J. Differential Equations 248 (2010) 722–744

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Soliton solutions for quasilinear Schrödinger equations with critical growth

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ARTICLE INFO

Article history:
Received 14 May 2008
Available online 16 December 2009

MSC:
35J60
35Q55
35B33

Keywords:
Schrödinger equations
Standing wave solutions
Variational methods
Minimax methods
Critical exponent

ABSTRACT

In this paper we establish the existence of standing wave solutions for quasilinear Schrödinger equations involving critical growth. By using a change of variables, the quasilinear equations are reduced to semilinear one, whose associated functionals are well defined in the usual Sobolev space and satisfy the geometric conditions of the mountain pass theorem. Using this fact, we obtain a Cerami sequence converging weakly to a solution v. In the proof that v is nontrivial, the main tool is the concentration–compactness principle due to P.L. Lions together with some classical arguments used by H. Brezis and L. Nirenberg (1983) in [9].

1. Introduction

Many recent studies have focused on quasilinear equations of the form

$$-\Delta u + V(x)u - \kappa\big[\Delta\big(u^2\big)\big]u = h(u) \quad \text{in } \mathbb{R}^N. \tag{1.1}$$

Such equations arise in various branches of mathematical physics and they have been the subject of extensive study in recent years. Part of the interest is due to the fact that solutions of (1.1) are

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[1] Partially supported by CNPq/Brazil and INCTmat/Brazil.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2009.11.030

related to the existence of solitary wave solutions for quasilinear Schrödinger equations of the form

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi + W(x)\psi - h\big(|\psi|^2\big)\psi - \kappa\big[\Delta\rho\big(|\psi|^2\big)\big]\rho'\big(|\psi|^2\big)\psi, \tag{1.2}$$

where $\psi : \mathbb{R}\times\mathbb{R}^N \to \mathbb{C}$, $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential, κ is a positive constant and $\rho, h : \mathbb{R}^+ \to \mathbb{R}$ are suitable functions.

Quasilinear Schrödinger equations of form (1.2) appear naturally in mathematical physics and have been derived as models of several physical phenomena corresponding to various types of nonlinear term ρ. The case $\rho(s) = s$ was used for the superfluid film equation in plasma physics by Kurihura in [22] (see also [23]). In the case $\rho(s) = (1+s)^{1/2}$, Eq. (1.2) models the self-channeling of a high-power ultra short laser in matter, see [7,8,11,36] and references in [13]. Eq. (1.2) also appears in plasma physics and fluid mechanics [3,21,34,40], in mechanics [18] and in condensed matter theory [28].

Recent mathematical studies have focused on the existence of solutions for (1.1) with $h(u) = |u|^{p-1}u$, with $4 \leqslant p+1 < 4N/(N-2)$, $N \geqslant 3$, for example, in [25,27,32]. The existence of a positive ground state solution has been proved by Poppenberg, Schmitt and Wang [32] and Liu and Wang [25] by using a constrained minimization argument, which gives a solution of (1.1) with an unknown Lagrange multiplier λ in front of the nonlinear term. In [27], by a change of variables the quasilinear problem was reduced to a semilinear one and an Orlicz space framework was used to prove the existence of a positive solution of (1.1) for every positive λ via mountain pass theorem. In [12], Colin and Jeanjean also made use of change of variables in order to reduce Eq. (1.1) to semilinear one. By using the Sobolev space $H^1(\mathbb{R}^N)$, they proved the existence of solutions from classical results given by Berestycki and Lions [6] when $N = 1$ or $N \geqslant 3$, and Berestycki, Gallouët and Kavian [5] when $N = 2$. For $N = 1$ and $N = 2$ we also cite [1,2,10,32,14], respectively.

It is worth pointing out that the related semilinear equations for $\kappa = 0$ have been extensively studied as in the subcritical case $p < 2^* - 1$, as in the critical case $p = 2^* - 1$. For the subcritical case see for example [6,15,35,38], and the references therein. For the critical case, after the pioneering paper by Brezis and Nirenberg [9] many authors have been worked in this subject improving or extending Brezis–Nirenberg work. We would like to cite papers by Noussair, Swanson and Yang [31], Miyagaki [29], García and Peral [16], Benci and Cerami [4] and the book of Willem [41].

Here we consider the case where $\rho(s) = s$, $\kappa = 1$ and our special interest is in the existence of *standing wave solutions*, that is, solutions of type $\psi(t,x) = \exp(-iEt)u(x)$, where $E \in \mathbb{R}$ and $u > 0$ is a real function. It is well known that ψ satisfies (1.2) if and only if the function $u(x)$ solves the equation of elliptic type (1.1), where $V(x) \doteq W(x) - E$ is the new potential.

As observed in [26], the number $2(2^*)$ behaves like a critical exponent for Eq. (1.1). In fact, by using a variational identity given by Pucci and Serrin [33], we can prove that (1.1) has no positive solution in $H^1(\mathbb{R}^N)$ with $u^2|\nabla u|^2 \in L^1(\mathbb{R}^N)$ if $p+1 \geqslant 2(2^*)$ and if V satisfies $\nabla V(x)\cdot x \geqslant 0$ for all $x \in \mathbb{R}^N$.

Thus, similar in spirit to [9], a natural question is whether adding a lower order term to $h(u) = |u|^{2(2^*)-1}u$ the solvability of (1.1) is regained.

The main purpose of the present paper is give affirmative answer for the following class of quasilinear equations

$$-\Delta u + V(x)u - \big[\Delta\big(u^2\big)\big]u = |u|^{q-1}u + |u|^{p-1}u \quad \text{in } \mathbb{R}^N, \tag{P}$$

where λ is a positive parameter, $3 < q < p \leqslant 2(2^*) - 1$ and $2^* = 2N/(N-2)$ is the critical Sobolev exponent (in dimension $N \geqslant 3$).

Next, for easy reference we state our assumptions in a more precise way. In order to deal with the convex term, we make the following assumptions on the potential V:

(V_1) The function $V : \mathbb{R}^N \to \mathbb{R}$ is continuous and uniformly positive, that is, there exists a constant $V_0 > 0$ such that

$$0 < V_0 \leqslant V(x) \quad \text{for all } x \in \mathbb{R}^N.$$

(V_2) There exists a constant V_∞ such that

$$\lim_{|x|\to\infty} V(x) = V_\infty \quad \text{and} \quad V(x) \leqslant V_\infty \quad \text{for all } x \in \mathbb{R}^N,$$

where the last inequality is strict on a subset of positive measure in $\mathbb{R}^N$.

(V_2') The function V is periodic in each variable of $x_1, \dots, x_N$.

The following theorem contains our main result:

Theorem 1.1. *Suppose $p = 2(2^*) - 1$ and $3 < q < 2(2^*) - 1$. In addition to (V_1), assume that either assumption (V_2) or (V_2') holds. Then (P) has a positive classic solution.*

Existence results for problem (1.1) involving critical exponent have been obtained by Moameni in [30] by assuming potential function $V(x)$ radial and satisfying some geometry conditions. However these conditions imply that the problem does not involve critical Sobolev exponent any more, because, in some sense, the Sobolev space considered is compactly embedded in L^s space for all $s > 2$. We observe that our proof does not require any geometric condition on the potential.

The underline idea for proving our main result: motivated by the argument used in [27] (see also [12]), we change of variable to reformulate the problem obtaining a semilinear problem involving a critical Sobolev exponent of the form:

$$-\Delta u + v(x)u = \phi(x,u) + |u|^{2^*-2}u, \quad u > 0 \text{ in } \mathbb{R}^N. \tag{1.3}$$

Even the study of this class of problem is new because in our case the nonlinear term ϕ satisfies

$$\lim_{u\to+\infty} \frac{\phi(x,u)}{u^{2^*-1}} = 0$$

instead of the usual subcritical condition $\phi(x,u) = o(|u|^r)$, $2 < r < 2^* - 1$ as $|u| \to \infty$. The associated functional is now well defined in the usual Sobolev space $H^1(\mathbb{R}^N)$ and it satisfies the geometric conditions of the mountain pass theorem. Then a bounded Cerami sequence (v_n) is obtained, which converges weakly to a weak solution v of problem (1.3). In order to prove that v is nontrivial, main tool is the concentration–compactness principle due to Lions [24] together with some classical arguments used by Brezis and Nirenberg in [9]. After changing variable v is a weak solution of the original problem (P).

The outline of the paper is as follows. In the forthcoming section is given the reformulation of the problem and some preliminary results. In Section 3, by using the mountain pass theorem we prove Theorems 1.1.

Notation. In this paper we make use of the following notation:

- $C, C_0, C_1, C_2, \dots$ denote positive (possibly different) constants.
- B_R denotes the open ball centered at origin and radius $R > 0$.
- $C_0^\infty(\mathbb{R}^N)$ denotes the functions infinitely differentiable with compact support in $\mathbb{R}^N$.
- For $1 \leqslant p \leqslant \infty$, $L^p(\mathbb{R}^N)$ denotes the usual Lebesgue space with norms

$$\|u\|_p \doteq \left[\int_{\mathbb{R}^N} |u|^p \, dx\right]^{1/p}, \quad 1 \leqslant p < \infty;$$

$$\|u\|_\infty \doteq \inf\{C > 0\colon \left|u(x)\right| \leqslant C \text{ almost everywhere on } \mathbb{R}^N\}.$$

- $H^1(\mathbb{R}^N)$ denotes the Sobolev spaces modeled in $L^2(\mathbb{R}^N)$ with norm

$$\|u\|_{H^1} = \left[\int\limits_{\mathbb{R}^N} \left(|\nabla u|^2 + |u|^2 \right) \mathrm{d}x \right]^{1/2}.$$

- By $\langle \cdot, \cdot \rangle$ we denote the duality pairing between X and its dual X^*.
- We denote the weak convergence in X and X' by "$\rightharpoonup$" and the strong convergence by "$\to$".

2. Reformulation of the problem and preliminaries

Notice that $u \equiv 0$ is a (trivial) solution of (P), our objective in this article is to apply minimax methods to study the existence of a positive solution for (P). We observe that formally (P) is the Euler–Lagrange equation associated of the natural energy functional

$$J(u) = \frac{1}{2} \int\limits_{\mathbb{R}^N} \left(1 + 2|u|^2\right) |\nabla u|^2 \,\mathrm{d}x + \frac{1}{2} \int\limits_{\mathbb{R}^N} V(x)|u|^2 \,\mathrm{d}x - \int\limits_{\mathbb{R}^N} H(u)\,\mathrm{d}x,$$

where

$$H(u) = \frac{\lambda}{q+1}|u|^{q+1} + \frac{1}{p+1}|u|^{p+1}.$$

From the variational point of view, the first difficulty we have to deal with (P) is to find an appropriate function space where the above functional is well defined. In the spirit of the argument developed by Liu, Wang and Wang in [27] (see also [12]), we make the change of variables $v = f^{-1}(u)$, where f is defined by

$$f'(t) = \frac{1}{(1 + 2f^2(t))^{1/2}} \quad \text{on } [0, +\infty),$$
$$f(-t) = -f(t) \quad \text{on } (-\infty, 0].$$

Therefore, after the change of variables, from $J(u)$ we obtain the following functional

$$I(v) = \frac{1}{2} \int\limits_{\mathbb{R}^N} \left[|\nabla v|^2 + V(x) f^2(v) \right] \mathrm{d}x - \int\limits_{\mathbb{R}^N} H\big(f(v)\big)\,\mathrm{d}x,$$

which is well defined on the usual Sobolev space $H^1(\mathbb{R}^N)$ under suitable assumptions on the potential $V(x)$ and the nonlinearity $H(s)$. Moreover, the positive critical points of the functional I correspond precisely to the positive weak solutions of the following equation

$$-\Delta v = \frac{1}{\sqrt{1 + 2f^2(v)}} \left[h\big(f(v)\big) - V(x) f(v) \right] \quad \text{in } \mathbb{R}^N. \tag{M}$$

For completeness we collect here some properties of the change of variable.

Lemma 2.1. *The function $f(t)$ enjoys the following properties*:

(1) *f is uniquely defined C^∞ function and invertible.*
(2) *$|f'(t)| \leqslant 1$ for all $t \in \mathbb{R}$.*

(3) $|f(t)| \leqslant |t|$ *for all* $t \in \mathbb{R}$.
(4) $f(t)/t \to 1$ *as* $t \to 0$.
(5) $f(t)/\sqrt{t} \to 2^{1/4}$ *as* $t \to +\infty$.
(6) $f(t)/2 \leqslant tf'(t) \leqslant f(t)$ *for all* $t \geqslant 0$.
(7) $|f(t)| \leqslant 2^{1/4}|t|^{1/2}$ *for all* $t \in \mathbb{R}$.
(8) *The function* $f^2(t)$ *is strictly convex.*
(9) *There exists a positive constant* C *such that*

$$\big|f(t)\big| \geqslant \begin{cases} C|t|, & |t| \leqslant 1, \\ C|t|^{1/2}, & |t| \geqslant 1. \end{cases}$$

(10) *There exist positive constants* C_1 *and* C_2 *such that*

$$|t| \leqslant C_1\big|f(t)\big| + C_2\big|f(t)\big|^2 \quad \textit{for all } t \in \mathbb{R}.$$

(11) $|f(t)f'(t)| \leqslant 1/\sqrt{2}$ *for all* $t \in \mathbb{R}$.

Proof. Properties (1), (2), (4), (5) and (6) were proved in [12] (see also [27]). Inequality (3) is a consequence of (2) and the fact that $f(t)$ is an odd and concave function for $t > 0$. To prove (7), we use (4), (5) and (6). Indeed, according to (4), we have

$$\lim_{t \to 0^+} \frac{f(t)}{\sqrt{t}} = 0,$$

and (6) implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{f(t)}{\sqrt{t}}\right) = \frac{2f'(t)t - f(t)}{2t\sqrt{t}} \geqslant 0 \quad \text{for all } t > 0.$$

Consequently, the function $f(t)/\sqrt{t}$ is nondecreasing for $t > 0$ and from (5) we conclude that

$$f(t)/\sqrt{t} \leqslant 2^{1/4} \quad \text{for all } t > 0.$$

This together with the fact that f is odd proves (7).

In order to prove (8) we notice that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\big[f^2(t)\big] = \frac{1}{(1 + 2f^2(t))^2} > 0.$$

Points (9) and (10) are immediate consequences of (4) and (5). Finally, estimate (11) follows directly from the definition of f and the lemma is proved. □

3. Existence results via mountain pass

We will achieve the existence result by using the well-known version of the mountain pass theorem which is a consequence of the Ekeland variational principle (see [20] and [39]).

3.1. Mountain pass geometry

Here we prove that the functional I exhibits the mountain pass geometry. For that matter, we first consider the set

$$\mathcal{S}(\rho) \doteq \{v \in H^1(\mathbb{R}^N): \|v\| = \rho\}.$$

Lemma 3.1. *There exist $\rho, \alpha > 0$, such that*

$$I(v) \geqslant \alpha \quad \textit{for all } v \in \mathcal{S}(\rho).$$

Proof. Since $|\nabla(f(v)^2)|^2 \leqslant 2|\nabla(v)|^2$, using Sobolev–Gagliardo–Nirenberg inequality we have

$$\|f^2(v)\|_{2^*} \leqslant C\|\nabla(f(v)^2)\|_2 \leqslant C\|\nabla v\|_2 \leqslant C\|v\|$$

for some positive constant C. Thus, for $v \in \mathcal{S}(\rho)$ we have

$$\int_{\mathbb{R}^N} |f(v)|^{2(2^*)}\,\mathrm{d}x \leqslant C\rho^{2^*}. \tag{3.1}$$

Setting $\alpha = (2(2^*) - (q+1))/((q+1)(2^* - 1))$ we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} |f(v)|^{q+1}\,\mathrm{d}x &= \int_{\mathbb{R}^N} |f^2(v)|^{(q+1)/2}\,\mathrm{d}x \\
&\leqslant \Big[\int_{\mathbb{R}^N} f^2(v)\,\mathrm{d}x\Big]^{\alpha(q+1)/2}\Big[\int_{\mathbb{R}^N} (f^2(v))^{2^*}\,\mathrm{d}x\Big]^{1-\alpha(q+1)/2} \\
&\leqslant C(\rho^2)^{\alpha(q+1)/2}\Big[\int_{\mathbb{R}^N} |\nabla(f^2(v))|^2\,\mathrm{d}z\Big]^{(1-\alpha(q+1)/2)2^*/2} \\
&= C\rho^{(2N+2(q+1))/(N+2)},
\end{aligned}$$

where $(2N + 2(q+1))/(N+2) > 2$ because $q + 1 > 4$. Therefore, for $v \in \mathcal{S}(\rho)$ we have

$$I(v) \geqslant C_1\rho^2 - C_2\rho^{(2N+2(q+1))/(N+2)} - C_3\rho^{2^*},$$

which implies the conclusion as required. □

Lemma 3.2. *There exists $v \in E$ such that $\|v\| > \rho$ and $I(v) < 0$.*

Proof. We are going to prove that there exists $\varphi \in H^1(\mathbb{R}^N)$ such that $I(t\varphi) \to -\infty$ as $t \to +\infty$, which proves our thesis if we take $v = t\varphi$ with t large enough. Consider $\varphi \in C_0^\infty(\mathbb{R}^N, [0,1])$ such that $\operatorname{supp}(\varphi) = \overline{B}_1$. Using property (6) in Lemma 2.1, it follows that $f(s)/s$ is decreasing for $s > 0$. Since $0 < t\varphi(x) \leqslant t$ for $x \in B_1$ and $t > 0$, we obtain $f(t\varphi(x)) \geqslant f(t)\varphi(x)$, which implies that

$$I(t\varphi) \leqslant \frac{t^2}{2}\left[\int_{B_1}\left(|\nabla\varphi|^2 + V(x)\varphi^2\right)\mathrm{d}x - C_1\frac{f(t)^{q+1}}{t^2}\int_{B_1}\varphi^{q+1}\,\mathrm{d}x - C_2\frac{f(t)^{2(2^*)}}{t^2}\int_{B_1}\varphi^{2(2^*)}\,\mathrm{d}x\right]$$
$$\to -\infty, \quad \text{as } t \to +\infty,$$

where we have used that for $s > 2$ we have $\lim_{t\to+\infty} f(t)^s/t^2 = +\infty$, which is a consequence of property (5) in Lemma 2.1. □

3.2. Cerami sequences

As a consequence of Lemmas 3.1 and 3.2 and of a version of Ambrosetti–Rabinowitz mountain pass theorem [37], for the constant

$$c_0 = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} I\big(\gamma(t)\big) > 0, \tag{3.2}$$

where

$$\Gamma = \big\{\gamma \in C\big([0,1], H^1\big(\mathbb{R}^N\big)\big);\ \gamma(0) = 0,\ \gamma(1) \neq 0,\ I\big(\gamma(1)\big) < 0\big\},$$

there exists a Cerami sequence (v_n) in $H^1(\mathbb{R}^N)$ at the level c_0, that is,

$$I(v_n) \to c_0 \quad \text{and} \quad \big(1 + \|v_n\|\big)\big\|I'(v_n)\big\| \to 0, \quad \text{as } n \to \infty.$$

Lemma 3.3. *The sequence (v_n) is bounded in $H^1(\mathbb{R}^N)$.*

Proof. Since (v_n) satisfies

$$I(v_n) = \frac{1}{2}\int_{\mathbb{R}^N}|\nabla v_n|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N}V(x)f^2(v_n)\,\mathrm{d}x - \int_{\mathbb{R}^N}H\big(f(v_n)\big)\,\mathrm{d}x \to c_0, \quad \text{as } n \to \infty, \tag{3.3}$$

and, for every $w \in H^1(\mathbb{R}^N)$,

$$\begin{aligned}&\big(1 + \|v_n\|\big)I'(v_n)w\\ &\quad = \big(1 + \|v_n\|\big)\left\{\int_{\mathbb{R}^N}\nabla v_n\cdot\nabla w\,\mathrm{d}x + \int_{\mathbb{R}^N}\big[f'(v_n)\big(V(x)f(v_n)w - h\big(f(v_n)\big)w\big)\big]\,\mathrm{d}x\right\}\\ &\quad = \varepsilon_n\|w\|,\end{aligned}$$

where $\varepsilon_n \to 0$ as $n \to \infty$, by choosing $w = w_n \equiv f(v_n)/f'(v_n)$ and inserting in (3.4) we obtain

$$\begin{aligned}&\big(1 + \|v_n\|\big)I'(v_n)w_n\\ &\quad = \big(1 + \|v_n\|\big)\left\{\int_{\mathbb{R}^N}\left[1 + \frac{2f^2(v_n)}{1 + 2f^2(v_n)}\right]|\nabla v_n|^2\,\mathrm{d}x + \int_{\mathbb{R}^N}\big[V(x)f^2(v_n) - h\big(f(v_n)\big)f(v_n)\big]\,\mathrm{d}x\right\}\\ &\quad = \varepsilon_n\|w_n\|.\end{aligned} \tag{3.4}$$

Notice that w_n verifies

$$|w_n|_2 \leqslant C|v_n|_2, \quad |\nabla w_n| \leqslant 2|\nabla v_n| \quad \text{and} \quad \|w_n\| \leqslant C\|v_n\|.$$

Consequently,

$$\begin{aligned} &I'(v_n)w_n \\ &= \int_{\mathbb{R}^N} \left[1 + \frac{2f^2(v_n)}{1+2f^2(v_n)}\right]|\nabla v_n|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} \left[V(x)f^2(v_n) - h\big(f(v_n)\big)f(v_n)\right]\mathrm{d}x \\ &= \varepsilon_n. \end{aligned} \tag{3.5}$$

Notice that

$$(q+1)H(s) - h(s)s = \left[\frac{q+1}{2(2^*)} - 1\right]|s|^{2(2^*)} < 0 \quad \text{for all } s \in \mathbb{R}. \tag{3.6}$$

Then, combining (3.3), (3.4) and (3.6), we infer that

$$\int_{\mathbb{R}^N} \left\{\frac{1}{2} - \frac{1}{q+1}\left[1 + \frac{2f^2(v_n)}{1+2f^2(v_n)}\right]\right\}|\nabla v_n|^2 \,\mathrm{d}x + \left(\frac{1}{2} - \frac{1}{q+1}\right)\int_{\mathbb{R}^N} V(x)f^2(x)\,\mathrm{d}x \leqslant c_0 + \delta_n + \varepsilon_n,$$

where δ_n is given in (3.3). Since $q+1 > 4$ we can conclude that the term

$$\int_{\mathbb{R}^N} \left[|\nabla v_n|^2 + V(x)f^2(v_n)\right]\mathrm{d}x$$

is bounded. Then, to conclude that (v_n) is bounded in $H^1(\mathbb{R}^N)$, it remains to show that (v_n) is bounded in $L^2(\mathbb{R}^N)$. To verify this we start splitting

$$\int_{\mathbb{R}^N} v_n^2 \,\mathrm{d}x = \int_{\{x:\ |v_n(x)| \leqslant 1\}} v_n^2 \,\mathrm{d}x + \int_{\{x:\ |v_n(x)| > 1\}} v_n^2 \,\mathrm{d}x.$$

Notice that there exists $C > 0$ such that $H(s) \geqslant Cs^{q+1}$, for every $s \geqslant 1$. Then, from (f_2) we have $H(f(s)) \geqslant Cs^{(q+1)/2}$, for every $s \geqslant 1$. Therefore

$$\int_{\{x:\ |v_n(x)| > 1\}} v_n^2 \,\mathrm{d}x \leqslant \frac{1}{C} \int_{\{x:\ |v_n(x)| > 1\}} H\big(f(v_n)\big)\,\mathrm{d}x \leqslant \frac{1}{C}\int_{\mathbb{R}^N} H\big(f(v_n)\big)\,\mathrm{d}x,$$

where we have used that $q > 3$. By using that $f(s) \geqslant Cs$, for some $C > 0$, we have

$$\int_{\{x:\ |v_n(x)| \leqslant 1\}} v_n^2 \,\mathrm{d}x \leqslant \frac{1}{C} \int_{\{x:\ |v_n(x)| \leqslant 1\}} f^2(v_n)\,\mathrm{d}x \leqslant \frac{1}{C}\int_{\mathbb{R}^N} f^2(v_n)\,\mathrm{d}x.$$

Hence v_n is bounded in $L^2(\mathbb{R}^N)$. This proves Lemma 3.3. □

We rewrite the functional I by

$$I(v)=\frac{1}{2}\int_{\mathbb{R}^N}\left[|\nabla v|^2+V(x)v^2\right]\mathrm{d}x-\frac{4^{1/(N-4)}}{2^*}\int_{\mathbb{R}^N}|v|^{2^*}\,\mathrm{d}x-\int_{\mathbb{R}^N}G(v)\,\mathrm{d}x,$$

where

$$G(v)=\frac{\lambda}{q+1}\left|f(v)\right|^{q+1}+\frac{1}{22^*}\left|f(v)\right|^{2(2^*)}+\frac{1}{2}V(x)v^2-\frac{1}{2}V(x)f(v)^2-\frac{4^{1/(N-2)}}{2^*}|v|^{2^*}$$

is the primitive of

$$g(v)=f'(v)\left[\lambda\left|f(v)\right|^{q-1}f(v)+\left|f(v)\right|^{2(2^*)-2}f(v)-V(x)f(v)\right]+V(x)v-4^{1/(N-2)}|v|^{2^*-2}v.$$

Notice that the functions G and g satisfy the following properties:

(G_1) $\lim_{s\to 0}\frac{G(s)}{s^2}=0$;
(G_2) $\lim_{s\to +\infty}\frac{G(s)}{s^{2^*}}=0$;
(G_3) $\lim_{s\to 0}\frac{g(s)}{s}=0$;
(G_4) $\lim_{s\to +\infty}\frac{g(s)}{s^{2^*-1}}=0$.

In fact, we must analyze the terms

$$\frac{f(v)^{q+1}}{v^2}=\left(\frac{f(v)}{v}\right)^2 f(v)^{q-1}\quad\text{and}\quad\frac{f(v)^{2(2^*)}}{v^2}=\left(\frac{f(v)}{v}\right)^2 f(v)^{2(2^*)-2}.$$

Since $q>1$, from Lemma 2.1(4), these two terms converge to zero, as $v\to 0$. Thus property (G_1) holds. Similarly we can prove property (G_3). Now, since $q+1<2(2^*)$, the term

$$\frac{f(v)^{q+1}}{v^{2^*}}=\left(\frac{f(v)}{\sqrt{v}}\right)^{q+1}v^{(q+1)/2-2^*}\to 0,\quad\text{as } v\to 0.$$

Also, from Lemma 2.1(3), we have

$$0\leqslant\frac{1}{2}V(x)\frac{v^2}{2^*}-\frac{1}{2}V(x)\frac{f(v)^2}{v^{2^*}}\leqslant\frac{1}{2}V(x)\frac{v^2}{v^{2^*}}.$$

So that

$$\frac{1}{2}V(x)\frac{v^2}{2^*}-\frac{1}{2}V(x)\frac{f(v)^2}{v^{2^*}}\to 0,\quad\text{as } v\to +\infty.$$

Now, from Lemma 2.1(5), the term

$$\frac{f(v)^{2(2^*)}}{v^{2^*}}=\left(\frac{f(v)}{\sqrt{v}}\right)^{22^*}\to\left(2^{1/4}\right)^{2(2^*)}=2^{N/(N-2)},\quad\text{as } v\to +\infty.$$

Therefore property (G_2) is verified.

Since

$$f'(v)\frac{|f(v)|^{2(2^*)-2}f(v)}{|v|^{2^*-1}} = f'(v)f(v)\left[\frac{|f(v)|}{\sqrt{v}}\right]^{2(2^*)-2}$$
$$= \frac{f(v)}{\sqrt{1+2f(v)}}\left[\frac{|f(v)|}{\sqrt{v}}\right]^{2(2^*)-2} \to 4^{1/(N-2)}, \quad \text{as } v \to +\infty,$$

we have

$$\lim_{v\to+\infty}\frac{g(v)}{v^{2^*-1}} = 0.$$

This proves (G_4). From (G_1) and (G_2) for all $\varepsilon > 0$ there exists a positive constant C_ε such that

$$0 \leqslant G(v) \leqslant \varepsilon\big(v^2 + v^{2^*}\big) + C_\varepsilon v^{(q+1)/2}. \tag{3.7}$$

Similarly, for $1 < (q+1)/2 < 2^*$, using properties (G_3) and (G_4) we have

$$g(v)v \leqslant \varepsilon\big(v^2 + v^{2^*}\big) + C_\varepsilon v^{(q+1)/2}. \tag{3.8}$$

Lemma 3.4. *The minimax level c_0 given in* (3.2) *satisfies*

$$c_0 < \frac{S^{N/2}}{2N}.$$

Proof. From the minimax characterization of c_0 we see that it is sufficient to show that there exists $v_0 \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that

$$\sup_{t\geqslant 0} I(tv_0) < \frac{S^{N/2}}{2N}.$$

Let $R > 0$ to be suitably chosen in the sequel, $\varepsilon > 0$ and $\psi_\varepsilon(x) \doteq \varphi(x)w_\varepsilon(x)$, where $\varphi \in C_0^\infty(\mathbb{R}^N, [0,1])$ is a standard cut-off function, such that $\varphi \equiv 1$ on $B_{R_\varepsilon}(0)$ and $\varphi \equiv 0$ on $\mathbb{R}^N \setminus B_{2R_\varepsilon}(0)$ with $R_\varepsilon = \varepsilon^\alpha$, $\alpha \in (\frac{1}{4}, \frac{1}{2})$, and

$$w_\varepsilon(x) = \big(N(N-2)\varepsilon\big)^{(N-2)/4}\frac{1}{(\varepsilon + |x|^2)^{(N-2)/2}}.$$

By definition, w_ε satisfies

$$\int_{\mathbb{R}^N} |\nabla w_\varepsilon|^2\,dx = \int_{\mathbb{R}^N} |w_\varepsilon|^{2^*}\,dx = S^{N/2},$$
$$\int_{B_{R_\varepsilon}(0)} |\nabla w_\varepsilon|^2\,dx \leqslant \int_{B_{R_\varepsilon}(0)} |w_\varepsilon|^{2^*}\,dx,$$
$$\int_{\mathbb{R}^N\setminus B_{R_\varepsilon}(0)} |\nabla w_\varepsilon|^2\,dx = O\big(\varepsilon^{(N-2)/2}\big), \quad \text{as } \varepsilon \to 0.$$

Thus, for

$$X_\varepsilon = \int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 \, dx \quad \text{and} \quad v_\varepsilon(x) = \frac{\psi_\varepsilon(x)}{(\int_{B_{2R_\varepsilon}(0)} |\psi_\varepsilon|^{2^*} \, dx)^{1/2^*}}$$

we have

$$X_\varepsilon = S + O\big(\varepsilon^\delta\big), \quad \text{where } \delta = \frac{N-2}{2}. \tag{3.9}$$

Assertion 1. *There exist $\varepsilon_0 > 0$ and positive constants C_1 and C_2, independent of ε, such that*

$$C_1 \leqslant \frac{f(v_\varepsilon)}{v_\varepsilon^{1/2}} \leqslant C_2 \quad \textit{for all } \varepsilon \in (0, \varepsilon_0) \textit{ and } x \in B_{R_\varepsilon}(0).$$

Proof. From Lemma 2.1(5), for $\eta \in (0, 2^{1/4})$ given, there exists $s_0 > 0$ such that

$$2^{1/4} - \eta < \frac{f(s)}{s^{1/2}} < 2^{1/4} + \eta \quad \text{for all } s \geqslant s_0. \tag{3.10}$$

For all $x \in B_{R_\varepsilon}(0)$, since $R_\varepsilon = \varepsilon^\alpha$ we obtain

$$\begin{aligned} v_\varepsilon(x) &\geqslant \frac{1}{S^{(N-2)/4}} \omega_\varepsilon(x) = \frac{1}{S^{(N-2)/4}} \big(N(N-2)\varepsilon\big)^{(N-2)/4} \frac{1}{(\varepsilon + |x|^2)^{(N-2)/2}} \\ &\geqslant \frac{1}{S^N} \frac{(N(N-2)\varepsilon)^{(N-2)/4}}{(\varepsilon + R_\varepsilon^2)^{(N-2)/2}} = \frac{1}{S^N} \frac{(N(N-2)\varepsilon)^{(N-2)/4}}{(\varepsilon + \varepsilon^{2\alpha})^{(N-2)/2}} \\ &= \frac{(N(N-2))^{(N-2)/4}}{S^N} \frac{\varepsilon^{(N-2)/4 - \alpha(N-2)}}{(1 + \varepsilon^{2\alpha})^{(N-2)/2}} \to \infty, \quad \text{as } \varepsilon \to 0, \end{aligned}$$

we have used that $(N-2)/4 - \alpha(N-2) < 0$ and $1/4 < \alpha < 1/2$. So that, there exists ε_0 (independent of x) such that

$$v_\varepsilon(x) \geqslant s_0 \quad \text{for all } \varepsilon \in (0, \varepsilon_0).$$

This inequality combined with (3.10) complete the proof of our assertion. □

Since $\lim_{t\to\infty} I(tv_\varepsilon) = -\infty$, there exists $t_\varepsilon > 0$ such that $I(t_\varepsilon v_\varepsilon) = \max_{t>0} I(tv_\varepsilon)$. Thus, $I'(t_\varepsilon v_\varepsilon) = 0$ and

$$t_\varepsilon\left(X_\varepsilon + \int_{\mathbb{R}^N} V(x) v_\varepsilon^2 \, dx\right) = \int_{\mathbb{R}^N} g(t_\varepsilon v_\varepsilon) v_\varepsilon \, dx + 4^{1/(N-2)} t_\varepsilon^{2^*-1} \geqslant 4^{1/(N-2)} t_\varepsilon^{2^*-1}$$

which implies that

$$0 < t_\varepsilon \leqslant t_0(\varepsilon) \doteq \frac{1}{4^{1/(N-2)}} \left(X_\varepsilon + \int_{\mathbb{R}^N} V(x) v_\varepsilon^2 \, dx\right)^{1/(2^*-2)}.$$

Now

$$I(t_\varepsilon v_\varepsilon) \leqslant \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 + V(x)v_\varepsilon^2\,dx - \frac{4^{1/(N-2)}t^{2^*}}{2^*}\int_{\mathbb{R}^N} |v_\varepsilon|^{2^*} - \int_{\mathbb{R}^N} G(t_\varepsilon v_\varepsilon)\,dx$$

$$\leqslant \frac{4^{1/(N-2)}t_0(\varepsilon)^{2^*}}{2} - \frac{4^{1/(N-2)}t^{2^*}}{2^*} - \int_{\mathbb{R}^N} G(t_\varepsilon v_\varepsilon)\,dx.$$

Since the function $t \mapsto (t^2/2)t_0^2(\varepsilon) - t^{2^*}/2^*$ is nondecreasing in $(0, t_0(\varepsilon))$ we get

$$I(t_\varepsilon v_\varepsilon) \leqslant \frac{4^{1/(N-2)}t_0(\varepsilon)^{2^*}}{N} - \int_{\mathbb{R}^N} G(t_\varepsilon v_\varepsilon)\,dx$$

$$= \frac{4^{1/(N-2)}}{N}\left[\frac{1}{4^{1/(N-2)}}\left(X_\varepsilon + \int_{\mathbb{R}^N} V(x)v_\varepsilon^2\,dx\right)\right]^{2^*/(2^*-2)} - \int_{\mathbb{R}^N} G(t_\varepsilon v_\varepsilon)\,dx.$$

From (3.9), we obtain

$$I(t_\varepsilon v_\varepsilon) \leqslant \frac{4^{1/(N-2)}}{N}\left[\frac{1}{4^{1/(N-2)}}\left(S + O\left(\varepsilon^\delta\right) + \int_{\mathbb{R}^N} V(x)v_\varepsilon^2\,dx\right)\right]^{N/2} - \int_{\mathbb{R}^N} G(t_\varepsilon v_\varepsilon)\,dx.$$

Noticing that $(b+c)^\alpha \leqslant b^\alpha + \alpha(b+c)^{\alpha-1}$ for all $b, c > 0$, we have

$$I(t_\varepsilon v_\varepsilon) \leqslant \frac{(4^{1/(N-2)})^{1-N/2}}{N}S^{N/2} + O\left(\varepsilon^\delta\right) + C\int_{\mathbb{R}^N} V(x)v_\varepsilon^2\,dx - \int_{\mathbb{R}^N} G(t_\varepsilon v_\varepsilon)\,dx$$

$$\leqslant \frac{S^{N/2}}{2N} + O\left(\varepsilon^\delta\right) + C_1\int_{\mathbb{R}^N} V(x)v_\varepsilon^2\,dx - C_2\int_{\mathbb{R}^N} G(v_\varepsilon)\,dx,$$

in the last estimate we have used the fact that $t_\varepsilon \geqslant K > 0$ and $G(t_\varepsilon v_\varepsilon) \geqslant G(K v_\varepsilon)$. Without loss of generality we choose $K = 1$.

Assertion 2. *The following limit holds*:

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{(N-2)/2}}\int_{B_{2R_\varepsilon}} \left[C_1 V(x)v_\varepsilon^2 - C_2 G(v_\varepsilon)\right]dx = -\infty.$$

Proof. Split the integral

$$\frac{1}{\varepsilon^\delta}\int_{B_{2R_\varepsilon}(0)} \left[C_1 V(x)v_\varepsilon^2 - C_2 G(v_\varepsilon)\right]dx = I_1 + I_2, \quad \text{with } \delta = \frac{N-2}{2},$$

where

$$I_1 = \int\limits_{B_{R_\varepsilon}(0)} \left[C_1 V(x) v_\varepsilon^2 - C_2 G(v_\varepsilon)\right] dx$$

and

$$I_2 = \int\limits_{B_{2R_\varepsilon}(0)\backslash B_{R_\varepsilon}(0)} \left[C_1 V(x) v_\varepsilon^2 - C_2 G(v_\varepsilon)\right] dx.$$

Let us estimate I_1. From Lemma 2.1(3), we have

$$I_1 = \int\limits_{B_{R_\varepsilon}(0)} \left\{C_1 V(x) v_\varepsilon^2 - C_2\left[\frac{\lambda}{q+1} f(v_\varepsilon)^{q+1} + \frac{1}{2(2^*)} f(v_\varepsilon)^{2(2^*)} + Z(x) - \frac{4^{1/(N-2)}}{2^*}|v_\varepsilon|^{2^*}\right]\right\} dx,$$

where

$$Z(x) \doteq \frac{V(x)}{2} v_\varepsilon^2 - \frac{V(x)}{2} f(v_\varepsilon)^2 \geqslant 0.$$

Thus,

$$\begin{aligned}
I_1 &\leqslant \frac{1}{\varepsilon^\delta} \int\limits_{B_{R_\varepsilon}(0)} \left\{C_1 V(x) v_\varepsilon^2 - C_2 \frac{\lambda}{q+1} f(v_\varepsilon)^{q+1} - C_2 \frac{1}{2(2^*)} f(v_\varepsilon)^{2(2^*)} + C_2 \frac{4^{1/(N-2)}}{2^*}|v_\varepsilon|^{2^*}\right\} dx \\
&= \frac{1}{\varepsilon^\delta} \int\limits_{B_{R_\varepsilon}(0)} \left\{C_1 V(x) v_\varepsilon^2 - C_2 \frac{\lambda}{q+1}\left[\frac{f(v_\varepsilon)}{v_\varepsilon^{1/2}}\right]^{q+1} v_\varepsilon^{(q+1)/2}\right. \\
&\quad \left. - C_2 \frac{1}{2(2^*)}\left[\frac{f(v_\varepsilon)}{v_\varepsilon^{1/2}}\right]^{2(2^*)} v^{2^*} + C_2 \frac{4^{1/(N-2)}}{2^*}|v_\varepsilon|^{2^*}\right\} dx.
\end{aligned}$$

Now, by assumption (V_1) and using Lemma 2.1 we get

$$\begin{aligned}
I_1 &\leqslant \frac{1}{\varepsilon^\delta} \int\limits_{B_{R_\varepsilon}(0)} \left[C_1 V_\infty v_\varepsilon^2 - \frac{\lambda C_3^{q+1}}{q+1} v_\varepsilon^{(q+1)/2} + \frac{C_2}{2^*}\left[4^{1/(N-2)} - \frac{2^{2^*/2}}{2}\right]\right] v_\varepsilon^{2^*} dx \\
&= \frac{1}{\varepsilon^\delta} \int\limits_{B_{R_\varepsilon}(0)} \left[C_1 V_\infty v_\varepsilon^2 - \frac{\lambda C_3^{q+1}}{q+1} v_\varepsilon^{(q+1)/2}\right] dx.
\end{aligned}$$

Thus, arguing as [29, Claim 2, p. 718], we can see that the last integral goes to $-\infty$ as ε converges to zero.

Similarly the estimate of integral I_2 is also delicate:

$$\begin{aligned}
I_2 \leqslant \frac{1}{\varepsilon^\delta} \int\limits_{B_{2R_\varepsilon}(0)\backslash B_{R_\varepsilon}(0)} &\left[C_1 V(x) v_\varepsilon^2 - \frac{\lambda C_2 \lambda}{q+1} f(v_\varepsilon)^{(q+1)}\right. \\
&\left. - \frac{C_2}{2(2^*)} f(v_\varepsilon)^{2(2)^*} - Z(x) + \frac{4^{1/(N-2)}}{2^*}|v_\varepsilon|^{2^*}\right] dx.
\end{aligned}$$

First of all, notice that

$$-\frac{C_2}{2(2^*)}f(v_\varepsilon)^{2(2)^\alpha}-Z(x)\leqslant 0,$$

then, by (V_1) we get

$$I_2\leqslant\frac{1}{\varepsilon^\delta}\int\limits_{B_{2R_\varepsilon}(0)\backslash B_{R_\varepsilon}(0)}\left[\left(C_1V_\infty+\frac{4^{1/(N-2)}}{2^*}\right)v_\varepsilon^{2^*}-\frac{C_2\lambda}{q+1}f(v_\varepsilon)^{q+1}\right]\mathrm{d}x.$$

Without loss of generality, I_2 can be estimated by

$$I_2\leqslant\frac{C}{\varepsilon^\delta}\int\limits_{B_{2R_\varepsilon}(0)\backslash B_{R_\varepsilon}(0)}\left[\omega_\varepsilon^2-f(\omega_\varepsilon)^{q+1}\right]\mathrm{d}x\quad\text{for some } C>0. \tag{3.11}$$

Notice, since $R_\varepsilon=\varepsilon^\alpha\geqslant\varepsilon$, we get

$$\omega_\varepsilon(x)=\frac{C\varepsilon^{(N-2)/4}}{(\varepsilon+|x|^2)^{(N-2)/2}}\geqslant\frac{C\varepsilon^{(N-2)/4}}{(\varepsilon+4\varepsilon^{2\alpha})^{(N-2)/2}}\geqslant\frac{C\varepsilon^{(N-2)/4}}{(\varepsilon^{2\alpha})^{(N-2)/2}}=C_1\varepsilon^{(N-2)/4-\alpha(N-2)}.$$

Then

$$f\big(\omega_\varepsilon(x)\big)\geqslant f\big(C_1\varepsilon^{(N-2)/4-\alpha(N-2)}\big).$$

Since $(N-2)/4-\alpha(N-2)<0$, and by Lemma 2.1(5), we obtain from (3.11),

$$\begin{aligned}I_2&\leqslant\varepsilon^{\alpha N}\frac{C}{\varepsilon^{2\alpha(N-2)}}-C\varepsilon^{\alpha N+[(N-2)/4-\alpha(N-2)](q+1)/2-(N-2)/2}\\&=\frac{C}{\varepsilon^{\alpha(N-4)}}-C\varepsilon^{[(N-2)/4-\alpha(N-2)](q+1)/2-(N-2)/2+\alpha N}\\&=\frac{C}{\varepsilon^{\alpha(N-4)}}\big[1-\varepsilon^{[(N-2)/4-\alpha(N-2)](q+1)/2-(N-2)/2+4\alpha}\big]\\&\doteq g(N,q,\alpha).\end{aligned}$$

Next, we analyze three cases:

Case: $N=3$. In this case we have

$$g(3,q,\alpha)=\frac{C}{\varepsilon^{-\alpha}}\big\{1-\varepsilon^{(1/4-\alpha)(q+1)/2-1/2+4\alpha}\big\}=C\big\{\varepsilon^\alpha-\varepsilon^{(1/4-\alpha)(q+1)/2-1/2+5\alpha}\big\}.$$

Notice that if $(1/4-\alpha)(q+1)/2-1/2+5\alpha\geqslant 0$, then $g(N,q,\alpha)$ is bounded. Thus $g(N,q,\alpha)\to 0$, as $\varepsilon\to 0$. Otherwise, if $(1/4-\alpha)(q+1)/2-1/2+5\alpha<0$, then $g(N,q,\alpha)\to-\infty$, as $\varepsilon\to 0$. Any way $g(N,q,\alpha)$ is bounded from above.

Case: $N=4$. In this case we obtain

$$g(4,q,\alpha)=1-\varepsilon^{(1/2-2\alpha)(q+1)/2-1+4\alpha}.$$

As above either $g(4,q,\alpha)\to 0$ or $g(4,q,\alpha)\to-\infty$, as $\varepsilon\to 0$.

Case: $N \geqslant 5$. Arguing as in the case $N = 3$, we get either

$$g(N,q,\alpha) \to 0 \quad \text{or} \quad g(N,q,\alpha) \to -\infty, \quad \text{as } \varepsilon \to 0.$$

Therefore, for all $N \geqslant 3$, we obtain either $I_2 \to -\infty$, as $\varepsilon \to 0$, or I_2 is bounded from above. Hence $I_1 + I_2 \to -\infty$, as $\varepsilon \to 0$. This proves Assertion 2. □

3.3. Proof of Theorem 1.1

From Lemma 3.3, there exists $v \in H^1(\mathbb{R}^N)$ such that $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^N)$ and $v_n \to v$ in $L^p_{loc}(\mathbb{R}^N)$ for all $p \in [2, 2^*)$. Then, $I'(v)\phi = 0$ for every $\phi \in C_0^\infty(\mathbb{R}^N)$, that is, v is a weak solution of problem (P). Notice that by L^p-regularity theory, see e.g. [17], we have $v \in H^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$. In order to complete the proof of Theorem 1.1, we must show v is nontrivial. The proof of this fact is delicate and it will be carried out in a series of steps. First, we suppose, by contradiction, that $v \equiv 0$.

The following result is a concentration of compactness result (see [41]).

Lemma 3.5. *There exist a sequence $(y_n) \subset \mathbb{R}^N$, and $\rho, \eta > 0$ such that*

$$\limsup_{n\to+\infty} \int_{B_\rho(y_n)} |v_n|^2 \, dx \geqslant \eta. \tag{3.12}$$

Proof. Suppose that (3.12) does not hold. Using [24, Lemma 1.1], it follows that

$$v_n \to 0 \quad \text{in } L^r\big(\mathbb{R}^N\big) \text{ for all } 2 < r < 2^*,$$

from which together with the estimates (3.7) and (3.8) we obtain that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} g(v_n)v_n \, dx = 0$$

and

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} G(v_n) \, dx = 0.$$

Therefore,

$$\begin{aligned} c_0 + o(1) &= I(v_n) \\ &= \frac{1}{2}\int_{\mathbb{R}^N} \big[|\nabla v_n|^2 + V v_n^2\big] \, dx - \frac{4^{1/(N-2)}}{2^*}\int_{\mathbb{R}^N} v_n^{2^*} \, dx - \int_{\mathbb{R}^N} G(v_n) \, dx. \end{aligned}$$

Setting

$$L \doteq \lim_{n\to\infty} \int_{\mathbb{R}^N} \big[|\nabla v_n|^2 + V v_n^2\big] \, dx \quad \text{and}$$

$$\ell \doteq \lim_{n\to\infty} \int_{\mathbb{R}^N} |v_n|^{2^*} \, dx,$$

we can write

$$c_0 = \frac{L}{2} - \frac{4^{1/(N-2)}\ell}{2^*}. \tag{3.13}$$

From

$$S\left(\int_{\mathbb{R}^N} |v_n|^{2^*}\,dx\right)^{2/2^*} \leqslant \int_{\mathbb{R}^N} |\nabla v_n|^2\,dx \leqslant \int_{\mathbb{R}^N} \left[|\nabla v_n|^2 + V v_n^2\right] dx$$

we get

$$S(\ell)^{2/2^*} \leqslant L. \tag{3.14}$$

Now, passing to the limit in

$$o(1) = I'(v_n)v_n = \int_{\mathbb{R}^N} \left[|\nabla v_n|^2 + V(x)v_n^2\right] dx - 4^{1/(N-2)} \int_{\mathbb{R}^N} |v_n|^{2^*}\,dx - \int_{\mathbb{R}^N} g(v_n)v_n\,dx \tag{3.15}$$

we have

$$L = 4^{1/(N-2)}\ell. \tag{3.16}$$

Using (3.14) and (3.17) we get

$$L \geqslant \left(\frac{S}{4^{1/N}}\right)^{N/2} = \frac{S^{N/2}}{2},$$

which together with (3.13) and (3.17) implies that

$$c_0 \geqslant \frac{1}{2N} S^{N/2}. \tag{3.17}$$

From Lemma 3.4, we obtain

$$I(tv_\varepsilon) \leqslant I(t_\varepsilon v_\varepsilon) < \frac{S^{N/2}}{2N} \quad \text{for all } \varepsilon \in (0, \varepsilon_0),$$

and for ε_0 sufficiently small. Hence,

$$\frac{S^{N/2}}{2N} \leqslant c_0 = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I\big(\gamma(t)\big) \leqslant \max_{t\geqslant 0} I(tv_\varepsilon) < \frac{S^{N/2}}{2N},$$

which is a contradiction. This proves Lemma 3.5. □

Case: (V_2'). The function V is periodic in each variable of $x_1, \dots, x_N$. We recall that $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^N)$, and v is a weak solution that we are supposing $v \equiv 0$. We can assume that the sequence (y_n) given in (3.12) is bounded. Setting $\omega_n(x) = v_n(x - y_n)$, we can assume that (ω_n) is also bounded Cerami sequence, then $\omega_n \rightharpoonup \omega$ weakly in $H^1(\mathbb{R}^N)$, and ω is a weak solution. From Lemma 3.5 follows that ω is nontrivial.

Case: (V_2). There exists a constant V_∞ such that $\lim_{|x|\to\infty} V(x) = V_\infty$ and $V(x) \leqslant V_\infty$ for all $x \in \mathbb{R}^N$.

In this case the sequence (v_n) is also a bounded Cerami sequence for the functional I_∞, where

$$I_\infty(v) = \frac{1}{2}\int_{\mathbb{R}^N} \left[|\nabla v|^2 + V_\infty v^2\right] dx - \frac{4^{1/(N-2)}}{2^*}\int_{\mathbb{R}^N} |v_n|^{2^*} dx - \int_{\mathbb{R}^N} G_\infty(v)\, dx$$

and

$$G_\infty(v) \doteq \frac{1}{q+1}\big(f(v)\big)^{q+1} + \frac{1}{2(2^*)}\big(f(v)\big)^{2(2^*)} + \frac{1}{2}V(x)v^2 - \frac{1}{2}V_\infty\big(f(v)\big)^2 - \frac{4^{1/(N-2)}}{2^*}|v|^{2^*}.$$

Indeed, from assumption (V_2), given $\varepsilon > 0$, there exists $R > 0$ such that

$$\left|V(x) - V_\infty\right| < \varepsilon \quad \text{for all } |x| \geqslant R.$$

Then

$$\begin{aligned}\left|I_\infty(v_n) - I(v_n)\right| &= \frac{1}{2}\int_{\mathbb{R}^N} \left|V_\infty - V(x)\right| f(v_n)^2\, dx \\ &= \int_{B_R(0)} \left|V_\infty - V(x)\right| f(v_n)^2\, dx + \int_{\mathbb{R}^N \setminus B_R(0)} \left|V_\infty - V(x)\right| f(v_n)^2\, dx.\end{aligned}$$

Thus,

$$\left|I_\infty(v_n) - I(v)\right| \to 0, \quad \text{as } n \to \infty.$$

Similarly, we obtain

$$\big(1 + \|v_n\|\big)\left[I'_\infty(v_n) - I'(v_n)\right] \to 0, \quad \text{as } n \to \infty,$$

that is,

$$\left\|I'_\infty(v_n) - I'(v_n)\right\| = \sup_{\|\phi\|\leqslant 1}\int_{\mathbb{R}^N} \big(V_\infty - V(x)\big) f(v_n) f'(v_n)\phi\, dx$$

goes to zero, as $n \to \infty$.

Define $\tilde{v}_n(x) = v_n(x - y_n)$, where (y_n) is the sequence given in Lemma 3.5. Then, $(\tilde{v}_n)$ is bounded in $H^1(\mathbb{R}^N)$ and

$$I_\infty(\tilde{v}_n) \to c_0 \quad \text{and} \quad I'_\infty(\tilde{v}_n)\big(1 + \|\tilde{v}_n\|\big) \to 0.$$

Therefore $\tilde{v}_n \to \tilde{v}$ weakly in $H^1(\mathbb{R})$ and $\tilde{v}$ is a critical point of I_∞. From Lemma 3.5 follows that $\tilde{v}$ is nontrivial. Hence

$$\int_{\mathbb{R}^N} \big(\nabla\tilde{v}\nabla w + V_\infty f(\tilde{v}) f'(\tilde{v}) w\big)\, dx = \int_{\mathbb{R}^N} h\big(x, f(\tilde{v})\big) f'(\tilde{v}) w\, dx \tag{3.18}$$

for all $w \in H^1(\mathbb{R}^N)$. Taking $w = -\tilde{v}^-$, where $v^- = \max\{-v, 0\}$, we get

$$\int_{\mathbb{R}^N} \left|\nabla \tilde{v}^-\right|^2 \mathrm{d}x + \int_{\mathbb{R}^N} V_\infty f'(\tilde{v}) f(\tilde{v}) \left(-\tilde{v}^-\right) \mathrm{d}x = 0.$$

Since $f(\tilde{v})(-\tilde{v}^-) \geqslant 0$ we obtain

$$\int_{\mathbb{R}^N} \left|\nabla \tilde{v}^-\right|^2 \mathrm{d}x = 0 \quad \text{and} \quad \int_{\mathbb{R}^N} \frac{V_\infty f(\tilde{v})(-\tilde{v}^-)}{\sqrt{1 + 2f^2(\tilde{v})}} \mathrm{d}x = 0.$$

Thus, $\tilde{v}^- = 0$ almost everywhere in $\mathbb{R}^N$ and therefore $\tilde{v} \geqslant 0$. By elliptic regularity theory we can assume that $\tilde{v} \in C^2(\mathbb{R}^N)$ (see [39, p. 245]). In order to prove that $\tilde{v} > 0$ in $\mathbb{R}^N$, we suppose, otherwise, that there exists $x_0 \in \mathbb{R}^N$ such that $\tilde{v}(x_0) = 0$. We observe that (P) can be written in the form

$$-\Delta \tilde{v} + c\tilde{v} = \left[h\left(f(\tilde{v})\right) - V_\infty f(\tilde{v})\right] f'(\tilde{v}) + c\tilde{v}$$

where $c \geqslant 0$ is such that the right term is nonnegative for all $x \in \mathbb{R}^N$. Applying the strong maximum principle for an arbitrary ball centered in x_0 we can conclude that $\tilde{v} \equiv 0$, which is impossible. Therefore $\tilde{v}$ has to be strictly positive and consequently $u = f(\tilde{v})$ is a positive classical solution of (P).

We also remark that

$$\tilde{v}(x) \to 0, \quad \text{as } |x| \to \infty. \tag{3.19}$$

Effectively, $\tilde{v}$ is a weak solution of

$$-\Delta v = g(v) \quad \text{in } \mathbb{R}^N,$$

where $h(s) \doteq (g(f(v)) - V_\infty f(v)) f'(v)$. Since $\tilde{v} \in L^q_{loc}(\mathbb{R}^N)$, $1 < q < \infty$, by the Sobolev embedding theorem, $g(\tilde{v}) \in L^{2^*}(\mathbb{R}^N)$. Thus, we infer by interior elliptic estimates that $\tilde{v} \in W^{2,2^*}_{loc}$ and moreover

$$\|\tilde{v}\|_{W^{2,2^*}(\Omega')} \leqslant C\left(\left|g(\tilde{v})\right|_{L^{2^*}(\Omega)} + |\tilde{v}|_{L^{2^*}(\Omega)}\right),$$

where $\Omega' \Subset \Omega$, Ω is an open bounded set of $\mathbb{R}^N$ and C depends only on the diameter of Ω and the measure of $\Omega \setminus \Omega'$.

Let $x_0 \in \mathbb{R}^N$ and denote by $B_r \subset \mathbb{R}^N$ the open ball of radius $r > 0$ centered at x_0. Then,

$$\|\tilde{v}\|_{W^{2,2^*}(B_1)} \leqslant C\left(\left|g(\tilde{v})\right|_{L^{2^*}(B_2)} + |\tilde{v}|_{L^{2^*}(B_2)}\right),$$

where C depends only on the diameter of B_2 and the measure of $B_2 \setminus B_1$. By bootstrap argument $W^{2,2^*}(B_2) \subset C(\overline{B}_1)$ we obtain

$$\|\tilde{v}\|_{L^\infty(B_1)} \leqslant C\left(\left|h(\tilde{v})\right|_{L^{2^*}(B_2)} + |\tilde{v}|_{L^{2^*}(B_2)}\right).$$

In particular,

$$\left|\tilde{v}(x_0)\right| \leqslant C\left(\left|h(\tilde{v})\right|_{L^{2^*}(B_2)} + |\tilde{v}|_{L^{2^*}(B_2)}\right)$$

and since $h(\tilde{v})$ and $\tilde{v}$ belong to $L^{2^*}(\mathbb{R}^2)$, we have

$$\big|h(\tilde{v})\big|_{L^{2^*}(B_2)} + |\tilde{v}|_{L^{2^*}(B_2)} \to 0, \quad \text{as } |x_0| \to \infty,$$

so that $|\tilde{v}(x)| \to 0$ as $|x| \to \infty$ and the verification of (3.19) is complete.

We assert now that

$$c_\infty \leqslant I_\infty(\tilde{v}) \leqslant c_0, \tag{3.20}$$

where c_∞ is the mountain pass level given by

$$c_\infty = \inf_{\gamma \in \Gamma_\infty} \sup_{t \in [0,1]} I_\infty\big(\gamma(t)\big),$$

and

$$\Gamma_\infty = \big\{\gamma \in C\big([0,1], H^1\big(\mathbb{R}^N\big)\big);\ \gamma(0) = 0,\ \gamma(1) \neq 0,\ I_\infty\big(\gamma(1)\big) < 0\big\}.$$

We start the verification of (3.20) showing that $I_\infty(\tilde{v}) \leqslant c_0$. Indeed, by Lemma 2.1(6) and Fatou lemma, we have

$$\begin{aligned}
c_0 &= \limsup_{n\to\infty} \left\{ I_\infty(\tilde{v}_n) - \frac{1}{2} I'_\infty(\tilde{v}_n)\tilde{v}_n \right\} \\
&= \limsup_{n\to\infty} \int_{\mathbb{R}^N} \left\{ \frac{1}{2}\big[\big(f^2(\tilde{v}_n) - f(\tilde{v}_n)f'(\tilde{v}_n)\tilde{v}_n\big)V_\infty\big] + \frac{1}{2} g\big(f(\tilde{v}_n)\big)f'(\tilde{v}_n)\tilde{v}_n - G\big(f(\tilde{v}_n)\big) \right\} dx \\
&\geqslant \int_{\mathbb{R}^N} \frac{1}{2}\big(f^2(\tilde{v}) - f(\tilde{v})f'(\tilde{v})\tilde{v}\big)V_\infty \, dx + \int_{\mathbb{R}^N} \frac{1}{2} g\big(f(\tilde{v})\big)f'(\tilde{v})\tilde{v} - G\big(f(\tilde{v})\big)\, dx \\
&= I_\infty(\tilde{v}) - \frac{1}{2} I'_\infty(\tilde{v})\tilde{v} = I_\infty(\tilde{v}).
\end{aligned}$$

Thus $I_\infty(\tilde{v}) \leqslant c_0$. Now, in order to show $c_\infty \leqslant I_\infty(\tilde{v})$, we slightly modify an argument used in [19] to get a path $\gamma : [0,1] \to H^1(\mathbb{R}^N)$ such that

$$\begin{cases}
\gamma(0) = 0, \quad I_\infty\big(\gamma(1)\big) < 0, \quad \tilde{v} \in \gamma\big([0,1]\big), \\
\gamma(t)(x) > 0 \quad \forall x \in \mathbb{R}^N,\ t \in (0,1], \\
\max_{t\in[0,1]} I_\infty\big(\gamma(t)\big) = I_\infty(\tilde{v}).
\end{cases} \tag{3.21}$$

Indeed, define

$$\tilde{v}_t(x) = \begin{cases} \tilde{v}(x/t) & \text{if } t > 0, \\ 0 & \text{if } t = 0. \end{cases}$$

Choose three points $t_o \in (0,1)$, $t_1 \in (1,\infty)$ and $\theta_1 > t_1$ such that the path γ defined by three pieces, namely,

$$\gamma_1 : [0,1] \to H^1(\mathbb{R}^N), \quad \gamma_1(\theta) = \theta \tilde{v}_{t_o},$$
$$\gamma_2 : [t_o, t_1] \to H^1(\mathbb{R}^N), \quad \gamma_2(t) = \tilde{v}_t,$$
$$\gamma_3 : [1, \theta_1] \to H^1(\mathbb{R}^N), \quad \gamma_3(\theta) = \theta \tilde{v}_{t_1},$$

it is desired path. Effectively, because of $\tilde{v}$ is a critical point of I_∞, the function $\tilde{v}$ is a weak positive solution of

$$-\Delta \tilde{v} = \psi(\tilde{v}) \quad \text{in } \mathbb{R}^N.$$

Then

$$\int_{\mathbb{R}^N} \psi(\tilde{v})\tilde{v}\,\mathrm{d}x = \|\nabla \tilde{v}\|^2 > 0,$$

where $\psi(s) = (g(f(s)) - V_\infty f(s))f'(s)$. Thus, there exists $\theta_1 > 0$ such that

$$\int_{\mathbb{R}^N} \psi(\theta\tilde{v})\tilde{v}\,\mathrm{d}x > 0, \quad \forall \theta \in [1, \theta_1]. \tag{3.22}$$

Let $\Phi(s) = \frac{\psi(s)}{s}$ for $s > 0$. By (3.22) we infer that

$$\int_{\mathbb{R}^N} \Phi(\theta\tilde{v})\tilde{v}^2\,\mathrm{d}x > 0, \quad \forall \theta \in [1, \theta_1]. \tag{3.23}$$

On the other hand, from

$$\frac{\mathrm{d}}{\mathrm{d}\theta} I_\infty(\theta\tilde{v}_t) = \theta\left(\|\nabla \tilde{v}\|_2^2 - t^2 \int_{\mathbb{R}^N} \Phi(\theta v_t)v^2\,\mathrm{d}x\right)$$

there exists $t_o \in (0,1)$ such that

$$\|\nabla \tilde{v}\|_2^2 - t_o^2 \int_{\mathbb{R}^N} \Phi(\theta\tilde{v}_t)\tilde{v}^2\,\mathrm{d}x > 0, \quad \forall \theta \in [0,1]. \tag{3.24}$$

From (3.23) there exists $t_1 > 1$ such that

$$\|\nabla \tilde{v}\|_2^2 - t_1^2 \int_{\mathbb{R}^N} \Phi(\theta\tilde{v}_t)\tilde{v}^2\,\mathrm{d}x < \frac{-2}{\theta_1^2 - 1}\|\nabla \tilde{v}\|_2^2, \quad \forall \theta \in [1, \theta_1]. \tag{3.25}$$

From (3.24), by along of the path γ_1, $I_\infty(\theta\tilde{v}_{t_o})$ decreases and it takes its maximum value at $\theta = 1$. Since $\int_{\mathbb{R}^N} \Psi(\tilde{v})\,\mathrm{d}x = 0$, where $\Psi(\tilde{s}) = \int_0^s \psi(t)\,\mathrm{d}t$, by Pohozaev identity we obtain

$$I_\infty(\tilde{v}_t) = I_\infty(\tilde{v}) = \frac{1}{2}\|\nabla \tilde{v}\|_2^2$$

along the path γ_2. From (3.25), $I_\infty(\theta\tilde{v}_{t_1})$ decreases along the path γ_3. Thus,

$$I_\infty\big(\gamma_1(t)\big) \leqslant I_\infty(\tilde{v}_t) = I_\infty(\tilde{v}),$$

on the other hand

$$I_\infty(\tilde{v}) = I_\infty(\tilde{v}_t) \geqslant I_\infty(\theta\tilde{v}_{t_1}), \quad \forall\theta \in [0,\theta_1].$$

Therefore

$$\max_{t\in[0,\theta_1]} I_\infty\big(\gamma(t)\big) = I_\infty(\tilde{v}).$$

Moreover, from (3.25) and the fact $I_\infty(\theta\tilde{v}_{t_1})$ decreases along γ_3 we have

$$\begin{aligned} I_\infty(\theta_1\tilde{v}_{t_1}) &= I_\infty(\tilde{v}_{t_1}) + \int_1^{\theta_1} \frac{\mathrm{d}}{\mathrm{d}\theta} I_\infty(\theta\tilde{v}_{t_1})\,\mathrm{d}\theta \\ &\leqslant \frac{1}{2}\|\nabla\tilde{v}\|_2^2 - \int_1^{\theta_1} \frac{2\theta}{\theta_1^2-1}\|\nabla\tilde{v}\|_2^2\,\mathrm{d}\theta \\ &= -\frac{1}{2}\|\nabla\tilde{v}\|_2^2 < 0. \end{aligned}$$

Hence we obtain the desired path (3.21).

The path (3.21) together with the definition of c_∞ imply that

$$c_\infty \leqslant \max_{t\in[0,1]} I_\infty\big(\gamma(t)\big) = I_\infty(\tilde{v}).$$

Thus, $c_\infty \leqslant I_\infty(\tilde{v})$ and the verification of (3.20) is complete.

Finally, we may conclude the proof of Theorem 1.1. Take again the path γ given by (3.21). Since $\gamma \in \Gamma_\infty \subset \Gamma$, $\gamma(t)(x) > 0$, and $V(x) \leqslant V_\infty$, with $V \not\equiv V_\infty$, from (3.20) we obtain

$$\begin{aligned} c_0 &\leqslant \sup_{t\in[0,1]} I\big(\gamma(t)\big) = I\big(\gamma(\bar{t})\big) \\ &< I_\infty\big(\gamma(\bar{t})\big) \leqslant \max_{t\in[0,1]} I_\infty\big(\gamma(t)\big) \\ &= I_\infty(\tilde{v}) \leqslant c_0, \end{aligned}$$

which is a contradiction. Therefore, v is nontrivial. Theorem 1.1 is proved.

Remark 3.1. By a similar argument we can prove a version of Theorem 1.1 in the asymptotic case to a periodic function V_p, that is, when V satisfies

$$V_p(x) \doteq \lim_{|x|\to\infty} V(x), \qquad V_p(x+1) = V_p(x), \quad \forall x \in \mathbb{R}^N, \quad \text{and}$$
$$V(x) \leqslant V_p(x), \quad \forall x \in \mathbb{R}^N,$$

where the last inequality is strict on a positive Lebesgue measure set of $\mathbb{R}^N$.

We can establish Theorem 1.1, in the compact-coercive case, that is, when $\lim_{|x|\to\infty} V(x) = +\infty$, and its proof follows easily because the map $v \to f(v)$ from $H^1(\mathbb{R}^N)$ into $L^q(\mathbb{R}^N)$ is compact for $2 \leqslant q < \infty$. (See [35] also [27].)

Theorem 1.1 still holds in the radially symmetric case, namely $V(x) = V(|x|)$, $\forall x \in \mathbb{R}^N$. The proof can be handled as above by using that the map $v \to f(v)$ from $H^1(\mathbb{R}^N)$ into $L^q(\mathbb{R}^N)$ is compact for $2 < q < \infty$. (See [38] also [27].)

Acknowledgments

Part of this work was done while the second named author was visiting the Universidade de São Paulo at São Carlos and Universidade Federal da Paraíba. He thanks all the faculty and staff of those Department of Mathematics for their support and kind hospitality.

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