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Construction of quasi-twisted codes and enumeration of defining polynomials

Research Article

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Abstract: Let $d_q(n, k)$ be the maximum possible minimum Hamming distance of a linear $[n, k]$ code over $\mathbb{F}_q$. Tables of best known linear codes exist for small fields and some results are known for larger fields. Quasi-twisted codes are constructed using $m \times m$ twistulant matrices and many of these are the best known codes. In this paper, the number of $m \times m$ twistulant matrices over $\mathbb{F}_q$ is enumerated and linear codes over $\mathbb{F}_{17}$ and $\mathbb{F}_{19}$ are constructed for k up to 5.

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1. Introduction

Let $\mathbb{F}_q$ denote the finite field of q elements, and $V(n, q)$ the vector space of n-tuples over $\mathbb{F}_q$. A linear $[n, k]$ code C of length n and dimension k over $\mathbb{F}_q$ is a k-dimensional subspace of $V(n, q)$. The elements of C are called codewords. The (Hamming) weight of a codeword is the number of non-zero coordinates, and the minimum distance of C is the smallest weight among all non-zero codewords of C. An $[n, k, d]$ code is an $[n, k]$ code with minimum distance d. Let A_i be the number of codewords of weight i in C. Then the numbers $A_0, A_1, \ldots, A_n$ are called the weight distribution of C.

A central problem in coding theory is that of optimizing one of the parameters n, k and d for given values of the other two. One can find $d_q(n, k)$, the largest value of d for which there exists an $[n, k, d]$ code over $\mathbb{F}_q$, or $n_q(k, d)$, the smallest value of n for which there exists an $[n, k, d]$ code over $\mathbb{F}_q$. A code which achieves either of these values is called *optimal*. Tables of best known linear codes exist for $q = 2$ to 9 [6], 11 [3] and 13 [4].

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The Griesmer bound is a well-known lower bound on $n_q(k,d)$

$$n_q(k,d) \geq g_q(k,d) = \sum_{j=0}^{k-1} \left\lceil \frac{d}{q^j} \right\rceil, \tag{1}$$

where $\lceil x \rceil$ denotes the smallest integer $\geq x$. For $k \leq 2$, there exist codes that attain equality in the Griesmer bound for all q and d. The Singleton bound [12] is a lower bound on $n_q(k,d)$ and is given by

$$n_q(k,d) \geq d + k - 1. \tag{2}$$

Codes that meet this bound are called maximum distance separable (MDS). MDS codes exist for all values of $n \leq q+1$. Thus, for $q = 17$ MDS codes exist for all lengths 18 or less, and for $q = 19$ all lengths 20 or less. Note that all MDS codes are optimal. For larger lengths and dimensions, far less is known about codes over $\mathbb{F}_{17}$ and $\mathbb{F}_{19}$.

MDS self-dual codes ($k = n/2$), of lengths 2, 4, 6, 8, 10 and 18 over $\mathbb{F}_{17}$ are known [1], as well as self-dual $[12,6,6]$, $[14,7,7]$, $[16,8,8]$, $[20,10,10]$, $[22,11,10]$ and $[24,12,10]$ codes. The $[14,7,8]$ and $[20,10,10]$ extended quadratic residue (QR) codes are given in [13]. Using Magma [2], it was determined that the next extended QR code has parameters $[44,22,18]$. MDS self-dual codes of lengths 4, 8, 12 and 20 over $\mathbb{F}_{19}$ are known [1], as well as self-dual $[16,8,8]$ and $[24,12,9]$ codes. The $[18,9,9]$ and $[32,16,14]$ extended QR codes are given in [13]. In this paper, codes over $\mathbb{F}_{17}$ and $\mathbb{F}_{19}$ for k up to 5 are presented. These codes establish lower bounds on the minimum distance. Many of these meet the Singleton and/or Griesmer bounds, and so are optimal.

2. Quasi-twisted codes

A constacyclic shift of an m-tuple

$$(x_0, x_1, \ldots, x_{m-1}),$$

is the m-tuple

$$(\lambda x_{m-1}, x_0, \ldots, x_{m-2}),$$

where $\lambda \in \mathbb{F}_q \backslash \{0\}$, and a constacyclic shift by p positions is the m-tuple

$$(\lambda x_{m-p}, \ldots, \lambda x_{m-1}, x_0, \ldots, x_{m-p-1}).$$

A linear code C is said to be quasi-twisted (QT) if a constacyclic shift of any codeword by p positions is also a codeword in C [8]. Note that quasi-twisted codes generalize the classes of constacyclic codes ($p = 1$), quasi-cyclic codes ($\lambda = 1$), cyclic codes ($\lambda = 1$, $p = 1$), and negacyclic codes ($\lambda = -1$, $p = 1$). The length of a QT code considered here is $n = mp$. With a suitable permutation of coordinates, many QT codes can be characterized in terms of $m \times m$ twistulant matrices. In this case, a QT code can be transformed into an equivalent code with generator matrix

$$G = [B_0 \; B_1 \; B_2 \; \ldots \; B_{p-1}], \tag{3}$$

where B_i, $i = 0, 1, \ldots, p-1$, is an $m \times m$ twistulant matrix (also known as a constacyclic matrix), over $\mathbb{F}_q$ of the form [9]

$$B = \begin{bmatrix} b_0 & b_1 & b_2 & \cdots & b_{m-1} \\ \lambda b_{m-1} & b_0 & b_1 & \cdots & b_{m-2} \\ \lambda b_{m-2} & \lambda b_{m-1} & b_0 & \cdots & b_{m-3} \\ \vdots & \vdots & \vdots & & \vdots \\ \lambda b_1 & \lambda b_2 & \lambda b_3 & \cdots & b_0 \end{bmatrix},$$

where $\lambda \in \mathbb{F}_q \backslash \{0\}$ and $b_i, 0 \leq i \leq m-1$, are elements of $\mathbb{F}_q$. When $\lambda = 1$, a twistulant matrix is a circulant matrix, and when $\lambda = -1$, a twistulant matrix is known as a negacirculant matrix [8].

The algebra of $m \times m$ twistulant matrices over $\mathbb{F}_q$ is isomorphic to the algebra of polynomials in the ring $\mathbb{F}_q[x]/(x^m - \lambda)$ if B_i is mapped onto the polynomial $b_i(x) = b_{0,i} + b_{1,i}x + b_{2,i}x^2 + \ldots + b_{m-1,i}x^{m-1}$ formed from the entries in the first row of B_i. The $b_i(x)$ associated with a QT code are called the *defining polynomials* [7]. The set $\{b_0(x), b_1(x), \ldots, b_{p-1}(x)\}$ defines an $[mp, m]$ QT code with $k \leq m$.

3. Defining polynomials

The construction of QT codes requires a representative set of defining polynomials. These are the equivalence class representatives of a partition of the set of polynomials of degree less than m. For defining polynomials, multiplication by a non-zero element of $\mathbb{F}_q$ does not change the weight and hence does not change the equivalence class. Thus, two polynomials $r_j(x)$ and $r_i(x)$ are said to be *equivalent* if

$$r_j(x) = \gamma x^l r_i(x) \bmod (x^m - \lambda),$$

for some integer $l \geq 0$ and scalar $\gamma \in \mathbb{F}_q \backslash \{0\}$.

A closed-form expression for the number of defining polynomials is now given. Let g be the permutation $(1, 2, \ldots, m)$ so that g maps i to $i+1$, for $1 \leq i \leq m-1$, and m to 1. Therefore, g^i, $1 \leq i \leq m$, is also a permutation and has order $\frac{m}{\gcd(m,i)}$ in the symmetric group of degree m. Thus, the action of g on the m-tuple $x = (x_1, x_2, \ldots, x_m)$ changes x to $(x_m, x_1, x_2, \ldots, x_{m-1})$ where $x_i \in \mathbb{F}_q$. Now let λg be such that the action of λg on the m-tuple changes x to $(\lambda x_m, x_1, x_2, \ldots, x_{m-1})$, the action of $(\lambda g)^2$ on x results in $(\lambda x_{m-1}, \lambda x_n, x_1, x_2, \ldots, x_{m-2})$, and similarly for other powers. Then the order of λg is $Ord(\lambda)m$, where $Ord(\lambda)$ is the order of λ in $\mathbb{F}_q$. Further, let $t(\lambda g)$, $t \in \mathbb{F}_q \setminus \{0\}$, be such that it changes x to $(t\lambda x_m, tx_1, tx_2, \ldots, tx_{m-1})$. The action of $t(\lambda g)^2$ changes x to $(t\lambda x_{m-1}, t\lambda x_m, tx_1, tx_2, \ldots, tx_{m-2})$, and similarly for other powers. The equivalence relation is induced by the action of the group consisting of the elements $t(\lambda g)^i$, $1 \leq i \leq Ord(\lambda)m$, $t \in \mathbb{F}_q \setminus \{0\}$. Distinct equivalence classes correspond to distinct orbits under the action of this group and so can be enumerated using Burnside's Lemma [5, 9].

Definition 3.1. *An ordered m-tuple (or word of length m), $x = (x_1, x_2, \ldots, x_m)$, is said to be fixed by $t(\lambda g)^i$, $t, \lambda \in \mathbb{F}_q \setminus \{0\}$, $1 \leq i \leq Ord(\lambda)m$, if the m-tuple x remains unchanged by the action of $t(\lambda g)^i$.*

Theorem 3.2. *The number of words of length m over the alphabet $\mathbb{F}_q$ fixed by $t(\lambda g)^i$ for some fixed $\lambda \in \mathbb{F}_q \setminus \{0\}$, $t \in \mathbb{F}_q \setminus \{0\}$, $1 \leq i \leq Ord(\lambda)m$, is $q^{\gcd(m,i)}$ if $t^{(\frac{m}{\gcd(m,i)})}\lambda^{(\frac{i}{\gcd(m,i)})} = 1$. Otherwise, it is 1.*

Proof. Let $x = (x_1, x_2, \ldots, x_m)$ be a word of length m over $\mathbb{F}_q$. Then the relation between the components of x before and after the action of $t(\lambda g)^i$ is

$$\begin{aligned}
x_1 &= t\lambda x_{m-i+1} \\
x_2 &= t\lambda x_{m-i+2} \\
x_3 &= t\lambda x_{m-i+3} \\
&\vdots \\
x_{i-1} &= t\lambda x_{m-1} \\
x_i &= t\lambda x_m \\
x_{i+1} &= tx_1 \\
x_{i+2} &= tx_2 \\
&\vdots \\
x_{m-1} &= tx_{m-i-1} \\
x_m &= tx_{m-i}
\end{aligned}$$

if $1 \leq i \leq m$. If $m+1 \leq i \leq 2m$, the relation between the components of x is

$$\begin{aligned}
x_1 &= t\lambda^2 x_{m-i+1} \\
x_2 &= t\lambda^2 x_{m-i+2} \\
x_3 &= t\lambda^2 x_{m-i+3} \\
&\vdots \\
x_{i-1} &= t\lambda^2 x_{m-1} \\
x_i &= t\lambda^2 x_m \\
x_{i+1} &= t\lambda x_1 \\
x_{i+2} &= t\lambda x_2 \\
&\vdots \\
x_{m-1} &= t\lambda x_{m-i-1} \\
x_m &= t\lambda x_{m-i}.
\end{aligned}$$

Thus in general, the relation between the components of x is

$$\begin{aligned}
x_1 &= t\lambda^j x_{m-i+1} \\
x_2 &= t\lambda^j x_{m-i+2} \\
x_3 &= t\lambda^j x_{m-i+3} \\
&\vdots \\
x_{i-1} &= t\lambda^j x_{m-1} \\
x_i &= t\lambda^j x_m \\
x_{i+1} &= t\lambda^{j-1} x_1 \\
x_{i+2} &= t\lambda^{j-1} x_2 \\
&\vdots \\
x_{m-1} &= t\lambda^{j-1} x_{m-i-1} \\
x &= t\lambda^{j-1} x_{m-i}
\end{aligned}$$

if $(j-1)m+1 \leq i \leq jm$, $1 \leq j \leq Ord(\lambda)$.

Let $\gcd(m,i) = h$. Then, from the expressions above, the orbit of x_m is

$$\begin{aligned}
x_m &= tx_{m-i} = t^2 x_{m-2i} = \cdots = t^{\lfloor \frac{m}{i} \rfloor} x_{m-\lfloor \frac{m}{i} \rfloor i} \\
&= t^{(\lfloor \frac{m}{i} \rfloor+1)} \lambda x_{2m-(\lfloor \frac{m}{i} \rfloor+1)i} = t^{(\lfloor \frac{m}{i} \rfloor+2)} \lambda x_{2m-(\lfloor \frac{m}{i} \rfloor+2)i} \\
&= \cdots = t^{(\lfloor \frac{2m}{i} \rfloor-1)} \lambda x_{2m-(\lfloor \frac{2m}{i} \rfloor-1)i} \\
&= t^{(\lfloor \frac{2m}{i} \rfloor)} \lambda x_{2m-(\lfloor \frac{2m}{i} \rfloor)i} = t^{(\lfloor \frac{2m}{i} \rfloor+1)} \lambda^2 x_{3m-(\lfloor \frac{2m}{i} \rfloor+1)i} \\
&= t^{(\lfloor \frac{2m}{i} \rfloor+2)} \lambda^2 x_{3m-(\lfloor \frac{2m}{i} \rfloor+2)i} = \cdots = t^{(\lfloor \frac{3m}{i} \rfloor-1)} \lambda^2 x_{3m-(\lfloor \frac{3m}{i} \rfloor-1)i} \\
&= t^{(\lfloor \frac{3m}{i} \rfloor)} \lambda^2 x_{3m-(\lfloor \frac{3m}{i} \rfloor)i} = t^{(\lfloor \frac{3m}{i} \rfloor+1)} \lambda^3 x_{4m-(\lfloor \frac{3m}{i} \rfloor+1)i} \\
&= t^{(\lfloor \frac{3m}{i} \rfloor+2)} \lambda^3 x_{4m-(\lfloor \frac{3m}{i} \rfloor+2)i} \\
&= \cdots = t^{(\lfloor \frac{4m}{i} \rfloor-1)} \lambda^3 x_{4m-(\lfloor \frac{4m}{i} \rfloor-1)i} \\
&= t^{(\lfloor \frac{4m}{i} \rfloor)} \lambda^3 x_{4m-(\lfloor \frac{4m}{i} \rfloor)i} = \cdots = t^{(\lfloor (\frac{i}{h}-1)\frac{m}{i} \rfloor+1)} \lambda^{(\frac{i}{h}-1)} x_{\frac{im}{h}-(\lfloor \frac{(\frac{i}{h}-1)m}{i} \rfloor+1)i} \\
&= t^{(\lfloor (\frac{i}{h}-1)\frac{m}{i} \rfloor+2)} \lambda^{(\frac{i}{h}-1)} x_{\frac{im}{h}-(\lfloor \frac{(\frac{i}{h}-1)m}{i} \rfloor+2)i} \\
&= \cdots = t^{(\lfloor (\frac{i}{h}\frac{m}{i}) \rfloor-1)} \lambda^{(\frac{i}{h}-1)} x_{\frac{im}{h}-(\lfloor \frac{\frac{im}{h}}{i} \rfloor-1)i} = t^{(\lfloor (\frac{i}{h}\frac{m}{i}) \rfloor)} \lambda^{(\frac{i}{h})} x_m,
\end{aligned}$$

and so we also have

$$\begin{aligned}
x_{m-1} &= tx_{m-1-i} = t^2 x_{m-1-2i} = \cdots = t^{\lfloor \frac{m}{i} \rfloor} x_{m-1-\lfloor \frac{m}{i} \rfloor i} \\
&= t^{(\lfloor \frac{m}{i} \rfloor +1)} \lambda x_{2m-1-(\lfloor \frac{m}{i} \rfloor +1)i} = t^{(\lfloor \frac{m}{i} \rfloor +2)} \lambda x_{2m-1-(\lfloor \frac{m}{i} \rfloor +2)i} \\
&= \cdots = t^{(\lfloor \frac{2m}{i} \rfloor -1)} \lambda x_{2m-1-(\lfloor \frac{2m}{i} \rfloor -1)i} = t^{(\lfloor \frac{2m}{i} \rfloor)} \lambda x_{2m-1-(\lfloor \frac{2m}{i} \rfloor)i} \\
&= t^{(\lfloor \frac{2m}{i} \rfloor +1)} \lambda^2 x_{3m-1-(\lfloor \frac{2m}{i} \rfloor +1)i} = t^{(\lfloor \frac{2m}{i} \rfloor +2)} \lambda^2 x_{3m-1-(\lfloor \frac{2m}{i} \rfloor +2)i} \\
&= \cdots = t^{(\lfloor \frac{3m}{i} \rfloor -1)} \lambda^2 x_{3m-1-(\lfloor \frac{3m}{i} \rfloor -1)i} = t^{(\lfloor \frac{3m}{i} \rfloor)} \lambda^2 x_{3m-1-(\lfloor \frac{3m}{i} \rfloor)i} \\
&= t^{(\lfloor \frac{3m}{i} \rfloor +1)} \lambda^3 x_{4m-1-(\lfloor \frac{3m}{i} \rfloor +1)i} = t^{(\lfloor \frac{3m}{i} \rfloor +2)} \lambda^3 x_{4m-1-(\lfloor \frac{3m}{i} \rfloor +2)i} \\
&= \cdots = t^{(\lfloor \frac{4m}{i} \rfloor -1)} \lambda^3 x_{4m-1-(\lfloor \frac{4m}{i} \rfloor -1)i} = t^{(\lfloor \frac{4m}{i} \rfloor)} \lambda^3 x_{4m-1-(\lfloor \frac{4m}{i} \rfloor)i} \\
&= \cdots = t^{(\lfloor (\frac{i}{h}-1)\frac{m}{i} \rfloor +1)} \lambda^{(\frac{i}{h}-1)} x_{\frac{im}{h}-1-(\lfloor \frac{(\frac{i}{h}-1)m}{i} \rfloor +1)i} \\
&= t^{(\lfloor (\frac{i}{h}-1)\frac{m}{i} \rfloor +2)} \lambda^{(\frac{i}{h}-1)} x_{\frac{im}{h}-1-(\lfloor \frac{(\frac{i}{h}-1)m}{i} \rfloor +2)i} \\
&= \cdots = t^{(\lfloor (\frac{i}{h}\frac{m}{i}) \rfloor -1)} \lambda^{(\frac{i}{h}-1)} x_{\frac{im}{h}-1-(\lfloor \frac{\frac{im}{h}}{i} \rfloor -1)i} \\
&= t^{(\lfloor (\frac{i}{h}\frac{m}{i}) \rfloor)} \lambda^{(\frac{i}{h})} x_{m-1}.
\end{aligned}$$

Similar expressions exist for $x_{m-2}, x_{m-3}, \ldots, x_{m-h+2}$, and in general

$$\begin{aligned}
x_{m-h+1} &= tx_{m-h+1-i} = t^2 x_{m-h+1-2i} = \cdots = t^{\lfloor \frac{m}{i} \rfloor} x_{m-h+1-\lfloor \frac{m}{i} \rfloor i} \\
&= t^{(\lfloor \frac{m}{i} \rfloor +1)} \lambda x_{2m-+1-(\lfloor \frac{m}{i} \rfloor +1)i} = t^{(\lfloor \frac{m}{i} \rfloor +2)} \lambda x_{2m-h+1-(\lfloor \frac{m}{i} \rfloor +2)i} \\
&= \cdots = t^{(\lfloor \frac{2m}{i} \rfloor -1)} \lambda x_{2m-h+1-(\lfloor \frac{2m}{i} \rfloor -1)i} = t^{(\lfloor \frac{2m}{i} \rfloor)} \lambda x_{2m-h+1-(\lfloor \frac{2m}{i} \rfloor)i} \\
&= t^{(\lfloor \frac{2m}{i} \rfloor +1)} \lambda^2 x_{3m-h+1-(\lfloor \frac{2m}{i} \rfloor +1)i} = t^{(\lfloor \frac{2m}{i} \rfloor +2)} \lambda^2 x_{3m-h+1-(\lfloor \frac{2m}{i} \rfloor +2)i} \\
&= \cdots = t^{(\lfloor \frac{3m}{i} \rfloor -1)} \lambda^2 x_{3m-h+1-(\lfloor \frac{3m}{i} \rfloor -1)i} = t^{(\lfloor \frac{3m}{i} \rfloor)} \lambda^2 x_{3m-h+1-(\lfloor \frac{3m}{i} \rfloor)i} \\
&= t^{(\lfloor \frac{3m}{i} \rfloor +1)} \lambda^3 x_{4m-h+1-(\lfloor \frac{3m}{i} \rfloor +1)i} = t^{(\lfloor \frac{3m}{i} \rfloor +2)} \lambda^3 x_{4m-h+1-(\lfloor \frac{3m}{i} \rfloor +2)i} \\
&= \cdots = t^{(\lfloor \frac{4m}{i} \rfloor -1)} \lambda^3 x_{4m-h+1-(\lfloor \frac{4m}{i} \rfloor -1)i} = t^{(\lfloor \frac{4m}{i} \rfloor)} \lambda^3 x_{4m-h+1-(\lfloor \frac{4m}{i} \rfloor)i} \\
&= \cdots = t^{(\lfloor (\frac{i}{h}-1)\frac{m}{i} \rfloor +1)} \lambda^{(\frac{i}{h}-1)} x_{\frac{im}{h}-h+1-(\lfloor \frac{(\frac{i}{h}-1)m}{i} \rfloor +1)i} \\
&= t^{(\lfloor (\frac{i}{h}-1)\frac{m}{i} \rfloor +2)} \lambda^{(\frac{i}{h}-1)} x_{\frac{im}{h}-h+1-(\lfloor \frac{(\frac{i}{h}-1)m}{i} \rfloor +2)i} \\
&= \cdots = t^{(\lfloor (\frac{i}{h}\frac{m}{i}) \rfloor -1)} \lambda^{(\frac{i}{h}-1)} x_{\frac{im}{h}-h+1-(\frac{im}{ih}-1)i} \\
&= t^{(\frac{i}{h}\frac{m}{i})} \lambda^{(\frac{i}{h})} x_{m-h+1}.
\end{aligned}$$

Thus, the orbits of x_m, x_{m-1}, $\ldots$, x_{m-h+1} are fixed by the action of $t(\lambda g)^i$ if and only if $t^{\frac{m}{h}} \lambda^{\frac{i}{h}} = t^{\frac{m}{\gcd(m,i)}} \lambda^{\frac{i}{\gcd(m,i)}} = 1$. Since there are $h = \gcd(m,i)$ independent orbits and each orbit can take on q values, $q^h = q^{\gcd(m,i)}$ words are fixed by $t(\lambda g)^i$. If $t^{\frac{m}{\gcd(m,i)}} \lambda^{\frac{i}{\gcd(m,i)}} \neq 1$, then there is only one orbit consisting of all m elements of the word. In this case, since $\lambda \neq 1$, the only word that is fixed is the zero word. □

Theorem 3.3. *The number of defining polynomials of length m over $\mathbb{F}_q$ is*

$$M_{q,\lambda}(m) = \frac{1}{(q-1)Ord(\lambda)m} \sum_{\substack{i=1 \\ t^{\frac{m}{\gcd(m,i)}} \lambda^{\frac{i}{\gcd(m,i)}} = 1}}^{Ord(\lambda)m} (q^{\gcd(m,i)} - 1) + 1 \tag{4}$$

Proof. There are $(q-1)Ord(\lambda)m$ permutations given by $t(\lambda g)^i$. Thus, by Burnside's lemma [5], the number of orbits of words of length m over an alphabet of size q is equal to the average number of words fixed by each $t(\lambda g)^i$, $1 \le i \le Ord(\lambda)m$, $t \in \mathbb{F}_q \setminus \{0\}$. Therefore, we have

$$M_{q,\lambda}(m) = \frac{1}{(q-1)Ord(\lambda)m} \sum_{\substack{i=1 \\ t\in\mathbb{F}_q\setminus\{0\}}}^{Ord(\lambda)m} |\mathrm{Fix}\ t(\lambda g)^i|,$$

where $|\mathrm{Fix}\ t(\lambda g)^i|$ denotes the number of words fixed by $t(\lambda g)^i$.

From Theorem 1, the number of words fixed by $t(\lambda g)^i$ is either $q^{\gcd(m,i)}$ or 1 depending on whether $t^{\frac{m}{\gcd(m,i)}}\lambda^{\frac{i}{\gcd(m,i)}} = 1$ or not. Therefore

$$\begin{aligned}
M_{q,\lambda}(m) &= \frac{1}{(q-1)Ord(\lambda)m}\left\{\sum_{\substack{i=1 \\ t^{\frac{m}{\gcd(m,i)}}\lambda^{\frac{i}{\gcd(m,i)}}=1}}^{Ord(\lambda)m} q^{\gcd(m,i)} + \left\{\sum_{\substack{i=1 \\ t^{\frac{m}{\gcd(m,i)}}\lambda^{\frac{i}{\gcd(m,i)}}\neq 1}}^{Ord(\lambda)m} 1\right\}\right\} \\
&= \frac{1}{(q-1)Ord(\lambda)m}\left\{\sum_{\substack{i=1 \\ t^{\frac{m}{\gcd(m,i)}}\lambda^{\frac{i}{\gcd(m,i)}}=1}}^{Ord(\lambda)m} q^{\gcd(m,i)}\right. \\
&\qquad \left. + \left\{(q-1)Ord(\lambda)m - \sum_{\substack{i=1 \\ t^{\frac{m}{\gcd(m,i)}}\lambda^{\frac{i}{\gcd(m,i)}}=1}}^{Ord(\lambda)m} 1\right\}\right\} \\
&= \frac{1}{(q-1)Ord(\lambda)m}\left\{\sum_{\substack{i=1 \\ t^{\frac{m}{\gcd(m,i)}}\lambda^{\frac{i}{\gcd(m,i)}}=1}}^{Ord(\lambda)m} (q^{\gcd(m,i)}-1) + (q-1)Ord(\lambda)m\right\} \\
&= \frac{1}{(q-1)Ord(\lambda)m} \sum_{\substack{i=1 \\ t^{\frac{m}{\gcd(m,i)}}\lambda^{\frac{i}{\gcd(m,i)}}=1}}^{Ord(\lambda)m} (q^{\gcd(m,i)}-1) + 1
\end{aligned}$$

□

Example 3.4. *Let* $q = 13$*,* $m = 2$*, and* $\lambda = 4$*. Since* 6 *is the least integer such that* $\lambda^6 = 4^6 = 1$*, we* *have* $Ord(\lambda) = Ord(4) = 6$. *Then*

$$\begin{aligned} M_{13,4}(2) &= \frac{1}{144} \sum_{\substack{i=1 \\ t\in\mathbb{F}_{13}\setminus\{0\}, t^{\frac{2}{\gcd(2,i)}} 4^{\frac{i}{\gcd(2,i)}}=1}}^{12} (13^{\gcd(2,i)} - 1) + 1 \\ &= \frac{1}{144}\left\{\left[(13^{\gcd(2,12)} - 1)\right] + \left[(13^{\gcd(2,5)} - 1) + (13^{\gcd(2,11)} - 1)\right]\right\} \\ &\quad + \frac{1}{144}\left\{\left[(13^{\gcd(2,8)} - 1)\right] + \left[(13^{\gcd(2,10)} - 1)\right]\right\} \\ &\quad + \frac{1}{144}\left\{\left[(13^{\gcd(2,3)} - 1) + (13^{\gcd(2,9)} - 1)\right] + \left[(13^{\gcd(2,1)} - 1) + (13^{\gcd(2,7)} - 1)\right]\right\} \\ &\quad + \frac{1}{144}\left\{\left[(13^{\gcd(2,1)} - 1) + (13^{\gcd(2,7)} - 1)\right]\right\} \\ &\quad + \frac{1}{144}\left\{\left[(13^{\gcd(2,3)} - 1) + (13^{\gcd(2,9)} - 1)\right]\right\} \\ &\quad + \frac{1}{144}\left\{\left[(13^{\gcd(2,4)} - 1)\right] + \left[(13^{\gcd(2,2)} - 1)\right]\right\} \\ &\quad + \frac{1}{144}\left\{\left[(13^{\gcd(2,5)} - 1) + (13^{\gcd(2,11)} - 1)\right] + \left[(13^{\gcd(2,6)} - 1),\right]\right\} + 1. \end{aligned}$$

because

$$\begin{aligned} & 1^{\frac{2}{\gcd(2,12)}} 4^{\frac{12}{\gcd(2,12)}} \\ = 2^{\frac{2}{\gcd(2,5)}} 4^{\frac{5}{\gcd(2,5)}} &= 2^{\frac{2}{\gcd(2,11)}} 4^{\frac{11}{\gcd(2,11)}} \\ &= 3^{\frac{2}{\gcd(2,8)}} 4^{\frac{8}{\gcd(2,8)}} \\ &= 4^{\frac{2}{\gcd(2,10)}} 4^{\frac{10}{\gcd(2,10)}} \\ = 5^{\frac{2}{\gcd(2,3)}} 4^{\frac{3}{\gcd(2,3)}} &= 5^{\frac{2}{\gcd(2,9)}} 4^{\frac{9}{\gcd(2,9)}} \\ = 6^{\frac{2}{\gcd(2,1)}} 4^{\frac{1}{\gcd(2,1)}} &= 6^{\frac{2}{\gcd(2,7)}} 4^{\frac{7}{\gcd(2,7)}} \\ = 7^{\frac{2}{\gcd(2,1)}} 4^{\frac{1}{\gcd(2,1)}} &= 7^{\frac{2}{\gcd(2,7)}} 4^{\frac{7}{\gcd(2,7)}} \\ = 8^{\frac{2}{\gcd(2,3)}} 4^{\frac{3}{\gcd(2,3)}} &= 8^{\frac{2}{\gcd(2,9)}} 4^{\frac{9}{\gcd(2,9)}} \\ &= 9^{\frac{2}{\gcd(2,4)}} 4^{\frac{4}{\gcd(2,4)}} \\ &= 10^{\frac{2}{\gcd(2,2)}} 4^{\frac{2}{\gcd(2,2)}} \\ = 11^{\frac{2}{\gcd(2,5)}} 4^{\frac{5}{\gcd(2,5)}} &= 11^{\frac{2}{\gcd(2,11)}} 4^{\frac{11}{\gcd(2,11)}} \\ &= 12^{\frac{2}{\gcd(2,6)}} 4^{\frac{6}{\gcd(2,6)}} = 1, \end{aligned}$$

and therefore

$$\begin{aligned} M_{13,4}(2) &= \frac{1}{144}[(13^2 - 1) + [(13-1) + (13-1)] + (13^2 - 1) + (13^2 - 1) + [(13-1) + (13-1)]] \\ &\quad + \frac{1}{144}[[(13-1) + (13-1)] + [(13-1) + (13-1)] + [(13-1) + (13-1)]] \\ &\quad + \frac{1}{144}[(13^2 - 1) + (13^2 - 1) + [(13-1) + (13-1)] + (13^2 - 1)] + 1 = 9. \end{aligned}$$

Note that setting $\lambda = 1$ in (4) gives the number of defining polynomials for quasi-cyclic codes [14]

$$M_{q,1}(m) = \frac{1}{(q-1)m} \sum_{i|m} \phi(i) \gcd(i, q-1)(q^{m/i} - 1) + 1. \tag{5}$$

Table 1 gives the number of defining polynomials over $\mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{F}_5$, $\mathbb{F}_7$, and $\mathbb{F}_{11}$ with $\lambda = 1$, Table 2 gives the number of defining polynomials over $\mathbb{F}_4$, $\mathbb{F}_8$, $\mathbb{F}_9$, and $\mathbb{F}_{16}$ with $\lambda = 1$, and Table 3 gives the number of defining over $\mathbb{F}_{13}$, $\mathbb{F}_{17}$, and $\mathbb{F}_{19}$ with $\lambda = 1$. To illustrate the effect of λ, Tables 4 and 5 give the number of defining polynomials over $\mathbb{F}_3$ and $\mathbb{F}_4$.

Table 1. The number of defining polynomials over $\mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{F}_5$, $\mathbb{F}_7$, and $\mathbb{F}_{11}$ with $\lambda = 1$

	q				
m	2	3	5	7	11
1	2	2	2	2	2
2	3	4	5	6	8
3	4	6	12	22	46
4	6	14	45	106	374
5	8	26	158	562	3226
6	14	68	665	3298	29576
7	20	158	2792	19610	278390
8	36	424	12255	120206	2679860
9	60	1098	54262	747330	26199450
10	108	2980	244301	4708486	259377496
11	188	8054	1109732	29959498	2593742462
12	352	22218	5086965	192243598	26153599626

Table 2. The number of defining polynomials over $\mathbb{F}_4$, $\mathbb{F}_8$, $\mathbb{F}_9$, and $\mathbb{F}_{16}$ with $\lambda = 1$

	q			
m	4	8	9	16
1	2	2	2	2
2	4	6	7	10
3	10	26	32	94
4	24	150	213	1098
5	70	938	1478	13986
6	238	6258	11107	186478
7	782	42806	85412	2556530
8	2744	299670	672825	35791946
9	9726	2130458	5380862	509033346
10	34990	15339642	43586287	7330084546
11	127102	111557594	356602952	106619309362
12	466198	818092242	2941985613	1563749966062

4. Quasi-twisted codes over $\mathbb{F}_{17}$ and $\mathbb{F}_{19}$

In this section, the defining polynomials given above are used to construct quasi-twisted codes over $\mathbb{F}_{17}$ and $\mathbb{F}_{19}$. The number of defining polynomials over $\mathbb{F}_{17}$ for $m = 1$ to 5 is given in Table 6 and over $\mathbb{F}_{19}$ for $m = 1$ to 5 in Table 7. Note that the zero polynomial is not considered in constructing codes.

Considering a code structure (i.e. QT), results in a search space that is smaller than for the general code design problem. The more restrictions on the structure, the smaller the search, but this creates a tradeoff since good codes may be missed if too much structure is imposed on the code. The QT codes presented here were constructed using a stochastic optimization algorithm, namely tabu search, which is similar to that in [10, 11, 14]. By restricting the search for good codes to the class of QT codes, and using a stochastic heuristic, codes with high minimum distance can be found with a reasonable amount of computational effort. Based on the results obtained here, this approach provides a good tradeoff.

The search for a (pm, m) QT code begins with a random set of p defining polynomials. A polynomial

Table 3. The number of defining polynomials over $\mathbb{F}_{13}$, $\mathbb{F}_{17}$, and $\mathbb{F}_{19}$ with $\lambda = 1$

	q		
m	13	17	19
1	2	2	2
2	9	11	12
3	64	104	130
4	605	1317	1822
5	6190	17750	27514
6	67117	251543	435760
7	747008	3663740	7094222
8	8497807	54499433	117943232
9	98189934	823526990	1991899630
10	1148826961	12599979635	34061506732
11	13576972684	194726683568	588334640902
12	161792326165	3034491071421	10246828768390

Table 4. The number of defining polynomials over $\mathbb{F}_3$ with $\lambda \in \{1, 2\}$

m	λ	Number
1	1,2	2
2	1	4
2	2	3
3	1,2	6
4	1	14
4	2	11
5	1,2	26
6	1	68
6	2	63
7	1,2	158
8	1	424
8	2	411
9	1,2	1098
10	1	2980
10	2	2955
11	1,2	8054
12	1	22218
12	2	22151

is replaced with a new polynomial if this results in an increase in the minimum distance. This process is repeated until a code with the desired minimum distance is found or an iteration threshold is reached. The search is restarted periodically to ensure good coverage of the search space. It is not necessary to check the weight of every codeword in a QT code in order to determine the minimum distance d. Only a subset of the codewords need be considered since the Hamming weights of the polynomials $r_j(x)$ and $r_i(x)$ are the same if

$$r_j(x) = \gamma x^l r_i(x) \bmod (x^m - \lambda),$$

for integer $l \geq 0$ and scalar $\gamma \in \mathbb{F}_q \backslash \{0\}$.

Table 5. The number of defining polynomials over $\mathbb{F}_4$ with $\lambda \in \{1, \alpha, \alpha^2\}$

m	λ	Number
1	1,α,α^2	2
2	1,α,α^2	4
3	1	10
3	α,α^2	8
4	1,α,α^2	24
5	1,α,α^2	70
6	1	238
6	α,α^2	232
7	1,α,α^2	782
8	1,α,α^2	2744
9	1	9726
9	α,α^2	9710
10	1,α,α^2	34990
11	1,α,α^2	127102
12	1	466198
12	α,α^2	466152

Table 6. The number of defining polynomials over $\mathbb{F}_{17}$

m	λ	Number
1	all	2
2	1, 2, 4, 8, 9, 13, 15, 16	11
2	3, 5, 6, 7, 10, 11, 12, 14	10
3	all	104
4	1, 4, 13, 16	1317
4	2, 8, 9, 15	1315
4	3, 5, 6, 7, 10, 11, 12, 14	1306
5	all	17750
6	1, 2, 4, 8, 9, 13, 15, 16	251543
6	3, 5, 6, 7, 10, 11, 12, 14	251440

The best QT codes found over $\mathbb{F}_{17}$ are given in Tables 8 to 10, and over $\mathbb{F}_{19}$ in Tables 11 to 13. The defining polynomials are listed with the lowest degree coefficient on the left, i.e. 7321 corresponds to the polynomial $x^3 + 2x^2 + 3x + 7$, with leading zeroes left out for brevity. The digits 10, 11,. . . , 18 are denoted by (10), (11), . . . , (18), respectively. As an example, consider the [24,4] code in Table 12 with $m = 4$, $\lambda = 1$ and $p = 6$ defining polynomials. These polynomials give the following generator matrix

$$G = \left[\begin{array}{c|c|c|c|c|c} 1129 & 1596 & 1632 & 01(14)(11) & 0016 & 169(12) \\ 9112 & 6159 & 2163 & (11)01(14) & 6001 & (12)169 \\ 2911 & 9615 & 3216 & (14)(11)01 & 1600 & 9(12)16 \\ 1291 & 5961 & 6321 & 1(14)(11)0 & 0160 & 69(12)1 \end{array}\right]$$

with weight distribution

Table 7. The number of defining polynomials over $\mathbb{F}_{19}$

m	λ	number
1	all	2
2	$1, 4, 5, 6, 7, 9, 11, 16, 17$	12
2	$2, 3, 8, 10, 12, 13, 14, 15, 18$	11
3	$1, 7, 8, 11, 12, 18$	130
3	$2, 3, 4, 5, 6, 9, 10, 13, 14, 15, 16, 17$	128
4	$1, 4, 5, 6, 7, 9, 11, 16, 17$	1822
4	$2, 3, 8, 10, 12, 13, 14, 15, 18$	1811
5	all	27514

i	W_i
0	1
20	6372
21	10944
22	28296
23	49680
24	35028

This code is optimal since it meets the Griesmer bound (1), and so establishes that $d_{19}(24, 4) = 20$. The codes that meet the Griesmer bound are indicated by * in the tables. The codes given in the tables are QC codes ($\lambda = 1$) when a QC code has the highest minimum distance among all QT codes with the same length and dimension. In two cases for $q = 19$ and $m = 4$, a code with $\lambda = -1$ was found with a minimum distance higher than the corresponding QC code.

5. Conclusion

Closed-form expressions for the number of twistulant matrices and corresponding defining polynomials were given. These polynomials were used in the construction of quasi-twisted codes, and several new optimal codes were obtained.

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Appendix A:

Table 8. QT codes over $\mathbb{F}_{17}$ with $m = 3$

code	λ	d	$b_i(x)$
[6,3]	1	4*	11, 1(13)8
[9,3]	1	7*	18(11), 1, 117
[12,3]	1	10*	14(15), 1(12), 16, 157
[15,3]	1	13*	1, 118, 125, 152, 164
[18,3]	1	16*	118, 125, 1, 152, 13(12), 164
[21,3]	1	18*	152, 14(10), 13(16), 17(12), 127, 172, 115
[24,3]	1	21*	11, 137, 11(10), 145, 17(16), 18(11), 15(16), 13
[27,3]	1	24*	116, 12(11), 12(16), 18(11), 118, 11(13), 12, 147, 16(10)
[30,3]	1	26	147, 11(14), 12(10), 16, 11, 164, 164, 1(13)8, 11(12), 182
[33,3]	1	29	184, 17(12), 14, 114, 17(10), 11(10), 19(16), 18, 12(13), 11(12), 1(13)4
[36,3]	1	32	18, 1(13)4, 12, 1(14), 1(12)8, 11(13), 14(13), 157, 18(12), 198, 1(10), 16(11)
[39,3]	1	35*	116, 124, 117, 1(13)8, 11(11), 11(10), 12, 145, 19, 12(14), 1(12), 1(10)4, 15(14)
[42,3]	1	38*	1(13), 15(12), 1, 152, 11, 125, 1(13)4, 12(11), 116, 175, 118, 198, 132, 137
[45,3]	1	40	1, 1(15), 118, 113, 1(12)8, 14(11), 124, 114, 19(16), 17, 126, 125, 154, 14, 11(16)
[48,3]	1	43	15, 126, 14(15), 1(10)4, 15(16), 17(16), 11(14), 125, 12(12), 13(14), 143, 11(10), 1(10)(11), 11(16), 16(15), 157
[51,3]	1	46	18, 126, 198, 11(13), 157, 13(14), 15(14), 13(15), 12(16), 12(14), 12, 152, 13(10), 11, 16(11), 16(14), 14(11)
[54,3]	1	49	11, 11(13), 154, 1, 135, 12, 152, 124, 143, 137, 14(11), 116, 16(14), 16(15), 13(12), 11(12), 1(10)8, 1(10)
[57,3]	1	52*	11, 13(16), 1(13)4, 11(15), 1(13)8, 12, 15(14), 16(16), 12(11), 11(12), 18(12), 14(14), 1, 15(16), 125, 115, 147, 145, 162
[60,3]	1	54	12, 1, 113, 152, 12(10), 11(14), 114, 16(11), 15(12), 135, 1(10)(11), 17, 16, 126, 11, 134, 16(14), 157, 125, 165
[63,3]	1	57	11(13), 11, 12(15), 154, 19(16), 12, 12(10), 114, 125, 12(12), 15(14), 15(12), 117, 113, 14(10), 11(15), 16(15), 162, 1(13), 13(16), 123
[66,3]	1	60	13, 145, 15(12), 13(13), 117, 1(15), 12(10), 13(10), 143, 116, 14(13), 168, 16(11), 11, 11(12), 13(15), 1(14), 13(14), 115, 182, 17, 142
[69,3]	1	63	11(13), 1(12), 154, 1, 19, 128, 12(10), 182, 12(16), 12(15), 15(14), 18(12), 112, 11(12), 117, 198, 15(11), 1(11), 13(16), 11(14), 138, 1(13), 145
[72,3]	1	65	15, 175, 11, 13(12), 1(10), 11(12), 118, 19(14), 17(12), 16(11), 182, 154, 137, 138, 132, 16(15), 145, 18, 11(15), 125, 142, 147, 116, 17(10)
[75,3]	1	68	1(10), 12, 12(13), 13(15), 126, 1(11), 12(16), 112, 11(13), 117, 12(12), 16(14), 14(13), 175, 1(10)4, 11, 14(11), 168, 11(16), 17(10), 17(16), 1(12), 1(12)8, 114, 127

Table 9. QT codes over $\mathbb{F}_{17}$ with $m = 4$

code	λ	d	$b_i(x)$
[8,4]	1	5^*	1, 1128
[12,4]	1	9^*	16(11)(14), 123, 1685
[16,4]	1	13^*	12(10)(16), 1275, 113, 11(16)7
[20,4]	1	16	18, 13(11)7, 113(12), 1179, 12(15)2
[24,4]	1	20^*	134, 1798, 18(11), 1(11)1(14), 126(10), 1(10)(11)(12)
[28,4]	1	23	1(11), 1(13)1, 11(10)6, 137(15), 1(15)(11), 1(11)1(16), 1(10)5
[32,4]	1	27	1, 124, 13(14)7, 1(10)(11)(12), 126(16), 1(15)(16), 1437, 13(13)2
[36,4]	1	30	13, 15(16)(11), 13(15)(14), 1(10)2, 16(16), 11, 112(14), 12(13)(16), 1734
[40,4]	1	34	14(10)7, 1453, 1317, 1, 1(13)1, 11(15)4, 1598, 12(15)4, 1(12)2, 118(10)
[44,4]	1	38	114(15), 12(10)(16), 11, 118(14), 113, 17(13)(16), 1(16)7, 14(15)(10), 1(12)7, 19(14), 147(16)
[48,4]	1	41	12(12), 12, 11(16)(14), 181, 14, 111(14), 1254, 12(10)(13), 1114, 1(10)5, 156(10), 147(12)
[52,4]	1	45	124(16), 14(13)3, 1114, 16, 1(15)(16), 12, 183(10), 118(16), 1192, 13(10), 12(14), 173(11), 14(15)(10)
			116(15), 14(10)2, 1(11)64, 14(14)7
[60,4]	1	52	11, 11(13), 13(16), 146(13), 153, 13, 192, 164(10), 1386, 137(13), 1372, 125(16), 11(10)5, 1272, 12(13)3
[64,4]	1	56	13, 158(14), 11(15)(14), 1982, 1(13)5, 1(10)1(16), 12(14)5, 1517, 13(13)(12), 103, 11(12)9, 1(12)(15)8, 116(16), 11(15)7, 1239, 1(12)(10)
[68,4]	1	59	12, 1(10)1(12), 12(13)(12), 1(11)2, 151(11), 17(12)(15), 16(11)(12), 1(15)(16), 1187, 1472, 124(15), 13(10)(13), 11(14), 1462, 1(11)3, 172(14), 11
[72,4]	1	63	1(15), 12(10)2, 11(16)(11), 11(16)7, 1(13), 15(16)5, 1838, 12(15), 14(15), 13(13)(11), 1682, 1(12)38, 1468, 135(15), 1795, 137(10), 14(10), 175
[76,4]	1	67	1(10), 12(16)7, 15(12)8, 1127, 13(15)5, 12(15)6, 1358, 126(16), 1123, 121(13), 14, 1387, 187, 1654, 11(12)(14), 1193, 163(11), 1(12)(13), 1468
[80,4]	1	70	11(13), 1, 13, 1(10)1(11), 15(12)4, 1268, 167, 11(12), 141(10), 184(15), 1517, 17(12)(11), 16(15)(14), 12(14)(15), 18(14), 1188, 17(10)(14), 138(12), 111(16), 176(16)
[84,4]	1	74	1418, 17, 1356, 135(15), 13(16)2, 12(11)(13), 136, 1162, 16(10)(15), 149(16), 147(13), 12(16)(13), 17(16), 11(14)5, 1164, 1213, 121(15), 1838, 11(11)5, 1547, 17(10)4
[88,4]	1	78	1(10), 12(11)(16), 1(11)(13), 15(14), 13(16)8, 138(11), 133, 172(15), 1568, 13(16)(10), 132(14), 1(13)(15), 1169, 1(11)1, 17(15), 1146, 14(13)2, 14(10)(11), 11(10)(11), 14(11)(15), 14(13)(14), 15(11)(12)
[92,4]	1	81	13, 11(16)2, 17(12)5, 16(15)(12), 11(10)(11), 1(13)(16), 1(10)(11), 119(11), 16(14), 163(11), 139, 1(12)2, 13(13)8, 15(16), 162, 128(13), 1652, 14(12), 16(16)5, 162(14), 118(15), 11(10)7, 1(10)1
[96,4]	1	85	1, 16(11)(14), 1145, 16(10)(14), 102, 14(11)(12), 1(10)8(11), 111, 11(11)2, 13(13)8, 176(10), 16(11)8, 1376, 12(11)5, 1(10)8(12), 187, 15(12)3, 1(12)3(11), 11(12)4, 17(12)(10), 111(12), 1(10)(12), 125(14), 1832
[100,4]	1	89	105, 1(12)3, 1235, 198(12), 13(15)3, 151(15), 134(10), 11, 146(10), 1(13)(15), 1(11)4, 11(12)(11), 184(15), 143(15), 113(14), 1158, 17(12)(10), 1173, 116(10), 183(10), 119(11), 19(15)4, 11(15)(13), 1584, 1425

Table 10. QT codes over $\mathbb{F}_{17}$ with $m = 5$

code	λ	d	$b_i(x)$
[10,5]	1	6*	18, 16(15)(16)2
[15,5]	1	10	11, 162(15)(14), 11(11)(15)(12)
[20,5]	1	14	131, 11(13)96, 1041, 11(11)
[25,5]	1	19	15, 11(11)(16)(12), 135(12)4, 1126(10), 1278(14)
[30,5]	1	23	11(13)(10)(15), 1(11)(10)(16), 115, 115(11)4, 112(11)8, 17(12)8(11)
[35,5]	1	28	111(16)6, 143(13)(16), 121, 106(13), 15(14)68, 119(16)6, 13(13)(16)7
[40,5]	1	32	19, 13415, 1576(16), 1547, 17(13)(15)(14), 12(12)(13)(12), 13(12)85, 12(13)(11)4
[45,5]	1	36	1(16)58, 131, 13(12)9, 1(15)7(16), 11(13)(15)(14), 12(15)3(15), 1(10)1(10), 13(14)3(12), 141(15)(12)
[50,5]	1	41	16, 121(15)(13), 15217, 172(13)(12), 11132, 17(10)1(16), 13(11)75, 14(14)2(15), 12(11)26, 11(15)7(14)
[55,5]	1	45	1, 12, 11(16)(14)(15), 1(15)(11)4, 1416(10), 11636, 19(10)7, 111(15)8, 11(13)7(16), 12495, 1748
[60,5]	1	50	19, 10(11)4, 1(10)83(11), 1(14)74, 11(13)(16)3, 11452, 13(10)(15)(14), 13(12)(14)(16), 12(12)34, 12(13)8(15), 17(10), 154(15)3
[65,5]	1	54	13, 11, 1058, 14(14)4, 126(14)8, 13(11)(12)(10), 121(11)8, 1389(12), 19(15)3, 129(12), 13(14)94, 12(15)27, 1212(13)
[70,5]	1	59	11, 13475, 1(12)(10)(14), 11(13)4, 18(12)(15)2, 14(12)4, 14(13)(16)3, 173(11)(12), 11235, 1586(11), 1(16)6(12), 18(14)(10), 15(12)1(12), 1(14)9(10)
[75,5]	1	63	1(10)(11), 1956, 19, 16(15)(12)(16), 19(16)4, 11(14)59, 112(10), 15(11)(12)8, 17(14)3, 17(16)42, 11(10)4(15), 11(11)3, 17(16)1(14), 12(15)1, 19(14)8
[80,5]	1	68	1, 1(16)(14)6, 14, 13(14)6(15), 163(11)(12), 111, 17595, 179(15)(14), 14382, 1(12)5(12)(10), 1(13)(14)9, 1(12)(15)6(16), 14(10)2, 12(14)15, 19(13)82, 1825(11)
[85,5]	1	72	1(10), 1327(16), 11, 1(11)11, 177, 15(10)4, 11513, 13263, 155(14), 12(16)8(14), 1342(11), 123(15)9, 125(13)(14), 12(14)38, 1(12)5(12), 17(10)4(10), 1641(14)
[90,5]	1	77	1(11), 119(13), 1933, 18198, 13714, 115(15)(15), 195(13), 17(11)3, 11475, 13(15)7, 156(10)2, 1(11)(16)(10), 15(16)2(15), 15(11)34, 113, 11(10)(12)3, 1(10)5(13), 11(14)(10)3
[95,5]	1	81	1, 12432, 1438(10), 13(11)1(12), 1(11)42, 131(10)(11), 14(11)1(15), 1425(11), 10(10), 1185(13), 1(12)18, 15982, 118(15)3, 1794(14), 12927, 1321(14), 13(14)(16)(14), 12(14)(12)(15), 1976(16)
[100,5]	1	86	16(14)(13)(11), 15(14)72, 11, 1, 1431(11), 12(10)(13)3, 113(12)(14), 1156(10), 115(11), 1752(16), 12(12)(10)(13), 1247, 173(13)4, 172(15)(16), 1(13)(11)(11), 116(11)(14), 146(12), 16(15)(16)8, 11(11)(15)3, 157(16)4
[105,5]	1	90	13(13)(10)5, 1(11)37, 11, 1(14)(11)3, 117(14)3, 13(11)(16)(12), 13, 123(16)7, 1195, 19(12)9, 15(16)(12), 11(14)39, 11(14)6(14), 1(12)8, 15932, 1517(10), 1(12)(11)9, 156(10)(12), 13(10)6(13), 1183(13), 11(10)(16)
[110,5]	1	95	12, 14, 13(12)9, 1(15)7(16), 11(13)(15)(14), 12(15)3(15), 1(10)1(10), 13(14)3(12), 141(15)(12), 15168, 11534, 1568(14), 118(15)(12), 1(10)44, 135(11)3, 163(11)(12), 11(15)(10)(16), 13(11)(15)(14), 169(16)(10), 117(11), 12(11)(14)(10), 14234
[115,5]	1	99	18, 1, 11(10)3(16), 1(11)(12)(15)8, 13(16)9(11), 13(16)95, 13(16)54, 1012, 1127(14), 15942, 131(16)2, 1187, 12(12)9(12), 1326, 15(14)1(14), 11112, 18(11)(12)8, 15(11)32, 164(10)8, 159(12)5, 1(14)(16)(13), 103(13), 17(11)4
[120,5]	1	104	14, 117(15)8, 1071, 15(13)(11)(15), 15(12)1, 13(12)(16)(13), 14(15)62, 134(10)(13), 15(16)85, 15(14)(16), 12, 19(11)(12), 1815, 112(14)(13), 14(11)2, 15(14)(10)8, 12(12)(11)(15), 1256(13), 1(12)3(12), 1452(16), 1243(10), 1166(14), 13(11)(10)4, 11(10)5(15)
[125,5]	1	109	1(10), 13184, 13(13)(16), 12(13)53, 1081, 1(11)(10)(10), 13(11)2, 15(14)(15)2, 11(14)(11)9, 127(16)(13), 1415(12), 12972, 15175, 11(11)(12)6, 1(13)(16)(13), 19832, 14295, 13(16)(11), 12(16)4(15), 132(14)(15), 156(14)2, 11(12)87, 131(11)4, 15(16)64, 1656

Table 11. QT codes over $\mathbb{F}_{19}$ with $m = 3$

code	λ	d	$b_i(x)$
[6,3]	1	4*	1, 112
[9,3]	1	7*	135, 11, 16(16)
[12,3]	1	10*	1, 15(12), 17(15), 138
[15,3]	1	13*	1(11), 154, 11(16), 149, 14(17
[18,3]	1	16*	17, 159, 11(16), 157, 145, 176
[21,3]	1	18	1, 114, 142, 12, 137, 159, 16(15)
[24,3]	1	21*	11, 15(12), 1(11)(18), 138, 15(17), 139, 1(10)6, 159
[27,3]	1	24*	12, 18(14), 116, 162, 189, 154, 14(14), 13(12), 11(14)
[30,3]	1	27*	1(11), 1, 15(11), 1(10)7, 1(13)6, 135, 11(13), 1(10)(18), 15(16), 128
[33,3]	1	29	112, 115, 1(12)(14), 12, 11, 15(12), 138, 1(17), 1(16), 128, 192
[36,3]	1	32	14(15), 15, 125, 117, 12(13), 158, 18(17), 11(13), 13(12), 1, 17(15), 11(17)
[39,3]	1	35	167, 16, 11(11), 1(14), 17(12), 14, 1(12)(14), 173, 17, 13(16), 146, 127, 152
[42,3]	1	38*	1(11), 11(17), 17(18), 114, 12, 12(12), 15(17), 1(13)6, 18(17), 162, 11(12), 19, 116 1(10)(18)
[45,3]	1	41*	14, 1(14)6, 16(14), 17, 11(11), 162, 17(12), 187, 16, 1(16), 1(10)(15), 1(10)6, 11(12) 178, 169
[48,3]	1	43	15(16), 1, 168, 12(12), 19(14), 129, 1(16), 12, 1(12)9, 154, 135, 189, 112, 1(10)9 1(10)8, 117
[51,3]	1	46	12, 18(17), 162, 15(11), 11(11), 137, 13(14), 15, 17(17), 11, 163, 1, 11(13), 12(15) 19(14), 135, 18(15)
[54,3]	1	49	1, 18, 176, 13(16), 162, 12(12), 1(10), 125, 154, 192, 147, 1(10)6, 135, 159, 134, 169 13(12), 17
[57,3]	1	52	14, 13, 1, 1(13)6, 129, 143, 1(10), 168, 118, 158, 124, 1(10)(18), 18(14), 19(14), 14(15) 1(11)(18), 159, 12(13), 14(14)
[60,3]	1	55	1(14), 1(10), 14, 1(17), 115, 126, 17(18), 154, 17, 1(13)6, 129, 12(15), 137, 11(16), 128 1(11)(18), 1(13)(18), 127, 113, 149
[63,3]	1	57	17, 1(12)9, 1(16), 1(14), 193, 135, 1(10)6, 12(16), 19, 1(18), 162, 116, 145, 126, 129 169, 14(17), 15(12), 11(16), 1(12), 1(14)6
[66,3]	1	60	1(12), 11(15), 15(18), 176, 137, 193, 138, 12(11), 1(10)9, 1(11)(18), 13(16), 135, 162 11(13), 16(15), 1(16)6, 17(18), 132, 17, 1(12)(14), 147, 125
[69,3]	1	63	1, 18, 1(16), 134, 1(13)(18), 14(15), 114, 1(14), 115, 11(11), 139, 186, 15(16), 12(15) 1(13), 15(17), 129, 1(10)8, 17(18), 14(11), 132, 14(14), 126
[72,3]	1	66	1(18), 12(15), 173, 117, 139, 1(15), 15(18), 1(12)9, 167, 1(13), 11, 127, 1(13)6, 11(16) 11(14), 13(16), 19, 168, 129, 13(17), 1(11)(18), 138, 14(11), 16(16)
[75,3]	1	69	15, 12(14), 13(17), 149, 1, 1(14), 193, 11(13), 134, 16(16), 1(13)(18), 15(16), 115, 1(13) 14(15), 12(17), 1(11)(18), 128, 116, 16(14), 17(17), 15(12), 186, 137, 11

Table 12. QT codes over $\mathbb{F}_{19}$ with $m = 4$

code	λ	d	$b_i(x)$
[8,4]	1	5*	12, 114
[12,4]	1	9*	1(10), 1193, 1(10)6(15)
[16,4]	-1	13*	12(12), 16, 134(12) , 111(14)
[20,4]	-1	17*	14(17), 115(17), 1281, 11(11)9, 1146
[24,4]	1	20*	1129, 1596, 1632, 1(14)(11), 16, 169(12)
[28,4]	1	23	18, 135(14), 19, 16(14)(17), 11(15), 145(11), 14(11)(15)
[32,4]	1	27	104, 112, 1534, 1629, 1737, 17(15)6, 12(16)6, 1356
[36,4]	1	30	18, 119(12), 12(16), 112, 15(11)(15), 13(16)3, 121(18), 15(18)(11), 1719
[40,4]	1	34	106, 14(14)3, 1627, 11(12)(18), 16(10)(14), 14(11)(12), 157(15), 152(16), 1(16)9 1923
[44,4]	1	38	16, 127(14), 12(14)(16), 15, 1(10)(18)7, 18(17)6, 149(15), 1398, 158, 1279, 141(18)
[48,4]	1	41	159(14), 11, 15(17), 121(13), 15(13)(16), 14(10)(17), 12(14)6, 124(18), 146(11), 13(11) 1168, 118(15)
[52,4]	1	45	1(12), 1(12)5, 14(11), 123(18), 14(10)(18), 13(15)(12), 12(14)(11), 15, 187(17) 163(17), 159(11), 17(12), 1174
[56,4]	1	49	13, 11(15)(13), 1(10)5(14), 1(11)59, 161(14), 1619, 13(10)(15), 1(12)(16), 11(12)(11) 1(10)(12), 12(15)(18), 13(16)6, 126(11), 1272
[60,4]	1	52	1843, 169, 17(17), 1153, 11(11)2, 1(11)8(14), 183(17), 16(16)2, 141(12), 12(14)(15) 11(15), 1416, 1576, 148, 162(17)
[64,4]	1	56	1(14), 153(15), 169(16), 11(12)(16), 1157, 19, 1(12)4, 1427, 124(11), 1144, 12(16)2 1(16)7, 14(17)9, 12(11)7, 169(15), 1188
[68,4]	1	60	14, 1283, 1642, 15(17)(16), 1(16)(10), 1(11)7, 1(12)29, 12, 1923, 1287, 17(15)8, 113 12(14)3, 131(18), 17(10), 1582, 149(12)
[72,4]	1	63	14, 15(14)6, 1(11)9, 1215, 19(15), 15(17)(18), 1, 168(18), 1278, 11(17)4, 13(12)8 1(11)7(15), 16(14)(15), 1192, 164, 1(11)(16)6, 111(12), 132
[76,4]	1	67	1(10), 152(17), 1(12)(13), 113(17), 1(11)4, 12(13)(14), 1(11)78, 191(14), 191, 13(12)3 14(12), 1358, 1696, 11(18)(15), 1(12)2(15), 1(12)8, 1(18)7, 1194, 1(13)(13)
[80,4]	1	71	15, 14(11)4, 12(18)2, 1(15)9, 1(13)6(14), 129(14), 11(13)(16), 1153, 12(14)(13), 128 1(16)(17), 167(12), 12(10)(16), 13, 11(11)(15), 14(14)(15), 12(17), 117(10), 1(12)2(15) 14(18)(12)
[88,4]	1	78	1(11), 1629, 11(11)(15), 178(12), 14(17)9, 113(11), 12(18)(16), 12(13)(15), 1923 113(12), 11(17), 14(15)4, 1287, 125(13), 14(10)5, 102, 1(12)78, 14(14)(17), 1(11)53 19(17), 154(17), 12(10)(12)
[92,4]	1	82	1(12), 13(18)(14), 13(13)6, 1319, 115(15), 106, 164, 12(17)7, 19(12)7, 14(14)3, 13(13)3 14(17)9, 19(12)(18), 112(17), 138(17), 1(11)(17), 1175, 15(13)(12), 168, 1(10)58, 181 16(13)3, 178(12)
[96,4]	1	86	17, 1(11)(10)8, 18(17)(14), 1(18)4, 145(17), 13(17)(18), 1(11)(15), 12(14)(15), 1293 11(16)9, 164(14), 1532, 1(10)78, 12(10)(11), 14(13)4, 14(14)2, 186(14), 111(16) 15(13)9, 1569, 132(17), 13, 11(16)(14), 12(11)
[100,4]	1	89	15, 1, 1(13)9, 131(15), 14(10)7, 1(10)5(18), 11(16)7, 159(16), 16(14), 138(12), 11(18)6 11(11)(13), 118(10), 12(12)7, 115, 19(11)2, 11(14)(15), 13, 12(17)4, 1(10)(18), 17(18)3 17(10)(17), 15(12)8, 1(11)78, 141(12)

Table 13. QT codes over $\mathbb{F}_{19}$ with $m = 5$

code	λ	d	$b_i(x)$
[10,5]	1	6*	104, 1(10)(18)59
[15,5]	1	11*	114, 11(15)(13)(10), 14(14)(10)3
[20,5]	1	15	105, 11(15)(18)(12), 12(18)(15)(18), 131(12)6
[25,5]	1	19	12, 117(10)(15), 198, 12(12)(11)(13), 1(15)(14)4
[30,5]	1	23	11, 11(15)8(15), 11(16)(13)9, 1(11)12, 11634, 1(16)(18)8
[35,5]	1	28	132(14)(15), 11, 10(13)(18), 11(13)(11)5, 135(12)(15), 159(12), 1(14)(18)1
[40,5]	1	32	1(13)1, 1185(15), 17637, 16(13)(11)8, 1(15)(13)2, 131(14)(11), 17(12)1(14) 1(18)(13)(14)
[45,5]	1	37	1(12), 1(16)(10)7, 13(11)(15)4, 17(10)53, 1567(18), 12(17)62, 1(11)71(15) 16(10)8(17), 119(16)3
[50,5]	1	41	1(15)(15)6, 1298(11), 13(18)4(14), 13, 143(18)9, 1(14)(13)(11), 11766, 1(13)9(12) 124(10)(16), 19(13)(11)2
[55,5]	1	46	1(16), 14(13)(17)(12), 143(18)(11), 11(12)7(10), 13(17)(16)(11), 12347, 1(10)(15)93 138(12)7, 138(14)(17), 11(13)(11)6, 1(15)(11)
[60,5]	1	50	1, 15(12)4(18), 11345, 1562(15), 12145, 12(10)5(17), 108, 18(16)5, 13(18)8(12) 12(16)(12)(16), 15(18)(13)(15), 12634
[65,5]	1	55	11(10), 11(12)8(10), 16843, 137(13)4, 18(14)(16)6, 1(10)2, 13(11)(10)5, 1237(15) 1287, 17(18)57, 13(17)95, 13(15)(18)8, 13235
[70,5]	1	59	13, 13(11)(18)8, 1146(13), 11524, 13739, 12(10)(13)(16), 108(10), 16953, 17(11)(17) 13(17)74, 1535, 18387, 16(13)(12)(11), 141(15)(12)
[75,5]	1	64	13, 1(13)(11)(13)(18), 12, 13(16)(18)4, 189(13)(17), 12(18)(15), 1177(17), 145(13) 12(13)(12)2, 15(17)(16)6, 1013, 11(17)(15)(11), 117(15), 11112, 14(14)57
[80,5]	1	68	128(17)3, 1934(18), 18, 132(10)(18), 184(14)6, 1(16)52, 133(16), 13(13)(15)2 12(15)43, 13(16)17, 14(14)15, 17586, 12(15)15, 11962, 17(14), 117(12)9
[85,5]	1	73	1(15), 13(16)2, 143(11)9, 12(16)(17)4, 1136(11), 13(18)46, 15(17)59, 1163(17), 1 126(15)6, 13(16)56, 161(12)8, 10(16), 15(18)5(18), 12(15)(16)(18), 135(14)(11) 17(17)(15)
[90,5]	1	77	101, 102, 11(11)(11)(18), 1418(17), 1274, 12(16)7, 13(17)(12)(18), 11(13)(16)(11) 1693, 13(18)(14)(12), 15(17)1(11), 11778, 1(15)1, 13854, 1457, 11(14)(17)4, 1135 12(13)(12)(14)
[95,5]	1	82	1, 12(14)3(17), 101, 11(17)7(12), 14(17)(14)(15), 137(17)(16), 12(10)49 1(13)(14)(16), 12(14)68, 114, 12(16)1(12), 134(16)5, 18(12)4, 18(13)2, 161(15)2 1(10)(18)(11), 11(14)4(14), 112(15)(14), 12(12)43
[100,5]	1	87	111, 14, 183(16), 192(13)(17), 17(17)(14)6, 11(15)38, 13(13)(18)(15), 13(18)7(12) 1(12)(16)1, 13(13)(15)5, 1354(14), 164(18)(11), 1585, 145(11)5, 1(11)(17)4 1(10)(18)(17), 11(15)(17)(11), 17(18)7(18), 117(18)(12), 13(14)46
[105,5]	1	91	10(14), 11, 14(12)(15), 1118(17), 12, 1(13)(14)3, 18(14)6(15), 14(13)46, 137(13)9 11(16)(17)6, 1247(12), 1354(16), 11(10)27, 115(14)8, 131(12)3, 12(18)78, 114(14)2 11(11)75, 1538, 1(16)(16)2, 16(11)4
[110,5]	1	96	1025, 14, 18, 193(10)(17), 12(10)4(11), 1(10)8(17)(14), 14(13)2(18), 1(18)(18)(16) 137(10)(14), 11(18)78, 158(11)(15), 111(17)6, 1(13)(11)(13)6, 12(18)42 13(12)(16)(12), 11484, 1383(15), 142(12)6, 12(15)2(17), 11279, 14(11)(16)(17) 1693(17)
[115,5]	1	100	12, 1(14)5, 1(11), 145(16)5, 131(14)(18), 11(16)(17)(10), 177(12), 198(13), 161(14)6 14(13)(10)(14), 181(13), 1239, 11(13)9(14), 18(11)7(15), 12(10)1(16), 13(12)3(15) 167(12)(18), 11277, 19(10)1, 1442, 115(13)(17), 1(17)(10)4, 15(11)(12)6
[120,5]	1	105	118, 1769(12), 1, 142(10)3, 1(14)(17)(14), 12(12)(17)(18), 14(10)8(17), 1616 1529(16), 1(10)(17)4(14), 1144(16), 135(12), 15(13)4(14), 11853, 1(12)(10)(17) 11(10), 153(13)8, 13(14)34, 18(14)(10)(14), 13(11)1(18), 12(17)5(14), 11(15)(17)(16) 1(17)42, 123(17)(16)
[125,5]	1	109	102, 17(16)7, 13894, 13(15)(18)(14), 13(18)(10)(15), 12(14)76, 11, 15696, 142(13)3 11(11)(12)9, 15376, 11(13)76, 115(10), 1(10)52, 112(17)(10), 1145(10), 18957 132(11)2, 132(10)6, 118(11)5, 12(16), 13(14)2(17), 13(13)(10)3, 15(18)32 134(11)(15)