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Computing the zero forcing number for generalized Petersen graphs*

Research Article

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Abstract: Let G be a simple undirected graph with each vertex colored either white or black, u be a black vertex of G, and exactly one neighbor v of u be white. Then change the color of v to black. When this rule is applied, we say u forces v, and write $u \to v$. A $zero\ forcing\ set$ of a graph G is a subset Z of vertices such that if initially the vertices in Z are colored black and remaining vertices are colored white, the entire graph G may be colored black by repeatedly applying the color-change rule. The zero forcing number of G, denoted $Z(G)$, is the minimum size of a zero forcing set.
In this paper, we investigate the zero forcing number for the generalized Petersen graphs (It is denoted by $P(n,k)$). We obtain upper and lower bounds for the zero forcing number for $P(n,k)$. We show that $Z(P(n,2)) = 6$ for $n \geq 10$, $Z(P(n,3)) = 8$ for $n \geq 12$ and $Z(P(2k+1,k)) = 6$ for $k \geq 5$.

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Keywords: Zero forcing number, Generalized Petersen graph, Colin de Verdière parameter

1. Introduction

Let $G = (V, E)$ be a simple undirected graph. Each vertex is colored either white or black. In such a case we say that G has a *coloring* and the set of all black vertices is called an *initial coloring* of G. The *color-change rule* is defined as follows: if u is a black vertex of G and exactly one neighbor v of u is white, then the color of v changes to black.

Given a coloring of G, let A be the set of all black vertices of G. The *derived coloring* of A, denoted $der(A)$, is the result of applying the color-change rule until no more changes are possible. The *zero forcing set* for a graph G (ZFS) is an initial coloring Z of G such that $der(Z) = G$. The $zero\ forcing\ number$ $Z(G)$ is the minimum size of all zero forcing sets of G. The concept of zero forcing set indicates one

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model of propagation in general networks. It was introduced in [4]. the associated terminology has been extended in [5, 7, 11, 12]. For example according to [4] if G is a path, an endpoint of G is the zero forcing set for G. If G is a cycle, each set of two adjacent vertices is a zero forcing set.

A *contraction* of a graph G is the graph obtained by identifying two adjacent vertices of G, and ignoring any loops or multiple edges occurred. A *minor* of G is a graph obtained by applying a sequence of deletions of edges, deletions of isolated vertices, and contraction of edges. A graph parameter ζ is called *minor monotone* if for any minor H of G, $\zeta(H) \leq \zeta(G)$.

Definition 1.1 ([9])**.** *The generalized Petersen graph $P(n,k)$ is defined to be the graph with the vertex set and edge set respectively as follows*

$$\begin{aligned} V(P(n,k)) &= \{u_1, \dots, u_n, v_1, \dots, v_n\} \\ E(P(n,k)) &= \{u_iu_{i+1},\, u_iv_i,\, v_iv_{i+k} : 1 \leq i \leq n\}. \end{aligned}$$

Here, the subscripts are assumed as integers modulo n such that $n \geq 5$. Note that, $P(n,k) \cong P(n, n-k)$. So, we assume $n \geq 2k+1$.

$P(n,k)$ is a 3-regular graph with $2n$ vertices. The generalized Petersen graph has been studied from several points of view, such as: hamiltonicity [1, 3, 15], crossing numbers [13, 14], spectrum [10] and vertex domination [9].

In Section 2, we turn to the zero forcing number of the generalized Petersen graphs. We present an upper bound for $Z(P(n,k))$. We show that $Z(P(n,2)) = 6$ for $n \geq 10$, $Z(P(n,3)) = 8$ for $n \geq 12$ and $Z(P(2k+1,k)) = 6$ for $k \geq 5$.

In Section 3, we show that $K_{k,[\frac{n}{k}]}$ is a minor of $P(n,k)$ (where $[x]$ is the maximum integer not greater than x). Using this, we conclude that:

$$\min\{k, [\frac{n}{k}]\} + 1 = \mu\Big(K_{k,[\frac{n}{k}]}\Big) \leq \mu(P(n,k)) \leq Z(P(n,k)).$$

The graph parameter μ is introduced by Colin de Verdiere in 1990 [6]. It is equal to the maximum nullity among all matrices satisfying several conditions. This conditions are stated in Section 3. It is the first parameter of Colin de Verdiere type parameters. There exist a relation between this parameter and the zero forcing number that we apply it for achieving the upper bound.
There exists a comparison between the zero forcing sets and dynamic monopolies in the last section. Note that, in all figures of this paper the vertex $v_i(u_i)$ is indicated by $[i]_v([i]_u)$.

2. Upper bounds and equalities for $Z(P(n,k))$

In the following theorem, we obtain an upper bound for $Z(P(n,k))$, where $k \nmid n$.

Theorem 2.1. *If $n = rk+s$, then $Z(P(n,k)) \leq r(s+2)$, where $1 \leq s \leq k-1$, $r, s \in \mathbb{N}$.*

Proof. Let $A = \{u_1, u_2, \cdots, u_{s+2}, u_{1+k}, u_{2+k}, \cdots, u_{s+2+k}, \cdots, u_{1+(r-1)k}, u_{2+(r-1)k}, \cdots, u_{s+2+(r-1)k}\}$ be an initial coloring of $P(n,k)$. The following vertices change to black by the color-change rule: $\{v_2, \dots, v_{s+1}, v_{2+k}, \dots, v_{s+1+k}, \dots, v_{2+(r-1)k}, \dots, v_{s+1+(r-1)k}\}$. Since, two neighbors of the vertices u_j, $2+ik \leq j \leq s+1+ik$ for $0 \leq i \leq r-1$ are black and the only white neighbor of them is v_j.
We also show that the vertices $\{v_1, v_{1+k}, \dots, v_{1+(r-1)k}\}$ are in der(A). Note that

$$s+1+(r-1)k = s+rk+1-k = n+1-k \equiv 1-k \ (\mathrm{mod(n)}).$$

We have the following adjacency:

The vertex v_1 is the only white neighbor of the black vertex $v_{s+1+(r-1)k}$ and the vertex $v_{s+1+(r-1)k}$ forces it.
The vertex v_{k+1} is the only white neighbor of the black vertex v_1 and the vertex v_1 forces it.
The vertex $v_{1+(r-1)k}$ is the only white neighbor of the black vertex $v_{1+(r-2)k}$ and the vertex $v_{1+(r-2)k}$ forces it.

Also, the color of the vertices $u_n, u_k, \ldots, u_{(r-1)k}, v_n, v_k, \ldots, v_{(r-1)k}$ change to black.

The vertex u_n is the only white neighbor of the black vertex u_1 and is forced by it.
For $i = 1, \cdots, r-1$, the vertex u_{ik} is the only white neighbor of the black vertex u_{ik+1} and is forced by it.
The vertex v_n is the only white neighbor of the black vertex $v_{(r-1)k+s}$ and is forced by it.
The vertex v_k is the only white neighbor of the black vertex v_n and is forced by it.
For $i = 2, \cdots, r-1$, the vertex v_{ik} is the only white neighbor of the black vertex $v_{(i-1)k}$ and is forced by it.
Now, for $i = 1, \ldots, r$:
The vertex u_{ik-1} is the only white neighbor of the black vertex u_{ik} and is forced by it.
Then the vertex v_{ik-1} is the only white neighbor of the black vertex u_{ik-1} and is forced by it.
Also for each $t < k$, it is one counter, we have:
The vertex u_{ik-t} is the only white neighbor of the black vertex u_{ik-t+1} and is forced by it.
the vertex v_{ik-t} is the only white neighbor of the black vertex u_{ik-t} and is forced by it.
This process continues until $ik-t = i(k-1)+s+3$ and $u_{i(k-1)+s+3}$ forced by u_{ik+s+4}. Then $v_{i(k-1)+s+2}$ is the only white neighbor of the black vertex $u_{i(k-1)+s+3}$ and is forced by it. So, all vertives of graph became black. □

In the next theorem we obtain an upper bound for $Z(P(n,k))$. This bound does not depend on n.

Theorem 2.2. $Z(P(n,k)) \leq 2k+2$.

Proof. Let $A = \{u_1, u_2, \ldots, u_{2k+2}\}$ be an initial coloring of $P(n,k)$ (see Figure 1).
The vertex v_j is the only white neighbor of the black vertex u_j for $2 \leq j \leq 2k+1$. It is forced by u_j. So, $v_j \in$ der(A). Now the vertex v_1 is the only white neighbor of the vertex u_1 and is forced by it. Therefore v_{2k+2} is the only white neighbor of the black vertex v_{k+2}. Hence, the vertices v_2, v_3,...,v_{2k+2} are in der(A). We continue by induction. Let $m \geq 2k+2$ and the color of vertices

$$\begin{cases} u_m, \ldots, u_2, u_1 \\ v_m, \ldots, v_2 \end{cases}$$

have been changed to black. It suffices to show that the color of the vertices u_{m+1} and v_{m+1} change to black. Note that $m \geq 2k+2$ hence $m \geq m+1-2k \geq 3$. Therefore, the vertex v_{m+1} is the only white neighbor of the black vertex v_{m+1-k} and u_{m+1} is the only white neighbor of the black vertex u_m. □

Corollary 2.3. *If $n = rk+s$, then $Z(P(n,k)) \leq \min\{r(s+2), 2k+2\}$, where $1 \leq s \leq k-1$.*

Remark 2.4. In [2] the authors proved that $Z(P(15r,2)) = 6$ and $Z(P(24r,5)) = 12$ for all $r \geq 1$. Also they proved Theorem 2.2 another way. They obtain the upper bound $Z(P(2k+1,k)) \leq 6$ that we conclude this upper bound from Theorem 2.1 and obtain the equality in Theorem 2.6.

Theorem 2.5. *If $n \geq 10$, then $Z(P(n,2)) = 6$.*

Proof. By Theorem 2.2, we have $Z(P(n,2)) \leq 6$. Hence it suffices to show that no initial coloring of the graph with five vertices can be a zero forcing set. Let A be such an initial coloring. By checking all of possible cases we show that $|\text{der(A)}| \leq 10 < 2n = |P(n,2)|$. We have illustrated all cases, unless the trivial or similar ones, in the following figures. In each figure the white vertices are the vertices that will change to black by A. The set A can consist some vertices of type u_i or v_i. Therefore, the following division is considered. Note that, r vertices can be belong to the inner cycle of generalized peterson graph

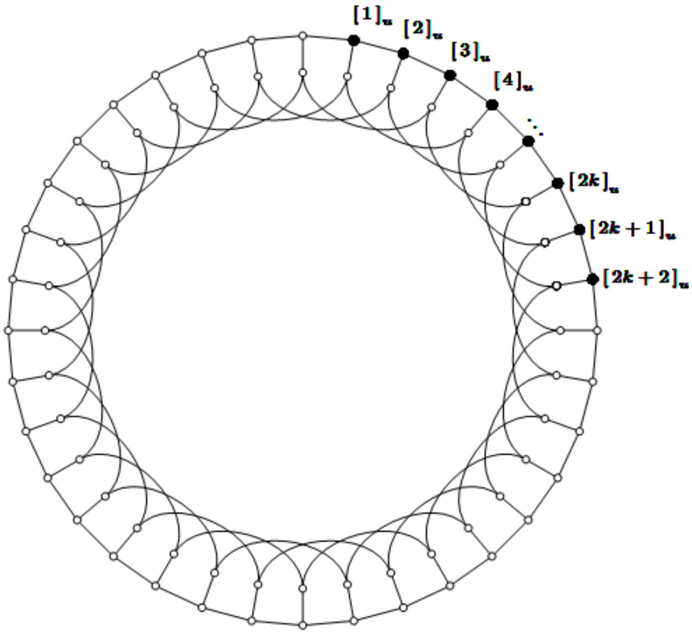
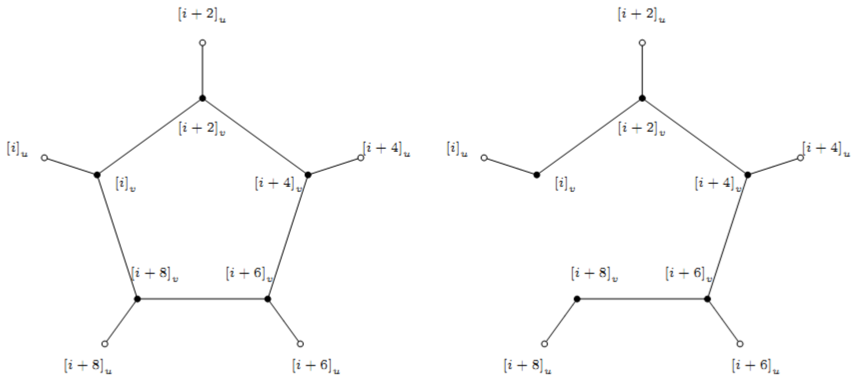
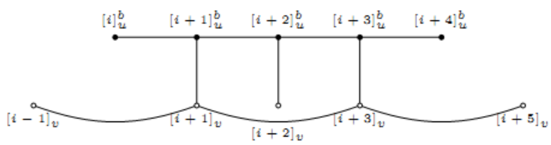


Figure 1. An initial coloring of $P(n, k)$ by $2k + 2$ vertices

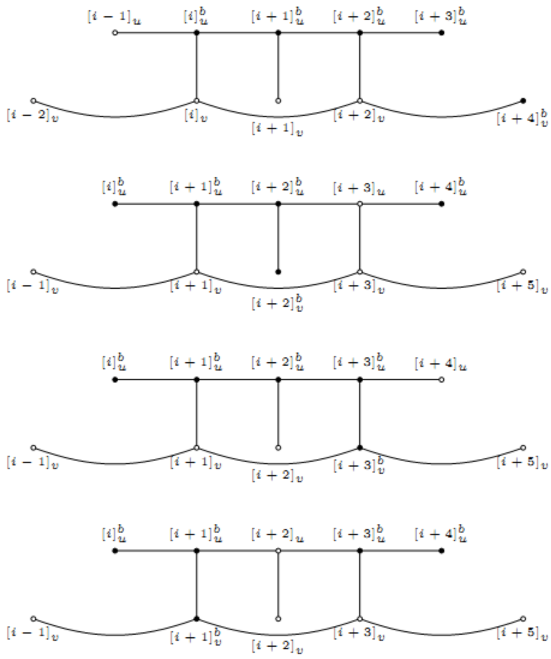
and $5 - r$ vertices must be belong to the outer cycle of it.
1. The set A consists of five v-vertices ($r = 5$):



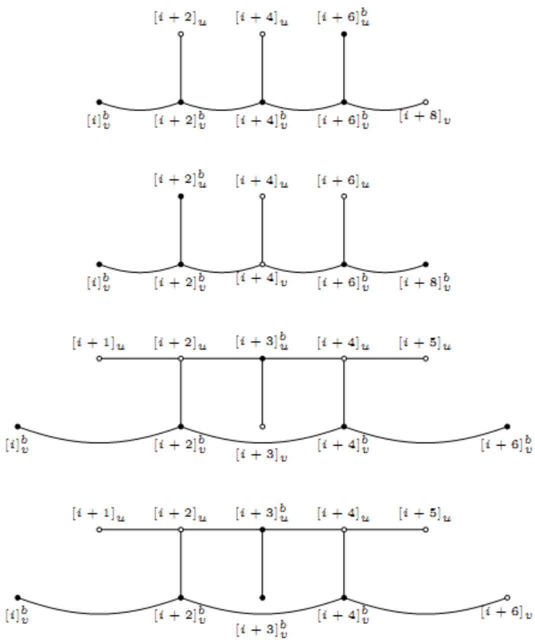
Note that, the vertex v_i and v_{i+8} are adjacent for $n = 10$.
2. The set A consists of five u-vertices ($r = 0$):



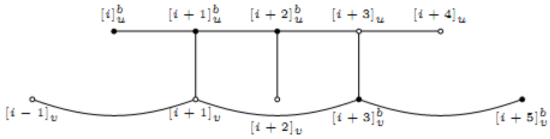
3. The set A consists of four u-vertices and one v-vertex ($r = 1$):

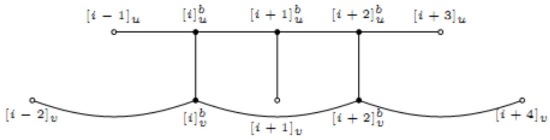


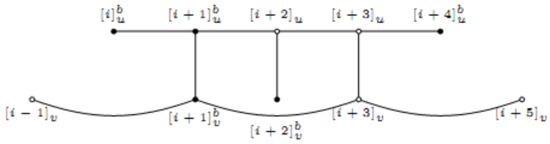
4. The set A consists of four v-vertices and one u-vertex ($r = 4$):

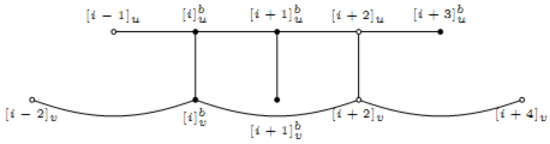


5. The set A consists of three u-vertices and two v-vertices ($r = 2$):

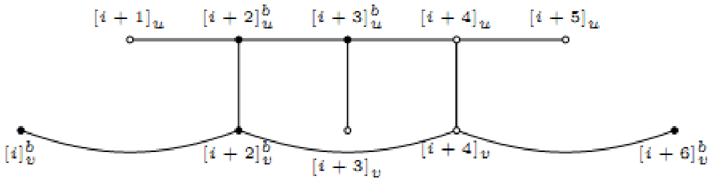


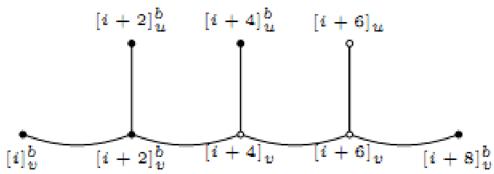


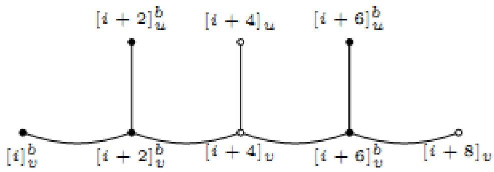


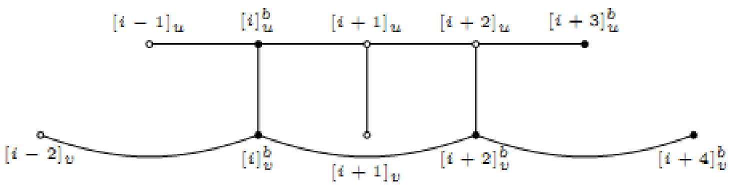


6. The set A consists of two u-vertices and three v-vertices ($r = 3$):







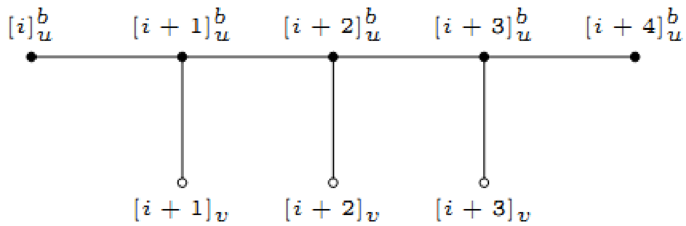


$\square$

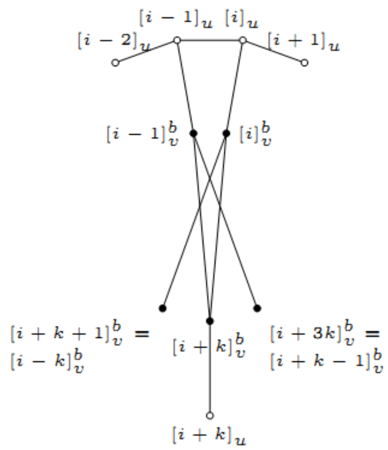
Theorem 2.6. *If $k \geq 5$, then $Z(P(2k+1,k)) = 6$.*

Proof. By Theorem 2.1, we have $Z(P(2k+1,k)) \leq 6$. By the same argument of Theorem 2.5, we show that, no initial coloring A with 5 vertices can be a zero forcing set for $P(n,k)$. The cases are essentially same as Theorem 2.5.

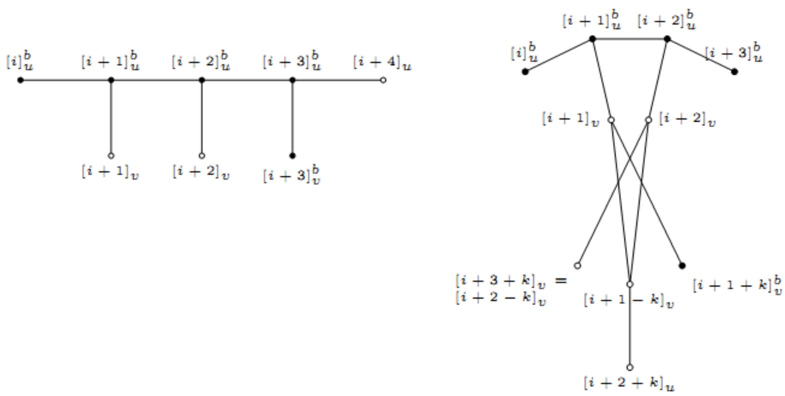
1. The set A consists of five u-vertices:



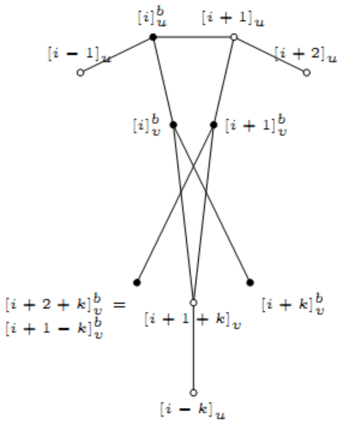
2. The set A consists of five v-vertices:

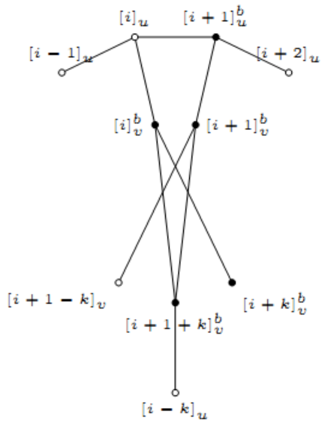


3. The set A consists of four u-vertices and one v-vertex:

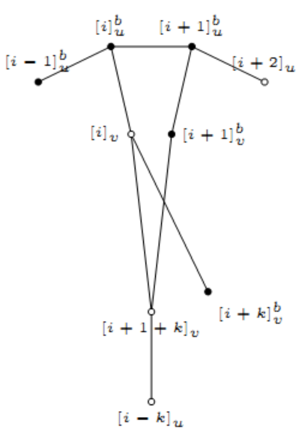


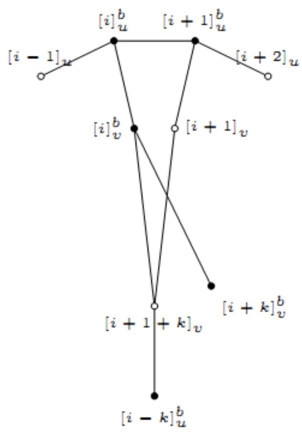
4. The set A consists of four v-vertices and one u-vertex:



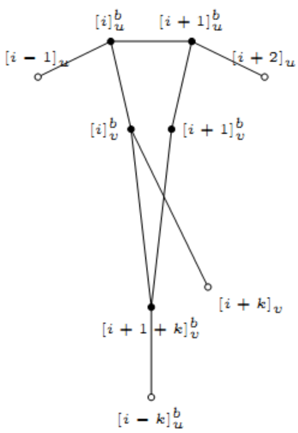


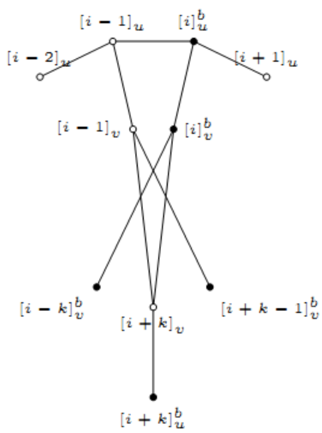
5. The set A consists of three u-vertices and two v-vertices:





6. The set A consists of two u-vertices and three v-vertices:





□

3. Lower bound for $Z(P(n,k))$

In this section, we obtain a lower bound for $Z(P(n,k))$. For this aim, we use the graph parameter $\mu(G)$. It has a monotonicity property, which proved first by Colin de Verdière in [8].

Definition 3.1. *[16] Let $G = (V,E)$ be an undirected graph, assuming (without loss of generality) that $V = \{1,\ldots,n\}$. Then parameter $\mu(G)$ is the largest corank of any matrix $M = (M_{i,j}) \in R^n$ such that:*
(M1) for all i,j with $i \neq j$: $M_{i,j} < 0$ if i and j are adjacent and $M_{i,j} = 0$ if i and j are nonadjacent;
(M2) M has exactly one negative eigenvalue, of multiplicity 1;
(M3) there is no nonzero matrix $X = (X_{i,j}) \in R^n$ such that $MX = 0$ and such that $X_{i,j} = 0$ whenever $i = j$ or $M_{i,j} \neq 0$.

In [6] it is stated that $\mu(G) \leq Z(G)$.

Theorem 3.2. *[8] If H is a minor of a graph G, then $\mu(H) \leq \mu(G)$.*

This property sometimes described as μ minor-monotone. For instance $\mu(K_n) = n-1$ and for $p \leq q$

$$\mu(K_{p,q}) = \begin{cases} p & \text{if } q \leq 2 \\ p+1 & \text{if } q \geq 3 \end{cases}$$

See [16] for more details.

Definition 3.3. *Let $G = (V,E)$ be a simple graph and A,B be none-empty subsets of $V(G)$. We say that A and B are adjacent if there exist vertices $x \in A$ and $y \in B$ such that $xy \in E$. In such a case we write $A \sim B$.*

Theorem 3.4. *If $k \geq 3$, then the graph $K_{k,[\frac{n}{k}]}$ is a minor of $P(n,k)$.*

Proof. Let $A_i = \{u_{(1+(i-1)k)}, u_{2+(i-1)k}, \ldots, u_{k+(i-1)k}\}$ for each $1 \leq i \leq r = \frac{n}{k}$. Put $B_j = \{v_j, v_{j+k}, \ldots, v_{j+(r-1)k}\}$ for each $1 \leq j \leq k$. It is easy to see that each A_i is adjacent to each B_j and B_i is not adjacent to B_j for $i \neq j$. Now proceed as follows:

1. Delete the edges between the vertices $1 + mk_u$ and mk_u for each $1 \leq m \leq r$.
2. Contract all the vertices of A_i in the vertex $u_{1+(i-1)k}$ (starting with the vertex $u_{k+(i-1)k}$ and contracting successively).
3. Contract all the vertices of B_j in the vertex v_j.
4. Delete all the remaining vertices and their edges.

Finally, we achieve the complete bipartite graph □

Now, we obtain the following lower bound.

Corollary 3.5. *If $k \geq 3$, then $\min\{k, \frac{n}{k}\} + 1 \leq Z(P(n,k))$.*

Theorem 3.6. *If $n \geq 12$, then $Z(P(n,3)) = 8$.*

Proof. Let A be a set of initial black vertices of the graph $P(n,3)$. By Corollary 3.5, $4 \leq |A|$. Let u_i be one white vertex on the outer cycle. The color of it can be forced by the vertex u_{i-1}, vertex u_{i+1} or the vertex v_i.

Assume the vertices $u_1, \cdots, u_{i-1} \in A$, then:
1) If the vertex u_{i-1} wants to force the vertex u_i, then it is necessary that $v_{i-1} \in A$.
2) If the vertex u_{i+1} wants to force the vertex u_i, then it is necessary that $v_{i+1}, u_{i+1}, u_{i+2} \in A$.
3) If the vertex v_i wants to force the vertex u_i, then it is necessary that $v_i, v_{i-k}, v_{i+k} \in A$.
Therefor the best case for the color-change processing in the vertices of the outer cycle is that the vertex u_{i-1} forces the vertex u_i. So, suppose $\{u_1, u_2, u_3, u_4\} \subseteq A$. This set can not change the color of all vertices. By a simple argument, we conclude that the set A be $\{u_1, u_2, u_3 \cdots, u_8\}$. □

4. A comparison between zero forcing sets and dynamic monopolies

In the last section, we compare the zero forcing sets with another propagation concept of graph theory. This concept is dynamic monopoly. It is studied by Zaker in [17].

Definition 4.1. *[17] By a threshold assignment for the vertices of G we mean any function $\tau : V(G) \to N \cup \{0\}$. A subset of vertices D is said to be a τ-dynamic monopoly of G or simply τ-dynamo of G, if for some nonnegative integer k, the vertices of G can be partitioned into subsets $D_0, D_1, ..., D_k$ such that $D_0 = D$ and for any $i, 1 \leq i \leq k$, the set D_i consists of all vertices v which has at least $\tau(v)$ neighbors in $D_0 \cup \ldots \cup D_{i-1}$. Denote the smallest size of any τ-dynamo of G by $dyn(G)$.*

It is obvious that each ZFS is a 1-dynamo. For $\tau = 1$, there does not exist any resistant subgraph. So, each subgraph can be a candidate for a dynamo of graph. [17] A resistant subgraph of G means each subgraph K such that for each vertex $v \in K$ one has $d_{K(v)} \geq d_{G(v)} t(v) + 1$, where $d_{G(v)}$ is the degree of v in G. Zaker proved that each dynamo of graph does not contain any resistant subgraph of it [17]. So, it is satisfy for the ZFS.

Example 4.2. *We know $Z(K_n) = n - 1$ and $Z(P_n) = 1$. The* ZFS *of complete graph K_n and path P_n are 1-dynamo too. For the complete graph K_4, $dyn(K_4) \neq Z(K_4)$. It is an interesting question that for what graphs there exist this equality. In this example the subsets D_0 and D_1 are ZFS.*

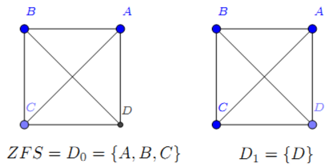


Figure 2. $Z(K_4) = 3$ **and** $dyn(K_4) = 1$

If we consider $D_0 = \{A\}$ and $D_1 = \{B, C, D\}$. Then the subset D_0 is a dynamo and it is not a ZFS.

There exists another question. A dynamo under what condition is a ZFS? The following lemma states this conditions.

Lemma 4.3. *Consider one Dynamo as $D_0, D_1, ..., D_k$. If for each vertex $u \in D_{i+1}$ there exist a vertex $v \in D_i$ such that $N(v) - \{u\} \subseteq (D_0 \cup D_1 \cup \cdots \cup D_i)$, then that dynamo is a ZFS.*

There exists one lower bound for $Z(G)$ which is obtained from the following results about dynamos.

Theorem 4.4. *[17] Let D be a dynamic monopoly of size k in G. Set $H = G \setminus D$ and let t_{max} be the maximum threshold among the vertices of H. Then:*
1) $\sum_{v \in H} t(v) \leq |E(G)| - |E(G[D])| - \delta(G) + t_{max}$.
2) $\sum_{v \in H} t(v) \leq |E(G)|$ *provided that $t(v) \leq d_{G(v)}$ for any vertex $v \in H$.*

We know that each ZFS is a 1-dynamo. So, we have the following corollary.

Corollary 4.5. *Let G be a graph with $1 \leq \delta(G)$, then:*
1) $|G| - |E(G)| + |E(G[ZFS])| + \delta - 1 \leq Z(G)$.
2) $|G| - |E(G)| \leq Z(G)$

Also, Corollary 2 from [17] confirms the second inequality. This first bound is equality for some graphs. For example, let G be a complete graph K_n. So $|G| - |E(G)| + |E(G[ZFS])| + \delta - 1 = n - \frac{(n)(n-1)}{2} + \frac{(n-1)(n-2)}{2} + (n-1) - 1 = n - 1$ and $Z(K_n) = n - 1$. Also, it is equality for path ($Z(P_t) = 1$). The characterization of all graphs that satisfy this bound will be interesting.

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