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Research Article

Loop Quantization of a 3D Abelian *BF* Model with σ-Model Matter

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The main goal of this work is to explore the symmetries and develop the dynamics associated with a 3D Abelian *BF* model coupled to scalar fields submitted to a sigma model like constraint, at the classical and quantum levels. Background independence, on which the model is founded, strongly constrains its nature. We adapt to the present model the techniques of Loop Quantum Gravity in order to construct its physical Hilbert space and its observables.

1. Introduction

The now quasi-hundred-year-old General Relativity as a theory of gravitation, despite its tremendous successes in accounting for predicting phenomena, still lacks a quantum version. Previous perturbative attempts have shown the non-renormalizability of the theory [1], whereas the pioneering nonperturbative approach of Wheeler and DeWitt [2–5] had its successes concentrated in reduced "minisuperspace" models dedicated to cosmology. However, very important progresses have been made in the last decades, especially in the framework of Loop Quantum Gravity (LQG) [6–9], based on the canonical Hamiltonian approach of Dirac [10, 11] applied to the Ashtekar-Barbero [12–14] parametrization of the theory. General Relativity, as a background independent theory—in the sense that no background geometry is given *a priori*, geometry being dynamical—is a fully constrained theory, its Hamiltonian being merely a sum of constraints generating the gauge invariances of the theory. The LQG program entails the difficult task of implementing the constraints of the theory as quantum operators in some predefined kinematic Hilbert space and to solve them, thus leaving as a subspace the physical Hilbert space in which act the self-adjoint operators representing the observables of the theory. Some of the constraints have been resolved, but the last one, the so-called scalar constraint. The latter has resisted up to now a complete solution, the most popular approach being that of "spin foams" [15, 16].

By contrast, the lower-dimensional gravitation theories are much more easy to handle, since they can be described as topological gauge theories, when not coupled to matter [17–23]. Coupling them to matter however lets them lose their topological character, excepted in some special cases, where a complete and rather simple loop quantization can be achieved [24, 25].

The purpose of this paper is to present the loop quantization of a background independent theory of the *BF* type [26, 27] with the Abelian group $U(1)$ as a gauge group. Background independence means that no metric is introduced in the manifold on which the theory is defined, the sole requirement being the invariance under diffeomorphisms, beyond the gauge invariance. The *BF* fields are coupled to a complex scalar "matter" field subject to a σ-model type of constraint. Background independence severely restricts the form of the action. It turns out that the topological nature of the *BF* theory persists in the sense that no local degrees of freedom are present. However, provided the topology of 2-dimensional space is nontrivial, global degrees of freedom are present in the theory. We consider spaces with point-like singularities, in which cases a nontrivial physical Hilbert space and global observables are explicitly constructed. A non-Abelian version is presently under study [25].

The model and its gauge invariances are presented in Section 2, its classical analysis is done in Section 3 together with the separation of the first and second class constraints and the definition of the Dirac brackets, and the quantization is presented in Section 4. Brief conclusions are given at last.

2. Formulation of the Model

2.1. The Gauge Invariances and the Action. The field content of the model is a $U(1)$ connection form $A = A_\mu(x)dx^\mu$, a "B" form $B = B_\mu(x)dx^\mu$, a complex scalar field $\phi(x)$, and a 3-form field $e = (1/3!)e_{\mu\nu\rho}dx^\mu dx^\nu dx^\rho$, transforming as (Wedge symbols $\wedge$ are not written explicitly; space-time indices $\mu,\nu,\ldots$ take the values 0, 1, and 2; later on, space indices will be denoted by the letters $a,b,\ldots$ taking the values 1, 2; A_μ and θ are taken as imaginary)

$$A'_\mu = A'_\mu + g^{-1}\partial_\mu g = A_\mu + \partial_\mu\theta, \qquad B' = B,$$
$$\phi' = g\phi, \qquad \bar{\phi}' = g^{-1}\bar{\phi}, \qquad e' = e, \tag{1}$$

under $U(1)$ gauge transformations (A_μ and θ are taken as purely imaginary; B_μ is real) $g(x) = \exp\theta(x)$.

The most general action, invariant under the gauge transformations (1) and background independent, reads

$$S_{\text{general}} = \int_{\mathscr{M}^3}\left(K\left(\bar{\phi}\phi\right)BF + \lambda\left(\bar{\phi}\phi\right)BD\bar{\phi}D\phi + e\left[\mu\left(\bar{\phi}\phi\right) - R\right]\right), \tag{2}$$

where K, λ, and μ are arbitrary real functions of $\bar{\phi}\phi$, assumed to be analytic in their argument and to fulfill the conditions

$$K(0) \neq 0 \qquad \lambda(0) \neq 0 \qquad \mu(0) = 0; \tag{3}$$

$F = dA$, and D denotes the covariant derivative (covariant with respect to the gauge transformation (1)):

$$D\phi = (d - A)\phi, \qquad D\bar{\phi} = (d + A)\bar{\phi}. \tag{4}$$

The integration is performed over some 3-dimensional differential manifold $\mathscr{M}^3$. The action is obviously invariant under the diffeomorphisms of $\mathscr{M}^3$. Having no metric at our disposal, it is clear that no other term, such as, *for example*, a potential term, can be added.

The parameter R can be taken equal to 1 through a renormalization of the field e, and one easily shows that one can reduce the function $\mu(\bar{\phi}\phi)$ to the form $\mu = \bar{\phi}\phi$ through a field redefinition $\phi \to \phi'(\phi,\bar{\phi}) = \phi f(\bar{\phi}\phi)$, with $f(\bar{\phi}\phi) = (\mu(\bar{\phi}\phi)/\bar{\phi}\phi)$, compatible with the gauge invariance (1). Having done this, we have the field equation $\bar{\phi}\phi = 1$ which implies that the functions K and λ can be replaced by constants. One of them, let us say λ, can be given the value 1 through a renormalization of the field B. We are thus left with only the constant K as independent parameter. We will take the valued $K = 1$ without loss of generality. The final action is then

$$S = \int_{\mathscr{M}^3}\left(BF + BD\bar{\phi}D\phi + e\left(\bar{\phi}\phi - 1\right)\right). \tag{5}$$

One recognizes in (5) a BF action coupled with scalar fields and a Lagrange multiplier field e assuring the σ-model type constraint $\bar{\phi}\phi = 1$.

It turns out that this action (5) has two more invariances under the gauge transformations

$$A' = A, \qquad B' = B + d\psi, \qquad \phi' = \phi, \qquad e' = e - d\psi F, \tag{6}$$

$$A' = A + d\eta, \qquad B' = B, \qquad \phi' = \phi, \qquad e' = e + d\eta\, dB, \tag{7}$$

where the scalars $\psi(x)$ and $\eta(x)$ are the transformation parameters. The transformation (6) coincides with the usual topological type transformations of the BF model, in the absence of the fields ϕ and e. Invariance under (7) is specific for the model.

In order to check the invariances of the action (up to boundary terms), as well as for all the manipulations involving partial integrations, it is useful to remember that the covariant derivative D, defined by $DX = dX - qAX$, where q is the $U(1)$ charge of the field X, obeys the Leibniz rule. The respective $U(1)$ charges of the basic fields A, B, ϕ, $\bar{\phi}$, and e are 0, 0, 1, −1, and 0. Let us also note the useful identity

$$D\bar{\phi}D\phi = d\bar{\phi}d\phi + Ad\left(\bar{\phi}\phi\right). \tag{8}$$

The field equations read

$$\frac{\delta S}{\delta B} = F + D\bar{\phi}D\phi \overset{*}{=} 0, \qquad \frac{\delta S}{\delta A} = dB - Bd\left(\bar{\phi}\phi\right) \overset{*}{=} 0,$$
$$\frac{\delta S}{\delta\bar{\phi}} = BF\phi - dBD\phi + e\phi \overset{*}{=} 0,$$
$$\frac{\delta S}{\delta\phi} = BF\bar{\phi} - dBD\bar{\phi} + e\bar{\phi} \overset{*}{=} 0, \tag{9}$$
$$\frac{\delta S}{\delta e} = \bar{\phi}\phi - 1 \overset{*}{=} 0,$$

where the symbol $\overset{*}{=}$ means "on shell" equality, that is, "equations of motion being fulfilled." The last equation is equivalent to

$$\phi(x) \overset{*}{=} e^{i\varphi(x)}, \quad \bar{\phi}(x) \overset{*}{=} e^{-i\varphi(x)}, \quad \varphi \text{ a real phase.} \tag{10}$$

This system of equations is equivalent to the simpler one:

$$F \overset{*}{=} 0, \qquad dB \overset{*}{=} 0, \qquad e \overset{*}{=} 0, \qquad \bar{\phi}\phi - 1 \overset{*}{=} 0. \tag{11}$$

2.2. Diffeomorphism Invariance. In the present theory, like in the topological theories of the Chern-Simons or BF type, the invariance under the diffeomorphisms is a consequence of the invariance under the gauge transformations (1), (6), and (7), up to field equations. Indeed, with the diffeomorphisms being generated by the Lie derivative $\mathscr{L}_\xi = i_\xi d + di_\xi$ along an infinitesimal vector field ξ when acting on forms (i_ξ is the interior derivative, with $i_\xi dx^\mu = \xi^\mu$), one checks that

$$\mathscr{L}_\xi A \stackrel{*}{=} d\left(i_\xi A\right), \qquad \mathscr{L}_\xi B = d\left(i_\xi B\right),$$
$$\mathscr{L}_\xi \phi \stackrel{*}{=} i\left(i_\xi d\varphi\right)\phi, \qquad \mathscr{L}_\xi \overline{\phi} \stackrel{*}{=} -i\left(i_\xi d\varphi\right)\overline{\phi}, \tag{12}$$
$$\mathscr{L}_\xi e \stackrel{*}{=} 0,$$

where φ is the phase of the field ϕ defined in (10). One sees that these infinitesimal diffeomorphisms are given, on-shell, by a combination of the three gauge invariances, with the respective field dependent infinitesimal parameters given by

$$\theta = i\left(i_\xi d\varphi\right), \qquad \psi = i_\xi B, \qquad \eta = i_\xi\left(A - i d\varphi\right). \tag{13}$$

3. Hamiltonian Analysis and Constraints

We apply here the canonical formalism of Dirac [10, 11] for systems with constraints. Supposing that the space-time manifold admits a "time" × "space" foliation $\mathscr{M}_3 = \mathbb{R} \times \Sigma$, where the space slice Σ is some two-dimensional manifold, we first rewrite the action as the time integral

$$S = \int dt L\left(A, \dot{A}, B, \dot{B}, \phi, \dot{\phi}, \overline{\phi}, \dot{\overline{\phi}}, e, \dot{e}\right) \tag{14}$$

of a Lagrangian function

$$L\left(A, \dot{A}, B, \dot{B}, \phi, \dot{\phi}, \overline{\phi}, \dot{\overline{\phi}}, e, \dot{e}\right)$$
$$= \int_\Sigma d^2x \left(\widetilde{B}^a \partial_t A_a - \widetilde{B}^a D_a \overline{\phi} \partial_t \phi \right. \tag{15}$$
$$\left. + \widetilde{B}^a D_a \phi \partial_t \overline{\phi} + A_t \mathscr{C}_1 + B_t \mathscr{C}_2 + \widetilde{e}\mathscr{C}_5\right),$$

where

$$\mathscr{C}_1 = \partial_a \widetilde{B}^a + \widetilde{B}^a \partial_a\left(\overline{\phi}\phi\right),$$
$$\mathscr{C}_2 = \widetilde{F} + \varepsilon^{ab} D_a \overline{\phi} D_b \phi, \tag{16}$$
$$\mathscr{C}_5 = \overline{\phi}\phi - 1,$$

$$\widetilde{F} = \frac{1}{2}\varepsilon^{ab} F_{ab}, \qquad \widetilde{B}^a = \varepsilon^{ab} B_b, \qquad \widetilde{e} = \frac{1}{3!}\varepsilon^{\mu\nu\rho} e_{\mu\nu\rho}.$$

Following the canonical procedure, we identify the conjugate momenta of each field X, $\Pi_X = \delta L / \delta \dot{X}$:

$$\Pi_{A_t} = 0, \qquad \Pi_{B_t} = 0, \qquad \Pi_{\widetilde{e}} = 0,$$
$$\Pi_\phi = -\widetilde{B}^a D_a \overline{\phi}, \qquad \Pi_{\overline{\phi}} = \widetilde{B}^a D_a \phi, \tag{17}$$
$$\Pi_{A_a} = \widetilde{B}^a, \qquad \Pi_{B_a} = 0,$$

satisfying together with the X's the equal time Poisson bracket relations

$$\left\{X^\alpha(\mathbf{x}), \Pi_{X^\beta}(\mathbf{y})\right\} = \delta^\alpha_\beta \delta^2(\mathbf{x}, \mathbf{y}),$$
$$\left\{X^\alpha(\mathbf{x}), X^\beta(\mathbf{y})\right\} = \left\{\Pi_{X^\alpha}(\mathbf{x}), \Pi_{X^\beta}(\mathbf{y})\right\} = 0, \tag{18}$$

where the indices α, β run over all components of all fields. The Legendre transform $H_c = -L + \sum_\alpha \int d^2x \Pi_{X^\alpha} \dot{X}^\alpha$ yields the canonical Hamiltonian

$$H_c = -\int d^2x \left(A_t \mathscr{C}_1 + B_t \mathscr{C}_2 + \widetilde{e}\mathscr{C}_5\right), \tag{19}$$

with the $\mathscr{C}$'s given in (16).

Noting that the velocities do not appear in any of (17) for the momenta, we conclude that all of these equations are (primary) constraints [10, 11]. The equality sign must be replaced by the "weak equality" sign $\approx$, meaning that the constraints are solved at the end, after all calculations involving Poisson brackets are done. We remark that the last two constraints in (17) are second class, their brackets being nonzero: $\{\Pi_{A_a}(\mathbf{x}) - \varepsilon^{ab} B_b(\mathbf{x}), \Pi_{B_c}(\mathbf{y})\} = \varepsilon^{ab}\delta^2(\mathbf{x}, \mathbf{y})$. These constraints can be solved as strong equalities

$$\Pi_{A_a} = \varepsilon^{ab} B_b = \widetilde{B}^a, \qquad \Pi_{B_a} = 0, \tag{20}$$

provided the Poisson brackets are replaced by the corresponding Dirac brackets, which read

$$\left\{A_a(\mathbf{x}), \widetilde{B}^b(\mathbf{y})\right\} = \delta^b_a \delta^2(\mathbf{x}, \mathbf{y}), \tag{21}$$

the other brackets being left unchanged. We use the same notation $\{\cdot, \cdot\}$ for these Dirac brackets.

We are left with the five constraints

$$\Pi_{A_t} \approx 0, \qquad \Pi_{B_t} \approx 0, \qquad \Pi_{\widetilde{e}} \approx 0, \tag{22}$$

$$\mathscr{C}_3(\mathbf{x}) = \Pi_\phi + \widetilde{B}^a D_a \overline{\phi} \approx 0,$$
$$\mathscr{C}_4(\mathbf{x}) = \Pi_{\overline{\phi}} - \widetilde{B}^a D_a \phi \approx 0. \tag{23}$$

The stability of the three constraints (22) under the Hamiltonian evolution requires the three secondary constraints

$$\mathscr{C}_1(\mathbf{x}) \approx 0, \qquad \mathscr{C}_2(\mathbf{x}) \approx 0, \qquad \mathscr{C}_5(\mathbf{x}) \approx 0, \tag{24}$$

with $\mathscr{C}_1$, $\mathscr{C}_2$, and $\mathscr{C}_5$ as given in (16). It will turn out convenient to replace $\mathscr{C}_1$ by the equivalent constraint:

$$\mathscr{C}'_1(\mathbf{x}) \approx 0, \text{ with}$$
$$\mathscr{C}'_1(\mathbf{x}) = \mathscr{C}_1 - \phi\mathscr{C}_3 + \overline{\phi}\mathscr{C}_4 = \partial_a \widetilde{B}^a - \phi\Pi_\phi + \overline{\phi}\Pi_{\overline{\phi}}. \tag{25}$$

The constraints (22) can be put strongly to zero, the corresponding fields A_t, B_t, and $\widetilde{e}$ playing now the roles of Lagrange multipliers λ_1, λ_2, and λ_5. Introducing also Lagrange multipliers fields for the primary constraints (23), we define the total Hamiltonian as

$$H_{\mathrm{T}} = \sum_{m=1}^{5} \mathscr{C}_m\left[\lambda_m\right], \tag{26}$$

where we have defined the functionals

$$\mathscr{C}_m\left[\lambda_m\right] = \int d^2x \lambda_m(\mathbf{x}) \mathscr{C}_m(\mathbf{x}), \tag{27}$$

considering the Lagrangian multiplier fields λ_m as smooth test functions.

Since this Hamiltonian is entirely made of constraints—a characteristic of theories with general covariance—the stability of our five constraints $\mathscr{C}_\alpha$, $\alpha = 1, \ldots, 5$, boils down to examine the matrix $M_{mn}(\mathbf{x}, \mathbf{y}) \approx \{\mathscr{C}_m(\mathbf{x}), \mathscr{C}_n(\mathbf{y})\}$ of their Poisson brackets, written up to constraints, hence the $\approx$ sign. Indeed, their stability condition reads (summation convention is assumed)

$$\dot{\mathscr{C}}_m = \{\mathscr{C}_m, H_{\mathrm{T}}\} = M_{mn}\lambda^n = 0. \tag{28}$$

This provides a system of equations for the λ's; which can be solved for some of the λ's in terms of the remaining ones. The matrix M reads

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi\mathscr{C}_2 & 0 \\ 0 & 0 & 0 & -\left(\phi\mathscr{C}_1' + \mathscr{C}_3\right) & 0 \\ 0 & -\phi\mathscr{C}_2 & \left(\phi\mathscr{C}_1' + \mathscr{C}_3\right) & 0 & -\phi \\ 0 & 0 & 0 & \phi & 0 \end{pmatrix} \times \delta^2(\mathbf{x} - \mathbf{y}), \tag{29}$$

where we have substituted the constraint $\mathscr{C}_3$ with the equivalent one

$$\mathscr{C}_3' = \phi\mathscr{C}_3 - \overline{\phi}\mathscr{C}_4. \tag{30}$$

One sees that the first three constraints, $\mathscr{C}_1$, $\mathscr{C}_2$, and $\mathscr{C}_3'$, are first class; *that is*, their Poisson brackets with any other constraint are constraints: they generate three gauge invariances of the theory. The last two, namely, $\mathscr{C}_4$ and $\mathscr{C}_5$, however are second class. Indeed, denoting them by χ_p $(p = 1, 2)$, their Poisson brackets form the matrix C_{pq} of nonvanishing determinant on the constraint surface:

$$C = \begin{pmatrix} 0 & -\phi \\ \phi & 0 \end{pmatrix}. \tag{31}$$

These second class constraints may be written as strong equalities, provided the Poisson brackets are substituted by the Dirac brackets [10, 11]

$$\{X, Y\}_{\mathrm{D}} = \{X, Y\} - \sum_{p,q} \left\{X, \chi_p\right\} \left(C^{-1}\right)^{pq} \left\{\chi_q, Y\right\}. \tag{32}$$

The second class constraints χ_p can be solved for $\overline{\phi}$ and $\Pi_{\overline{\phi}}$ in terms of the now independent fields A_a, $\widetilde{B}^a$, ϕ, and Π_ϕ:

$$\overline{\phi} = \frac{1}{\phi}, \qquad \Pi_{\overline{\phi}} = \widetilde{B}^a D_a \phi. \tag{33}$$

The independent fields obey the Dirac bracket relations

$$\begin{gathered} \left\{A_a(\mathbf{x}), \widetilde{B}^b(\mathbf{y})\right\}_{\mathrm{D}} = \delta_a^b \delta^2(\mathbf{x} - \mathbf{y}), \\ \left\{\phi(\mathbf{x}), \Pi_\phi(\mathbf{y})\right\}_{\mathrm{D}} = \delta^2(\mathbf{x} - \mathbf{y}), \\ \left\{A_a(\mathbf{x}), \Pi_\phi(\mathbf{y})\right\}_{\mathrm{D}} = D_a\left(\frac{1}{\phi}\right)\delta^2(\mathbf{x} - \mathbf{y}), \\ \left\{\widetilde{B}^a(\mathbf{x}), \Pi_\phi(\mathbf{y})\right\}_{\mathrm{D}} = -\widetilde{B}^a \frac{1}{\phi}\delta^2(\mathbf{x} - \mathbf{y}), \\ (\text{other brackets vanishing}). \end{gathered} \tag{34}$$

This system can be diagonalized through the redefinition

$$\Pi = \Pi_\phi - \widetilde{B}^a D_a\left(\frac{1}{\phi}\right), \tag{35}$$

with the result

$$\begin{gathered} \left\{A_a(\mathbf{x}), \widetilde{B}^b(\mathbf{y})\right\}_{\mathrm{D}} = \delta_a^b \delta^2(\mathbf{x} - \mathbf{y}), \\ \{\phi(\mathbf{x}), \Pi(\mathbf{y})\}_{\mathrm{D}} = \delta^2(\mathbf{x} - \mathbf{y}), \\ (\text{other brackets vanishing}). \end{gathered} \tag{36}$$

Finally, the remaining three constraints read, taking (35) into account,

$$\mathscr{C}_1 = \partial_a \widetilde{B}^a, \qquad \mathscr{C}_2 = \widetilde{F}, \qquad \mathscr{C}_3 = \Phi\Pi. \tag{37}$$

They are first class (their Dirac brackets are indeed zero) and generate the three gauge invariances defined by $\delta_i X = \{X, \mathscr{C}_i[\epsilon_i]\}$ $(i = 1, 2, 3)$ using the functional notation (27):

$$\begin{gathered} \delta_1 A_a = -\partial_a \epsilon_1, \qquad \delta_2 A_a = 0, \qquad \delta_3 A_a = 0, \\ \delta_1 \widetilde{B}^a = 0, \qquad \delta_2 \widetilde{B}^a = -\epsilon^{ab}\partial_b \epsilon_2, \qquad \delta_3 \widetilde{B}^a = 0, \\ \delta_1 \phi = 0, \qquad \delta_2 \phi = 0, \qquad \delta_3 \phi = \epsilon_3 \phi, \\ \delta_1 \Pi = 0, \qquad \delta_2 \Pi = 0, \qquad \delta_3 \Pi = -\epsilon_3 \Pi. \end{gathered} \tag{38}$$

We see that the $U(1)$ gauge invariance is split in two invariances generated by $\mathscr{C}_1$ and $\mathscr{C}_3$, corresponding to the invariances (1) and (7) of the Lagrangian formalism. The invariance generated by $\mathscr{C}_2$ corresponds to the topological type invariance (6).

4. Quantization

4.1. Kinematical Hilbert Space. The constraints $\mathscr{C}_1$ and $\mathscr{C}_3$ will be solved at the quantum level in this section, whereas the last one, $\mathscr{C}_2$, is left for the next section. Following the lines of Loop Quantum Gravity [6–9], we will construct a kinematical Hilbert space $\mathscr{H}_{\mathrm{kin}}$ whose vectors $|\Psi\rangle$ are subjected to the constraints $\mathscr{C}_1$ and $\mathscr{C}_3$ in the forms $\widehat{\mathscr{C}}_1|\Psi\rangle = 0$ and $\widehat{\mathscr{C}}_3|\Psi\rangle = 0$, where $\widehat{\mathscr{C}}_i$ are operators representing the classical $\mathscr{C}_i$. Choosing the fields A_a and ϕ as configuration space coordinates, our task will be to define wave functionals (we use the "bra" and "ket" Dirac notation, with $\langle A,\phi \mid \Psi\rangle = \Psi[A,\phi]$) $\Psi[A,\phi]$ and the scalar product $\langle \Psi \mid \Psi'\rangle$. The fields are now promoted to operators $\widehat{A}_a$, $\widehat{\phi}$, $\widehat{B}^a$, and $\widehat{\Pi}$ obeying the canonical commutation relations corresponding to the classical Dirac brackets (36):

$$\left[\widehat{A}_a(\mathbf{x}), \widehat{B}^b(\mathbf{y})\right] = i\hbar\delta_a^b\delta^2(\mathbf{x}-\mathbf{y}),$$
$$\left[\widehat{\phi}(\mathbf{x}), \widehat{\Pi}(\mathbf{y})\right] = i\hbar\delta^2(\mathbf{x}-\mathbf{y}), \tag{39}$$
$$(\text{other brackets vanishing}).$$

$\widehat{A}$ and $\widehat{\phi}$ act multiplicatively and $\widehat{B}$ and $\widehat{\Pi}$ as functional derivatives:

$$\widehat{B}^a(\mathbf{x})\Psi[A,\phi] = -i\hbar\frac{\delta\Psi[A,\phi]}{\delta A_a(\mathbf{x})},$$
$$\widehat{\Pi}(\mathbf{x})\Psi[A,\phi] = -i\hbar\frac{\delta\Psi[A,\phi]}{\delta\phi(\mathbf{x})}. \tag{40}$$

Everything up to now is purely formal since we have still no proper Hilbert space. But we can already solve the constraint $\widehat{\mathscr{C}}_3(\mathbf{x})\Psi[\phi,A] = -i\hbar\widehat{\phi}\delta\Psi[\phi,A]/\delta\phi(\mathbf{x}) = 0$: the wave functional only depends on A, $\Psi = \Psi[A]$.

In order to construct a scalar product defined by an appropriate integration measure in configuration space, we first restrict the space of wave functionals to the set of functions of finite numbers of holonomies of the connection A—the "cylindrical functions." If γ is an orientated curve in Σ (a "link"), the holonomy of A on γ is defined as the exponentiated line integral

$$h_\gamma[A] = \exp\int_\gamma A. \tag{41}$$

Given a "graph," *that is*, a finite set $\Gamma = \{\gamma_1,\ldots,\gamma_N\}$ of links, a "cylindrical function" $\Psi_{\Gamma,\psi}[A]$ is function ψ of the holonomies of Γ:

$$\Psi_{\Gamma,\psi}[A] = \psi\left(h_{\gamma_1}[A],\ldots,h_{\gamma_N}[A]\right). \tag{42}$$

The cylindrical functions associated with all graphs on Σ form the vectorial space Cyl, in which we can define a sesquilinear scalar product using the Haar measure $d\mu(g)$ of the gauge group. For $U(1)$, the (normalized) measure is given by $(1/2\pi)\int d\theta f(g(\theta))$ for g parameterized as $g(\theta) = \exp(i\theta)$. First, for two cylindrical functions defined on the same graph,

$$\left\langle \Gamma,\psi \mid \Gamma,\psi'\right\rangle = \int_{G^{\otimes N}}\int d\mu(g_1)\cdots d\mu(g_N)\left(\psi(g_1,\ldots,g_N)\right)^* \times\psi'(g_1,\ldots,g_N). \tag{43}$$

Next, for two cylindrical functions corresponding to two different graphs Γ and Γ', one defines

$$\left\langle \Gamma,\psi \mid \Gamma',\psi'\right\rangle = \int_{G^{\otimes N}}\int d\mu(g_1)\cdots d\mu(g_{\widehat{N}})\left(\psi(g_1,\ldots,g_N)\right)^* \times\psi'(g_1,\ldots,g_{N'}), \tag{44}$$

where $\widehat{\Gamma}$ is the union graph $\Gamma\cup\Gamma'$ consisting of $\widehat{N} \le (N+N')$ links.

With this scalar product in hands we dispose of a norm so one can define a Hilbert space $\mathscr{H}_{\text{Cyl}}$ through the Cauchy completion of Cyl.

An orthonormal basis of $\mathscr{H}_{\text{Cyl}}$ may be defined using the Peter-Weyl theorem—which in the Abelian $U(1)$ case is nothing but the Fourier series theorem. Basis elements are the cylindrical functions

$$\Psi_{\Gamma,\vec{n}}[A] = \chi_{n_1}\left(h_{\gamma_1}[A]\right)\cdots\chi_{n_N}\left(h_{\gamma_N}[A]\right),$$
$$\text{where } \vec{n} = (n_1,\ldots,n_N),\quad n_k\in\mathbb{Z},\ n_k\neq 0, \tag{45}$$

and $\chi_n(g)$ is the character of the irreducible unitary representation of "charge" $n\in\mathbb{Z}$. In the parametrization $g = \exp(i\theta)$, $\chi_n(g) = \exp(in\theta)$. The orthonormality condition

$$\left\langle \Gamma,\vec{n} \mid \Gamma',\vec{n}'\right\rangle \tag{46}$$

is an obvious consequence of the theory of Fourier series. The prescription of nonvanishing charges n_k avoids an overcounting of the basis vectors which would otherwise occur since a graph with a zero charge link would give the same function as the graph with this link omitted. Therefore, the basis must be completed with the zero charge function $\Psi_\emptyset$ corresponding to the empty set $\emptyset$. These basis vectors $|\Gamma,\vec{n}\rangle$ will be called "charge networks" in analogy with the spin networks of Loop Quantum Gravity [6–9]. A particular consequence of these definitions is that vectors corresponding to different graph are orthogonal, and thus the Hilbert space $\mathscr{H}_{\text{Cyl}}$ is the infinite direct sum of spaces $\mathscr{H}_{\text{Cyl},\Gamma}$, each of them being associated with a single graph Γ. With this sum being performed over the noncountable set of all graphs, $\mathscr{H}_{\text{Cyl}}$ is a nonseparable Hilbert space.

Let us now turn to the constraint $\mathscr{C}_1$ in (37), which corresponds to the invariance under the $U(1)$ gauge transformations δ_1 of (38). It will be fulfilled by demanding the gauge invariance of the basis cylindrical functions (45). Under a gauge transformation $A_a' = A_a' + \partial_a\omega$, the holonomy (41) transforms as

$$h_\gamma[A]' = h_\gamma[A]\exp\left(\omega\left(\mathbf{x}_f\right) - \omega\left(\mathbf{x}_i\right)\right), \tag{47}$$

where $\mathbf{x}_i$ and $\mathbf{x}_f$ are the coordinates of the initial and end points of the link γ, respectively. Thus gauge invariance of a charge network functional $\Psi_{\Gamma,\vec{n}}$ follows from the requirement of a "charge conservation law"; *that is*, the sum of charges entering a vertex of Γ (point of intersection of links) must be zero, with the convention that the charge entering a vertex is

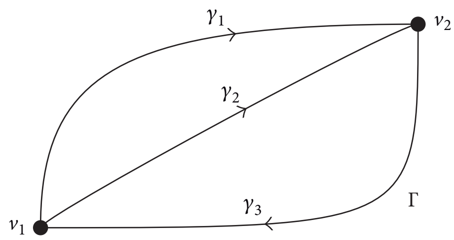

Figure 1: Closed graph Γ with three links and two vertices. Link γ_k carrying a charge n_k, charge conservation at vertex v_1 (or v_2) amounts to $-n_1 - n_2 + n_3 = 0$.

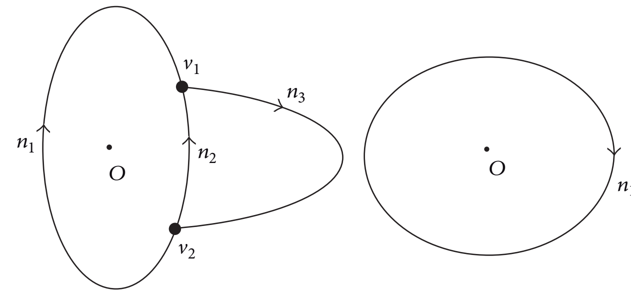

Figure 2: Two charge network graphs with the singular point O "inside."

positive if the vertex lies at the end of the link and negative if it lies at the beginning. This requires in particular that the graphs must be closed since no zero charge links are allowed. An example is depicted in Figure 1.

The vectors of $\mathscr{H}_{\text{Cyl}}$ obeying the condition of gauge invariance span the nonseparable "kinematical" Hilbert space $\mathscr{H}_{\text{kin}} \subset \mathscr{H}_{\text{Cyl}}$.

4.2. Physical Hilbert Space. The last constraint to be imposed is the curvature constraint $\mathscr{C}_2$ in (37), whose quantum expression is $\widehat{F}|\Psi\rangle = 0$. Its general solution is given by a wave functional $\Psi[A]$ whose argument A is a connection with null curvature. It is sufficient to impose this condition on the basis vectors of $\mathscr{H}_{\text{kin}}$ (charge networks), which will select the basis of the physical Hilbert space $\mathscr{H}_{\text{phys}} \subset \mathscr{H}_{\text{kin}}$.

The condition of null curvature means that, *locally*, there exists a scalar function φ such that

$$A_a = \partial_a \varphi. \tag{48}$$

The rest of the discussion depends on the topology of the space sheet Σ.

Let us begin with the case where the topology of Σ is that of $\mathbb{R}^2$. Then (48) holds globally, with the result that the holonomy associated with any link γ with initial and final end points $\mathbf{x}_i$ and $\mathbf{x}_f$ takes the form

$$h_\gamma [A] = \exp\left(\varphi\left(\mathbf{x}_f\right) - \varphi\left(\mathbf{x}_i\right)\right). \tag{49}$$

Together with the fact that the graph associated with any charge network $|\Gamma, \vec{n}\rangle$ is closed and that the charge conservation condition must hold at each vertex, one easily sees that its wave functional $\Psi_{\Gamma,\vec{n}}$ is equal to 1. In other words, the graph Γ shrinks to a single point, and we are left with the sole vector $|\emptyset\rangle$. The physical Hilbert space is reduced to a trivial 1-dimensional space.

The next case is that with the topology of $\mathbb{R}^2 \setminus \{O\}$, the 2-dimensional plane with one point O suppressed. There are now two classes of closed graphs, those with O inside and those with O outside. Two examples of the former class are shown in Figure 2.

Applying the charge conservation condition as in the previous case shows that any charge network graph with the point O "outside" reduces to a point with the resulting wave functional equal to 1, defining the empty state described by the vector $|\emptyset\rangle$. On the other hand, any charge network graph with the point O "inside" is equivalent to a single loop γ with O inside, with the resulting wave functional equal to a unimodular complex number

$$\langle A \mid n\rangle = \Psi_n [A] = \exp(inQ), \tag{50}$$

with $n \in \mathbb{Z}$ the charge of the loop. The value of the "flux" Q, given by

$$\exp(iQ) = h_C [A], \tag{51}$$

where C is a closed positively oriented loop around the singular point O, is independent of the form and size of the loop, and the value of n is computed using the charge conservation condition. Figure 2 shows an example of two such equivalent graphs. The basis of the physical Hilbert space $\mathscr{H}_{\text{phys}}$ then consists of the vectors $|n\rangle$, $n \in \mathbb{Z}$, with $\langle n \mid n'\rangle = \delta_{nn'}$. For $n = 0$, one has $|0\rangle = |\emptyset\rangle$, corresponding to the former class of graphs. One notes that the integer number n can be interpreted also as a winding number of the loop: to wind n times around the singular point with charge 1 or to wind 1 time the singular point with charge n yields the same wave functional.

The generalization to a plane with N singular points, $\mathbb{R}^2 \setminus \{O_1, \ldots, O_N\}$, is straightforward. The basis vectors of $\mathscr{H}_{\text{phys}}$ read $|\vec{n}\rangle = |n_1, \ldots, n_N\rangle$ where n_k is the charge (or winding number) of a loop encircling the kth singular point, all the other singular points remaining outside of it. The corresponding wave functional is explicitly given by

$$\langle A \mid \vec{n}\rangle = \Psi_{\vec{n}} [A] = \exp\left(i \sum_{k=1}^{N} n_k Q_k\right), \tag{52}$$

where Q_k is the flux associated with the kth singular point, defined by

$$\exp\left(iQ_k\right) = h_{C_k} [A], \tag{53}$$

where

$$\begin{aligned} C_k = {} & \text{closed loop encircling positively} \\ & \text{one time the singular point } O_k \\ & \text{and leaving aside all the other ones.} \end{aligned} \tag{54}$$

The orthonormality relations are

$$\left\langle n_1, \dots, n_N \mid n'_1, \dots, n'_{N'} \right\rangle = \delta_{NN'} \prod_{k=1}^{N} \delta_{n_k n'_k}. \quad (55)$$

$\mathscr{H}_{\text{phys}}$ is separable.

One remarks that diffeomorphism invariance, which in the classical theory is a consequence of its gauge invariances, is explicit in the quantum theory constructed here, once all constraints are fulfilled. Note that the states of the (nonseparable) kinematical Hilbert space, which still do not obey the curvature constraint $\mathscr{C}_2$, are not diffeomorphism invariant since they depend on the location and form of the associated graphs.

4.3. Observables. It follows from the above discussion that no nontrivial observables do exist in the case of a trivial topology such as that of $\mathbb{R}^2$. On the other side, with a nontrivial topology such as that of $\mathbb{R}^2$ with N singular points O_k, there is a set of N observables $\widehat{L}_k$, $k = 1, \dots, N$, simultaneously diagonalized in the basis (52) of $\mathscr{H}_{\text{phys}}$:

$$\widehat{L}_k \left|\vec{n}\right\rangle = n_k \left|\vec{n}\right\rangle, \quad k = 1, \dots, N. \quad (56)$$

They are explicitly given by

$$\widehat{L}_k = \int_\Sigma d^2x X_a^{(k)}(\mathbf{x}) \widehat{B}^a(\mathbf{x}), \quad (57)$$

where $X_a^{(k)}$ is a closed 1-form ($dX^{(k)} = 0$), such that its integral on a loop C_k as defined by (54) takes the value $i/\hbar$, whereas its integral on a loop C_l around another singular point O_l vanishes. Explicitly:

$$\int_{C_k} X^{(l)} = \frac{i}{\hbar} \delta_{kl}, \quad (58)$$

the result depending only on the homotopy class of C_k. In a polar coordinate frame (r, θ) centred in O_k, a particular solution (a "physical" interpretation may be to view $-iX$ as the analogue of a 2-dimensional magnetic field whose source is a point current of magnitude $1/\hbar$ located in O_k) for the 1-form $X^{(k)}$ is given by $A_r = 0$ and $A_\theta = i/(2\pi\hbar)$. The result (56) follows from the expression (52) for the basis vector functionals, together with (53) and the differentiation formula (taking into account the support property of $X^{(k)}$)

$$\int_\Sigma d^2x X_a^{(k)}(\mathbf{x}) \frac{\delta}{\delta A_a(\mathbf{x})} h_{C_k}[A] = \left(\int_{C_k} X \right) h_{C_k}[A]. \quad (59)$$

The operators $\widehat{L}_k$ thus defined are obviously self-adjoint in $\mathscr{H}_{\text{phys}}$ and form a complete commutative set of observables.

5. Conclusions

What we have shown, using the Dirac canonical scheme together with the LQG quantization procedure, is that the three-dimensional Abelian *BF* model, minimally coupled to a scalar field obeying a σ-model type of constraint, has the same degrees of freedom as the pure *BF* model. These degrees of freedom are nonlocal, of purely topological nature, characterized by the topological nature of space. They are represented by a complete set of N commuting observables $\widehat{L}_k$ in the case of the space topology being that of $\mathbb{R}^2$ with N points omitted (N "punctures").

Two main conclusions can be drawn. First, the model we have presented is a simple example of how restrictive is the assumption of background invariance. It eliminates from the action an infinity of terms which otherwise would be present if one only postulates $U(1)$ gauge invariance, such as potential terms, for instance. It is also interesting to note that invariance under both gauge transformations (6) and (7) is a consequence of background invariance and $U(1)$ gauge invariance.

The second main conclusion is that we have succeeded to implement the loop quantization scheme up to the construction of the physical Hilbert space and its observables. The implementation has turned out to be rather simple, in contrast to the difficulties which one encounters in 4-dimensional gravity [6–9]. This simplicity, in our case, originates from the topological character of the theory, where the diffeomorphism constraints reduce to a very simple curvature constraint. Moreover, we have been able to show explicitly how different topologies of space lead to different Hilbert spaces and sets of observables, which is a very hard problem in genuine quantum gravity.

Similar achievements for 3-dimensional theories, non-Abelian but without coupling with matter, may be found in [28, 29]. The generalization to a non-Abelian *BF* theory coupled with matter is not straightforward [25].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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