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Minimal sets and chaos in planar piecewise smooth vector fields

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Abstract. Some aspects concerning chaos and minimal sets in discontinuous dynamical systems are addressed. The orientability dependence of trajectories sliding trough some variety is exploited and new phenomena emerging from this situation are highlighted. In particular, although chaotic flows and nontrivial minimal sets are not allowed for smooth vector fields in the plane, the existence of such objects for some classes of vector fields is verified. A characterization of chaotic flows in terms of orientable minimal sets is also provided. The main feature of the dynamical systems under study is related to the non uniqueness of trajectories in some zero measure region as well as the orientation of orbits reaching such region.

Keywords: vector fields, piecewise smooth vector fields, chaos, minimal sets.

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1 Introduction

Dynamical systems have become one of the most promising areas of mathematics since its strong development started by Poincaré (see [22]). The main reason for this is due to the fact that several applied sciences from economy and biology to engineering and statistical mechanics benefited of dynamical systems' tools. In the last case, for instance, ergodic theory plays an important role, we mention for short Poincaré recurrence theorem as well as the concepts of chaos and entropy. In fact, while a mathematical object models a concrete phenomena, such modeling is in fact no more than an theoretical approximation of an real event and invariably ignores some important features of it. Is therefore mandatory to search for news methods and tools that are not only more realistic but also feasible in theory.

In this direction have emerged within the theory of dynamical system a set of methods which is now widely known by *piecewise smooth vector fields* (PSVFs, for short). For a formal introduction to PSVFs see [14]. The main advantage of PSVFs over the classical theory of

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dynamical system is the fact that they provide a more accurate approach by allowing non smoothness or discontinuities of the vector field defining the system. Indeed, several problems involving impact, friction or abrupt changes of certain regime can be modeled or at least approximated by PSVFs, in the sense that the transition from one kind of behavior to another one can be idealized as a discrete and instantaneous transition. A non exhaustive list of applications of such theory involves the relay systems, the control theory, the stick-slip process, the dynamics of a bouncing ball and the antilock braking system (ABS), see those and other applications in [2–4,11,12,15,16,18,19] and references therein.

The main aspect of PSVFs concerns non uniqueness of solutions on some zero measure variety and consequent amalgamation of orbits under such region, which split the phase portrait into two or more pieces. That leads to the behavior known as *sliding motion*, characterized by the collapse of distinct trajectories which combine to slide on the common frontier of each dynamic. Under this scenario some behavior strange to the classical theory of dynamical systems may occur, so the study of new objects and the validation of known results is mandatory when one investigate PSVFs. For instance, we mention the Peixoto's Theorem (see [21]), the Closing Lemma (see [8]) and the Poincaré–Bendixson Theorem (see [5]), which posses analogues version in the context of PSVFs (see also [10,13,17]). We also mention that the study of PSVFs may take into account orientability of trajectories. This is because the collision of any particular trajectory to the boundary region and subsequent sliding occurs in different ways when considering forward or backward time.

This paper is addressed to some particular features of PSVFs. Indeed, we take into account aspects of chaotic PSVFs and how this concept relates to minimal sets. To do this, the definitions of both chaos e minimal sets are refined to consider the role of orientation and we provide a definitive characterization of chaotic PSVFs involving such objects.

Let $\text{med}(W)$ be the Lebesgue measure of a set W. The first main result of the paper states that *a PSVF Z is chaotic on the set W if, and only if, Z is positive chaotic and negative chaotic on W.* The second main result of the paper states that *if Z is chaotic on the set W and* $\text{med}(W) > 0$ *then W is positive minimal and negative minimal.* In order to prove these results we present and prove some other results which are indispensable to main results but also important on their own. For instance, we provide a sufficient condition for a Lebesgue measure subset of $\mathbb{R}^2$ to be chaotic, which elucidates the richness of PSVFs. Other considerations and results are presented timely throughout the text.

The paper is organized as follows: in Section 2 we provide the first statements around the subject of PSVFs, particularly considering minimal sets for PSVFs and their chaotic behavior. In Section 3 we state and prove the main results of the paper and some consequences of them. In Section 4 we provide a discussion around the results of the paper and present some examples and counterexamples contextualizing the results.

2 Preliminaries

2.1 Piecewise smooth vector fields

Consider two smooth vector fields X and Y and a codimension one manifold $\Sigma \subset \mathbb{R}^2$ that separates the plane in two regions $\Sigma+$ and $\Sigma-$. A PSVF Z is a vector field defined in $\mathbb{R}^2$ and given by

$$Z(x,y) = \begin{cases} X(x,y), & \text{for } (x,y) \in \Sigma^+, \\ Y(x,y), & \text{for } (x,y) \in \Sigma^-. \end{cases} \tag{2.1}$$

Since Σ is a codimension one manifold, there exists a function f such that $\Sigma = f^{-1}(0)$ and 0 is a regular value of f. As consequence, $\Sigma^+ = \{q \in \mathbb{R}^2 \,|\, f(q) \geq 0\}$ and $\Sigma^- = \{q \in \mathbb{R}^2 \,|\, f(q) \leq 0\}$. The trajectories of Z are solutions of $\dot{q} = Z(q)$ and we accept it to be multi-valued at points of Σ. We will call Ω the set of all PSVFs defined in $\mathbb{R}^2$. The basic results of differential equations in this context were stated by Filippov in [14], that we summarize next. Indeed, consider the Lie derivatives $X.f(p) = \langle \nabla f(p), X(p) \rangle$ and $X^i.f(p) = \langle \nabla X^{i-1}.f(p), X(p) \rangle$, $i \geq 2$, where $\langle .,. \rangle$ is the usual inner product in $\mathbb{R}^2$. We distinguish the following regions on the discontinuity set Σ:

(i) $\Sigma^c \subseteq \Sigma$ is the *sewing region* if $(X.f)(Y.f) > 0$ on Σ^c. Moreover, when $X.f(p) > 0$ and $Y.f(p) > 0$, we say that $p \in \Sigma^{c+}$ and when $X.f(p) < 0$ and $Y.f(p) < 0$, we say that $p \in \Sigma^{c-}$.

(ii) $\Sigma^e \subseteq \Sigma$ is the *escaping region* if $(X.f) > 0$ and $(Y.f) < 0$ on Σ^e.

(iii) $\Sigma^s \subseteq \Sigma$ is the *sliding region* if $(X.f) < 0$ and $(Y.f) > 0$ on Σ^s.

The *sliding vector field* associated to $Z \in \Omega$ is the vector field Z^s tangent to Σ^s and defined at $q \in \Sigma^s$ by $Z^s(q) = m - q$ with m being the point of the segment joining $q + X(q)$ and $q + Y(q)$ such that $m - q$ is tangent to Σ^s. It is clear that if $q \in \Sigma^s$ then $q \in \Sigma^e$ for $(-Z)$ and we can define the *escaping vector field* Z^e on Σ^e associated to Z by $Z^e = -(-Z)^s$. We will use the notation Z^Σ to both, Z^s and Z^e.

We say that $q \in \Sigma$ is a Σ*-regular point* if it is a sewing point or a regular point of the Filippov vector field. Lastly, any point $q \in \Sigma^p$ is called a *pseudo-equilibrium of* Z and it is characterized by $Z^\Sigma(q) = 0$. Any $q \in \Sigma^t$ is called a *tangential singularity* (or also *tangency point*) and it is characterized by $(X.f(q))(Y.f(q)) = 0$. If there exist an orbit of the vector field $X|_{\Sigma^+}$ (respec. $Y|_{\Sigma^-}$) reaching $q \in \Sigma^t$ in a finite time, then such tangency is called a *visible tangency* for X (resp. Y); otherwise we call q an *invisible tangency* for X (resp. Y).

We may also distinguish a particular tangential singularity called *two-fold*, which is a common tangency q of both X and Y (that is, $X.f(q) = Y.f(q) = 0$) satisfying $X^2.f(q)), Y^2.f(q)) \neq 0$. A two-fold is called visible if it is a visible tangency for X and Y. A visible two fold singularity is called a *singular tangency point* and all other $p \in \Sigma^t$ is called a *regular tangency point*.

Definition 2.1. The **local trajectory (orbit)** $\phi_Z(t,p)$ of a PSVF given by (2.1) through $p \in \mathbb{R}^2$ is defined as follows:

(i) For $p \in \Sigma^+ \backslash \Sigma$ and $p \in \Sigma^- \backslash \Sigma$ the trajectory is given by $\phi_Z(t,p) = \phi_X(t,p)$ and $\phi_Z(t,p) = \phi_Y(t,p)$ respectively, where $t \in I$: the maximal interval of existence of the corresponding trajectory before it hits Σ.

(ii) For $p \in \Sigma^{c+}$ and taking the origin of time at p, the trajectory is defined as $\phi_Z(t,p) = \phi_Y(t,p)$ for $t \in I \cap \{t \leq 0\}$ and $\phi_Z(t,p) = \phi_X(t,p)$ for $t \in I \cap \{t \geq 0\}$. For the case $p \in \Sigma^{c-}$ the definition is the same reversing time. Again, I is the maximal interval of existence of the corresponding trajectory before it hits Σ again.

(iii) For $p \in \Sigma^e$ and taking the origin of time at p, the trajectory is defined as $\phi_Z(t,p) = \phi_{Z^\Sigma}(t,p)$ for $t \in I \cap \{t \leq 0\}$ and $\phi_Z(t,p)$ is either $\phi_X(t,p)$ or $\phi_Y(t,p)$ or $\phi_{Z^\Sigma}(t,p)$ for $t \in I \cap \{t \geq 0\}$. For $p \in \Sigma^s$ the definition is the same reversing time. Here, I is the maximal interval of existence of the corresponding trajectory of $\phi_X(t,p)$ or $\phi_Y(t,p)$ before it hits Σ again or $\phi_{Z^\Sigma}(t,p)$ before it leaves Σ.

(iv) For p a regular tangency point and taking the origin of time at p, the trajectory is defined as $\phi_Z(t,p) = \phi_1(t,p)$ for $t \in I \cap \{t \leq 0\}$ and $\phi_Z(t,p) = \phi_2(t,p)$ for $t \in I \cap \{t \geq 0\}$, where each ϕ_1, ϕ_2 is either ϕ_X or ϕ_Y or ϕ_{Z^T}. Here, I is the maximal interval of existence of the corresponding trajectory of $\phi_X(t,p)$ or $\phi_Y(t,p)$ before it hits Σ again or $\phi_{Z^\Sigma}(t,p)$ before it leaves Σ.

(v) For p a singular tangency point, $\phi_Z(t,p) = p$ for all $t \in \mathbb{R}$.

Definition 2.2. Let ϕ_Z^1 and ϕ_Z^2 two distinct local trajectories. Suppose that there exists a common point $q \in \phi_Z^1 \cap \phi_Z^2$. We say that $\phi_Z^1 \cup \phi_Z^2$ **preserves orientation** if there exists an interval I, with $0 \in I$, such that: (i) $\phi_Z^1(0,q) = \phi_Z^2(0,q)$, (ii) $\phi_Z^1(t,.)$ is well defined for $t \in I \cap \{t \leq 0\}$ and (iii) $\phi_Z^2(t,.)$ is well defined for $t \in I \cap \{t \geq 0\}$.

Remark 2.3. Note that the point q of the previous definition is such that $q \in \Sigma$. In fact, it is enough to observe that there is uniqueness of trajectories in points belonging to $\mathbb{R}^2 \backslash \Sigma$.

Definition 2.4. A **global trajectory (orbit)** $\Gamma_Z(t,p_0)$ of $Z \in \Omega$ passing through p_0 when $t = 0$, is a union $\Gamma_Z(t,p_0) = \cup_{i \in \Theta} \{\sigma_i(t,p_i); t_i \leq t \leq t_{i+1}\}$ of preserving-orientation local trajectories $\sigma_i(t,p_i)$ satisfying $\sigma_i(t_i,p_i) = p_i \in \Sigma$ and $\sigma_i(t_{i+1},p_i) = p_{i+1} \in \Sigma$, here $\Theta \subset \mathbb{Z}$. A **maximal trajectory** $\Gamma_Z(t,p_0)$ is a maximal trajectory that cannot be extended to any others global trajectories by joining local ones, that is, if $\widetilde{\Gamma}_Z$ is a global trajectory containing Γ_Z then $\widetilde{\Gamma}_Z = \Gamma_Z$. In this case, we call $I = (\tau^-(p_0), \tau^+(p_0))$ the **maximal interval of the solution** Γ_Z. A maximal trajectory is a **positive** (respectively, **negative**) **maximal trajectory** if we restrict the previous definition to $t \geq 0$ (resp. $t \leq 0$).

Definition 2.5. A maximal trajectory $\Gamma_Z(t,p_0)$ has a **positive** (respectively, **negative**) **periodic trajectory** passing through p_0 if there exists $T_+ > 0$ (respectively, $T_- > 0$) such that $\phi_Z(t + k\,T_+, p_0) = \phi_Z(t,p_0)$ for all integer $k > 0$ (respectively, $\phi_Z(t + k\,T_-, p_0) = \phi_Z(t,p_0)$ for all integer $k < 0$). A maximal trajectory $\Gamma_Z(t,p_0)$ has a **periodic trajectory** passing through p_0 if it has coincident positive and negative periodic trajectories passing through p_0 in such a way that $T_+ = T_-$.

Definition 2.6. Consider $Z = (X,Y) \in \Omega$. A closed (connected) union of trajectories Δ of Z is a:

(i) **pseudo-cycle** if $\Delta \cap \Sigma \neq \emptyset$ and it does not contain neither equilibrium nor pseudo-equilibrium.

(ii) **pseudo-graph** if $\Delta \cap \Sigma \neq \emptyset$ and it is a union of equilibria, pseudo equilibria and orbit-arcs of Z joining these points.

2.2 Minimal sets and chaotic PSVFs

One of the most important facts concerning PSVFs is the orientation of its trajectories. Indeed, it is very important, for instance, for the concept of invariance or defining the flow associated to the Filippov vector field. In the smooth theory of vector fields this distinction does not play an important role since we have uniqueness of trajectories. In this direction, we should verify if such distinction is also necessary when defining minimal sets and chaotic PSVFs. Indeed, these concepts do not play the same role by considering positive and negative times. As far as the authors know, the role of orientability under this context have not be treated in literature about PSVFs, although the concept of chaos and minimality have been discussed before, for instance, in [5], [6] and [10]. We start doing some adaptations to the definitions of invariance and minimality.

Definition 2.7. A set $A \subset \mathbb{R}^2$ is **positive invariant** (respectively, **negative invariant**) if for each $p \in A$ and all positive maximal trajectory $\Gamma_Z^+(t,p)$ (respectively, negative maximal trajectory $\Gamma_Z^-(t,p)$) passing through p it holds $\Gamma_Z^+(t,p) \subset A$ (respectively, $\Gamma_Z^-(t,p) \subset A$). A set $A \subset \mathbb{R}^2$ is **invariant** for Z if it is positive and negative invariant.

Definition 2.8. Consider $Z \in \Omega$. A non-empty set $M \subset \mathbb{R}^2$ is **minimal** (respectively, either **positive minimal** or **negative minimal**) for Z if it is compact, invariant (respectively, either positive invariant or negative invariant) for Z and does not contain proper compact invariant (respectively, either does not contain proper compact positive invariant or proper compact negative invariant) subsets.

Next we present the definitions concerning chaotic PSVFs. As commented before, we need to distinguish between forward and backward time or assuming both possibilities. The notion of chaos we take into account is that based on Devaney. So, the first aspect to be considered is related to topological transitivity.

Definition 2.9. System (2.1) is **topologically transitive** on an invariant set W if for every pair of nonempty, open sets U and V in W, there exist $q^+, q^- \in U$, $\Gamma_Z^+(.,q^+), \Gamma_Z^-(.,q^-)$ maximal trajectories and $t_0^+ > 0 > t_0^-$ such that $\Gamma_Z^+(t_0^+,q^+)$ and $\Gamma_Z^-(t_0^-,q^-) \in V$.

Definition 2.10. System (2.1) is **topologically positive transitive** (respectively, **topologically negative transitive**) on a positive invariant (respectively, negative invariant) set W if for every pair of nonempty, open sets U and V in W, there exist $q \in U$, $\Gamma_Z^+(t,q)$ a positive (respectively, $\Gamma_Z^-(t,q)$ a negative) maximal trajectory and $t_0 > 0$ (resp., $t_0 < 0$) such that $\Gamma_Z^+(t_0,q) \in V$ (resp., $\Gamma_Z^-(t_0,q) \in V$).

Remark 2.11. A direct consequence of the two previous definitions is that:

Z is topologically transitive on W if, and only if, Z is simultaneously topologically positive transitive and topologically negative transitive on W.

Analogously to the definition of topologically transitive systems, the definition of sensitive dependence for PSVFs is inspired in the classical Devaney concept of chaos.

Definition 2.12. System (2.1) exhibits **sensitive dependence** on a compact invariant set W if there is a fixed $r > 0$ satisfying $r < \operatorname{diam}(W)$ such that for each $x \in W$ and $\varepsilon > 0$ there exist $y^+, y^- \in B_\varepsilon(x) \cap W$ and maximal trajectories Γ_x^+, Γ_x^-, $\Gamma_{y^+}^+$ and $\Gamma_{y^-}^-$ passing through x, y^+ and y^-, respectively, satisfying

$$d_H(\Gamma_x^+(t), \Gamma_{y^+}^+(t)) = \sup_{a \in \Gamma_x^+(t), b \in \Gamma_{y^+}^+(t)} d(a,b) > r,$$

$$d_H(\Gamma_x^-(t), \Gamma_{y^-}^-(t)) = \sup_{a \in \Gamma_x^-(t), b \in \Gamma_{y^-}^-(t)} d(a,b) > r,$$

where $\operatorname{diam}(W)$ is the diameter of W and d is the Euclidean distance.

Associated to the previous definition we give the next one, where the orientation of the trajectories of Z is also considered:

Definition 2.13. System (2.1) exhibits **sensitive positive dependence** (resp., **sensitive negative dependence**) on a compact positive invariant (resp., negative invariant) set W if there is a fixed $r > 0$ satisfying $r < \text{diam}(W)$ such that for each $x \in W$ and $\varepsilon > 0$ there exist a $y \in B_\varepsilon(x) \cap W$ and positive (resp., negative) maximal trajectories Γ_x^+ and Γ_y^+ (resp., Γ_x^- and Γ_y^-) passing through x and y, respectively, satisfying

$$d_H(\Gamma_x^+(t), \Gamma_y^+(t)) = \sup_{a \in \Gamma_x^+(t), b \in \Gamma_y^+(t)} d(a,b) > r,$$

$$(\text{resp., } d_H(\Gamma_x^-(t), \Gamma_y^-(t)) = \sup_{a \in \Gamma_x^-(t), b \in \Gamma_y^-(t)} d(a,b) > r),$$

where $\text{diam}(W)$ is the diameter of W and d is the Euclidean distance.

Remark 2.14. A direct consequence of the two previous definitions is that:

Z exhibits sensitive dependence on W if, and only if, Z exhibits simultaneously sensitive positive dependence and sensitive negative dependence on W.

In this paper we will consider the notations stated in the following table.

Table of abbreviations	
Topologically transitive	TT
Topologically positive transitive	TPT
Topologically negative transitive	TNT
Sensitive dependence	SD
Sensitive positive dependence	SPD
Sensitive negative dependence	SND

We should mention, as observed in [10], that Definitions 2.9 and 2.12 coincide with the definitions of topological transitivity and sensible dependence of smooth vector fields for single-valued flows, so these definitions are natural extension for a set-valued flow. Lastly, in what follows we introduce the definition of chaos and orientable chaos in the piecewise smooth context. Note that the concept for chaos in the paper is inspired by Devaney for a deterministic flow, but the systems of differential equations discussed in the article define non-deterministic flows:

Definition 2.15. System (2.1) is **chaotic** (resp., either **positive chaotic** or **negative chaotic**) on a compact invariant (resp., either positive invariant or negative invariant) set W if it is TT and exhibits SD (resp., either TPT and exhibits SPD or TNT and exhibits SND) on W.

Remark 2.16. A direct consequence of the previous definition is that:

A PSVF Z is chaotic on W if, and only if, Z is positive chaotic and negative chaotic on W.

3 Main results

In this Section we present and prove the main results of the paper.

Proposition 3.1. *Let $\mathcal{A}$ be the set of pseudo cycles Γ of $Z = (X, Y)$ such that $\Gamma \cap (\overline{\Sigma^e} \cup \overline{\Sigma^s}) = \emptyset$ and Γ has at least a visible two-fold singularity. The elements Γ of $\mathcal{A}$ are chaotic for Z.*

In Figure 4.5 we exhibit an element $\Gamma \subset \mathcal{A}$. In fact, the elements $\Gamma \subset \mathcal{A}$ are obtained by the concatenation of orbits of X and Y, without using orbits of Z^{Σ}.

Proof of Proposition 3.1. Let A, B open sets relative to Γ. Since Γ is a pseudo-cycle, given points $p_A \in A$ and $p_B \in B$, there exists a trajectory of Z connecting them (for positive and negative times). So Γ is topologically transitive.

On the other hand, given $x, y \in A$, there exists a trajectory passing through x and another trajectory passing through y such that each one of them follows a distinct path after the visible two-fold singularity of Γ. So Γ has sensitive dependence.

Therefore, Γ is chaotic. □

Remark 3.2. By the previous proposition, we conclude the existence of trivial minimal sets presenting chaotic behavior.

Remark 3.3. An analogous of result of Remarks 2.11, 2.14 and 2.16 does not hold for minimal sets. Indeed, while sets which are both positive and negative minimal are also minimal, the converse is not true. The Example 2 of [6] exemplify this situation.

The most part of the results obtained in [5] and [6] takes into account sets having positive Lebesgue measure. Indeed, in almost every approach concerning ergodic aspects of PSVFs, this is the interesting case. We cite, for instance, the existence of non-trivial minimal sets and planar chaotic PSVFs, as shown in the papers cited previously. In this direction we state the next result.

Lemma 3.4. *Let $K \subset \mathbb{R}^2$ be a compact invariant set and Z a PSVF presenting a finite number of critical points and a finite number of tangency points with Σ in K. If $\operatorname{med}(K) = 0$ and $K \notin \mathcal{A}$ then Z is not chaotic on K.*

We recall that $\mathcal{A}$ is the set of pseudo-cycles having a visible two-fold singularity which does not connect to any sliding or escaping segment (see Proposition 3.1). Also, the *saturation* of a set M by a vector field W is the set

$$W(M) = \{\phi_W(t, p) \mid p \in M \text{ and } t \in I\}$$

where I is the maximal interval of existence of the W-trajectory passing through p.

Proof. First, suppose that $K \cap \Sigma \subset \Sigma^c \cup \Sigma^t$ and take $p \in K$. Consequently, $\phi_Z(t, p) \xrightarrow{t \to \infty} L \in \omega(p) \subset K$, since K is compact. Here $\omega(p)$ denotes the ω-limit set of the point p. Thus, by using the Poincaré–Bendixson Theorem for PSVFs (see [5]) we get that L is a (pseudo-)equilibrium, a (pseudo-)graph or (pseudo-)cycle which does not belongs to $\mathcal{A}$ since $L \subset K$ and $K \notin \mathcal{A}$ by hypothesis. In any case, it is trivial to see that Z is not chaotic on K since Z does not exhibits SD on K.

Now consider the case where $K \cap (\Sigma^s \cup \Sigma^e) \neq \emptyset$ and suppose that there exist a PSVF Z which is chaotic on K. Take $p \in K \cap (\Sigma^s \cup \Sigma^e)$ and $V_p \subset \mathbb{R}^2$ a neighborhood of p. Consider the sets $V_p^+ = \{\phi_t^+(p) \cap V_p \,|\, \phi_t^+$ is a positive trajectory of Z passing through $p\}$ and V_p^- defined analogously for the negative trajectory. Observe that $\mathrm{med}(V_p^+ \cup V_p^-) > 0$, since using the Definition 2.1, in this case the saturation of $K \cap (\Sigma^s \cup \Sigma^e)$ (for either positive or negative times) contain an open set $U \subset V_p$ satisfying $0 < \mathrm{med}(U) < \mathrm{med}(V_p^+ \cup V_p^-)$. Consequently there exist a point $q \in V_p^+ \cup V_p^-$ such that $q \notin K$, because otherwise $V_p^+ \cup V_p^- \subset K$ and then $\mathrm{med}(K) > \mathrm{med}(V_p^+ \cup V_p^-) > 0$ (see Figure 3.1). As consequence, K is not invariant, producing a contradiction. □

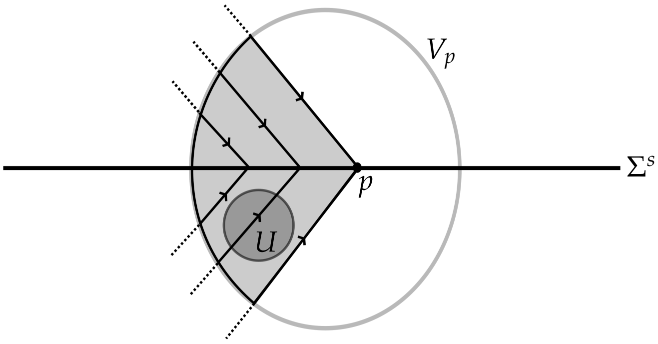

Figure 3.1: The neighborhood V_p of p. The filled region correspond to V_p^-, and in this case $V_p^+ = V_p \cap \Sigma$. Observe that it has positive Lebesgue measure.

In the proof of the next Theorem 3.6 we will use the following remark.

Remark 3.5. A direct consequence of Definition 2.15 is that

Let Z a chaotic PSVF on W. Then Z is chaotic on every compact invariant proper subset $\widetilde{W} \subset W$.

In [6], among other results, the authors prove that, if a compact invariant set W satisfying $\mathrm{med}(W) > 0$ is simultaneously positive and negative minimal for a PSVF Z, then Z is chaotic on W. Now, we prove the converse of this important theorem. Observe that, due to Lemma 3.4, we must impose a condition demanding the positive Lebesgue measure of the considered set.

Theorem 3.6. *If Z is chaotic on the compact invariant set W, $\mathrm{med}(W) > 0$, Z has a finite number of critical points and a finite number of tangency points with Σ in W, then W is positive minimal and negative minimal for Z.*

Proof. According to Remark 2.16, Z is positive chaotic on W. So, W is compact, non-empty and positive invariant. Suppose that W is not positive minimal. In this case, there exists a proper subset $\widetilde{W}$ of W with the previous three properties. Moreover, by Remark 3.5 and Lemma 3.4, we get $\mathrm{med}(\widetilde{W}) > 0$ or $\mathrm{med}(\widetilde{W}) = 0$ and $\widetilde{W} \subset \mathcal{A}$. Of course $\widetilde{W}$ is not dense in W since $\widetilde{W}$ is compact and $\widetilde{W} \neq W$. Therefore there exists an open set $A \subset W$ such that $A \cap \widetilde{W} = \emptyset$. First suppose that $\mathrm{med}(\widetilde{W}) > 0$ and let $B \subset \widetilde{W}$ be an open set of W. In this case, using the open sets

A and B, we have that Z is not TPT. But this is a contradiction with the fact that Z is chaotic on W. On the other hand, if $\text{med}(\widetilde{W}) = 0$ we get $\widetilde{W} \subset \mathcal{A}$ and therefore $\widetilde{W}$ is a curve on W. Let $I(\widetilde{W})$ the region delimited by $\widetilde{W}$ which is clearly invariant and notice that $\text{med}(I(\widetilde{W})) > 0$ since $\text{med}(\widetilde{W}) = 0$. So we can take open sets $B \subset I(\widetilde{W})$ and $A \subset W \setminus \big(\widetilde{W} \cup I(\widetilde{W})\big)$ to lead again to a contradiction with the fact that Z is chaotic on W. Therefore, W is positive minimal for Z.

An analogous argument proves that W is negative minimal for Z. □

Next corollary is a straightforward consequence of Theorem 3.6, but it is very important once it provides a ultimate answer about the relation between chaotic systems and minimal sets.

Corollary 3.7. *If Z is chaotic on W, $\text{med}(W) > 0$, Z has a finite number of critical points and a finite number of tangency points with Σ in W, then W is minimal for Z.*

Proof. It is enough to use Theorem 3.6 and Definition 2.8. □

We remark that the converse is not true, as observed in [6].

The next two corollaries are also consequences of Theorem 3.6. Their proofs, analogously, are quite trivial although the results can find applications.

Corollary 3.8. *If $\text{med}(W) > 0$, Z has a pseudo equilibria on W and a finite number of tangency points with Σ in W then Z is not chaotic on W.*

Proof. It is not difficult to see that a pseudo equilibria is neither positive nor negative minimal for Z since there exists trajectories of X and Y hitting it in finite (positive or negative) time. So, W is not positive or negative minimal. Therefore the proof follows straightforward from Theorem 3.6. □

Remark 3.9. A consequence of the proof of Theorem 3.6 is that

If Z is positive (resp. negative) chaotic on W, $\text{med}(W) > 0$, Z has a finite number of tangency points with Σ in W then W is positive (resp. negative) minimal.

The next result provide a sufficient condition in order to a PSVF Z be chaotic on an invariant compact set W. Additionally, it guarantee that under suitable hypotheses the periodic trajectories of Z are dense in W.

Theorem 3.10. *Let Z be a PSVF and W a compact positive (resp. negative) invariant set satisfying $\text{med}(W) > 0$. Given $x, y \in W$, assume that there exist a positive (resp. negative) trajectory ϕ_t^+ (resp. ϕ_t^-) connecting x and y. Then Z is positive (resp. negative) chaotic on W and the positive (resp. negative) periodic trajectories of Z are dense in W.*

We shall prove the last result in forward time, obtaining positive chaos and dense trajectories. The proof for trajectories in backward time is completely similar.

Proof. Since $\text{med}(W) > 0$, let U and V be nonempty open sets in W and p_U, p_V points of U and V, respectively. By hypotheses there exist a positive trajectory ϕ_t^+ connecting p_U and p_V in forward time. Since U and V are arbitrary it follows that W is topologically positive transitive. On the other hand, let d_W be the diameter of W and take $r = d_W/2$, so clearly there exists $a, b \in W$ such that $d(a, b) > r$. Now consider $x \in W$, $\varepsilon > 0$ and fix $y \in B_\varepsilon(x) \cap W$. Again, by hypotheses there exists positive trajectories $\phi_a^+(t, x)$ and $\phi_b^+(t, x)$ satisfying $\phi_a^+(0, x) =$

$\phi_b^+(0,x) = x$ and values $t_a, t_b > 0$ such that $\phi_a^+(t_a,x) = a$ and $\phi_b^+(t_b,x) = b$ so Z exhibits sensitive positive dependence on W. At last, the density of positive periodic trajectories is straightforward from the fact that any point $x \in W$ can be connected to itself by a positive trajectory. □

Theorem 3.10 leads to the next corollary.

Corollary 3.11. *Let Z be a PSVF and W satisfying $med(W) > 0$ a compact invariant set on which any two points can be connected simultaneously by positive and negative trajectories. Then Z is chaotic on W and its periodic trajectories are dense in W.*

Proof. Since every pair of points in W can be connected simultaneously by positive and negative trajectories of Z, by Theorem 3.10, the PSVF Z is both positive and negative chaotic on Z. So, by Remark 2.16, we get that Z is chaotic on W. Moreover, since the positive and negative periodic trajectories of Z are dense in W, the density of the periodic trajectories of Z on W is straightforward. □

4 Discussions

We observed throughout the paper a closed relation between PSVFs presenting minimal sets or chaotic behavior. However, in order to observe the richness of such relation we introduced new concepts by considering the orientation of the trajectories in time. By one hand, according to Theorem 14 of [6], every PSVF having a positive and negative non trivial minimal set K is chaotic on K. On the other hand, in this paper, due to Remark 2.16 and 3.6 we get the equivalence. Putting those and other results of this paper together, we get the following diagram:

Z is pos. and neg. chaotic on W	$\Leftrightarrow$	Z is chaotic on W	$\Leftrightarrow$	W is pos. and neg. min. for Z	$\Rightarrow$	W is min. for Z

We note by observing the previous diagram that it could exist some minimal set which is not chaotic for the PSVF, as the authors observed in [5]. Other aspects of that diagram are presented in what follows:

Orientable chaotic sets which are not chaotic: Consider the PSVF:

$$ccZ_\epsilon(x,y) = (\dot{x},\dot{y}) = \frac{1}{2}\Big((-1, -2x - x^2(4x+3) + (1+\epsilon)x(3x+2)) \\ + \operatorname{sgn}(y)\,(3, -2x + x^2(4x+3) - (1+\epsilon)x(3x+2))\Big) \tag{4.1}$$

or, equivalently,

$$Z_\epsilon(x,y) = \begin{cases} X(x,y) = (1,-2x) & \text{if } y \geq 0, \\ Y_\epsilon(x,y) = (-2, -x^2(4x+3) + (1+\epsilon)x(3x+2)) & \text{if } y \leq 0, \end{cases} \tag{4.2}$$

with $\epsilon \in \mathbb{R}$ an arbitrarily small parameter. In [6] the authors proved that Z_0 has a chaotic set given (see Figure 4.1) by

$$\Lambda = \{(x,y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1 \text{ and } x^4/2 - x^2/2 \leq y \leq 1 - x^2\}. \tag{4.3}$$

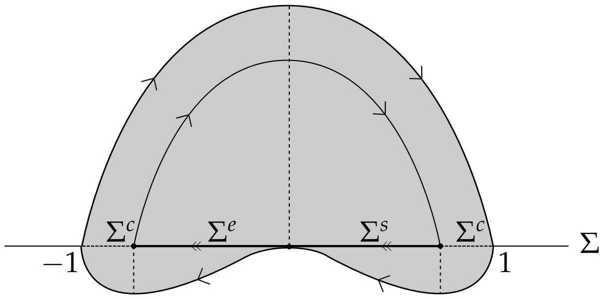

Figure 4.1: Chaotic set Λ.

Taking $\epsilon < 0$ (resp., $\epsilon > 0$) in (4.2) the PSVF Z_ϵ has a negative chaotic (resp., positive chaotic) set $\widetilde{\Lambda}$. We construct such a set for the case $\varepsilon < 0$. Indeed call p_2 the two-fold located at the origin and p_1 the first intersection of the backward trajectory of p_2 with Σ. From p_1 it can be concatenated a regular arc of trajectory of X which again intersects Σ in backward time at a point p_4. Finally, call p_3 the continuation of p_4 through the trajectory of Y until reaching Σ. Hence the set $\widetilde{\Lambda}$ is the region bounded by $\widehat{p_1\, p_2} \cup \widehat{p_2\, p_3} \cup \widehat{p_3\, p_4} \cup \widehat{p_4\, p_1}$, where $\widehat{a\, b}$ is the orbit-arc connecting the points a and b, see the shadowed region in Figure 4.2 (resp., Figure 4.3). Moreover, when $\epsilon \neq 0$, $\widetilde{\Lambda}$ is not a chaotic set. This happens because $\widetilde{\Lambda}$ is not an invariant set; it is only negative invariant (resp., positive invariant).

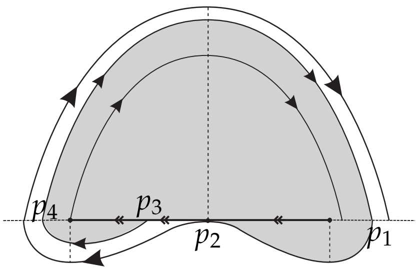

Figure 4.2: Negative chaotic set $\widetilde{\Lambda}$.

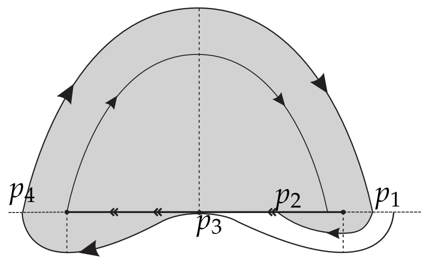

Figure 4.3: Positive chaotic set $\widetilde{\Lambda}$.

Remark 4.1. The previous paragraph remains true if we change the word chaotic by the word minimal. A complete bifurcation analysis of the family (4.2) is given in [8].

The sets given in Figures 4.2 and 4.3 are orientable chaotic and orientable minimal sets. Despite of this, it is easy to exhibit examples of orientable minimal sets that are not orientable chaotic.

Orientable chaotic sets and orientable minimality: Consider the PSVF

$$Z(x,y) = (X(x,y), Y(x,y)) = ((-1, 3x^2 - 3), (1, -(9/4) + 3(-1+x)x)).$$

Such PSVF has a periodic orbit (see Figure 4.4) which is a negative minimal set. However, Z is not a negative chaotic PSVF on the periodic orbit since it does not present SPD.

Observe that, in the last example the Lebesgue measure of the periodic orbit is null. However, it is not difficult to exhibit a minimal set W for some PSVF, with $\text{med}(W) > 0$, in such way that W is neither positive chaotic nor negative chaotic. Indeed, Example 2 of [6] satisfies these properties. In other words, in general minimality does not imply chaoticity. The converse, on the other hand, is true, as proved in Section 3.

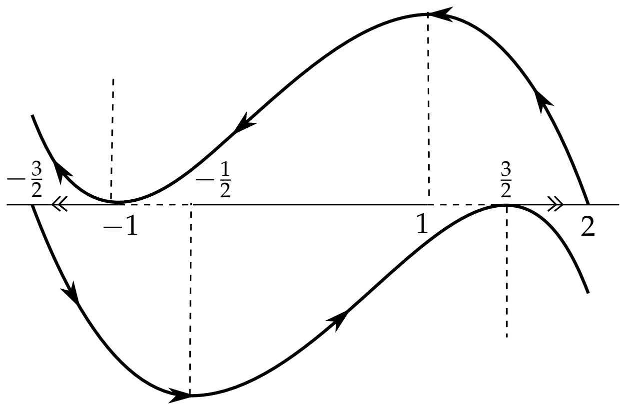

Figure 4.4: Periodic orbit (for positive time).

Trivial chaos: In PSVFs the route to chaos is not hard. In fact, here we show that a chaotic behavior can be achieved by trivial minimal sets.

Consider the PSVF $Z = (X, Y)$ where $X(x, y) = (1, 4x(1-x^2))$ and $Y(x, y) = (-1, 4x(1-x^2))$. The phase portrait is pictured in Figure 4.5. Take $\Lambda = \Lambda_1 \cup \Lambda_2$, where Λ_1 (respectively, Λ_2) is the trajectory of X (respectively, Y) passing through $p_1 = (-\sqrt{2}, 0)$. It is easy to see that Λ is a trivial minimal set (a pseudo-cycle) and it is a chaotic set for Z.

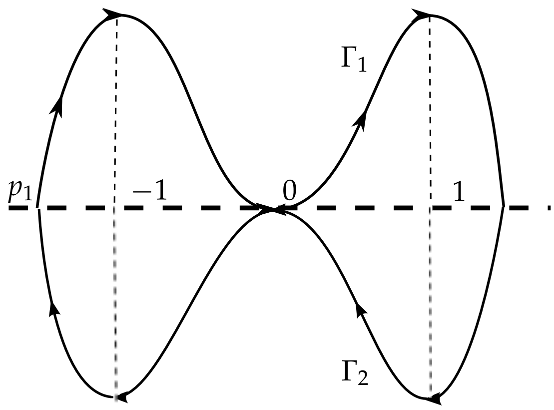

Figure 4.5: Trivial minimal set which is chaotic for Z.

The previous example illustrates a more general result, stated in Proposition 3.1.

We finish this section highlighting two particular conclusions from the results of the paper:

(i) although the chaoticity of a PSVF Z under a set W implies that W is minimal for Z, the converse is false according to Example 2 of [6];

(ii) if Z is positive chaotic on W then W is positive minimal for Z (see Remark 3.9), but the converse is false since we can exhibit positive minimal sets that are not positive chaotic (see Example 4 in [5]). Analogously for negative chaotic/minimal.

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