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Game chromatic number of Cartesian and corona product graphs

Research Article

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Abstract: The game chromatic number χ_g is investigated for Cartesian product $G\square H$ and corona product $G\circ H$ of two graphs G and H. The exact values for the game chromatic number of Cartesian product graph of $S_3\square S_n$ is found, where S_n is a star graph of order $n+1$. This extends previous results of Bartnicki et al. [1] and Sia [5] on the game chromatic number of Cartesian product graphs. Let P_m be the path graph on m vertices and C_n be the cycle graph on n vertices. We have determined the exact values for the game chromatic number of corona product graphs $P_m\circ K_1$ and $P_m\circ C_n$.

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1. Introduction

We consider the following well-known graph coloring game, played on a simple graph G with a color set C of cardinality k. Two players, Alice and Bob, alternately color an uncolored vertex of G with a color from C such that no adjacent vertices receive the same color (such a coloring of a graph G is known as proper coloring). Alice has the first move and the game ends when no move is possible any more. If all the vertices are properly colored, Alice wins, otherwise Bob wins. The game chromatic number of G, denoted by $\chi_g(G)$, is the least cardinality k of the set C for which Alice has a winning strategy. In other words, necessary and sufficient conditions for k to be the game chromatic number of a graph G are:
i) Bob has winning strategy for $k-1$ colors or less and
ii) Alice has winning strategy for k colors.

This parameter is well defined, since it is easy to see that Alice always wins if the number of colors is larger than the maximum degree of G. Clearly, $\chi_g(G)$ is at least as large as the ordinary chromatic number $\chi_(G)$, but it can be considerably more. For example, let G be a complete bipartite graph $K_{n,n}$

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minus a perfect matching M and consider the following strategy for Bob. If Alice colors vertex v with color c then Bob responds by coloring the vertex u matched with v in the matching M with the same color c. Note that now c cannot be used on any other vertex in the graph. Therefore, if the number of colors is less than n, Bob wins the game. This shows that there are bipartite graphs with arbitrarily large game chromatic number and thus there is no upper bound on $\chi_g(G)$ as a function of $\chi_(G)$.

The obvious bounds for the game chromatic number are:

$$\chi(G) \leq \chi_g(G) \leq \Delta(G) + 1 \tag{1}$$

where $\chi(G)$ is the chromatic number and $\Delta(G)$ is the maximum degree of the graph G.

A lot of attempts have been made to determine the game chromatic number for several classes of graphs. This work was first initiated by Faigle et al. [3]. It was proved by Kierstead and Trotter [4] that the maximum of the game chromatic number of a forest is 4, also that 33 is an upper bound for the game chromatic number of planar graphs. Bodlaender [2], found that the game chromatic number of Cartesian product is bounded above by constant in the family of planar graphs. Later, Bartnicki et al. [1] determine the exact values of $\chi_g(G\Box H)$ when G and H belong to certain classes of graphs, and showed that, in general, the game chromatic number $\chi_g(G\Box H)$ is not bounded from above by a function of game chromatic numbers of the graphs G and H. After that, Zhu [6] established a bound for game coloring number and acyclic chromatic number for Cartesian product of two graphs H and S. Sia [5] determined the exact values for the Cartesian product of different families of graph like $S_m\Box P_n$, $S_m\Box C_n$, $P_2\Box W_n$, where P_n, C_n, denotes the path graph and cycle graph with n vertices, respectively. While S_m is a star graph with $m+1$ vertices and W_n is a wheel graph with $n+1$ vertices.

In this work, we have extended the study of game chromatic number and found the exact values of the game chromatic number of $S_3\Box S_n$, $P_m \circ K_1$ and $P_m \circ C_n$, where $\circ$ denote corona product of graphs.

2. Exact value of $\chi_g(S_3\Box S_n)$

A *Cartesian product* of two graphs G and H, denoted by $G\Box H$, is the graph with vertex set $V(G) \times V(H)$, where two vertices (u, u') and (v, v') are adjacent if and only if $u = v$ and $u'v' \in E(H)$ or $u' = v'$ and $uv \in E(G)$. Note that the Cartesian product operation is both commutative and associative up to isomorphism.

Before stating our results, we introduce some definitions and notational conventions. Suppose that Alice and Bob plays the coloring game with k colors. We say that there is a *threat* to an uncolored vertex v if there are $k-1$ colors in the neighborhood of v, and it is possible to color a vertex adjacent to v with the last color, so that all k colors would then appear in the neighborhood of v. The threat to the vertex v is said to be *blocked* if v is subsequently assigned a color, or it is no longer possible for v to have all k colors in its neighborhood. We shall also use the convention that color numbers are only used to differentiate distinct colors, and should not be regarded as ascribed to particular colors. For example, if only colors 1 and 2 have been used so far and we introduce a new color, color 3, then color 3 can refer to any color that is not the same as color 1 or color 2. Finally, we label figures in the following manner: vertices are labeled in the form "$color(player)_{turn}$", with the information in parentheses being omitted if the same configuration can be attained in multiple ways.

In the following result, we have found the exact value of $\chi_g(S_3\Box S_n)$.

Theorem 2.1. *For integer $n \geq 3$, $\chi_g(S_3\Box S_n) = 4$.*

Proof. First, we show that Bob has a winning strategy with three or fewer colors. We give Bob's winning strategy when there are exactly three colors, it will be easy to see that Bob can win using the same strategy when there are fewer than three colors. Denote the vertices of $S_3\Box S_n$ by $a_1, a_2, ..., a_{n+1}, b_1, b_2, ..., b_{n+1}, c_1, c_2, ..., c_{n+1}, d_1, d_2, ...d_{n+1}$. The vertex a_1 has degree $n+3$, three vertices b_1, c_1 and d_1 have degree $n+1$. The vertices $a_i's$ for $i \geq 2$ has degree 4. While all the remaining vertices have degree two.

The graph shown in Figure 1 is isomorphic to $S_3 \square S_n$.

Bob's winning strategy with three colors:

Case 1: Let $D = \{1, 2, 3\}$ be set of colors. If Alice begins by playing color 1 in the vertex a_1 of highest degree. Bob should plays color 2 in any of the vertex d_k of degree two (say d_2), see Figure 1. Alice can not block the threats to both the vertices d_1 and a_2, so Bob wins.

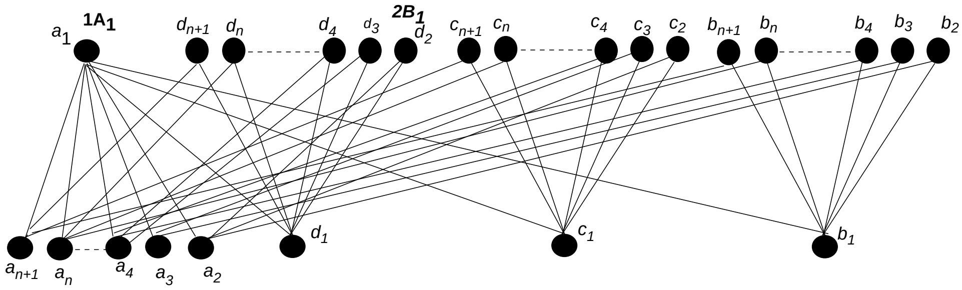

Figure 1. Bob's winning strategy in Case 1 for the graph $S_3 \square S_n$

Case 2: If Alice start the game by playing color 1 in any of the vertex b_i, c_j or d_k $2 \leq i, j, k \leq n$ (say b_2), Bob respond with color 2 in the vertex a_1, see Figure 2. Now there are two vertices a_2 and b_1, which are under threat. Alice can save only one of these vertices and Bob wins the game.

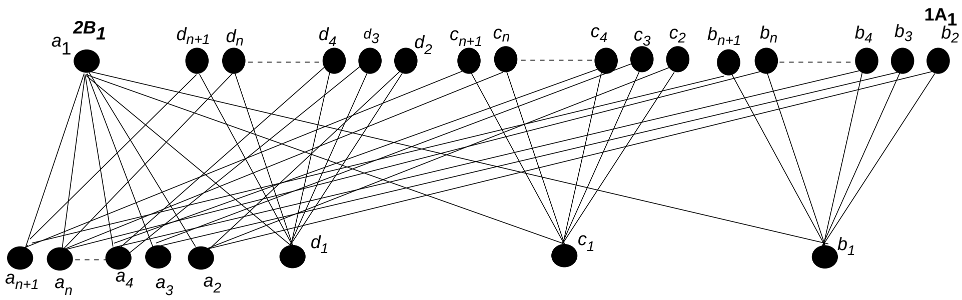

Figure 2. Bob's winning strategy in Case 2 for the graph $S_3 \square S_n$

Case 3: If Alice begins the game by assigning color to any of the vertices a_l, $2 \leq l \leq n$. Suppose, she plays a color 1 in the vertex a_2. In response, Bob plays color 2 in the vertex a_i, where $i \neq 2$ and $i > 2$ say a_3. This forces Alice to assign color 3 to the vertex a_1, because this is the only way, she can block the threat to this vertex. After that Bob plays color 1 in the neighbor of $a_l's$ $(l \geq 4)$ and wins the games.

Case 4: If Alice start the game by playing color 1 in vertex b_1, c_1 or d_1 (say b_1). Bob should plays color 2 in any of the vertices a_i, where $2 \leq i \leq n$, for simplicity, say a_2. There are two distinct colors in the neighbor of vertex a_1, so Alice must block the threat to this vertex by assigning it the color 3. In the next move Bob plays color 1 in d_k $(k \neq 2)$ say d_3. Now Alice can't block the threat of d_1 and a_3 and Bob wins the game.

Thus $\chi_g(S_3 \square S_n) \geq 4$. Next, Alice's winning strategy is given for four colors, which proves that $\chi_g(S_3 \square S_n) \leq 4$.

Alice's winning strategy with four colors: Alice in her strategy, first assigns colors to all the vertices of degree $n+1$, because in this way, she can prevent these vertices from threats. Now a_l, where $l \geq 1$ are

the only vertices which can come under threat. Since $a_l \sim b_i$ $(l = i)$, $a_l \sim c_j$ $(l = j)$ and $a_l \sim d_k$ $(l = k)$. So, whenever, Bob plays a color in a vertex adjacent to a_i, say c_j, Alice will play the color of vertex a_1 or color of the vertex c_j in b_i (adjacent to a_l). In this way, Alice can prevent these vertices from threats, since every vertex has two same colors in its neighbor. So, in order to win the game, she need only to assign colors in the vertices of degree greater than 4. Below, we have given the Alice strategy to color these vertices with four given colors.
Alice makes her first move by playing color 1 in any of three vertices having degree $n+1$, say b_1. We, now consider all the cases for Bob's first move.

Case 1: If Bob plays color 2 in any of the vertex having degree four, say a_2. Alice makes her second move by playing color 2 in the vertex c_1. Now Bob has two possibilities for his second move.

Case 1.1: If Bob wants that the vertex a_1 be under threat in his second move and assigns color 3 to the vertex a_3. Alice, then has to play color 4 in a_1, in order to prevent this vertex from threat. Alice is safe to assign suitable color in the vertex d_1 in her fourth move. Now all the vertices of degree greater than four have assigned color. After that whenever Bob plays a color in a vertex b_i or c_j say c_j (adjacent to a_l), Alice will play the color of c_j in b_i where $l = i = j$ and win the game. (strategy is shown in Figure 3)

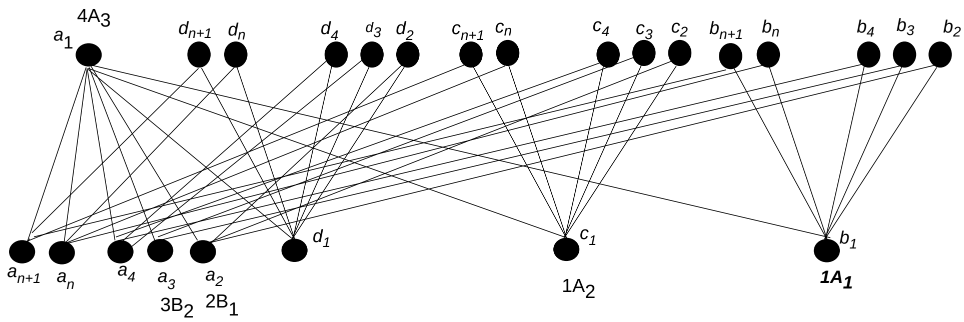

Figure 3. Alice's winning strategy in Case 1 for the graph $S_3 \Box S_n$

Case 1.2**:** Bob plays color 1 in $d_k's$, say d_3 which is adjacent to d_1. In respond, Alice assigns color 2 in a_1. In his third move Bob plays according to the following strategy.
i: If Bob plays color 2 in the vertex $b_i's$ or $c_j's$, Alice then plays color of the vertex c_j in b_i $(i = j)$. Now vertex a_l $(i = l = j)$ has two same colors in its neighbors. In this way Alice protects all the vertices of degree four from threat. Alice will play only color in a_1, whenever Bob plays color in $a_i's$.

Case 2: In this case, Bob plays his first move in the vertices of degree two and assigns color 1 in d_k or c_j say d_2. Alice then plays color 1 in c_1. Now we consider all the cases for Bob's second move.

Case 2.1**:** If Bob plays again in the neighbor of d_1 say d_3 by assigning it color 2. Alice, in respond plays color 3 in d_1. Now there are following possibilities of Bob's third move. We distinguish 3 subcases.
i: If Bob wants that the vertex a_1 be under threat and plays color 2 in a_l say a_2. Alice then should play color 4 in a_1. Now all the vertices of degree greater than four have assigned color. After that Alice follow the same strategy as in case 1.

ii: If Bob plays in the neighbor of that a_l which has already two distinct colors say a_2 and assigns color 2 in b_i or c_j say c_2. Alice can prevent this vertex from threat by playing color of vertex c_2 in vertex b_2. After that Alice follows the same strategy as in case 1.
iii: If Bob wants to plays in the neighbor of that a_l which has has no color assigned say a_4 and assigns color 1 in d_4 (d_4 is adjacent to a_4), then Alice plays color 2 or color 4 in the vertex a_1. In his next move if, Bob again plays in the neighbor of a_4 and assigns color 3 in c_4. Alice then should play color of c_4 in b_4. Now a_4 has two same colors in its neighbor. Whenever, Bob plays color 1 in any one of the remaining uncolored $d_{k's}$ say d_5, Alice then plays color of vertex a_1 in c_5. Now there are two same colors in the

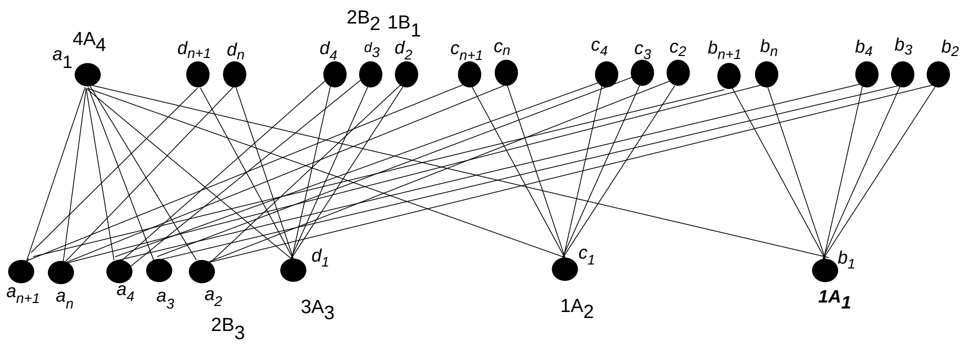

Figure 4. Alice's winning strategy in Case 2 **for the graph** $S_3 \Box S_n$

neighbor of a_5. After that Alice follow the same strategy as in case 1.

Case 2.2**:** Now vertices d_1 and a_2 have one color in its neighbor. If Bob make his second move by playing the color 2 in b_i or c_j say c_2, then Alice assigns color of vertex c_2 in b_2. In this way, Alice can block the threat of vertex a_2. After that Alice follows the same strategies as in case 1 and case 2.1.

Case 2.3**:** If Bob plays suitable color in c_j or b_i say c_j $(k \neq j)$, then Alice plays color of that c_j in b_i $(i = j)$. After that Alice follows the same strategies as in above cases.

Case 3**:** If Alice assigns color 1 in b_1 in her first move and Bob plays color 2 in c_1. Alice then plays color in d_1. Now all the vertices of degree $n+1$ have colored. After that Alice will follow the strategy of case 1 and case 2 in her next moves to win the game.

Case 4**:** Bob plays color 2 in the vertex a_1 in his first move. Alice then plays color 1 in c_1. Now, irrespective of Bob's next move. Alice plays color 3 in d_1. In this way, Alice can succeed to assign color to all the vertices of degree greater than $n+1$. Now Bob will in b_i or c_j or d_k to create threat in a_l. Alice will respond by following the strategies of case 1 and case 2 and she wins the game. $\square$

3. Exact values of $\chi_g(P_m \circ K_1)$ and $\chi_g(C_m \circ P_n)$

Let G and H be two graphs of order n_1 and n_2, respectively. The *corona product* $G \circ H$ is defined as the graph obtained from G and H by taking one copy of G and n_1 copies of H and joining by an edge each vertex from the $i^{th}-$copy of H with the $i^{th}-$vertex of G. We will denote by $V = \{v_1, v_2, \ldots, v_n\}$ the set of vertices of G and by $H_i = (V_i, E_i)$ the copy of H such that $v_i \sim v$ for every $v \in V_i$. It is important to note that the corona product operation is not commutative. In this section we prove the exact values of $\chi_g(P_m \circ K_1)$ and $\chi_g(C_m \circ P_n)$.

Theorem 3.1. *The game chromatic number of* $P_m \circ K_1 = 3$*, where* $m \geq 2$.

Proof. It is obvious that $\chi_g(P_m \circ K_1) \geq 3$. In order to show that $\chi_g(P_m \circ K_1) \leq 3$, Alice winning strategy is given.
Alice's winning strategy with 3 **colors:** Alice starts the game by playing color 1 in the vertex w_1. Suppose, in his first move, Bob plays one of the available color in vertex v_i, then Alice responds by playing the same color in w_{i+1} or w_{i-1} . In this way she can block the threat to the vertex v_i, because it has two same colors in its neighbor. Similarly if Bob plays some color in vertex w_i, Alice responds by playing the same color in v_{i+1} or v_{i-1}. If during the game the following two cases arises, then Alice will play according to the following strategy.

Case 1: Suppose the vertex v_j is already colored and Bob plays a color in the vertex w_i. In this situation, If d(v_i, v_m) is odd then Alice assigns the same color in v_{i-1} (if $i > j$) or v_{i+1} (if $i < j$), otherwise she plays any available color in the vertex v_i.

Case 2: If there are two vertices say v_m and v_n that are already colored and Bob plays a color 1 in the vertex w_i where $m < i < n$. In response, Alice plays the same color 1 in v_{i-1}, (if d(v_i, v_m) is even) or in v_{i+1} (if d(v_i, v_n) is even). In case when v_i is at odd distance to both v_m and v_n, she will play any available color in the vertex v_i.
Similarly if Bob plays some color in v_i then Alice plays the same color in w_{i-1} or w_{i+1} according to the above two cases.
If during the game, the following situations arise then Alice modifies her strategy according to the situation.

Situation 1: A part of graph $P_m \circ K_1$ is shown in Figure 5.
If Bob plays the color 3 on w_{i+2}. Suppose, Bob already assigned the color 3 in the vertex v_{i+4}. Then

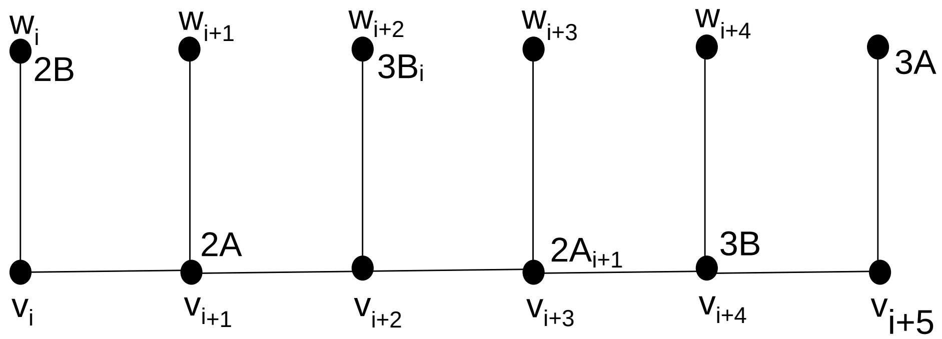

Figure 5. $P_3 \circ K_1$

according to her strategy Alice can not assign the same color in the vertex v_{i+3}. In this case Alice plays color 2 in v_{i+3}, because this is the only way she can block the threat to the vertex v_{i+2}.

Situation 2: If Bob plays a color (say 3) in the vertex w_{i+2} and the vertices v_{i+4} and v_{i-1} are already colored with colors 3 and 2 respectively. In this situation Alice plays the same color as of the vertex v_{i-1} in w_i.

□

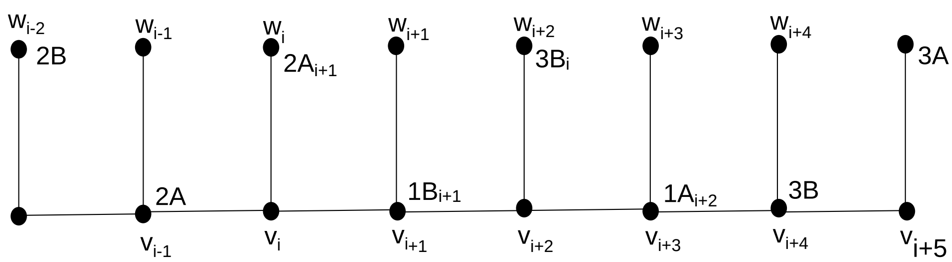

Figure 6. $P_3 \circ K_1$

Theorem 3.2. *For any integer $n \geq 2$ and $m \geq 3$ we have that $\chi_g(C_m \circ P_n) = 4$.*

Proof. The graph $C_8 \circ P_n$ is shown in the Figure 7.
Bob's winning strategy for 3 colors:

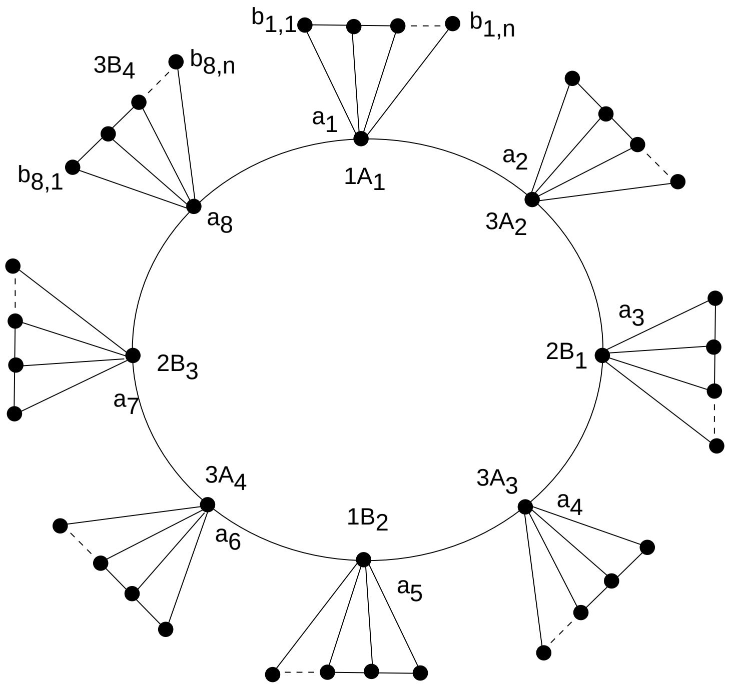

Figure 7. $C_8 \circ P_n$

Case 1: Alice makes her first move by playing color 1 in any of the vertex from the inner cycle, say a_1. In response Bob plays the color 2 in the vertex a_3. In her second move Alice will play color 3 in the vertex a_2 because this is the only way she can block the threat to this vertex. In his next moves Bob plays suitable color in the alternative vertices $a_5, a_7, a_9, \ldots$, respectively, forcing Alice to play suitable color in the vertices $a_4, a_6, a_8, \ldots$. Continuing in this way, Bob plays his last move according to the following two situations:
i: If the number of vertices in inner cycle are even then Bob plays any suitable color in the vertex a_{n-1}. Alice cannot block the threat to both the vertices a_{n-2} and a_n, and Bob wins.
ii: If the number of vertices in inner cycle are odd then Bob in his second last move plays a color in any of the vertex $b_{i,n-1}$, which is different to that of a_1. In response Alice is forced to play color in the vertex a_{n-1} because this is the only way to block the threat to this vertex. By playing a color different to both a_1 and a_{n-1} in any of the vertex $b_{i,n-1}$, Bob can succeed to assign three distinct colors in the neighbor of the vertex a_n.

Case 2: Alice makes her first move by playing color 1 in any of the outer vertex $b_{i,n}$, say $b_{1,1}$. In response Bob plays color 2 in the vertex $b_{2,1}$. In her second move Alice has to play color 3 in the vertex a_1 because this is the only way she can block the threat to this vertex. In his next moves, Bob plays in the alternative vertices $a_3, a_5, a_7, \ldots$, respectively, forcing Alice to play in the vertices $a_2, a_4, a_6, \ldots$ Continuing in this way, Bob plays his last move according to Case 1.
Hence $\chi_g(C_m \circ P_n) \geq 4$.
Alice's winning strategy with 4 **colors:**
The graph $C_m \circ P_n$ have $m + mn$ vertices. The vertices of degree greater than four are only the vertices of the inner cycle, as shown in the above Figure. Alice, in her strategy assigns color to the vertices $a_1, a_2, ..., a_m$ and consequently blocks the threat to these vertices. Since Alice has to start the game,

suppose she makes her first move by playing color 1 in the vertex a_1. In respond, Bob can play the game in the following two ways.

Case 1: Bob plays in the vertices of inner cycle say a_i, for some $2 \leq i \leq m$. Alice responds by playing any color from the set of available colors in the uncolored vertices of inner cycle.

Case 2: Bob plays in the vertices $b_{i,j}$, then Alice assigns a suitable color in the corresponding a_i.

Bob can only win if he succeeded to assign 4 distinct colors in the neighborhood of any vertex. But whenever he plays in the outer vertices, $b_{i,j}$, Alice assigns a suitable color in the corresponding a_i. In this way Alice can color all the vertices in the inner cycle and she wins the game.
From both these strategies we conclude that $\chi_g(C_m \circ P_n) = 4$. □

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