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Existence of infinitely many radial nodal solutions for a Dirichlet problem involving mean curvature operator in Minkowski space

Man Xu ✉ and **Ruyun Ma**

Department of Mathematics, Northwest Normal University, Lanzhou 730070, P. R. China

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Abstract. In this paper, we show the existence of infinitely many radial nodal solutions for the following Dirichlet problem involving mean curvature operator in Minkowski space

$$\begin{cases} -\mathrm{div}\left(\frac{\nabla y}{\sqrt{1-|\nabla y|^2}}\right) = \lambda h(y) + g(|x|, y) \quad \text{in } B, \\ y = 0 \quad \text{on } \partial B, \end{cases}$$

where $B = \{x \in \mathbb{R}^N : |x| < 1\}$ is the unit ball in $\mathbb{R}^N$, $N \geq 1$, $\lambda \geq 0$ is a parameter, $h \in C(\mathbb{R})$ and $g \in C(\mathbb{R}^+ \times \mathbb{R})$. By bifurcation and topological methods, we prove the problem possesses infinitely many component of radial solutions branching off at $\lambda = 0$ from the trivial solution, each component being characterized by nodal properties.

Keywords: infinitely many radial solutions, mean curvature operator, Minkowski space, topological method, bifurcation.

2020 Mathematics Subject Classification: 35J65, 34B18, 34C23, 35B40, 34B10.

1 Introduction

The purpose of this paper is to deal with radial nodal solutions for the following 0-Dirichlet problem with mean curvature operator in the Minkowski space

$$\begin{gathered} -\mathrm{div}\left(\frac{\nabla y}{\sqrt{1-|\nabla y|^2}}\right) = \lambda h(y) + g(|x|, y) \quad \text{in } B, \\ y = 0 \quad \text{on } \partial B, \end{gathered} \tag{1.1}$$

where $B = \{x \in \mathbb{R}^N : |x| < 1\}$ is the unit ball in $\mathbb{R}^N$, $N \geq 1$, $\lambda \geq 0$ is a parameter, $h(y) \simeq |y|^{q-2}y$, $1 < q < 2$ near $y = 0$ and g is of higher order with respect to h at $y = 0$. This kind of problems are originated from differential geometry or classical relativity.

✉ Corresponding author. Email: xmannwnu@126.com

For example, let

$$\mathbb{L}^{N+1} := \{(x,t) : x \in \mathbb{R}^N, t \in \mathbb{R}\}$$

be the flat Minkowski space, endowed with the Lorentzian metric

$$\sum_{j=1}^{N} dx_j^2 - dt^2.$$

It is known (see [4,28]) that the study of spacelike submanifolds of codimension one in $\mathbb{L}^{N+1}$ with prescribed mean extrinsic curvature leads to Dirichlet problems of the type

$$\begin{aligned} -\operatorname{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) &= H(x,u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^N$ and the nonlinearity $H : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous.

There are a large amount of papers in the literature on the existence, multiplicity and qualitative properties of solutions for this type of problems, see [1–3,7,11,12,14,16,25,26,31]. It is worth pointing out that the starting point of this type of problems is the seminal paper [9] which prove the Bernstein's property for entire solutions of the maximal (i.e., zero mean curvature) hypersurface equation. Bartnik and Simon [4] proved the existence of one strictly spacelike solution when $\lambda = 1$ and H is bounded, this always can be seen as an important universal existence result of (1.2). For the case $N = 1$, the existence and multiplicity of positive solutions of the Dirichlet problem for the quasilinear ordinary differential equation

$$\begin{aligned} -\left(\frac{u'}{\sqrt{1-u'^2}}\right)' &= H(x,u), \qquad x \in (0,1), \\ u(0) = u(1) &= 0 \end{aligned}$$

have been extensively studied by Coelho et al. [10] via variational or topological methods. For the special case Ω is a ball, by using upper and lower solutions, Leray–Schauder degree arguments and critical point theory for convex, lower semicontinuous perturbations of C^1-functionals, Bereanu, Jebelean, and Torres [5,6] obtained some nonexistence, existence and multiplicity results of classical positive radial solutions of (1.2). Ma, Gao and Lu [24] concerned with the global structure of radial positive solutions of (1.2) by using global bifurcation techniques, and extended the results of [5,6] to more general cases, all results, depending on the behavior of nonlinear term H near 0. Later, Ma and Xu [27] studied the global behavior of positive solutions of (1.2) with Ω is a general domain in $\mathbb{R}^N$.

However, few results on the existence of radial nodal solutions [15], even positive solutions, have been established for problem with mean curvature operator on general domain. In this paper, we will show an existence result of infinitely many radial nodal solutions for Dirichlet problem (1.1) by bifurcation and topological methods. For the applications of nodal solutions, see Kurth [20] and Lazer and McKenna [21].

Our study is motivated by some recent works on one-dimensional prescribed mean curvature problems with concave-convex nonlinearities, see [19,34].

Setting, as usual $|x| = r$ and $y(x) = u(r)$, the problem (1.1) reduces to the mixed boundary value problem

$$\begin{aligned} Au &= \lambda h(u) + g(r,u), \qquad r \in (0,1), \\ u'(0) &= u(1) = 0, \end{aligned} \tag{1.3}$$

where

$$Au = -\frac{1}{r^{N-1}}(r^{N-1}\phi_1(u'))', \tag{1.4}$$

and

$$\phi_1(s) = \frac{s}{\sqrt{1-s^2}}, \qquad s \in \mathbb{R},$$

note that $\phi_1 : (-1,1) \to \mathbb{R}$ is an odd, increasing homeomorphism and $\phi_1(0) = 0$. Throughout we assume $\lambda \geq 0$, $h \in C(\mathbb{R})$, $g \in C(\mathbb{R}^+ \times \mathbb{R})$ and satisfy the following conditions:

(A1) $h \in C(\mathbb{R}, \mathbb{R})$ with $sh(s) > 0$ for $s \neq 0$, $\lim\limits_{s\to 0} \frac{h(s)}{s} = \infty$;

(A2) $\lim\limits_{s\to 0} \frac{g(r,s)}{s} = 0$ uniformly for $r \in [0,1]$.

Let $X = \{u \in C^1[0,1] : u'(0) = u(1) = 0\}$ with the norm $\|u\| := \|u'\|_\infty$, and let $E = \mathbb{R} \times X$. In the sequel by a solution of (1.1) we mean a pair $(\lambda, u) \in E$, such that $u \in C^1[0,1]$, $\max_{r\in[0,1]} |u'(r)| < 1$, $r^{N-1}\phi_1(u') \in C^1[0,1]$, and satisfies (1.1). These are strong strictly spacelike solutions of (1.1) according to the terminology of [4,9,18,31].

The main result of this paper is the following.

Theorem 1.1. *Let (A1) and (A2) hold. Then the point $(\lambda, u) = (0,0)$ is a bifurcation point for problem (1.1). More precisely, there are infinitely many unbounded component (i.e., closed connected sets) $\Gamma_k \subset E$ of solutions of (1.1) branching off from $(0,0)$, such that*

(i) If $(\lambda, u) \in \Gamma_k$ and $\lambda > 0$, then $u \not\equiv 0$.

(ii) If $(\lambda, u) \in \Gamma_k$, then u has exactly $k-1$ simple zeros in the interval $(0,1)$.

(iii) There exists a constant $\rho_0 \in (0,1/2)$ such that if $\rho \in (0,\rho_0]$, and $(\lambda, u) \in \Gamma_k$ with $\|u\| = \rho$, then $\lambda > \lambda(\rho) > 0$.

As an immediate consequence we get:

Corollary 1.2. *There exists $\lambda_* > 0$ such that problem (1.1) has infinitely many radial nodal solutions for any $\lambda \in (0, \lambda_*)$.*

Remark 1.3. It is easy to find that (A2) yields that

$$g(r,0) = 0 \quad \text{uniformly for } r \in [0,1].$$

Otherwise, from the continuity of g, we get $\lim_{s\to 0} \frac{g(r,s)}{s} = \infty$ for some $r \in [0,1]$, this is a contradiction.

Remark 1.4. Let (λ, u) be a solution of (1.3), then it follows from $|u'(r)| < 1$ that

$$\|u\|_\infty < 1.$$

This leads to the bifurcation diagrams mainly depend on the behavior of $h = h(s)$ and $g = g(r,s)$ near $s = 0$. This is a significant difference between the Minkowski-curvature problems and the p-Laplacian problems.

Remark 1.5. If $g(r,s) \equiv 0$ for all $r \in [0,1]$, then

$$\lim_{s\to 0} \frac{g(r,s)}{s} = 0 \quad \text{uniformly for } r \in [0,1].$$

Clearly, Theorem 1.1 improves some well-known existence results of positive solutions [5] and radial nodal solutions [15] for related problems.

The rest of the paper is arranged as follows. In Section 2, we show the property of the superior limit of a sequence of components and obtain a topological degree jumping result. Finally in Section 3, we prove our main result and give an example to illustrate our main result.

2 Some preliminary results

2.1 Superior limit and component

The following results are somewhat scattered in Ma and An [22,23].

Definition 2.1 ([22,23])**.** Let X be a Banach space and $\{C_n : n = 1,2,\dots\}$ be a family of subsets of X. Then the the *superior limit* $\mathcal{D}$ of $\{C_n\}$ is defined by

$$\mathcal{D} := \limsup_{n\to\infty} C_n = \{x \in X : \text{ there exist } \{n_i\} \subset \mathbb{N} \text{ and } x_{n_i} \in C_{n_i} \text{ such that } x_{n_i} \to x\}.$$

Definition 2.2 ([22,23])**.** A component of a set M means a maximal connected subset of M.

Lemma 2.3 ([22, Lemma 2.4], [23, Lemma 2.2])**.** *Assume that*

(i) there exist $z_n \in C_n$, $n = 1,2,\dots$, and $z^* \in X$, such that $z_n \to z^*$;

(ii) $\lim_{n\to\infty} r_n = \infty$, where $r_n = \sup\{\|x\| : x \in C_n\}$;

(iii) for every $R > 0$, $\left(\cup_{n=1}^{\infty} C_n\right) \cap B_R$ is a relative compact set of X, where

$$B_R = \{x \in X : \|x\| \le R\}.$$

Then there exists an unbounded component $\mathcal{C}$ in $\mathcal{D}$ with $z^* \in \mathcal{C}$.

2.2 Topological degree jumping result

Let us introduce the eigenvalue problem

$$\begin{aligned} -(r^{N-1}u')' &= \lambda r^{N-1} u, \qquad r \in (0,1), \\ u'(0) &= u(1) = 0. \end{aligned} \tag{2.1}$$

From [29] with $p = 2$ or [32, p. 269], we have the following result.

Lemma 2.4. *Problem (2.1) has infinitely many simple real eigenvalues, which can be arranged in the increasing order*

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \to +\infty \quad \text{as } k \to +\infty,$$

and no other eigenvalues. Moreover, the algebraic multiplicity of λ_k is 1, and the eigenfunction φ_k has exactly $k-1$ simple zeros in $(0,1)$.

For any $t \in (0,1]$, we consider the following auxiliary problem

$$\begin{aligned} -\frac{1}{r^{N-1}}\left(r^{N-1}\frac{u'}{\sqrt{1-tu'^2}}\right)' &= f(r), \qquad r \in (0,1), \\ u'(0) = u(1) &= 0 \end{aligned} \tag{2.2}$$

for a given $f \in C[0,1]$. Letting $v = \sqrt{t}u$, problem (2.2) is equivalent to

$$\begin{aligned} -\frac{1}{r^{N-1}}\left(r^{N-1}\frac{v'}{\sqrt{1-v'^2}}\right)' &= \sqrt{t}f(r), \qquad r \in (0,1), \\ v'(0) = v(1) &= 0. \end{aligned} \tag{2.3}$$

By Theorem 3.6 of [4], we know that there exists a unique strictly spacelike solution $v \in C^1[0,1]$ to problem (2.3) which is denoted by $\psi(\sqrt{t}f)$. So $u = \frac{v}{\sqrt{t}}$ is the unique solution of problem (2.2).

For a given $b \in C[0,1]$, we also consider the following auxiliary problem

$$\begin{aligned} -\frac{1}{r^{N-1}}\left(r^{N-1}u'\right)' &= b(r), \qquad r \in (0,1), \\ u'(0) = u(1) &= 0. \end{aligned} \tag{2.4}$$

It is well known that problem (2.4) has a solution u for every given $b \in C[0,1]$. Let $\phi(b)$ denote the unique solution to problem (2.4). It is easy to check that $\phi : C[0,1] \to X$ is linear and completely continuous.

Therefore, for any given $f \in C[0,1]$, let us define $G : [0,1] \times C[0,1] \to X$ by

$$G(t,f) = \begin{cases} \frac{\psi(\sqrt{t}f)}{\sqrt{t}}, & t \in (0,1], \\ \phi(f), & t = 0. \end{cases} \tag{2.5}$$

From the Lemma 2.3 of [14], we have G is completely continuous.

For any fixed λ, consider the following problem

$$\begin{aligned} -\frac{1}{r^{N-1}}\left(r^{N-1}\frac{u'}{\sqrt{1-u'^2}}\right)' &= \lambda u, \qquad r \in (0,1), \\ u'(0) = u(1) &= 0. \end{aligned} \tag{2.6}$$

Clearly, problem (2.6) is equivalent to the operator equation

$$u = \psi(\lambda u) := \psi_\lambda(u).$$

From Lemma 2.3 of [14], we see that $\psi_\lambda : X \to X$ is completely continuous. And we can also obtain the following topological degree jumping result.

Lemma 2.5. *For any $r > 0$, we have that*

$$\deg(I - \psi_\lambda, B_r(0), 0) = \begin{cases} 1, & \text{if } \lambda \in (0,\lambda_1), \\ (-1)^k, & \text{if } \lambda \in (\lambda_k, \lambda_{k+1}),\ k \in \mathbb{N}. \end{cases}$$

Proof. It is not difficult to show that $I-\psi_\lambda$ is a nonlinear compact perturbation of the identity. Thus, the Leray–Schauder degree $\deg(I-\psi_\lambda, B_r(0), 0)$ is well defined for arbitrary r-ball $B_r(0)$ and $\lambda \neq \lambda_k$. From the invariance of the degree under homotopies we obtain that

$$\begin{aligned}\deg(I-\psi_\lambda, B_r(0), 0) &= \deg(I-G(1,\lambda\cdot), B_r(0), 0)\\ &= \deg(I-G(0,\lambda\cdot), B_r(0), 0)\\ &= \deg(I-\lambda\phi, B_r(0), 0).\end{aligned}$$

Since ϕ is compact and linear, by [13, Lemma 3.1] or [17, Theorem 8.10], we have that

$$\deg(I-\lambda\phi, B_r(0), 0) = \begin{cases}1 & \text{if } \lambda\in(0,\lambda_1),\\ (-1)^k & \text{if } \lambda\in(\lambda_k,\lambda_{k+1}),\ k\in\mathbb{N},\end{cases}$$

and accordingly,

$$\deg(I-\psi_\lambda, B_r(0), 0) = \begin{cases}1 & \text{if } \lambda\in(0,\lambda_1),\\ (-1)^k & \text{if } \lambda\in(\lambda_k,\lambda_{k+1}),\ k\in\mathbb{N}.\end{cases}$$

□

3 Proof of the main result

Before proving the Theorem 1.1, we state the following lemmas.

Lemma 3.1. *Assume that (A1) and (A2). Let (λ, u) be a solution of problem (1.3). If u has a double zero, then $u\equiv 0$.*

Proof. Assume on the contrary that there exists a solution (λ, u), $\lambda>0$, of (1.3) and u has a double zero. Let $\tau\in[0,1]$ be a double zero of u. Integrating the equation of (1.3) over $[\tau, r]$, we have

$$\frac{u'(r)}{\sqrt{1-(u'(r))^2}} = -\frac{1}{r^{N-1}}\int_\tau^r s^{N-1}(\lambda h(u(s))+g(s,u(s)))ds.$$

If $\tau=0$, then for $r\in[0,1]$, from (A1) and the fact

$$|u'(r)|<1,$$

it follows that

$$|u'(r)| \le \frac{1}{r^{N-1}}\int_0^r s^{N-1}|g(s,u)|ds \le \frac{r}{N}|g(s,u)|.$$

Recalling (A2), there exists a constant $M>0$ such that $|g(s,u)|\le M|u|$ for any $s\in[0,1]$ and $u\in[-1,1]$. Using the boundary conditions $u'(0)=u(1)=0$, we get

$$|u'(r)| \le \frac{Mr}{N}|u| \le \frac{Mr}{N}\int_1^r |u'(s)|ds.$$

By the Gronwall–Bellman inequality [8], we obtain $u'(r)\equiv 0$ on $[0,1]$. Therefore, $u(r)\equiv 0$ on $[0,1]$.

If $\tau>0$, we first assume that $r\in[0,\tau]$. Since

$$u(r) = -\int_\tau^r \phi_1^{-1}\left(\frac{1}{t^{N-1}}\int_\tau^t s^{N-1}(\lambda h(u(s))+g(s,u(s)))ds\right)dt$$

for all $r \in [0, \tau]$, where ϕ_1^{-1} is the inverse function of ϕ_1, namely

$$\phi_1^{-1}(s) = \frac{s}{\sqrt{1+s^2}}, \qquad s \in \mathbb{R}.$$

It is easy to check that ϕ_1^{-1} is increasing. Hence, by (A1), we have

$$\begin{aligned} u(r) &= \int_r^\tau \phi_1^{-1}\left(\frac{1}{t^{N-1}}\int_\tau^t s^{N-1}\left(\lambda\, h(u(s)) + g(s,u(s))\right) ds\right) dt \\ &= \int_r^\tau \phi_1^{-1}\left(\frac{1}{t^{N-1}}\int_t^\tau s^{N-1}\left(-\lambda\, h(u(s)) - g(s,u(s))\right) ds\right) dt \\ &\le \int_r^\tau \phi_1^{-1}\left(\frac{1}{t^{N-1}}\int_\tau^t s^{N-1} g(s,u(s)) ds\right) dt \\ &= \int_r^\tau \frac{\frac{1}{t^{N-1}}\int_\tau^t s^{N-1} g(s,u(s))ds}{\sqrt{1+\left(\frac{1}{t^{N-1}}\int_\tau^t s^{N-1} g(s,u(s))ds\right)^2}} dt, \end{aligned}$$

since $0 \le \frac{r}{t} \le 1$ and $N \ge 1$, this implies

$$r^{N-1}|u(r)| \le \int_r^\tau \int_\tau^t s^{N-1}|g(s,u(s))|dsdt \le M\int_r^\tau s^{N-1}|u(s)|ds.$$

By Gronwall–Bellman inequality, we have $r^{N-1}|u(r)| \equiv 0$ on $[0,\tau]$. And accordingly, $u(r) \equiv 0$ on $(0,\tau]$. This fact together with the continuity of u, we conclude that $u(r) \equiv 0$ on $[0,\tau]$.

Similarly, if $\tau > 0$ and $r \in [\tau, 1]$, then by Gronwall–Bellman inequality again, we can get $u(r) \equiv 0$ on $[\tau, 1]$ and the proof is completed. □

Lemma 3.2. *There exists $\rho_0 > 0$ such that any nontrivial solution u of*

$$\begin{aligned} Au &= g(r,u), \qquad r \in (0,1), \\ u'(0) &= u(1) = 0 \end{aligned} \tag{3.1}$$

satisfies $\|u\| > \rho_0$.

Proof. Assume, by contradiction, that there is a sequence $\{u_n\}$ of solutions of (3.1) and such that $u_n \neq 0$ and $\|u_n\| \to 0$. For all $n \in \mathbb{N}$, let $v_n = \frac{u_n}{\|u_n\|}$. Then $\|v_n\| = \|v_n'\|_\infty = 1$, consequently, $\|v_n\|_\infty$ is bounded. By the Ascoli–Arzelà theorem, there exists a subsequence of $\{v_n\}$ which uniformly converges to $v \in C[0,1]$. We again denote the subsequence by $\{v_n\}$. For any u_n, we have

$$\begin{aligned} -\frac{1}{r^{N-1}}\left(r^{N-1}\frac{u_n'}{\sqrt{1-u_n'^2}}\right)' &= g(r,u_n), \qquad r \in (0,1), \\ u_n'(0) &= u_n(1) = 0. \end{aligned} \tag{3.2}$$

Multiplying both sides of (3.2) by $\|u_n\|^{-1}$, we have

$$\begin{aligned} -\frac{1}{r^{N-1}}\left(r^{N-1}\frac{v_n'}{\sqrt{1-u_n'^2}}\right)' &= \frac{g(r,u_n)}{u_n}v_n, \qquad r \in (0,1), \\ v_n'(0) &= v_n(1) = 0. \end{aligned}$$

Since $\|u_n\| \to 0$ implies $\|u_n\|_\infty \to 0$. From (A2) and Lebesgue's dominated convergence theorem, we conclude that

$$-\frac{1}{r^{N-1}}\left(r^{N-1}v'\right)' = 0, \qquad r \in (0,1),$$
$$v'(0) = v(1) = 0,$$

which means that $v \equiv 0$ contradicting with $\|v\| = 1$. □

Proof of Theorem 1.1. Theorem 1.1 cannot be proved using standard bifurcation techniques by linearization. Actually, from (A1), we have known the nonlinear term h has infinite derivative at $u = 0$. To overcome this problem we shall employ a limiting procedure. Let us define a function $\tilde{h} : \mathbb{R} \to \mathbb{R}$ by setting

$$\tilde{h}(s) = \begin{cases} h(s), & 0 \le |s| \le 1, \\ \text{linear}, & 1 < |s| < 2, \\ 0, & |s| \ge 2, \end{cases}$$

and define a function $\tilde{g} : [0,1] \times \mathbb{R} \to \mathbb{R}$ by setting, for $r \in [0,1]$,

$$\tilde{g}(r,s) = \begin{cases} g(r,s), & 0 \le |s| \le 1, \\ \text{linear}, & 1 < |s| < 2, \\ 0, & |s| \ge 2. \end{cases}$$

Observe that, within the context of positive solutions, problem (1.3) is equivalent to the same problem with h, g replaced by $\tilde{h}, \tilde{g}$. Indeed, if u is a positive solution, then $\|u'\|_\infty < 1$ and hence $\|u\|_\infty < 1$. Clearly, $\tilde{h}$ and $\tilde{g}$ satisfy all the properties assumed in the statement of the theorem. In the sequel, we shall replace h, g with $\tilde{h}$ and $\tilde{g}$, however, for the sake of simplicity, the modified functions $\tilde{h}$, $\tilde{g}$ will still be denoted by h, g. Next, for any $\delta \in (0,1)$, let us define h_δ by setting

$$h_\delta(s) = \begin{cases} \dfrac{h(\delta)}{\delta} s, & 0 \le |s| \le \delta, \\ h(s), & |s| > \delta. \end{cases}$$

Obviously,

$$\lim_{\delta \to 0} h_\delta(s) = h(s), \qquad (h_\delta)_0 = \lim_{s \to 0} \frac{h_\delta(s)}{s} = \frac{h(\delta)}{\delta} > 0. \tag{3.3}$$

This together with (A1) implies that

$$\lim_{\delta \to 0} (h_\delta)_0 = \infty. \tag{3.4}$$

Let us consider the approximated problems

$$\begin{aligned} Au &= \lambda h_\delta(u) + g(r,u), \qquad r \in (0,1), \\ u'(0) &= u(1) = 0, \end{aligned} \tag{3.5}$$

where A is given by (1.4).

Define

$$F_\delta(\lambda, u) = \lambda h_\delta(u) + g(r,u) + \frac{1}{r^{N-1}}\left(r^{N-1}\frac{u'}{\sqrt{1-u'^2}}\right)'$$

for any $(\lambda, u) \in \mathbb{R} \times X$ and fixed $\delta > 0$. Then, from Remark 1.3, and by a simple calculation, we have that

$$\begin{aligned}(F_\delta)_u(\lambda,0)v &= \lim_{t\to 0}\frac{F_\delta(\lambda,tv)-F_\delta(\lambda,0)}{t}\\ &= \lambda\frac{h(\delta)}{\delta}v+\frac{1}{r^{N-1}}\left(r^{N-1}v'\right)'.\end{aligned} \tag{3.6}$$

Let $\lambda_{k,\delta} = \lambda_k \cdot \frac{\delta}{h(\delta)}$. Then from (3.6), it follows that if $(\lambda_{k,\delta}, 0)$ is a bifurcation point of problem (3.5), then λ_k is an eigenvalue of problem (2.1).

For any $\gamma \in [0,1]$, we consider the following problem

$$\begin{aligned}Au &= \lambda h_\delta(u)+\gamma g(r,u), \qquad r\in(0,1),\\ u'(0) &= u(1)=0.\end{aligned} \tag{3.7}$$

Then problem (3.7) is equivalent to

$$u = \psi(\lambda h_\delta(u)+\gamma g(r,u)) := F_{\delta,\lambda}(\gamma,u).$$

From [14, Lemma 2.3], it follows that $F_{\delta,\lambda} : [0,1]\times X \to X$ is completely continuous. In particular, $H_{\delta,\lambda} := F_{\delta,\lambda}(1,\cdot) : X \to X$ is completely continuous.

By (A2) and an argument similar to that of Lemma 2.5, we can show that the Leray–Schauder degree $\deg(I - F_{\delta,\lambda}(\gamma,\cdot), B_r(0), 0)$ is well defined for $\lambda \in (0,\infty)\setminus\{\lambda_k\}$. From the invariance of the degree under homotopies we obtain that

$$\begin{aligned}\deg(I-H_{\delta,\lambda},B_r(0),0) &= \deg(I-F_{\delta,\lambda}(1,\cdot),B_r(0),0)\\ &= \deg(I-F_{\delta,\lambda}(0,\cdot),B_r(0),0)\\ &= \deg\left(I-\psi\left(\lambda\frac{h(\delta)}{\delta}\cdot\right),B_r(0),0\right).\end{aligned}$$

So by Lemma 2.5, we have that

$$\deg(I-H_{\delta,\lambda},B_r(0),0)=\begin{cases}1, & \text{if } \lambda\in\left(0,\dfrac{\delta}{h(\delta)}\lambda_1\right),\\ (-1)^k, & \text{if } \lambda\in\left(\dfrac{\delta}{h(\delta)}\lambda_k,\dfrac{\delta}{h(\delta)}\lambda_{k+1}\right),\ k\in\mathbb{N}.\end{cases}$$

Denote

$$\mathscr{F}_\delta = \overline{\{(\lambda,u) : (\lambda,u)\in[0,\infty)\times X,\ u \text{ is a solution of (3.5)}\}}^{\mathbb{R}\times X}.$$

Then by a variant of the global bifurcation theorem of Rabinowitz [30], or index jump principle of Zeidler [33], for any $\delta > 0$, there exists a maximal closed connected set $S_{k,\delta}$ in $\mathscr{F}_\delta$ such that $(\lambda_{k,\delta}, 0) \in S_{k,\delta}$ and at least one of the following conditions holds:

(i) $S_{k,\delta}$ is unbounded in $\mathbb{R}\times X$;

(ii) $S_{k,\delta} \cap (\mathbb{R}\backslash\{\lambda_{k,\delta}\}\times\{0\}) \neq \emptyset$.

Since $(0,0)$ is the only solution of (3.5) for $\lambda = 0$ and 0 is not the eigenvalue of eigenvalue problem (2.1), therefore $S_{k,\delta} \cap (\mathbb{R}\backslash\{\lambda_{k,\delta}\} \times \{0\}) = \emptyset$. Recalling Remark 1.4, we get $S_{k,\delta}$ is unbounded in λ-direction for each fixed δ.

Combining this and (3.3) and (3.4) and using Lemma 2.3, it follows that for each $k \in \mathbb{N}$, there exists a component Γ_k in $\limsup S_{k,\delta}$ which joins $(0,0)$ to infinity in λ-direction.

In the following, we will prove the properties (i)–(iii) of Theorem 1.1, respectively.

(i) Let δ_0 be a positive constant such that $\lambda \frac{h(\delta_0)}{\delta_0} > \lambda_1$. Let us consider $(\lambda, u) \in S_{1,\delta}$, with $\lambda > 0$ and $\delta \in (0, \delta_0]$.

Fixing $\varepsilon > 0$ small, from (A1) and (A2), we obtain there exists $c = c(\lambda) > 0$ such that

$$\lambda h_\delta(s) + g(r,s) > (\lambda_1 + \varepsilon)s, \qquad \forall s \in (0, c].$$

Hence, we obtain if $\|u_1\|_\infty \le c$, then u_1 satisfies

$$Au_1 > (\lambda_1 + \varepsilon)u_1.$$

From [6], we have u_1 is an upper solution of the eigenvalue problem

$$Au = (\lambda_1 + \varepsilon)s. \tag{3.8}$$

On the other hand, it is easy to verify that $u_2 \equiv 0$ is a lower solution of (3.8). Therefore, [6, Proposition 1] yields the existence of a positive solution $u \in X$ of the eigenvalue problem (3.8). However, this is a contradiction, because $\lambda_1 + \varepsilon$ is not the first eigenvalue of (2.1).

This shows that if $(\lambda, u) \in S_{1,\delta}$, with $\lambda > 0$ and $\delta \in (0, \delta_0]$, then $\|u\|_\infty > c(\lambda)$. Passing to the limit as $\delta \to 0$ it follows that if $(\lambda, u) \in \Gamma_1$ then $\|u\|_\infty \ge c(\lambda)$.

When we consider Γ_k with $k > 1$ the argument is similar. If $(\lambda, u) \in S_{k,\delta}$, then there exists at least one interval I_k with length $1/k$ where u has constant sign. Therefore if we restrict the discussion to the interval I_k and replace λ_1 by the first eigenvalue of (2.1) on the interval I_k, then we can get the same contradiction as before.

(ii) From (i), we have for any $(\lambda, u) \in \Gamma_k$, if $\lambda > 0$, then $u \neq 0$.

Let $\{(\lambda_n, u_n)\} \subseteq S_{k,n}$ be a sequence, converging to (λ, u) in $\mathbb{R} \times X$. First, if $k = 1$, then we have $u_n > 0$ in $[0,1)$, therefore $u \ge 0$, moreover, the strong Maximum Principle yields that $u > 0$ in $[0,1)$.

Next, if $k > 1$, then let $\{x_n\}$ and $\{y_n\}$ be two consecutive zeros of u_n with $x_n \to \xi$ and $y_n \to \eta$. Obviously, $u(\xi) = u(\eta) = 0$. We claim that $\xi \neq \eta$. Otherwise, there exists a third sequence $\{z_n\}$ such that $u_n'(z_n) = 0$ and $\lim_{n\to\infty} z_n = \xi$. Therefore, we can find a u, it is a solution of

$$Au = \lambda h(u) + g(r,u),$$

and satisfies

$$u(\xi) = u'(\xi) = 0.$$

However, from Lemma 3.1, we know this is impossible. Therefore, we conclude that for any $(\lambda, u) \in \Gamma_k$ and $\lambda > 0$, u has exactly $k-1$ simple zeros in the interval $(0,1)$.

(iii) Suppose on the contrary that there exists a sequence $\{(\lambda_n, u_n)\} \subseteq S_{k,n}$ such that $\lambda_n \to 0$, $u_n \to u$ and $\|u_n\| = \rho \leq \rho_0$. Passing to the limit we find that $u \neq 0$ is a solution of (3.1) and u satisfies $\|u\| \leq \rho_0$, however this contradicts Lemma 3.2. □

Example 3.3. Let us consider the following Dirichlet problem with mean curvature operator in the Minkowski space

$$\begin{cases} -\text{div}\left(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) = \lambda h(u) + g(r,u), & r = |x| < 1, \\ u = 0, & r = |x| = 1, \end{cases} \tag{3.9}$$

where

$$h(u) = \begin{cases} \sqrt{u}, & u \geq 0, \\ -\sqrt{-u}, & u < 0, \end{cases}$$

and

$$g(r,u) = \begin{cases} u^2, & u \geq 0, \\ -u^2, & u < 0. \end{cases}$$

Obviously, $q = \frac{3}{2}$ and all assumptions of Theorem 1.1 are valid. Therefore, from Theorem 1.1, we know there are infinitely many unbounded component of radial nodal solutions of (3.9) branching off from $(0,0)$.

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