CHROMATICITY OF CERTAIN BIPARTITE GRAPHS

By

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Thesis Submitted to the School of Graduate Studies,
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Dedication

Specially Dedicated to My Wife,

My Daughter and My Parents

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirements for the degree of Doctor of Philosophy

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January 2005

Chairman: **Professor Peng Yee Hock, PhD**

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Since the introduction of the concepts of chromatically unique graphs and chromatically equivalent graphs, numerous families of such graphs have been obtained. The purpose of this thesis is to continue with the search of families of chromatically unique bipartite graphs.

In Chapters 1 and 2, we define the concept of graph colouring, the associated chromatic polynomial and some properties of a chromatic polynomial. We also give some necessary conditions for graphs that are chromatically unique or chromatically equivalent. We end this chapter by stating some known results on the chromaticity of bipartite graphs, denoted as K(p,q).

Let $\mathcal{K}^{-s}(p,q)$(resp. $\mathcal{K}_2^{-s}(p,q)$) denote the set of connected (resp. $2-$connected) bipartite graphs which can be obtained from $K(p,q)$ by deleting a set of s edges. For a bipartite graph $G=(A,B;E)$ with bipartition A and B and edge set E, let $G'=(A',B';E')$ be the bipartite graph induced by the edge set $E'=\{xy \mid xy \notin E,\ x \in A, y \in B\ \}$, where $A' \subseteq A$ and $B' \subseteq B$. We write $G'=K(p,q)-G$, where $p=|A|$ and $q=|B|$. Let $\triangle(G')$ denote the maximum degree of G'.

In Chapter 3, we study the chromatic uniqueness of any $G \in \mathcal{K}_2^{-s}(p,q)$ where $p \geq q \geq 3$, $9 \leq s \leq q-1$ and $\triangle(G') = s-3$. In Chapter 4, we give a similar result by examining the chromatic uniqueness of any $G \in \mathcal{K}_2^{-s}(p,q)$ where $p \geq q \geq 3$, $11 \leq s \leq q-1$ and $\triangle(G') = s-4$.

Let $\alpha(G,k)$ denote the number of $k-$independent partitions in G. Define $\alpha'(G,3) = \alpha(G,3) - (2^{|A|-1} + 2^{|B|-1} - 2)$. For $t = 0, 1, 2, \ldots,$ let $\mathcal{B}(p,q,s,t)$ denote the set of graphs $G \in \mathcal{K}^{-s}(p,q)$ with $\alpha'(G,3) = s+t$. It is known that if G is $2-$connected graph in $\mathcal{B}(p,q,s,t)$ for $0 \leq t \leq 4$ or $t = 2^s - s - 1$, then G is chromatically unique. In Chapter 5, we examine the chromatic uniqueness of a 2-connected graph G in $\mathcal{B}(p,q,s,5) \cup \mathcal{B}(p,q,s,6)$. We continue this work and prove the chromatic uniqueness of every 2-connected graph G in $\mathcal{B}(p,q,s,7)$ in Chapter 6.

In Chapter 7, we investigate the chromatic uniqueness of the graphs in $\mathcal{K}_2^{-s}(p,q)$, where $p \geq q \geq 6$ and $5 \leq s \leq \min\{q-1, 7\}$. In the final chapter, we study the chromatic uniqueness of the graphs in $\mathcal{K}_2^{-s}(p,q)$, where $p \geq q = 5$ and $s = 5$. We also present a short discussion and some open problems for futher research.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia
sebagai memenuhi keperluan untuk ijazah Doktor Falasafah

KEKROMATIKAN GRAF BIPARTIT TERTENTU

Oleh

ROSLAN BIN HASNI @ ABDULLAH

January 2005

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Sejak konsep graf unik kromatik dan setara kromatik diperkenalkan, banyak famili graf yang unik kromatik dan setara kromatik telah diperolehi. Tesis ini bertujuan meneruskan pencarian famili graf bipartit yang unik kromatik.

Dalam Bab 1 dan 2, kami takrifkan konsep pewarnaan graf, polinomial kromatik yang berkaitan dan beberapa ciri polinomial kromatik. Kami juga kemukakan beberapa syarat perlu supaya sesuatu graf itu unik kromatik atau setara kromatik. Kami mengakhiri bab ini dengan menyatakan beberapa keputusan yang telah diketahui tentang kekromatikan graf bipartit, yang ditandakan dengan K(p,q).

Biarkan $\mathcal{K}^{-s}(p,q)$(atau $\mathcal{K}_2^{-s}(p,q)$) menandai famili graf bipartit terkait (atau terkait-2) yang diperolehi daripada $K(p,q)$ dengan menyingkirkan suatu set sisi s. Bagi graf bipartit $G=(A,B;E)$ dengan bipartisi A dan B dan set sisi E, biarkan $G'=(A',B';E')$ adalah graf bipartit yang diaruhkan oleh set sisi $E'=\{xy \mid xy \notin E, x \in A, y \in B\}$, dengan $A' \subseteq A$ dan $B' \subseteq B$. Kita tulis $G'=K(p,q)-G$, dengan $p=|A|$ dan $q=|B|$. Biarkan $\triangle(G')$ menandakan darjah maksimum bagi G'. Dalam Bab 3, kami mengkaji keunikan kromatik

bagi sebarang $G \in \mathcal{K}_2^{-s}(p,q)$ dengan $p \geq q \geq 3$, $9 \leq s \leq q-1$ dan $\triangle(G') = s-3$. Dalam Bab 4, kami mengemukakan keputusan serupa dengan memeriksa keunikan kromatik bagi sebarang $G \in \mathcal{K}_2^{-s}(p,q)$ dengan $p \geq q \geq 3$, $11 \leq s \leq q-1$ dan $\triangle(G') = s-4$.

Biarkan $\alpha(G,k)$ menandakan bilangan partisi tak bersandar-k dalam G. Takrifkan $\alpha'(G,3) = \alpha(G,3) - (2^{|A|-1} + 2^{|B|-1} - 2)$. Untuk $t = 0,1,2,\ldots,$ biarlah $\mathcal{B}(p,q,s,t)$ menandakan famili graf $G \in \mathcal{K}^{-s}(p,q)$ dengan $\alpha'(G,3) = s+t$. Diketahui bahawa jika G adalah graf terkait-2 dalam $\mathcal{B}(p,q,s,t)$ bagi $0 \leq t \leq 4$ atau $t = 2^s - s - 1$, maka G adalah unik kromatik. Dalam Bab 5, kami memeriksa keunikan kromatik bagi graf G yang terkait-2 dalam $\mathcal{B}(p,q,s,5) \cup \mathcal{B}(p,q,s,6)$. Kajian diteruskan dengan membuktikan keunikan kromatik bagi graf G yang terkait-2 dalam $\mathcal{B}(p,q,s,7)$ dalam Bab 6.

Dalam Bab 7, kami menyiasat keunikan kromatik bagi graf dalam $\mathcal{K}_2^{-s}(p,q)$, dengan $p \geq q \geq 6$ dan $5 \leq s \leq \min\{q-1, 7\}$. Dalam bab terakhir, kami mengkaji keunikan kromatik bagi graf dalam $\mathcal{K}_2^{-s}(p,q)$, dengan $p \geq q = 5$ dan $s = 5$. Kami juga mengemukakan perbincangan ringkas dan beberapa masalah terbuka untuk kajian selanjutnya.

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I certify that an Examination Committee met on 26th Jan 2005 to conduct the final examination of Roslan bin Hasni @ Abdullah on his Doctor of Philosophy thesis entitled "Chromaticity of Certain Bipartite Graphs" in accordance with Universiti Pertanian Malaysia (Higher Degree) Act 1980 and Universiti Pertanian Malaysia (Higher Degree) Regulations 1981. The Committee recommends that the candidate be awarded the relevant degree. Members of the Examination Committee are as follows:

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DECLARATION

I hereby declare that the thesis is based on my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously or concurrently submitted for any other degree at UPM or other institutions.

Roslan bin Hasni@Abdullah
Date:

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CHAPTER 1

INTRODUCTION

1.1 Introduction

Throughtout this thesis, a graph G is a finite non-empty *vertex set* $V(G)$ together with an (possibly empty) *edge set* $E(G)$ (disjoint from $V(G)$) of two-element subsets of distinct elements of $V(G)$. We denote by n and m respectively the order and size of G where $n = |V(G)|$ and $m = |E(G)|$ unless otherwise stated. We denote uv for the edge $e = \{u, v\}$. An edge $e = uv$ is said to *join* the vertices u and v. If $e = uv$ is an edge of a graph G, then u and v are *adjacent* vertices while u and e are *incident*, as are v and e. The vertices contained in an edge are its *endpoints*. The repeated edges and edges with the same endpoints are called *multiple edges* and *loops*, respectively. A graph is *simple* if it has no loops and no multiple edges. A *directed* graph is a finite nonempty set $V(G)$ with a set $E(G)$ of ordered pairs of distinct elements of $V(G)$, where set $E(G)$ is disjoint from $V(G)$. All graphs considered here are finite, undirected, simple and loopless. We shall refer to [1] for all notations and terminologies not explained in this thesis.

The *four-colour* problem has played a role of the utmost importance in the development of graph theory as we know it today. It was young Francis Guthrie who conjectured, while colouring the district map of England, that four colours were sufficient to colour the world map so that adjacent countries receive distinct colours. Ever since the conjecture was first published in 1852, many eminent mathematicians, especially graph theorists, attempted to settle the conjecture. In 1912, Birkhoff [2] initiated a new quantitative approach to attack the problem. For a given positive integer λ and a given map M, he introduced the symbol $P(\lambda)$ to denote the number of ways of colouring the regions of M when λ colours are available such that adjacent regions have different colours. The function for number of ways of such colouring, $P(\lambda)$ was then proved to be a polynomial in λ.

In 1932, Whitney [28] extended Birkhoff's idea of map colourings to vertex colouring of a graph G. The general problem of colouring graphs had been mentioned earlier by Kempe [15] but little work had been done on this problem prior to 1930. Whitney used the notation $M(\lambda)$ to denote the number of ways of colouring the vertices of a graph G when λ colours are available; as in the case of a map, the function M is a polynomial function of λ. This function $P(G, \lambda)$ now known as *chromatic polynomial*, was then used by Birkhoff and Lewis [3] in 1946 trying to solve the four-colour conjecture by characterizing what polynomials were chromatic polynomials of maps. They also proved that $P(G, \lambda)$ is a polynomial for any graph G. The minimum integer λ such that $P(G, \lambda)$ is nonzero is called the *chromatic number* of G, denoted $\chi(G)$. For the development and more information about the chromatic polynomial, see [20], [21], [22] and [26].

The problem of characterizing the chromatic polynomial is remaining unsolved until today. However, it leads to the concept of chromatically equivalence and chromatically unique graphs. The concept of chromatically unique graphs was first introduced by Chao and Whitehead [5] in 1978. Since then, various families of chromatically unique graphs have been found successively (see [17] and [18]).

1.2 Organization of Thesis

Since the introduction of the concepts of chromatically unique graphs and chromatically equivalent graphs, numerous families of such graphs have been obtained. The purpose of this thesis is to continue the search of chromatically unique bipartite graphs.

In Chapter 2, we give a brief literature review of the works done on the chromaticity of graphs and in particular, the chromaticity of bipartite graphs.

Dong et al. [9] proved that if $G \in \mathcal{K}_2^{-s}(p,q)$ with $p \geq q \geq 3$, where $5 \leq s \leq q-1$ and $\triangle(G') = s-1$, or $7 \leq s \leq q-1$ and $\triangle(G') = s-2$, then G is $\chi-$unique. We shall study the chromatic uniqueness of any $G \in \mathcal{K}_2^{-s}(p,q)$ with $p \geq q \geq 3$, where $9 \leq s \leq q-1$ and $\triangle(G') = s-3$ in Chapter 3. In Chapter 4, we give a similar result by examining the chromatic uniqueness of any $G \in \mathcal{K}_2^{-s}(p,q)$ with $p \geq q \geq 3$, where $11 \leq s \leq q-1$ and $\triangle(G') = s-4$.

Dong et al. [11] have shown that if G is $2-$connected graphs in $\mathcal{B}(p,q,s,0) \cup \mathcal{B}(p,q,s,2^s-s-1)$, then G is chromatically unique. The chromatic uniqueness of $2-$connected graphs $G \in \mathcal{B}(p,q,s,t)$ for $1 \leq t \leq 4$, was later on studied by Dong et al. in [10]. We shall examine the chromatic uniqueness of 2-connected graphs G in $\mathcal{B}(p,q,s,5) \cup \mathcal{B}(p,q,s,6)$ in Chapter 5. In Chapter 6, we continue this work and prove the chromatic uniqueness of 2-connected graphs G in $\mathcal{B}(p,q,s,7)$.

Dong et al. [10] also showed that any $G \in \mathcal{K}_2^{-s}(p,q)$ is $\chi-$unique if $p \geq q \geq 3$ and $1 \leq s \leq min\{q-1,4\}$. In Chapter 7, we investigate the chromatic uniqueness of the graphs in $\mathcal{K}_2^{-s}(p,q)$, where $p \geq q \geq 6$ and $5 \leq s \leq min\{q-1, 7\}$.

In the final chapter, we investigate the chromatic uniqueness of the graphs in $\mathcal{K}_2^{-s}(p,q)$, where $p \geq q = 5$ and $s = 5$. We also present a short discussion and some open problems for further research.

CHAPTER 2

LITERATURE REVIEW

2.1 The Fundamental Results on Chromatic Polynomial

We first give the formal definition of the chromatic polynomial of a graph. An assignment of at most λ colours to the vertices of a graph G is a $\lambda-$*colouring* of G, and such a colouring of G is *proper* if no two adjacent vertices are assigned the same colour. More precisely, a proper $\lambda-$colouring of G is a mapping

$$f : V(G) = \{ v_1, v_2, \ldots, v_n \} \to \{ 1, 2, \ldots, \lambda \}$$

such that $f(v_i) \neq f(v_j)$ whenever $v_iv_j \in E(G)$. Two proper $\lambda-$colourings f and g of G are considered *different* if $f(v_i) \neq g(v_i)$ for some vertex v_i in G. Let $P(G, \lambda)$ (or simply $P(G)$ if there is no danger of confusion) denote the number of different proper $\lambda-$colourings of G.

An *empty* graph is a graph with no edges while a complete graph is a graph in which each pair of distinct vertices is joined by an edge. Thus, for instance, if O_n is the empty graph of order n, then $P(O_n, \lambda) = \lambda^n$; and if K_n is the *complete* graph of order n, then $P(K_n, \lambda) = \lambda(\lambda - 1) \ldots (\lambda - n + 1)$. Observe that $P(O_n, \lambda)$ and $P(K_n, \lambda)$ are polynomials in λ. It turns out (see Theorem 2.5) that for any graph G, $P(G,\lambda)$ is in fact a polynomial in λ, called the *chromatic polynomial* of G.

The following result is very useful in determining $P(G,\lambda)$ or in showing that certain graphs are chromatically unique.

Theorem 2.1 (Fundamental Reduction Theorem)(Whitney [29]) *Let G be a graph and e an edge in G. Then*

$$P(G) = P(G-e) - P(G \cdot e)$$

where $G-e$ is the graph obtained from G by deleting e, and $G \cdot e$ is the graph obtained from G by contracting the two vertices incident with e and removing all but one of the multiple edges, if they arise.

By means of Theorem 2.1, the chromatic polynomial of a graph can be expressed in terms of the chromatic polynomials of a graph with an extra less, and another with one fewer vertices. When applying this theorem repeatedly, we can express chromatic polynomials as a sum of the chromatic polynomials of empty graphs.

The Fundamental Reduction Theorem can also be used in another way. Let $v_i, v_j \in V(G)$ such that $v_iv_j \notin E(G)$. Then

$$P(G,\lambda) = P(G + v_iv_j, \lambda) + P(G \cdot v_iv_j, \lambda)$$

where $G+v_iv_j$ is the graph obtained from G by adding the edge v_iv_j and $G \cdot v_iv_j$ is the graph obtained from G by identifying the vertices v_i and v_j. In this way, one can express $P(G,\lambda)$ as a sum of the chromatic polynomials of complete graphs.

Suppose G_1 and G_2 are the graphs each containing a complete subgraph $K_r (r \geq 1)$. Let G be the graph obtained from the union of G_1 and G_2 by identifying the two subgraphs K_r in arbitrary way, then G is called a *K_r-gluing* of G_1 and G_2. Note that K_1-gluing and K_2-gluing are called a *vertex-gluing* and *edge-gluing* of G_1 and G_2, respectively. The following two lemmas will provide a shortcut for computing $P(G,\lambda)$.

Lemma 2.1 (Zykov [32]) *Let G be a K_r-gluing of graphs G_1 and G_2. Then*

$$P(G) = \frac{P(G_1)P(G_2)}{P(K_r)} = \frac{P(G_1)P(G_2)}{\lambda(\lambda-1)\cdots(\lambda-r+1)}.$$

Lemma 2.2 (Read [20]) *If a graph G has connected components $G_1, G_2, \ldots, G_k$, then*

$$P(G) = P(G_1)P(G_2)\ldots P(G_k).$$

We shall now list some properties of the chromatic polynomial $P(G,\lambda)$ of a graph G.

Theorem 2.2 (Read [20]) *Let G be a graph of order n and size m. Then $P(G,\lambda)$ is a polynomial in λ such that*

(i) $deg(P(G,\lambda)) = n$;

(ii) all the coefficients are integers;

(iii) the leading term is λ^n;

(iv) the constant term is zero;

(v) the coefficient alternate in sign;

(vi) the absolute value of the coefficient of λ^{n-1} is the number of edges of G;

(vii) either $P(G,\lambda) = \lambda^n$ or the sum of the coefficients in $P(G,\lambda)$ is zero.

The following two results for determining $P(G,\lambda)$ are due to Whitney, which can be proved by using the Principle of Inclusion and Exclusion.

Theorem 2.3 (Whitney [28]) *Let G be a graph of order n and size m. Then*

$$P(G,\lambda)=\sum_{k=1}^{n}\Big(\sum_{r=0}^{m}(-1)^r N(k,r)\Big)\lambda^k$$

where $N(k,r)$ denotes the number of spanning subgraphs of G having exactly k components and r edges.

Let G be a graph with an arbitrary bijection $\beta : E(G) \to \{\, 1,2,\ldots,m \,\}$. Let C be any cycle in G and e be an edge in C such that $\beta(e) \geq \beta(x)$ for each edge x in C. Then the path $C-e$ is called a *broken cycle* in G induced by β. Thus we have the following theorem.

Theorem 2.4 (Broken-Cycle Theorem)(Whitney [28]) *Let G be a graph of order n and size m, and let $\beta : E(G) \to \{\, 1,2,\ldots,m \,\}$ be a bijection. Then*

$$P(G,\lambda)=\sum_{i=0}^{n-1}(-1)^i h_i \lambda^{n-i}$$

where h_i is the number of spanning subgraphs of G that have exactly i edges and that contain no broken cycles induced by β.

Let G be a graph of order n. By using Theorems 2.3 and 2.4, we then can derive the coefficient of λ^i, where $n-3 \leq i \leq n$, expressed in terms of the numbers of certain simple subgraphs of G. Let $t_1(G)$, $t_2(G)$ and $t_3(G)$ denote respectively the number of triangles K_3, the number of cycles of order 4 without chords and the number of K_4 in G.

Theorem 2.5 (Farrell [12]) *Let G be a graph of order n and size m. Then in the polynomial $P(G, \lambda)$, the coefficient of*

(i) λ^n *is 1;*

(ii) λ^{n-1} *is* $-m$*;*

(iii) λ^{n-2} *is* $\binom{m}{2} - t_1(G)$*;*

(iv) λ^{n-3} *is* $-\binom{m}{3} + (m-2)t_1(G) + t_2(G) - 2t_3(G)$*.*

2.2 Chromatically Unique Graphs and Chromatically Equivalent Graphs

It can be proved by Lemma 2.1 that for any tree T of order n, $P(T, \lambda) = \lambda(\lambda - 1)^{n-1}$. Thus there exists non-isomorphic graphs which have the same chromatic polynomial. On the other hand, there are graphs like the complete graph K_n and the empty graph O_n such that no other graphs will have the same chromatic polynomial as K_n or O_n. These observations lead to the following definitions.

Let $P(G, \lambda)$ be the chromatic polynomial of a graph G. Two graphs G and H are *chromatically equivalent* or simply $\chi-$*equivalent*, symbolically $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. A graph G is *chromatically unique* or simply $\chi-$*unique* if $G \cong H$ for any graph H such that $H \sim G$. Trivially, the relation $\sim$ is an equivalence relation on the class of graphs. We shall denote by $[G]$ the chromatic equivalence class determined by G under $\sim$; indeed, $[G]$ is the set of all graphs having the same chromatic polynomial $P(G, \lambda)$. Clearly, G is $\chi-$unique if and only if $[G] = \{G\}$ (up to isomorphism).

Recall that $t_1(G)$, $t_2(G)$ and $t_3(G)$ denote respectively the number of triangles K_3, the number of cycles of order 4 without chords and the number of K_4 in G. Let $\chi(G)$ denote the chromatic number of G. Then $\chi(G)$ is the smallest integer λ such that $P(G, \lambda) > 0$. The following lemma can be derived from Theorem 2.5.

Lemma 2.3 *Let G and H be graphs such that $G \sim H$. Then*

(i) G and H have the same order;

(ii) G and H have the same size;

(iii) $t_1(G) = t_1(H)$;

(iv) $t_2(G) - 2t_3(G) = t_2(H) - 2t_3(H)$;

(v) $\chi(G) = \chi(H)$;

(vi) G is connected if and only if H is connected.

Since there are no general methods for constructing families of $\chi-$unique graphs, it is very helpful to know as many as possible necessary conditions for two graphs to be $\chi-$equivalent. Thus, the above lemma is just the necessary conditions for two graphs G and H to be $\chi-$equivalent.

The following result is obvious.

Lemma 2.4 *Let G be a graph of size m. Then $m \geq 1$ if and only if* $\lambda(\lambda - 1)|P(G, \lambda)$.

A *block* of a graph G is a maximal subgraph of G which contains no cut-vertices. Whitehead and Zhao [27] proved the following results.

Theorem 2.6 *Let G be a graph. The multiplicity of the root 1 in $P(G,\lambda)$ is the number of non-trivial blocks of G.*

Corollary 2.1 *Let G be a connected graph of order n. Then G contains a cut-vertex if and only if $(\lambda-1)^2|P(G,\lambda)$.*

We say a graph G is $k-$connected if we need to remove at least k vertices of G in order to get a disconnected graph from G.

Corollary 2.2 *Let G and H be two graphs such that $G \sim H$. Then G is $2-$connected if and only if H is $2-$connected.*

We shall now give some typical examples of $\chi-$unique graphs.

(a) The empty graph O_n of order n is $\chi-$unique and $P(O_n,\lambda) = \lambda^n$.

(b) The complete graph K_n of order n is $\chi-$unique and $P(K_n,\lambda) = \lambda(\lambda-1)\ldots(\lambda-n+1)$.

(c) Let C_n be the cycle of order n, $n \geq 3$. Then $P(C_n,\lambda) = (\lambda-1)^n + (-1)^n(\lambda-1)$. Chao and Whitehead [5] proved that every cycle is $\chi-$unique.

(d) A $\theta-$graph denoted by $\theta(p,q)$, consists of two cycles C_p and C_q with a an edge in common. Then

$$P(\theta(p,q),\lambda) = \frac{P(C_p,\lambda)P(C_q,\lambda)}{\lambda(\lambda-1)}.$$

Chao and Whitehead [5] proved that $\theta(p,q)$ is $\chi-$unique.

2.3 Chromaticity of Bipartite Graphs

In this section, we give a brief survey of the works done on the chromaticity of bipartite graphs. A bipartite graph is a graph whose vertex set can be partitioned into two subsets A and B such that every edge of the graph joins a vertex in A to a vertex in B. It is called a *complete bipartite* graph if every vertex in A is adjacent to every vertex in B. For any two positive integers p and q, let $K(p,q)$ denote the complete bipartite graph with $|A| = p$ and $|B| = q$.

Salzberg et al. [23] characterized bipartite graphs by their chromatic polynomials as follows.

Theorem 2.7 *A graph G is a bipartite graph if and only if $(\lambda - 2) \nmid P(G,\lambda)$.*

By applying Whitney's Broken-Cycle Theorem, Hong [14] gave a different characterization of bipartite graphs.

Theorem 2.8 *Let G be a graph of order n. Let*

$$P(G,\lambda) = \sum_{i=1}^{n-1} (-1)^i h_i \lambda^{n-i}$$

be the chromatic polynomial of G where h_i is the number of spanning subgraphs of G that have i edges and that contain no broken cycles induced by a bijection $\beta : E(G) \to \{\ 1, 2, \ldots, m\ \}$. Then G is a bipartite graph if and only if h_{n-1} is an odd number.

Farrell [12] has given explicit expressions in terms of the number of certain subgraphs of the graph for the first five coefficients of the chromatic polynomial of a graph. As a consequence of his result and Lemma 2.3, the following result is obtained.

Theorem 2.9 (Peng [19], Salzberg et al. [23]) *Let G be a bipartite graph. If H is a graph such that $H \sim G$, then H is also a bipartite graph having the same number of vertices, edges, cycles of order 4 and complete bipartite subgraphs $K(2,3)$ as G.*

A complete bipartite graph $K(p,q)$ has $p+q$ vertices and pq edges. Therefore, according to Theorem 2.9, the following result is obtained.

Theorem 2.10 *Suppose that $H \sim K(p,q)$, then H is isomorphic with the complete bipartite graph $K(p+k, q-k)$ with $(q-p)k - k^2$ edges deleted where $0 \leq k \leq (q-p)/2$.*

In [6], Chao and Novacky proved that the graphs $K(p,p)$ and $K(p,p-1)$ are chromatically unique. In 1978, Chao conjectured (see [23]) that the graph $K(p,q)$ is chromatically unique if $p \geq 2$ and $0 \leq p-q \leq 2$. This was later confirmed by Salzberg et al. in [23]. From Theorems 2.9 and 2.10 and analysis of some particular cases, Salzberg et al. [23] proved more generally that the graph $K(p,q)$ is chromatically unique if $p \geq 2$ and $0 \leq q-p \leq max\{\ 5, \sqrt{2p}\ \}$; and conjectured further that the graph $K(p,q)$ is chromatically unique for all p, q with $p \geq q \geq 2$. Through the study of some extremal properties of $3-$colourings of certain bipartite graphs, Tomescu [25] improved the above result slightly by showing that the graph $K(p,q)$ is chromatically unique if $p \geq 2$ and $0 \leq q-p \leq 2\sqrt{p+1}$. In spite of this, the conjecture due to Salzberg et al. [23] still remains unsettled.

By studying the *simplified adjacency matrix* of a bipartite graph, Teo and Koh [24] eventually settled the conjecture.

For a graph G containing a cycle, the girth $g(G)(= g)$ of G is the length of a shortest cycle in G. Let $\sigma_g(G)$ denote the number of cycles of length $g(G)$ in G. Teo and Koh in [16] established the following result.

Theorem 2.11 *Let G be a 2-connected graph of order n, size m, and girth g. Then*

(i) $\sigma_g(G) \geq \begin{cases} \frac{m}{g}(m-n+1) & \text{if } g \text{ is even,} \\ \frac{n}{g}(m-n+1) & \text{if } g \text{ is odd.} \end{cases}$

(ii) For even g, the equality in (i) holds if and only if every two edges of G are contained in a common shortest cycle C_g.

(iii) For odd g, the equality in (i) holds if and only if every vertex and every edge of G are contained in common shortest cycle C_g.

By means of Theorem 2.11, Teo and Koh [16] showed that the graph $K(p,q)$ is $\chi-$unique for all $p \geq q \geq 2$. This result was also obtained by Dong [8] using an idea similar to but not as general as that given in Theorem 2.11. In [31], Xu by considering the number of induced complete bipartite subgraphs of G, also proved that $K(p,q)$ is $\chi-$unique for all $p \geq q \geq 2$ but unfortunately the proof is found invalid (see [30]).

Theorem 2.12 (Teo and Koh [16, 24])(see also [8], [30] and [31]) *Every complete bipartite graph $K(p,q)$ is $\chi-$unique for all p, q with $p \geq q \geq 2$.*

For integers p,q,s with $p \geq q \geq 2$ and $s \geq 0$, let $\mathcal{K}^{-s}(p,q)$(resp. $\mathcal{K}_2^{-s}(p,q)$) denote the set of connected (resp. 2$-$connected) bipartite graphs which can be obtained from $K(p,q)$ by deleting a set of s edges. Salzberg et al. [23] proved that every graph in $\mathcal{K}^{-1}(p,q)$ is $\chi-$unique if $p \geq 3$ and $0 \leq q-p \leq 1$. Teo and Koh in [24] then proved every graph in $\mathcal{K}^{-1}(p,q)$ is $\chi-$unique if $p \geq q \geq 3$.

The case $s \geq 2$ has been studied by Giudici and Lima de Sa [13], Peng [19], Borowiecki and Drgas-Burchardt [4]. Their typical results are of the following:

(i) If $2 \leq s \leq 4$ and $p-q$ is *small enough*, then each graph in $\mathcal{K}^{-s}(p,q)$ is $\chi-$unique;

(ii) If $G \in \mathcal{K}^{-s}(p,q)$, where $0 \leq p-q \leq 1$, such that the set of s edges deleted forms a matching, then G is $\chi-$unique.

Chen [7] showed that if $G \in \mathcal{K}^{-s}(p,q)$, where $3 \leq s \leq p-q$ and

$$q \geq \max \left\{ \frac{1}{2}(p-q)(s-1)+\frac{3}{2},\ \frac{8}{27}(p-q)^2+\frac{1}{3}(p-q)+5s+6 \right\},$$

and the set of s edges deleted forms a matching or a star (among other possibilities), then G is $\chi-$unique.

Recall that A and B are the partite sets of $K(p,q)$ with $|A|=p$ and $|B|=q$. For $1 \leq s \leq p-1$, let $H_1(p,q,s)$ denote the graph obtained from $K(p,q)$ by deleting a set of s edges that induces a star with the center in B; let $H_2(p,q,s)$ denote the graph obtained from $K(p,q)$ by deleting a set of s edges that induces a star with the center in A; and for $1 \leq s \leq q$, let $H_3(p,q,s)$ denote the graph obtained from $K(p,q)$ by deleting a set of s edges that forms a matching of $K(p,q)$. The following result was obtained in [11].

Theorem 2.13 *For positive integers p,q,s with $p \geq q \geq 3$,*

(i) when $1 \leq s \leq q-2$, $H_i(p,q,s)$ is $\chi-$unique for $i = 1, 2$;

(ii) when $q-1 \leq s \leq p-2$, $H_1(p,q,s)$ is $\chi-$unique; and

(iii) when $0 \leq s \leq q-1$, $H_3(p,q,s)$ is $\chi-$unique.

For a bipartite graph $G = (A, B; E)$ with bipartition A and B and edge set E, let $G' = (A', B'; E')$ be the bipartite graph induced by the edge set $E' = \{xy \mid xy \notin E,\ x \in A, y \in B\ \}$, where $A' \subseteq A$ and $B' \subseteq B$. We write $G' = K(p,q) - G$, where $p = |A|$ and $q = |B|$. Let $\triangle(G')$ denote the maximum degree of G'.

Partition $\mathcal{K}^{-s}(p,q)$ into the following subsets:

$$\mathcal{D}_i(p,q,s) = \left\{\ G \in \mathcal{K}^{-s}(p,q)\ \middle|\ \triangle(G') = i\ \right\}, \quad i = 1, 2, \ldots, s.$$

Dong et al. [9] established the following result.

Theorem 2.14 *(1) For any $G \in \mathcal{K}_2^{-s}(p,q)$ with $p \geq q \geq s+1 \geq 6$, if $\triangle(G') = s-1$, then G is $\chi-$unique.*

(2) For any $G \in \mathcal{K}_2^{-s}(p,q)$ with $p \geq q \geq s+1 \geq 8$, if $\triangle(G') = s-2$, then G is $\chi-$unique.

Motivated by the work done by Dong et al. [9] above, we shall investigate the chromatic uniqueness of any $G \in \mathcal{K}_2^{-s}(p,q)$ when $\triangle(G') \geq s-3$.

For a graph G and a positive integer k, a partition $\{\ A_1, A_2, \ldots, A_k\ \}$ of $V(G)$ is called a *k-independent partition* in G if each A_i is a non-empty independent set of G. Let $\alpha(G,k)$ denote the number of $k-$independent partitions in G. For any bipartite graph $G=(A,B;E)$ with bipartition A and B and edge set E, define

$$\alpha'(G,3)=\alpha(G,3)-(2^{|A|-1}+2^{|B|-1}-2).$$

In [11], the authors found the following sharp bounds for $\alpha'(G,3)$.

Theorem 2.15 *For $G\in\mathcal{K}_2^{-s}(p,q)$ with $p\geq q\geq 3$ and $0\leq s\leq q-1$,*

$$s\leq\alpha'(G,3)\leq 2^s-1,$$

where $\alpha'(G,3)=s$ iff $\Delta(G')=1$, and $\alpha'(G,3)=2^s-1$ iff $\Delta(G')=s$.

For $t=0,1,2,\ldots,$ let $\mathcal{B}(p,q,s,t)$ denote the set of graphs $G\in\mathcal{K}^{-s}(p,q)$ with $\alpha'(G,3)=s+t$. Thus, $\mathcal{K}^{-s}(p,q)$ is partitioned into the following subsets:

$$\mathcal{B}(p,q,s,0),\quad \mathcal{B}(p,q,s,1),\quad \ldots,\mathcal{B}(p,q,s,2^s-s-1).$$

Assume that $\mathcal{B}(p,q,s,t)=\emptyset$ for $t>2^s-s-1$.

Dong et al. [10] then obtained the following two results.

Theorem 2.16 *Let p,q and s be integers with $p\geq q\geq 3$ and $0\leq s\leq q-1$. For every $G\in\mathcal{B}(p,q,s,t)$ for $1\leq t\leq 4$, if G is 2-connected, then G is $\chi-$unique.*

Theorem 2.17 *For any $G\in\mathcal{K}_2^{-s}(p,q)$ with $p\geq q\geq 3$ and $0\leq s\leq min\{\ q-1,4\ \}$, G is $\chi-$unique.*

Motivated by Theorems 2.16 and 2.17 above, we shall examine the chromatic uniqueness of $2-$connected graphs $G\in\mathcal{B}(p,q,s,t)$ for $t\geq 5$ and the chromatic uniqueness of any $G\in\mathcal{K}_2^{-s}(p,q)$ if $5\leq s\leq q-1$.

CHAPTER 3

CHROMATIC UNIQUENESS OF COMPLETE BIPARTITE GRAPHS WITH CERTAIN EDGES DELETED

3.1 Introduction

Recall that Dong et al. [9] have proved the following results.

(1) For any $G \in \mathcal{K}_2^{-s}(p,q)$ with $p \geq q \geq s+1 \geq 6$, if $\triangle(G') = s-1$, then G is $\chi-$unique.

(2) For any $G \in \mathcal{K}_2^{-s}(p,q)$ with $p \geq q \geq s+1 \geq 8$, if $\triangle(G') = s-2$, then G is $\chi-$unique.

In this chapter, we shall study the chromatic uniqueness of every $G \in \mathcal{K}_2^{-s}(p,q)$, where $9 \leq s \leq q-1$ and $\triangle(G') = s-3$; and the main result will be presented in Section 3.3. In Section 3.2, we give some known results and notations which will be used to prove our main result. In Section 3.4, we give the detailed proof of this result by using the same approach introduced by Dong et al. [9]. The chromatic uniqueness of any $G \in \mathcal{K}_2^{-s}(p,q)$, where $11 \leq s \leq q-1$ and $\triangle(G') = s-4$, will be discussed in the next chapter.

3.2 Preliminary Results and Notations

For any graph G of order n, we have (see [21]):

$$P(G,\lambda) = \sum_{k=1}^{n} \alpha(G,k)\lambda(\lambda-1)\cdots(\lambda-k+1).$$

Thus, we have

Lemma 3.1 *If $G \sim H$, then $\alpha(G,k) = \alpha(H,k)$ for $k = 1, 2, \ldots$.*

For a set $\mathcal{G}$ of graphs, if $[G] \subseteq \mathcal{G}$ for every $G \in \mathcal{G}$, then $\mathcal{G}$ is said to be $\chi-$*closed*. For two sets $\mathcal{G}_1$ and $\mathcal{G}_2$ of graphs, if $P(G_1,\lambda) \neq P(G_2,\lambda)$ for every $G_1 \in \mathcal{G}_1$ and $G_2 \in \mathcal{G}_2$, then $\mathcal{G}_1$ and $\mathcal{G}_2$ are said to be *chromatically disjoint*, or simply $\chi-$*disjoint*.

Recall that $\mathcal{K}^{-s}(p,q)$ can be partitioned into the following subsets:

$$\mathcal{D}_i(p,q,s) = \left\{ G \in \mathcal{K}^{-s}(p,q) \;\middle|\; \triangle(G') = i \right\}, \quad i = 1, 2, \ldots, s.$$

The above partition then will be used to study the chromaticity of bipartite graphs. Thus we need the following known result which is obtained in [9].

Theorem 3.1 *Let $p \geq q \geq 3$ and $1 \leq s \leq q-1$.*

(i) *$\mathcal{D}_1(p,q,s)$ is $\chi-$closed.*

(ii) *$\cup_{2 \leq i \leq (s+3)/2} \mathcal{D}_i(p,q,s)$ is $\chi-$closed for $s \geq 2$.*

(iii) *$\mathcal{D}_i(p,q,s)$ is $\chi-$closed for each i with $\lceil (s+3)/2 \rceil \leq i \leq min\{s, q-2\}$.*

(iv) *$\mathcal{D}_{q-1}(p,q,s) \cap \mathcal{K}_2^{-s}(p,q)$ is $\chi-$closed for $s = q-1$.*

Recall that for any bipartite graph $G = (A, B; E)$ with bipartition A and B and edge set E, we have

$$\alpha'(G, 3) = \alpha(G, 3) - (2^{|A|-1} + 2^{|B|-1} - 2). \tag{3.1}$$

For a bipartite graph $G = (A, B; E)$, let $\mathcal{I}(G)$ be the set of independent sets in G and let

$$\Omega(G) = \{\ Q \in \mathcal{I}(G) \mid Q \cap A \neq \emptyset, Q \cap B \neq \emptyset\ \}.$$

The following result is then obtained.

Lemma 3.2 (Dong et al. [11]) *For $G \in \mathcal{K}^{-s}(p, q)$,*

$$\alpha'(G, 3) = |\Omega(G)| \geq 2^{\triangle(G')} + s - 1 - \triangle(G').$$

For a bipartite graph $G = (A, B; E)$, the number of $4-$independent partitions $\{\ A_1, A_2, A_3, A_4\ \}$ in G with $A_i \subseteq A$ or $A_i \subseteq B$ for all $i = 1, 2, 3, 4$ is

$$\begin{aligned} &(2^{|A|-1} - 1)(2^{|B|-1} - 1) + \frac{1}{3!}(3^{|A|} - 3 \cdot 2^{|A|} + 3) + \frac{1}{3!}(3^{|B|} - 3 \cdot 2^{|B|} + 3) \\ &= (2^{|A|-1} - 2)(2^{|B|-1} - 2) + \frac{1}{2}(3^{|A|} + 3^{|B|}) - 2. \end{aligned}$$

Define

$$\alpha'(G, 4) = \alpha(G, 4) - \left\{\ (2^{|A|-1} - 2)(2^{|B|-1} - 2) + \frac{1}{2}(3^{|A|} + 3^{|B|}) - 2\ \right\}.$$

Observe that for $G, H \in \mathcal{K}^{-s}(p, q)$,

$$\alpha(G, 4) = \alpha(H, 4) \quad \text{if and only if} \quad \alpha'(G, 4) = \alpha'(H, 4).$$

The following lemmas will be used to prove our main result.

Lemma 3.3 (Dong et al. [10]) *For $G = (A, B; E) \in \mathcal{K}^{-s}(p, q)$ with $|A| = p$ and $|B| = q$,*

$$\alpha'(G, 4) = \sum_{Q \in \Omega(G)} (2^{p-1-|Q\cap A|} + 2^{q-1-|Q\cap B|} - 2) + |\ \{\ \{Q_1, Q_2\} \mid Q_1, Q_2 \in \Omega(G),\ Q_1 \cap Q_2 = \emptyset\ \}\ |.$$

Lemma 3.4 (Dong et al. [10]) *For a bipartite graph $G = (A, B; E)$, if uvw is a path in G' with $d_{G'}(u) = 1$ and $d_{G'}(v) = 2$, then for any $k \geq 2$,*

$$\alpha(G, k) = \alpha(G + uv, k) + \alpha(G - \{u, v\}, k - 1) + \alpha(G - \{u, v, w\}, k - 1).$$

The following lemma is an extension of Lemma 3.4 which is also useful to prove certain case in our main result.

Lemma 3.5 *For a bipartite graph $G = (A, B; E)$, if uvw, uvy and wvy are three paths in G' with $d_{G'}(u) = 1$ and $d_{G'}(v) = 3$, then for any $k \geq 2$,*

$$\begin{aligned}\alpha(G, k) &= \alpha(G + uv, k) + \alpha(G - \{u, v\}, k - 1) + \alpha(G - \{u, v, w\}, k - 1) + \\ &\quad \alpha(G - \{u, v, y\}, k - 1) + \alpha(G - \{u, v, w, y\}, k - 1).\end{aligned}$$

Proof Since $P(G, \lambda) = P(G + uv, \lambda) + P(G \cdot uv, \lambda)$, we have

$$\alpha(G, k) = \alpha(G + uv, k) + \alpha(G \cdot uv, k).$$

Let x be the vertex in $G \cdot uv$ produced by identifying u and v, and z the vertex in $G{\cdot}uv{\cdot}xw$ produced by identifying x and w. Notice that x is adjacent to all vertices in $V(G \cdot uv) - \{x, w, y\}$ and z is adjacent to all vertices in $V(G(uv \cdot xw) - \{z, y\}$. Thus

$$G \cdot uv + xw + xy = K_1 + (G - \{u, v\}),$$

$$\begin{aligned}(G\cdot uv + xw)\cdot xy &= K_1 + (G - \{u,v,y\}),\\ G\cdot uv\cdot xw + zy &= K_1 + (G - \{u,v,w\}) \quad \text{and}\\ G\cdot uv\cdot xw\cdot zy &= K_1 + (G - \{u,v,w,y\}).\end{aligned}$$

We also observe that for any graph H, $\alpha(K_1+H,k) = \alpha(H,k-1)$, since $P(K_1+H,\lambda) = \lambda P(H,\lambda-1)$. Hence

$$\begin{aligned}\alpha(G\cdot uv,k) &= \alpha(G\cdot uv + xw,k) + \alpha(G\cdot uv\cdot xw,k)\\ &= \alpha(G\cdot uv + xw + xy,k) + \alpha((G\cdot uv + xw)\cdot xy,k) +\\ &\qquad \alpha(G\cdot uv\cdot xw + zy,k) + \alpha(G\cdot uv\cdot xw\cdot zy,k)\\ &= \alpha(K_1 + (G-\{u,v\}),k) + \alpha(K_1 + (G-\{u,v,y\}),k) +\\ &\qquad \alpha(K_1 + (G-\{u,v,w\}),k) + \alpha(K_1 + (G-\{u,v,w,y\}),k)\\ &= \alpha(G-\{u,v\},k-1) + \alpha(G-\{u,v,y\},k-1) +\\ &\qquad \alpha(G-\{u,v,w\},k-1) + \alpha(G-\{u,v,w,y\},k-1).\end{aligned}$$

The lemma is then obtained. $\square$

3.3 Main Result

In [11], Dong et al. proved that every 2−connected graph in $\mathcal{D}_s(p,q,s)$ is χ−unique. Then, Dong et al. in [9] also proved that G is χ−unique for every $G \in \mathcal{D}_{s-1}(p,q,s)$, where $s \geq 5$, and that G is χ−unique for every $G \in \mathcal{D}_{s-2}(p,q,s)$, where $s \geq 7$. In this section, we shall study the chromaticity of all the graphs in $\mathcal{D}_{s-3}(p,q,s)$, where $s \geq 9$. We first have the following lemma which can be easily proved by construction.

Lemma 3.6 *For any G in $\mathcal{D}_{s-3}(p,q,s)$, where $s \geq 9$, G' is one of the 21 graphs in Figure 3.1 and $G' = K(p,q) - G$.*

By Lemma 3.6 above, $\mathcal{D}_{s-3}(p,q,s)$ contains 48 graphs, which are named as V_1, V_2, ..., V_{48} (see Table 3.1). Note that Figure 3.1 and Table 3.1 are shown at the end of this chapter.

We now calculate the values of $\alpha'(V_i,3)$ for these 48 graphs by using Lemma 3.2 and these values are in column three of Table 3.1. Thus we have the following observations.

(i) $\alpha'(V_i,3) = 2^{s-3} + 13$, for i=1,2;

(ii) $\alpha'(V_i,3) = 2^{s-3} + 12$, for i=3,4;

(iii) $\alpha'(V_i,3) = 2^{s-3} + 9$, for i=5,6,7,8;

(iv) $\alpha'(V_i,3) = 2^{s-3} + 7$, for i=9,10,...,16;

(v) $\alpha'(V_i,3) = 2^{s-3} + 6$, for i=17,18,...,24;

(vi) $\alpha'(V_i,3) = 2^{s-3} + 5$, for i=25,26,...,30;

(vii) $\alpha'(V_i,3) = 2^{s-3} + 4$, for i=31,32,...,40;

(viii) $\alpha'(V_i,3) = 2^{s-3} + 3$, for i=41,42,...,46;

(ix) $\alpha'(V_i,3) = 2^{s-3} + 2$, for i=47,48.

We then group these graphs according to their $\alpha'(V_i,3)$. Hence we have the following classification of the graphs.

$$\mathcal{T}_1 = \{ V_1, V_2 \}$$

$$\mathcal{T}_2 = \{ V_3, V_4 \}$$

$$\mathcal{T}_3 = \{ V_5, V_6, V_7, V_8 \}$$

$$\mathcal{T}_4 = \{ V_9, V_{10}, V_{11}, V_{12}, V_{13}, V_{14}, V_{15}, V_{16} \}$$

$$\mathcal{T}_5 = \{ V_{17}, V_{18}, V_{19}, V_{20}, V_{21}, V_{22}, V_{23}, V_{24} \}$$

$$\mathcal{T}_6 = \{ V_{25}, V_{26}, V_{27}, V_{28}, V_{29}, V_{30} \}$$

$$\mathcal{T}_7 = \{ V_{31}, V_{32}, V_{33}, V_{34}, V_{35}, V_{36}, V_{37}, V_{38}, V_{39}, V_{40} \}$$

$$\mathcal{T}_8 = \{ V_{41}, V_{42}, V_{43}, V_{44}, V_{45}, V_{46} \}$$

$$\mathcal{T}_9 = \{ V_{47}, V_{48} \}.$$

We also calculate the values of $\alpha'(V_i, 4)$ by using Lemma 3.3 and we list them in column four of Table 3.1.

We now present our main result in the following theorem.

Theorem 3.2 *For any $G \in \mathcal{K}_2^{-s}(p,q)$, with $p \geq q \geq s+1 \geq 10$, if $\triangle(G') = s-3$, then G is $\chi-$unique.*

Proof Since $s \geq 9$, then $(s+3)/2 \leq s-3$. Thus by Theorem 3.1, $\mathcal{D}_{s-3}(p,q,s)$ is $\chi-$closed. Observe that for any i, j with $1 \leq i < j \leq 9$, $\alpha'(V_{i_1}, 3) > \alpha'(V_{j_1}, 3)$ if $V_{i_1} \in \mathcal{T}_i$ and $V_{j_1} \in \mathcal{T}_j$. Thus by Lemma 3.1 and Equation (3.1), $\mathcal{T}_i$ and $\mathcal{T}_j$ $(1 \leq i < j \leq 9)$ are $\chi-$disjoint; and since $\mathcal{D}_{s-3}(p,q,s)$ is $\chi-$closed, each $\mathcal{T}_i$ $(1 \leq i \leq 9)$ is $\chi-$closed. Hence, for each i, to show that all graphs in $\mathcal{T}_i$ are $\chi-$unique, it suffices to show that for any two graphs, V_{i_1}, $V_{i_2} \in \mathcal{T}_i$, if $V_{i_1} \ncong V_{i_2}$, then either $\alpha'(V_{i_1}, 4) \neq \alpha'(V_{i_2}, 4)$ or $\alpha(V_{i_1}, 5) \neq \alpha(V_{i_2}, 5)$. Generally, we use a method similar to that of Dong et al. in [9], where we shall compare every two graphs in each $\mathcal{T}_i$ $(1 \leq i \leq 9)$. The detailed proof will be presented in Section 3.4. $\square$