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New extremal singly even self-dual codes of lengths 64 and 66*

Research Article

Damyan Anev, Masaaki Harada, Nikolay Yankov

Abstract: For lengths 64 and 66, we construct six and seven extremal singly even self-dual codes with weight enumerators for which no extremal singly even self-dual codes were previously known to exist, respectively. We also construct new 40 inequivalent extremal doubly even self-dual $[64, 32, 12]$ codes with covering radius 12 meeting the Delsarte bound. These new codes are constructed by considering four-circulant codes along with their neighbors and shadows.

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1. Introduction

A (binary) $[n, k]$ *code* C is a k-dimensional vector subspace of $\mathbb{F}_2^n$, where $\mathbb{F}_2$ denotes the finite field of order 2. All codes in this note are binary. The parameter n is called the *length* of C. The *weight* $\mathrm{wt}(x)$ of a vector x is the number of non-zero components of x. A vector of C is a *codeword* of C. The minimum non-zero weight of all codewords in C is called the *minimum weight* of C. An $[n, k]$ code with minimum weight d is called an $[n, k, d]$ code. The *dual code* $C^\perp$ of a code C of length n is defined as $C^\perp = \{x \in \mathbb{F}_2^n \mid x \cdot y = 0 \text{ for all } y \in C\}$, where $x \cdot y$ is the standard inner product. A code C is called *self-dual* if $C = C^\perp$. A self-dual code C is *doubly even* if all codewords of C have weight divisible by four, and *singly even* if there is at least one codeword x with $\mathrm{wt}(x) \equiv 2 \pmod 4$. It is known that a self-dual code of length n exists if and only if n is even, and a doubly even self-dual code of length n exists if and only if n is divisible by 8.

Let C be a singly even self-dual code. Let C_0 denote the subcode of C consisting of codewords x with $\mathrm{wt}(x) \equiv 0 \pmod 4$. The *shadow* S of C is defined to be $C_0^\perp \setminus C$. Shadows for self-dual codes

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Damyan Anev, Nikolay Yankov; Faculty of Mathematics and Informatics, Konstantin Preslavski University of Shumen, Shumen, 9712, Bulgaria (email: damian_anev@mail.bg, jankov_niki@yahoo.com).
Masaaki Harada (Corresponding Author); Research Center for Pure and Applied Mathematics, Graduate School of Information Sciences, Tohoku University, Sendai 980–8579, Japan (email: mharada@tohoku.ac.jp).

were introduced by Conway and Sloane [6] in order to give the largest possible minimum weight among singly even self-dual codes, and to provide restrictions on the weight enumerators of singly even self-dual codes. The largest possible minimum weights among singly even self-dual codes of length n were given for $n \le 72$ in [6]. The possible weight enumerators of singly even self-dual codes with the largest possible minimum weights were given in [6] and [7] for $n \le 72$. It is a fundamental problem to find which weight enumerators actually occur for the possible weight enumerators (see [6]). By considering the shadows, Rains [13] showed that the minimum weight d of a self-dual code of length n is bounded by $d \le 4\lfloor \frac{n}{24} \rfloor + 6$ if $n \equiv 22 \pmod{24}$, $d \le 4\lfloor \frac{n}{24} \rfloor + 4$ otherwise. A self-dual code meeting the bound is called *extremal*.

The aim of this note is to construct extremal singly even self-dual codes with weight enumerators for which no extremal singly even self-dual codes were previously known to exist. More precisely, we construct extremal singly even self-dual $[64, 32, 12]$ codes with weight enumerators $W_{64,1}$ for $\beta = 35$, and $W_{64,2}$ for $\beta \in \{19, 34, 42, 45, 50\}$ (see Section 2 for $W_{64,1}$ and $W_{64,2}$). These codes are constructed as self-dual neighbors of extremal four-circulant singly even self-dual codes. We construct extremal singly even self-dual $[66, 33, 12]$ codes with weight enumerators $W_{66,1}$ for $\beta \in \{7, 58, 70, 91, 93\}$, and $W_{66,3}$ for $\beta \in \{22, 23\}$ (see Section 2 for $W_{66,1}$ and $W_{66,3}$). These codes are constructed from extremal singly even self-dual $[64, 32, 12]$ codes by the method given in [14]. We also demonstrate that there are at least 44 inequivalent extremal doubly even self-dual $[64, 32, 12]$ codes with covering radius 12 meeting the Delsarte bound.

All computer calculations in this note were done with the help of the algebra software Magma [1] and the computer system Q-extensions [2].

2. Weight enumerators of extremal singly even self-dual codes of lengths 64 and 66

The possible weight enumerators $W_{64,i}$ and $S_{64,i}$ of extremal singly even self-dual $[64, 32, 12]$ codes and their shadows are given in [6]:

$$\begin{cases} W_{64,1} = & 1 + (1312 + 16\beta)y^{12} + (22016 - 64\beta)y^{14} + \cdots, \\ S_{64,1} = & y^4 + (\beta - 14)y^8 + (3419 - 12\beta)y^{12} + \cdots, \end{cases}$$

$$\begin{cases} W_{64,2} = & 1 + (1312 + 16\beta)y^{12} + (23040 - 64\beta)y^{14} + \cdots, \\ S_{64,2} = & \beta y^8 + (3328 - 12\beta)y^{12} + \cdots, \end{cases}$$

where β are integers with $14 \le \beta \le 104$ for $W_{64,1}$ and $0 \le \beta \le 277$ for $W_{64,2}$. Extremal singly even self-dual codes with weight enumerator $W_{64,1}$ are known for

$$\beta \in \left\{ \begin{array}{l} 14, 16, 18, 20, 22, 24, 25, 26, 28, 29, 30, 32, \\ 34, 36, 38, 39, 44, 46, 53, 59, 60, 64, 74 \end{array} \right\}$$

(see [4], [10], [11] and [16]). Extremal singly even self-dual codes with weight enumerator $W_{64,2}$ are known for

$$\beta \in \left\{ \begin{array}{l} 0, 1, \ldots, 41, 44, 48, 51, 52, 56, 58, 64, 65, 72, \\ 80, 88, 96, 104, 108, 112, 114, 118, 120, 184 \end{array} \right\} \setminus \{19, 31, 34, 39\}$$

(see [4], [10], [16] and [18]).

The possible weight enumerators $W_{66,i}$ and $S_{66,i}$ of extremal singly even self-dual $[66, 33, 12]$ codes and their shadows are given in [7]:

$$\begin{cases} W_{66,1} = 1 + (858 + 8\beta)y^{12} + (18678 - 24\beta)y^{14} + \cdots, \\ S_{66,1} = \beta y^9 + (10032 - 12\beta)y^{13} + \cdots, \end{cases}$$

$$\begin{cases} W_{66,2} = 1 + 1690y^{12} + 7990y^{14} + \cdots, \\ S_{66,2} = y + 9680y^{13} + \cdots, \end{cases}$$

$$\begin{cases} W_{66,3} = 1 + (858 + 8\beta)y^{12} + (18166 - 24\beta)y^{14} + \cdots, \\ S_{66,3} = y^5 + (\beta - 14)y^9 + (10123 - 12\beta)y^{13} + \cdots, \end{cases}$$

where β are integers with $0 \le \beta \le 778$ for $W_{66,1}$ and $14 \le \beta \le 756$ for $W_{66,3}$. Extremal singly even self-dual codes with weight enumerator $W_{66,1}$ are known for

$$\beta \in \{0, 1, \dots, 92, 94, 100, 101, 115\} \setminus \{4, 7, 58, 70, 91\}$$

(see [5], [8], [10], [17] and [18]). Extremal singly even self-dual codes with weight enumerator $W_{66,2}$ are known (see [8] and [15]). Extremal singly even self-dual codes with weight enumerator $W_{66,3}$ are known for

$$\beta \in \{24, 25, \dots, 92\} \setminus \{65, 68, 69, 72, 89, 91\}$$

(see [9], [10], [11] and [12]).

3. Extremal four-circulant singly even self-dual $[64, 32, 12]$ codes

An $n \times n$ circulant matrix has the following form:

$$\begin{pmatrix} r_0 & r_1 & r_2 & \cdots & r_{n-1} \\ r_{n-1} & r_0 & r_1 & \cdots & r_{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ r_1 & r_2 & r_3 & \cdots & r_0 \end{pmatrix},$$

so that each successive row is a cyclic shift of the previous one. Let A and B be $n \times n$ circulant matrices. Let C be a $[4n, 2n]$ code with generator matrix of the following form:

$$\begin{pmatrix} & I_{2n} & & \begin{matrix} A & B \\ B^T & A^T \end{matrix} \end{pmatrix}, \tag{1}$$

where I_n denotes the identity matrix of order n and A^T denotes the transpose of A. It is easy to see that C is self-dual if $AA^T + BB^T = I_n$. The codes with generator matrices of the form (1) are called *four-circulant*.

Two codes are *equivalent* if one can be obtained from the other by a permutation of coordinates. In this section, we give a classification of extremal four-circulant singly even self-dual $[64, 32, 12]$ codes. Our exhaustive search found all distinct extremal four-circulant singly even self-dual $[64, 32, 12]$ codes, which must be checked further for equivalence to complete the classification. This was done by considering all pairs of 16×16 circulant matrices A and B satisfying the condition that $AA^T + BB^T = I_{16}$, the sum of the weights of the first rows of A and B is congruent to 1 (mod 4) and the sum of the weights is greater than or equal to 13. Since a cyclic shift of the first rows gives an equivalent code, we may assume without loss of generality that the last entry of the first row of B is 1. Then our computer search shows that the above distinct extremal four-circulant singly even self-dual $[64, 32, 12]$ codes are divided into 67 inequivalent codes.

Proposition 3.1. *Up to equivalence, there are* 67 *extremal four-circulant singly even self-dual* $[64, 32, 12]$ *codes.*

We denote the 67 codes by $C_{64,i}$ ($i = 1, 2, \dots, 67$). For the 67 codes $C_{64,i}$, the first rows r_A (resp. r_B) of the circulant matrices A (resp. B) in generator matrices (1) are listed in Table 1. We verified that the codes $C_{64,i}$ have weight enumerator $W_{64,2}$, where β are also listed in Table 1.

Table 1. Extremal four-circulant singly even self-dual [64, 32, 12] codes

Codes	r_A	r_B	β
$C_{64,1}$	(0000001100111111)	(0001011010101111)	0
$C_{64,2}$	(0000010101111101)	(0010011010111011)	0
$C_{64,3}$	(0000011001101111)	(0010110101011011)	0
$C_{64,4}$	(0000000001011111)	(0001001100101011)	8
$C_{64,5}$	(0000000010101111)	(0011011011110111)	8
$C_{64,6}$	(0000000011010111)	(0000100110011011)	8
$C_{64,7}$	(0000000011010111)	(0000101100010111)	8
$C_{64,8}$	(0000000011010111)	(0011101110101111)	8
$C_{64,9}$	(0000000110111111)	(0101101111111111)	8
$C_{64,10}$	(0000001001011101)	(0001000101011011)	8
$C_{64,11}$	(0000001100011111)	(0010101011011111)	8
$C_{64,12}$	(0000001100011111)	(0010111011011011)	8
$C_{64,13}$	(0000001100111011)	(0001101011101111)	8
$C_{64,14}$	(0000001101111111)	(0011101111011111)	8
$C_{64,15}$	(0000010000111101)	(0010111011011111)	8
$C_{64,16}$	(0000010001011111)	(0001110101101111)	8
$C_{64,17}$	(0000010110111011)	(0001101110001111)	8
$C_{64,18}$	(0000000100011111)	(0010111111110011)	16
$C_{64,19}$	(0000000100111101)	(0000101011000111)	16
$C_{64,20}$	(0000000110010111)	(0001001111111111)	16
$C_{64,21}$	(0000000111001111)	(0010101110111101)	16
$C_{64,22}$	(0000000111001111)	(0010110110111011)	16
$C_{64,23}$	(0000001000101111)	(0011101011110111)	16
$C_{64,24}$	(0000001011100011)	(0010101111110111)	16
$C_{64,25}$	(0000001011100011)	(0011011011111011)	16
$C_{64,26}$	(0000010010011111)	(0010110011101111)	16
$C_{64,27}$	(0000011001101111)	(0001001011011111)	16
$C_{64,28}$	(0000011011011111)	(0010010101011101)	16
$C_{64,29}$	(0000011011100111)	(0001011111001011)	16
$C_{64,30}$	(0000011101111111)	(0101101110110111)	16
$C_{64,31}$	(0000101110111111)	(0011101011110111)	16
$C_{64,32}$	(0000000000100111)	(0001011101101011)	24
$C_{64,33}$	(0000000001011011)	(0010010101101011)	24
$C_{64,34}$	(0000000100111111)	(0001001000101011)	24
$C_{64,35}$	(0000000101001011)	(0010010110011011)	24
$C_{64,36}$	(0000000101001011)	(0010011001011011)	24
$C_{64,37}$	(0000000110111111)	(0000001000100111)	24
$C_{64,38}$	(0000001001111111)	(0010101111001011)	24
$C_{64,39}$	(0000001100011111)	(0001010011111111)	24
$C_{64,40}$	(0000001100011111)	(0001110011110111)	24
$C_{64,41}$	(0000010001011111)	(0010101111001111)	24
$C_{64,42}$	(0000010001101111)	(0011001110101111)	24
$C_{64,43}$	(0000010011101111)	(0001011101100111)	24
$C_{64,44}$	(0000010101010111)	(0001010111101111)	24
$C_{64,45}$	(0000010101010111)	(0010110011111011)	24
$C_{64,46}$	(0000010101110111)	(0000101111110011)	24
$C_{64,47}$	(0000010101110111)	(0001011101101011)	24
$C_{64,48}$	(0000011011110111)	(0101101110111111)	24
$C_{64,49}$	(0000000001001011)	(0000111010110111)	32
$C_{64,50}$	(0000000001100111)	(0001001111100011)	32

Table 1. Extremal four-circulant singly even self-dual $[64, 32, 12]$ codes (continued)

Codes	r_A	r_B	β
$C_{64,51}$	(0000001010111011)	(0001011111100111)	32
$C_{64,52}$	(0000010101011111)	(0001101111000111)	32
$C_{64,53}$	(0000010101111101)	(0010110010110111)	32
$C_{64,54}$	(0000011010111111)	(0000101110011101)	32
$C_{64,55}$	(0000101011101011)	(0001011111001011)	32
$C_{64,56}$	(0000000000100111)	(0001011010111011)	40
$C_{64,57}$	(0000000010101101)	(0001001011011011)	40
$C_{64,58}$	(0000001000011101)	(0000100101111011)	40
$C_{64,59}$	(0000001110011111)	(0001010111101101)	40
$C_{64,60}$	(0000011000111111)	(0001010111101101)	40
$C_{64,61}$	(0000011011001111)	(0000101010111111)	40
$C_{64,62}$	(0000100111011111)	(0001010101011011)	40
$C_{64,63}$	(0000001001101011)	(0001010011001101)	48
$C_{64,64}$	(0000000001011011)	(0001011000101111)	56
$C_{64,65}$	(0000010111011111)	(0010100101011011)	56
$C_{64,66}$	(0000101110011101)	(0001000101111111)	64
$C_{64,67}$	(0000000001011111)	(0001011111110111)	72

4. Extremal self-dual $[64, 32, 12]$ neighbors of $C_{64,i}$

Two self-dual codes C and C' of length n are said to be *neighbors* if $\dim(C \cap C') = n/2 - 1$. Any self-dual code of length n can be reached from any other by taking successive neighbors (see [6]). Since every self-dual code C of length n contains the all-one vector $\mathbf{1}$, C has $2^{n/2-1} - 1$ subcodes D of codimension 1 containing $\mathbf{1}$. Since $\dim(D^{\perp}/D) = 2$, there are two self-dual codes rather than C lying between $D^{\perp}$ and D. If C is a singly even self-dual code of length divisible by 8, then C has two doubly even self-dual neighbors (see [3]). In this section, we construct extremal self-dual $[64, 32, 12]$ codes by considering self-dual neighbors.

For $i = 1, 2, \ldots, 67$, we found all distinct extremal singly even self-dual neighbors of $C_{64,i}$, which are equivalent to none of the 67 codes. Then we verified that these codes are divided into 385 inequivalent codes $D_{64,i}$ ($i = 1, 2, \ldots, 385$). These codes $D_{64,i}$ are constructed as

$$\langle (C_{64,j} \cap \langle x \rangle^{\perp}), x \rangle.$$

To save space, the values j, the supports $\mathrm{supp}(x)$ of x, the values (k, β) in the weight enumerators $W_{64,k}$ are listed in "http://www.math.is.tohoku.ac.jp/~mharada/Paper/64-SE-d12.txt" for the 385 codes. For extremal singly even self-dual $[64, 32, 12]$ codes with weight enumerators for which no extremal singly even self-dual codes were previously known to exist, j, $\mathrm{supp}(x)$ and (k, β) are list in Table 2. Hence, we have the following:

Proposition 4.1. *There is an extremal singly even self-dual* $[64, 32, 12]$ *code with weight enumerator* $W_{64,1}$ *for* $\beta = 35$*, and* $W_{64,2}$ *for* $\beta \in \{19, 34, 42, 45, 50\}$.

Now we consider the extremal doubly even self-dual neighbors of $C_{64,i}$ ($i = 1, 2, 3$). Since the shadow has minimum weight 12, the two doubly even self-dual neighbors $\mathcal{C}^1_{64,i}$ and $\mathcal{C}^2_{64,i}$ are extremal doubly even self-dual $[64, 32, 12]$ codes with covering radius 12 (see [4]). Thus, six extremal doubly even self-dual $[64, 32, 12]$ codes with covering radius 12 are constructed. In addition, among the 385 codes $D_{64,i}$ ($i = 1, 2, \ldots, 385$), the 19 extremal singly even self-dual codes $D_{64,j}$ have shadow of minimum weight 12, where

$$j \in \{1, 2, 12, 19, 22, 33, 44, 58, 66, 68, 84, 95, 108, 115, 136, 143, 191, 240, 254\}.$$

Table 2. Extremal singly even self-dual $[64, 32, 12]$ neighbors

Codes	j	supp(x)	(k, β)
$D_{64,138}$	24	$\{1, 2, 3, 38, 42, 43, 45, 46, 48, 54, 56, 57\}$	$(2, 19)$
$D_{64,270}$	49	$\{1, 2, 8, 32, 38, 41, 48, 49, 50, 53, 55, 61\}$	$(1, 35)$
$D_{64,283}$	52	$\{1, 2, 4, 33, 36, 37, 41, 43, 46, 51, 61, 64\}$	$(2, 42)$
$D_{64,293}$	56	$\{3, 7, 9, 10, 11, 37, 43, 53, 57, 58, 62, 64\}$	$(2, 34)$
$D_{64,314}$	64	$\{6, 8, 26, 37, 38, 40, 43, 46, 48, 59, 61, 63\}$	$(2, 50)$
$D_{64,329}$	65	$\{1, 6, 8, 9, 37, 47, 50, 52, 57, 60, 63, 64\}$	$(2, 45)$
$D_{64,1}$	1	$\{4, 7, 9, 34, 38, 40, 45, 46, 47, 50, 51, 53\}$	$(2, 0)$
$D_{64,2}$	1	$\{3, 37, 38, 47, 48, 50, 52, 53, 54, 59, 60, 63\}$	$(2, 0)$
$D_{64,12}$	4	$\{2, 4, 5, 16, 17, 38, 40, 46, 56, 57, 60, 62\}$	$(2, 0)$
$D_{64,19}$	4	$\{2, 3, 6, 7, 9, 35, 41, 49, 55, 56, 57, 61\}$	$(2, 0)$
$D_{64,22}$	4	$\{2, 33, 34, 35, 38, 39, 42, 45, 48, 52, 61, 62\}$	$(2, 0)$
$D_{64,33}$	6	$\{8, 9, 10, 16, 17, 33, 44, 45, 54, 55, 59, 61\}$	$(2, 0)$
$D_{64,44}$	6	$\{1, 3, 6, 33, 36, 38, 39, 45, 47, 55, 57, 59\}$	$(2, 0)$
$D_{64,58}$	8	$\{1, 3, 5, 16, 17, 35, 36, 38, 42, 44, 54, 59\}$	$(2, 0)$
$D_{64,66}$	8	$\{4, 6, 9, 34, 36, 39, 41, 42, 48, 51, 57, 63\}$	$(2, 0)$
$D_{64,68}$	8	$\{3, 6, 9, 33, 36, 37, 38, 49, 56, 57, 60, 62\}$	$(2, 0)$
$D_{64,84}$	13	$\{1, 4, 5, 35, 37, 38, 41, 44, 53, 60, 61, 62\}$	$(2, 0)$
$D_{64,95}$	13	$\{2, 4, 9, 34, 35, 40, 42, 47, 49, 52, 59, 64\}$	$(2, 0)$
$D_{64,108}$	15	$\{2, 16, 17, 37, 43, 48, 49, 52, 54, 57, 58, 64\}$	$(2, 0)$
$D_{64,115}$	16	$\{1, 3, 6, 7, 8, 41, 45, 46, 49, 50, 57, 60\}$	$(2, 0)$
$D_{64,136}$	21	$\{3, 16, 17, 33, 34, 37, 42, 44, 47, 51, 52, 56\}$	$(2, 0)$
$D_{64,143}$	26	$\{1, 2, 9, 34, 37, 38, 41, 48, 57, 58, 59, 64\}$	$(2, 0)$
$D_{64,191}$	35	$\{1, 2, 6, 8, 10, 33, 37, 46, 54, 59, 60, 63\}$	$(2, 0)$
$D_{64,240}$	47	$\{2, 4, 7, 9, 13, 16, 17, 44, 56, 59, 62, 64\}$	$(2, 0)$
$D_{64,254}$	48	$\{1, 2, 5, 7, 8, 35, 36, 37, 45, 47, 49, 63\}$	$(2, 0)$
$D_{64,14}$	4	$\{1, 7, 8, 35, 36, 37, 41, 43, 46, 49, 51, 53\}$	$(1, 14)$
$D_{64,383}$	67	$\{1, 33, 34, 36, 37, 38, 40, 41, 47, 49, 50, 53, 55, 59, 61, 63\}$	$(2, 40)$

The constructions of the 19 codes $D_{64,j}$ are listed in Table 2. Their two doubly even self-dual neighbors $\mathcal{D}^1_{64,j}$ and $\mathcal{D}^2_{64,j}$ are extremal doubly even self-dual $[64, 32, 12]$ codes with covering radius 12. We verified that there are the following equivalent codes among the four codes in [4], the six codes $\mathcal{C}^1_{64,i}$, $\mathcal{C}^2_{64,i}$ and the 38 codes $\mathcal{D}^1_{64,j}$, $\mathcal{D}^2_{64,j}$, where

$$\mathcal{D}^2_{64,22} \cong \mathcal{D}^2_{64,68}, \mathcal{D}^2_{64,33} \cong \mathcal{D}^2_{64,84}, \mathcal{D}^2_{64,44} \cong \mathcal{D}^2_{64,95}, \mathcal{D}^2_{64,136} \cong \mathcal{D}^2_{64,143},$$

where $C \cong D$ means that C and D are equivalent, and there is no other pair of equivalent codes. Therefore, we have the following proposition.

Proposition 4.2. *There are at least* 44 *inequivalent extremal doubly even self-dual* $[64, 32, 12]$ *codes with covering radius* 12 *meeting the Delsarte bound.*

In order to distinguish two doubly even neighbors $\mathcal{D}^1_{64,i}$ and $\mathcal{D}^2_{64,i}$ $(i = 68, 84, 95, 143)$, we list in Table 3 the supports supp(x) for the 8 codes, where $\mathcal{D}^1_{64,i}$ and $\mathcal{D}^2_{64,i}$ are constructed as $\langle (D_{64,i} \cap \langle x \rangle^\perp), x \rangle$.

Table 3. Extremal doubly even self-dual [64, 32, 12] neighbors

Codes	supp(x)
$\mathcal{D}^1_{64,68}$	$\{1, 4, 7, 34, 35, 36, 47, 54, 55, 58, 60, 63\}$
$\mathcal{D}^2_{64,68}$	$\{1, 4, 5, 6, 30, 42, 45, 47, 54, 56, 58, 64\}$
$\mathcal{D}^1_{64,84}$	$\{16, 17, 33, 39, 43, 46, 48, 49, 51, 54, 58, 64\}$
$\mathcal{D}^2_{64,84}$	$\{1, 2, 6, 33, 35, 38, 40, 42, 52, 57, 59, 60\}$
$\mathcal{D}^1_{64,95}$	$\{1, 2, 6, 33, 35, 38, 40, 42, 52, 57, 59, 60\}$
$\mathcal{D}^2_{64,95}$	$\{3, 33, 38, 41, 45, 47, 51, 53, 58, 60, 62, 64\}$
$\mathcal{D}^1_{64,143}$	$\{1, 4, 10, 40, 43, 46, 52, 54, 58, 61, 62, 63\}$
$\mathcal{D}^2_{64,143}$	$\{1, 31, 34, 42, 44, 45, 46, 50, 51, 52, 54, 62\}$

5. Four-circulant singly even self-dual [64, 32, 10] codes and self-dual neighbors

Using an approach similar to that given in Section 3, our exhaustive search found all distinct four-circulant singly even self-dual [64, 32, 10] codes. Then our computer search shows that the distinct four-circulant singly even self-dual [64, 32, 10] codes are divided into 224 inequivalent codes.

Proposition 5.1. *Up to equivalence, there are* 224 *four-circulant singly even self-dual* [64, 32, 10] *codes.*

We denote the 224 codes by $E_{64,i}$ $(i = 1, 2, \ldots, 224)$. For the codes, the first rows r_A (resp. r_B) of the circulant matrices A (resp. B) in generator matrices (1) can be obtained from "http://www.math.is.tohoku.ac.jp/~mharada/Paper/64-4cir-d10.txt".

The following method for constructing self-dual neighbors was given in [4]. For $C = E_{64,i}$ $(i = 1, 2, \ldots, 224)$, let M be a matrix whose rows are the codewords of weight 10 in C. Suppose that there is a vector x of even weight such that

$$Mx^T = \mathbf{1}^T. \tag{2}$$

Then $C^0 = \langle x\rangle^\perp \cap C$ is a subcode of index 2 in C. We have self-dual neighbors $\langle C^0, x\rangle$ and $\langle C^0, x + y\rangle$ of C for some vector $y \in C \setminus C^0$, which have no codeword of weight 10 in C. When C has a self-dual neighbor C' with minimum weight 12, there is a vector x satisfying (2) and we can obtain C' in this way. For $i = 1, 2, \ldots, 224$, we verified that there is a unique vector satisfying (2) and C has two self-dual neighbors, where C^0 is a doubly even [64, 31, 12] code. In this case, the two neighbors are automatically doubly even. Hence, we have the following:

Proposition 5.2. *There is no extremal singly even self-dual* [64, 32, 12] *neighbor of* $E_{64,i}$ *for* $i = 1, 2, \ldots, 224$.

6. Extremal singly even self-dual [66, 33, 12] codes

The following method for constructing singly even self-dual codes was given in [14]. Let C be a self-dual code of length n. Let x be a vector of odd weight. Let C^0 denote the subcode of C consisting of all codewords which are orthogonal to x. Then there are cosets C^1, C^2, C^3 of C^0 such that ${C^0}^\perp = C^0 \cup C^1 \cup C^2 \cup C^3$, where $C = C^0 \cup C^2$ and $x + C = C^1 \cup C^3$. It was shown in [14] that

$$C(x) = (0, 0, C^0) \cup (1, 1, C^2) \cup (1, 0, C^1) \cup (0, 1, C^3) \tag{3}$$

is a self-dual code of length $n+2$. In this section, we construct new extremal singly even self-dual codes of length 66 using this construction from the extremal singly even self-dual $[64, 32, 12]$ codes obtained in Sections 3 and 4.

Our exhaustive search shows that there are 1166 inequivalent extremal singly even self-dual $[66, 33, 12]$ codes constructed as the codes $C(x)$ in (3) from the codes $C_{64,i}$ $(i = 1, 2, \ldots, 67)$. 1157 codes of the 1166 codes have weight enumerator $W_{66,1}$ for $\beta \in \{7, 8, \ldots, 92\} \setminus \{9, 11\}$, 3 of them have weight enumerator $W_{66,3}$ for $\beta \in \{30, 49, 54\}$, and 6 of them have weight enumerator $W_{66,2}$. Extremal singly even self-dual $[66, 33, 12]$ codes with weight enumerator $W_{66,1}$ for $\beta \in \{7, 58, 70, 91\}$ are constructed for the first time. For the four weight enumerators W, as an example, codes $C_{66,i}$ with weight enumerators W are given $(i = 1, 2, 3, 4)$. We list in Table 4 the values β in W, the codes C and the vectors $x = (x_1, x_2, \ldots, x_{32})$ of $C(x)$ in (3), where $x_j = 1$ $(j = 33, \ldots, 64)$.

Table 4. Extremal singly even self-dual $[66, 33, 12]$ codes

Codes	β	W	C	$(x_1, \ldots, x_{32})$
$C_{66,1}$	7	$W_{66,1}$	$C_{64,1}$	(01101101101010010111111010101100)
$C_{66,2}$	58	$W_{66,1}$	$C_{64,56}$	(00001101100000011000110000011100)
$C_{66,3}$	70	$W_{66,1}$	$C_{64,66}$	(00100110011011001001011100000010)
$C_{66,4}$	91	$W_{66,1}$	$C_{64,67}$	(00001110110111110000011101000010)
$D_{66,1}$	22	$W_{66,3}$	$D_{64,14}$	(10100011100100110111101010011111)
$D_{66,2}$	23	$W_{66,3}$	$D_{64,14}$	(10111100111100000100101000100011)
$D_{66,3}$	93	$W_{66,1}$	$D_{64,383}$	(10100101011110010011001101001101)

By applying the construction given in (3) to $D_{64,i}$, we found more extremal singly even self-dual $[66, 33, 12]$ codes $D_{66,j}$ with weight enumerators for which no extremal singly even self-dual codes were previously known to exist. For the codes $D_{66,j}$, we list in Table 4 the values β in the weight enumerators W, the codes C and the vectors $x = (x_1, x_2, \ldots, x_{32})$ of $C(x)$ in (3), where $x_i = 1$ $(i = 33, \ldots, 64)$. Hence, we have the following:

Proposition 6.1. *There is an extremal singly even self-dual* $[66, 33, 12]$ *code with weight enumerator* $W_{66,1}$ *for* $\beta \in \{7, 58, 70, 91, 93\}$*, and weight enumerator* $W_{66,3}$ *for* $\beta \in \{22, 23\}$*.*

Remark 6.2. *The code* $D_{66,1}$ *has the smallest value* β *among known extremal singly even self-dual* $[66, 33, 12]$ *codes with weight enumerator* $W_{66,3}$*.*

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