J. Algebra Comb. Discrete Appl.
6(2) • 95–103

Received: 9 December 2018
Accepted: 16 April 2019

Journal of Algebra Combinatorics Discrete Structures and Applications

Coretractable modules relative to a submodule

Research Article

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Abstract: Let R be a ring and M a right R-module. Let N be a proper submodule of M. We say that M is N-coretractable (or M is coretractable relative to N) provided that, for every proper submodule K of M containing N, there is a nonzero homomorphism $f : M/K \to M$. We present some conditions that a module M is coretractable if and only if M is coretractable relative to a submodule N. We also provide some examples to illustrate special cases.

2010 MSC: 16D10, 16D40, 16D80

Keywords: Coretractable module, N-coretractable module

1. Introduction

Throughout this paper R will denote an arbitrary associative ring with identity and all modules will be unitary right R-modules unless stated otherwise. Let M be an R-module. We use $End_R(M)$, $ann_r(M)$ (in the case M is a right R-module), $ann_l(M)$ (in the case M is a left R-module) to denote the ring of endomorphisms of M, the right annihilator in R of M and the left annihilator in R of M, respectively. Let M be a module and K a submodule of M. Then K is essential in M denoted by $K \leq_e M$, if $L \cap K \neq 0$ for every nonzero submodule L of M. Dually, K is small in M ($K \ll M$), in case $M = K + L$ implies that $L = M$. A submodule N of M is called *supplement*, if there is a submodule K of M such that $M = N + K$ and $N \cap K \ll N$. A module M is called *supplemented* if every submodule of M has a supplement in M. For any unexplained terminology we refer to [3], [9] and [11].

Khuri in [5] introduced the concept of a retractable module. Let M be a module. Then M is retractable in case for every nonzero submodule N of M, there is a nonzero homomorphism $f : M \to N$, i.e $Hom_R(M, N) \neq 0$. In the literature, there are some works about retractable modules (see [6, 12, 14]). Amini, Ershad and Sharif in [2] defined a dual notation namely coretractable modules. A module M is coretractable provided that, $Hom_R(M/N, M) \neq 0$ for every proper submodule N of M. There are also some papers whose main subject is to study and investigate coretractable modules. We refer readers to [1, 4, 13] for more information about coretractable modules.

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In [10], the author introduced a generalization of coretractable modules via the cosingular submodule. Following [10], a module M is called $\overline{Z}(M)$-coretractable in case, for every proper submodule N of M containing $\overline{Z}(M)$, there is a nonzero homomorphism $f : M/N \to M$. It is proved in [10, Theorem 2.11] that a ring R is $\overline{Z}(R_R)$-coretractable if and only if every finitely generated free right R-module F is $\overline{Z}(F)$-coretractable. Also, a characterization of commutative semiperfect Kasch rings is presented via $\overline{Z}$-coretractablity ([10, Corollary 2.14]). Inspiring by [10], we are interested to study coretractablity of modules relative to their submodules. If in the definition of a coretractable module M, we fix a submodule N and focus just on nonzero homomorphisms from M/K to M where $K \neq M$ contains N, we have a special generalization of coretractable modules. We may choose special submodules of a module M such as $Soc(M)$, $Rad(M)$ and some others. We present some necessary conditions to prove that when two concepts coretractable and coretractable relative to a submodule coincide. Among them, we show that for a small or a semisimple submodule N of M, M is coretractable if and only if M is N-coretractable. It is also shown that if M is N-coretractable and N is coretractable, then M is coretractable. For a right ideal I of R, we show that R_R is I-coretractable if and only if every simple right R-module that is annihilated by I, can be embedded in R_R. As a consequence, R_R is coretractable if and only if R is right Kasch.

2. Coretractable modules relative to a submodule

In this section we introduce a new generalization of coretractable modules via submodules.

Recall that a module M is coretractable, in case for every proper submodule N of M, there exists a nonzero homomorphism $f : M/N \to M$.

Definition 2.1. *Let M be a module and N a proper submodule of M. We say M is N-coretractable in case for every proper submodule K of M containing N, there is a nonzero homomorphism $f : M/K \to M$. Note that a module M is coretractable if and only if M is $\{0\}$-coretractable.*

Let M be a module and N a proper submodule of M. It is not hard to verify that M is N-coretractable if and only if for every proper essential submodule K of M containing N, there is a nonzero homomorphism from M/K to M.

Note that if a module M is N-coretractable, then for every submodule $T \subseteq N$, there is a nonzero homomorphism $g : M/T \to M$. In fact, if M is N-coretractable, then for every submodule T of M, either contained in N or containing N, there will be a nonzero homomorphism from M/T to M.

Recall from [7], a ring R is right (left) Kasch in case every simple right (left) R-module can be embedded in R_R (${}_RR$). In [2, Theorem 2.14], the authors proved that R is right Kasch if and only if R_R is coretractable.

Let R be a right Kasch ring which is not left perfect. Then by [4, Proposition 2.9], there is a right ideal I of R such that R/I is not coretractable while R_R is coretractable as R is a Kasch ring (see also [4, Example 2.10]).

Lemma 2.2. (1) *Let $N, K, N_i < M$. Let M be N-coretractable. If $K \supseteq N$, then M is K-coretractable. In particular, if M is N_i-coretractable for each $i \in I$, then M is $(\sum_{i\in I} N_i)$-coretractable.*

(2) *Let M be N-coretractable. If $K \leq N$ such that K contains no nonzero image of any endomorphism of M, then M/K is N/K-coretractable. In a special case, if M is N-coretractable such that for every $f \in End(M)$, $Imf \nsubseteq N$, then M/N is coretractable (see [4, Proposition 2.11]).*

Proof. (1) This is straightforward.

(2) Let T/K be a proper submodule of M/K containing N/K. Then $N \subseteq T \subset M$. Since M is N-coretractable, there exists a nonzero homomorphism $g : M/T \to M$. Now define $h : \frac{M/K}{T/K} \to M/K$ by $h(x + K + \frac{T}{K}) = g(x + T) + K$ for every $x \in M$. If $Imh = 0$, then $Img \subseteq K$. Now, K contains

the image of the endomorphism $go\pi$ of M where $\pi : M \to M/T$ is the natural epimorphism, this gives a contradiction. Therefore, M/K is N/K-coretractable. □

Let R be a right Noetherian ring and M be a N-coretractable module where N is a finitely generated proper submodule of M. Then by Lemma 2.2(2), M/N is coretractable (see [4, Corollary 2.13]).

Proposition 2.3. *Let M be a module and $K \leq N < M$. If M/K is N/K-coretractable and M/K can be embedded in M, then M is N-coretractable. In particular, if $M = K \oplus K'$ and N is any submodule of M such that K' is $(N \cap K')$-coretractable, then M is N-coretractable.*

Proof. Let T be a proper submodule of M containing N. Then T/K is a proper submodule of M/K containing N/K. By assumption, there is a nonzero homomorphism $g : \frac{M/K}{T/K} \cong M/T \to M/K$. There also exists a monomorphism $h : M/K \to M$. Now, the homomorphism $hog : M/T \to M$ is the required one. □

Corollary 2.4. *Let M be a module and $N < M$ such that M/N is coretractable. If M/N can be embedded in M, then M is N-coretractable. In particular, if M is supplemented with $Rad(M)$ a direct summand of M, then M is $Rad(M)$-coretractable.*

Proof. This is a special case of Proposition 2.3. The last part follows from the fact that for a supplemented module M, the module $M/Rad(M)$ is coretractable since $M/Rad(M)$ is semisimple. In this case M is $Rad(M)$-coretractable. □

Example 2.5. (1) *Let M be a coretractable module and $N < M$. Then M is N-coretractable. In particular, every cogenerator M in the category of right R-modules is coretractable relative to every $N < M$.*

(2) *Let M be a module such that for every submodule K of M we have $M/K \cong M$. Then M is coretractable relative to each $N < M$.*

(3) *Let M be a module and $N < M$. If every proper submodule of M containing N, is contained in a proper summand of M, then M is N-coretractable.*

(4) *Let M be an uniserial module. If M is coretractable relative to a proper submodule N, then M is coretractable.*

The following introduces a N-coretractable module which is not coretractable. In fact, the class of relative coretractable modules properly contains the class of coretractable modules.

Example 2.6. *Let P be the set of all prime numbers and $M = \prod_{p \in P} \mathbb{Z}_p$ as an $\mathbb{Z}$-module. Take $N = \{0\} \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \ldots$ which is a maximal submodule of M, since $M/N \cong \mathbb{Z}_2$. Consider $g : \mathbb{Z}_2 \to M$ defined by $g(x) = (x, 0, 0, \ldots)$. Then g is a nonzero homomorphism indicating that M is N-coretractable. Note that by [2, Example 2.9], M is not a coretractable $\mathbb{Z}$-module.*

Remark 2.7. *Let M be a module and $N < M$. If there is not a nonzero homomorphism from M/N to M, then M is not N-coretractable. For example, let M be a nonsingular module and N be a proper submodule of M such that M/N is singular. So there does not exist any nonzero homomorphism from M/N to M. Now, M is not N-coretractable (for example, $\mathbb{Z}$-modules $\mathbb{Q}$ and $\mathbb{Z}$ can not be $n\mathbb{Z}$-coretractable).*

We shall consider some conditions under which the two concepts coretractable and N-coretractable coincide.

Lemma 2.8. *Let M be a module and $N < M$. In each of the following cases M is N-coretractable if and only if M is coretractable.*

(1) *N is a small submodule of M.*

(2) *N is a coretractable module.*

Proof. (1) Let M be N-coretractable where $N \ll M$ and K be a proper submodule of M. Since N is small in M, we have $N + K \neq M$. Now since M is N-coretractable, then there is a nonzero homomorphism $f : M/(N+K) \to M$. So that $Hom_R(M/K, M) \neq 0$. It follows that M is coretractable. The converse is clear.

(2) Let K be a proper submodule of M. Then either $K + N \neq M$ or $K + N = M$. If $K + N \neq M$, then similarly to (1) we have $Hom_R(M/K, M) \neq 0$. Now suppose that $K + N = M$. Then there is an isomorphism $h : M/K \to N/(N \cap K)$ induced from $M = N + K$. Since N is coretractable, there is a nonzero homomorphism $g : N/(N \cap K) \to N$. Therefore, $jogoh : M/K \to M$ is a nonzero homomorphism where $j : N \to M$ is the inclusion. □

Recall that a module M is hollow, provided every proper submodule of M is small in M.

Corollary 2.9. (1) *Let M be a hollow module and $N < M$. Then M is N-coretractable if and only if M is coretractable.*

(2) *Let M be a finitely generated module. Then M is $Rad(M)$-coretractable if and only if M is coretractable.*

(3) *Let N be a semisimple submodule of M. Then M is N-coretractable if and only if M is coretractable.*

(4) *Let M be a module. Then M is $Soc(M)$-coretractable if and only if M is coretractable.*

Let M be a module and N a submodule of M. Following [15], N is δ-small in M (denoted by $N \ll_\delta M$), in case $M = N + K$ with M/K singular implies that $M = K$. Note that by definitions, every small submodule of M is δ-small in M. The sum of all δ-small submodules of M is denoted by $\delta(M)$. Also $\delta(M)$ is the reject of the class of all simple singular modules in M.

Proposition 2.10. *Let M be a module and N be a proper δ-small submodule of M. Then M is N-coretractable if and only if M is coretractable.*

Proof. Let M be N-coretractable and K be a proper submodule of M. Suppose that $M \neq N+K$. Since M is N-coretractable, there is a nonzero homomorphism $f : M/(N+K) \to M$. So that $fo\pi : M/K \to M$ is the required homomorphism where $\pi : M/K \to M/(N + K)$ is natural epimorphism. Otherwise, $M = N + K$. Now from [15, Lemma 1.2], there is a decomposition $M = Y \oplus K$ where Y is a semisimple projective submodule of N. Therefore, there is a monomorphism from M/K to M since K is a direct summand of M. It follows that M is coretractable. □

Proposition 2.11. *Let M be a module and N be a proper submodule of M. If M is N-coretractable and M/N has a maximal submodule, then $Soc(M) \neq 0$. In particular, if M is finitely generated and N-coretractable, then $Soc(M) \neq 0$.*

Proof. Let K/N be a maximal submodule of M/N. Then K is a maximal submodule of M. So there is a nonzero homomorphism $h : M/K \to M$. It follows that Imh is a simple submodule of M. This completes the proof. □

The following is an immediate consequence of last proposition.

Corollary 2.12. *Let R be a ring such that every cyclic right R-module is coretractable relative to at least one of its submodules. Then R is semi-Artinian.*

Let R be a ring. Then R is called a right V-ring in case every simple right R-module is injective. As a generalization of V-rings, R is a right generalized V-ring (GV-ring for short), if every simple singular right R-module is injective ([11]).

Proposition 2.13. *Let R be a ring and M be an indecomposable right R-module with $Rad(M) \neq M$. If each of the following statements holds, then M is $Rad(M)$-coretractable if and only if M is simple.*

(1) *R is a right GV-ring.*

(2) *M is noncosingular.*

Proof. (1) Let M be $Rad(M)$-coretractable. Then for each maximal submodule K of M there is a monomorphism $g : M/K \to M$. It follows that Img is a simple submodule of M. Then Img is either singular or projective. If Img is projective, then K is a direct summand of M and hence $K = 0$ or $K = M$. So that $K = 0$. If Img is singular, it will be injective as R is right GV. Therefore, Img is a summand of M and since $g \neq 0$ we conclude that $Img = M$. In both cases, M is simple. The converse is obvious.

(2) It follows from (1) and the fact that every homomorphic image of M is noncosingular. □

Corollary 2.14. *Let R be a right V-ring and M an indecomposable right R-module. Then M is coretractable if and only if M is simple.*

Following [8], a module M is dual Rickart provided that for every $f \in End(M)$, Imf is a direct summand of M.

Remark 2.15. *Let M be an indecomposable dual Rickart module with $Rad(M) \neq M$. Then M is ($Rad(M)$-)coretractable if and only if M is simple. Let K be a maximal submodule of M. Then there is a monomorphism $g : M/K \to M$. Consider the endomorphism $h = go\pi : M \to M$ where $\pi : M \to M/K$ is the natural epimorphism. Then $Imh = Img$ is a summand of M. So $Img = M$ as M is indecomposable. It follows that M is simple.*

Proposition 2.16. *Let M be a module and L a proper submodule of M such that L has a supplement K in M. If M is L-coretractable and K is fully invariant in M, then K is coretractable.*

Proof. Let K be a supplement of L in M. Then $M = K + L$ and $K \cap L \ll K$. Let N be a proper submodule of K. Then $N + L$ is a proper submodule of M. For if, $N + L = M$, by modular law $N + (K \cap L) = K$, which implies that $N = K$, a contradiction. Since M is L-coretractable, there is a nonzero homomorphism $f : M/(N + L) \to M$. Since K is a fully invariant submodule of M, we have $fo\pi(K) \subseteq K$ where $\pi : M \to M/(N + L)$ is the natural epimorphism. Now consider $h : K/N \to K$ by $h(x + N) = f(x + N + L)$ for every $x \in K$. It is not hard to verify that h is well-defined. Now, there is $y \in M$ such that $y \notin N + L$ and $f(y + N + L) \neq 0$. Now there exists $k \in K$ and $l \in L$ such that $y = k + l$. It is easy to see that $h(k + N) = f(k + l + N + L) = f(y + L) \neq 0$. It follows that h is nonzero. □

Corollary 2.17. *([2, Proposition 2.5]) Every fully invariant direct summand of a coretractable module is coretractable.*

Let M be a module. Then M is called a duo module provided every submodule of M is fully invariant.

Corollary 2.18. *Let M be a duo module. If M is coretractable relative to each direct summand of M, then every direct summand of M is coretractable.*

Proposition 2.19. *Let $M = M_1 \oplus \ldots \oplus M_n$ and $N < M$. If each M_i is $N \cap M_i$-coretractable, then M is N-coretractable. Especially a finite direct sum of coretractable modules is coretractable.*

Proof. The proof is exactly similar to proof of [2, Proposition 2.6]. □

Proposition 2.20. *Let R be a right max ring and $M = \bigoplus_{i \in I} M_i$ be a direct sum of $N \cap M_i$-coretractable right R-modules where $N < M$. Then M is N-coretractable. In particular, an arbitrary direct sum of coretractable right R-modules is coretractable.*

Proof. Similar to the proof of [2, Proposition 2.7]. □

Let M be an R-module. A submodule K of M is said to be dense in M if, for any $y \in M$ and $0 \neq x \in M$, there exists $r \in R$ such that $xr \neq 0$ and $yr \in K$. Obviously, any dense submodule of M is essential in M. From [7, Proposition 8.6], K is dense in M if and only if $Hom_R(P/K, M) = 0$ for every submodule $P \supseteq K$.

Remark 2.21. *Let M be a module and $N < M$. If N is dense in M, then M is not N-coretractable. In fact for a N-coretractable module M, we have N is not dense in M. This follows from the fact that if M is N-coretractable, then there is a nonzero homomorphism from M/N to M.*

Proposition 2.22. *Let M be a module and N a proper submodule of M. If M is quasi-injective or every proper submodule of M is contained in a maximal submodule, then M is N-coretractable if and only if every proper submodule of M containing N is not dense in M.*

Proof. (1) Let M be a quasi-injective module such that every proper submodule of M containing N is not dense in M. Suppose that K is a proper submodule of M containing N. Since K is not dense in M, there is a $f : P/K \to M$ where P is a submodule of M containing K. It follows that $fo\pi : P \to M$ is a nonzero homomorphism where $\pi : P \to P/K$ is the natural epimorphism. Consider the inclusion homomorphism $j : P \to M$. Since M is quasi-injective, there exists $h : M \to M$ such that $hoj = fo\pi$. By defining $\overline{h} : M/K \to M$ with $\overline{h}(m + K) = h(m)$ we conclude that M is N-coretractable. Note that $\overline{h}$ is nonzero. Conversely, if M is N-coretractable and $N \subseteq K < M$, then there is a homomorphism $g : M/K \to M$ which shows that K is not dense in M.

(2) Suppose that every submodule of M is contained in a maximal submodule of M. Let $N \subseteq K < M$. Then there is a maximal submodule L of M such that $K \leq L$. Since L is not dense in M, there is a nonzero homomorphism $h : M/L \to M$. As $f : M/K \to M/L$ with $f(x + K) = x + L$ is a nonzero homomorphism, then hof is nonzero. It follows that M is N-coretractable. The converse is the same as (1). □

The following presents a characterization of I-coretractable rings.

Theorem 2.23. *Let R be a ring and I be a proper right ideal of R. Then the following are equivalent:*

(1) *R_R is I-coretractable;*

(2) *Every n-generated free right R-module is $I^{(n)}$-coretractable;*

(3) *For every right ideal $T \supseteq I$, $ann_l(T) \neq 0$.*

Proof. (1) ⇔ (2) Follows from Proposition 2.19.

(1) ⇒ (3) Let T be a right ideal of R containing I. Since R_R is I-coretractable, there is a nonzero homomorphism $f : R/T \to R$. Consider the endomorphism $g = fo\pi : R \to R$ where π is the natural epimorphism from R to R/T. Then there is an element $0 \neq a \in R$ such that $g(x) = ax$. Let $y \in T$. Then $g(y) = ay = 0$ as $T \subseteq Kerg$. This shows that $0 \neq a \in ann_l(T)$.

(3) ⇒ (1) Let T be a right ideal of R containing I. Since $ann_l(T) \neq 0$, there exists an element of R such as a that $aT = 0$ and $a \neq 0$. Define $f : R/T \to R$ by $f(x + T) = ax$. It is easy to check that f is an R-homomorphism and in particular $f \neq 0$. □

Remark 2.24. *Let R be a ring and $I \leq R_R$ with $ann_l(I) = 0$. Then R_R is not I-coretractable. For example, let $R = \begin{bmatrix} K & K \\ 0 & K \end{bmatrix}$ be the ring of 2×2 upper triangular matrices over a field K. Let $I = \begin{bmatrix} 0 & K \\ 0 & K \end{bmatrix}$ which is a right ideal of R. Then $ann_l(I) = 0$. Hence, R_R is not I-coretractable. In other words, $R/J(R)$ is coretractable relative to each of its ideals as $R/J(R)$ is a semisimple ring. Note that $J(R) = \begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix}$.*

Theorem 2.25. *Let R be a ring and I be a proper two-sided ideal of R. Then the following statements are equivalent:*

(1) *R_R is I-coretractable;*

(2) *Every simple right R-module that is annihilated by I can be embedded in R_R.*

Proof. (1) ⇒ (2) Let $M \cong R/K$ be a simple right R-module such that $MI = 0$. It follows that $I \subseteq K$. Since R_R is I-coretractable, there is a nonzero homomorphism $f : R/K \to R$.

(2) ⇒ (1) Let T be a right ideal of R containing I. Now there exists a right maximal ideal K of R such that $I \subseteq T \subseteq K$. Consider the simple right R-module $M = R/K$. Since $MI = 0$, there is a nonzero homomorphism $g : R/K \to R$ by assumption. As T is a submodule of K, there exists $f : R/T \to R/K$ defined by $f(x + T) = x + K$. Hence gof is the desired homomorphism. □

For a ring R, Theorem 2.25 implies that R_R is ($J(R)$-)coretractable if and only if R is a right Kasch ring.

In [2, Proposition 4.4], it is shown that if R is a von Neumann regular ring then R is right Kasch if and only if R is semisimple. In the following we shall investigate a more general version.

Proposition 2.26. *Let R be a right GV-ring. Then the following are equivalent:*

(1) *R is right Kasch;*

(2) *R is semisimple.*

Proof. (1) ⇒ (2) Let R be right Kasch. So R_R is $J(R)$-coretractable. Now suppose that K is an arbitrary maximal right ideal of R. Then there is a monomorphism $g : R/K \to R$. It follows that $R/K \cong Img$ is a simple right R-module. So, Img is either singular or projective. In first case Img should be injective as R is right GV. Therefore, Img is a direct summand of R_R. Now Img is singular projective which implies that $Img = 0$, a contradiction. So that Img and hence every simple right R-module will be projective. This shows that R is semisimple.

(2) ⇒ (1) It is obvious. □

Corollary 2.27. *Let R be a right V-ring. Then R is a Kasch ring if and only if R is semisimple.*

Example 2.28. (1) *Let $R = \begin{bmatrix} K & K \\ 0 & K \end{bmatrix}$ where K is a field. Then $J(R) = \begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix}$. It is easy to check that R is a semilocal ring as $R/J(R) \cong K \times K$ which is a semisimple ring. Now by [3, Exercise 10, Page 113], $Soc(_RR) = \begin{bmatrix} K & K \\ 0 & 0 \end{bmatrix}$. However, $Soc(R_R) = \begin{bmatrix} 0 & K \\ 0 & K \end{bmatrix}$. Set $m_1 = Soc(_RR)$ and $m_2 = Soc(R_R)$. Then both m_1 and m_2 are maximal left and right ideals of R. A quick calculation shows that $ann_l(m_1) = m_2$, $ann_l(m_2) = 0$, $ann_r(m_1) = 0$ and $ann_r(m_2) = m_1$. Now by Theorem 2.23, R_R is m_1-coretractable while R_R is not m_2-coretractable. Also left version of Theorem 2.23, implies that $_RR$ is m_2-coretractable but it is not m_1-coretractable. Since the simple right R-module R/m_2 can not be embedded in R_R and the simple left R-module R/m_1 can not be embedded in $_RR$, the ring R is neither right Kasch nor left Kasch (note that since R is right GV which is not a V-ring, it can not be Kasch from Proposition 2.26).*

(2) *Let K be a division ring and $R = \{A = \begin{bmatrix} a & 0 & b & c \\ 0 & a & 0 & d \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & e \end{bmatrix} \mid a, b, c, d, e \in K\}$. Then $J(R) = \{A \in R \mid a = 0 = e\}$, $Soc(R_R) = ann_l(J(R)) = \{A \in R \mid a = 0\}$, $Soc(_RR) = ann_r(J(R)) = J(R)$. Since $R/J(R) \cong K \times K$, R is a semilocal ring. Now $Soc(R_R) = \{A \in R \mid a = 0\}$ and $Soc(_RR) = J(R)$. From [7, Example 8.29], $Soc(R_R)$ is a left and right maximal ideal of R. Since $ann_r(Soc(R_R)) = \{A \in R \mid a = e = 0\} = J(R) \neq 0$, it follows from [7, Corollary 8.28], $R/Soc(R_R)$ can be embedded in $_RR$ (see also Theorem 2.23). Therefore, $_RR$ is $Soc(R_R)$-coretractable while $_RR$ is not $Soc(_RR)$-coretractable (see also Corollary 2.9). Now an easy computation shows that $ann_l(Soc(R_R)) = \{A \in R \mid a = c = d = e = 0\} \neq 0$. So $R/Soc(R_R)$ can be embedded in R_R by [7, Corollary 8.28]. As $Soc(R_R)$ is a maximal right ideal of R, then R_R is $Soc(R_R)$-coretractable. Also from [7, Example 8.29], R is a right Kasch ring while it is not a left Kasch ring.*

(3) *Let K be a field and $R = \prod_{i=1}^{\infty} K$. It is well-known that R is a Von Neumann regular V-ring. Consider the ideal $T_i = K \times K \times \ldots \times K \times 0 \times K \times K \times \ldots$. It is clear that T_i for each $i \in \mathbb{N}$ is a maximal*

ideal of R. It is easy to see that $ann(T_i) = 0 \times 0 \times \ldots \times 0 \times K \times 0 \times \ldots$ which is nonzero. Therefore, from Theorem 2.23, R is I-coretractable for each $I \subseteq T_i$. Now consider the ideal $L = \bigoplus_{i=1}^{\infty} K$ of R. Then $ann(L) = 0$ and of course $ann(m) = 0$ for every maximal ideal m of R containing L. Hence the simple R-module R/m can not be embedded in R (see [7, Corollary 8.28]). Therefore, R is not coretractable relative to L. This means that R is not a Kasch ring.

Proposition 2.29. *Let R be a ring and I a right ideal of R such that every free right R-module $R^{(A)}$ is $(I^{(A)})$-coretractable. Then for every right R-module M with $I \subseteq ann_r(M)$, $Hom_R(M, R) \neq 0$.*

Proof. Let M be a right R-module such that $I \subseteq ann_r(M)$. Then there is a free right R-module F and a submodule K of F such that $M \cong F/K$. Since $MI = 0$, we have $I^{(A)} \subseteq K$ where A is an indexed set. By assumption, there is a nonzero homomorphism $f : F/K \to F$. Then the homomorphism $\pi o f : M \to R$ is the required one where $\pi : F \to R$ is the natural epimorphism. □

Proposition 2.30. *Let R be a ring having a radical right R-module M with $MI \neq M$ where $I \leq R_R$. If for every right ideal T of R, $Rad(T) \neq T$, then there is a free right R-module $R^{(A)}$ which is not $I^{(A)}$-coretractable.*

Proof. Let $Rad(M) = M$ such that MI is a proper submodule of M. There exists a free right R-module $F = R^{(A)}$ and a submodule K of F such that $M/MI \cong F/K$. Being M radical implies that M/MI is radical. So, $Hom_R(M/MI, R) = 0$. Since $(F/K)I = 0$, $I^{(A)} \subseteq K$. It follows that $Hom_R(F/K, F) = 0$ which implies F is not $I^{(A)}$-coretractable. □

Proposition 2.31. *Let R be a right max ring and $I \leq R_R$ such that every cyclic R-module N is NI-coretractable. Then every right R-module M is MI-coretractable. In particular, if R is a (semiperfect) right perfect ring with all cyclic right R-modules coretractable, then every (finitely generated) right R-module is coretractable.*

Proof. Let M be a right R-module. Suppose that K is a proper submodule of M containing MI. Since R is a right max ring, K is contained in a maximal submodule L of M. For every $x \in M \setminus L$, we know $M/L \cong xR/(xR \cap L)$ as $xR + L = M$. Note that $MI \subseteq L$. So that $(xR/(xR \cap L))I = 0$. It follows that $(xR)I \subseteq xR \cap L$. Being xR a $(xR)I$-coretractable module implies that $Hom_R(xR/(xR \cap L), xR) \neq 0$. Hence there is a nonzero homomorphism $f : M/L \to M$. Therefore, $Hom_R(M/K, M) \neq 0$ as $K \subseteq L$. □

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