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Renormalization and forcing of horseshoe orbits

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In this paper we deal with the Boyland forcing of horseshoe orbits. We prove that there exists a set $\mathcal{R}$ of renormalizable horseshoe orbits containing only quasi-one-dimensional orbits, that is, for these orbits the Boyland order coincides with the unimodal order.

1. Introduction

In [2], Boyland introduced the forcing relation between periodic orbits of the disk D^2. Given two periodic orbits P and R, we say that P *forces* R, denoted by $P \geqslant_2 R$, if every homeomorphism of D^2 containing the braid type of P must contain the braid type of R. The set of periodic orbits forced by P is denoted by Σ_P. In this paper we are concerned with the forcing of Smale horseshoe periodic orbits. A horseshoe orbit P is called *quasi-one-dimensional* if P forces all orbit R such that $P \geqslant_1 R$, where $\geqslant_1$ is the unimodal order. In [4], Hall gave a set of quasi-one-dimensional horseshoe orbits, called NBT orbits (Non-Bogus Transition orbits) which are in bijection with $\mathbb{Q} \cap (0, \frac{1}{2})$ and have the property that their thick interval map induced has minimal periodic orbit structure, that is, if P is an NBT orbit then every braid type of a periodic orbit of its thick interval map θ_P is forced by the braid type of P.

In this paper we obtain a type of orbits which are quasi-one-dimensional too although their associated thick interval maps are reducible in the sense of Thurston [6], that is, they are isotopic to reducible homeomorphisms which have an invariant set of non-homotopically trivial disjoint curves $\{C_1, \cdots, C_n\}$. Restricted to the components of $D^2 \setminus \{C_1, \cdots, C_n\}$, these reducible maps or one of its power have minimal periodic orbit structure.

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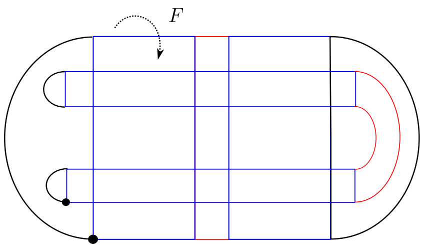


Fig. 1. Dynamics of F.

Theorem 1. *There exists a set* $\mathcal{R} \supset \mathrm{NBT}$ *of quasi-one-dimensional horseshoe orbits, that is, if* $P \in \mathcal{R}$ *then* $\Sigma_P = \{R : P \geqslant_1 R\}$.

These orbits are defined using the renormalization operator which was introduced in [3] as the $*$-product.

2. Preliminaries

2.1. Boyland partial order

Let D_n be the punctured disk. Let $\mathrm{MCG}(D_n)$ be the group of isotopy classes of homeomorphisms of D_n, which is called the *mapping class group of* D_n. Given a homeomorphism $f : D^2 \to D^2$ of the disk D^2 with a periodic orbit P, the *braid type* of P, denoted by $\mathrm{bt}(P, f)$ is defined as follows: Take an orientation preserving homeomorphism $h : D^2 \setminus P \to D_n$ then $\mathrm{bt}(P, f)$ is the conjugacy class $[h \circ f \circ h^{-1}] \in \mathrm{MCG}(D_n)$ of $h \circ f \circ h^{-1} : D_n \to D_n$.

Let BT be the union of all the periodic braid types and let $\mathrm{bt}(f)$ be the set formed by the braid types of the periodic orbits of f. We will say that $f : D^2 \to D^2$ *exhibits* a braid type β if there exists an n-periodic orbit P for f with $\beta = \mathrm{bt}(P, f)$. Now we can define the relation $\geqslant_2$ on BT. We say that β_1 *forces* β_2, denoted by $\beta_1 \geqslant_2 \beta_2$, if every homeomorphism exhibiting β_1, exhibits β_2 too. Then we will say that a periodic orbit P *forces* another periodic orbit R, denoted by $P \geqslant_2 R$, if $\mathrm{bt}(P) \geqslant_2 \mathrm{bt}(R)$.

In [2], P. Boyland proved the following theorem.

Theorem 2. *([2, Theorem 9.1]) The relation* $\geqslant_2$ *is a partial order.*

2.2. Smale horseshoe

The Smale horseshoe is a map $F : D^2 \to D^2$ of the disk which acts as in Fig. 1. The set $\Omega = \bigcap_{j \in \mathbb{Z}} F^j(V_0 \cup V_1)$ is F-invariant and $F|_\Omega$ is conjugated to the shift σ on the sequence space of two symbols 0 and 1, $\Sigma_2 = \{0, 1\}^{\mathbb{Z}}$, where

$$\sigma\big((s_i)_{i \in \mathbb{Z}}\big) = (s_{i+1})_{i \in \mathbb{Z}}. \tag{1}$$

The conjugacy $h : \Omega \to \Sigma_2$ is defined by

$$\big(h(x)\big)_i = \begin{cases} 0 & \text{if } F^i(x) \in V_0 \\ 1 & \text{if } F^i(x) \in V_1 \end{cases} \tag{2}$$

To compare horseshoe orbits it is necessary to define the *unimodal order*. It is a total order in $\Sigma^+ = \{0, 1\}^{\mathbb{N}}$ given by the following rule: Let $s = s_0 s_1 \ldots$ and $t = t_0 t_1 \ldots$ be sequences in Σ^+ such that $s_i = t_i$ for $i \le k$ and $s_{k+1} \ne t_{k+1}$, then $s < t$ if

(O1) $\sum_{i=0}^{k} s_i$ is even and $s_{k+1} < t_{k+1}$, or
(O2) $\sum_{i=0}^{k} s_i$ is odd and $s_{k+1} > t_{k+1}$.

We say that $s \geqslant_1 t$ if either $s = t$ or $s > t$.

Every n-periodic orbit $P \in \Omega$ of F has a *code* denoted by $c_P \in \Sigma_2$. It is obtained from $h(p) = c_P^{\infty}$ where p is a point of P and c_P satisfies $\sigma^i(c_P) \leqslant_1 c_P$, that is, c_P is *maximal* in the unimodal order $\geqslant_1$. We say $P \geqslant_1 R$ if $\sigma^n(R) \leqslant_1 c_P$, $\forall n \geq 1$. For every orbit P, there exists a homeomorphism θ_P that realizes the combinatorics of P. This is obtained fatting the line diagram of P and it is called the *tick map induced by* P. See [4].

2.3. Renormalized horseshoe orbits

Let P and Q be two horseshoe periodic orbits with codes $c_P = Aa_{n-1}$, where $A = a_0 a_1 \cdots a_{n-2}$, and $c_Q = b_0 b_1 \cdots b_{m-2} b_{m-1}$ and periods n and m, respectively.

Definition 3 *(Renormalization operator).* We will write $P * Q$ for the nm-periodic orbit with code

$$c_{P*Q} = \begin{cases} Ab_1 Ab_2 \cdots Ab_{m-2} Ab_{m-1} & \text{if } \epsilon(A) \text{ is even} \\ A\overline{b_1} A\overline{b_2} \cdots A\overline{b_{m-2}} A\overline{b_{m-1}} & \text{if } \epsilon(A) \text{ is odd} \end{cases} \tag{3}$$

where $\epsilon(A) = \sum_{i=0}^{n-2} a_i$ and $\overline{b_i} = 1 - b_i$.

The orbit $P * Q$ is called the *renormalization of* P *and* Q. If an orbit S satisfies $S = P * Q$ for some $P, Q \in \Sigma_2$, it is said that S is *renormalizable*. Also we will denote

$$P_1 * P_2 * \cdots * P_k = \big(\cdots\big((P_1 * P_2) * P_3\big)\cdots\big).$$

Example 4. If P and Q have codes $c_P = 101$ and $c_Q = 1001$ then $P * Q$ has code $c_{P*Q} = 10\mathbf{0}10\mathbf{1}10\mathbf{1}10\mathbf{0}$.

2.4. NBT orbits

There are a type of horseshoe orbits for which the Boyland partial order is well-understood. They are constructed in the following way. Given a rational number $q = \frac{m}{n} \in \widehat{\mathbb{Q}} := \mathbb{Q} \cap (0, \frac{1}{2})$, let L_q be the straight line segment joining $(0,0)$ and (n,m) in $\mathbb{R}^2$. Then construct a finite word $c_q = s_0 s_1 \cdots s_n$ as follows:

$$s_i = \begin{cases} 1 & \text{if } L_q \text{ intersects some line } y = k,\ k \in \mathbb{Z},\ \text{for } x \in (i-1, i+1) \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

It follows that c_q is palindromic and has the form:

$$c_q = 10^{\mu_1} 1^2 0^{\mu_2} 1^2 \cdots 1^2 0^{\mu_{m-1}} 1^2 0^{\mu_m} 1. \tag{5}$$

We will denote P_q to the periodic orbits of period $n+2$ which have the codes $c_q{}^1_0$, when the distinction is not important and let $\mathrm{NBT} = \{P_q : q \in \widehat{\mathbb{Q}}\}$. In [4], Hall proved the following result.

Theorem 5. *Let* $q, q' \in \widehat{\mathbb{Q}}$. *Then*

(i) P_q *is quasi-one-dimensional, that is,* $P_q \geqslant_1 R \Longrightarrow P_q \geqslant_2 R$.
(ii) $q \leqslant q' \Longleftrightarrow (c_q{}^0_1)^{\infty} \geqslant_1 (c_{q'}{}^0_1)^{\infty} \Longleftrightarrow (c_q{}^0_1)^{\infty} \geqslant_2 (c_{q'}{}^0_1)^{\infty}$

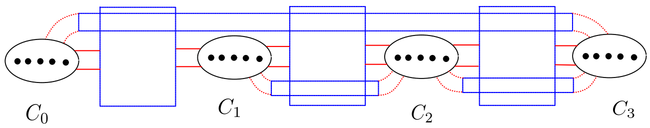


Fig. 2. The image $\theta_{P*Q}(C_i)$ when $P = 1001$.

So theorem above says that the Boyland order restricted to the NBT orbits is equal to the unimodal order.

3. Forcing of renormalizable orbits

For proving Theorem 1 we shall need the following result:

Theorem 6. *Let $P = A_1^0$ and Q be periodic orbits. Then*

$$\Sigma_{P*Q} = \{R : P \geqslant_2 R\} \cup \{P * R : Q \geqslant_2 R\}. \tag{6}$$

To prove the result above it will be needed two lemmas whose proofs are left to the reader.

Lemma 7. *Let $i, j \in \{1, \cdots, n-1\}$ be positive integers with $i \neq j$ and $T_M = P * Q$ and $T_m = \sigma^n(T_M)$. Then*

(a) If $\epsilon(A)$ is even then $A0^\infty \leqslant_1 T_m \leqslant_1 T_M \leqslant_1 A1^\infty$,
(b) If $\epsilon(A)$ is odd then $A1^\infty \leqslant_1 T_m \leqslant_1 T_M \leqslant_1 A0^\infty$,
(c) $\sigma^i(P) \leqslant_1 \sigma^j(P) \iff [\sigma^i(T_M) \leqslant_1 \sigma^j(T_M)$ and $\sigma^i(T_m) \leqslant_1 \sigma^j(T_m)]$.

Lemma 8. *Let $P = A_1^0$ and Q be two periodic orbits. If $i, j \in \{0, \cdots, m-1\}$, with $i \neq j$, then*

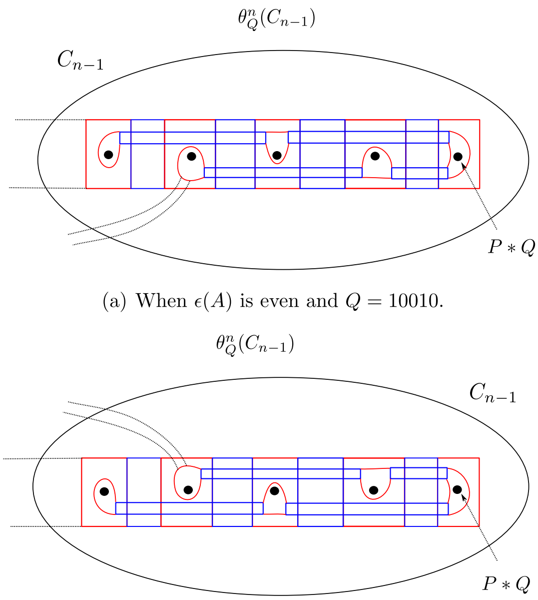
$$\sigma^i(Q) >_1 \sigma^j(Q) \iff \sigma^{in}(P * Q) >_1 \sigma^{jn}(P * Q).$$

Proof of Theorem 6. Let θ_{P*Q} be the thick map induced by $P * Q$. First we see that the only iterates of $P * Q$ satisfying $T_m \leqslant_1 \sigma^i(P * Q) \leqslant_1 T_M$ are the iterates $\sigma^{in}(P * Q)$, with $0 \le i \le m - 1$; so there exists a curve C_{n-1} containing these orbits disjoint from the others and bounding a region D_{n-1}. By Lemma 7(c) and noting that $\sigma^i(T_m)$ and $\sigma^i(T_M)$ have the same initial symbol for $i \in \{1, \cdots, n-1\}$, it follows that $\{\theta^i_{P*Q}(C_{n-1})\}_{i=1}^{n-1}$ has the same combinatorics as P. For $i = 0, \cdots, n-2$, let $C_i = \theta^{i+1}_{P*Q}(C_{n-1})$ be a curve which bounds a domain D_i. It is possible to define θ_{P*Q} such that $\theta^n_{P*Q}(C_{n-1}) = C_{n-1}$. Then the line diagram of $\{D_0, \cdots, D_{n-1}\}$ is as the line diagram of P and then θ_{P*Q} has the same behaviour than θ_P in the exterior of $\cup D_i$. Since θ_{P*Q} can be reduced by a family of curves, we will need study the Thurston representative of θ_{P*Q} restricted to $D^2 \setminus \cup C_i$. As θ_P and θ_{P*Q} have the same combinatorics in the exterior of $\cup D_i$, they have the same Thurston representative in the exterior of $\cup D_i$. So P and $P * Q$ force the same periodic orbits in the exterior of $\cup D_i$. Then $\{R : P \geqslant_2 R\} \subset \Sigma_{P*Q}$. See Fig. 2.

It is clear that to find what orbits are forced by $P * Q$ in $\cup D_i$, it is enough to study θ^n_{P*Q} restricted to D_{n-1}. By Lemma 8, the line diagram of θ^n_{P*Q} inside D_{n-1} is the same as the line diagram of Q when $\epsilon(A)$ is even, and it is flipped when $\epsilon(A)$ is odd. See Fig. 3. So θ^n_{P*Q} has the same combinatorics than θ_Q.

As in Lemma 8, we can prove that $Q \geqslant_2 R \iff P * Q \geqslant_2 P * R$. So $\Sigma_{P*Q} = \{R : P \geqslant_2 R\} \cup \{P * R : Q \geqslant_2 R\}$. $\square$

Remark 9. Theorem 6 says us that to look for the orbits that are forced by $P * Q$ it is enough to look for the orbits that are forced by P and the orbits that are forced by Q. So we can study the thick maps induced



(a) When $\epsilon(A)$ is even and $Q = 10010$.

(b) When $\epsilon(A)$ is odd and $Q = 10010$.

Fig. 3. The image $\theta^n_{P*Q}(C_{n-1})$.

by P and Q separately. Every of these thick maps can be reduced using methods to determine its minimal representative, e.g. [1,5].

Corollary 10. *Let $P_1, P_2, \cdots, P_k$ be NBT orbits. Then*

(a) $\Sigma_{P_1*\cdots*P_k} = \bigcup_{j=1}^{k}\{P_1 * \cdots * P_{j-1} * R : P_j \geqslant_1 R\}$, *and*
(b) $\Sigma_{P_1*\cdots*P_k} = \{R : P_1 * \cdots * P_k \geqslant_1 R\}$.

Proof. Item (a) follows directly from Theorem 6. For item (b) it is enough to prove that if P and Q are quasi-one-dimensional horseshoe orbits then $P * Q$ is a quasi-one-dimensional orbit too. Suppose that $\epsilon(A)$ is even. From Theorem 6,

$$\Sigma_{P*Q} = \{R : P \geqslant_1 R\} \cup \{P * R : Q \geqslant_1 R\}. \tag{7}$$

Hence it follows that $\Sigma_{P*Q} \subset \{S : P * Q \geqslant_1 S\}$. We have to prove the inclusion $\{S : P * Q \geqslant_1 S\} \subset \Sigma_{P*Q}$. If $P \geqslant_1 S$ then $S \in \Sigma_{P*Q}$. Let S with $c_S = s_0 s_1 \cdots s_{k-1}$ be a periodic orbit with $P \leqslant_1 S \leqslant_1 P * Q$. By Lemma 7(a), $(A0)^\infty \leqslant_1 c_S^\infty \leqslant_1 (A1)^\infty$. This implies that $c_S = A s_{n-1} s_n \cdots$ and

$$(0A)^\infty \leqslant_1 \sigma^{n-1}\big(c_S^\infty\big) = s_{n-1} s_n \cdots \leqslant_1 (1A)^\infty,$$

and $\sigma^n({c_S}^\infty) \geqslant_1 (A0)^\infty$. In the other hand $\sigma^n({c_S}^\infty) \leqslant_1 {c_{P*Q}}^\infty$. Then $\sigma^n({c_S}^\infty) = A s_{2n-1} \cdots$ and then $c_S = A s_n A s_{2n-1} \cdots$. Continuing this process, it follows that $S = P * R$ where $c_R = s_{n-1} s_{2n-1} \cdots$. So $P * R \leqslant_1 P * Q$ which implies that $R \leqslant_1 Q$. So $S \in \{P * R : Q \geqslant_1 R\}$ and the proof is finished. □

Now we proceed to prove Theorem 1.

Proof of Theorem 1. Let $\{P_j\}_{j\in\mathbb{N}}$ be the set of NBT orbits and consider the space $\mathbb{N}^{\mathbb{N}}$ of sequences of positive integers. Take a sequence $\mathcal{J} = (j_1, j_2, \cdots, j_n, \cdots) \in \mathbb{N}^{\mathbb{N}}$ and define

$$\mathcal{R}_{\mathcal{J}} = \bigcup_{k=1}^{\infty} \{P_{j_1} * \cdots * P_{j_k}\} \tag{8}$$

and $\mathcal{R} = \bigcup_{\mathcal{J} \in \mathbb{N}^{\mathbb{N}}} \mathcal{R}_{\mathcal{J}}$. By Corollary 10(b), every orbit of $\mathcal{R}$ is quasi-one-dimensional. □

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