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PERFECT POWERS IN AN ALTERNATING SUM OF CONSECUTIVE CUBES

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Abstract. In this paper, we consider the problem about finding out perfect powers in an alternating sum of consecutive cubes. More precisely, we completely solve the Diophantine equation $(x+1)^3 - (x+2)^3 + \cdots - (x+2d)^3 + (x+2d+1)^3 = z^p$, where p is prime and x, d, z are integers with $1 \le d \le 50$.

1. Introduction

In 1964, Leveque ([11]) proved that, if $f(x) \in \mathbb{Z}[x]$ is a polynomial of degree $k \ge 2$ with at least two simple roots, and $n \ge \max\{2, 5-k\}$ is an integer, then the superelliptic equation

$$f(x) = z^n \tag{1.1}$$

has at most finitely many solutions in integers x and z. In 1976, this result was extended by Schinzel and Tijdeman ([17]). They proved that the equation (1.1) has at most finitely many solutions in integers x, z and variable $n \ge \max\{2, 5-k\}$ through an application of lower bounds for linear forms in logarithms.

Earlier in 1875, Lucas ([12]) considered the Diophantine equation

$$1^2 + 2^2 + \cdots + x^2 = y^2, \tag{1.2}$$

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and claimed that $(1,1)$ and $(24,70)$ are the only solutions in positive integers (x,y) to the equation (1.2). In 1918, Watson ([22]) completely solved the equation (1.2).

In 1956, Schäffer ([16]) studied the more general equation

$$1^k + 2^k + \cdots + x^k = y^n. \tag{1.3}$$

It is easy to see that for every k and n, $(x,y)=(1,1)$ is a solution of (1.3). Schäffer ([16]) proved that if $k \geq 1$ and $n \geq 2$ are fixed, then (1.3) has only finitely many solutions except the following cases

$$(k,n) \in \{(1,2),(3,2),(3,4),(5,2)\}. \tag{1.4}$$

In the same paper Schäffer gave a conjecture regarding the integral solutions of (1.3). He conjectured that, for $k \geq 1$ and $n \geq 2$ with (k,n) not in the set (1.4), equation (1.3) has only one non-trivial solution, namely $(k,n,x,y) = (2,2,24,70)$. There are some results, at least in principle, to determine all solutions of (1.3).

Jacobson, Pintér, Walsh ([8]) confirmed the conjecture for $n = 2$ and k even with $k \leq 58$. Recently, Bennett, Győry, Pintér ([1]) proved completely the Schäffer conjecture for arbitrary n and $k \leq 11$. As an extension of [1], Pintér ([14]) proved Schäffer conjecture for odd values of k with $1 \leq k \leq 170$ and even values of n.

Zhang and Bai ([24]) generalized the equation (1.3) and considered the more general equation

$$(x+1)^k + (x+2)^k + \cdots + (x+d)^k = y^n, \tag{1.5}$$

for $k \geq 2$. They completely solved the equation (1.5) for $k = 2$ and $d = x$. For $k = 2$, they also proved that for a prime $p \equiv \pm 5 \pmod{12}$ with $p \mid d$ and $\nu_p(d) \not\equiv 0 \pmod{n}$, the equation (1.5) has no integer solution.

Recently, Soydan ([20]) considered the equation (1.5) for $k \geq 2$ and $d = lx$ for some integer $l \geq 2$. He proved that all solutions of the equation (1.5) in integers $x, y \geq 1$ and $n \geq 2$ satisfy $n < C$, where C is an effectively computable constant depending only on l and k. He also proved that for $k \neq 3$ all solutions of the equation (1.5) in integers x, y, n with $x, y \geq 1, n \geq 2$ and $l \equiv 0 \pmod 2$ satisfy $\max\{x,y,n\} < C_1$ where C_1 is an effectively computable constant depending only on l and k.

Cassels ([5]) solved the equation (1.5) completely for $n = 2, d = 3$ and $k = 3$. Zhang ([25]) determined the perfect powers in sum of three consecutive cubes by rewriting the equation (1.5) for $k = d = 3$ as

$$(x-1)^3 + x^3 + (x-1)^3 = y^n. \tag{1.6}$$

Stroeker ([21]) completely solved the equation (1.5) for $k = 3, n = 2$ and $2 \leq d \leq 50$ using linear forms in elliptic logarithms. Recently, Bennett, Patel and Siksek ([3]) extended the result of Stroeker for $n \geq 3$.

Several generalizations of (1.3) have been considered by different authors. For example Dilcher ([7]) studied the equation

$$\chi(1)1^k + \chi(2)2^k + \cdots + \chi(xf)(xf)^k = by^n, \tag{1.7}$$

where χ is a primitive quadratic residue class character with conductor f and $k, b \neq 0$ are fixed integers. This may be viewed as a *character-twisted* analogue of a classic equation of Schäffer. Recently, Bennett ([2]) completely solved the equation

$$1^k - 3^k + 5^k - \cdots + (4x-3)^k - (4x-1)^k = -y^n \tag{1.8}$$

for $3 \leq k \leq 6$.

In this paper we consider the following Diophantine equation

$$(x+1)^3 - (x+2)^3 + \cdots + (-1)^{m-1}(x+m)^3 = z^p, \tag{1.9}$$

where m, x, z are integers with $m \geq 2$ and p is any prime number.

We note that for a fixed ordered tuple (m, p), it is easy to conclude that equation (1.9) has only finitely many solutions. Since we are dealing with an infinite collection of tuples (m, p) in our case it is not obvious that there are finitely many solutions. On top of that we also find the precise solutions of equation (1.9) using a combination of both classical and modern techniques in Diophantine analysis.

Simplifying (1.9), for odd m we obtain

$$\left(x + \frac{m+1}{2}\right)\left\{\left(x + \frac{m+1}{2}\right)^2 + 3\frac{m^2-1}{4}\right\} = z^p. \tag{1.10}$$

Putting $m = 2d+1$ for some positive integer d, we have

$$(x+d+1)\left\{(x+d+1)^2 + 3d(d+1)\right\} = z^p. \tag{1.11}$$

From the equation (1.11), we can see that $\gcd((x+d+1), (x+d+1)^2+3d(d+1))$ divides $3d(d+1)$. Hence

$$x+d+1 = \alpha {z_1}^p \quad \text{and} \quad (x+d+1)^2 + 3d(d+1) = \beta {z_2}^p \tag{1.12}$$

for some integers z_1, z_2 and rationals α, β with $\alpha\beta = 1$ and $z_1 z_2 = z$. The denominator and the numerator of α and β are composed of prime divisors of $3d(d+1)$. From (1.11) and (1.12), we deduce the following ternary equation

$$\beta {z_2}^p - \alpha^2 z_1^{2p} = 3d(d+1). \tag{1.13}$$

If $\beta < 0$, then from the equation (1.12), we have $z_2 < 0$. Also $\alpha < 0$ as $\alpha\beta = 1$. Hence, $(\pm z_1, z_2)$ is an integral solution of equation (1.13) corresponding to (α, β) if and only if $(\pm z_1, -z_2)$ is an integral solution of equation (1.13) corresponding to $(-\alpha, -\beta)$. Therefore it is enough to solve the equation (1.13) for $\beta > 0$.

Suppose S_d is the set of such pairs of positive rationals (α, β). We need to solve the equation (1.13) for each $(\alpha, \beta) \in S_d$ with $1 \leq d \leq 50$. Clearing denominators we can rewrite the equation (1.13) as

$$rz_2{}^p - sz_1^{2p} = t, \tag{1.14}$$

where r, s, t are positive integers and $\gcd(r, s, t) = 1$.

Now we state our main theorem as follows.

THEOREM 1.1. *For $m = 2d + 1$ with $1 \leq d \leq 50$, the integral solutions (x, z, p) of the equation* (1.9) *are given in the Table 1.*

REMARK 1.2. If $z = 0$, then from the equation (1.11), we have $x = -(d+1)$ as $(x+d+1)^2+3d(d+1) > 0$ for any d. Therefore, $(x, z, p) = (-d-1, 0, p)$ are the trivial solutions of the equation (1.9) for any d.

REMARK 1.3. From Theorem 1.1, it is clear that for $p > 7$, there is no integral solution for the equation (1.9). For $p = 5$, $(d, x, z) = (20, -15, 6)$ is the only integral solution for the equation (1.9). For $p = 7$, $(d, x, z) \in \{(4, -3, 2), (15, -13, 3), (27, 26, 6)\}$.

2. Preliminaries

We use well known tools such as linear forms in two logarithms, variation of Kraus criterion, modular method, local solubility, descent to prove Theorem 1.1. In this section we provide the necessary details for these methods.

Linear forms in 2 *logarithms:* We state a special case of the following well known result of Laurent ([10]).

PROPOSITION 2.1 ([10, Corollary 2]). *Let α_1 and α_2 be two positive real, multiplicatively independent algebraic numbers and $\log \alpha_1, \log \alpha_2$ be any fixed determinations of the logarithms that are real and positive. Write $D = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}]$ and*

$$b' = \frac{b_1}{D \log A_2} + \frac{b_2}{D \log A_1},$$

where b_1, b_2 are positive integers and A_1, A_2 are real numbers greater than one such that

$$\log A_i \geq \max\left\{h(\alpha_i), \frac{|\log \alpha_i|}{D}, \frac{1}{D}\right\}, \quad i = 1, 2$$

with

$$h(\alpha) = \frac{1}{d}\left(\log |a| + \sum_{i=1}^{d} \log \max(1, |\alpha^{(i)}|)\right),$$

where a is the leading coefficient of the minimal polynomial of α and the $\alpha^{(i)}$'s are the conjugates of α in $\mathbb{C}$.

Let $\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1$. *Then*

$$\log |\Lambda| \geq -25.2D^4(\max\{\log b' + 0.38, 10/D, 1\})^2 \log A_1 \log A_2.$$

Variation of Kraus Criterion: Now we state the following variation of Kraus criterion for the non-existence of integral solutions to the equation (1.14) for given r, s, t and p.

d	(x, z, p)
d	$(-d-1, 0, p)$
2	$(0, \pm 9, 2), (3, \pm 18, 2), (69, \pm 612, 2)$
4	$(-3, 2, 7), (1, \pm 24, 2), (5, \pm 40, 2), (235, \pm 3720, 2)$
5	$(34, \pm 260, 2)$
6	$(0, \pm 35, 2), (11, \pm 90, 2)$
7	$(-7, \pm 13, 2), (160, \pm 2184)$
8	$(16, \pm 145, 2)$
11	$(36, \pm 360, 2)$
12	$(-9, \pm 44, 2), (0, \pm 91, 2), (23, \pm 252, 2), (104, \pm 1287, 2), (195, \pm 3016, 2)$
15	$(-13, 3, 7)$
16	$(83, \pm 1040, 2)$
19	$(-16, \pm 68, 2), (-14, \pm 84, 2), (34, \pm 468, 2),$ $(170, \pm 2660, 2), (265, \pm 4845, 2), (5746, \pm 437844, 2)$
20	$(-15, 6, 5), (0, \pm 189, 2), (39, \pm 540, 2)$
26	$(-39, -30, 3), (-36, -27, 3), (-18, 27, 3), (-15, 30, 3)$
27	$(-10, \pm 216, 2), (-46, -36, 3), (-34, -24, 3), (-22, 24, 3), (-10, 36, 3)$ $(84, \pm 1288, 2), (98, \pm 1512, 2), (39734, \pm 7928712, 2), (26, 6, 7)$
28	$(13, \pm 420, 2), (29, \pm 580, 2)$
29	$(-24, \pm 126, 2), (405, \pm 9135, 2)$
30	$(-21, \pm 170, 2), (0, \pm 341, 2), (59, \pm 990, 2),$ $(248, \pm 4743, 2), (1179, \pm 42130, 2), (5208, \pm 379223, 2)$
32	$(-24, \pm 171, 2), (319, \pm 6688, 2)$
34	$(16, \pm 561, 2), (35, \pm 770, 2), (14245, \pm 1706460, 2)$
36	$(-91, -72, 3), (-39, -20, 3), (-35, 20, 3), (17, 72, 3)$
38	$(2811, \pm 152190, 2)$
39	$(-31, \pm 207, 2), (81, \pm 1529, 2), (480, \pm 11960, 2)$
42	$(-124, -99, 3), (0, \pm 559, 2), (83, \pm 1638, 2), (38, 99, 3)$
45	$(8, \pm 702, 2), (69, \pm 1495, 2), (440, \pm 10854, 2)$
47	$(-36, \pm 288, 2), (516, \pm 13536, 2)$
49	$(230, \pm 4900, 2), (-95, -75, 3), (-5, 75, 3)$

TABLE 1. The integral solutions to equation (1.9) for $m = 2d + 1$ with $1 \leq d \leq 50$ and p is prime.

Lemma 2.2 ([3, Lemma 6.1]). *Consider the equation* (1.14) *for* $p \geq 3$. *Also let* $q = 2kp + 1$ *be a prime that does not divide* r. *Define*

$$\mu(p,q) = \{\eta^{2p} : \eta \in \mathbb{F}_q\} = \{0\} \cup \{\zeta \in \mathbb{F}_q^* : \zeta^k = 1\} \tag{2.1}$$

and

$$B(p,q) = \{\zeta \in \mu(p,q) : ((s\zeta + t)/r)^{2k} \in \{0,1\}\}. \tag{2.2}$$

If $B(p,q) = \emptyset$, *then the equation* (1.14) *does not have any integral solution.*

Modular method: Before going to our problem we would like to give a brief description about modular method. Let E be an elliptic curve over $\mathbb{Q}$ of conductor N and $\#E(\mathbb{F}_q)$ be the number of points on E over the finite field $\mathbb{F}_q$ for a good prime q. Let $a_q(E) = q + 1 - \#E(\mathbb{F}_q)$. By a newform f of level N, we mean a normalized cusp form of weight 2 for the congruence subgroup $\Gamma_0(N)$. Write $f = q + \sum_{i\geq 2} c_i q^i$ and $K := \mathbb{Q}(c_1, c_2, \cdots)$ is the totally real number field generated by the Fourier coefficients of f.

We say that the curve E arises modulo p from the newform f (and write $E \sim_p f$) if there is a prime ideal $\mathfrak{p}$ of K above p such that for all but finitely many primes q, we have $a_q(E) \equiv c_q \pmod{\mathfrak{p}}$. If f is a rational newform, then f corresponds to some elliptic curve F(say). If E arises modulo p from f, then also we say that E arises modulo p from F. In this regard we have the following result.

Proposition 2.3 ([6, Prop 15.2.2]). *Let* E *and* F *be elliptic curves over* $\mathbb{Q}$ *with conductors* N *and* N' *respectively. Suppose that* E *arises modulo* p *from* F. *For all primes* q

1. *if* $q \nmid NN'$, *then* $a_q(E) \equiv a_q(F)$ $(mod\ p)$ *and*
2. *if* $q \nmid N'$ *and* $q \| N$, *then* $q + 1 \equiv \pm a_q(F)$ $(mod\ p)$.

The following result provides a bound for the exponent p.

Proposition 2.4 ([6, Prop 15.4.1]). *Let* $E/\mathbb{Q}$ *be an elliptic curve of conductor* N *with* $h \mid \#E(\mathbb{Q})_{tors}$ *for some integer* h. *Suppose* f *is a newform of level* N' *and* q *be a prime with* $q \nmid N'$, $q^2 \nmid N$. *Also let*

$$T_q = \{a \in \mathbb{Z} : -2\sqrt{q} \leq a \leq 2\sqrt{q},\ a \equiv q + 1 \pmod{h}\}.$$

Let c_q *be the* q*-th coefficient of* f *and define*

$$B_q^{'}(f) := \mathrm{Norm}_{K/\mathbb{Q}}\left((q+1)^2 - {c_q}^2\right) \prod_{a \in T_q} \mathrm{Norm}_{K/\mathbb{Q}}(a - c_q)$$

and

$$B_q(f) = \begin{cases} q \cdot B_q^{'}(f) & \text{if } f \text{ is irrational,} \\ B_q^{'}(f) & \text{if } f \text{ is rational.} \end{cases}$$

If $E \sim_p f$, *then* $p | B_q(f)$.

Descent: Following well known method is very useful to eliminate possible integral solution for a Diophantine equation of certain type.

Consider the equation in integers R, X, S, Y, T,

$$RY^p - SX^{2p} = T \tag{2.3}$$

with R, S, T pairwise coprime integers.

For a prime q, we define

$$S^{'} := \prod_{\mathrm{ord}_q(S) \text{ is odd}} q.$$

Then $SS^{'} = v^2$ for some integer v. Take $RS^{'} = u$ and $TS^{'} = mn^2$ for some integers u, m and n with m squarefree. Substituting these values in the equation (2.3), we have

$$(vX^p + n\sqrt{-m})(vX^p - n\sqrt{-m}) = uY^p.$$

Let $K = \mathbb{Q}(\sqrt{-m})$ and $\mathcal{O}$ be its ring of integers. Let P be the set of prime ideals of $\mathcal{O}$ which divide u and $2n\sqrt{-m}$. The p-Selmer group is given by

$$K(P, p) = \{\epsilon \in K^*/K^{*p} : \mathrm{ord}_{\mathcal{P}}(\epsilon) \equiv 0 \pmod{p} \text{ for } \mathcal{P} \notin P\}$$

and this is a $\mathbb{F}_p$–vector space of finite dimension. Let

$$\Theta = \{\epsilon \in K(P, p) : \mathrm{Norm}(\epsilon)/u \in \mathbb{Q}^{*p}\}.$$

Now it is easy to see that

$$vX^p + n\sqrt{-m} = \epsilon Z^p, \tag{2.4}$$

where $\epsilon \in \Theta$ and $Z \in K^*$.

Lemma 2.5 ([3, Lemma 9.1]). *Let $\mathfrak{q}$ be a prime ideal of K. Suppose one of the following holds:*

1. $\mathrm{ord}_{\mathfrak{q}}(v), \mathrm{ord}_{\mathfrak{q}}(n\sqrt{-m}), \mathrm{ord}_{\mathfrak{q}}(\epsilon)$ *are pairwise distinct modulo p;*
2. $\mathrm{ord}_{\mathfrak{q}}(2v), \mathrm{ord}_{\mathfrak{q}}(\epsilon), \mathrm{ord}_{\mathfrak{q}}(\bar{\epsilon})$ *are pairwise distinct modulo p;*
3. $\mathrm{ord}_{\mathfrak{q}}(2n\sqrt{-m}), \mathrm{ord}_{\mathfrak{q}}(\epsilon), \mathrm{ord}_{\mathfrak{q}}(\bar{\epsilon})$ *are pairwise distinct modulo p.*

Then there is no $X \in \mathbb{Z}$ and $Z \in K$ satisfying the equation (2.4).

Lemma 2.6 ([3, Lemma 9.2]). *Let $q = 2kp + 1$ be a prime. Suppose $q\mathcal{O} = \mathfrak{q}_1\mathfrak{q}_2$ where $\mathfrak{q}_1, \mathfrak{q}_2$ are distinct prime ideals in $\mathcal{O}$, such that* $\mathrm{ord}_{\mathfrak{q}_j}(\epsilon) = 0$ *for $j = 1, 2$. Let*

$$\chi(p, q) = \{\eta^p : \eta \in \mathbb{F}_q\}.$$

Let

$$C(p, q) = \{\zeta \in \chi(p, q) : ((v\zeta + n\sqrt{-m})/\epsilon)^{2k} \equiv 0 \text{ or } 1 \pmod{\mathfrak{q}_j} \text{ for } j = 1, 2\}.$$

Suppose $C(p, q) = \emptyset$. Then there is no $X \in \mathbb{Z}$ and $Z \in K$ satisfying the equation (2.4).

LEMMA 2.7 ([3, Lemma 9.3]). *Suppose*

1. $\mathrm{ord}_{\mathfrak{q}}(n\sqrt{-m}) < p$ *for all prime ideals* $\mathfrak{q}$ *of* $\mathcal{O}$;
2. *the polynomial* $U^p + (\rho - U)^p - 2$ *has no roots in* $\mathcal{O}$ *for* $\rho = 1, -1, -2$;
3. *the only root of the polynomial* $U^p + (2 - U)^p - 2$ *in* $\mathcal{O}$ *is* $U = 1$.

Then, for $\epsilon = n\sqrt{-m}$, *the only solution to equation* (2.4) *with* $X \in \mathbb{Z}$ *and* $Z \in K$ *is* $X = 0$ *and* $Z = 1$.

3. Proof of Theorem 1.1 for $p \geq 5$

At first, we use lower bounds for linear forms in two logarithms to bound the exponent p appearing in (1.13).

LEMMA 3.1. *Let* $p > 19$. *Consider*

$$\alpha_1 = \beta/\alpha^2 \quad \text{and} \quad \alpha_2 = z_1^2/z_2 \ (\neq 1) \tag{3.1}$$

with $|z_1| \geq 2$ *and* $z_2 \geq 2$.

Then α_1 *and* α_2 *are positive and multiplicatively independent. Moreover, if we write*

$$\Lambda = \log \alpha_1 - p \log \alpha_2, \tag{3.2}$$

then

$$0 < \Lambda < \frac{3d(d+1)}{\alpha^2 z_1^{2p}}. \tag{3.3}$$

PROOF. One can see that α_1 and α_2 are positive as $\beta > 0$ and $z_2 > 0$. From the equations (1.13),(3.1) and (3.2), we have

$$e^{\Lambda} - 1 = \frac{\beta z_2^p}{\alpha^2 z_1^{2p}} - 1 = \frac{3d(d+1)}{\alpha^2 z_1^{2p}} > 0.$$

Therefore $0 < \Lambda < \frac{3d(d+1)}{\alpha^2 z_1^{2p}}$ since $e^x - 1 > x$ for any positive real number x.

Now we want to prove that α_1 and α_2 are multiplicatively independent. On contrary, let us suppose that α_1 and α_2 are not multiplicatively independent i.e., there exist coprime positive integers a and b such that $\alpha_1^a = \alpha_2^b$. Clearly $\alpha_1 \neq 1$. Then $a \,\mathrm{ord}_l(\alpha_1) = b \,\mathrm{ord}_l(\alpha_2)$ for all prime l. Hence $b | \mathrm{ord}_l(\alpha_1)$.

Let $g = \gcd\{\mathrm{ord}_l(\alpha_1) : \ l \text{ is prime}\}$. From (3.2), we have

$$\Lambda = \log \alpha_1 \left(1 - p\frac{\log \alpha_2}{\log \alpha_1}\right) = |\log \alpha_1| \left|1 - p\frac{a}{b}\right|. \tag{3.4}$$

Hence from (3.3) and (3.4), we have

$$0 < \frac{1}{g} \leq \left|1 - p\frac{a}{b}\right| < \frac{3d(d+1)}{|\log \alpha_1| \alpha^2 z_1^{2p}}, \tag{3.5}$$

as $b \mid g$.

Since $|z_1| \geq 2$, from the equation (3.5), it follows that

$$4^p \leq z_1^{2p} < \frac{3d(d+1)g}{|\log \alpha_1|\alpha^2}.$$

Therefore,

$$p \leq \log\left(\frac{3d(d+1)g}{|\log \alpha_1|\alpha^2}\right) / \log 4. \tag{3.6}$$

We wrote a Magma script ([4]) to compute this bound on p for $1 \leq d \leq 50$. The maximum possible value for the right-hand side of (3.6) is 18.11 corresponding to $d = 48$ and $(\alpha, \beta) = (1/7056, 7056)$, which is not possible as $p > 19$. This completes the proof of lemma. □

Lemma 3.2. *Let $p > 1000$. Consider*

$$\alpha_1 = \beta/\alpha^2 \quad \textit{and} \quad \alpha_2 = z_1^2/z_2 \ (\neq 1)$$

with $|z_1| \geq 2$ and $z_2 \geq 2$. Then we have

$$\frac{\log z_2}{\log z_1^2} \leq 1.01.$$

Proof. From the equations (3.1), (3.2) and (3.3), we have

$$\log \alpha_1 - p(\log z_1^2 - \log z_2) < \frac{3d(d+1)}{\alpha^2 4^p}. \tag{3.7}$$

Hence

$$\begin{aligned} \frac{\log z_2}{\log z_1^2} &\leq 1 + \frac{1}{p \log z_1^2}\left(\frac{3d(d+1)}{\alpha^2 4^p} + |\log \alpha_1|\right) \\ &\leq 1 + \frac{1}{1000 \log 4}\left(\frac{3d(d+1)}{\alpha^2 4^{1000}} + |\log \alpha_1|\right), \end{aligned} \tag{3.8}$$

where $p > 1000$ and $z_2 \geq 2$. We write a Magma script ([4]) to find the maximum possible value of the right-hand side which is 1.01, corresponding to $d = 50$ and $(\alpha, \beta) = (7650, 1/7650)$. This completes the proof. □

Now we are ready to apply Proposition 2.1 to find an upper bound for p.

Lemma 3.3. *Let (z_1, z_2) be an integral solution of the equation* (1.13) *with $|z_1|, z_2 \geq 2$ and $z_1^2 \neq z_2$, where $1 \leq d \leq 50$ and $(\alpha, \beta) \in S_d$. Then we have $p < 4 \times 10^4$.*

Proof. Let $A_1 = \max\{H(\alpha_1), e\}$, where $H(a/b) = \max\{|a|, |b|\}$ for $\alpha_1 = \frac{a}{b}$. Let $A_2 = \max\{{z_1}^2, z_2\}$. From Lemma 3.1, it is clear that the hypothesis of Proposition 2.1 is satisfied for our choices of $\alpha_1, \alpha_2, A_1, A_2$ with $D = 1$. Let

$$b' = \frac{1}{\log A_2} + \frac{p}{\log A_1}.$$

For $p > 1000$, we have $b' > \frac{1000}{\log A_1}$. For $1 \le d \le 50$ and $(\alpha, \beta) \in S_d$, the lower bound for $1000/\log A_1$ is 37.27 corresponding to $d = 50$ and $(\alpha, \beta) = (7650, 1/7650)$. Now applying Proposition 2.1, we have

$$\log|\Lambda| \ge -25.2 \left(\max\{\log b' + 0.38, 10, 1\}\right)^2 \log A_1 \log A_2. \tag{3.9}$$

Further, this gives

$$\begin{aligned} -\log \Lambda &\le 25.2 \log A_1 \log A_2 (\log b')^2 \\ &\le 25.2 \log A_1 \log A_2 \log^2 \left(\frac{p}{\log A_1} + \frac{1}{\log 4} \right). \end{aligned} \tag{3.10}$$

From equation (3.3), we conclude

$$p < \frac{1}{\log z_1^2} \left\{ \log \left(\frac{3d(d+1)}{\alpha^2} \right) + 25.2 \log A_1 \log A_2 \log^2 \left(\frac{p}{\log A_1} + \frac{1}{\log 4} \right) \right\}. \tag{3.11}$$

As $|z_1| \ge 2$, from Lemma 3.2 we have

$$p < \frac{1}{\log 4} \left\{ \log \left(\frac{3d(d+1)}{\alpha^2} \right) + 26 \log A_1 \log^2 \left(\frac{p}{\log A_1} + \frac{1}{\log 4} \right) \right\}.$$

We wrote a Magma script ([4]) to obtain $p < 4 \times 10^4$. This completes the proof of the lemma. □

Let z_1 and z_2 be integral solutions of (1.13). Then by Lemmas 3.1, 3.2 and 3.3, we found

$$p < 4 \times 10^4, \text{ for } |z_1| \ge 2 \text{ and } z_2 \ge 2 \text{ with } z_1^2 \ne z_2.$$

When $z_1^2 = z_2$, we determine all the possible solutions for $1 \le d \le 50$ and these solutions (z_1, z_2) are not satisfying the equation (1.13). Similarly, if $z_1 \in \{-1, 0, 1\}$ or $z_2 = 1$, we determine all the possible solutions for $1 \le d \le 50$ and we observe that $(20, -15, 6, 5), (27, 26, 6, 7)$ are the only integral solutions for (d, x, z, p) satisfying the equation (1.11). Hence we conclude that the equation (1.11) has no integral solution for $p > 4 \times 10^4$.

For $1 \le d \le 50, (\alpha, \beta) \in S_d$ and $5 \le p \le 4 \times 10^4$, we wrote a Magma script ([4]) with $k \le 765$, that searches for a prime q satisfying $q = 2kp + 1 \nmid r$ such that $B(p, q) = \emptyset$.

We note that if there exist such a prime q with $B(p, q) = \emptyset$, then by Lemma 2.2 the equation (1.13) has no solution for exponent p. This criterion fails when $\beta = 3d(d+1)$ (equivalently $r = t$) for which we have the trivial solution $(z_1, z_2) = (0, 1)$. In addition, for $\beta \ne 3d(d+1)$ (equivalently $r \ne t$) we found 1716 quintuples (d, p, r, s, t) which fail to satisfy this criterion.

Now, to complete the proof of Theorem 1.1 for $p \ge 5$, we are remaining with the following cases.

1. $r = t$ and $p < 4 \times 10^4$;

2. $r \neq t$ and $p < 4 \times 10^4$ consisting 1716 quintuples (d, p, r, s, t).

To solve the equation (1.14) for $r = t$ and $p < 4 \times 10^4$ we want to apply modular method. Here we use the recipes of Kraus ([9]) due to Wiles ([23]), Ribet ([15]) and Mazur ([13]).

In the case $r = t$, the equation (1.13) has a solution $(z_1, z_2) = (0, 1)$. In fact, we want to show that $(z_1, z_2) = (0, 1)$ is the only solution.

Since $r = t$, we have $\alpha = 1/3d(d+1)$ and thus the equation (1.13) will reduce to

$$z_2^p - \frac{1}{(3d(d+1))^3} z_1^{2p} = 1. \tag{3.12}$$

Let $R = \text{Rad}\ (3d(d+1))$. Since z_1 and z_2 are integers, we have $R \mid z_1$. Hence $z_1 = Rz_3$ for some integer z_3. Then from the equation (3.12), we have

$$z_2^p - \frac{R^{2p}}{(3d(d+1))^3} z_3^{2p} = 1.$$

Take $T = \frac{R^{2p}}{(3d(d+1))^3}$ then the above equation becomes

$$z_2^p - T z_3^{2p} = 1. \tag{3.13}$$

It is easy to see that $\text{Rad}(T) = R$. Further we assume that

$$2p > 3 \cdot \text{ord}_q(3d(d+1)) \tag{3.14}$$

for all odd primes q. We want to show that $z_1 = 0$ for the equation (3.12). On contrary, let us assume that $z_1 \neq 0$, which implies $z_3 \neq 0$. Also $z_2 \neq 0$. The equation (3.13) can be written in the following form

$$Ax^p + By^p + Cz^p = 0,$$

where $A = -1, B = -T, C = 1, x = 1, y = z_3^2, z = z_2$ and also

$$Ax^p \equiv -1 \pmod 4), By^p \equiv 0 \pmod 2).$$

Now we associate a solution (z_2, z_3) to the Frey Curve

$$E: \quad Y^2 = X(X+1)(X - T z_3^{2p}). \tag{3.15}$$

The Weierstrass model given in (3.15) is smooth as $z_2 z_3 \neq 0$. Let $E \sim_p f$, where f is a weight 2 newform of level N_p with N_p is defined as follows:

$$N_p = \begin{cases} R & \text{if } \text{ord}_2(T) = 0 \text{ or } \geq 5, \\ \frac{R}{2} & \text{if } \text{ord}_2(d(d+1)) = 2 \text{ and } p = 5, \\ R & \text{if } \text{ord}_2(d(d+1)) = 3, \ \ p = 5 \text{ and } z_3 \text{ even}, \\ R & \text{if } \text{ord}_2(d(d+1)) = 4, \ p = 7 \text{ and } z_3 \text{ even}, \\ 2^2 R & \text{if } \text{ord}_2(d(d+1)) = 4, \ p = 7 \text{ and } z_3 \text{ odd}, \\ 2^4 R & \text{if } \text{ord}_2(d(d+1)) = 3, \ p = 5 \text{ and } z_3 \text{ odd}. \end{cases} \tag{3.16}$$

Suppose f is rational and hence we get an elliptic curve F of conductor N_p. Now we choose a prime $q = 2kp+1$ such that $q \nmid N_p$ and E has multiplicative reduction at q. Then by Proposition 2.3, $q+1 \equiv \pm a_q(F) \pmod p$ and this implies $4 \equiv (a_q(F))^2 \pmod p$ as $q \equiv 1 \pmod p$.

Suppose that f is irrational. Since $c_q \notin \mathbb{Q}$ for infinitely many coefficients of f, we have $B_q(f) \neq 0$ for infinitely many primes q. Then Proposition 2.4 allows us to obtain a bound for p. In fact, this bound is very small. Here we improve this bound by choosing a set of primes $\mathcal{P} = \{q_1, \ldots, q_n\}$ such that $q_i \nmid N_p$ for all i and $B_{\mathcal{P}}(f) = \gcd(B_q(f) : q \in \mathcal{P})$. Thus, if $E \sim_p f$ then $p \mid B_{\mathcal{P}}(f)$.

From the above observations, the following lemma is very helpful to eliminate newforms of level N_p.

LEMMA 3.4. *Let* $1 \leq d \leq 50$. *Also let* $p \geq 5$ *be a prime which satisfies the inequality* (3.14) *for all primes* q. *Let* N_p *be given in* (3.16). *Suppose for each irrational newform* f *of weight* 2 *and level* N_p *there is a set of primes* $\mathcal{P}$ *not dividing* N_p *such that* $p \nmid B_{\mathcal{P}}(f)$. *Suppose for every elliptic curve* F *of conductor* N_p *there is a prime* $q = 2kp+1, q \nmid N_p$, *such that*

1. $B(p,q) = \{\bar{0}\}$, *where* $B(p,q)$ *is in statement of Lemma 2.2;*
2. $p \nmid (a_q(F)^2 - 4)$.

Then the equation (1.11) *has only one solution with*

$$(\alpha, \beta) = \left(\frac{1}{3d(d+1)}, 3d(d+1)\right)$$

satisfying $x = -(d+1)$.

PROOF. If $z_1 = 0$, then we see that $x = -(d+1)$. Let us assume that, $z_1 \neq 0$. We know that, there is a newform f of level N_p such that $E \sim_p f$, where E is the Frey-Hellegouarch curve. If f is irrational, then $p \mid B_{\mathcal{P}}(f)$, which is a contradiction to our hypothesis. Hence f is rational and so $E \sim_p F$, where F is an elliptic curve of conductor N.

From equation (1.14), we see that ${z_2}^p = \frac{s{z_1}^{2p}+t}{r}$. Hence

$$\left(\frac{s{z_1}^{2p}+t}{r}\right)^{2k} \equiv {z_2}^{2kp} \equiv {z_2}^{q-1} \pmod q.$$

Since ${z_2}^{q-1} \equiv 0 \text{ or } 1 \pmod q$, by the definition of $B(p,q)$ we have $\bar{z_1} \in B(p,q)$. Thus, by condition (1), $\bar{z_1} = \bar{0}$. Hence we see that, $q \mid z_1$. Since $z_1 = Rz_3$, we have $q \mid z_3$. Thus, it follows that E has multiplicative reduction at q. Hence $q+1 \equiv \pm a_q(F) \pmod p$. Since $q \equiv 1 \pmod p$, we observe that $(a_q(F))^2 \equiv 4 \pmod p$, which is a contradiction by condition (2). ∎

Now we wrote a Magma script ([4]) for each $1 \leq d \leq 50$ which computes the newforms of weight 2 and level N_p. Here we assume that $\mathcal{P}$ is the set of primes < 100 which do not divide N_p. Then for each irrational newform we compute $B_{\mathcal{P}}(f)$.

For every prime $5 \le p < 4 \times 10^4$ that does not divide $B_{\mathcal{P}}(f)$, satisfies the inequality (3.14) and for every isogeny class of elliptic curves F of conductor N_p, we search for the primes $q = 2kp + 1, q \nmid N_p$ with $k \le 765$ such that conditions (1) and (2) of Lemma 3.4 hold.

If we find such a prime then the equation (1.11) has no solution with $r = t$. The criterion holds for all values of p except for few small values of p. When $N_p = R$, there are 55 cases where either p does not satisfy the inequality (3.14), or it divides $B_{\mathcal{P}}(f)$ for some irrational newform f, or q does not satisfy conditions (1) and (2) of Lemma 3.4.

For other special cases of N_p we are remaining with 3 equations, which do not satisfy the above conditions. The largest value of p among the 58 quintuples is $p = 19$ with

$$d = 37, \alpha = 1/4218, \beta = 4218, r = t = 75044648232, s = 1.$$

Now we have total $1716 + 55 + 3 = 1774$ remaining equations, which can not be eliminated by Lemma (2.2) and modular approach. These equations are of the form (1.14) with r, s and t positive integers and $\gcd(r, s, t) = 1$. There is a possibility that r, s and t may not be pairwise coprime. We apply the procedure mentioned in [3, section 9.1] which is nothing but a repetitive way of clearing out the common factor to get an equation of the form

$$RY^p - SX^{2p} = T \tag{3.17}$$

where R, S, T are pairwise coprime and X, Y are divisors of z_1, z_2 respectively.

If there exist a solution for the equation(3.17), then $-ST$ is a square modulo q for any odd prime $q \nmid R$. Also we check for local solubility at the primes dividing R, S, T, and the primes $q \le 19$. Applying these above tests, we are remaining with 175 equations after elimination. For these remaining equations we apply descent.

By applying Lemmas 2.5 and 2.6 to the remaining equations, which were left after local solubility, we eliminate $\epsilon \in \Theta$. But we know that if $r = t$ then the equation (1.14) has a solution, i.e., $(z_1, z_2) = (0, 1)$. For $r = t$, the reduction process leads to $R = T = 1$. Thus the solution $(z_1, z_2) = (0, 1)$ in (1.14) corresponds to $(X, Y) = (0, 1)$ in (3.17). Also

$$n\sqrt{-m}(K^*)^p \in \Theta.$$

Hence using Lemmas 2.5 and 2.6, we eliminate all ϵ except the case $\epsilon = n\sqrt{-m}$ as the equation (2.4) has a solution $(X, Z) = (0, 1)$.

For the case $\epsilon = n\sqrt{-m}$, the equation (3.17) has only one solution $(X, Y) = (0, 1)$ by Lemma 2.7. If $X = 0$ then $z_1 = 0$ and hence, $x = -(d+1)$. If Lemmas 2.5, 2.6 and 2.7 allow us to conclude $X = 0$, then we can eliminate (r, s, t) as we can consider $x \neq -(d + 1)$. We write a Magma script ([4]) for above procedure and we eliminate 164 equations. Now we have to solve only 11 remaining equations by Thue approach. By writing $V = Y^2$ in (3.17), we obtain the Thue equation

$$RY^p - SV^p = T. \tag{3.18}$$

Using Thue equation solver in Magma ([4]), we solve the remaining equations. Finally we have the following solutions.

$$\begin{aligned} &27^3 - 28^3 - \cdots - 80^3 + 81^3 = 6^7,\\ &(-2)^3 - (-1)^3 + \cdots - 5^3 + 6^3 = 2^7,\\ &(-12)^3 - (-11)^3 + \cdots - 17^3 + 18^3 = 3^7,\\ &(-14)^3 - (-13)^3 + \cdots - 25^3 + 26^3 = 6^5. \end{aligned} \tag{3.19}$$

This concludes the proof of Theorem 1.1 for $p \geq 5$.

4. Proof of Theorem 1.1 for $p = 2$

Putting $x + d + 1 = u$ and $p = 2$ in the equation (1.11), we have

$$z^2 = u^3 + 3d(d+1)u. \tag{4.1}$$

This represents a family of elliptic curves. For $1 \leq d \leq 50$, we obtain the integral solutions of the equation (4.1) by Magma ([4]). These solutions give rise to all the integral solutions of (1.11) and those are given explicitly in Table 1.

5. Proof of Theorem 1.1 for $p = 3$

In this case the required equation is

$$z^3 = u^3 + 3d(d+1)u. \tag{5.1}$$

Let $\alpha = \gcd(u, 3d(d+1))$, then

$$u = \alpha u_1 \ \text{ and } u^2 + 3d(d+1) = \alpha u_2, \tag{5.2}$$

where $\gcd(u_1, u_2) = 1$. Let $\mathrm{Ord}_2(u_1 u_2) = l$ and $\mathrm{Ord}_3(u_1 u_2) = m$, for some non-negative integers l, m. Then we can write

$$\begin{aligned} &u_1 = 2^l \cdot u_3 \ \text{ and } u_2 = 3^m \cdot u_4,\\ \text{or } &u_1 = 3^l \cdot u_3 \ \text{ and } u_2 = 2^m \cdot u_4,\\ \text{or } &u_1 = 2^l \cdot 3^m \cdot u_3 \ \text{ and } \ u_2 = u_4,\\ \text{or } &u_1 = u_3 \ \text{ and } u_2 = 2^l \cdot 3^m \cdot u_4, \end{aligned} \tag{5.3}$$

where u_3 and u_4 are integers with $\gcd(u_3, u_4) = 1$.

Also write $\alpha = 2^{\delta_2} \cdot 3^{\delta_3} \cdot \alpha_1$ for some integer α_1 with $\delta_2 := \mathrm{ord}_2(\alpha)$ and $\delta_3 := \mathrm{ord}_3(\alpha)$. As $\alpha u_1 \cdot \alpha u_2 = z^3$, we have $\alpha_1^2 u_3 u_4 = z_1^3$ for some integer z_1. Since $1 \leq d \leq 50$, for any prime $q \mid \alpha_1, \mathrm{ord}_q(\alpha_1) \leq 2$. Therefore, we can conclude that, if $\alpha_1^2 \mid z_1^3$ then $\alpha_1 \mid z_1$. Write $z_1 = \alpha_1 \cdot z_2$ for some integer z_2, hence we have $u_3 \cdot u_4 = \alpha_1 z_2^3$. Since $\gcd(u_3, u_4) = 1$, we can write

$$u_3 = \alpha_2 \cdot z_3^3 \text{ and } u_4 = \alpha_3 \cdot z_4^3, \tag{5.4}$$

for some integers $\alpha_2, \alpha_3, z_3, z_4$ with $\alpha_2\alpha_3 = \alpha_1$ and $z_3 z_4 = z_2$. Rewriting the equation (5.2), we have

$$\alpha \cdot u_2 - \alpha^2 \cdot u_1^2 = 3d(d+1). \tag{5.5}$$

Now from equations (5.3),(5.4) and (5.5), we will have a set of Thue equations as follows:

$$\begin{aligned} &\alpha \cdot \alpha_3 \cdot 3^m \cdot z_4^3 - \alpha^2 \cdot \alpha_2^2 \cdot 2^{2l} \cdot (z_3^2)^3 = 3d(d+1), \\ \text{or } &\alpha \cdot \alpha_3 \cdot 2^m \cdot z_4^3 - \alpha^2 \cdot \alpha_2^2 \cdot 3^{2l} \cdot (z_3^2)^3 = 3d(d+1), \\ \text{or } &\alpha \cdot \alpha_3 \cdot z_4^3 - \alpha^2 \cdot \alpha_2^2 \cdot 2^{2l} \cdot 3^{2m} \cdot (z_3^2)^3 = 3d(d+1), \\ \text{or } &\alpha \cdot \alpha_3 \cdot 2^{2l} \cdot 3^{2m} \cdot z_4^3 - \alpha^2 \cdot \alpha_2^2 \cdot (z_3^2)^3 = 3d(d+1). \end{aligned} \tag{5.6}$$

Now, for $1 \leq d \leq 50$ we have written a Magma script ([4]) to solve these four Thue equations. The theory about solving these Thue equations is discussed in [19]. Using backward calculations from these solutions we find all solutions for the equation (5.1) and these are given explicitly in Table 1.

6. CONCLUDING REMARK

For $m = 2d$ the equation (1.9) becomes

$$d[3x^2 + 3(2d+1)x + d(4d+3)] = (-z)^p.$$

Since the polynomial $3dx^2+3d(2d+1)x+d^2(4d+3)$ is an irreducible polynomial over $\mathbb{Q}$ for $1 \leq d \leq 50$, by [6, Theorem 12.11.2, p. 437], we conclude that the equation (1.9) has finitely many solutions for even m.

For $x > 50$, we are not able to conclude anything about getting perfect powers in alternating sums of consecutive cubes of even length. Though for $x \leq 50$ we see that $(2, 2, -2, 7)$, $(3, 12, -3, 7)$ and $(6, 14, -6, 5)$ are solutions for (d, x, z, p) in the equation (1.9). In general, when m is even in the equation (1.9), we conjecture the following.

CONJECTURE 6.1. *Let $m = 2d$ with $1 \leq d \leq 50$ and $p \geq 5$ be a prime. Then the only integer solutions of the equation* (1.9) *are given by*

$$(d, x, z, p) \in \{(2, 2, -2, 7), (3, 12, -3, 7), (6, 14, -6, 5), (27, 215, -9, 7)\}.$$

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References

[1] M. Bennett, K. Győry and Á. Pintér, *On the diophantine equation* $1^k + 2^k + \cdots + x^k = y^n$, Compos. Math. **140** (2004), 1417–1431.

[2] M. Bennett, *A superelliptic equation involving alternating sums of powers*, Publ. Math. Debrecen **79** (2011), 317–324.

[3] M. Bennett, V. Patel and S. Siksek, *Perfect powers that are sums of consecutive cubes*, Mathematika **63** (2017), 230–249.

[4] W. Bosma, J. Cannon and C. Playoust, *The Magma algebra system. I. The user language,* J. Symboli Comput. **24** (1997), 235–265.

[5] J. W. S. Cassels, *A Diophantine equation,* Glasgow Math. J. **27** (1985), 11–88.

[6] H. Cohen, Number theory vol. II: analytic and modern tools, Springer, New York, 2007.

[7] K. Dilcher, *On a Diophantine equation involving quadratic characters,* Compos. Math. **57** (1986), 383–403.

[8] M. Jacobson, Á. Pintér, P. G. Walsh, *A computational approach for solving* $y^2 = 1^k + 2^k + \cdots + x^k$*,* Math Comp. **72** (2003), 2099–2110.

[9] A. Kraus, *Majorations effectives pour l'équation de Fermat généralisée,* Canad. J. Math. **49** (1997), 1139–1161.

[10] M. Laurent, *Linear forms in two logarithms and interpolation determinants II,* Acta. Arith. **133** (2008), 325–348.

[11] W. J. Leveque, *On the equation* $y^m = f(x)$*,* Acta. Arith. **9** (1964), 209–219.

[12] É. Lucas, *Problem 1180,* Nouvelle Ann. Math. **14** (1875), 336.

[13] B. Mazur, *Rational isogenies of prime degree*, Invent. Math. **44** (1978), 129–162.

[14] Á. Pintér, *On the power values of power sums,* J. Number Theory **125** (2007), 412–423.

[15] K. Ribet, *On modular representations of* $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ *arising from modular forms,* Invent. Math. **100** (1990), 431–476.

[16] J. J. Schäffer, *The equation* $1^p + 2^p + \cdots + n^p = m^q$*,* Acta Math. **95** (1956), 155–189.

[17] A. Schinzel and R. Tijdeman, *On the equation* $y^m = f(x)$*,* Acta Arith. **31** (1976), 199–204.

[18] S. Siksek, *Modular approach to Diophantine equations,* in: Explicit methods in number theory, Panor. Synthèses 36, Soc. Math. France, Paris, 2012, 151—179.

[19] N. P. Smart, The algorithmic resolution of Diophantine equations, Cambridge University Press, 1997.

[20] G. Soydan, *On the Diophantine equation* $(x+1)^k + (x+2)^k + \ldots + (lx)^k = y^n$, Publ. Math. Debrecen **91** (2017), 369–382.

[21] R. J. Stroeker, *On the sum of consecutive cubes being a square,* Compos. Math. **97** (1995), 295–307.

[22] G. N. Watson, *The problem of the square pyramid,* Messenger of Math. **48** (1918), 1–22.

[23] A. Wiles, *Modular elliptic curves and Fermat's Last Theorem,* Ann. Math. (2) **141** (1995), 443–551.

[24] Z. Zhang and M. Bai, *On the diophantine equation* $(x+1)^2+(x+2)^2+\cdots+(x+d)^2 = y^n$, Funct. Approx. Comment. Math. **49** (2013), 73–77.

[25] Z. Zhang, *On the Diophantine equation* $(x-1)^k + x^k + (x+1)^k = y^n$, Publ. Math. Debrecen **85** (2014), 93–100.

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