* \cap \mathcal{K}$. Moreover, we also see (4.2) holds for $\Gamma \cap \mathcal{C}$ if and only if (4.6) holds for $\Gamma^* \cap \mathcal{K}$.

Ellipticity:

$$\begin{aligned} &\frac{\partial \mathcal{W}_f}{\partial \kappa_i} > 0 \text{ in } \Gamma^* \cap \mathcal{K} \text{ if and only if } \frac{\partial f}{\partial \lambda_i} > 0 \text{ in } \Gamma \cap \mathcal{C}. \\ &\frac{\partial f_{\mathcal{W}}}{\partial \lambda_i} > 0 \text{ in } \Gamma \cap \mathcal{C} \text{ if and only if } \frac{\partial \mathcal{W}}{\partial \kappa_i} > 0 \text{ in } \Gamma^* \cap \mathcal{K}. \end{aligned} \tag{4.14}$$

So, the above discussion allows us to announce:

Theorem 4.1. *There is an one-to-one correspondence between elliptic Weingarten equations for admissible horospherically concave hypersurfaces in hyperbolic space $\mathbb{H}^{n+1}$ and elliptic Yamabe problems for realizable conformal metrics on $\mathbb{S}^n$.*

We continue relating certain properties of the functionals:

Homogeneity:

Homogeneity of symmetric functions is not preserved under this correspondence. In fact, scaling on the metric side corresponds to deforming along the normal flow in hypersurface side.

Concavity:

The concavity on the other hand is preserved under this correspondence from f to $\mathcal{W}$, but not necessarily from $\mathcal{W}$ to f. The concavity of a function is understood to be the nonpositivity of the Hessian matrix. One may simply calculate that

$$\frac{\partial^2 \mathcal{W}_f}{\partial \kappa_i \partial \kappa_j} = \frac{1}{(1+\kappa_i)^2(1+\kappa_j)^2} \frac{\partial^2 f}{\partial \lambda_i \partial \lambda_j} - \frac{2\delta_{ij}}{(1+\kappa_i)^3} \frac{\partial f}{\partial \lambda_i}$$

and

$$\frac{\partial^2 f_{\mathcal{W}}}{\partial \lambda_i \partial \lambda_j} = \frac{1}{(\frac{1}{2}-\lambda_i)^2(\frac{1}{2}-\lambda_j)^2} \frac{\partial^2 \mathcal{W}}{\partial \kappa_i \partial \kappa_j} + \frac{2\delta_{ij}}{(\frac{1}{2}-\lambda_i)^3} \frac{\partial \mathcal{W}}{\partial \kappa_i}.$$

Hence, instead, the convexity is preserved under this correspondence from $\mathcal{W}$ to f.

Admissible constants:

In (4.4), a positive constant C is admissible for a given curvature function f if $f(\bar{\lambda}_0, \bar{\lambda}_0, \cdots, \bar{\lambda}_0) = C$, $\frac{\partial f}{\partial x_i}(\bar{\lambda}_0, \cdots, \bar{\lambda}_0) > 0$, and $\bar{\lambda}_0 > \lambda_0$, while in (4.8), a positive constant K is admissible for a given curvature function $\mathcal{W}$ if $\mathcal{W}(\bar{\kappa}_0, \cdots, \bar{\kappa}_0) = K$, $\frac{\partial \mathcal{W}}{\partial x_i}(\bar{\kappa}_0, \cdots, \bar{\kappa}_0) > 0$, and $\bar{\kappa}_0 > \kappa_0$, where $\bar{\lambda}_0 = \frac{1}{2} - \frac{1}{1+\bar{\kappa}_0}$. Geometrically it means that the horospherical metric of a geodesic sphere of principal curvature $\kappa > 1$ is of constant sectional curvature < 1; the horospherical metric of a horosphere is of zero sectional curvature; and the horospherical metric of a hypersphere of principal curvature $\kappa \in [0, 1)$ is of negative constant sectional curvature.

Scalar curvature vs mean curvature:

In the context of solving elliptic problems one typically assumes

$$\Gamma \subset \Gamma_1 = \{(x_1, \cdots, x_n) : \sum_{i=1}^n x_i \geq 0\}$$

and

$$\Gamma^* \subset \Gamma_1^* = \{(x_1, \cdots, x_n) : \sum_{i=1}^n x_i \geq n\}.$$

In contrast to $\mathcal{T}(\Gamma_n^*) = \Gamma_n \cap \mathcal{C}$, we only have

$$\mathcal{T}^{-1}(\Gamma_1) \subset \Gamma_1^*. \tag{4.15}$$

Therefore, we only have $\Gamma \cap \mathcal{C} \subset \Gamma_1$ implies $\Gamma^* \cap \mathcal{K} \subset \Gamma_1^*$, but not necessarily the converse.

Before we end this subsection we would like to give a proof of (4.15). This turns out to be a consequence of the following simple algebraic fact.

Lemma 4.1. *Let a_i be real numbers so that $a_i > -1$ for all $i = 1, \cdots, n$. Set $b_i = \dfrac{a_i - 1}{a_i + 1}$. Then*

$$\sum_{i=1}^{n} b_i \leq 2\sum_{i=1}^{n} a_i - n.$$

Proof. Set $Q(t) = \sum_{i=1}^{n} \dfrac{a_i}{1 + a_i t}$ for $0 \leq t \leq 1$. Then,

$$Q'(t) = -\sum_{i=1}^{n} \frac{a_i^2}{(1 + a_i t)^2} \leq 0,$$

which implies $Q(1) \leq Q(0) = \sum_{i=1}^{n} a_i$. On the other hand,

$$\begin{aligned}\sum_{i=1}^{n} b_i &= \sum_{i=1}^{n} \frac{a_i - 1}{a_i + 1} = 2\sum_{i=1}^{n} \frac{a_i}{a_i + 1} - \sum_{i=1}^{n} \frac{a_i + 1}{a_i + 1} \\ &= 2Q(1) - n.\end{aligned}$$

Therefore, the lemma is easily seen. $\square$

Let $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$ be a horospherically concave hypersurface with the canonical orientation. Then, λ_i and κ_i are related by

$$2\lambda_i = \frac{\kappa_i - 1}{\kappa_i + 1}, \tag{4.16}$$

or equivalently,

$$\kappa_i = \frac{1 + 2\lambda_i}{1 - 2\lambda_i}. \tag{4.17}$$

Set $a_i := -2\lambda_i$ and $b_i := -\kappa_i$. Note that since $\lambda_i < 1/2$, then $1 - 2\lambda_i > 0$, that is, $a_i > -1$. Therefore, from (4.16) it follows

$$-\sum_{i=1}^{n} \kappa_i \leq -4\sum_{i=1}^{n} \lambda_i - n,$$

so that

$$\sum_{i=1}^{n} \lambda_i \geq 0 \quad \text{implies} \quad \sum_{i=1}^{n} \kappa_i \geq n, \tag{4.18}$$

which in turn implies (4.15).

4.2. Obata theorem vs Alexandrov theorem

Here we establish an explicit relationship between a famous theorem in conformal geometry and a famous theorem in hypersurface theory. Namely, the Obata theorem and the Alexandrov theorem (cf. [19]). First, let us state the aforementioned results. For conformal metrics, we have the following:

Obata theorem [38,24]: *Let g be a metric conformal to the standard round metric $g_{\mathbb{S}^n}$ on $\mathbb{S}^n$ with constant positive scalar curvature. Then, there exists a conformal diffeomorphism $\Phi : \mathbb{S}^n \to \mathbb{S}^n$ and a positive constant $a > 0$ such that $g = a\,\Phi^* g_{\mathbb{S}^n}$.*

Its generalization to fully nonlinear elliptic functions (f, Γ) is as follows:

Generalized Obata theorem [33]: *Let g be a metric conformal to the standard round metric $g_{\mathbb{S}^n}$ on $\mathbb{S}^n$. Suppose that (f, Γ) is elliptic in the sense that it satisfies (4.1)–(4.3) and that*

$$f(\lambda(\mathrm{Sch}_g)) = C,\ \lambda(\mathrm{Sch}_g) \in \Gamma$$

for a positive constant C. Then, there exists a conformal diffeomorphism $\Phi : \mathbb{S}^n \to \mathbb{S}^n$ and a positive constant a such that $g = a\,\Phi^* g_{\mathbb{S}^n}$.

For hypersurfaces in hyperbolic space we have the following:

Alexandrov theorem [3,27]: *Let $\Sigma \subset \mathbb{H}^{n+1}$ be a compact (without boundary) embedded hypersurface with constant mean curvature. Then, Σ is a totally umbilical round sphere.*

Its generalization to elliptic Weingarten hypersurfaces $(\mathcal{W}, \Gamma^*)$ is as follows:

Generalized Alexandrov theorem [29]: *Let $\Sigma \in \mathbb{H}^{n+1}$ be compact (without boundary) embedded hypersurface. Suppose that $(\mathcal{W}, \Gamma^*)$ satisfies (4.5)–(4.7) and that*

$$\mathcal{W}(\kappa_1, \cdots, \kappa_n) = K \text{ and } (\kappa_1, \cdots, \kappa_n) \in \Gamma^*$$

on Σ, where K is a positive constant. Then Σ is totally umbilical round sphere.

In the light of the correspondence observed in Theorem 2.1 of [19] and the discussions in Section 4.1 we obtain a new Alexandrov type theorem for horospherical ovaloids as an equivalent statement of the generalized Obata theorem of Li–Li [33,34] above (notice that a conformal metric on $\mathbb{S}^n$ is always realizable). But due to our Corollary 3.4, it can be seen as a consequence of the generalized Alexandrov theorem of Korevaar [29]. Thus, such new Alexandrov type theorem becomes the bridge connecting the two sides and it is interesting to see that the generalized Alexandrov theorem of Korevaar implies the generalized Obata theorem of Li–Li [33,34], instead of the other way around as given in [19].

Theorem 4.2. *Suppose that* $(\mathcal{W}, \Gamma^*)$ *satisfies* (4.5)–(4.7)*. Then a horospherical ovaloid in* $\mathbb{H}^{n+1}$ *with the canonical orientation satisfying* (4.8) *for a positive constant is a geodesic sphere in* $\mathbb{H}^{n+1}$*. Equivalently, suppose that* (f, Γ) *satisfies* (4.1)–(4.3)*. Then any conformal metric on* $\mathbb{S}^n$ *satisfying* (4.4) *for a positive constant is isometric to a round metric on* $\mathbb{S}^n$*.*

Proof. According to Theorem 3.6, the horospherical ovaloid Σ_t along the (past) normal flow of the given horospherical ovaloid Σ becomes embedded when $t \geq t_0$ for some t_0. Moreover,

$$\kappa_i = \frac{\kappa_i^t - \tanh(t)}{1 - \kappa_i^t \tanh(t)} \quad \text{and} \quad \kappa_i^t = \frac{\tanh(t) + \kappa_i}{1 + \kappa_i \tanh(t)}.$$

Hence, Σ_t is still an elliptic Weingarten hypersurface for all $t > t_0$. To see this, we let

$$\mathcal{W}^t(x_1, \cdots, x_n) := \mathcal{W}\left(\frac{x_1 - \tanh(t)}{1 - x_1 \tanh(t)}, \cdots, \frac{x_n - \tanh(t)}{1 - x_n \tanh(t)}\right).$$

Therefore, $\mathcal{W}^t$ is a symmetric function of n-variables with $\mathcal{W}^t(1, \cdots, 1) = 0$. Let

$$T(x_1, \cdots, x_n) = \left(\frac{\tanh(t) + x_1}{1 + x_1 \tanh(t)}, \cdots, \frac{\tanh(t) + x_n}{1 + x_n \tanh(t)}\right).$$

We then have

$$\Gamma_t^* \cap \mathcal{K} = T(\Gamma^* \cap \mathcal{K}).$$

Similar to the case of the map $\mathcal{T}$, we in fact have

$$T((x_1, \cdots, x_n) + \Gamma_n) = T(x_1, \cdots, x_n) + \Gamma_n$$

for all $(x_1, \cdots, x_n) \in \mathcal{K}$. For ellipticity we easily calculate

$$\frac{\partial \mathcal{W}^t}{\partial x_i} = \frac{1 - \tanh^2(t)}{(1 + x_i \tanh(t))^2} \frac{\partial \mathcal{W}}{\partial y_i}.$$

Therefore, $(\mathcal{W}^t, \Gamma_t^*)$ satisfies (4.5)–(4.7). Thus, from the above generalized Alexandrov theorem of Korevaar [29], Σ_t is a totally umbilical round sphere for $t \geq t_0$, and therefore so is Σ. □

4.3. Liouville theorem vs Bernstein theorem

Next to compact hypersurfaces in $\mathbb{H}^{n+1}$, the simplest noncompact hypersurfaces in $\mathbb{H}^{n+1}$ are those that have a single point at the infinity $\mathbb{S}^n$. Their corresponding domains in $\mathbb{S}^n$ are punctured spheres $\mathbb{S}^n \setminus \{\mathbf{n}\}$. In this context, we establish an explicit relationship between another pair of celebrated theorems in conformal geometry and in hypersurface theory: Liouville type theorems and Bernstein type theorems. We focus on the cases where the positive constants in (4.4) and (4.8) are admissible and elliptic equations are non-degenerate. First, let us state the aforementioned results.

> **Liouville theorem [7]:** *The only complete conformal metrics on $\mathbb{S}^n \setminus \{\mathbf{n}\}$ with non-negative constant scalar curvature is the Euclidean metric.*

Its generalization to fully nonlinear non-degenerate elliptic functions (f, Γ) is as follows:

> **Generalized Liouville theorem [34]:** *Suppose that (f, Γ) is elliptic in the sense that it satisfies (4.1)–(4.3). Then the only possible complete conformal metric of nonnegative scalar curvature on $\mathbb{S}^n \setminus \{\mathbf{N}\}$ satisfying (4.4) for an admissible constant is the Euclidean metric.*

We remark that the above theorems are simplified versions of Theorem 1.4 in [33] and Theorem 1.3 in [34]. In fact, the hypothesis about the completeness of the metric is not necessary (see [34]). We have preferred continue to keep this simplified version since geometrically is more natural.

For hypersurfaces in hyperbolic space we have the following:

> **Bernstein theorem [4,12]:** *The only properly embedded, complete, constant mean curvature $H \geq n$ hypersurfaces with one point at infinity in $\mathbb{H}^{n+1}$ are horospheres.*

As far as we know, the above result has been generalized for special Weingarten surfaces in $\mathbb{H}^3$ (see [39,40] and [1]), but not for higher dimensions. In fact, we can prove:

Theorem 4.3. *Suppose that $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$ is an immersed, complete, horospherically concave hypersurface with constant mean curvature $H = \sum_{i=1}^n \kappa_i$ and*

$$\sum_{i=1}^{n} \frac{2}{1+\kappa_i} \leq n. \tag{4.19}$$

Then it is a horosphere if its boundary at infinity is a single point in $\mathbb{S}^n$.

The condition (4.19) says that the hyperbolic Gauss map is injective (Theorem 3.3), and therefore, Theorem 3.6 and Theorem 4.1 imply that ϕ_t, for some t big enough, is an embedded horospherically concave hypersurface whose boundary at infinity is one point and satisfies an elliptic Weingarten equation. Therefore, proof follows from the following generalized Bernstein theorem:

Theorem 4.4 *(Generalized Bernstein theorem). Suppose that* $(\mathcal{W}, \Gamma^*)$ *is an elliptic function satisfying (4.5)–(4.7). Then the only possible properly embedded, complete hypersurface in* $\mathbb{H}^{n+1}$ *satisfying (4.8) for an admissible constant with only one point at infinity is a horosphere.*

Proof. The proof is more or less the same as the proof of Theorem A given in Section 2 of [12]. The readers are referred to [12] for more details. It is particularly helpful to use Figs. 1 and 2 in Section 2 of [12]. However, we would like to take this opportunity to clarify that our assumptions (4.5)–(4.7) are sufficient for the argument.

Suppose that there is such surface Σ in $\mathbb{H}^{n+1}$. We use the half space model for hyperbolic space $\mathbb{H}^{n+1}$ so that the single point infinity of the surface Σ is at the infinity of $\mathbb{R}^{n+1}_+$. Let γ be any vertical line in the half space $\mathbb{R}^{n+1}_+$. Then γ is a complete geodesic in $\mathbb{H}^{n+1}$. Let P_t denote the foliation of totally geodesic hyperplanes orthogonal to γ passing through $\gamma(t)$. Since Σ is properly embedded with only one point at its boundary at infinity, $\mathbb{H}^{n+1} \setminus \Sigma$ has two connected components, one component $\mathcal{U}$ containing the set

$$\partial_\infty \mathcal{U} = \left\{(x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} \ : \ x_{n+1} = 0\right\}$$

at its boundary at infinity, and the other component $\mathcal{O}$ containing the point at infinity of Σ on its boundary at infinity, $\partial_\infty \mathcal{O} = \partial_\infty \Sigma$. Let us see now that $\mathcal{O}$ is the connected component where the normal vector field η points. Now, since Σ is horospherically concave and the image of the hyperbolic Gauss map $G : \Sigma \to \partial_\infty \mathbb{H}^{n+1}$ is equals to

$$G(\Sigma) = \left\{(x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} \ : \ x_{n+1} = 0\right\},$$

then η must point into $\mathcal{O}$.

Moreover, since Σ is properly embedded, there exists t_0 so that $\Sigma \cap P_t = \emptyset$ for all $t < t_0$ and t_0 is the first time that P_t touches Σ. At this first point of contact, Σ is locally a graph over P_{t_0}. We can raise P_t further for $t > t_0$ to obtain $\Sigma_-(t) := \Sigma \cap \bigcup_{s<t} P_s$, which is a graph of bounded slope over P_t at least for t close to t_0. Now we reflect $\Sigma_-(t)$ with respect to P_t and denote the reflection by $\tilde{\Sigma}_-(t)$. Note that $\tilde{\Sigma}_-(t) \subset \mathcal{O}$ for all $t > t_0$.

The proof of Theorem A in [12] is based on the fact that the reflection $\tilde{\Sigma}_-(t)$ can never touch the rest of the surface $\Sigma_+(t) := \Sigma \setminus \Sigma_-(t)$, not even at the boundary of the reflection $\tilde{\Sigma}_-(t)$. Here, by touch, we mean either their normal vector fields coincide at an interior contact point or their normals and conormals coincide at a boundary tangent point.

Assumption (4.5) allows us to define admissible constants K. Assume otherwise the reflection $\tilde{\Sigma}_-(t)$ does touch the rest $\Sigma_+(t)$. When $\tilde{\Sigma}_-(t)$ first touches $\Sigma_+(t)$, with respect to the normal η to Σ, $\tilde{\Sigma}_-(t)$ is above $\Sigma_+(t)$. Let us denote the principal curvatures $(\kappa_1^-, \cdots, \kappa_n^-)$ and $(\kappa_1^+, \cdots, \kappa_n^+)$ of $\tilde{\Sigma}_-(t)$ and $\Sigma_+(t)$, respectively, at the touch point. Recall that, from (4.5), we have

$$(\kappa_1^+, \cdots, \kappa_n^+) \in \Gamma^* \cap ((\kappa_1^-, \cdots, \kappa_n^-) + \Gamma_n).$$

Then, from (4.6), one may connect the principal curvature of the two surface at the contact point by a curve whose velocity stays in the positive cone all the time and conclude that the two principal curvatures at the contact point are the same $(\kappa_1^-, \cdots, \kappa_n^-) = (\kappa_1^+, \cdots, \kappa_n^+)$ due to (4.7) and $\mathcal{W}(\kappa_1^-, \cdots, \kappa_n^-) = \mathcal{W}(\kappa_1^+, \cdots, \kappa_n^+)$.

Then, using the openness of the domain of the curvature function, one simply need to work with surfaces whose principal curvatures are lying in the convex open neighborhood of the principal curvature at the contact point inside the domain of the curvature function. One may assume the two surfaces near by the contact point are graphs over the tangent space at the contact point (interior or boundary) and apply the usual Maximum Principle. Then the Hopf maximum principle is applicable and gives the fundamental proposition of the Alexandrov reflection method due to the assumption (4.7) (cf. [29]).

Having explained the use of our assumptions (4.5)–(4.7), we now briefly recapture the idea in the proof of Theorem A in [12]. First one proves that the reflection $\tilde{\Sigma}_-(t)$ can never touch the rest of the surface $\Sigma_+(t)$. Second one observes that if the surface is not a horosphere in $\mathbb{H}^{n+1}$ (i.e. a horizontal hyperplane in $\mathbb{R}^{n+1}_+$), then the incident that the reflection touches the rest of the surface at the boundary for the first time always happens for some vertical line (cf. Fig. 2 in Section 2 of [12]). This completes the proof. $\square$

As a consequence of Theorem 3.1 and Theorem 3.6, we have the following:

Corollary 4.1. *Suppose that $(\mathcal{W}, \Gamma^*)$ is an elliptic function satisfying (4.5)–(4.7). Then the only possible admissible hypersurface in $\mathbb{H}^{n+1}$ with the canonical orientation and a single point at infinity satisfying (4.8) for an admissible constant is a horosphere.*

Equivalently, suppose that (f, Γ) satisfies (4.1)–(4.3). Then the only possible realizable metric on $\mathbb{S}^n \setminus \{p\}$ satisfying (4.4) for an admissible constant is the Euclidean metric.

4.4. General symmetry

In this subsection we derive a slight extension of [31, Theorem 2.1]. As a consequence, we will derive Delaunay type theorems for admissible hypersurafces as well as realizable metrics.

First we introduce some notation. Let us denote the group of conformal transformations on $\mathbb{S}^n$ by $\mathrm{Con}(\mathbb{S}^n)$ and the group of isometries of $\mathbb{H}^{n+1}$ by $\mathrm{Iso}(\mathbb{H}^{n+1})$. Let g be a conformal metric on a domain $\Omega \subset \mathbb{S}^n$. We say that g is Φ-invariant if

$$\Phi : \Omega \to \Omega \text{ and } g = \Phi^* g$$

for $\Phi \in \mathrm{Con}(\mathbb{S}^n)$. And we say that a hypersurface Σ in $\mathbb{H}^{n+1}$ is I-invariant if

$$I : \Sigma \to \Sigma$$

for $I \in \mathrm{Iso}(\mathbb{H}^{n+1})$. Remember that a conformal transformation $\Phi \in \mathrm{Con}(\mathbb{S}^n)$ induces a unique isometry $I \in \mathrm{Iso}(\mathbb{H}^{n+1})$ and vice versa (cf. [11], for instance). The following fact can be verified and appeared in [18]:

Lemma 4.2. *Let $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$ be an admissible hypersurface in $\mathbb{H}^{n+1}$ and g be corresponding realizable metric on $G(\mathrm{M})$. Let $I \in \mathrm{Iso}(\mathbb{H}^{n+1})$ be an isometry and $\Phi \in \mathrm{Con}(\mathbb{S}^n)$ be the associated conformal transformation. Then ϕ is I-invariant if and only if g is Φ-invariant.*

From this fact we know that symmetries are preserved under the correspondence between admissible hypersurfaces and realizable metrics. Our issue here is to retain the symmetry for a complete hypersurface in $\mathbb{H}^{n+1}$ from that of its boundary at infinity or equivalently to retain the symmetry for a complete conformal metric on a domain in $\mathbb{S}^n$ from that of the domain. The proof of the following slight extension of [31, Theorem 2.1] is readily seen from the original proof in [31] and from the proof of Theorem 4.4. To state our theorem we introduce some more notation. Let E be the equator in $\mathbb{S}^n$ and P be the totally geodesic hyperplane whose boundary is E. Let R stand for the reflection in $\mathbb{H}^{n+1}$ with respect to the hyperplane P, and $\Phi : \mathbb{S}^n \to \mathbb{S}^n$ the unique conformal transformation induced by R.

Theorem 4.5. *Suppose that $(\mathcal{W}, \Gamma^*)$ satisfies (4.5)–(4.7). Let $\Sigma \subset \mathbb{H}^{n+1}$ be a properly embedded hypersurface whose boundary $\partial_\infty \Sigma$ at the infinity is in E. Assume that Σ is an elliptic Weingarten hypersurface in the sense that Eq. (4.8) holds on Σ for an admissible constant K. Then $\partial_\infty \Sigma$ cannot be all of E and the surface Σ is R-invariant.*

Hence, from the above result, we conclude the following general Alexandrov reflection principle for both admissible hypersurfaces and realizable metrics:

Theorem 4.6. *Suppose that $(\mathcal{W}, \Gamma^*)$ satisfies (4.5)–(4.7). Let $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$ be an admissible hypersurface with the canonical orientation satisfying (4.8), whose boundary $\partial_\infty \phi(\mathrm{M}^n)$ at the infinity is a disjoint union of smooth compact submanifolds with no boundary in E. Then $\partial_\infty \phi(\mathrm{M}^n)$ cannot be E and the surface ϕ is R-invariant.*

Equivalently, suppose that (f, Γ) satisfies (4.1)–(4.3). Let g be a realizable metric satisfying (4.4) on Ω such that $\partial\Omega \subset E$ is a disjoint union of smooth compact submanifolds with no boundary. Then $\partial\Omega$ cannot be E and g is Φ-invariant.

There are many consequences of Theorem 4.6. In particular, when the boundary at infinity consists of exactly two points, we obtain the following Delaunay type theorem.

Corollary 4.2. *Suppose that* $(\mathcal{W}, \Gamma^*)$ *satisfies* (4.5)–(4.7). *Let* $\phi : \mathrm{M}^n \to \mathbb{H}^{n+1}$ *be an admissible hypersurface with the canonical orientation satisfying* (4.8), *whose boundary* $\partial_\infty \phi(\mathrm{M}^n)$ *at the infinity consists of exactly two points. Then the surface* ϕ *is rotationally symmetric with respect to the geodesic joining the two points at the infinity of* ϕ.

Equivalently, suppose that (f, Γ) *satisfies* (4.1)–(4.3). *Let* g *be a realizable metric satisfying* (4.4) *on* $\Omega = \mathbb{S}^n \setminus \{p, q\}$. *Then* g *is cylindric with respect to the geodesic joining the two points in* $\partial\Omega$.

In the theory of hypersurfaces in hyperbolic space, Delaunay type theorems were established in [28,31] for constant mean curvature surfaces, and in [1,39,40] for special Weingarten surfaces in $\mathbb{H}^3$. Also Corollary 4.2 should be compared with Theorem 1.2 in [32], where the scalar curvature is assumed to be nonnegative.

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