Coupled out of plane vibrations of spiral beams

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An analytical method is proposed to calculate the natural frequencies and corresponding mode shape functions of an Archimedean spiral beam. The deflection of the beam is due to both bending and torsion, which makes the problem coupled in nature. The governing partial differential equation and the boundary conditions are derived using Hamilton's principle. The vibration problem of a constant radius curved beam is solved using a general exponential solution with complex coefficients. Two factors make the vibrations of spirals different from oscillations of constant radius arcs. The first is the presence of terms with derivatives of the radius in the governing equations of spirals and the second is the fact that variations of radius of the beam causes the coefficients of the differential equations to be variable. It is demonstrated, using perturbation techniques that the R' terms have negligible effect on the structure's dynamics. The spiral is then approximated with many merging constant-radius curved sections joint together to consider the slow change of radius along the spiral. The natural frequencies and mode shapes of two spiral structures have been calculated for illustration.

Nomenclature

EI_x: Bending stiffness

GJ: Torsional stiffness

$k = GJ/EI_x$: Stiffness parameter

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$m(\theta)$: Mass per unit length of spiral

M_x : Bending moment

M_z: Twist torque

$R(\theta)$: Radius of the curved beam

s : Position coordinate along the arc

$v(\theta, t)$: Out of plane deflection

$x' = \partial x/\partial\theta$; x is an arbitrary parameter

$\dot{x} = \partial x/\partial t$; x is an arbitrary parameter

α= Total length of the arc

$\beta(\theta, t)$: Twist angle

θ: Angular position in polar coordinate system

I. Introduction

We are motivated to look at the vibrations of spiral shaped structures as a prelude to energy harvesting using the piezoelectric effect. Because of the unique coupled bending-twist mechanics of spiral beams our hope is to create a MEMS scale energy harvesting device, which will have natural frequencies in a useable region for a MEMS energy harvester. The idea of using spiral structures to achieve low frequency vibrational energy harvesters was first proposed in [1]. However the vibrational analysis of curved beam with varying radius (spirals) is missing in the literature [2]. This paper attempts to solve the free vibrations of spiral beams and pave the way to the modeling of spiral MEMS harvesting devices.

Out-of plane vibrations of circular curved beams have been studied by many investigators. Here we retreat our attention to just out-of-plane vibrations of curved beams having variable radius of curvature. In historical order: Love[3] derived the equations of curved beams of arbitrary geometry. Chang and Volterra [4] obtained the upper and lower bounds of the first four natural frequencies of elastic clamped arcs of which the center lines were in the forms of circles, cycloids, catenaries, and parabola by using a method based on differential operator theory. Wang [5] employed the Rayleigh-Ritz method to predict the natural frequency of clamped elliptic arc. Irie et al [6] determined the steady state response of a Timoshenko curved beam with circular, elliptical, catenary and parabolical neutral axes driven at the free end by use of the transfer matrix approach. Huang et al [7] presented the dynamic responses of non-circular Timoshenko curved beam with viscous damping by using a numerical Laplace transform approach. They considered the numerical examples for a two-span elliptic beam subjected to a rectangular impulse. Then, Huang et al [8] developed a dynamic stiffness matrix by using the Laplace transform technique for both the free vibration and forced vibration of non-uniform parabolic curved beams with various ratios of rise to span. Tufekci and Dogruer [9] obtained the exact solution of the differential equations for the static behavior of an arch with varying curvature and cross section including the shear deformation effect by using the initial value method.

Despite the long history of attempts to solve for the vibrations of arcs, a straight forward solution of the vibration of *spiral* beams, suitable for a design procedure, is still unavailable. In this paper we study the dynamic behavior of spiral beams. The fact that the radius of the beam *slowly* changes along the spiral validates use of perturbation methods and discretization of the beam. The governing differential equations and corresponding boundary conditions are derived simultaneously from Hamilton's principle. A new general solution is proposed for the curved beam with constant radius. Next, the effect of terms including R' is shown to be negligible using the multiple scales method. Subsequently, to tackle change of radius in spirals, the beam is approximated with several merged constant radius arcs and the vibrations of the approximation is studied. The natural frequency and mode shapes of a clamped-free full turn spiral as well as the mode shapes of a five turn spiral are finally calculated and plotted.

II. Governing Equations

The moment-displacement relationships for the spiral depicted in Fig.1 are given by Love [3] as follows

$$M_x = E I_x \kappa \quad , \quad M_z = G J \tau \tag{1,2}$$

where

$$\kappa = \frac{\beta}{R} - \frac{\partial^2 v}{\partial s^2} \quad , \quad \tau = \frac{\partial \beta}{\partial s} + \frac{1}{R}\frac{\partial v}{\partial s} \tag{3,4}$$

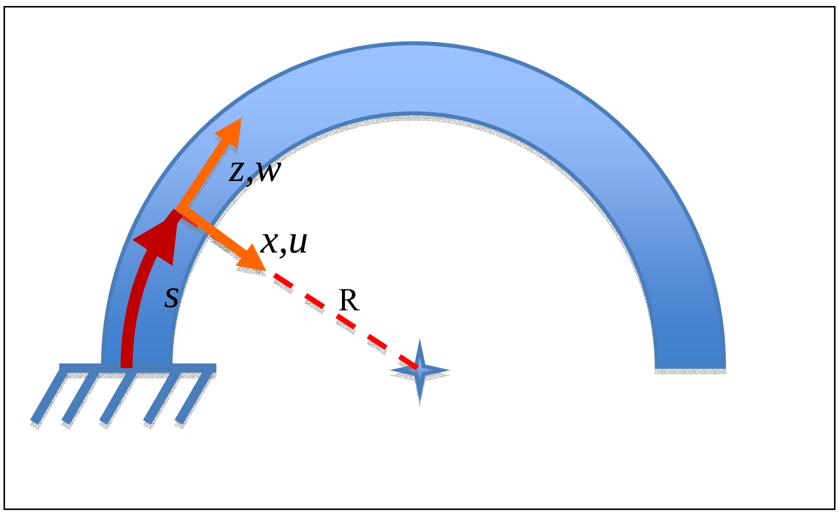

Fig1. Coordinates

The strain energy U and the kinetic energy T of the curved beam for out-of-plane motion are

$$U = \frac{1}{2}\int_0^{S_L} (M_x \kappa + M_z \tau - Y v - \Phi \beta)\, ds \tag{5}$$

$$T = \frac{1}{2}\int_0^{S_L} \left(m\, \dot{v}^2 + \imath\, \dot{\beta}^2\right) ds \tag{6}$$

Here, $\imath$ is the mass moment of inertia per unit length of the curved beam. In Eq.(5), Y and Φ are external force and twisting moment, respectively. For the free vibration problem, neglecting rotational inertia, Hamilton's principle, Eqs. (5-6) along with Eqs. (1-4) give the equations of motion as

$$\frac{\partial^2}{\partial s^2}\left[E\,I_x\left(\frac{\beta}{R}-\frac{\partial^2 v}{\partial s^2}\right)\right]+\frac{\partial}{\partial s}\left[\frac{G\,J}{R}\left(\frac{\partial \beta}{\partial s}+\frac{1}{R}\frac{\partial v}{\partial s}\right)\right]=m\ddot{v} \tag{7-a}$$

$$\frac{E\,I_x}{R}\left(\frac{\partial^2 v}{\partial s^2}-\frac{\beta}{R}\right)+\frac{\partial}{\partial s}\left[G\,J\left(\frac{\partial \beta}{\partial s}+\frac{1}{R}\frac{\partial v}{\partial s}\right)\right]=0 \tag{7-b}$$

The boundary conditions associated with equations of motion are:

$$\frac{E\,I_x}{R}\left(\frac{\partial^2 v}{\partial s^2}-\frac{\beta}{R}\right)\delta\left(\frac{\partial v}{\partial s}\right)\Bigg|_0^{S_L}=0 \tag{8}$$

$$G\,J\left(\frac{\partial \beta}{\partial s}+\frac{v}{R}\right)\delta\beta\,\Bigg|_0^{S_L}=0 \tag{9}$$

$$\left\{\frac{\partial}{\partial s}\left[E\,I_x\left(\frac{\beta}{R}-\frac{\partial^2 v}{\partial s^2}\right)\right]+\frac{G\,J}{R}\left(\frac{\partial \beta}{\partial s}+\frac{1}{R}\frac{\partial v}{\partial s}\right)\right\}\delta v\,\Bigg|_0^{S_L}=0 \tag{10}$$

In the above equations, $s = R\theta, \frac{\partial}{\partial s} = \frac{1}{R}\frac{\partial}{\partial \theta}, \frac{\partial^2}{\partial s^2} = \frac{1}{R^2}\frac{\partial^2}{\partial \theta^2}$ and so on.

III. Vibration of a constant radius curved beam

Next Consider finding the natural frequencies and mode shapes of a curved beam with *constant radius*. The solution is based on the one presented by Ojalvo [10] with a different choice of general solution. Assuming R is constant in Eq. (7), one gets the following governing differential equations.

$$v^{iv} - R\beta'' - k(v'' + R\beta'') = -\frac{\rho A R^4}{E I_x}\ddot{v} \tag{11-a}$$

$$v'' - R\beta + k(R\beta'' + v'') = 0 \tag{11-b}$$

Evaluating v'' from (11-b) and substituting that in (11-a) we can derive,

$$\beta^{(4)} + 2\beta'' + \beta = \frac{\rho A R^4}{GJ}\frac{1+k}{R}\ \ddot{v} \tag{12}$$

The separation of variables is now used to solve the above partial differential equation by assuming

$$\beta(\theta, t) = B(\theta)T(t),\ v(\theta, t) = V(\theta)T(t) \tag{13}$$

Substituting for β and v from (13) into (12) and grouping the terms which depend on the same variable together results,

$$\frac{B^{(4)}+2B''+B}{V} = \frac{\rho A R^4}{GJ}\frac{1+k}{R}\frac{\ddot{T}}{T} = \gamma \tag{14}$$

γ is a constant not depending on eighter β or t. The right side of Eq. (14) prescribes a harmonic variation for T, $T = C\cos(\omega_n\, t + \phi)$. The natural frequency depends on the value of γ given by

$$\omega_n^2 = -\frac{\ddot{T}}{T} = -\frac{GJ}{\rho A R^4}\frac{K+1}{R}\ \gamma \tag{15}$$

Evaluating V in terms of B from (11-b) and then using the left side of Eq. (14) gives:

$$B^{(6)} + 2B^{(4)} + \left(1 + \frac{kR\gamma}{k+1}\right)B'' - \frac{R\gamma}{k+1}B = 0$$

Here we deviate from [10] as we consider an exponential form of general solution:

$$B = \sum_{i=1}^{6} A_i e^{s_i\theta} \tag{16}$$

Where s_i is one of the complex roots of the characteristic equation, Eq. (17). Each A_i is a constant complex number. Thus

$$s^6 + 2s^4 + \left(1 + \frac{kR\gamma}{k+1}\right)s^2 - \frac{R\gamma}{k+1} = 0 \tag{17}$$

Using Eq. (11-b) V can also be written in corresponding exponential form.

$$V = \sum_{i=1}^{6}\frac{R}{k+1}\ \left(\frac{1}{S_i^2} - k\right)A_i e^{s_i\theta} \tag{18}$$

The boundary conditions can be written in terms of B, V and their derivatives. The resulting six equations can be written in a matrix form:

$$[M]_{6\times6}\begin{bmatrix}A_1\\ \vdots\\ A_6\end{bmatrix}=0_{6\times1} \quad (19)$$

Here the matrix $[M]$ depends on the values of the roots, s_i, which are in turn related to γ by Eq. (17). Eq. (19) yields the trivial solution, $[A_1 \ldots A_6]^T=0$, unless $\det([M])=0$. For specific eigenvalues, $\gamma=\gamma_n$, $\det([M])$ vanishes and the problem can have nontrivial solutions. The natural frequencies of the system are determined by these eigenvalues using Eq. (15) [11].

The main distinction of this method compared to [10] is that we do not assume that each of the general solutions are real. The s_i are complex and the same is true for any $A_i e^{s_i}$. When all added up the solution will be real. This makes it unnecessary to categorize the solutions of Eq. (17) based on the sign of each root and consider them separately (as done in [10]).

The complex approach here was validated finding the natural frequencies of a clamped-clamped constant radius arc. The results precisely matched with those in [10].

IV. The effect of terms containing R' on the vibration of spirals

The radius of a spiral beam varies along the spiral. The radius of Archimedean spiral varies linearly with the polar angle, θ: $R=R_0+\epsilon\theta$, as depicted in Fig.2. Substituting for R from $R=R_0+R'\theta$ in Eq.(7) and conducting separation of variables, similar to previous section, results in

$$RV^{(4)}-4R'V^{(3)}-kRV''+3kR'V'-(k+1)R^2B''+2(k+1)RR'B'=\gamma R^5V \quad \text{(20-a)}$$

$$kR^2B''-kRR'B'-R^2B+(k+1)RV''-2kR'V'=0 \quad \text{(20-b)}$$

$$\omega_n^2=\frac{EI}{\rho A}\gamma$$

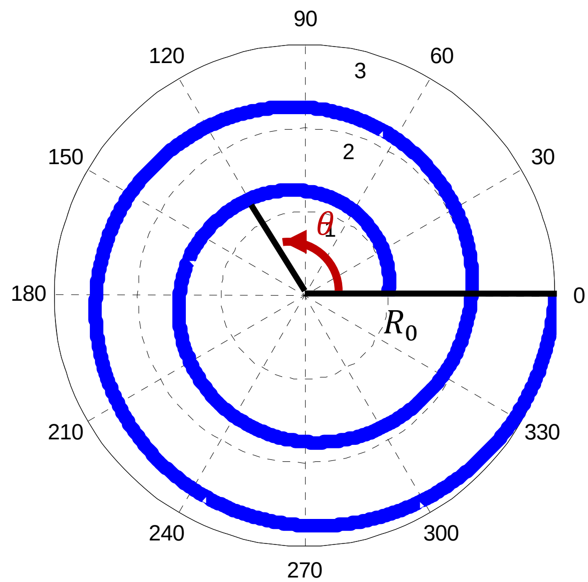

Fig. 2 Archimedean Spiral

Eqs. (20) have two major differences with Eqs. (11). The first is the existence of terms including R'. The second, which is quite fundamental, is that R in Eqs. (20) is no longer a constant and varies as a function of θ. We tackle these complexities one by one. In this section we study how the terms with R' affect the natural frequencies while assuming R is still a constant. Physically this study corresponds to very short spirals, in which the radius varies but since the spiral is too short the radius will be *almost* the same everywhere. Considering an Archimedean spiral Eqs. (20) become,

$$RV^{(4)} - 4\epsilon V^{(3)} - kRV'' + 3k\epsilon V' - (k+1)R^2B'' + 2(k+1)R\epsilon B' = \gamma R^5 V \quad \text{(21-a)}$$

$$kR^2B'' - kR\epsilon B' - R^2B + (k+1)RV'' - 2k\epsilon V' = 0 \quad \text{(21-b)}$$

The rate of change of radius, ϵ, is small so we use Multiple Scales perturbation method [12] to solve Eqs. (21). First we expand the solution of our system as bellow.

$$V = v_0 + \epsilon v_1 + \cdots, B = B_0 + \epsilon B_1 + \cdots \quad \text{(22)}$$

v_0 and B_0 are the solution of Eqs. (21) provided $\epsilon = 0$. The terms ϵV_1 and ϵB_1 can be interpreted as the effect of small perturbations on the answer. In fact, the dependency of V and B on ϵ and θ occurs over different scales namely θ, $\epsilon\theta$, $\epsilon^2\theta$ and so on. $\epsilon\theta$ is a slower scale than θ since it becomes notable only when θ is big. We determine V and B as functions of $T_0 = \theta$ and $T_1 = \epsilon\theta$ rather than a function of ϵ and θ. All the derivatives with respect to θ must be reevaluated using the chain rule in terms of derivatives of T_0 and, T_1 resulting in

$$-R^5\gamma v_0 - R^5\gamma\epsilon v_1 + 2R\epsilon B_0{}^{(1,0)} + 2kR\epsilon B_0{}^{(1,0)} + 3k\epsilon v_0{}^{(1,0)} - 2R^2\epsilon B_0{}^{(1,1)} - 2kR^2\epsilon B_0{}^{(1,1)} - 2kR\epsilon v_0{}^{(1,1)} - R^2 B_0{}^{(2,0)} - kR^2 B_0{}^{(2,0)} - R^2\epsilon B_1{}^{(2,0)} - kR^2\epsilon B_1{}^{(2,0)} - kRv_0{}^{(2,0)} - kR\epsilon v_1{}^{(2,0)} - 4\epsilon v_0{}^{(3,0)} + 4R\epsilon v_0{}^{(3,1)} + Rv_0{}^{(4,0)} + R\epsilon v_1{}^{(4,0)} = 0$$

$$-R^2 B_0 - R^2\epsilon B_1 - kR\epsilon B_0{}^{(1,0)} - 2k\epsilon v_0{}^{(1,0)} + 2kR^2\epsilon B_0{}^{(1,1)} + 2R\epsilon v_0{}^{(1,1)} + 2kR\epsilon v_0{}^{(1,1)} + kR^2 B_0{}^{(2,0)} + kR^2\epsilon B_1{}^{(2,0)} + Rv_0{}^{(2,0)} + kRv_0{}^{(2,0)} + R\epsilon v_1{}^{(2,0)} + kR\epsilon v_1{}^{(2,0)} = 0$$

The derivatives of parameters with respect to T_0 and T_1 are denoted in order in the parenthesis. For example: $v_0^{(3,1)} = \frac{\partial^4 v_0}{\partial^3 T_0\, \partial T_1}$. Equating coefficients of like powers of ϵ in the above equation gives

$$-R^5\gamma v_0 - R^2 B_0{}^{(2,0)} - kR^2 B_0{}^{(2,0)} - kRv_0{}^{(2,0)} + Rv_0{}^{(4,0)} = 0 \qquad \text{(23-a1)}$$

$$-R^5\gamma v_1 - R^2 B_1{}^{(2,0)} - kR^2 B_1{}^{(2,0)} - kRv_1{}^{(2,0)} + Rv_1{}^{(4,0)} = -2RB_0{}^{(1,0)} - 2kRB_0{}^{(1,0)} - 3kv_0{}^{(1,0)} + 2R^2 B_0{}^{(1,1)} + 2kR^2 B_0{}^{(1,1)} + 2kRv_0{}^{(1,1)} + 4v_0{}^{(3,0)} - 4Rv_0{}^{(3,1)} \qquad \text{(23-a2)}$$

$$-R^2 B_0 + kR^2 B_0{}^{(2,0)} + Rv_0{}^{(2,0)} + kRv_0{}^{(2,0)} = 0 \qquad \text{(23-b1)}$$

$$-R^2 B_1 + kR^2 {B_1}^{(2,0)} + R{v_1}^{(2,0)} + kR{v_1}^{(2,0)} = kR{B_0}^{(1,0)} + 2k{v_0}^{(1,0)} - 2kR^2{B_0}^{(1,1)} - 2R{v_0}^{(1,1)} - 2kR{v_0}^{(1,1)} \quad (23\text{-b2})$$

We can solve Eqs. (23-a1) and (23-b1) for v_0 and B_0. The equations are the same as Eqs. (11) and they result in the following

$$B_0(T_0,\ T_1) = \sum_{i=1}^{6} A_i(T_1) e^{s_i T_0},\ \ v_0(T_0,\ T_1) = \sum_{i=1}^{6} \frac{R}{k+1}\left(\frac{1}{S_i^2} - k\right) A_i(T_1)\, e^{s_i T_0} \quad (24)$$

We then substitute (24) into (23-a2) and (23-b3) and eliminate the secular terms to get:

$$A_i(T_1) = a_i e^{\sigma_i T_1} \quad (25\text{-a})$$

$$\sigma_i = \frac{(2kR^5\gamma + k(-3+R^5\gamma - k(3-2R+R^5\gamma)))S_i^2 + (2+k^2(-3+R) - k^3(-1+R) - 2k(1+R))S_i^4 + k(-4+k(-4+R)-R)S_i^6)}{2(1+k)R(R^5\gamma + k(-1+R)S_i^2 - (-1+2k+R)S_i^4 - 2kS_i^6)} \quad (25\text{-b})$$

a_is are neither a function of T_0 or T_1. Substituting (24) and (25) in (22) and neglecting higher orders of ϵ results,

$$B(\theta, \epsilon) = \sum_{i=1}^{6} a_i e^{(s_i + \sigma_i\,\epsilon)\theta},\ \ V(\theta, \epsilon) = \sum_{i=1}^{6} \frac{R}{k+1}\left(\frac{1}{s_i^2} - k\right)\, a_i\, e^{(s_i + \sigma_i\,\epsilon)\theta} \quad (26)$$

Eq. (26) is the result of perturbation analysis. It gives an estimate of the effect of the rate of change of radius, ϵ, on the solution of the system. Comparing Eq. (26) with Eq.(24) reveals that the rate of change of radius shifts the roots of the characteristic equation. It remains to satisfy the boundary conditions to get the natural frequencies. This last step is similar to that of the previous section. The value of matrix M is different from before due to the changes in general solution. A summary of the effect of radius change on the first eigenvalues of a clamped-clamped spiral with $k = 1$ is presented in Table 1.

It can be inferred from Table1 that the derivative terms have only a very small effect on the eigenvalues and consequently on the natural frequencies. We therefore conclude that it is proper to simply ignore their effect in calculating the natural frequencies of spiral beams.

R	α	$R' = \frac{\Delta R}{2\pi}$	γ_1 of Constant Radius	γ_1 of Changing Radius	$\frac{\Delta\gamma_1/\gamma_n}{R'/R}$
10	2π	1	1.91551E-05	1.91550E-05	0.0001
10	2π	5	1.91551E-05	1.91434E-05	0.0012
1	2π	0.1	1.91551E-01	1.92956E-01	0.0733
1	2π	0.5	1.91551E-01	2.20461E-01	0.3019
0.1	2π	0.01	1.91551E+03	1.91481E+03	0.0037
0.1	2π	0.05	1.91551E+03	1.89835E+03	0.0179
0.001	2π	0.0001	1.91551E+11	1.91528E+11	0.0012
0.001	2π	0.0005	1.91551E+11	1.90970E+11	0.0061

Table 1: Effect of derivative terms on eigenvalues

V. The effect of the slow variation of *R* on the vibration of the spiral beam

The above analysis allows us to exclude the effect of derivative terms in spiral beam vibrations. It now remains to evaluate the effect of the slow variation of R on the natural frequencies of the spiral. Excluding the terms which contain R' in Eq. (20) yields

$$RV^{(4)} - kRV'' - (k+1)R^2B'' = \gamma R^5 V \tag{27-a}$$

$$kR^2B'' - R^2B + (k+1)RV'' = 0 \tag{20-b}$$

Considering that the rate of change of radius is small, we can divide the spiral into a number of segments and say that the radius is almost the same (constant) throughout each segment.

The general solution for each of the segments is already derived in Section III. For the i[th] element we have:

$$B_i = \sum_{p=1}^{6} A_{ip} e^{s_{ip}\theta}, V_i = \sum_{p=1}^{6} \frac{R}{k+1}\left(\frac{1}{S_{ip}^2} - k\right) A_{ip} e^{s_{ip}\theta}$$

s_i is one of the six roots of characteristic equation for each of the pieces.

$$s^6 + 2s^4 + (1 - kR^4\mu)s^2 + \mu R^4 = 0, \quad \mu = \frac{\rho A}{GJ}\omega_n^2 = -\frac{\gamma}{(k+1)\,R^3} \tag{28}$$

The above equation can be rewritten in terms of three dimensionless parameters, k, η and r.

$$s^6 + 2s^4 + (1 - kr^4\eta)s^2 + r^4\eta = 0 \tag{29}$$

$$\eta = \frac{\rho A}{GJ} R_0^4 \omega_n^2 \, , r = \frac{R}{R_0} \quad (30)$$

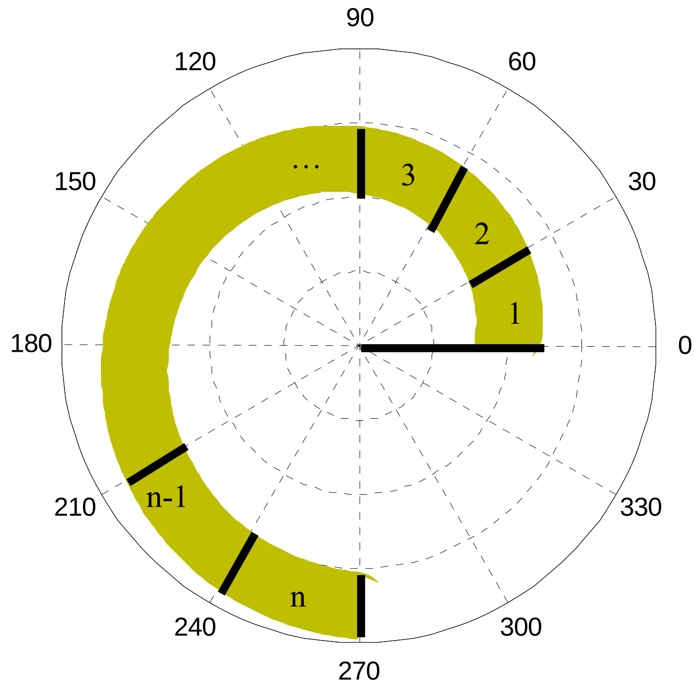

Fig. 3 Discretization of Spiral

The boundary conditions of the spiral are the same as those of constant radius arc, meanwhile some matching conditions between elements are needed to relate the solution in each of the segments to the next. Considering the problem from a degree of freedom aspect, each of the elements has 6 DOFs, namely the A_{ip}s. If the spiral is composed of n elements, there will be $6n$ unknowns. There are six boundary conditions and 6 equilibrium and continuity equations can be written for each of the $n-1$ element interfaces. Therefore the number of equations will be the same as the number of unknowns. The continuity and equilibrium conditions at the piece interfaces are as follows.

Deflection continuity $\quad V_i(\theta_{end}) = V_{i+1}(\theta_{start})$

Slope continuity $\quad V'_i(\theta_{end}) = V'_{i+1}(\theta_{start})$

Twist angle continuity $\beta_i(\theta_{end}) = \beta_{i+1}(\theta_{start})$

Torsion torque equilibrium $M_{z_i}(\theta_{end}) = M_{z_{i+1}}(\theta_{start})$

Bending moment equilibrium $M_{x_i}(\theta_{end}) = M_{x_{i+1}}(\theta_{start})$

Shear force equilibrium $Q_i(\theta_{end}) = Q_{i+1}(\theta_{start})$

The expressions for effort terms in terms of displacements can be distinguished from the boundary conditions expressed in Eqs. 8 to 10:

$$M_z = \frac{GJ}{R^2}(V' + R\beta')\ ,\ \ M_x = \frac{EI}{R^2}(R\beta - V'')\ ,\ Q = EI\left(\frac{\beta'}{R^2}(1+k) + \frac{-V^{(3)}+kV'}{R^3}\right)$$

We end up with the following matrix equation to satisfy the boundary conditions and equilibrium and continuity conditions.

$$[M]_{6n\times 6n}[A_{11}, \dots, A_{16}, A_{21}, \dots, A_{26}, \dots, A_{n1}, \dots, A_{n6}]^T_{6n\times 1} = 0_{6n\times 1} \quad (31)$$

Eq. (31) yields trivial solution unless the determinant of M is zero. The matrix M is a function of s_{ip} which are in turn functions of η. The eigenvalues, η_n, which make the determinant zero, are proportional to second power of natural frequencies of the system (refer to Eq.(28)).

The value of the first eigenvalue, η_1, of single turn spiral beams with varying radius for different beam radii and different number of elements are given in Table 2.

R	$R' = \frac{\Delta R}{2\pi}$	η_1, constant R	η_1, two elements	η_1, three elements	η, four elements	η_1, five elements	η_1, ten elements	η_1, twenty elements
1	0.1	0.1916	0.2365	0.2364	0.2358	0.2357	0.2356	0.2355

Table 2. Effect of radius variations and number of elements on eigenvalues

Since the formulation is nondimensionalized, the results in Table 2 are independent of R_0.

The resulted mode shapes for $k = 1, R'/R_0 = -0.3/2\pi$, $\alpha = 2\pi$ and $n = 10$ is depicted in Figs.4-6. The boundary condition of the spiral is clamped-free.

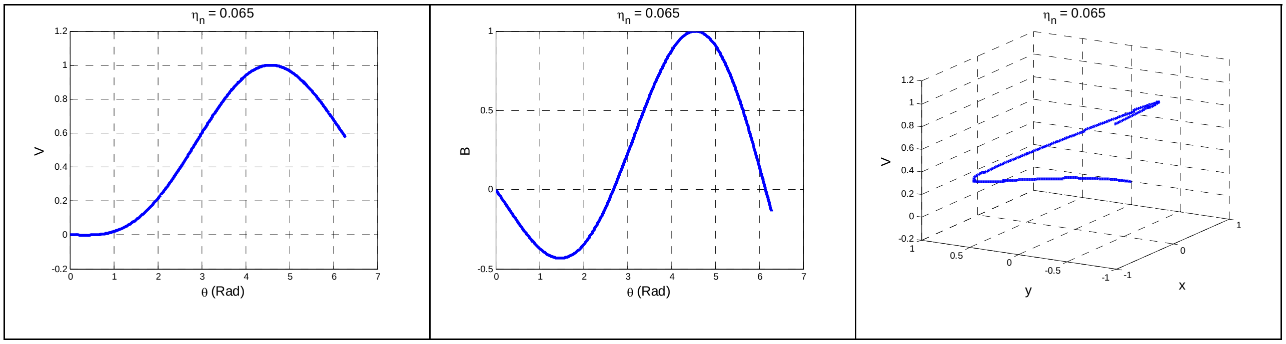

Fig. 4: First Mode shape

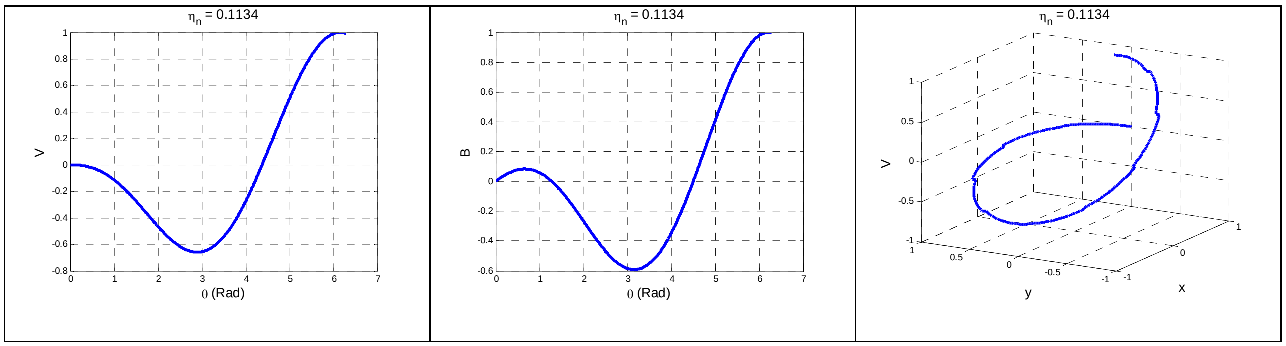

Fig. 5: Second Mode shape

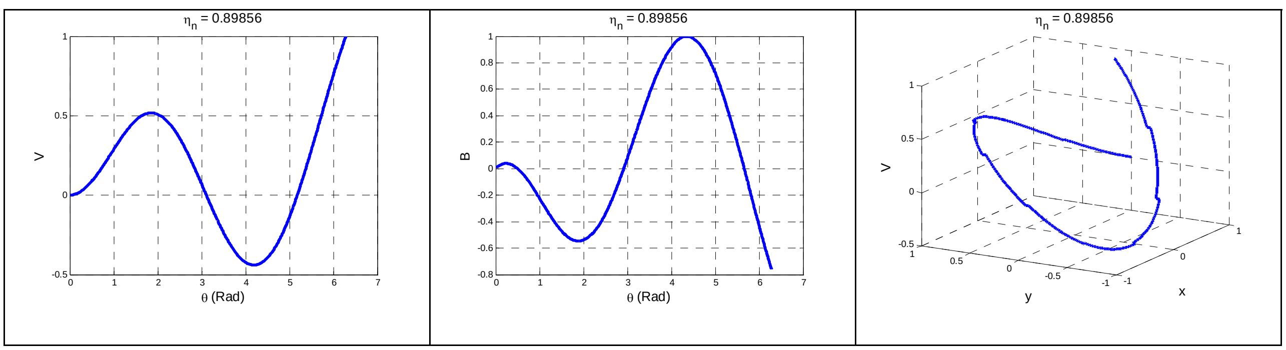

Fig 6: Fourth Mode shape

Finally the mode shapes of a five turn clamped-free spiral with $k = 2, R'/R_0 = -0.1/2\pi$, $\alpha = 10\pi$ and $n = 10$ are depicted in Figs. 7-9.

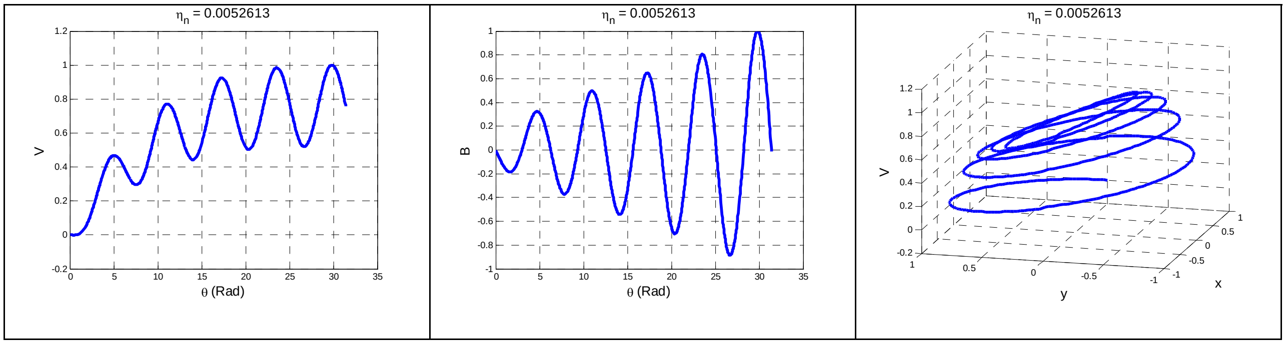

Fig. 7: First Mode shape

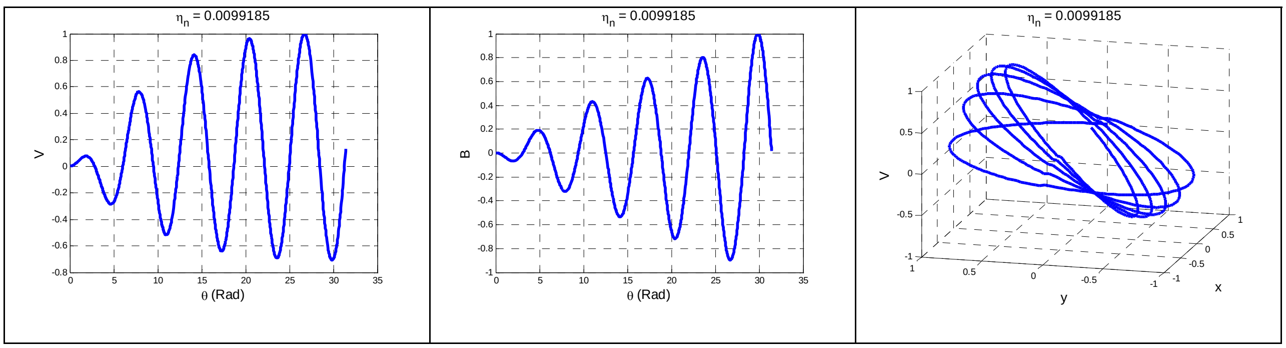

Fig. 8: Second Mode shape

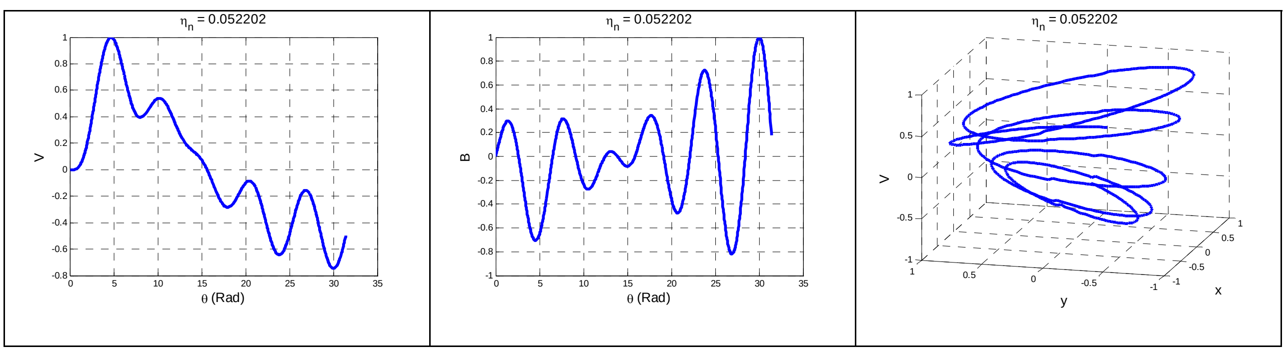

Fig 9: Fourth Mode shape

The relationship between the length of the spiral and the fundamental natural frequency is depicted in Fig. 10. The *y*-axis of Fig.10 indicates the normalized fundamental natural frequency, which is the first resonance frequency divided by that of a spiral beam of the arc length π. In this specific problem radius decreases 10% per each turn and the stiffness parameter, *k*, is 2 [13]. Since the frequency is normalized the trend is identical for all cross sections and materials with the same *k* and for any initial radius of the beam. The fundamental natural frequency exponentially decreases as the length grows. The first resonance frequency drops 12.5 db as the containing angle of spiral increases by a decade.

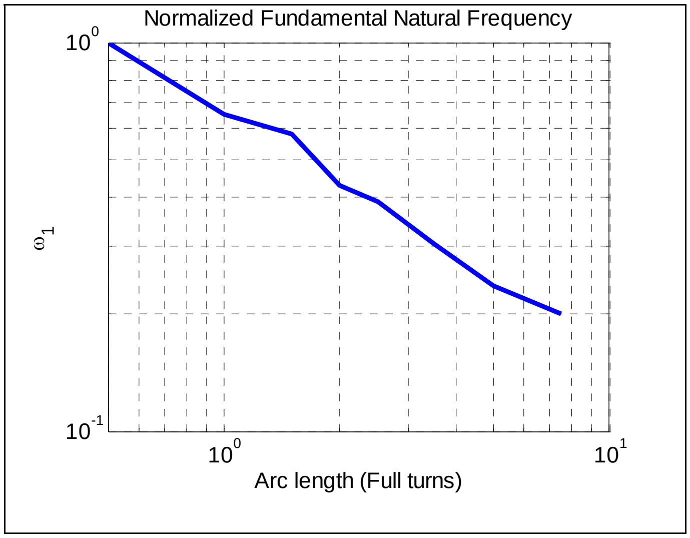

Fig. 10: The effect of length on natural frequency

VI. Conclusion

An analytical method is proposed to calculate the natural frequencies and mode shapes of spiral beams. It has been shown that the effect of derivative terms on the structural dynamics can be neglected. Thus the structure is approximated by many constant radius sections joint together, to tackle the problem of slowly changing coefficients. The piecewise continuous solution converges with relatively few number of elements. The eigenvalues and correspondingly natural frequencies depend on the stiffness ratio the radius, the rate of change of radius and the total angle. The resulted mode shapes of two different spirals are depicted for illustration.

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