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Antisymmetric solutions for a class of quasilinear defocusing Schrödinger equations

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Abstract. In this paper we consider the existence of antisymmetric solutions for the quasilinear defocusing Schrödinger equation in $H^1(\mathbb{R}^N)$:

$$-\Delta u + \frac{k}{2} u \Delta u^2 + V(x)u = g(u),$$

where $N \geq 3$, $V(x)$ is a positive continuous potential, $g(u)$ is of subcritical growth and k is a non-negative parameter. By considering a minimizing problem restricted on a partial Nehari manifold, we prove the existence of antisymmetric solutions via a deformation lemma.

Keywords: quasilinear Schrödinger equation, antisymmetric solutions, Nehari manifold.

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1 Introduction and main results

In this paper we are interested in the existence of antisymmetric solutions in $H^1(\mathbb{R}^N)$ for the modified quasilinear Schrödinger equation

$$-\Delta u + \frac{k}{2} u \Delta u^2 + V(x)u = g(u) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $V : \mathbb{R}^N \to (0, \infty)$ is a continuous and positive potential function, $g : \mathbb{R} \to \mathbb{R}$ is a continuous and subcritical function, $k \geq 0$ is a parameter. The existence of solutions for (1.1) is closely related to study of standing waves $\omega(x,t) = u(x)e^{-(iEt)/\hbar}$ for the superfluid film equation arising in the plasma physics (see [9]),

$$i\hbar\partial_t \omega = -\Delta\omega + W(x)\omega - \widetilde{h}(|\omega|^2)\omega + \frac{k}{2}\omega\Delta\omega^2, \tag{1.2}$$

where $W(x)$ is a given potential and $\widetilde{h}(u^2)u = g(u)$ is a real function. So, $\omega(x,t)$ will be a such solution of (1.2) if and only if $u(x)$ solves equation (1.1) with $V(x) = W(x) - E$.

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For the case $k = 0$, equation (1.1) becomes a semilinear Schrödinger equation. The existence of positive ground states or least action nodal solutions for the semilinear Schrödinger equation has been studied widely, we refer the readers to [3,8,24,26] and the references therein for the literature on nodal solutions of the semilinear Schrödinger equation.

For $k = -1$, the modified quasilinear Schrödinger equation has received a lot of attention. The appearance of the quasilinear part $u\Delta u^2$ makes the problem much more complicated, it is quite difficult to study the associated energy functional directly in the Sobolev space $H^1(\mathbb{R}^N)$ and requires one to develop new techniques to apply variational methods. The existence of a positive ground state solution of equation (1.1) has been proved in [16] and [25] by introducing a parameter λ in front of the nonlinear term. In [17], by a change of variables, the authors studied the quasilinear problem was transformed to a semilinear one and the existence of a positive solution was proved using the Mountain-Pass Lemma in an Orlicz space. Different from the change of variable methods, in [20] the authors introduced new perturbation techniques and also proved the existence of solutions for a new kind of critical problems for the modified quasilinear Schrödinger equation in [21].

The existence of sign-changing solution is an interesting topic i.e. looking for solutions u with $u^+, u^- \neq 0$, where $u^+(x) = \max\{u(x), 0\} \geq 0$, and $u^-(x) = \min\{u(x), 0\} \leq 0$, $x \in \mathbb{R}^N$. In [18] the authors proved the existence of sign-changing ground state solution for (1.1) with $k = -1$ and $g(s) = |s|^{p-2}s$, $s \in \mathbb{R}$ with $3 \leq p < 22^* - 1$, that is, g having subcritical growth (22^* plays the role of critical exponent here), and V is a continuous function such that $0 < V_0 = \inf_{\mathbb{R}^N} V(x) \leq \lim_{|x|\to\infty} V(x) = V_\infty$ with $V(x) \leq V_\infty - A/(1 + |x|^m)$, for $|x| \geq M$, for some real constants $A, M, m > 0$. The perturbation arguments in [21] was successfully applied to study the existence of multiple nodal solutions for a general class of sub-critical quasilinear Schrödinger equation in [19].

Also, we would also like to mention [10, 11, 13, 15, 18] and references therein for some recent progress of the study of the quasilinear Schrödinger equation for $k < 0$. However, in [12, 14], the nonlinearity g is permitted to behave in a critical way, under the more restrictive assumption that V is symmetric radially positive and differentiable continuous function with $V'(r) \geq 0$ for $r \geq 0$. Their approach was based on Mountain Pass Theorem on Nehari manifolds.

But, for the case $k > 0$, it seems that there are few work about this type of problems. The existence results of solutions, we like to mention [1] and the existence of sign-changing solutions, we like to mention [2].

The existence of τ-antisymmetric solutions, in [5] and [6], the authors proved existence of τ-antisymmetric solutions for the problem

$$-\Delta u + V(x)u = g(u) \quad \text{in } \mathbb{R}^N,$$

by considering the limit problem

$$-\Delta u + V_\infty u = g(u) \quad \text{in } \mathbb{R}^N.$$

In [7], the authors showed the existence of τ-antisymmetric solutions for the system

$$\begin{cases} -\Delta u + u = |u|^{2p-2}u + \beta(x)|v|^p|u|^{p-2}u, & \text{in } \mathbb{R}^N, \\ -\Delta v + \omega^2 v = |v|^{2p-2}v + \beta(x)|u|^p|v|^{p-2}v, & \text{in } \mathbb{R}^N \end{cases}$$

under suitable assumptions by considering the limit problem

$$\begin{cases} -\Delta u + u = |u|^{2p-2}u + \beta_\infty|v|^p|u|^{p-2}u, & \text{in } \mathbb{R}^N, \\ -\Delta v + \omega^2 v = |v|^{2p-2}v + \beta_\infty|u|^p|v|^{p-2}v, & \text{in } \mathbb{R}^N, \end{cases}$$

and other additional conditions.

However, for the case $k \neq 0$, it seems that the existence results of solutions of τ-antisymmetric solutions to equation (1.1) has not been considered yet. Thus the aim of the present paper is to study the existence of τ-antisymmetric solution for a quasilinear defocusing Schrödinger equation.

To state the main results, we may assume that the potential function V is continuous such that $V(x) \geq V_0 > 0$ for all $x \in \mathbb{R}^N$, and:

(V_1) $V(\tau x) = V(x)$, where $\tau : \mathbb{R}^N \to \mathbb{R}^N$ is a nontrivial orthogonal involution that is a linear orthogonal transformation on $\mathbb{R}^N$ such that $\tau \neq \mathrm{Id}$ and $\tau^2 = \mathrm{Id}$;

(V_2) V is 1-periodic in x_i, $1 \leq i \leq N$;

(V_3) V is radially symmetric, i.e. $V(x) = V(|x|)$ and $V \in L^\infty(\mathbb{R}^N)$;

(V_4) $\lim_{|x|\to\infty} V(x) = \infty$.

The nonlinearity g is supposed to satisfy:

(G_1) $g \in C(\mathbb{R}, \mathbb{R})$ is such that $g(0) = 0$ and odd;

(G_2) $\lim_{|t|\to 0} \frac{g(t)}{t} = 0$ and $\limsup_{|t|\to\infty} \frac{g(t)}{|t|^{q-1}} < \infty$ for some $q \in (2, 2^*)$;

(G_3) $0 < \theta G(s) \leq s g(s)$, $s \neq 0$ for some $2 < \theta < 2^*$, where $G(u) = \int_0^u g(t)dt$;

(G_4) $t \longmapsto \frac{g(t)}{t^\rho}$, $t > 0$ is non-decreasing for some $\rho > 1$.

Our principal result shows the existence of a τ-antisymmetric solution, that is u satisfies (1.1) and $u(\tau x) = -u(x)$.

Theorem 1.1. *Suppose that (V_1) holds and one of (V_2), (V_3) and (V_4) is satisfied and the conditions (G_1)–(G_4) hold. Then there exists $k_0 > 0$ such that for each $k \in (0, k_0)$ equation (1.1) has at least one τ-antisymmetric solution $u \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with*

$$\max_{x\in\mathbb{R}^N} |u(x)| \leq \frac{\sigma}{\sqrt{k}}, \quad \text{where } \sigma = \left[\left(4 - \frac{1}{\rho} - \sqrt{\frac{1}{\rho^2} + \frac{8}{\rho}}\right)/8\right]^{1/2}. \tag{1.3}$$

The antisymmetric solution found in Theorem 1.1 minimizes the energy functional among all possible solutions for (1.1), and so we can call it the least action antisymmetric solution.

This work contributes to the literature of modified quasilinear defocusing Schrödinger equation in the two senses: on the hand, we found an τ-antisymmetric solution instead of a limit problem, we used several different conditions of the function V; on the other hand, we just need the function g to be continuous, so we can not use directly Ekeland's variational principle.

The paper is organized as follows. In Section 2, we introduce the variational framework for the quasilinear defocusing Schrödinger equation. In Section 3, establishing some auxiliary lemmas and build a homeomorphism between sphere and Nehari manifold. Finally in Section 4, we prove the existence of τ-antisymmetric solution for (1.1) with subcritical growth and obtaining a L^∞-estimate.

Notation

We will use the following notations frequently:
- $C, C_0, C_1, C_2, \ldots$ denote positive (possibly different) constants.
- B_R denotes the open ball centered at the origin with radius $R > 0$.
- $C_0^\infty(\mathbb{R}^N)$ denotes functions infinitely differentiable with compact support in $\mathbb{R}^N$.
- For $1 \le s \le \infty$, $L^s(\mathbb{R}^N)$ denotes the usual Lebesgue space with the norms

$$|u|_s := \Big(\int_{\mathbb{R}^N} |u|^s \Big)^{1/s}, \quad 1 \le s < \infty;$$
$$|u|_\infty := \inf\{C > 0 : |u(x)| \le C \text{ almost everywhere in } \mathbb{R}^N\}.$$

- $H^1(\mathbb{R}^N)$ denotes the Sobolev spaces with usual norm

$$\|u\|_{1,2} := \big(|\nabla u|_2^2 + |u|_2^2\big)^{1/2}.$$

- The weak convergence in $H^1(\mathbb{R}^N)$ is denoted by $\rightharpoonup$, and the strong convergence by $\to$.

2 The modified problem

Formally, this equation has a variational structure, that is, by considering

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} (1 - k|u|^2)|\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u|^2 - \int_{\mathbb{R}^N} G(u),$$

a function $u \in H^1(\mathbb{R}^N)$ is said to be a weak solution of equation (1.1) if it satisfies

$$\int_{\mathbb{R}^N} (1 - k|u|^2)\nabla u \nabla \varphi - k\int_{\mathbb{R}^N} |\nabla u|^2 u\varphi + \int_{\mathbb{R}^N} V(x)u\varphi = \int_{\mathbb{R}^N} g(u)\varphi$$

for all $\varphi \in H^1(\mathbb{R}^N)$, which means $\langle I'(u), \varphi\rangle = 0$ for all $\varphi \in H^1(\mathbb{R}^N)$.

First, we point out that, under the hypothesis $V(x) \ge V_0 > 0$ for all $x \in \mathbb{R}^N$, the subset

$$E = \left\{ u \in H^1(\mathbb{R}^N) \;\Big|\; \int_{\mathbb{R}^N} V(x)u^2(x) < \infty \right\}$$

is a closed subspace of $H^1(\mathbb{R}^N)$. Moreover,

$$\|u\|_E^2 = \int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} V(x)u^2(x)$$

defines a norm on E. However, the presence of the second order nonhomogeneous term $u\Delta u^2$ prevents us to work directly with the functional I, because it is not even well defined in general in $H^1(\mathbb{R}^N)$.

In order to prove the main results, we first establish the existence of nontrivial solution for a modified quasilinear Schrödinger equation. More precisely, we will show the existence of sign changing solutions for the following quasilinear Schrödinger equations

$$-\,div(l^2(u)\nabla u) + l(u)l'(u)|\nabla u|^2 + V(x)u = g(u), \qquad x \in \mathbb{R}^N \tag{2.1}$$

with $l(t) = \sqrt{1 - kt^2}$ for $|t| < \sigma/\sqrt{k}$ for $k > 0$, where $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function and $\sigma > 0$ was chosen in (1.3). Clearly, when $l(t) = \sqrt{1 - kt^2}$, we derive that (2.1) turns into

(1.1). Then, by using Morse type L^∞-estimate, we will prove that there exist k_0 such that for all $k \in [0, k_0)$ the solution found verifies the estimate $\max_{\mathbb{R}^N} |u| < \sigma/\sqrt{k}$. After that, we conclude that the solutions obtained are solutions of the original equation (1.1).

For the equation (2.1), we will consider $l : \mathbb{R} \to \mathbb{R}$ defined by

$$l(t) = \begin{cases} \sqrt{1-kt^2}, & \text{if } 0 \le t < \frac{\sigma}{\sqrt{k}}, \\ \dfrac{\sigma^3\sqrt{k}}{kt\sqrt{1-\sigma^2}} + \sqrt{\dfrac{1}{\rho}}, & \text{if } t \ge \frac{\sigma}{\sqrt{k}}, \end{cases}$$

and $l(t) = l(-t)$ for all $t \le 0$. So, it follows from the choice of $\sigma = \sigma(\rho) > 0$ for $\rho > 1$ in (1.3) that $l \in C^1(\mathbb{R}, (\sqrt{1/\rho}, 1))$ is an even function and it increases in $(-\infty, 0)$ and decreases in $[0, +\infty)$.

Note that (2.1) is the Euler–Lagrange equation associated to the energy functional

$$I_k(u) = \frac{1}{2}\int_{\mathbb{R}^N} l^2(u)|\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u|^2 - \int_{\mathbb{R}^N} G(u) \tag{2.2}$$

for $|u| < \sigma/\sqrt{k}$.

In the sequel, we will prove the existence of nontrivial antisymmetric critical points u of (2.2) satisfying $\sup_{x\in\mathbb{R}^N} |u(x)| \le \sigma/\sqrt{k}$. This means that it is a nontrivial antisymmetric solution of (2.1) with $l(u) = \sqrt{1-ku^2}$, and so, a nontrivial antisymmetric solution of (1.1) can be got from the function l.

In what follows, we set

$$L(t) = \int_0^t l(s)ds, \qquad t \in \mathbb{R}.$$

By a simple computation, we see that the inverse function $L^{-1}(t)$ exists and it is an odd function. Moreover, it is very important to note that $L, L^{-1} \in C^2(\mathbb{R})$. The lemma below shows some important properties of the functions l and L^{-1} that will be used in the later part of the paper.

Remark 2.1. From assumption (G_4), if $\rho_2 > \rho_1 > 1$ and $g(t)/t^{\rho_2}$ is non-decreasing, then $g(t)/t^{\rho_1}$ is non-decreasing as well. Thus, if $g(t)/t^\rho$ is non-decreasing for some $\rho > 1$, we can assume that ρ is sufficiently close to 1, satisfying

$$4 + \frac{1}{\rho} + \sqrt{\frac{1}{\rho^2} + \frac{8}{\rho}} > \frac{8}{\sqrt{\rho}} \quad \text{and} \quad 2 < 2\sqrt{\rho} < \theta. \tag{2.3}$$

Throughout the paper, we need the following lemma. Its proof can be found in [1] and [2].

Lemma 2.2. *The functions l and L^{-1} satisfy:*

(1) $\lim_{t\to 0} \frac{L^{-1}(t)}{t} = 1$;

(2) $\lim_{t\to\infty} \frac{L^{-1}(t)}{t} = \sqrt{\rho}$;

(3) $\sqrt{\frac{1}{\rho}}t \le l(t)t \le L(t) \le t$ *and* $t \le L^{-1}(t) \le \sqrt{\rho}t$, *for all* $t \ge 0$;

(4) $-\frac{\sigma^2}{1-\sigma^2} \le \frac{t}{l(t)}l'(t) \le 0$, *for all* $t \ge 0$;

(5) $\frac{[L^{-1}(t)]^\delta}{l(L^{-1}(t))t}$, $t > 0$ *is increasing for* $\delta > 1$ *and non-decreasing for* $\delta = 1$,

(6) $\frac{L^{-1}(t)}{l(L^{-1}(t))t^\rho}$, $t > 0$ *is decreasing for* $\rho > 1$ *close to* 1 *and* $\frac{L^{-1}(t)}{t}$, $t > 0$ *is non-decreasing.*

Now, changing variable by

$$v = L(u) = \int_0^u l(s)ds,$$

we can observe that the functional I_k can be rewritten in the form

$$J_k(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|L^{-1}(v)|^2 - \int_{\mathbb{R}^N} G(L^{-1}(v)).$$

From Lemma 2.2, J_k is well defined in $H^1(\mathbb{R}^N)$ and $J_k \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ with

$$\langle J_k'(v), \phi \rangle = \int_{\mathbb{R}^N} \left[\nabla v \nabla \phi + V(x)\frac{L^{-1}(v)}{l(L^{-1}(v))}\phi - \frac{g(L^{-1}(v))}{l(L^{-1}(v))}\phi \right], \tag{2.4}$$

for all $v, \phi \in H^1(\mathbb{R}^N)$.

Lemma 2.3. *If $v \in H^1(\mathbb{R}^N)$ is a critical point of J_k, then $u = L^{-1}(v) \in H^1(\mathbb{R}^N)$ and additionally it is a weak solution for (2.1) if $\sup_{x\in\mathbb{R}^N} |u(x)| \leq \sigma/\sqrt{k}$.*

Proof. See [2]. □

The following embedding result plays an important role in showing that the minimizing function on the partial Nehari manifold are non-trivial functions.

Proposition 2.4. *The function L^{-1} is such that:*

1. *the map $v \longmapsto L^{-1}(v)$ from $(E, \|\cdot\|_E)$ to $(L^s(\mathbb{R}^N), |\cdot|_s)$ is continuous for $2 \leq s \leq 2^*$.*
2. *under (V_4), the above map is compact for $2 \leq s < 2^*$, and under (V_3) with $N \geq 2$, this map is compact for $2 < s < 2^*$.*

Proof. See [2]. □

3 Auxiliary results

Before stating the auxiliary results, let us point out some consequences of our hypotheses.

Remark 3.1. From assumption (G_2), there exists $c_\epsilon > 0$ such that

$$g(t)t \leq \epsilon|t|^2 + c_\epsilon|t|^q \qquad \forall\, t \in \mathbb{R}$$

for each $\epsilon > 0$ given.

Remark 3.2. From assumption (G_3), there exists a constant $K > 0$ such that

$$G(t) \geq K|t|^\theta \quad \text{for all } |t| > \delta$$

for each $\delta > 0$ given.

After these, let us associate to the functional J_k the Nehari manifold

$$\mathcal{N} = \{v \in E\backslash\{0\} \mid \langle J_k'(v), v\rangle = 0\}.$$

In order to find τ-antisymmetric solutions, we look for critical points of the functional J_k on

$$\mathcal{N}^\tau = \{v \in \mathcal{N} \mid v(\tau x) = -v(x)\} \subset \mathcal{N}.$$

The involution τ on $\mathbb{R}^N$ induces an involution $T_\tau : E \to E$ given by

$$T_\tau(v(x)) := -v(\tau(x)).$$

We denote by $E^\tau := \{u \in E : T_\tau(v(x)) = v(x)\}$ the subspace of τ-invariant functions of E, we have

$$\mathcal{N}^\tau = \mathcal{N} \cap E^\tau.$$

Now, we are going to introduce the differentiable continuous function $h_k^v : [0, \infty) \to \mathbb{R}$ by setting $h_k^v(t) = J_k(tv)$, that is,

$$h_k^v(t) := \frac{1}{2}\int_{\mathbb{R}^N} |t\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|L^{-1}(tv)|^2 - \int_{\mathbb{R}^N} G(L^{-1}(tv)),$$

for each $v \in E$ with $v \neq 0$.

Lemma 3.3. *Assume that (G_1)–(G_3) hold. If $v \in E^\tau$ with $v \neq 0$, then there exist $\alpha > 0$ such that*

$$\langle J_k'(\alpha v), v\rangle = 0,$$

that is, $\alpha v \in \mathcal{N}^\tau$, and $\alpha \in (0, \infty)$ is a critical point of h_k^v.

Proof. It follows from the definition of h_k^v, that

$$\begin{aligned}\frac{\partial h_k^v(t)}{\partial t} &= t\int_{\mathbb{R}^N} |\nabla v|^2 + \int_{\mathbb{R}^N} V(x)\frac{L^{-1}(tv)}{l(L^{-1}(tv))}v - \int_{\mathbb{R}^N} \frac{g(L^{-1}(tv))}{l(L^{-1}(tv))}v \\ &= \langle J_k'(tv), v\rangle.\end{aligned} \tag{3.1}$$

So, it follows from Remark 3.1 and (3) of Lemma 2.2, that

$$\begin{aligned}\langle J_k'(tv), tv\rangle &\geq t^2\int_{\mathbb{R}^N} |\nabla v|^2 - \int_{\mathbb{R}^N} \frac{g(L^{-1}(tv))}{l(L^{-1}(tv))}tv \\ &\geq t^2\int_{\mathbb{R}^N} |\nabla v|^2 - \int_{\mathbb{R}^N} \frac{\epsilon|L^{-1}(tv)|^2 + c_\epsilon|L^{-1}(tv)|^q}{\sqrt{1/\rho}|L^{-1}(tv)|}|tv| \\ &\geq t^2|\nabla v|_2^2 - \rho\epsilon t^2|v|_2^2 - \sqrt{\rho}^q c_\epsilon t^q|v|_q^q,\end{aligned}$$

which means there exists $t_m > 0$ sufficiently small such that

$$\langle J_k'(t_m v), t_m v\rangle > 0,$$

since $q > 2$.

On the other hand, it follows from Hypothesis (G_3) that

$$\langle J_k'(tv), tv\rangle \leq t^2\int_{\mathbb{R}^N} |\nabla v|^2 + \int_{\mathbb{R}^N} V(x)\frac{L^{-1}(tv)}{l(L^{-1}(tv))}(tv) - \theta\int_{\mathbb{R}^N} \frac{G(L^{-1}(tv))}{l(L^{-1}(tv))L^{-1}(tv)}(tv).$$

Set $\delta > 0$ such that the set

$$\mathcal{A} = \{x \in \mathbb{R}^N;\ |v(x)| \geq \delta\} \subset \mathbb{R}^N$$

is not empty. By Remark 3.2; $l(t) > 1/\sqrt{\rho}$, $t > 0$; and (3) of Lemma 2.2, we get

$$\langle J_k'(tv), tv\rangle \leq t^2\int_{\mathbb{R}^N} |\nabla v|^2 + \sqrt{\rho}t^2\int_{\mathbb{R}^N} V(x)v^2 - \theta K t^\theta\int_{\mathcal{A}} |v|^\theta$$

for $t > 0$.

As a consequence, we obtain $t_M > 0$ sufficiently large such that

$$\langle J_k'(t_M v), t_M v\rangle < 0,$$

since $\theta > 2$. Hence, the lemma follows from intermediate value theorem. $\square$

Lemma 3.4. *If $v \in \mathcal{N}$ and (G_4) hold, then*

$$\frac{\partial h_k^v}{\partial t}(t) > 0 \quad \textit{for } 0 < t < 1, \qquad \frac{\partial h_k^v}{\partial t}(t) < 0 \quad \textit{for } t > 1,$$

In particular, $h_k^v(t) < h_k^v(1) = J_k(v)$ for all $t \geq 0$ such that $t \neq 1$.

Proof. By the facts of l being even and L odd, it is sufficiently to prove the case of that $v \geq 0$. First, it follows from (3.1) that

$$\frac{\partial h_k^v(t)}{\partial t} = t^\rho \left\{ \int_{\mathbb{R}^N} \frac{|\nabla v|^2}{t^{\rho-1}} - \int_{\mathbb{R}^N} \left[\frac{g(L^{-1}(tv))}{l(L^{-1}(tv))(tv)^\rho} - \frac{V(x)L^{-1}(tv)}{l(L^{-1}(tv))(tv)^\rho} \right] v^{\rho+1} \right\}.$$

Now, by using (G_4), (5), (6) of Lemma 2.2, and the monotonicity of l, L^{-1}, we obtain

$$\begin{aligned}
&\frac{g(L^{-1}(tv))}{l(L^{-1}(tv))(tv)^\rho} - \frac{V(x)L^{-1}(tv)}{l(L^{-1}(tv))(tv)^\rho} \\
&= \frac{g(L^{-1}(tv))}{(L^{-1}(tv))^\rho} \left[\frac{(L^{-1}(tv))}{tv} \right]^\rho \frac{1}{l(L^{-1}(tv))} - V(x) \frac{L^{-1}(tv)}{l(L^{-1}(tv))(tv)^\rho} \\
&< \frac{g(L^{-1}(v))}{(L^{-1}(v))^\rho} \left[\frac{(L^{-1}(v))}{v} \right]^\rho \frac{1}{l(L^{-1}(v))} - V(x) \frac{L^{-1}(v)}{l(L^{-1}(v))(v)^\rho} \\
&= \frac{g(L^{-1}(v))}{l(L^{-1}(v))(v)^\rho} - V(x) \frac{L^{-1}(v)}{l(L^{-1}(v))(v)^\rho}
\end{aligned}$$

for $0 < t < 1$, and in a similar way, we obtain

$$\frac{g(L^{-1}(tv))}{l(L^{-1}(tv))(tv)^\rho} - \frac{V(x)L^{-1}(tv)}{l(L^{-1}(tv))(tv)^\rho} > \frac{g(L^{-1}(v))}{l(L^{-1}(v))(v)^\rho} - \frac{V(x)L^{-1}(v)}{l(L^{-1}(v))(v)^\rho}$$

for $t > 1$.

So, it follows from above informations, and the hypothesis $v \in \mathcal{N}$, that

$$\frac{\partial h_k^v}{\partial t}(t) > 0 \quad \text{for } 0 < t < 1, \quad \text{and} \quad \frac{\partial h_k^v}{\partial t}(t) < 0 \quad \text{for } t > 1. \tag{3.2}$$

That is, $h_k^v(t) < h_k^v(1) = J_k(v)$. So, the lemma is proved. $\square$

It follows from above informations, that:

Remark 3.5. If $v \in \mathcal{N}$, then 1 is an unique critical point of h_k^v.

Remark 3.6. If $v \in E$ with $v \neq 0$, then the critical point $\alpha = \alpha_v \in (0, +\infty)$ of h_k^v, given by Lemma 3.3, is unique.

In fact, by Lemma 3.3 there is $\alpha > 0$ such that α is a critical point of h_k^v. Finally, assume that α_1 and α_2 are two critical points of h_k^v, then

$$\frac{\alpha_2}{\alpha_1}(\alpha_1 v) = \alpha_2 v.$$

Since $\alpha_1 v \in \mathcal{N}$, then by the Remark 3.5, we have $\alpha_2/\alpha_1 = 1$, and so $\alpha_1 = \alpha_2$.

The following two lemmas are important to prove our theorem, the proofs can be found in [2]

Lemma 3.7. *Assume that V is continuous such that $V(x) \geq V_0 > 0$ for all $x \in \mathbb{R}^N$ and (G_1)–(G_3) hold. Then:*

(i) for all $v \in \mathcal{N}$, we have

$$J_k(v) \geq \frac{\theta - 2\sqrt{\rho}}{2\theta}\left(\int_{\mathbb{R}^N} |\nabla v|^2 + \int_{\mathbb{R}^N} V(x)|L^{-1}(v)|^2\right),$$

(ii) there is $\gamma > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla v|^2 + \int_{\mathbb{R}^N} V(x)|L^{-1}(v)|^2 \geq \gamma, \quad \textit{for all } v \in \mathcal{N}.$$

Lemma 3.8. *Assume the same hypotheses of Lemma 3.7, and (v_n) being a sequence in $\mathcal{N}$. Then*

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} |L^{-1}(v_n)|^q dx > 0$$

for some $q \in (2, 2^*)$.

Remark 3.9. By Lemma 3.8 and (3) of Lemma 2.2, there exists a constant $\gamma_1 > 0$ such that

$$\int_{\mathbb{R}^N} |v_n|^q \geq \gamma_1 > 0.$$

Lemma 3.10. *Assume that (G_4) hold. If $\mathcal{V} \subset S^\tau$ is a compact subset of E^τ, then there exists $R > 0$ such that $J_k \leq 0$ on $(\mathbb{R}^+\mathcal{V}) \setminus B_R(0)$, where $S^\tau := \{u \in E^\tau;\ \|u\|_E = 1\}$.*

Proof. Arguing by contradiction, suppose there exits $u_n \in \mathcal{V}$ and $w_n = t_n u_n$ such that $J_k(w_n) \geq 0$ and $t_n \to \infty$ as $n \to \infty$.

By the definition of J_k and (3) of Lemma 2.2 have

$$J_k(w_n) \leq \frac{\rho}{2}\|w_n\|_E - \int_{\mathbb{R}^N} G(L^{-1}(w_n)) = \frac{\rho}{2}t_n^2 - \int_{\mathbb{R}^N} G(L^{-1}(w_n)).$$

Using (G_4), we have $t \longmapsto \frac{G(t)}{t^{\rho+1}}$, $t > 0$ is non-decreasing for some $\rho > 1$ and

$$\frac{G(L^{-1}(w))}{L^{-1}(w)^2} \to \infty \quad \text{uniformly in } x \text{ as } |w| \to \infty. \tag{3.3}$$

Passing to a subsequence, we may assume that $u_n \to u \in S^\tau$. Since $|w_n(x)| \to \infty$ if $u(x) \neq 0$, it follows from (3) of Lemma 2.2, (3.3) and Fatou's lemma that

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{G(L^{-1}(w_n))}{t_n^2} &= \int_{\mathbb{R}^N} \frac{G(L^{-1}(w_n))u_n^2}{w_n^2} \\ &= \int_{\mathbb{R}^N} \frac{G(L^{-1}(w_n))}{L^{-1}(w_n)^2} \frac{L^{-1}(w_n)^2}{w_n^2} u_n^2 \to \infty\end{aligned}$$

Hence

$$0 \leq J_k(w_n) \leq t_n^2 \left[\frac{\rho}{2} - \int_{\mathbb{R}^N} \frac{G(L^{-1}(w_n))}{t_n^2} \right] \to -\infty,$$

a contradiction. □

Recall that S is the unit sphere in E and define the mapping $m : S \to \mathcal{N}$ by setting

$$m(w) := t_w w,$$

where t_w is as α in Lemma 3.3. Moreover, $\|m(w)\|_E = t_w$.

Recall that S^τ is the unit sphere in E^τ, and consider the mapping $m^\tau : S^\tau \to \mathcal{N}^\tau$ by setting

$$m^\tau := m|_{S^\tau}.$$

We shall consider the functional

$$\psi_k^\tau(w) := J_k(m^\tau(w)).$$

By Lemma 3.3, Lemma 3.4, Remark 3.5, Lemma 3.7 and Lemma 3.10, we have the following two lemmas, similar to the results in [23].

Lemma 3.11. *The mapping m^τ is a homeomorphism between S^τ and $\mathcal{N}^\tau$, and the inverse of m^τ is given by $(m^\tau)^{-1}(u) = \frac{u}{\|u\|_E}$.*

Lemma 3.12.

(1) $\psi_k^\tau \in C^1(S^\tau, \mathbb{R})$ and

$$\langle (\psi_k^\tau)'(w), z \rangle = \|m^\tau(w)\|_E \langle J_k'(m^\tau(w)), z \rangle \quad \text{for all } z \in T_w(S^\tau) \subset E^\tau.$$

(2) If (w_n) is a Palais–Smale sequence for ψ_k^τ, then $(m^\tau(w_n))$ is a Palais–Smale sequence for J_k. If $(u_n) \subset \mathcal{N}^\tau$ is a bounded Palais–Smale sequence for J_k, then $((m^\tau)^{-1}(u_n))$ is a Palais–Smale sequence for ψ_k^τ.

(3) w is a critical point of ψ_k^τ if and only if $m^\tau(w)$ is a nontrivial critical point of $J_k|_{E^\tau}$. Moreover, the corresponding values of ψ_k^τ and J_k coincide and $\inf_{S^\tau} \psi_k^\tau = \inf_{\mathcal{N}^\tau} J_k$.

(4) If J_k is even, then so is ψ_k^τ.

4 Proof of Theorem 1.1

Now, we are ready to prove Theorem 1.1 by applying the auxiliary results in Section 3.

Proof of Theorem 1.1. It follows from Lemma 3.7 that there exists $c_0 > 0$ such that

$$c_0 = \inf_{w \in \mathcal{N}^\tau} J_k(w).$$

Moreover, if $u_0 \in \mathcal{N}^\tau$ satisfies $J_k(u_0) = c_0$, then $(m^\tau)^{-1}(u_0) \in S^\tau$ is a minimizer of ψ_k^τ and therefore a critical point of ψ_k^τ, so that u_0 is a critical point of J_k in E^τ by Lemma 3.12. We will show that there exists a minimizer $v \in \mathcal{N}^\tau$ of $J_k|_{\mathcal{N}^\tau}$. By Ekeland's variational principle [27], there exists a sequence $(w_n) \subset S^\tau$ with $\psi_k^\tau(w_n) \to c_0$ and $(\psi_k^\tau)'(w_n) \to 0$ as $n \to \infty$. Put $u_n = m^\tau(w_n) \in \mathcal{N}^\tau$ for $n \in \mathbb{N}$. Then $J_k(u_n) \to c_0$ and $J_k'(u_n) \to 0$ as $n \to \infty$ by Lemma 3.12 (2).

<u>Claim:</u> $(u_n) \subset E^\tau$ is bounded.

In fact, assume by contradiction that $\|u_n\| \to +\infty$ up to subsequence, that is,

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 + \int_{\mathbb{R}^N} V(x) u_n^2 = \|u_n\|_E^2 \to \infty.$$

So, at least one of the two terms goes to infinity. If

$$\left(\int_{\mathbb{R}^N} |\nabla u_n|^2 \right)^{1/2} \to \infty,$$

it would follow from Lemma 3.7 that

$$J_k(u_n) \geq \frac{\theta - 2\sqrt{\rho}}{2\theta} \int_{\mathbb{R}^N} |\nabla u_n|^2 \to \infty,$$

which is a contradiction, because $(J_k(u_n)) \subset \mathbb{R}$ is bounded. Now, if

$$\int_{\mathbb{R}^N} V(x) u_n^2 \to \infty,$$

then it would follow from Lemma 3.7 again and (3) of Lemma 2.2, that

$$\begin{aligned} J_k(u_n) &\geq \frac{\theta - 2\sqrt{\rho}}{2\theta} \int_{\mathbb{R}^N} V(x) |L^{-1}(u_n)|^2 \\ &\geq \frac{\theta - 2\sqrt{\rho}}{2\theta} \int_{\mathbb{R}^N} V(x) u_n^2 \to \infty, \end{aligned}$$

which is a contradiction again. Hence $u_n \rightharpoonup v$ after passing to a subsequence.

<u>Claim:</u> $v \neq 0$ and $J_k'(v) = 0$ in E^τ.

If (V_2) is fulfilled, then let $y_n \in \mathbb{R}^N$ satisfy

$$\int_{B_1(y_n)} u_n^2 dx = \max_{y \in \mathbb{R}^N} \int_{B_1(y)} u_n^2 dx.$$

Using once more that J_k and $\mathcal{N}^\tau$ are invariant under translations of the form $u \longmapsto u(\cdot - k)$ with $k \in \mathbb{Z}^N$, we may assume that (y_n) is bounded in $\mathbb{R}^N$. If

$$\int_{B_1(y_n)} u_n^2 dx \to 0 \quad \text{as } n \to \infty, \tag{4.1}$$

then $u_n \to 0$ in $L^p(\mathbb{R}^N)$, $2 < p < 2^*$, by Lemma 1.21 in [27]. From Proposition 2.4 and (G_2), we infer that

$$\int_{\mathbb{R}^N} \frac{g(L^{-1}(u_n))u_n}{l(L^{-1}(u_n))}dx = o(\|u_n\|_E)$$

as $n \to \infty$, hence

$$\begin{aligned} o(\|u_n\|_E) = J_k'(u_n)u_n &= \int_{\mathbb{R}^N} |\nabla u_n|^2 + \int_{\mathbb{R}^N} V(x)\frac{L^{-1}(u_n)u_n}{l(L^{-1}(u_n))} - \int_{\mathbb{R}^N} \frac{g(L^{-1}(u_n))u_n}{l(L^{-1}(u_n))}dx \\ &= \int_{\mathbb{R}^N} |\nabla u_n|^2 + \int_{\mathbb{R}^N} V(x)\frac{L^{-1}(u_n)u_n}{l(L^{-1}(u_n))} - o(\|u_n\|_E) \end{aligned}$$

and therefore $\|u_n\|_E \to 0$, contrary to Lemma 3.7. It follows that (4.1) cannot hold, so $u_n \rightharpoonup v \neq 0$ and $J_k'(v) = 0$.

Suppose that (V_3) or (V_4) is satisfied. Then it follows from Proposition 2.4, that

$$L^{-1}(u_n) \to L^{-1}(v) \quad \text{in } L^\gamma(\mathbb{R}^N) \text{ for all } \gamma \in (2, 2^*).$$

Then by Lemma 3.8, we conclude that $v \neq 0$ and $J_k'(v) = 0$ in E^τ.

Hence, we conclude that $v \in \mathcal{N}^\tau$ is a critical point of J_k in E^τ. Now we will show that $J_k(v) = c_0$. By Lemma 2.2, Fatou's lemma and since $(u_n) \subset E^\tau$ is bounded,

$$\begin{aligned} c_0 + o(1) &= J_k(u_n) - \frac{1}{\theta}\langle J_k'(u_n), u_n\rangle \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|L^{-1}(u_n)|^2 dx - \int_{\mathbb{R}^N} G(L^{-1}(u_n))dx \\ &\quad - \frac{1}{\theta}\int_{\mathbb{R}^N} |\nabla u_n|^2 dx - \frac{1}{\theta}\int_{\mathbb{R}^N} V(x)\frac{L^{-1}(u_n)u_n}{l(L^{-1}(u_n))}dx + \frac{1}{\theta}\int_{\mathbb{R}^N} \frac{g(L^{-1}(u_n))u_n}{l(L^{-1}(u_n))}dx \\ &= \frac{\theta-2}{2\theta}\int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|L^{-1}(u_n)|^2 dx - \frac{\sqrt{\rho}}{\theta}V(x)|L^{-1}(u_n)|^2 dx \\ &\quad + \frac{\sqrt{\rho}}{\theta}V(x)|L^{-1}(u_n)|^2 dx - \frac{1}{\theta}\int_{\mathbb{R}^N} V(x)\frac{L^{-1}(u_n)u_n}{l(L^{-1}(u_n))}dx \\ &\quad + \int_{\mathbb{R}^N}\left[\frac{1}{\theta}\frac{g(L^{-1}(u_n))u_n}{l(L^{-1}(u_n))} - G(L^{-1}(u_n))\right]dx \\ &= \frac{\theta-2}{2\theta}\int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \frac{\theta-2\sqrt{\rho}}{2\theta}\int_{\mathbb{R}^N} V(x)|L^{-1}(u_n)|^2 dx \\ &\quad + \frac{1}{\theta}\int_{\mathbb{R}^N} V(x)\left[\sqrt{\rho}|L^{-1}(u_n)|^2 - \frac{L^{-1}(u_n)u_n}{l(L^{-1}(u_n))}\right]dx \\ &\quad + \int_{\mathbb{R}^N}\left[\frac{1}{\theta}\frac{g(L^{-1}(u_n))u_n}{l(L^{-1}(u_n))} - G(L^{-1}(u_n))\right]dx \\ &\geq \frac{\theta-2}{2\theta}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{\theta-2\sqrt{\rho}}{2\theta}\int_{\mathbb{R}^N} V(x)|L^{-1}(v)|^2 dx \\ &\quad + \frac{1}{\theta}\int_{\mathbb{R}^N} V(x)\left[\sqrt{\rho}|L^{-1}(v)|^2 - \frac{L^{-1}(v)v}{l(L^{-1}(v))}\right]dx \\ &\quad + \int_{\mathbb{R}^N}\left[\frac{1}{\theta}\frac{g(L^{-1}(v))v}{l(L^{-1}(v))} - G(L^{-1}(v))\right]dx + o(1) \\ &= J_k(v) - \frac{1}{\theta}\langle J_k'(v), v\rangle + o(1) = J_k(v) + o(1). \end{aligned}$$

On the other hand, since $J_k(v) \geq c_0$, hence $J_k(v) = c_0$.

Now, by using a quantitative deformation lemma and adapting the arguments in [4, 11], we are going to show $J'_k(v) = 0$ in E.

Suppose, by contradiction, that $J'_k(v) \neq 0$. Then there exist $\delta > 0$ and $\nu > 0$ such that

$$\|J'_k(w)\| \geq \nu \quad \text{for every } w \in E \ \text{ with } \|w - v\| \leq 2\delta.$$

Since $v \neq 0$, we can take $L = \|v\|_E > 0$ and, without loss of generality, we may assume $6\delta < L$.

Let $I = \left[\frac{1}{2}, \frac{3}{2}\right]$. Since, $\langle J'_k(v), v\rangle = 0$ and by Lemma 3.4,

$$J_k(tv) < J_k(v) = c_0,$$

holds for $t \in I$ with $t \neq 1$, we obtain that

$$\tilde{c} = \max_{\partial I} J_k(tv) < c_0.$$

Applying Theorem A.4 in [28] with $\epsilon = \min\{(c_0 - \tilde{c})/2, \nu\delta/8\}$ and $S = B(v, \delta)$, there exists $\eta \in C([0,1] \times E, E)$ such that

(i) $\eta(\theta, u) = u$ if $\theta = 0$ or if $u \notin J_k^{-1}[c_0 - 2\epsilon, c_0 + 2\epsilon] \cap B(v, 2\delta)$;

(ii) $\eta(1, J_k^{c_0+\epsilon}) \cap B(v, \delta) \subset J_k^{c_0-\epsilon}$;

(iii) $J_k(\eta(1, w)) \leq J_k(w)$ for every $w \in E$, where $J_k^a = \{w \in E;\ J_k(w) \leq a\}$,

(iv) $\eta(t, u)$ is odd in u.

Consequently, we have

$$\max_{t \in I} J_k(\eta(1, tv)) < c_0. \tag{4.2}$$

On the other hand, we claim that there exists $t_0 \in I$ such that

$$\eta(1, t_0 v) \in \mathcal{N}^\tau.$$

In fact, by (iv) for η, we know $\eta(1, tv) \in E^\tau$ for each t. Now we will prove that there exists $t_0 \in I$ such that $t_0 v \in \mathcal{N}$. Define $\varphi(t) = \eta(1, tv)$ and

$$\Psi(t) = \langle J'_k(\varphi(t)), \varphi(t)\rangle$$

for $t > 0$. Since,

$$\|v - tv\|_E = |1 - t|\|v\|_E = |1 - t|L \geq 6\delta|1 - t| > 2\delta \tag{4.3}$$

if only if $t < \frac{2}{3}$ or $t > \frac{4}{3}$. It follows from property (i) for η and inequality (4.3) that $\varphi(t) = \eta(1, tv) = tv \in E^\tau$ if $t \in [\frac{1}{2}, \frac{2}{3}) \cup (\frac{4}{3}, \frac{3}{2}]$.

Thus,

$$\Psi(\tfrac{1}{2}) = \left\langle J'_k\left(\varphi\left(\tfrac{1}{2}\right)\right), \varphi(\tfrac{1}{2})\right\rangle = \left\langle J'_k\left(\tfrac{1}{2}v\right), \tfrac{1}{2}v\right\rangle,$$

and it follows from (3.2) that

$$\left\langle J'_k\left(\tfrac{1}{2}v\right), \tfrac{1}{2}v\right\rangle = \frac{1}{2}\frac{\partial h_k^v}{\partial t}\left(\tfrac{1}{2}\right) > 0. \tag{4.4}$$

On the other hand,

$$\Psi(\tfrac{3}{2}) = \langle J'_k(\varphi(\tfrac{3}{2})), \varphi(\tfrac{3}{2})\rangle = \langle J'_k(\tfrac{3}{2}v), \tfrac{3}{2}v\rangle,$$

and it follows from (3.2) that

$$\left\langle J_k'\left(\tfrac{3}{2}v\right),\tfrac{3}{2}v\right\rangle = \tfrac{3}{2}\tfrac{\partial h_k^v}{\partial t}\left(\tfrac{3}{2}\right) < 0. \tag{4.5}$$

Noting that the function Ψ is continuous on I and taking (4.4) and (4.5) into account, we can apply the intermediate value theorem again to conclude that there exists $t_0 \in I$ such that $\Psi(t_0) = 0$. This and (4.2) lead to a contradiction. Hence, we conclude that v is a critical point of J_k. So, by Lemma 2.3, we just need to show that $|u|_\infty = |L^{-1}(v)|_\infty \le \sigma/\sqrt{k}$ holds to conclude that u is a solution of problem (1.1).

Now, set $\varphi = L^{-1}(v)l(L^{-1}(v))$. It follows from Lemma 2.2 that

$$|\varphi| = |L^{-1}(v)l(L^{-1}(v))| \le |v|, \quad \text{and} \quad |\nabla\varphi| = \left|1 + \frac{L^{-1}(v)l'(L^{-1}(v))}{l(L^{-1}(v))}\right| |\nabla v| \le |\nabla v|,$$

that is, $\varphi \in H^1(\mathbb{R}^N)$. So, by taking φ as a test function in (2.4), we obtain

$$\int_{\mathbb{R}^N} \left[1 + \frac{L^{-1}(v)l'(L^{-1}(v))}{l(L^{-1}(v))}\right] |\nabla v|^2 + V(x)|L^{-1}(v)|^2 - g(L^{-1}(v))L^{-1}(v) = 0.$$

As a consequence of (4) of Lemma 2.2, we have

$$\int_{\mathbb{R}^N} |\nabla v|^2 + V(x)|L^{-1}(v)|^2 - g(L^{-1}(v))L^{-1}(v) \ge 0.$$

Since v is a critical point of J_k, it follows that

$$\begin{aligned}\theta c_0 &= \theta J_k(v) - \langle J_k'(v), L^{-1}(v)l(L^{-1}(v))\rangle \\ &\ge \frac{\theta - 2}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + V(x)|L^{-1}(v)|^2.\end{aligned}$$

Then, by (3) of Lemma 2.2,

$$\|v\|_E^2 \le \frac{2\theta c_0}{\theta - 2}. \tag{4.6}$$

For each $m \in \mathbb{N}$ and $\beta > 1$ given, define

$$A_m = \{x \in \mathbb{R}^N;\ |v|^{\beta-1} \le m\} \text{ and } B_m = \mathbb{R}^N \backslash A_m,$$

and

$$v_m = \begin{cases} v|v|^{2(\beta-1)} & \text{in } A_m, \\ m^2 v & \text{in } B_m. \end{cases}$$

We know $v_m \in H^1(\mathbb{R}^N)$, $v_m \le v_{m+1}$, $v_m \le |v|^{2\beta-1}$, and

$$\nabla v_m = \begin{cases} (2\beta-1)|v|^{2(\beta-1)}\nabla v & \text{in } A_m, \\ m^2 \nabla v & \text{in } B_m, \end{cases}$$

that is, v_m can be used as a test function. Besides this, we have

$$\int_{\mathbb{R}^N} \nabla v \nabla v_m = (2\beta - 1)\int_{A_m} |v|^{2(\beta-1)}|\nabla v|^2 + m^2 \int_{B_m} |\nabla v|^2. \tag{4.7}$$

Letting

$$w_m = \begin{cases} v|v|^{\beta-1} & \text{in } A_m, \\ mv & \text{in } B_m, \end{cases}$$

we obtain $w_m^2 = vv_m \le |v|^{2\beta}$, $w_m \le w_{m+1}$, and

$$\nabla w_m = \begin{cases} \beta |v|^{\beta-1}\nabla v & \text{in } A_m, \\ m\nabla v & \text{in } B_m. \end{cases}$$

So,

$$\int_{\mathbb{R}^N} |\nabla w_m|^2 = \beta^2 \int_{A_m} |v|^{2(\beta-1)}|\nabla v|^2 + m^2 \int_{B_m} |\nabla v|^2. \tag{4.8}$$

As a consequence of (4.7) and (4.8), we obtain

$$\int_{\mathbb{R}^N} [|\nabla w_m|^2 - \nabla v \nabla v_m] = (\beta-1)^2 \int_{A_m} |v|^{2(\beta-1)}|\nabla v|^2. \tag{4.9}$$

Taking v_m as a test function, it follows from (4.7) and (4.9) that

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla w_m|^2 \quad & +\beta^2 \int_{\mathbb{R}^N} V(x)\frac{L^{-1}(v)}{l(L^{-1}(v))}v_m \\
& = (\beta-1)^2 \int_{A_m} |v|^{2(\beta-1)}|\nabla v|^2 + \int_{\mathbb{R}^N} \nabla v \nabla v_m + \beta^2 \int_{\mathbb{R}^N} V(x)\frac{L^{-1}(v)}{l(L^{-1}(v))}v_m \\
& \le \left[\frac{(\beta-1)^2}{2\beta-1} + 1\right] \int_{\mathbb{R}^N} \nabla v \nabla v_m + +\beta^2 \int_{\mathbb{R}^N} V(x)\frac{L^{-1}(v)}{l(L^{-1}(v))}v_m \\
& \le \beta^2 \int_{\mathbb{R}^N} [\nabla v \nabla v_m + V(x)\frac{L^{-1}(v)}{l(L^{-1}(v))}v_m] \\
& = \beta^2 \int_{\mathbb{R}^N} \frac{g(L^{-1}(v))}{l(L^{-1}(v))}v_m.
\end{aligned}$$

Now, it follows from Remark 3.1 that

$$\begin{aligned}
& \int_{\mathbb{R}^N} |\nabla w_m|^2 + \beta^2 \int_{\mathbb{R}^N} V(x)\frac{L^{-1}(v)}{l(L^{-1}(v))}v_m \\
& \qquad \le \beta^2 \int_{\mathbb{R}^N} \frac{\epsilon|L^{-1}(v)|^2}{|l(L^{-1}(v))L^{-1}(v)|}|v_m| + \beta^2 \int_{\mathbb{R}^N} \frac{c_\epsilon|L^{-1}(v)|^q}{|l(L^{-1}(v))L^{-1}(v)|}|v_m|,
\end{aligned}$$

that is,

$$\beta^2 \int_{\mathbb{R}^N} V(x)\frac{L^{-1}(v)}{l(L^{-1}(v))}v_m \ge \beta^2 \int_{\mathbb{R}^N} \frac{\epsilon|L^{-1}(v)|^2}{|l(L^{-1}(v))L^{-1}(v)|}|v_m|,$$

for $\epsilon > 0$ sufficiently small. So, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla w_m|^2 & \le \beta^2 \int_{\mathbb{R}^N} \frac{c_\epsilon|L^{-1}(v)|^q}{|l(L^{-1}(v))L^{-1}(v)|}|v_m| \\
& \le \beta^2 \int_{\mathbb{R}^N} c_\epsilon \rho^{\frac{q}{2}}|v|^{q-2}w_m^2.
\end{aligned}$$

Then, it follows from the Sobolev inequality that

$$\begin{aligned}
\left(\int_{A_m} |w_m|^{2^*}\right)^{\frac{N-2}{N}} & \le S \int_{\mathbb{R}^N} |\nabla w_m|^2 \\
& \le S\beta^2 \int_{\mathbb{R}^N} c_\epsilon \rho^{\frac{q}{2}}|v|^{q-2}w_m^2.
\end{aligned}$$

The Hölder inequality implies that

$$\left(\int_{A_m} |w_m|^{2^*}\right)^{\frac{N-2}{N}} \le c_\epsilon \rho^{\frac{q}{2}} S\beta^2 |v|_{2^*}^{q-2} \left(\int_{\mathbb{R}^N} |w_m|^{2r_1}\right)^{1/r_1},$$

where $1/r_1 + (q-2)/2^* = 1$.

Since, $|w_m| \leq |v|^\beta$ in $\mathbb{R}^N$ and $|w_m| = |v|^\beta$ in A_m, we have

$$\left(\int_{A_m} |v|^{\beta 2^*}\right)^{\frac{N-2}{N}} \leq c_\epsilon \rho^{\frac{q}{2}} S\beta^2 |v|_{2^*}^{q-2} \left(\int_{\mathbb{R}^N} |v|^{2\beta r_1}\right)^{1/r_1},$$

which implies, by the Monotone Convergence Theorem, that

$$|v|_{\beta 2^*} \leq \beta^{1/\beta} (c_\epsilon \rho^{\frac{q}{2}} S |v|_{2^*}^{q-2})^{1/2\beta} |v|_{2\beta r_1} \tag{4.10}$$

So, taking $\sigma = 2^*/(2r_1)$ and set $\beta = \sigma^i$, $i = 1, 2, \ldots$, in an iterative way in (4.10), we get

$$|v|_{\sigma^i 2^*} \leq \sigma^{(\sum_{j=1}^i j/\sigma^j)} (c_\epsilon \rho^{\frac{q}{2}} S |v|_{2^*}^{q-2})^{(1/2 \sum_{j=1}^i 1/\sigma^j)} |v|_{2^*},$$

that is, by doing $i \to \infty$ and using the limitation of $\|v\|_E$, given by (4.6), together with the Sobolev inequality, we get $|v|_\infty \leq C_0$, where $C_0 > 0$ is a real constant independent of $k > 0$.

Now, it follows from Lemma 2.2-(3) that

$$|u|_\infty = |L^{-1}(v)|_\infty \leq \sqrt{\rho}|v|_\infty \leq \sqrt{\rho} C_0 \leq \sigma/\sqrt{k}$$

holds for all $k \in (0, k_0)$, where $k_0 > 0$ is such that $\sqrt{\rho}C_0 \leq \sigma/\sqrt{k_0}$. Thus, Lemma 2.3 implies that problem (1.1) admits a solution. $\square$

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