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Critical singular problems via concentration-compactness lemma

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Abstract

In this work we consider existence and multiplicity results of nontrivial solutions for a class of quasilinear degenerate elliptic equations in $\mathbb{R}^N$ of the form

$$-\operatorname{div}\big[|x|^{-ap}|\nabla u|^{p-2}\nabla u\big]+\lambda|x|^{-(a+1)p}|u|^{p-2}u=|x|^{-bq}|u|^{q-2}u+f, \tag{P}$$

where $x\in\mathbb{R}^N$, $1<p<N$, $q=q(a,b)\equiv Np/[N-p(a+1-b)]$, λ is a parameter, $0\leqslant a<(N-p)/p$, $a\leqslant b\leqslant a+1$, and $f\in(L_b^q(\mathbb{R}^N))^*$. We look for solutions of problem (P) in the Sobolev space $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$ and we prove a version of a concentration-compactness lemma due to Lions. Combining this result with the Ekeland's variational principle and the mountain-pass theorem, we obtain existence and multiplicity results.

Keywords: Degenerate quasilinear equation; p-Laplacian; Variational methods; Compactness-concentration

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1. Introduction and main results

In this work we consider existence and multiplicity results of nontrivial solutions for a class of quasilinear degenerate elliptic equations in $\mathbb{R}^N$ of the form

$$-\operatorname{div}\left[|x|^{-ap}|\nabla u|^{p-2}\nabla u\right]+\lambda|x|^{-(a+1)p}|u|^{p-2}u=|x|^{-bq}|u|^{q-2}u+f, \tag{P}$$

where $x\in\mathbb{R}^N$, $1<p<N$, $q=q(a,b)\equiv Np/[N-p(a+1-b)]$, λ is a parameter, $0\leqslant a<(N-p)/p$, $a\leqslant b\leqslant a+1$, and $f\in(L_b^q(\mathbb{R}^N))^*$, dual space of

$$L_b^q(\mathbb{R}^N)\equiv\left\{u:\mathbb{R}^N\to\mathbb{R}\colon 5\left||x|^{-b}u\right|_q^q=\int\limits_{\mathbb{R}^N}|x|^{-bq}|u|^q\,dx<\infty\right\}.$$

Equations of this form arise in several models (see, e.g., [2,4,14,17,31]). For another version of problem (P), we cite Clément et al. [15], who proved, for example, the Brézis and Nirenberg's result [7] for the operator in the radial form. (See also Clément et al. [16].)

We look for solutions of problem (P) in the Sobolev space $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$ defined as the completion of the space $C_0^\infty(\mathbb{R}^N)$ endowed with the norm $\|u\|\equiv[\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u|^p\,dx]^{1/p}$.

The starting point for the variational approach to these problems is the well known Caffarelli, Kohn and Nirenberg's inequality [9]. (See also Catrina and Wang [12].)

We begin by treating existence results of positive solutions for problem (P) with $f\equiv 0$, which has a variational formulation for the parameters in the specified intervals; specifically, we can formulate the following minimization problem with constraints:

$$S(a,b,\lambda)\equiv\inf_{0\neq u\in\mathcal{D}_a^{1,p}(\mathbb{R}^N)}\left\{E(a,b,\lambda,u)\equiv\left||x|^{-a}\nabla u\right|_p^p+\lambda\left||x|^{-(a+1)}u\right|_p^p\colon\ \left||x|^{-b}u\right|_q^q=1\right\}. \tag{1}$$

Using [9] we can guarantee that $S(a,b,\lambda)$ is a positive constant.

The first result is presented in the following theorem. In its statement, we use the notations: $S(a,b)\equiv S(a,b,0)$, and given a function $v(x)$, we define the dilation by $v^t(x)\equiv t^k v(tx)$, where $k\equiv[N-(a+1)p]/p$.

Theorem 1.1. *Let $1<p<N$, $0\leqslant a<(N-p)/p$ and $q=q(a,b)\equiv Np/[N-p(a+1-b)]$. Then there exists a minimum $u\in\mathcal{D}_a^{1,p}(\mathbb{R}^N)$ for $S(a,b,\lambda)$ provided that one of the conditions below holds*:

(i) $a\leqslant b<a+1$ *and* $-S(a,a+1)<\lambda\leqslant 0$,
(ii) $a<b<a+1$ *and* $0<\lambda$,
(iii) $0<a=b$ *and* $0<\lambda$ *small.*

After the pioneering work of Brézis and Nirenberg [7], several researchers have dedicated to study variants of problem (P) with $f\equiv 0$ among which we cite [3,5,19,22,24]. For the singular problems in bounded domains we would like to mention [20]. In $\mathbb{R}^N$, Lions [23] and Lieb [21] proved the existence of a minimum to $S(a,b)$ in the case $p=2$, $a=0$, and $0<b<1$. Chou and Chu [13] studied the existence of a minimum for $S(a,b)$ in the case $p=2$, $a\leqslant b<a+1$, and $\lambda=0$. On the other hand, both proved that the minimum is not attained in the case $p=2$, and $b=a+1$. Lions [22] treated the existence of a minimum in the case $p=2$, $a=0$, $b=0$ and $-S(0,1)<\lambda<0$, while Wang and Willem [31] considered the singular problem (P) with $f\equiv 0$

and $p = 2$. They solved completely the problem of compactness of the minimizing sequences for $S(a, b)$ and they obtained a precise estimate to the noncompactness of the minimizing sequences. We remark that our result does not follow directly from the case $p = 2$, because we obtained only an inequality (Lemma 2.2) for the estimate of the noncompactness of the minimizing sequences for $S(a, b, \lambda)$, and by a result of Smets [27, Example 2.3] there is no equality. However, even with a weaker estimate it is still possible to prove the relative compactness of the minimizing sequences. Our result generalizes the approach of Wang and Willem [31].

Remark 1.1. For $S(a, b)$ as well as for $S(a, b, \lambda)$ the ground state solutions are positive in $\mathbb{R}^N$ and are differentiable everywhere except the origin. These facts follow from the classical regularity theory of elliptic equations.

For our next result, given a function $f \in (L^q_b(\mathbb{R}^N))^*$, we prove the existence of two nontrivial solutions for problem (P) with $\lambda = 0$. We recall a result of Pohozaev that, for $a = 0$, $b = 0$, $q = 2N/(N-2)$ and $f \equiv 0$, in general this problem does not have solution in star-shaped domains. However, for $a = 0$, $b = 0$, and $f \not\equiv 0$ problem (P) with $\lambda = 0$ always has a solution in bounded domains by a result of Brézis and Nirenberg [8]. Tarantello [30] extended the results in [8], obtaining existence of two positive solutions for problem (P) with $\lambda = 0$, still in bounded domains. For unbounded domains see, e.g., [1,11] and references therein. For the singular operators, Rădulescu and Smets [26] treated the case $0 < a < 2$, $b = 0$, and $p = 2$ in unbounded conic domains, presenting a different type of noncompactness, as mentioned by Caldiroli and Musina [10]. Finally we mention the paper [25] for some multiplicity results for the subcritical singular problem in bounded domains.

Theorem 1.2. *Suppose that* $1 < p < N$, $0 \leqslant a < (N-p)/p$ *and* $a \leqslant b < a + 1$. *Then, for every function* $g \in (L^q_b(\mathbb{R}^N))^*$ *and* $g \geqslant 0$, *there exists a real number* $\varepsilon_0 > 0$ *such that, for every* $0 < \varepsilon \leqslant \varepsilon_0$, *problem* (P) *with* $\lambda = 0$ *and* $f = \varepsilon g$ *has at least two positive solutions.*

In our case we treat problems involving exponent p, not necessarily $p = 2$, and we consider problem (P) with $\lambda = 0$ and singularities in the operator as well as in the nonlinearity. Technically, there are several difficulties to prove existence and multiplicity of solutions of problem (P) with $f \equiv 0$ or $\lambda = 0$, because the usual methods of the calculus of variations do not apply directly. The first difficulty is associated to the space $\mathcal{D}^{1,p}_a(\mathbb{R}^N)$, which is not a Hilbert space in the case $p \neq 2$. Moreover, the differential equation involves the critical Hardy–Sobolev exponent, bringing the question of the lack of compactness in the immersion $\mathcal{D}^{1,p}_a(\mathbb{R}^N) \hookrightarrow L^q_b(\mathbb{R}^N)$.

Addendum. After completing this paper we learned that related results with Theorem 1.1 have been independently obtained by Tan and Yang [29].

2. Minimizing sequences for $S(a, b, \lambda)$

To prove the existence of solution to the problems stated in Theorem 1.1, we have to show the existence of a minimum for the Lagrange multipliers $S(a, b)$ and $S(a, b, \lambda)$. However, since $S(a, b) \equiv S(a, b, 0)$, it suffices to treat the existence of a minimum for $S(a, b, \lambda)$.

In order to prove that $S(a, b, \lambda)$ is attained, we consider an arbitrary minimizing sequence $(u_n) \subset \mathcal{D}^{1,p}_a(\mathbb{R}^N)$ for (1). Since $(u_n) \subset \mathcal{D}^{1,p}_a(\mathbb{R}^N)$ is bounded, we can suppose that

$u_n \rightharpoonup u$ weakly in $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$ and $u_n \to u$ a.e. $\mathbb{R}^N$ for some $u \in \mathcal{D}_a^{1,p}(\mathbb{R}^N)$. Moreover, we have $E(a,b,\lambda,u) \leqslant \liminf_{n\to\infty} E(a,b,\lambda,u_n) \to S(a,b,\lambda)$.

Clearly, the problem of finding minimizers to $S(a,b,\lambda)$ is invariant by dilation. The next step consists in proving that the sequence $(u_n) \subset \mathcal{D}_a^{1,p}(\mathbb{R}^N)$ is relatively compact up to dilation. Before we do this, however, we need some preliminary results.

The proof of the following lemma can be adapted from the similar result presented in [31].

Lemma 2.1. *Let $a \in \mathbb{R}$ be such that $0 \leqslant a < (N-p)/p$. We define the function $g:[0,(N-p)/p)) \to \mathbb{R}$ by $g(a) \equiv E(a,a,0,\bar{u})$, where $\bar{u} \equiv u/||x|^{-a}u|_q$, $u(x) \equiv [1+|x|^{p/(p-1)}]^{-(N-p)/p}$ and $q = q(a,a) = Np/(N-p) \equiv p^*$ (the critical Sobolev exponent). Then $g'(a) < 0$ for $a \in (0,(N-p)/p)$ and $g'(0^+) = 0$.*

The following lemma is crucial for our work. To state it, we denote by $\mathcal{M}(\mathbb{R}^N)$ the space of positive, bounded measures in $\mathbb{R}^N$

Lemma 2.2. *Let $1 < p < N$, $0 \leqslant a < (N-p)/p$, $a \leqslant b \leqslant a+1$, $-S(a,a+1) < \lambda$ and $q = q(a,b) \equiv Np/[N-p(a+1-b)]$. Let a sequence $(u_n) \subset \mathcal{D}_a^{1,p}(\mathbb{R}^N)$ be such that are valid the following convergences*:

(1) *$u_n \rightharpoonup u$ weakly in $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$,*
(2) *$||x|^{-a}\nabla(u_n - u)|^p + \lambda||x|^{-(a+1)}(u_n-u)|^p \rightharpoonup \gamma$ weakly in $\mathcal{M}(\mathbb{R}^N)$,*
(3) *$||x|^{-b}(u_n-u)|^q \rightharpoonup \nu$ weakly in $\mathcal{M}(\mathbb{R}^N)$,*
(4) *$u_n \to u$ a.e. in $\mathbb{R}^N$.*

We also define the measures of concentration at infinity

$$\nu_\infty \equiv \lim_{R\to\infty} \limsup_{n\to\infty} \int\limits_{|x|\geqslant R} |x|^{-bq}|u_n|^q\,dx,$$

$$\gamma_\infty \equiv \lim_{R\to\infty} \limsup_{n\to\infty} \left[\int\limits_{|x|\geqslant R} |x|^{-ap}|\nabla u_n|^p\,dx + \lambda \int\limits_{|x|\geqslant R} |x|^{-(a+1)p}|u_n|^p\,dx \right].$$

Then

$$\|\nu\|^{p/q} \leqslant \left[S(a,b,\lambda)\right]^{-1}\|\gamma\|, \tag{2}$$

$$\nu_\infty^{p/q} \leqslant \left[S(a,b,\lambda)\right]^{-1}\gamma_\infty, \tag{3}$$

$$\limsup_{n\to\infty}\left||x|^{-a}\nabla u_n\right|_p^p + \lambda\left||x|^{-(a+1)}u_n\right|_p^p \geqslant \left||x|^{-a}\nabla u\right|_p^p + \lambda\left||x|^{-(a+1)}u\right|_p^p + \|\gamma\| + \gamma_\infty, \tag{4}$$

$$\limsup_{n\to\infty}\left||x|^{-b}u_n\right|_q^q = \left||x|^{-b}u\right|_q^q + \|\nu\| + \nu_\infty. \tag{5}$$

Moreover, for $u(x) \equiv 0$, if $b < a+1$ and $\|\nu\|^{p/q} = [S(a,b,\lambda)]^{-1}\|\gamma\|$, then the measures ν and γ are concentrated at a single point.

Proof. Suppose initially that $u \equiv 0$. Choosing $h \in C_0^\infty(\mathbb{R}^N)$ we have $(hu_n) \subset \mathcal{D}_a^{1,p}(\mathbb{R}^N)$.

Arguing as in [31], and using inequality

$$|x+y|^p \leqslant (1+\varepsilon)|x|^p + C(\varepsilon,p)|y|^p, \tag{6}$$

valid for $x, y \in \mathbb{R}^+$ and $1 < p < \infty$ with $\varepsilon > 0$ fixed, we obtain

$$\begin{aligned}
&\left[\int_{\mathbb{R}^N} |x|^{-bq}|hu_n|^q\,dx\right]^{p/q} \\
&\quad\leqslant \frac{1}{S(a,b,\lambda)}\left[\int_{\mathbb{R}^N} |x|^{-ap}|h\nabla u_n|^p\,dx + \lambda\int_{\mathbb{R}^N} |x|^{-(a+1)p}|hu_n|^p\,dx\right] \\
&\qquad + \frac{C(\varepsilon,p)}{S(a,b,\lambda)}\int_{\mathbb{R}^N} |x|^{-ap}|u_n\nabla h|^p\,dx + \frac{\varepsilon}{S(a,b,\lambda)}\int_{\mathbb{R}^N} |x|^{-ap}|h\nabla u_n|^p\,dx. \qquad (7)
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, passing to the limit we obtain inequality (2).

To prove inequality (3) and that the last claim of the lemma, we follow the arguments in [31] and use the same cutoff function used there.

Now we consider the general case, in which possibly $u \not\equiv 0$; in this case we define $v_n \equiv u_n - u$ and so $v_n \rightharpoonup 0$ weakly in $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$. Here our result differs from that in [31], because for $p \neq 2$, in general we do not have equality. Also, we follow some ideas of Smets [27].

From Brézis–Lieb lemma applied to a nonnegative function $h \in C_0^\infty(\mathbb{R}^N)$, we have

$$|x|^{-bq}|u_n|^q \rightharpoonup \nu + |x|^{-bq}|u|^q \quad \text{weakly in } \mathcal{M}(\mathbb{R}^N). \qquad (8)$$

Using these weak convergences in the space $\mathcal{M}(\mathbb{R}^N)$, the inequality (2) in the general case follows from the correspondent inequality for the sequence $(v_n) \subset \mathcal{D}_a^{1,p}(\mathbb{R}^N)$.

Following up, we have

$$\begin{aligned}
&\left|\int_{|x|>R} |x|^{-ap}|\nabla v_n|^p\,dx + \lambda\int_{|x|>R} |x|^{-(a+1)p}|v_n|^p\,dx\right. \\
&\qquad \left. - \int_{|x|>R} |x|^{-ap}|\nabla u_n|^p\,dx - \lambda\int_{|x|>R} |x|^{-(a+1)p}|u_n|^p\,dx\right| \\
&\quad\leqslant \varepsilon\left[\int_{|x|>R} |x|^{-ap}|\nabla u_n|^p\,dx + \lambda\int_{|x|>R} |x|^{-(a+1)p}|u_n|^p\,dx\right] \\
&\qquad + C(\varepsilon,p)\left[\int_{|x|>R} |x|^{-ap}|\nabla u|^p\,dx + \lambda\int_{|x|>R} |x|^{-(a+1)p}|u|^p\,dx\right]
\end{aligned}$$

where we used inequality (6). Taking the limit at the expression above, we have

$$\lim_{R\to\infty}\limsup_{n\to\infty}\left[\int_{|x|>R} |x|^{-ap}|\nabla v_n|^p\,dx + \lambda\int_{|x|>R} |x|^{-(a+1)p}|v_n|^p\,dx\right] = \gamma_\infty.$$

Using Brézis–Lieb lemma, we have

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|>R} |x|^{-bq}|v_n|^q\,dx = \nu_\infty.$$

This way, inequality (3) follows from the correspondent inequality verified for the sequence $(v_n) \subset \mathcal{D}_a^{1,p}(\mathbb{R}^N)$.

Now we prove inequality (4). Since ν is a finite measure, the set

$$D \equiv \{x \in \mathbb{R}^N \mid \nu(\{x\}) > 0\}$$

is at most denumerable. Let $\psi_j \in C_0^\infty(B(r_j, x))$ be a positive function such that $\psi_j(x) = 1 = \sup_{\mathbb{R}^N} \psi_j$, where $r_j \to 0$ as $j \to \infty$.

Given $x \in D$ and using once more inequality (6), we obtain

$$\begin{aligned}\gamma(\{x\}) = \lim_{j\to\infty} \gamma(\psi_j) &= \lim_{j\to\infty} \limsup_{n\to\infty} \Bigg[\int_{\mathbb{R}^N} |x|^{-ap} \big|\nabla\psi_j(u_n - u)\big|^p \, dx \\ &\quad + \lambda \int_{\mathbb{R}^N} |x|^{-(a+1)p} \big|\psi_j(u_n - u)\big|^p \, dx\Bigg] \\ &\geqslant S(a,b,\lambda)\Bigg[\lim_{j\to\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^N} |x|^{-bq} \big|\psi_j(u_n - u)\big|^q \, dx\Bigg]^{p/q} \\ &= S(a,b,\lambda)\nu(\{x\})^{p/q}.\end{aligned}$$

Define some positive, finite measure $\tilde{\gamma} \in \mathcal{M}(\mathbb{R}^N)$ such that

$$\big||x|^{-a}\nabla u_n\big|^p + \lambda\big||x|^{-(a+1)}u_n\big|^p \rightharpoonup \tilde{\gamma} \quad \text{weakly in } \mathcal{M}(\mathbb{R}^N).$$

For the function $\psi_j \in C_0^\infty(B(r_j, x))$, we have

$$\begin{aligned}&\Bigg|\int_{\mathbb{R}^N} |x|^{-ap} \big|\psi_j \nabla(u_n - u)\big|^p \, dx + \lambda \int_{\mathbb{R}^N} |x|^{-(a+1)p} \big|\psi_j(u_n - u)\big|^p \, dx \\ &\qquad - \int_{\mathbb{R}^N} |x|^{-ap} |\psi_j \nabla u_n|^p \, dx - \lambda \int_{\mathbb{R}^N} |x|^{-(a+1)p} |\psi_j u_n|^p \, dx\Bigg| \\ &\quad \leqslant \varepsilon\Bigg[\int_{\mathbb{R}^N} |x|^{-ap} \psi_j |\nabla u_n|^p \, dx + \lambda \int_{\mathbb{R}^N} |x|^{-(a+1)p} \psi_j |u_n|^p \, dx\Bigg] \\ &\qquad + C(\varepsilon, p)\Bigg[\int_{\mathbb{R}^N} |x|^{-ap} \psi_j |\nabla u|^p \, dx + \lambda \int_{\mathbb{R}^N} |x|^{-(a+1)p} \psi_j |u|^p \, dx\Bigg].\end{aligned}$$

Letting $r_j \to 0$, we obtain

$$\gamma(\{x\}) = \tilde{\gamma}(\{x\}), \quad x \in D.$$

Since the application $v \mapsto \int_{\mathbb{R}^N} h|x|^{-ap}|v|^p \, dx$ is convex in $L^p(\mathbb{R}^N)$ for a positive $h \in C_0^\infty(\mathbb{R}^N)$, it follows that it is also weakly sequentially lower semicontinuous. Hence, $\tilde{\gamma} \geqslant |x|^{-ap}|\nabla u|^p + \lambda||x|^{-(a+1)}u|^p$. Using the orthogonality of $|x|^{-ap}|\nabla u|^p$ with respect to the Dirac measures, we obtain

$$\tilde{\gamma} \geqslant |x|^{-ap}|\nabla u|^p + \lambda\big||x|^{-(a+1)}u\big|^p + \|\gamma\|.$$

This way,

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}\big[|x|^{-ap}|\nabla u_n|^p\big](1-\psi_R)\,dx+\lambda\int_{\mathbb{R}^N}\big[|x|^{-(a+1)p}|u_n|^p\big](1-\psi_R)\,dx$$
$$\geqslant \int_{\mathbb{R}^N}\big[|x|^{-ap}|\nabla u|^p\big](1-\psi_R)\,dx+\lambda\int_{\mathbb{R}^N}|x|^{-(a+1)p}|u|^p(1-\psi_R)\,dx+\|\gamma\|, \tag{9}$$

where, for $R>1$, we define the cutoff function $\psi_R\in C^\infty(\mathbb{R}^N)$ such that $\psi_R(x)\equiv 1$ for $|x|>R+1$, $\psi_R(x)\equiv 0$ for $|x|<R$, and furthermore, $0\leqslant\psi_R(x)\leqslant 1$ for $x\in\mathbb{R}^N$.

Hence, we get

$$\limsup_{n\to\infty}\bigg[\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u_n|^p\,dx+\lambda\int_{\mathbb{R}^N}|x|^{-(a+1)p}|u_n|^p\,dx\bigg]$$
$$\geqslant \limsup_{n\to\infty}\bigg[\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u_n|^p\psi_R\,dx+\lambda\int_{\mathbb{R}^N}|x|^{-(a+1)p}|u_n|^p\psi_R\,dx\bigg]$$
$$+\lim_{n\to\infty}\bigg[\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u_n|^p[1-\psi_R]\,dx+\lambda\int_{\mathbb{R}^N}|x|^{-(a+1)p}|u_n|^p[1-\psi_R]\,dx\bigg]$$
$$=\limsup_{n\to\infty}\bigg[\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u_n|^p\psi_R\,dx+\lambda\int_{\mathbb{R}^N}|x|^{-(a+1)p}|u_n|^p\psi_R\,dx\bigg]+\tilde{\gamma}[1-\psi_R].$$

Passing to the limit as $R\to\infty$, we have

$$\limsup_{n\to\infty}\bigg[\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u_n|^p\,dx+\lambda\int_{\mathbb{R}^N}|x|^{-(a+1)p}|u_n|^p\,dx\bigg]$$
$$=\lim_{R\to\infty}\limsup_{n\to\infty}\bigg[\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u|^p\psi_R\,dx+\lambda\int_{\mathbb{R}^N}|x|^{-(a+1)p}|u|^p\psi_R\,dx\bigg]$$
$$+\lim_{R\to\infty}\tilde{\gamma}(1-\psi_R)$$
$$=\gamma_\infty+\|\tilde{\gamma}\|\geqslant\gamma_\infty+|x|^{-ap}|\nabla u|^p+\lambda|x|^{-(a+1)p}|u|^p+\|\gamma\|.$$

From this, it follows that

$$\limsup_{n\to\infty}\big||x|^{-a}\nabla u_n\big|_p^p+\lambda\big||x|^{-(a+1)}u_n\big|_p^p\geqslant\big||x|^{-a}\nabla u\big|_p^p+\lambda\big||x|^{-(a+1)}u\big|_p^p+\|\gamma\|+\gamma_\infty$$

and the inequality (4) is proved.

Finally, we prove equality (5). For every real number $R>1$, using Brézis–Lieb lemma we have

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}|x|^{-bq}|u_n|^q\,dx=\limsup_{n\to\infty}\bigg[\int_{\mathbb{R}^N}\psi_R|x|^{-bq}|u_n|^q\,dx+\int_{\mathbb{R}^N}(1-\psi_R)|x|^{-bq}|u_n|^q\,dx\bigg]$$
$$+\lim_{n\to\infty}\int_{\mathbb{R}^N}(1-\psi_R)|x|^{-bq}|u|^q\,dx.$$

Letting $R \to \infty$ in the expression above, and using Lebesgue theorem, we obtain

$$\lim_{R\to\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^N} |x|^{-bq}|u_n|^q \, dx = \nu_\infty + \|\nu\| + \big||x|^{-b}|u|\big|_q^q,$$

which implies equality (5). This concludes the proof of the lemma. □

3. Conclusion of the proof of Theorem 1.1

Proof of Theorem 1.1(i). Let $(u_n) \subset \mathcal{D}_a^{1,p}(\mathbb{R}^N)$ be a minimizing sequence for $S(a,b,\lambda)$. Let $B(x,r)$ denote the open ball with radius r centered at $x \in \mathbb{R}^N$. For every number $n \in \mathbb{N}$, there exists a number $t_n \in \mathbb{R}^+$ such that

$$\int_{B(0,t_n)} |x|^{-bq}|u_n|^q \, dx = \int_{B(0,1)} |x|^{-bq}|v_n|^q \, dx = \frac{1}{2}, \tag{10}$$

where we used the dilation $v_n(x) \equiv u_n^{t_n}(x)$.

By hypotheses and using the invariance of the problem by dilation, we have

$$\big||x|^{-b}v_n\big|_q = \big||x|^{-b}u_n\big|_q = 1$$

and

$$\begin{aligned}\big||x|^{-a}\nabla v_n\big|_p^p + \lambda\big||x|^{-(a+1)}v_n\big|_p^p &= \big||x|^{-a}\nabla u_n\big|_p^p + \lambda\big||x|^{-(a+1)}u_n\big|_p^p \\ &\to S(a,b,\lambda) \quad \text{as } n \to \infty.\end{aligned}$$

Since the sequence $(v_n) \subset \mathcal{D}_a^{1,p}(\mathbb{R}^N)$ is bounded, passing to a subsequence, still denoted in the same way, we can suppose that there exists a function $v \in \mathcal{D}_a^{1,p}(\mathbb{R}^N)$ such that are valid the hypotheses of Lemma 2.2.

By Lemma 2.2, we have

$$S(a,b,\lambda) \geqslant \big||x|^{-a}\nabla v\big|_p^p + \lambda\big||x|^{-(a+1)}v\big|_p^p + \|\gamma\| + \gamma_\infty, \tag{11}$$

$$1 = \big||x|^{-b}v\big|_q^q + \|\nu\| + \nu_\infty. \tag{12}$$

From inequalities (2), (3), (11) and from the definition of $S(a,b,\lambda)$ we deduce that

$$S(a,b,\lambda) \geqslant S(a,b,\lambda)\left\{\left[\int_{\mathbb{R}^N} |x|^{-bq}|v|^q \, dx\right]^{p/q} + \|\nu\|^{p/q} + \nu_\infty^{p/q}\right\}.$$

Using equality (12) we obtain three mutually excluding situations.

By equality (10), it follows that $\nu_\infty = 0$.

Suppose now that $v = 0$; we will get a contradiction. In fact, equality (12) implies that $\|\nu\| = 1$.

From inequality (11), we have

$$1 = \|\nu\| = \|\nu\|^{p/q} \leqslant \frac{1}{S(a,b,\lambda)}\|\gamma\| \leqslant \frac{1}{\|\gamma\| + \gamma_\infty}\|\gamma\| \leqslant 1$$

and this means that $\gamma_\infty = 0$ and $S(a,b,\lambda) = \|\gamma\|$.

Supposing that $b < a + 1$ and applying Lemma 2.2 once more, we deduce that the measures ν and γ are concentrated at a single point $x_0 \in \mathbb{R}^N$. Such point is not the origin, because of equality (10).

From this point on, we divide our argument in two cases.

Case $a < b$. In this case we have $q < p^*$. By the Rellich theorem we conclude that $\|\nu\| = 0$. But we have already established that $\|\nu\| = 1$. The contradiction leads to the situation in which $\|\nu\| = 0$ and $||x|^{-b}v|_q^q = 1$.

Case $a = b > 0$. In this case we have $q = p^*$. Given $r \in \mathbb{R}^+$, we define the expression

$$A \equiv \lim_{n\to\infty} \frac{\int_{B(x_0,r)} |x|^{-ap}|\nabla v_n|^p\,dx + \lambda \int_{B(x_0,r)} |x|^{-(a+1)p}|v_n|^p\,dx}{[\int_{B(x_0,r)} |x|^{-ap^*}|v_n|^{p^*}\,dx]^{p/p^*}}.$$

Then $A = \|\gamma\| = S(a, a, \lambda)$. Let $\eta \in C_0^\infty(B(x_0, r))$ be a function such that $\eta \equiv 1$ in $B(x_0, r/2)$ for $r \in \mathbb{R}^+$ sufficiently small. Then

$$A = \lim_{n\to\infty} \frac{\int_{B(x_0,r)} |\nabla \eta v_n|^p\,dx + \lambda \int_{B(x_0,r)} |\eta v_n|^p\,dx}{[\int_{B(x_0,r)} |\eta v_n|^{p^*}\,dx]^{p/p^*}} \geqslant S \equiv S(0,0),$$

because

$$\lim_{n\to\infty} \int_{B(x_0,r)} |x|^{-(a+1)}|v_n|^p\,dx = 0.$$

It follows that $S(a, a, \lambda) = A \geqslant S$. We recall that S is the best constant in Sobolev inequality [28].

On the other hand, by Lemma 2.1 we know that $S = g(0) > g(a) = S(a, a) \geqslant S(a, a, \lambda)$ if $-S(a, a + 1) < \lambda \leqslant 0$. The contradiction leads again to the situation $\|\nu\| = 0$ and $||x|^{-b}v|_q^q = 1$.

In any case there exists a minimum to $S(a, b, \lambda)$. This proves item (i) of Theorem 1.1. The proof of item (ii) is similar. □

Proof of Theorem 1.1(iii). Following the same ideas of the previous proof, also for $0 < a = b$ and $\lambda > 0$ we obtain three mutually excluding situations. In this case we proceed as we did in item (i) of Theorem 1.1 and we obtain

$$S \leqslant \|\gamma\| = S(a, a, \lambda).$$

On the other hand, since $S(a, a, 0) < S$, there exists $0 < \varepsilon < 1$ such that, for $0 < \lambda < \varepsilon$, we still have $S(a, a, \lambda) < S$.

The only possibility left is $\nu_\infty = 0$, $\nu = 0$ and $||x|^{-b}v|_q = 1$. Hence, $v \in \mathcal{D}_a^{1,p}(\mathbb{R}^N)$ is a minimum to $S(a, b, \lambda)$ and $v_n \to v$ in $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$. □

4. Nonautonomous perturbation problems: the first solution

In this section, we are going to use variational techniques. This way, associated to the problem (P) with $\lambda = 0$ we have the Euler–Lagrange functional $I : \mathcal{D}_a^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$I(u) \equiv \frac{1}{p}\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p\,dx - \frac{1}{q}\int_{\mathbb{R}^N} |x|^{-bq}|u|^q\,dx - \int_{\mathbb{R}^N} fu\,dx, \tag{13}$$

which is well defined for the parameters in the previously specified intervals.

Using the duality product, we define a weak solution of problem (P) with $\lambda = 0$ as a critical point for the functional I, that is, as a function $u \in \mathcal{D}_a^{1,p}(\mathbb{R}^N)$ such that

$$0 = \langle I'(u), \phi \rangle = \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^{p-2} \nabla u \nabla \phi \, dx - \int_{\mathbb{R}^N} |x|^{-bq} |u|^{q-2} u \phi \, dx - \int_{\mathbb{R}^N} f \phi \, dx, \quad \forall \phi \in C_0^\infty(\mathbb{R}^N).$$

Lemma 4.1. *Let $(u_n) \subset \mathcal{D}_a^{1,p}(\mathbb{R}^N)$ be a Palais–Smale sequence for the functional I at the level $c \in \mathbb{R}$ ($(PS)_c$, in short), that is, a sequence such that*

$$\lim_{n\to\infty} I(u_n) = c \quad \text{and} \quad \lim_{n\to\infty} \| I'(u_n) \|_{\mathcal{D}_a^{1,p}(\mathbb{R}^N)^*} = 0. \tag{14}$$

If $u_n \rightharpoonup u_0$ weakly in $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$ for some u_0, then u_0 is a weak solution for problem (P) *with $\lambda = 0$.*

Proof. We consider an arbitrary function $\zeta \in C_0^\infty(\mathbb{R}^N)$ and denote its support by ω. Then

$$\langle I'(u_n), \zeta \rangle \to 0 \quad \text{as } n \to \infty. \tag{15}$$

Claim 1. $|x|^{-a} \nabla u_n \to |x|^{-a} \nabla u$ a.e. in $\mathbb{R}^N$.

We are going to postpone the verification of this claim.

Since the sequence $(|x|^{-ap} |\nabla u|^p \nabla u_n) \subset L^{p'}(\mathbb{R}^N)$ is bounded $(1/p + 1/p' = 1)$, by Claim 1 we have

$$\lim_{n\to\infty} \int_\omega |x|^{-ap} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \zeta \, dx = \int_\omega |x|^{-ap} |\nabla u_0|^{p-2} \nabla u_0 \cdot \nabla \zeta \, dx, \tag{16}$$

because $|x|^{-a} \nabla \zeta \in L^p(\mathbb{R}^N)$.

On the other hand, the boundedness of the sequence $(u_n) \subset \mathcal{D}_a^{1,p}(\mathbb{R}^N)$ and the Caffarelli, Kohn and Nirenberg's inequality imply that $|x|^{-b(q-1)} |u_n|^{q-2} u_n$ is bounded in $L^{q'}(\mathbb{R}^N)$, where $1/q + 1/q' = 1$. Passing to a subsequence (still denoted in the same way), we have

$$\lim_{n\to\infty} \int_\omega |x|^{-bq} |u_n|^{q-2} u_n \zeta \, dx = \int_\omega |x|^{-bq} |u_0|^{q-2} u_0 \zeta \, dx, \tag{17}$$

because $|x|^{-b} \zeta \in L^q(\mathbb{R}^N)$ and $u_n \to u_0$ a.e. in $\mathbb{R}^N$.

Combining Eqs. (15), (16) and (17), it follows that $\langle I'(u_0), \zeta \rangle = 0$ for every function $\zeta \in C_0^\infty(\mathbb{R}^N)$. By using a density argument the lemma is proved. □

Proof of Claim 1. The proof was partially inspired in the works of Boccardo and Murat [6], and Ghoussoub and Yuan [20]. We begin by defining the family of functions

$$\tau_k(s) \equiv \begin{cases} s & \text{if } |s| \leqslant k, \\ ks/|s| & \text{if } |s| > k. \end{cases}$$

Affirmative 1. There exists a constant $C \in \mathbb{R}^+$ such that the following inequality holds:

$$\int_{\mathbb{R}^N} |x|^{-bq}\big[|u_n|^{q-2}u_n - |u|^{q-2}u\big]\tau_k(u_n-u)\,dx \leqslant Ck^q. \tag{18}$$

The proof of this affirmative follows from the Hölder's inequality and by combining the boundedness of the sequence $(u_n) \subset \mathcal{D}_a^{1,p}(\mathbb{R}^N)$ and the continuity of the inclusion $\mathcal{D}_a^{1,p}(\mathbb{R}^N) \hookrightarrow L_b^q(\mathbb{R}^N)$.

Passing to a subsequence, if necessary, still denoted in the same way, we get $u_n \rightharpoonup u$ weakly in $L_b^q(\mathbb{R}^N)$. Since $f \in (L_b^q(\mathbb{R}^N))^*$, it follows that

$$\begin{aligned} o(1) &= \big\langle I'(u_n) - I(u), \tau_k(u_n-u)\big\rangle \\ &= \int_{\mathbb{R}^N} \big\langle |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n - |x|^{-ap}|\nabla u|^{p-2}\nabla u, \nabla\tau_k(u_n-u)\big\rangle_e \\ &\quad - \int_{\mathbb{R}^N} |x|^{-bq}\big(|u_n|^{q-2}u_n - |u|^{q-2}u\big)\tau_k(u_n-u)\,dx - \int_{\mathbb{R}^N} f\tau_k(u_n-u)\,dx, \end{aligned}$$

where $\langle\cdot,\cdot\rangle_e$ denotes the usual inner product in $\mathbb{R}^N$. Passing to the limit and using inequality (18), we have

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} \big\langle |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n - |x|^{-ap}|\nabla u|^{p-2}\nabla u, \nabla\tau_k(u_n-u)\big\rangle_e\,dx \leqslant Ck^q.$$

Now we define the sequence of functions

$$e_n \equiv \big\langle |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n - |x|^{-ap}|\nabla u|^{p-2}\nabla u, \nabla\tau_k(u_n-u)\big\rangle_e.$$

It follows that $e_n(x) \geqslant 0$ by a well-known inequality. (See Ghoussoub and Yuan [20, Lemma 4.1].)

Affirmative 2. For every $n \in \mathbb{N}$ we have $\int_{\mathbb{R}^N} e_n(x)\,dx < \infty$.

The proof of this affirmative follows by applying the Hölder's inequality in

$$\int_{\mathbb{R}^N} \big\langle |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n - |x|^{-ap}|\nabla u|^{p-2}\nabla u, \nabla\tau_k(u_n-u)\big\rangle_e\,dx.$$

Given $m \in \mathbb{N}$, we denote $\Omega_m \equiv B(0,m)$ and we write $\mathbb{R}^N = \bigcup_{m=1}^{\infty}\Omega_m$. For $0<\theta<1$ and $k \in \mathbb{R}$ fixed, we split Ω_m in

$$A_n^k \equiv \big\{x \in \Omega_m \mid |u_n-u| \leqslant k\big\} \quad\text{and}\quad B_n^k \equiv \big\{x \in \Omega_m \mid |u_n-u| > k\big\}.$$

For $k \in \mathbb{R}$ fixed, from the convergence in measure we have

$$\lim_{n\to\infty}\big|B_n^k\big| = 0. \tag{19}$$

By the uniform boundedness of the sequence $(e_n) \subset L^1(\mathbb{R}^N)$, we have

$$\limsup_{n\to\infty}\int_{\Omega_m} e_n^\theta\,dx \leqslant (Ck)^\theta|\Omega_m|^{1-\theta}.$$

Fixing $m \in \mathbb{N}$ and letting $k \to 0$, it follows that $e_n^\theta \to 0$ in $L^1(\Omega_m)$. Finally, from the well-known inequality [20, Lemma 4.1], passing to the diagonal sequence it follows that

$$|x|^{-a}\nabla u_n \to |x|^a \nabla u \quad \text{a.e. in } \mathbb{R}^N.$$

This concludes the proof of the claim. □

Now we prove the existence of the first solution.

Lemma 4.2. *There exists a real number $\varepsilon_1 > 0$ such that problem* (P) *with $\lambda = 0$ has at least one solution u_0 if $f \not\equiv 0$ is such that $\|f\|_{(L_b^q(\mathbb{R}^N))^*} < \varepsilon_1$ with $I(u_0) < 0$. Furthermore, if $f \geqslant 0$, then u_0 is a positive solution.*

Proof. Fixing $\varepsilon \in (0,1)$, from Young's as well as Caffarelli, Kohn and Nirenberg's inequalities, we write

$$I(u) \geqslant \left(\frac{1}{p} - \frac{\varepsilon^p}{p}\right)\|u\|^p - C\|u\|^q - C_\varepsilon \|f\|_{(L_b^q(\mathbb{R}^N))^*}.$$

Hence there exist real numbers $R > 0$, $\varepsilon_1 > 0$ and $\delta > 0$ such that if $\|u\| = R$ and $\|f\|_{(L_b^q(\mathbb{R}^N))^*} < \varepsilon_1$, then $I(u) \geqslant \delta$.

Defining

$$c_0 \equiv \inf\bigl\{I(u) \mid u \in \mathcal{D}_a^{1,p}(\mathbb{R}^N) \text{ and } \|u\| \leqslant R\bigr\}, \tag{20}$$

and using $f \not\equiv 0$, it follows that $c_0 < I(0) = 0$.

Applying Ekeland's Variational Principle there exists a bounded $(PS)_{c_0}$ sequence $(u_n) \subset \mathcal{D}_a^{1,p}(\mathbb{R}^N)$ such that $\|u_n\| \leqslant R$, and for some $u_0 \in \mathcal{D}_a^{1,p}(\mathbb{R}^N)$,

$$u_n \rightharpoonup u_0 \quad \text{weakly in } \mathcal{D}_a^{1,p}(\mathbb{R}^N) \quad \text{and} \quad u_n \to u_0 \quad \text{a.e. in } \mathbb{R}^N. \tag{21}$$

Furthermore, from Lemma 4.1 it follows that u_0 is a weak solution for problem (P) with $\lambda = 0$.

Using $I'(u_0) = 0$ and Fatou lemma, we obtain

$$c_0 = \liminf_{n\to\infty} I(u_n) \geqslant \left(\frac{1}{p} - \frac{1}{q}\right)\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u_0|^p\,dx - \left(1 - \frac{1}{q}\right)\int_{\mathbb{R}^N} f u_0\,dx = I(u_0).$$

Since $\|u_0\| \leqslant R$, it follows that $I(u_0) = c_0$. Finally, if $f \geqslant 0$, the function u_0 can be replaced by $|u_0|$, and we get a positive solution. This concludes the proof. □

5. The existence of the second solution

Let the functional $J : \mathcal{D}_a^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ be defined by

$$J(u) \equiv \frac{1}{p}\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p\,dx - \frac{1}{q}\int_{\mathbb{R}^N} |x|^{-bq}|u|^{-q}\,dx. \tag{22}$$

We also define the Nehari manifold $V = \{u \in \mathcal{D}_a^{1,p}(\mathbb{R}^N) \mid \langle J'(u), u\rangle = 0\}$, which is nonempty.

Indeed, let $v_0 \in \mathcal{D}_a^{1,p}(\mathbb{R}^N) \setminus \{0\}$ be fixed and $\lambda \in \mathbb{R}^+$; we define the function $h(\lambda) \equiv \langle J'(\lambda v_0), \lambda v_0\rangle$. Since $p < q$, we have that for λ big enough it holds $h(\lambda) < 0$; on the other hand, for λ near zero it holds $h(\lambda) > 0$. Then, there exists $\lambda_0 \in \mathbb{R}^+$ such that $h(\lambda_0) = 0$.

Denoting by J_∞ the infimum of the functional J in V, that is, $J_\infty \equiv \inf\{J(u) \mid u \in V\}$, we have the following result, whose proof follows by using some arguments of Ding and Ni [18].

Lemma 5.1. *There exists $\bar{u} \in V$ such that $J_\infty = \sup_{t\geqslant 0} J(t\bar{u}) = J(\bar{u}) = \left(\frac{1}{p} - \frac{1}{q}\right)[S(a,b)]^{q/(q-p)}$.*

Proof. Initially we will show that

$$J_\infty \geqslant \left(\frac{1}{p} - \frac{1}{q}\right)\left[S(a,b)\right]^{q/(q-p)}. \tag{23}$$

Fixing $\phi \in \mathcal{D}_a^{1,p}(\mathbb{R}^N) \setminus \{0\}$, we define the function

$$k(t) \equiv J(t\phi) = \frac{t^p}{p}\int_{\mathbb{R}^N} |x|^{-ap}|\nabla\phi|^p\,dx - \frac{t^q}{q}\int_{\mathbb{R}^N} |x|^{-bq}|\phi|^q\,dx$$

which has a global maximum at t_0. It follows that

$$\inf_{0\neq\phi\in\mathcal{D}_a^{1,p}(\mathbb{R}^N)} \sup_{t\geqslant 0} J(t\phi) = \left(\frac{1}{p} - \frac{1}{q}\right)\left[S(a,b)\right]^{q/(q-p)}. \tag{24}$$

We also note that for every $u \in V$ we have $t_0 = t_0(u) = 1$.

So,

$$J_\infty = \inf_{u\in V}\sup_{t\geqslant 0} J(tu) \geqslant \inf_{0\neq\phi\in\mathcal{D}_a^{1,p}(\mathbb{R}^N)} \sup_{t\geqslant 0} J(t\phi) = \left(\frac{1}{p} - \frac{1}{q}\right)\left[S(a,b)\right]^{q/(q-p)}.$$

Using Theorem 1.1, we can guarantee that $S(a,b)$ defined in (1) is attained by a function $U \in \mathcal{D}_a^{1,p}(\mathbb{R}^N)$. Defining the function $\bar{u}(x) \equiv [S(a,b)]^{1/(q-p)}U(x)$, we have $\bar{u} \in V$ and

$$J_\infty \leqslant J(\bar{u}) = \left(\frac{1}{p} - \frac{1}{q}\right)\left[S(a,b)\right]^{q/(q-p)}, \tag{25}$$

which concludes the proof of the lemma. □

Next we state an alternative description for Palais–Smale sequences.

Lemma 5.2. *Suppose that $(u_n) \subset \mathcal{D}_a^{1,p}(\mathbb{R}^N)$ is a Palais–Smale sequence for the functional I at the level $c \in \mathbb{R}$. If $u_n \rightharpoonup u_0$ weakly in $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$ for some u_0, then one of the following alternatives holds*:

(1) $u_n \to u_0$ *in* $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$.
(2) $c \geqslant I(u_0) + J_\infty$.

Proof. Let $(u_n) \subset \mathcal{D}_a^{1,p}(\mathbb{R}^N)$ be a Palais–Smale sequence for the functional I at the level c. We define $v_n \equiv u_n - u_0$. It follows that $v_n \rightharpoonup 0$ weakly in $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$, then

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} f v_n\,dx = 0$$

and

$$I(v_n) = J(v_n) + o(1). \tag{26}$$

Using Caffarelli, Kohn and Nirenberg's inequality and Brézis–Lieb lemma, as well as equality (26) and Lemma 4.1, we get

$$c + o(1) = I(u_n) = I(u_0) + I(v_n) + o(1) = I(u_0) + J(v_n) + o(1) \tag{27}$$

and also

$$o(1) = \langle I'(u_n), u_n\rangle = \langle I'(u_0), u_0\rangle + \langle I'(v_n), v_n\rangle + o(1) = \langle J'(v_n), v_n\rangle + o(1). \tag{28}$$

Now we have two possibilities. If $v_n \to 0$ strongly in $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$, then $u_n \to u_0$ strongly in $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$ and also

$$c = \lim_{n\to\infty} I(u_n) = I(u_0).$$

In this case, the lemma is proved.

On the other hand, if $v_n \not\to 0$ in $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$, then from the weak convergence $v_n \rightharpoonup 0$ in $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$, we can suppose that $\|v_n\| \to \rho > 0$ (possibly after passage to a subsequence, still denoted in the same way). So, using the limit (27), we get

$$c = I(u_0) + J(v_n) + o(1). \tag{29}$$

It is easy to see that the following claim implies the lemma.

Claim. $J(v_n) \geqslant J_\infty + o(1)$.

To prove the claim we define

$$\alpha_n \equiv \int_{\mathbb{R}^N} |x|^{-ap}|\nabla v_n|^p\,dx = \|v_n\|^p \quad \text{and} \quad \beta_n \equiv \int_{\mathbb{R}^N} |x|^{-bq}|v_n|^q\,dx \geqslant 0,$$

and we write

$$\mu_n \equiv \langle J'(v_n), v_n\rangle = \alpha_n - \beta_n \to 0 \quad \text{as } n\to\infty.$$

Let $t\in\mathbb{R}^+$; then there exists a sequence $(t_n)\subset\mathbb{R}^+$ such that

$$\lim_{n\to\infty} t_n = 1 \quad \text{and} \quad \langle J'(t_n v_n), t_n v_n\rangle = 0. \tag{30}$$

Indeed, writing $t = 1+\tau$ where $\tau > 0$ is small enough and using the definitions of μ_n, α_n, and β_n, we have

$$\langle J'(tv_n), tv_n\rangle = \alpha_n(1+\tau)^p - \beta_n(1+\tau)^q = \alpha_n(p-q)\tau + \alpha_n o(\tau) + \mu_n(1+\tau)^q.$$

Since by hypothesis $\lim_{n\to\infty}\alpha_n = \rho^p > 0$, it follows that, for n big enough we can define the sequence

$$\tau_n \equiv \frac{2\mu_n}{\alpha_n(q-p)} \to 0 \quad \text{as } n\to\infty.$$

So,

$$\langle J'(1+\tau_n)v_n, (1+\tau_n)v_n\rangle < 0 \quad \text{and} \quad \langle J'(1-\tau_n)v_n, (1-\tau_n)v_n\rangle > 0. \tag{31}$$

In fact, rewriting the Gâteaux derivative of the functional J, we get

$$\begin{aligned}\langle J'(1+\tau_n)v_n, (1+\tau_n)v_n\rangle &= -2|\mu_n| + \mu_n + \frac{2q}{\alpha_n(q-p)}|\mu_n|\mu_n + \alpha_n o(\tau_n) + \mu_n o(\mu_n)\\ &\equiv K_n.\end{aligned}$$

If $\mu_n > 0$, then $K_n < 0$. Similarly, if $\mu_n < 0$, then $K_n > 0$.

This proves the first part of inequality (31). The other one is similar.

In this way, we can choose $t_n \in (1-\tau_n, 1+\tau_n)$ and we get a sequence $(t_n) \subset \mathbb{R}$ verifying (30). Using this sequence, it follows that

$$J(v_n) = J(t_n v_n) + \left(\frac{1-t_n^p}{p}\right)\alpha_n - \left(\frac{1-t_n^q}{q}\right)\beta_n = J(t_n v_n) + o(1) \geqslant J_\infty + o(1)$$

and this proves the claim. □

Our next lemma compares the minimum obtained previously with a minimax type level.

Fix $\bar{u} \in \mathcal{D}_a^{1,p}(\mathbb{R}^N)$ such that the conclusion of Lemma 5.1 holds.

Since $p < q$, there exists $\tau_0 \in \mathbb{R}^+$ such that

$$J(t\bar{u}) < 0 \quad \text{and} \quad I(t\bar{u}) < 0 \quad \text{if } t \geqslant \tau_0.$$

We define

$$c_1 \equiv \inf_{\gamma \in \mathcal{P}} \sup_{u \in \gamma} I(u), \tag{32}$$

where

$$\mathcal{P} = \left\{\gamma \in C\big([0,1];\ \mathcal{D}_a^{1,p}(\mathbb{R}^N)\big) \,\big|\, \gamma(0) = 0 \text{ and } \gamma(1) = \tau_0\bar{u}\right\}.$$

Lemma 5.3. *Let c_0 and c_1 be defined by* (20) *and* (32)*, respectively. Given a function $g \geqslant 0$ such that $\|g\|_{(L_b^q(\mathbb{R}^N))^*} = 1$, there exist real numbers $R > 0$ and $\varepsilon_2 = \varepsilon_2(R)$ such that $c_1 < c_0 + J_\infty$ for every function $f = \varepsilon g$ such that $\varepsilon \leqslant \varepsilon_2$.*

Proof. First of all we claim that

$$J_\infty + c_0 > 0 \tag{33}$$

if the real numbers $\varepsilon_1 > 0$ and $R > 0$ given at the proof of Lemma 4.2 are small enough.

Indeed, let u_0 be a solution of problem (P) with $\lambda = 0$ obtained from Lemma 4.2. Applying Hölder's and Young's inequalities to the expression of c_0 in terms of u_0, we have

$$c_0 \geqslant \left(\frac{1}{p} - \frac{1}{q}\right) \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u_0|^p \, dx - \left(1 - \frac{1}{q}\right) \|f\|_{(L_b^q(\mathbb{R}^N))^*} \|u_0\| \tag{34}$$

$$\geqslant \frac{\lambda^p}{p} \|u_0\|^p + \frac{\left(1-\frac{1}{q}\right)^{p'}}{p'\lambda^{p'}} \|f\|_{(L_b^q(\mathbb{R}^N))^*}^{p'}, \tag{35}$$

where $\lambda = (1 - p/q)^{1/p}$. Then we get

$$c_0 \geqslant \left[\frac{N(p-1) + p - p(b-a)}{pN}\right]^{p/(p-1)} \frac{(p-1)}{p} \left[1 - \frac{p}{q}\right]^{1/(1-p)} \|f\|_{(L_b^q(\mathbb{R}^N))^*}^{p'}. \tag{36}$$

So, inequality (33) holds for $\|f\|_{(L_b^q(\mathbb{R}^N))^*} < \varepsilon_1$, where $\varepsilon_1 > 0$ is small enough.

To conclude the proof of the lemma it is enough to use the definition of c_1 and the following result.

Claim. $\sup_{t \geqslant 0} I(t\bar{u}) < J_\infty + c_0$ for $\|f\|_{(L_b^q(\mathbb{R}^N))^*} > 0$ small enough.

Indeed, using the continuity of the functional I and $I(0)=0$, as well as inequality (33), we get $\varepsilon'>0$ and $M\in\mathbb{R}$ such that

$$J_\infty + c_0 > \sup_{t\in[0,M]} I(t\bar u) \quad \text{if } \|f\|_{(L^q_b(\mathbb{R}^N))^*} < \varepsilon' < \varepsilon_1.$$

Note that

$$\sup_{t\geqslant M} I(t\bar u) \leqslant \sup_{t\geqslant 0} J(t\bar u) - M\int_{\mathbb{R}^N} f\bar u\,dx = J_\infty - M\int_{\mathbb{R}^N} f\bar u\,dx.$$

Since $\int_{\mathbb{R}^N} fu\,dx$ is linear in ε and c_0 has a term of degree p' in ε, we have

$$\sup_{t\geqslant M} I(t\bar u) < J_\infty + c_0$$

and this concludes the proof of the lemma. □

Conclusion of the proof of Theorem 1.2. Let $\varepsilon_0\equiv\min\{\varepsilon_1,\varepsilon_2\}$. By Lemma 4.2 we get a positive solution $u_0\in\mathcal{D}_a^{1,p}(\mathbb{R}^N)$ for the problem (P) with $\lambda=0$ such that $c_0=I(u_0)$.

On the other hand, since $I(|u|)\leqslant I(u)$ for every function $f\geqslant 0$, the mountain-pass theorem without Palais–Smale condition guarantees the existence of a positive Palais–Smale sequence $(u_n)\subset\mathcal{D}_a^{1,p}(\mathbb{R}^N)$ for the functional I at the level c_1.

This implies that

$$\begin{aligned} c_1 + \frac{1}{q}\big\|I'(u_n)\big\|_{(\mathcal{D}_a^{1,p}(\mathbb{R}^N))^*}\|u_n\| + o(1) &\geqslant I(u_n) - \frac{1}{q}\big\langle I'(u_n),u_n\big\rangle \\ &\geqslant \left(\frac{1}{p}-\frac{1}{q}\right)\|u_n\|^p - \left(1-\frac{1}{q}\right)\|f\|_{(L^q_b(\mathbb{R}^N))^*}\|u_n\|. \end{aligned}$$

Hence, $(u_n)\subset\mathcal{D}_a^{1,p}(\mathbb{R}^N)$ is a bounded sequence. This way, passing to a subsequence (still denoted in the same way), we can suppose that there exists a positive function $u_1\in\mathcal{D}_a^{1,p}(\mathbb{R}^N)$ such that

$$u_n \rightharpoonup u_1 \quad \text{weakly in } \mathcal{D}_a^{1,p}(\mathbb{R}^N),\ \text{as } n\to\infty.$$

Lemma 4.1 implies that u_1 is a solution of problem (P) with $\lambda=0$.

We will show now that $u_0\neq u_1$; to do this, we will prove that $I(u_0)\neq I(u_1)$.

In fact, by Lemma 5.2 there exist two possibilities: if $u_n\to u_1$ strongly in $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$, then

$$I(u_1)=\lim_{n\to\infty} I(u_n)=c_1>0>c_0=I(u_0),$$

that is, $u_1\neq u_0$. On the other hand, if $I(u_1)=I(u_0)=c_0$ and

$$c_1=\lim_{n\to\infty} I(u_n)\geqslant I(u_1)+J_\infty,$$

then

$$c_1=\lim_{n\to\infty} I(u_n)\geqslant I(u_1)+J_\infty=I(u_0)+J_\infty=c_0+J_\infty,$$

which is a contradiction to Lemma 5.3. The theorem is proved. □

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