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RESEARCH ARTICLE

A maximally-graded invertible cubic threefold that does not admit a full exceptional collection of line bundles

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Abstract
We show that there exists a cubic threefold defined by an invertible polynomial that, when quotiented by the maximal diagonal symmetry group, has a derived category that does not have a full exceptional collection consisting of line bundles. This provides a counterexample to a conjecture of Lekili and Ueda.

Contents

1. Introduction

Let $\mathbb{C}$ be the complex numbers. We say a polynomial $w \in \mathbb{C}[x_1, \dots, x_n]$ is *invertible* if it is of the form

$$w = \sum_{i=1}^{n} \prod_{j=1}^{n} x_j^{a_{ij}}$$

where $A = (a_{ij})_{i,j=1}^n$ is a non-negative integer-valued matrix satisfying the following conditions:

A. The matrix A is invertible over $\mathbb{Q}$;
B. The polynomial w is quasihomogeneous: that is, there exist *positive* integers q_j such that $d := \sum_{j=1}^{n} q_j a_{ij}$ is constant for all i; and
C. The polynomial w is quasi-smooth: that is, the map $w : \mathbb{C}^n \to \mathbb{C}$ has a unique critical point at the origin.

Let $\mathbb{G}_m$ be the multiplicative torus. Consider the following group:

$$\Gamma_w := \{(t_1, \ldots, t_{n+1}) \in \mathbb{G}_m^{n+1} \mid w(t_1 x_1, \ldots, t_n x_n) = t_{n+1} w(x_1, \ldots, x_n)\}. \tag{1}$$

This group Γ_w acts on $\mathbb{A}^n$ by projecting onto its first n coordinates and then acting diagonally. Lekili and Ueda made the following conjecture concerning the bounded derived category associated to the polynomial w and the group Γ_w.

Conjecture 1.1 (Conjecture 1.3 of [20]). *For any invertible polynomial w, the bounded derived category $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X_w)$ of coherent sheaves on the stack*

$$X_w := [(\mathrm{Spec}(\mathbb{C}[x_1, \ldots, x_n]/(w)) \setminus 0/\Gamma_w]$$

has a tilting object, which is a direct sum of line bundles.

In this paper, we show that

$$w = x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_4 + x_4^2 x_5 + x_5^2 x_1 \tag{2}$$

provides a counterexample to this conjecture. In fact, the maximal length of any exceptional collection of line bundles on $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X_w)$ is 24. On the other hand, we calculate that 54 line bundles would be required in any full exceptional collection, let alone a tilting object.

1.1. Relation to current literature and mirror symmetry

The result above is analogous to the case of toric varieties. It was asked by King if the derived category of a smooth projective toric variety admits a tilting object that is a direct sum of line bundles. This later became known as *King's conjecture*. The first counterexamples to King's conjecture were provided by Hille-Perling [10] and then later by Efimov [2] in the Fano case. Nevertheless, in [15], Kawamata proved that the derived category of any smooth projective toric Deligne-Mumford stack has a full exceptional collection. It just need not consist of line bundles (or sheaves, for that matter, see [16, Remark 7]).

The Landau-Ginzburg B-model analogue to $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X_w)$ given by the singularity category of $(\mathbb{C}^n, \Gamma_w, w)$ is well-studied in the context of homological mirror symmetry. At present, it is known to have a full exceptional collection [3]. It is also known to have a full strong exceptional collection in certain cases: for example, when $n \leq 3$ [18] or when w can be written as the Thom-Sebastiani sum of Fermat and chain polynomials [12]. This has been desirable in order to establish homological mirror symmetry for mirror pairs of (gauged) Landau-Ginzburg models [4, 5, 8, 13, 14, 20, 21].

1.2. Plan of paper

In Section 2, we show that the Picard group of the stack X_w is isomorphic to $\mathbb{Z} \times \mathbb{Z}/11\mathbb{Z}$. In Section 3, we calculate that the Chen-Ruan cohomology of X_w is 54-dimensional. This implies that the cardinality of any full exceptional collection for $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X_w)$ must be 54 (Corollary 3.2). On the other hand, in Section 4, we find a sharp upper bound of 24 on the cardinality of an exceptional collection for $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X_w)$ consisting of line bundles.

2. Line bundles on X_w

To address Conjecture 1.1, we first require an explicit description of the Picard group of X_w.

2.1. The group Γ_w

First, we define the group of diagonal automorphisms of the invertible polynomial w to be

$$G_w := \{(t_1, \dots, t_n) \in \mathbb{G}_m^n \mid w(t_1x_1, \dots, t_nx_n) = w(x_1, \dots, x_n)\}. \tag{5}$$

This sits in an exact sequence

$$0 \longrightarrow G_w \longrightarrow \Gamma_w \xrightarrow{\chi_{n+1}} \mathbb{G}_m \to 0 \tag{6}$$

where χ_{n+1} is the projection onto the $(n+1)^{\text{th}}$ term of Γ_w. Indeed, we know that χ_{n+1} is surjective, as, given $\lambda \in \mathbb{G}_m$, we have that $(\lambda^{q_1/d}, \dots, \lambda^{q_n/d}, \lambda) \in \Gamma_w$.

By Lemma 1.6(B) of [17] for a loop polynomial

$$w = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \dots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}x_1,$$

we have $G_w \cong \mathbb{Z}/(a_1 \cdots a_n + (-1)^{n+1})\mathbb{Z}$ with generator $(e^{2\pi i \varphi_1}, \dots, e^{2\pi i \varphi_n})$, where

$$\varphi_j := \frac{(-1)^{n+1-j} a_1 \cdots a_{j-1}}{a_1 \cdots a_n + (-1)^{n+1}}. \tag{7}$$

Recall that w is quasi-homogeneous: that is, we can choose q_i such that $d := \sum_{j=1}^n q_j a_{ij}$ is constant for all i and such that $\gcd(q_1, \dots, q_n) = 1$. This yields a subgroup $J_w \cong \mathbb{G}_m$ defined by

$$f : J_w \to \Gamma_w; \quad f(\lambda) = (\lambda^{q_1}, \dots, \lambda^{q_n}, \lambda^d)$$

known as the *exponential grading operator* in the literature.

Furthermore, the inclusion f gives rise to a split short exact sequence

$$0 \longrightarrow J_w \longrightarrow \Gamma_w \longrightarrow \overline{G_w} \longrightarrow 0 \tag{8}$$

where $\overline{G_w} := G_w/(J_w \cap G_w)$ is the quotient group. Since $\gcd(q_1, \dots, q_n) = 1$, there exist b_i with $\sum_{i=1}^n b_i q_i = 1$, which gives rise to the splitting of the exact sequence given by

$$g : \Gamma_w \to J_w; \quad g(\lambda_1, \dots, \lambda_n, \lambda_{n+1}) = \prod_{i=1}^n \lambda_i^{b_i}.$$

Hence $\Gamma_w \cong J_w \times \overline{G_w}$.

The isomorphism $\Gamma_w \cong J_w \times \overline{G_w}$ gives rise to an intermediate quotient stack associated to J_w,

$$Z_w = [(\mathrm{Spec}(\mathbb{C}[x_1, \dots, x_n]/(w)) \setminus 0)/J_w],$$

which is a hypersurface in the weighted projective stack

$$[(\mathrm{Spec}(\mathbb{C}[x_1, \dots, x_n]) \setminus 0)/J_w] = \mathbb{P}(q_1 : \dots : q_n).$$

This allows us to identify X_w with the quotient $[Z_w/\overline{G_w}]$.

Example 2.1. *Let* $w = x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_5 + x_5^2x_1$, *as in* (2)*. Then* $G_w = \mathbb{Z}/33\mathbb{Z}$ *with generator*

$$g = (\zeta, \zeta^{-2}, \zeta^4, \zeta^{-8}, \zeta^{16}) \tag{9}$$

where ζ *is a primitive 33rd root of unity. Here, the intersection* $J_w \cap G_w$ *is generated by* $g^{11} = (\zeta^{11}, \zeta^{11}, \zeta^{11}, \zeta^{11}, \zeta^{11})$*. Hence* $\overline{G_w}$ *can be identified with the symmetry group generated by* $(\xi, \xi^9, \xi^4, \xi^3, \xi^5)$*, where* ξ *is a primitive 11th root of unity.*

2.2. *The Picard group of* X_w

The Grothendieck–Lefschetz theorem allows us to calculate the Picard group of X_w as follows.

Proposition 2.2. *Let* w *be an invertible polynomial with* $n \geq 5$ *and* $q_1 = \ldots = q_n = 1$*. The Picard group of* X_w *is isomorphic to* $\mathbb{Z} \times \widehat{\overline{G_w}}$*, where* $\widehat{\overline{G_w}}$ *is the group of characters of* $\overline{G_w}$*.*

Proof. Since $X_w = [Z_w/\overline{G_w}]$ is a global quotient stack, $\mathrm{Pic}(X_w)$ is nothing more than the $\overline{G_w}$-equivariant Picard group of Z_w. Note that there is a (surjective) pullback map

$$\mathrm{Pic}(X_w) \xrightarrow{f} \mathrm{Pic}(Z_w)$$

that just forgets the equivariant structure. By the Grothendieck–Lefschetz theorem (see, for example, [9, Corollary 3.2]), $\mathrm{Pic}(Z_w) \cong \mathbb{Z}$: that is, any line bundle is of the form $\mathcal{O}(n)$. As $\mathcal{O}(n)$ admits an equivariant structure, the forgetful map f is surjective.

Furthermore, as any two equivariant structures differ by a character of $\overline{G_w}$, we get a short exact sequence

$$0 \longrightarrow \widehat{\overline{G_w}} \longrightarrow \mathrm{Pic}(X_w) \xrightarrow{f} \mathbb{Z} \longrightarrow 0.$$

Since $\mathbb{Z}$ is a projective $\mathbb{Z}$-module, this splits to give the desired isomorphism. □

Example 2.3. *Let* $w = x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_5 + x_5^2x_1$ *so that* $\overline{G_w} = \mathbb{Z}/11\mathbb{Z}$*. Then by Proposition* 2.2*, we have* $\mathrm{Pic}(X_w) \cong \mathbb{Z} \times (\mathbb{Z}/11\mathbb{Z})$*.*

3. Dimension of the Hochschild homology of $\mathrm{D^b}(\mathrm{coh}\, X_w)$

In this section, we compute the dimension of the Chen–Ruan cohomology of X_w to be 54. This implies that any full exceptional collection for $\mathrm{D^b}(\mathrm{coh}\, X_w)$ must have 54 objects.

Proposition 3.1. *Let* $w = x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_5 + x_5^2x_1$*. Then* $\dim(H^*_{CR}(X_w; \mathbb{C})) = 54$.

Proof. As vector spaces, the (ungraded) Chen–Ruan cohomology of X_w is the direct sum of ordinary cohomology groups of twisted sectors

$$H^*_{CR}(X_w; \mathbb{C}) = \bigoplus_{\gamma \in \Gamma_w} H^*(\{w = 0\}_\gamma / \Gamma_w; \mathbb{C})$$

where $\{w = 0\}_\gamma := \{x \in \{w = 0\}_{\mathbb{C}^5 \setminus \{0\}} \mid \gamma \cdot x = x\}$ [1, Section 3].

First, note that if $\gamma = (\lambda_1, \ldots, \lambda_5)$ so that $\lambda_i \neq 1$ for all i, then $\gamma \cdot x \neq x$ for all $x \in \mathbb{C}^5 \setminus \{0\}$. This implies that the twisted sector corresponding to γ contributes the cohomology of the empty set: that is, nothing.

First, we address the twisted sector associated to the identity element $\gamma = e$. Note that $H^*(\{w = 0\}/\Gamma_w; \mathbb{C}) = H^*(Z_w; \mathbb{C})^{\bar{G}_w}$, so we must see how $\overline{G_w}$ acts on the cohomology of Z_w. Recall that the

Hodge diamond of the cubic Z_w is of the form

$$\begin{matrix} &&&1&&& \\ &&0&&0&& \\ &0&&1&&0& \\ 0&&5&&5&&0 \\ &0&&1&&0& \\ &&0&&0&& \\ &&&1&&& \end{matrix}$$

This is computed using the Griffiths' residue map [6], which also allows us to describe the action of $\overline{G_w}$. Namely, any element $H^{2,1}(Z_w)$ can be written as the residue of a 4-form

$$\varphi = \frac{Q}{w}\Omega_0, \qquad \Omega_0 = \sum_{i=1}^{5}(-1)^i x_i \,\mathrm{d}x_1 \wedge \ldots \wedge \widehat{\mathrm{d}x_i} \wedge \ldots \wedge \mathrm{d}x_5$$

where Q is a degree1 polynomial in $\mathbb{C}[x_1,\ldots,x_5]$. By looking at the action by the generator ρ of $\overline{G_w}$, we can see that w and Ω_0 are invariant under its action; however, no degree 1 polynomial is invariant, so the $\overline{G_w}$-invariant subspace of $H^{2,1}(Z_w;\mathbb{C})$ is zero. Similarly, the $\overline{G_w}$-invariant subspace of $H^{1,2}(Z_w;C)$ is zero. The hyperplane classes, on the other hand, are all invariant cycles, so

$$\dim H^*(\{w=0\}/\Gamma_w;\mathbb{C}) = 4.$$

Lastly, there are 50 non-identity elements

$$S := \{(\rho\tau^{-1})^a, (\rho\tau^{-9})^a, (\rho\tau^{-4})^a, (\rho\tau^{-3})^a, (\rho\tau^{-5})^a \mid 1 \le a \le 10\} \subseteq \Gamma_w$$

with a fixed point where $\rho := (\xi,\xi^9,\xi^4,\xi^3,\xi^5)$ is the generator of $\overline{G_w}$ and $\tau = (\xi,\xi,\xi,\xi,\xi)$. In fact, each has a single fixed point and hence contributes one dimension to the Chen-Ruan cohomology.

We conclude that $\dim(H^*_{CR}(X_w;\mathbb{C})) = 4 + |S| = 4 + 50 = 54$. □

This proposition implies the following corollary.

Corollary 3.2. *For w as defined in (2), we have that* $\dim(HH_*(\mathrm{D^b}(\mathrm{coh}\, X_w))) = 54$. *In particular, any full exceptional collection for* $\mathrm{D^b}(\mathrm{coh}\, X_w)$ *has precisely 54 objects.*

Proof. By an unpublished result of Toën (reproven in [7, Proposition 3.16]),

$$\dim(\mathrm{HH}_*(\mathrm{D^b}(\mathrm{coh}\, X_w))) = \dim(H^*_{CR}(X_w;\mathbb{C})) = 54.$$

The fact that any full exceptional collection must have 54 objects follows from the additivity of Hochschild homology under semi-orthogonal decomposition. □

Remark 3.3. In [3, Theorem 1.1], the authors prove that there is a strong exceptional collection for the singularity category $\mathrm{D}[\mathbb{A}^5,\Gamma_w,w]$. It is of length 32, the Milnor number of its mirror LG-model. By the equivariant version of Orlov's theorem (proven by Hirano [11, Theorem 1.3]), it follows that $\mathrm{D^b}(\mathrm{coh}\, X_w)$ has a full exceptional collection of length $32 + 2(11) = 54$. From this, it also follows that any full exceptional collection must have 54 objects.

4. Computations of Ext between line bundles on X_w

By Corollary 3.2, any full exceptional collection for $\mathrm{D^b}(\mathrm{coh}\, X_w)$ has 54 objects. However, in this section, we show that an exceptional collection consisting of line bundles on X_w has at most 24 objects (and remark that this bound is achieved).

Table 1. *The (a, b)th entry is an (a, b)-bigraded monomial in $\mathbb{C}[x_1, x_2, x_3, x_4, x_5]/(w)$.*

	$\mathbb{Z}/11\mathbb{Z}$-grading										
$\mathbb{Z}$-grading	0	1	2	3	4	5	6	7	8	9	10
0	1										
1		x_1		x_4	x_3	x_5				x_2	
2		x_2x_4	x_1^2	x_5x_2	x_1x_4	x_1x_3	x_1x_5	x_3x_4	x_3^2	x_3x_5	x_1x_2
3	$x_1^2x_2$	x_3^3	$x_1x_2x_4$	x_1^3	$x_1x_2x_5$	$x_1^2x_4$	$x_1^2x_3$	$x_1^2x_5$	$x_1x_3x_4$	$x_1x_3^2$	$x_1x_3x_5$

Lemma 4.1. *For $a \geq 0$, $\mathrm{Hom}(\mathcal{O}, \mathcal{O}(a, b)) \neq 0$ unless $a = 0$ and $b \neq 0$ or*

$$(a, b) \in \mathbb{X} := \{(1, 0), (1, 2), (1, 6), (1, 7), (1, 8), (1, 10), (2, 0)\}.$$

Proof. Observe that $\mathrm{Hom}(\mathcal{O}, \mathcal{O}(a, b))$ is the space of bidegree $(a, b) \in \mathbb{Z}\times\mathbb{Z}/11\mathbb{Z}$ polynomials in $\mathbb{C}[x_1, x_2, x_3, x_4, x_5]/(w)$. By Example 2.1, $\overline{G}_w = \langle(\xi, \xi^9, \xi^4, \xi^3, \xi^5)\rangle \cong \mathbb{Z}/11\mathbb{Z}$, where ξ is a primitive 11th root of unity. Hence,

$$\deg(x_1) = (1, 1),\ \ \deg(x_2) = (1, 9),\ \ \deg(x_3) = (1, 4),\ \ \deg(x_4) = (1, 3),\ \ \deg(x_5) = (1, 5).$$

So Table 1 exhibits an element in $\mathrm{Hom}(\mathcal{O}, \mathcal{O}(a, b))$ for $1 \leq a \leq 3$, unless $(a, b) \in \mathbb{X}$. We conclude that $\mathrm{Hom}(\mathcal{O}, \mathcal{O}(a, b))$ is non-zero for $a \geq 3$ by multiplying any monomial in $\mathrm{Hom}(\mathcal{O}, \mathcal{O}(3, b - a + 3))$ by x_1^{a-3}. □

Lemma 4.2. *For $a \geq 2$, we have that $\mathrm{Ext}^3(\mathcal{O}(a, b), \mathcal{O}) \neq 0$ unless $a = 2$ and $b \neq 0$ or*

$$(a, b) \in \mathbb{X}' := \{(3, 0), (3, 2), (3, 6), (3, 7), (3, 8), (3, 10), (4, 0)\}.$$

Proof. By adjunction, the canonical bundle is $\mathcal{O}(-2, 0)$. Therefore by Serre duality,

$$\begin{aligned}\mathrm{Ext}^i(\mathcal{O}(a, b), \mathcal{O}) &\overset{\text{Serre}}{\cong} \mathrm{Ext}^{3-i}(\mathcal{O}, \mathcal{O}(a, b) \otimes_{\mathcal{O}} \mathcal{O}(-2, 0))^* \\ &\cong \mathrm{Ext}^{3-i}(\mathcal{O}, \mathcal{O}(a-2, b)))^*.\end{aligned}$$

The result follows from Lemma 4.1. □

Proposition 4.3. *An exceptional collection of line bundles in $\mathrm{D^b}(\mathrm{coh}\, X_w)$ has at most 24 objects and hence cannot be full (by Corollary 3.2).*

Proof. By Example 2.3, any line bundle on X_w is of the form $\mathcal{O}(a, b)$ for $(a, b) \in \mathbb{Z}\times\mathbb{Z}/11\mathbb{Z}$. Let $\mathcal{E}$ denote an exceptional collection of line bundles, and take the minimal a such that $\mathcal{O}(a, b) \in \mathcal{E}$ for some $b \in \mathbb{Z}/11\mathbb{Z}$. Since $\mathcal{E} \otimes \mathcal{O}(-a, -b)$ is an exceptional collection, we can assume $(a, b) = (0, 0)$.

Notice that $\mathcal{E}$ cannot have an object of the form $\mathcal{O}(a, b)$ for $a \geq 5$, as, by Lemma 4.1, $\mathcal{O}(a, b)$ receives a non-zero map from $\mathcal{O}$ and, by Lemma 4.2, there is a non-trivial 3-extension of $\mathcal{O}$ by $\mathcal{O}(a, b)$.

By Table 1, observe that if $b \neq b'$, then for any a, one has non-zero elements

$$f_1 \in \mathrm{Hom}(\mathcal{O}(a, b), \mathcal{O}(a + 2, b')) \qquad \text{and} \qquad f_2 \in \mathrm{Hom}(\mathcal{O}(a, b'), \mathcal{O}(a + 2, b)).$$

Therefore, denoting by $\mathbb{S}(a, b)$ the Serre functor applied to the identity map on $\mathcal{O}(a, b)$, one has a loop:

$$\mathcal{O}(a, b) \xrightarrow{f_1} \mathcal{O}(a + 2, b') \xrightarrow{\mathbb{S}(a+2,b')} \mathcal{O}(a, b') \xrightarrow{f_2} \mathcal{O}(a + 2, b) \xrightarrow{\mathbb{S}(a+2,b)} \mathcal{O}(a, b).$$

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We conclude that $\mathcal{E}$ cannot have a quadruple of objects

$$\{\mathcal{O}(a,b), \mathcal{O}(a,b'), \mathcal{O}(a+2,b), \mathcal{O}(a+2,b')\}.$$

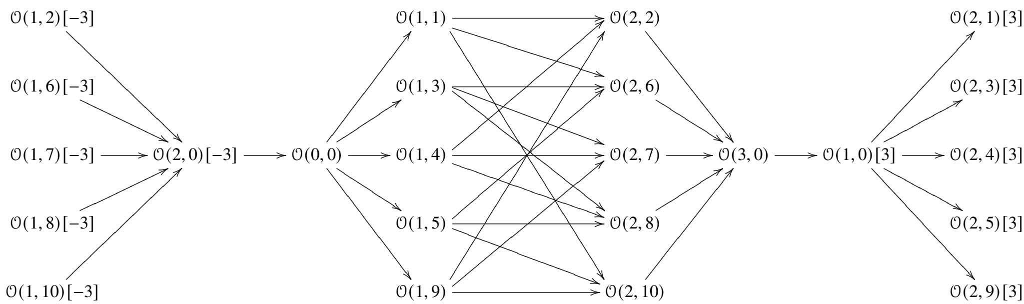
For example, taking $a = 0$ (respectively, $a = 1$), $\mathcal{E}$ cannot have multiple objects with $a = 0$ and $a = 2$ (respectively, $a = 1$ and $a = 3$). This forces there to be at most 12 line bundles in $\mathcal{E}$ with $a = 0, 2$ and $a = 1, 3$, respectively.

Now, again by Lemma 4.2, $\mathcal{E}$ cannot have an object of the form $\mathcal{O}(a,b)$ for $a \geq 4$ except $(a,b) = (4,0)$. Hence, we can have at most 1 more object. But if $\mathcal{O}(4,0) \in \mathcal{E}$, Lemma 4.2 also forces $\mathcal{O}(0,b) \notin \mathcal{E}$ for $b \neq 0$. Hence, if we already have 12 line bundles in $\mathcal{E}$ with $a = 0, 2$ then $\mathcal{O}(2,b) \in \mathcal{E}$ for all b. This gives a contradiction, as $\mathcal{O}, \mathcal{O}(2,0), \mathcal{O}(4,0)$ also form a loop

$$\mathcal{O} \xrightarrow{x_1^2 x_3 x_5} \mathcal{O}(4,0) \xrightarrow{\mathbb{S}(4,0)} \mathcal{O}(2,0) \xrightarrow{\mathbb{S}(2,0)} \mathcal{O}$$

and therefore cannot be in the same exceptional collection. We conclude that this 1 additional object cannot take us beyond 24 exceptional objects. □

Remark 4.4. The upper bound of 24 exceptional objects is sharp. It is achieved by the exceptional collection drawn below. This exceptional collection is not strong, however; we only draw the degree-0 maps for aesthetic simplicity. The required vanishing can be checked using Lemmas 4.1 and 4.2 and the fact that $\mathrm{Ext}^1, \mathrm{Ext}^2$ vanish for line bundles on a 3-fold hypersurface in projective space (for example, using the long exact sequence for the divisor).



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