

University of Texas Rio Grande Valley

ScholarWorks @ UTRGV

---

Mathematical and Statistical Sciences Faculty  
Publications and Presentations

College of Sciences

---

2-2020

## A (2+1)-dimensional sine-Gordon and sinh-Gordon equations with symmetries and kink wave solutions

Gangwei Wang

Kaitong Yang

Haicheng Gu

*The University of Texas Rio Grande Valley*

Fei Guan

A. H. Kara

Follow this and additional works at: [https://scholarworks.utrgv.edu/mss\\_fac](https://scholarworks.utrgv.edu/mss_fac)



Part of the [Mathematics Commons](#)

---

### Recommended Citation

Wang, Gangwei; Yang, Kaitong; Gu, Haicheng; Guan, Fei; and Kara, A. H., "A (2+1)-dimensional sine-Gordon and sinh-Gordon equations with symmetries and kink wave solutions" (2020). *Mathematical and Statistical Sciences Faculty Publications and Presentations*. 29.

[https://scholarworks.utrgv.edu/mss\\_fac/29](https://scholarworks.utrgv.edu/mss_fac/29)

This Article is brought to you for free and open access by the College of Sciences at ScholarWorks @ UTRGV. It has been accepted for inclusion in Mathematical and Statistical Sciences Faculty Publications and Presentations by an authorized administrator of ScholarWorks @ UTRGV. For more information, please contact [justin.white@utrgv.edu](mailto:justin.white@utrgv.edu), [william.flores01@utrgv.edu](mailto:william.flores01@utrgv.edu).



# A (2+1)-dimensional sine-Gordon and sinh-Gordon equations with symmetries and kink wave solutions

Gangwei Wang<sup>a,b,\*</sup>, Kaitong Yang<sup>a</sup>, Haicheng Gu<sup>c</sup>, Fei Guan<sup>a</sup>,  
A.H. Kara<sup>d</sup>

<sup>a</sup> School of Mathematics and Statistics, Hebei University of Economics and Business, Shijiazhuang 050061, PR China

<sup>b</sup> School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, PR China

<sup>c</sup> School of Mathematical and Statistical Sciences, University of Texas-Rio Grande Valley, Edinburg, TX 78539, USA

<sup>d</sup> School of Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa

Received 27 October 2019; received in revised form 31 January 2020; accepted 7 February 2020

Available online 11 February 2020

Editor: Hubert Saleur

## Abstract

In this paper, a (2+1)-dimensional sine-Gordon equation and a sinh-Gordon equation are derived from the well-known AKNS system. Based on the Hirota bilinear method and Lie symmetry analysis, kink wave solutions and traveling wave solutions of the (2+1)-dimensional sine-Gordon equation are constructed. The traveling wave solutions of the (2+1)-dimensional sinh-Gordon equation can also be provided in a similar manner. Meanwhile, conservation laws are derived.

© 2020 The Authors. Published by Elsevier B.V. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>). Funded by SCOAP<sup>3</sup>.

## 1. Introduction

It is well-known that the classical (1 + 1)-dimensional sine-Gordon (sG) equation

$$u_{tt} = u_{xx} + \sin u \tag{1}$$

or equivalent form

\* Corresponding author at: School of Mathematics and Statistics, Hebei University of Economics and Business, Shijiazhuang 050061, PR China.

E-mail address: [wanggangwei@heuet.edu.cn](mailto:wanggangwei@heuet.edu.cn) (G. Wang).

$$u_{xt} = \sin u \tag{2}$$

appears in many scientific fields [1,2,12,5-7,34], such as quantum-field and differential geometry theory [1,2,12,10,24,5-7]. Many mathematicians and physicists studied this well-known equation from different aspects. The authors in [2] discussed the sG equation through using the inverse scattering method. Leibbrandt [12] studied solutions of the sine-Gordon equation in higher dimensions. Klein [10] considered geometric interpretation as surfaces of constant negative curvature. Rubinstein [24] presented a model of field theory and studied in detail. Gu and Hu [5] provided explicit solutions to the intrinsic generalization for the wave and sine-Gordon equations, and Hu [7] investigated the relationship between soliton and differential Geometry through the sG equation. In [25], the authors studied symmetry groups of the intrinsic generalized wave and sine-Gordon equations. A quantum-mechanical system is constructed over a Fock space of particles in [31] based on N-soliton solutions.

In paper [35], one of the authors derived a (2+1)-dimensional KdV and mKdV equation from positive case. This paper is the continuation of that one. In this paper, from the extend AKNS system, we derive a (2+1)-dimensional sine-Gordon equation as well as a (2+1)-dimensional sinh-Gordon equation. Kink wave solutions and their interactional wave propagation are constructed for the (2+1)-dimensional sine-Gordon equation. Furthermore, Lie symmetries approach is employed to reduce the (2+1)-dimensional sine-Gordon and sinh-Gordon equations so that their traveling wave solutions are obtained.

## 2. Derivation of (2+1)-dimensional sine-Gordon and sinh-Gordon equations

It is well-known that the AKNS system [1] is one of the classical well-known integrable systems from which a great many of nonlinear evolution equations can be derived, such as the famous KdV equation, the MKdV equation, the nonlinear Schrödinger equation (NLS), the Burgers equation, the (1+1)-dimensional sine-Gordon equation, etc. Based on the AKNS system, let us consider the following (2+1)-dimensional zero curvature equation [1,33,11,34,35],

$$X_t - X_x + T_x - T_y + [X, T] = 0, \tag{3}$$

where  $[X, T] = XT - TX$ ,

$$X = \begin{pmatrix} -i\zeta & q \\ r & i\zeta \end{pmatrix}, T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{4}$$

Here in Eq. (4),  $i^2 = -1$ ,  $\zeta$  is an eigenparameter independent of time  $t$  (i.e.  $\zeta_t = 0$ ),  $q, r$  are two potential functions of  $x, t$ , and  $A, B, C, D$  are the functions to be determined. Substituting Eq. (4) into Eq. (3) leads to the following equations

$$\begin{cases} A_x - A_y = rB - qC, \\ B_x - B_y = q_x - q_t + 2iB\zeta + Aq - qD, \\ C_x - C_y = r_x - r_t - 2iC\zeta + rD - rA, \\ D_x - D_y = qC - rB. \end{cases} \tag{5}$$

From the first and the last equations, we can choose  $D = -A$ . Hence, Eq. (5) becomes

$$\begin{cases} A_x - A_y = rB - qC, \\ B_x - B_y = q_x - q_t + 2iB\zeta + 2Aq, \\ C_x - C_y = r_x - r_t - 2iC\zeta - 2rA. \end{cases} \tag{6}$$

In order to solve for  $A, B, C$ , let us target at expanding  $A, B, C$  in the form of truncated power series with regard to the eigenvalue  $\zeta$ . Since the positive cases have already been studied in the literature, here in our paper we do the negative case. The negative order of integrable equations originated from the work in [15,16], and thereafter some interesting integrable equations with the properties of generalized Lax representations and algebraic structure were developed from the negative hierarchy [32,17,19,21,22]. The negative case may generate some new equations which have different physical meanings [18,20,23].

Therefore let us try employing the following expansions

$$A = \sum a_n(x, y, t)\zeta^{-n}, \quad B = \sum b_n(x, y, t)\zeta^{-n}, \quad C = \sum c_n(x, y, t)\zeta^{-n}. \tag{7}$$

Substituting Eq. (7) into Eq. (6), it immediately generates

$$\begin{cases} a_{nx} - a_{ny} = rb_n - qc_n, \\ b_{n-1,x} - b_{n-1,y} = 2ib_n + 2qa_{n-1}, \\ c_{n-1,x} - c_{n-1,y} = -2ic_n - 2ra_{n-1}, \\ q_x - q_t = b_{nx} - b_{ny} - 2a_nq, \\ r_x - r_t = c_{nx} - c_{ny} + 2a_nr. \end{cases} \tag{8}$$

In the special case  $n = 1$ , we have

$$A = a_1(x, y, t)\zeta^{-1}, \quad B = b_1(x, y, t)\zeta^{-1}, \quad C = c_1(x, y, t)\zeta^{-1}, \tag{9}$$

and

$$\begin{cases} a_{1x} - a_{1y} = rb_1 - qc_1, \\ b_{1x} - b_{1y} = 2a_1q, \\ q_x - q_t = -2ib_1, \\ c_{1x} - c_{1y} = -2a_1r, \\ r_x - r_t - 2ic_1 = 0. \end{cases} \tag{10}$$

Eq. (10) admits the following special solutions:

$$\begin{cases} a_1 = \frac{i}{4} \cos u, \\ b_1 = c_1 = \frac{i}{4} \sin u, \\ q = -r = \frac{u_x - u_y}{2}, \end{cases} \tag{11}$$

and subsequently yields the following (2+1)-dimensional sine-Gordon equation

$$u_{xx} - u_{xy} - u_{xt} + u_{yt} = \sin u. \tag{12}$$

In a similar way, we may select the following special solutions of Eq. (10)

$$\begin{cases} a_1 = \frac{i}{4} \cosh u, \\ b_1 = -c_1 = \frac{i}{4} \sinh u, \\ q = r = \frac{u_x - u_y}{2}, \end{cases} \tag{13}$$

to get the (2+1)-dimensional sinh-Gordon equation below

$$u_{xx} - u_{xy} - u_{xt} + u_{yt} = \sinh u. \tag{14}$$

**Remark.**

1. One may obtain other integrable (2+1)-dimensional equations through choosing different solutions of Eq. (10). Here in our paper, we just focus on those which have physical applications.
2. Under the transformation  $\xi = x + y + t, \eta = y, \tau = t$ , Eq. (12) and Eq. (14) equivalent to Eqs.  $U_{\eta\tau} = \sin U$  and  $U_{\eta\tau} = \sinh U$ , this is just the case of (1+1)-dimensional.

**3. Multi-kink wave solutions**

Let us consider the following transformation [4]

$$u(x, y, t) = 2i \ln \frac{f^*}{f}, \tag{15}$$

where  $f^*$  is the complex conjugate of function  $f$ . Since  $\sin u = \frac{e^{iu} - e^{-iu}}{2i}$ , substituting Eq. (15) into Eq. (12) yields

$$\begin{aligned} & 2 \left( \frac{f_{xx}^* f^* - f_x^* f_x^*}{(f^*)^2} - \frac{f_{xx} f - f_x f_x}{f^2} \right) - 2 \left( \frac{f_{xy}^* f^* - f_y^* f_x^*}{(f^*)^2} - \frac{f_{yx} f - f_y f_x}{f^2} \right) \\ & - 2 \left( \frac{f_{tx}^* f^* - f_t^* f_x^*}{(f^*)^2} - \frac{f_{tx} f - f_t f_x}{f^2} \right) + 2 \left( \frac{f_{ty}^* f^* - f_t^* f_y^*}{(f^*)^2} - \frac{f_{ty} f - f_t f_y}{f^2} \right) \\ & = \frac{1}{2} \left( \frac{(f^*)^2 - f^2}{(f^*)^2} - \frac{f^2 - (f^*)^2}{f^2} \right), \end{aligned} \tag{16}$$

which implies the following bilinear forms [8]

$$D_x D_x f \cdot f - D_y D_x f \cdot f - D_t D_x f \cdot f + D_t D_y f \cdot f = \frac{f^2 - (f^*)^2}{2}, \tag{17}$$

where the operator  $D$  is defined by

$$D_{x_1}^{n_1} \cdots D_{x_l}^{n_l} f \cdot g = \left( \partial_{x_1} - \partial_{x_1'} \right)^{n_1} \cdots \left( \partial_{x_l} - \partial_{x_l'} \right)^{n_l} f(x_1, \dots, x_l) g(x_1', \dots, x_l') \Big|_{x_i' = x_i, \dots, x_l' = x_l}. \tag{18}$$

Let us assume that  $f$  can be expanded in the power of  $\varepsilon$  as follows,

$$f = 1 + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} + \dots. \tag{19}$$

Substituting Eq. (19) into Eq. (17), we have

coefficients ( $\varepsilon$ ):

$$f_{xx}^{(1)} - f_{yx}^{(1)} - f_{tx}^{(1)} + f_{ty}^{(1)} = \frac{1}{2} \left( f^{(1)} - (f^{(1)})^* \right), \tag{20}$$

coefficients ( $\varepsilon^2$ ):

$$\begin{aligned} & 2 \left( f_{xx}^{(2)} - f_{yx}^{(2)} - f_{tx}^{(2)} + f_{ty}^{(2)} \right) \\ & = - \left( D_x D_x f^{(1)} \cdot f^{(1)} - D_y D_x f^{(1)} \cdot f^{(1)} - D_t D_x f^{(1)} \cdot f^{(1)} + D_t D_y f^{(1)} \cdot f^{(1)} \right) \\ & + f^{(2)} - f^{(2*)} + \frac{1}{2} \left( (f^{(1)})^2 - ((f^{(1)})^*)^2 \right), \end{aligned} \tag{21}$$

coefficients ( $\varepsilon^3$ ):

$$2 \left( f_{xx}^{(3)} - f_{yx}^{(3)} - f_{tx}^{(3)} + f_{ty}^{(3)} \right)$$

$$= -2 \left( D_x D_x f^{(1)} \cdot f^{(2)} - D_y D_x f^{(1)} \cdot f^{(2)} - D_t D_x f^{(1)} \cdot f^{(2)} + D_t D_y f^{(1)} \cdot f^{(2)} \right)$$

$$+ f^{(3)} - f^{(3*)} + (f^{(1)})(f^{(2)}) - (f^{(1)})^*(f^{(2)})^*, \tag{22}$$

...

In order to get some exact solutions, let us set up

$$f^{(1)} = i e^{\xi_1}, \quad \xi_1 = c_1 t + k_1 x + l_1 y, \tag{23}$$

with the following constraint condition

$$k_1^2 - k_1 c_1 - k_1 l_1 + l_1 c_1 = 1. \tag{24}$$

Then, we have

$$f_1(x, y, t) = 1 + i e^{\xi_1}, \tag{25}$$

which gives us the following single kink wave solution

$$u = 2i \ln \frac{1 + i e^{\xi_1}}{1 - i e^{\xi_1}} = 4 \arctan e^{\xi_1}. \tag{26}$$

Because of its linearity, Eq. (20) admits the following solutions

$$f^{(1)} = i e^{\xi_1} + i e^{\xi_2}, \quad \xi_1 = c_1 t + k_1 x + l_1 y, \quad \xi_2 = c_2 t + k_2 x + l_2 y, \tag{27}$$

where  $c_j, k_j, l_j (j = 1, 2)$  are constants. Apparently, substituting Eq. (27) into Eq. (21) produces

$$f^{(2)} = -e^{\xi_1 + \xi_2 + A_{12}}, \tag{28}$$

where

$$e^{A_{12}} = \frac{(k_1 - k_2)(l_1 - l_2) + (c_1 - c_2)(k_1 - k_2) - (k_1 - k_2)^2 - (c_1 - c_2)(l_1 - l_2)}{(k_1 + k_2)^2 - (k_1 + k_2)(l_1 + l_2) - (c_1 + c_2)(k_1 + k_2) + (c_1 + c_2)(l_1 + l_2)}. \tag{29}$$

That is to say,

$$f_2(x, y, t) = 1 - e^{\xi_1 + \xi_2 + A_{12}} + i e^{\xi_1} + i e^{\xi_2}. \tag{30}$$

Therefore, two kink wave solution is given by

$$u = 4 \arctan \frac{e^{\xi_1} + e^{\xi_2}}{1 - e^{\xi_1 + \xi_2 + A_{12}}}. \tag{31}$$

Adopting the same procedure shown above, we could obtain a 3-kink wave solution

$$u = 2i \arctan \frac{e^{\xi_1} + e^{\xi_2} + e^{\xi_3} - e^{\xi_1 + \xi_2 + \xi_3 + A_{12} + A_{13} + A_{23}}}{1 - (e^{\xi_1 + \xi_2 + A_{12}} + e^{\xi_1 + \xi_3 + A_{13}} + e^{\xi_2 + \xi_3 + A_{23}})}, \tag{32}$$

where

$$e^{A_{ij}} = \frac{(k_i - k_j)(l_i - l_j) + (c_i - c_j)(k_i - k_j) - (k_i - k_j)^2 - (c_i - c_j)(l_i - l_j)}{(k_i + k_j)^2 - (k_i + k_j)(l_i + l_j) - (c_i + c_j)(k_i + k_j) + (c_i + c_j)(l_i + l_j)} \tag{33}$$

( $i < j, \quad i, j = 1, 2, 3$ ).

Repeated the similar procedure  $N$  times, we can construct  $N$ -kink wave solution:

$$u = 2i \ln \frac{\sum_{\mu=0,1} \exp\left(\sum_{i=1}^N \mu_i \left(\xi_i - i\frac{\pi}{2}\right) + \sum_{1 \leq i < j}^N \mu_i \mu_j A_{ij}\right)}{\sum_{\mu=0,1} \exp\left(\sum_{i=1}^N \mu_i \left(\xi_i + i\frac{\pi}{2}\right) + \sum_{1 \leq i < j}^N \mu_i \mu_j A_{ij}\right)}. \tag{34}$$

#### 4. Determinant representation of the $N$ -kink wave solution

The  $N$ -kink wave solution could be represented in the terms of determinants. Let us consider the following determinant associated with a parameter  $\lambda$

$$p(\lambda) = \det \left( \lambda \delta_{ij} + \frac{4A_i B_j}{(A_i + A_j)(B_i + B_j)} e^{\frac{\xi_i + \xi_j - i\pi}{2}} \right), \tag{35}$$

where  $\delta_{ij}$  is a characteristic function

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}; \tag{36}$$

and  $A_i = c_i - k_i$ ,  $A_j = c_j - k_j$ ,  $B_i = k_i - l_i$ ,  $B_j = k_j - l_j$ . Apparently,  $p(\lambda)$  is a  $N$ -th degree polynomial. So, let

$$p(\lambda) = \lambda^N + a_1 \lambda^{N-1} + a_2 \lambda^{N-2} + \dots + a_N, \tag{37}$$

where  $a_1, a_2, \dots, a_N$  are  $N$  coefficients. Obviously,

$$a_N = p(0) = \begin{vmatrix} e^{\xi_1 - \frac{i\pi}{2}} & \frac{4A_1 B_1}{(A_1 + A_2)(B_1 + B_2)} e^{\frac{\xi_1 + \xi_2 - i\pi}{2}} & \dots & \frac{4A_1 B_1}{(A_1 + A_N)(B_1 + B_N)} e^{\frac{\xi_1 + \xi_N - i\pi}{2}} \\ \frac{4A_2 B_2}{(A_2 + A_1)(B_2 + B_1)} e^{\frac{\xi_2 + \xi_1 - i\pi}{2}} & e^{\xi_2 - \frac{i\pi}{2}} & \dots & \frac{4A_2 B_2}{(A_2 + A_N)(B_2 + B_N)} e^{\frac{\xi_2 + \xi_N - i\pi}{2}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{4A_N B_N}{(A_N + A_1)(B_N + B_1)} e^{\frac{\xi_N + \xi_1 - i\pi}{2}} & \frac{4A_N B_N}{(A_N + A_2)(B_N + B_2)} e^{\frac{\xi_N + \xi_2 - i\pi}{2}} & \dots & e^{\xi_N - \frac{i\pi}{2}} \end{vmatrix} \tag{38}$$

$$= (-4i)^N A_1 B_1 A_2 B_2 \dots A_N B_N e^{\xi_1 + \xi_2 + \dots + \xi_N} \tag{39}$$

$$\times \begin{vmatrix} \frac{1}{4A_1 B_1} & \frac{1}{(A_1 + A_2)(B_1 + B_2)} & \dots & \frac{1}{(A_1 + A_N)(B_1 + B_N)} \\ \frac{1}{(A_2 + A_1)(B_2 + B_1)} & \frac{1}{4A_2 B_2} & \dots & \frac{1}{(A_2 + A_N)(B_2 + B_N)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(A_N + A_1)(B_N + B_1)} & \frac{1}{(A_N + A_2)(B_N + B_2)} & \dots & \frac{1}{4A_N B_N} \end{vmatrix} \tag{40}$$

$$= (-4i)^N A_1 B_1 A_2 B_2 \dots A_N B_N e^{\xi_1 + \xi_2 + \dots + \xi_N} \left( \frac{1}{4A_1 B_1} \prod_{i=2}^N \frac{(A_i - A_j)(B_i - B_j)}{(A_i + A_j)(B_i + B_j)} \right) \tag{41}$$

$$\times \begin{vmatrix} 1 & \frac{1}{(A_1+A_2)(B_1+B_2)} & \cdots & \frac{1}{(A_1+A_N)(B_1+B_N)} \\ 0 & \frac{1}{4A_2B_2} & \cdots & \frac{1}{(A_2+A_N)(B_2+B_N)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{1}{(A_N+A_2)(B_N+B_2)} & \cdots & \frac{1}{4A_NB_N} \end{vmatrix} \tag{42}$$

$$= \dots \tag{43}$$

$$= (-4i)^N A_1 B_1 A_2 B_2 \cdots A_N B_N e^{\xi_1 + \xi_2 + \cdots + \xi_N} \times \left( \frac{1}{4^N A_1 B_1 A_2 B_2 \cdots A_N B_N} \prod_{1 \leq i < j}^N \frac{(A_i - A_j)(B_i - B_j)}{(A_i + A_j)(B_i + B_j)} \right) \tag{44}$$

$$= (-i)^N e^{\xi_1 + \xi_2 + \cdots + \xi_N} \prod_{1 \leq i < j}^N \frac{(A_i - A_j)(B_i - B_j)}{(A_i + A_j)(B_i + B_j)}. \tag{45}$$

Repeating the above procedure, we shall obtain

$$\begin{aligned} a_N &= p(0), \\ a_{N-1} &= p'(0), \\ a_{N-2} &= \frac{1}{2} p''(0), \\ &\dots, \end{aligned} \tag{46}$$

where / stands for  $\frac{dp(\lambda)}{d\lambda}$ .

Comparing all the coefficients (46) with the numerator of (34), we can readily find that

$$\sum_{\mu=0,1} \exp \left( \sum_{i=1}^N \mu_i (\xi_i - i \frac{\pi}{2}) + \sum_{1 \leq i < j}^N \mu_i \mu_j A_{ij} \right) = a_N + a_{N-1} + a_{N-2} + \cdots + a_1 = p(1), \tag{47}$$

and

$$u = 2i \ln \frac{\det \left( \lambda \delta_{ij} + \frac{4A_i B_i}{(A_i + A_j)(B_i + B_j)} e^{\frac{\xi_i + \xi_j - i\pi}{2}} \right)}{\det \left( \lambda \delta_{ij} + \frac{4A_i B_i}{(A_i + A_j)(B_i + B_j)} e^{\frac{\xi_i + \xi_j + i\pi}{2}} \right)}, \tag{48}$$

where  $\delta_{ij}$  is defined in (36).

### 5. Interaction of kink waves

Now, let us consider the two-kink wave solution

$$u = 2i \ln \frac{1 - e^{\xi_1 + \xi_2 + A_{12}} - i(e^{\xi_1} + e^{\xi_2})}{1 - e^{\xi_1 + \xi_2 + A_{12}} - i(e^{\xi_1} + e^{\xi_2})}$$

where

$$\xi_j = c_j t + k_j x + l_j y, \quad k_j^2 - k_j c_j - k_j l_j + l_j c_j = 1, \quad j = 1, 2,$$



and investigate the interaction of two-kink waves solutions. Without loss of a generality, let us assume that  $k_1 = mk_2$ ,  $l_1 = ml_2$ , where  $m$  is a non-zero real number, then  $c_1 = \frac{k_1^2 - k_1 l_1 - 1}{k_1 - l_1}$  and  $c_2 = \frac{m^2 k_1^2 - m^2 k_1 l_1 - 1}{m(k_1 - l_1)}$ . Obviously,  $\xi_2$  can be rewritten in terms of  $\xi_1$ , that is,

$$\xi_2 = m\xi_1 - \frac{m - 1}{m(k_1 - l_1)}t. \tag{49}$$

Therefore, in the orbit of constant  $\xi_1$ , when  $t \rightarrow -\infty$  both  $e^{\xi_1 + \xi_2 + A_{12}}$  and  $e^{\xi_1}$  are approaching 0 and

$$u \sim 2i \ln \frac{1 - ie^{\xi_1}}{1 + ie^{\xi_1}} = 4 \arctan e^{\xi_1}. \tag{50}$$

When  $t \rightarrow \infty$ , apparently  $ie^{\xi_1}$ ,  $e^{\xi_1 + \xi_2 + A_{12}}$  and  $e^{\xi_2}$  are dominant terms. Hence, we have

$$u \sim 2i \ln \frac{e^{\xi_1 + \xi_2 + A_{12}} - ie^{\xi_1}}{e^{\xi_1 + \xi_2 + A_{12}} + ie^{\xi_1}} = 4 \arctan e^{-(\xi_1 + A_{12})}. \tag{51}$$

Adopting the similar procedure as above and playing the same scenario in the orbit of constant  $\xi_2$ , we then know that  $\xi_1$  can be rewritten by  $\xi_2$  as

$$\xi_1 = \frac{\xi_2}{m} + \frac{1 - m^2}{m^2(k_1 - l_1)}t. \tag{52}$$

Therefore, in the orbit of constant  $\xi_2$ , when  $t \rightarrow -\infty$  and  $t \rightarrow \infty$ , we obtain the following two asymptotic formulations

$$u \sim 2i \ln \frac{e^{\xi_1 + \xi_2 + A_{12}} - ie^{\xi_2}}{e^{\xi_1 + \xi_2 + A_{12}} + ie^{\xi_2}} = 4 \arctan e^{-(\xi_2 + A_{12})} \tag{53}$$

and

$$u \sim 2i \ln \frac{1 - ie^{\xi_2}}{1 + ie^{\xi_2}} = 4 \arctan e^{\xi_2}, \tag{54}$$

respectively.

In light of the above asymptotic analysis, we can conclude that when the kink waves travel alone with the  $x$ -axis, the one on the left travels faster and interacts with the one on the right. After their interactions, two kink waves interchange their position.

### 6. Lie symmetries analysis and traveling wave solutions of the (2+1)-dimensional sine-Gordon equation (12)

Obviously, the following simple transformation

$$v = e^{iu}, \tag{55}$$

sends

$$\sin u = \frac{v - v^{-1}}{2i}, \cos u = \frac{v + v^{-1}}{2}, \tag{56}$$

and

$$u = \arccos \frac{v + v^{-1}}{2}. \tag{57}$$

Substituting Eqs. (55) and (56) into Eq. (12) leads Eq. (12) to the following equation

$$-i \frac{v_{xx}v - v_x^2}{v^2} + i \frac{v_{xt}v - v_x v_t}{v^2} + i \frac{v_{xy}v - v_x v_y}{v^2} - i \frac{v_{yt}v - v_y v_t}{v^2} = \frac{v - v^{-1}}{2i}, \tag{58}$$

which can be reduced to

$$2 \left( v_{xx}v - v_x^2 - v_{xt}v + v_x v_t - v_{xy}v + v_x v_y + v_{yt}v - v_y v_t \right) - v^3 + v = 0. \tag{59}$$

As per the Lie group method shown in [13,14,3,9,26,29,30,27,28], Eq. (59) has the following vector fields,

$$V = \xi_t(x, y, t, u) \frac{\partial}{\partial t} + \xi_x(x, y, t, u) \frac{\partial}{\partial x} + \xi_y(x, y, t, u) \frac{\partial}{\partial y} + \xi_u(x, y, t, u) \frac{\partial}{\partial u}. \tag{60}$$

Then a direct but lengthy computation generates the following results

$$\xi_t = (x + y)F_2 + F_3, \quad \xi_x = (-x - 2y)F_2 + F_4, \quad \xi_y = yF_2 + F_1, \quad \eta_v = 0, \tag{61}$$

where  $F_1, F_2, F_3, F_4$  are arbitrary functions of  $x, y, t$ . Thus, Eq. (59) has the following symmetries

$$V_1 = \frac{\partial}{\partial t}, \quad V_2 = \frac{\partial}{\partial x}, \quad V_3 = \frac{\partial}{\partial y}, \quad V_4 = (x + y)F_2 \frac{\partial}{\partial t} + (-x - 2y)F_2 \frac{\partial}{\partial x} + yF_2 \frac{\partial}{\partial y}, \tag{62}$$

the first three of which are apparently the basic geometry symmetries. It is clear that Eq. (59) definitely has traveling wave solutions. Substituting the traveling wave setting  $v(\xi) = f(k_1x + k_2y - k_3t)$  into Eq. (12) yields the following ordinary differential equation

$$Af''f - Af'^2 - f^3 + f = 0, \tag{63}$$

where  $A = 2k_1^2 + 2k_1k_3 - 2k_1k_2 - 2k_2k_3$ . Let us assume that Eq. (63) admits special solutions in the form of

$$f = a_0 + a_1\phi + a_2\phi^2, \tag{64}$$

where  $a_0, a_1, a_2$  are constants and  $\phi$  satisfies the following well-known Riccati equation

$$\phi' = R + \phi^2, \tag{65}$$

with the following solutions:

$$\phi = -\sqrt{-R} \tanh \sqrt{-R}\xi, \quad \phi = -\sqrt{-R} \coth \sqrt{-R}\xi, \quad R < 0 \tag{66}$$

and

$$\phi = \sqrt{R} \tan \sqrt{R}\xi, \quad \phi = -\sqrt{R} \cot \sqrt{R}\xi, \quad R > 0. \tag{67}$$

Substituting Eqs. (64) and (65) into Eq. (63), we have

$$a_2 = \pm \frac{1}{R}, \quad A = \pm \frac{1}{2R}, \quad a_0 = a_1 = 0. \tag{68}$$

So, we obtain the traveling wave solutions of the (2+1)-dimensional sine-Gordon equation (12):

$$u = \arccos \frac{v + v^{-1}}{2}. \tag{69}$$

Case i) when  $R < 0$ ,

$$v = \mp \tanh^2 \sqrt{-R}\xi, \quad v = \mp \coth^2 \sqrt{-R}\xi, \tag{70}$$

and

$$u = \arccos \left[ \frac{\mp(\cosh(4\sqrt{-R}\xi) - 3)}{2 \sinh^2(2\sqrt{-R}\xi)} \right] \tag{71}$$

where  $\xi = k_1x + k_2y - k_3t$ .

Case ii) when  $R > 0$ ,

$$v = \pm \tan^2 \sqrt{R}\xi, \quad v = \pm \cot^2 \sqrt{R}\xi, \tag{72}$$

and

$$u = \arccos \left[ \frac{\pm(\cos(4\sqrt{R}\xi) - 3)}{2 \sin^2(2\sqrt{R}\xi)} \right] \tag{73}$$

where  $\xi = k_1x + k_2y - k_3t$ .

**Remark.** Let

$$v = e^u, \tag{74}$$

then

$$\sinh u = \frac{v - v^{-1}}{2}, \quad \cosh u = \frac{v + v^{-1}}{2}, \tag{75}$$

and

$$u = \operatorname{arccosh} \frac{v + v^{-1}}{2}. \tag{76}$$

Substituting Eqs. (74) and (76) into the (2+1)-dimensional sinh-Gordon equation (12) yields the same equation as Eq. (59)

$$2 \left( v_{xx}v - v_x^2 - v_{xt}v + v_xv_t - v_{xy}v + v_xv_y + v_{yt}v - v_yv_t \right) - v^3 + v = 0. \tag{77}$$

So, just substituting (62) and (64) into

$$u = \operatorname{arccosh} \frac{v + v^{-1}}{2} = \begin{cases} \operatorname{arccosh} \left[ \frac{\mp(\cosh(4\sqrt{-R}\xi) - 3)}{2 \sinh^2(2\sqrt{-R}\xi)} \right], & R < 0 \\ \operatorname{arccosh} \left[ \frac{\pm(\cos(4\sqrt{R}\xi) - 3)}{2 \sin^2(2\sqrt{R}\xi)} \right], & R > 0 \end{cases}, \tag{78}$$

which is the exact traveling wave solutions of the (2+1)-dimensional sinh-Gordon equation (14), with  $\xi = k_1x + k_2y - k_3t$ .

**7. Conservation laws**

Below, we present the multipliers  $Q$  with the corresponding conserved forms  $T^t dx dy + T^y dx dt + T^x dy dt$  (where  $(T^t, T^x, -T^y)$  is the conserved vector). The computed multipliers are up to first order in derivatives of  $u$ . Given here, there are infinitely many as all functions  $f_i(x + y + t)$ ,  $i = 1, 2, 3$ , of Eq. (12) viz.,

$$Q = [(x+t)u_y - (2t+x)u_x + tu_t]f_1(x+y+t) \\ + (u_t - u_y)f_2(x+y+t) + (u_x - u_y)f_3(x+y+t) + ku_y,$$

where  $k$  is a constant.

We construct the conserved forms for some special cases in which the term  $T^t dx dy$  leads to the ‘conserved density’.

1.  $Q_1 = u_x$ :

$$[\frac{1}{4}u_x u_y - \frac{1}{4}u_x^2 - \frac{1}{4}u u_{xy} + \frac{1}{4}u u_{xx}] dx dy \\ + [-\frac{1}{4}u_t u_x + \frac{1}{4}u_x^2 + \frac{1}{4}u u_{xt} - \frac{1}{4}u u_{xx}] dx dt \\ + [-1 - \frac{1}{4}u_t u_x - \frac{1}{4}u_x u_y + \frac{1}{2}u_x^2 - \frac{1}{4}u u_{xt} - \frac{1}{4}u u_{xy} + \frac{1}{2}u u_{yt} + \cos u] dy dt.$$

2.  $Q_2 = u_y$ :

$$[\frac{1}{4}u_y^2 - \frac{1}{4}u_x u_y - \frac{1}{4}u u_{yy} + \frac{1}{4}u u_{xy}] dx dy \\ + [1 - \frac{1}{4}u_t u_y + \frac{1}{4}u_x u_y - \frac{1}{4}u u_{yt} + \frac{1}{4}u u_{xy} - \frac{1}{2}u u_{xx} + \frac{1}{2}u u_{xt} - \cos u] dx dt \\ + [-\frac{1}{4}u_t u_y - \frac{1}{4}u_y^2 + \frac{1}{2}u_x u_y + \frac{1}{4}u u_{yt} + \frac{1}{4}u u_{yy} - \frac{1}{2}u u_{xy}] dy dt.$$

3.  $Q_3 = u_t$ :

$$[-1 + \frac{1}{4}u_t u_y - \frac{1}{4}u_t u_x + \frac{1}{4}u u_{yt} - \frac{1}{4}u u_{xt} + \frac{1}{2}u u_{xx} - \frac{1}{2}u u_{xy} + \cos u] dx dy \\ + [-\frac{1}{4}u_t^2 + \frac{1}{4}u_t u_x + \frac{1}{4}u u_{tt} - \frac{1}{4}u u_{xt}] dx dt \\ + [-\frac{1}{2}u_t^2 + \frac{3}{4}u_t u_x + \frac{1}{2}u u_{tt} - \frac{3}{4}u u_{xt} - \frac{1}{4}u_t u_y + \frac{1}{4}u u_{yt}] dy dt.$$

For the following special cases, we only present the conserved density component of the conserved form,  $T^t$  due to the cumbersome nature of it.

4.  $Q_4 = (x+y+t)(u_x - u_y)$ :

$$\frac{1}{2}u_x u_y x - \frac{1}{4}u_y^2 x + \frac{1}{2}u_y y u_x - \frac{1}{4}y u_y^2 \\ + \frac{1}{2}u_x u_y t - \frac{1}{4}u_y^2 t - \frac{1}{4}u_x^2 x - \frac{1}{4}y u_x^2 - \frac{1}{4}u_x^2 t \\ - \frac{1}{2}u_x u_{xy} - \frac{1}{2}u_y u_{xy} - \frac{1}{2}u t u_{xy} + \frac{1}{4}u u_{yy} x \\ + \frac{1}{4}u u_{yy} y + \frac{1}{4}u u_{yy} t + \frac{1}{4}u_x u_{xx} + \frac{1}{4}u_y u_{xx} + \frac{1}{4}u t u_{xx}.$$

$$5. Q_5 = (x + y + t)(u_t - u_y):$$

$$\begin{aligned} & -x - t - y + \frac{1}{4}u_x u_y x + \frac{1}{4}u_y y u_x + \frac{1}{4}u_x u_y t - \frac{3}{4}u_x u_x y \\ & - \frac{3}{4}u_y u_x y - \frac{3}{4}u_t u_x y + \frac{1}{4}u u_{yy} x + \frac{1}{4}u u_{yy} y \\ & + \frac{1}{4}u u_{yy} t + \frac{1}{2}u_x u_{xx} + \frac{1}{2}u_y u_{xx} + \frac{1}{2}u_t u_{xx} \\ & - \frac{1}{4}u_y^2 x - \frac{1}{4}u_y u_y^2 - \frac{1}{4}u_y^2 t + y \cos u + x \cos u + t \cos u - \frac{1}{4}u_t u_x t \\ & + \frac{1}{4}u_t u_y t - \frac{1}{4}u_y u_{xt} - \frac{1}{4}u_x u_{xt} - \frac{1}{4}u_t u_x t \\ & + \frac{1}{4}u_t u_y x - \frac{1}{4}u_t u_x x - \frac{1}{4}u_t y u_x + \frac{1}{4}u_t y u_y \\ & + \frac{1}{4}u_x u_{yt} + \frac{1}{4}u_y u_{yt} + \frac{1}{4}u_t u_{yt}. \end{aligned}$$

$$6. Q_6 = -u_y x - u_y t + 2u_x t + u_x x - t u_t:$$

$$\begin{aligned} & -t + \frac{1}{4}u_y^2 x + \frac{1}{4}u_y^2 t + \frac{1}{4}u_x^2 x + \frac{1}{2}u_x^2 t \\ & + \frac{1}{4}u u_y - \frac{1}{4}u u_x - \frac{1}{2}u_x u_y x + t \cos u \\ & - \frac{3}{4}u_x u_y t - \frac{1}{4}u u_{yy} x - \frac{1}{4}u u_{yy} t - \frac{1}{4}u_x u_{xx} \\ & + \frac{1}{4}u_t u_y t - \frac{1}{4}u_t u_x t - \frac{1}{4}u_t u_{xt} + \frac{1}{4}u_t u_{yt} \\ & + \frac{1}{2}u_x u_{xy} + \frac{1}{4}u_t u_{xy}. \end{aligned}$$

## 8. Conclusions

In this paper, under the extended Lax pair (3) for the (2+1)-dimensions, we derived a (2+1)-dimensional sine-Gordon and a (2+1)-dimensional sinh-Gordon equation. Kink wave solutions, multi-kink wave interactions, and traveling wave solutions are derived. At last, some conservation laws are presented for some special cases. In this paper, we got the (2+1)-dimensional sin-Gordon equation and sinh-Gordon equation. Conservation laws and kink wave solutions are derived, these results also provides a good basis for the effectiveness of some numerical methods, such as invariant discretization schemes [36,37], structure-preserving method [38–45] and so on. It is worth mention that, at this point, there are some issues need to be studied further, such as nonlocal symmetry, symmetry reductions, more exact solutions as well as their versions with variable coefficients. They will be reported in future works.

## Declaration of competing interest

The authors declare that they have no conflict of interest.

## Acknowledgements

This work is supported by Natural Science Foundation of Hebei Province of China (No. A2018207030, F2017207010) and Youth Key Program of Hebei University of Economics and Business (2018QZ07).

## References

- [1] M.J. Ablowitz, D.J. Kaup, A.C. Newell, H. Segur, Method for solving the sine-Gordon equation, *Phys. Rev. Lett.* 30 (25) (1973) 1262.
- [2] M.J. Ablowitz, H. Segur, *Solitons and the Inverse Scattering Transform*, SIAM, Philadelphia, 1981.
- [3] G.W. Bluman, S. Kumei, *Symmetries and Differential Equations*, Springer, New York, 1989.
- [4] D.Y. Chen, *Introduction of Solitons*, Science Press, Beijing, 2003.
- [5] C.H. Gu, H.S. Hu, Explicit solutions to the intrinsic generalization for the wave and sine-Gordon equations, *Lett. Math. Phys.* 29 (1993) 1–11.
- [6] C.H. Gu, H.S. Hu, Z.X. Zhou, *Darboux Transformations in Integrable Systems*, Springer, Berlin, 2005.
- [7] H.S. Hu, Soliton and differential geometry, in: *Soliton Theory and Its Applications*, Springer, Berlin, 1995, pp. 297–336.
- [8] R. Hirota, *The Direct Method in Soliton Theory*, Cambridge University Press, 2004.
- [9] N.H. Ibragimov (Ed.), *CRC Handbook of Lie Group Analysis of Differential Equations*, vols. 1–3, CRC Press, Boca Raton, FL, 1994.
- [10] J.J. Klein, Geometrical interpretation of the solutions of the sine-Gordon equation, *J. Math. Phys.* 26 (1985) 2181.
- [11] M.M. Latha, C.C. Vasanthi, An integrable model of (2+1)-dimensional Heisenberg ferromagnetic spin chain and soliton excitations, *Phys. Scr.* 89 (2014) 065204.
- [12] G. Leibbrandt, R. Morf, S. Wang, Solutions of the sine-Gordon equation in higher dimensions, *J. Math. Phys.* 21 (1980) 1613–1624.
- [13] P.J. Olver, *Application of Lie Group to Differential Equation*, Springer, New York, 1986.
- [14] L.V. Ovsiannikov, *Group Analysis of Differential Equations*, Academic Press, New York, 1982.
- [15] Z.J. Qiao, Commutator representations of three isospectral equation hierarchies, *Chin. J. Contemp. Math.* 14 (1993) 41–51.
- [16] Z.J. Qiao, A general approach for getting the commutator representations of the hierarchies of nonlinear evolution equations, *Phys. Lett. A* 195 (1994) 319–329.
- [17] Z.J. Qiao, Generation of the hierarchies of solitons and generalized structure of the commutator representation, preprint 1992, *Acta Appl. Math. Sin.* 18 (1995) 287–301.
- [18] Z.J. Qiao, Two new hierarchies containing the sine-Gordon and sinh-Gordon equation, and their Lax representations, *Physica A* 243 (1997) 141–151.
- [19] Z.J. Qiao, Generalized structure of lax representations for nonlinear evolution equation, *Appl. Math. Mech.* 18 (1997) 671–677.
- [20] Z.J. Qiao, W. Strampp, Negative order MKdV hierarchy and a new integrable Neumann-like system, *Physica A* 313 (2002) 365–380.
- [21] Z.J. Qiao, C.W. Cao, W. Strampp, Category of nonlinear evolution equations, algebraic structure, and r-matrix, *J. Math. Phys.* 44 (2003) 701–722.
- [22] Z.J. Qiao, J.B. Li, Negative order KdV equation with both solitons and kink wave solutions, *Europhys. Lett.* 94 (2011) 50003.
- [23] Z.J. Qiao, E. Fan, Negative-order Korteweg-de Vries equations, *Phys. Rev. E* 86 (2012) 016601.
- [24] J. Rubinstein, Sine-Gordon equation, *J. Math. Phys.* 11 (1970) 258.
- [25] K. Tenenblat, P. Winternitz, On the symmetry groups of the intrinsic generalized wave and sine-Gordon equations, *J. Math. Phys.* 34 (1993) 3527.
- [26] G.W. Wang, A.H. Kara, K. Fakhar, Symmetry analysis and conservation laws for the class of time-fractional nonlinear dispersive equation, *Nonlinear Dyn.* 82 (2015) 281–287.
- [27] G.W. Wang, A.H. Kara, Nonlocal symmetry analysis, explicit solutions and conservation laws for the fourth-order Burgers equation, *Chaos Solitons Fractals* 81 (2015) 290–298.
- [28] G.W. Wang, K. Fakhar, Lie symmetry analysis, nonlinear self-adjointness and conservation laws to an extended (2+1)-dimensional Zakharov-Kuznetsov-Burgers equation, *Comput. Fluids* 119 (2015) 143–148.
- [29] G.W. Wang, Symmetry analysis and rogue wave solutions for the (2+1)-dimensional nonlinear Schrödinger equation with variable coefficients, *Appl. Math. Lett.* 56 (2016) 56–64.
- [30] G.W. Wang, A.H. Kara, K. Fakhar, J. Vega-Guzman, A. Biswas, Group analysis, exact solutions and conservation laws of a generalized fifth order KdV equation, *Chaos Solitons Fractals* 86 (2016) 8–15.
- [31] Y. Zarmi, Nonlinear quantum-mechanical system associated with Sine-Gordon equation in (1+2) dimensions, *J. Math. Phys.* 55 (2014) 103510.
- [32] R.G. Zhou, Hierarchy of negative order equation and its Lax pair, *J. Math. Phys.* 36 (1995) 4220.

- [33] Z. Zhou, Finite dimensional Hamiltonians and almost-periodic solutions for 2+1 dimensional three-wave equations, *J. Phys. Soc. Jpn.* 71 (2002) 1857–1863.
- [34] G.W. Wang, *Studies on Symmetry Group, Invariant Solutions and Conservation Laws of Nonlinear Partial Differential Equations*, Beijing Institute of Technology, Beijing, 2017.
- [35] G.W. Wang, A.H. Kara, A (2+1)-dimensional KdV equation and mKdV equation: symmetries, group invariant solutions and conservation laws, *Phys. Lett. A* 383 (2019) 728–731.
- [36] A. Bihlo, et al., The Korteweg–de Vries equation and its symmetry-preserving discretization, *J. Phys. A, Math. Theor.* 48 (2015) 055201.
- [37] V.A. Dorodnitsyn, *Application of Lie Groups to Difference Equations*, vol. 8, Chapman, Hall, Florida, 2010.
- [38] W. Hu, et al., Generalized multi-symplectic integrators for a class of Hamiltonian nonlinear wave PDEs, *J. Comput. Phys.* 235 (2013) 394–406.
- [39] W. Hu, et al., Chaos in an embedded single-walled carbon nanotube, *Nonlinear Dyn.* 72 (2013) 389–398.
- [40] J.E. Macías-Díaz, A structure-preserving method for a class of nonlinear dissipative wave equations with Riesz space-fractional derivatives, *J. Comput. Phys.* 351 (15) (2017) 40–58.
- [41] W. Hu, et al., Almost structure-preserving analysis for weakly linear damping nonlinear Schrodinger equation with periodic perturbation, *Commun. Nonlinear Sci. Numer. Simul.* 42 (2017) 298–312.
- [42] T.J. Bridges, Multi-symplectic structures and wave propagation, *Math. Proc. Camb. Philos. Soc.* 121 (1997) 147–190.
- [43] W. Hu, Z. Deng, Non-sphere perturbation on dynamic behaviors of spatial flexible damping beam, *Acta Astron.* 152 (2018) 196–200.
- [44] T.J. Bridges, S. Reich, Multi-symplectic integrators: numerical schemes for Hamiltonian PDEs that conserve symplecticity, *Phys. Lett. A* 284 (2001) 184–193.
- [45] W. Hu, et al., Symmetry breaking of infinite-dimensional dynamic system, *Appl. Math. Lett.* 103 (2020) 106207.