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Non–existence of some 4–dimensional Griesmer codes over finite fields*

Research Article

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Abstract: We prove the non–existence of $[g_q(4,d),4,d]_q$ codes for $d = 2q^3 - rq^2 - 2q + 1$ for $3 \le r \le (q+1)/2$, $q \ge 5$; $d = 2q^3 - 3q^2 - 3q + 1$ for $q \ge 9$; $d = 2q^3 - 4q^2 - 3q + 1$ for $q \ge 9$; and $d = q^3 - q^2 - rq - 2$ with $r = 4, 5$ or 6 for $q \ge 9$, where $g_q(4,d) = \sum_{i=0}^{3} \lceil d/q^i \rceil$. This yields that $n_q(4,d) = g_q(4,d) + 1$ for $2q^3 - 3q^2 - 3q + 1 \le d \le 2q^3 - 3q^2$, $2q^3 - 5q^2 - 2q + 1 \le d \le 2q^3 - 5q^2$ and $q^3 - q^2 - rq - 2 \le d \le q^3 - q^2 - rq$ with $4 \le r \le 6$ for $q \ge 9$ and that $n_q(4,d) \ge g_q(4,d) + 1$ for $2q^3 - rq^2 - 2q + 1 \le d \le 2q^3 - rq^2 - q$ for $3 \le r \le (q+1)/2$, $q \ge 5$ and $2q^3 - 4q^2 - 3q + 1 \le d \le 2q^3 - 4q^2 - 2q$ for $q \ge 9$, where $n_q(4,d)$ denotes the minimum length n for which an $[n,4,d]_q$ code exists.

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1. Introduction

An $[n,k,d]_q$ code $\mathcal{C}$ is a linear code of length n, dimension k and minimum Hamming weight d over $\mathbb{F}_q$, the field of q elements. A fundamental problem in coding theory is to find $n_q(k,d)$, the minimum length n for which an $[n,k,d]_q$ code exists. The Griesmer bound gives a lower bound on $n_q(k,d)$ as

$$n_q(k,d) \ge g_q(k,d) = \sum_{i=0}^{k-1} \lceil d/q^i \rceil ,$$

where $\lceil x \rceil$ denotes the smallest integer $\ge x$. An $[n,k,d]_q$ code is called *Griesmer* if $n = g_q(k,d)$. The values of $n_q(k,d)$ are determined for all d only for some small values of q and k, see [22]. For $k = 4$,

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the exact value of $n_q(4,d)$ is known for all d only for $q = 2,3,4$. Recently, one of the open cases for $(q,k) = (5,4)$ was solved in [16]. For general q, see [18] and [11] for known results on $n_q(4,d)$. We have recently proved the following.

Theorem 1.1 ([12]). *There exists no $[g_q(4,d),4,d]_q$ code for*

(1) $d = q^3/2 - q^2 - 2q + 1$ *for* $q = 2^h$, $h \geq 3$,

(2) $d = 2q^3 - 3q^2 - 2q + 1$ *for* $q \geq 5$,

(3) $d = 2q^3 - rq^2 - q + 1$ *for* $3 \leq r \leq q - q/p$, $q = p^h$ *with* p *prime.*

As a continuation on the non–existence of Griesmer codes for $k = 4$, we prove the following four theorems.

Theorem 1.2. *There exists no $[g_q(4,d),4,d]_q$ code for $d = 2q^3 - rq^2 - 2q + 1$ for $3 \leq r \leq (q+1)/2$, $q \geq 5$.*

Theorem 1.3. *There exists no $[g_q(4,d),4,d]_q$ code for $d = 2q^3 - 3q^2 - 3q + 1$ for $q \geq 9$.*

Theorem 1.4. *There exists no $[g_q(4,d),4,d]_q$ code for $d = 2q^3 - 4q^2 - 3q + 1$ for $q \geq 9$.*

Theorem 1.5. *There exists no $[g_q(4,d),4,d]_q$ code for $d = q^3 - q^2 - rq - 2$ with $4 \leq r \leq 6$ for $q \geq 9$.*

Theorem 1.2 is a generalization of the non–existence of Griesmer $[209,4,166]_5$ codes. We note that the existence of a $[g_q(4,d),4,d]_q$ code for $d = 2q^3 - 3q^2 - 3q + 1$ is known for $q = 4$ but unknown and still open for $q = 5,7,8$. The existence of a $[g_q(4,d),4,d]_q$ code for $d = 2q^3 - 4q^2 - 3q + 1$ is known for $q = 5$ but unknown and still open for $q = 7,8$. For the non–existence of $[g_q(4,d),4,d]_q$ codes for $q^3 - q^2 - 4q + 1 \leq d \leq q^3 - q^2 - q$, see [23]. The non–existence of $[g_q(4,d),4,d]_q$ codes for $d = q^3 - q^2 - rq - 2$ is known for $(q,r) = (8,4)$ but unknown and still open for $(q,r) = (8,5)$ and $(q,r) = (8,6)$.

While the existence of a $[g_q(4,d)+1,4,d]_q$ code for $d = 2q^3 - rq^2 - q$ with $4 \leq r \leq q-1$ and for $d = 2q^3 - 4q^2 - 2q$ is unknown in general, such a code exists for $d = 2q^3 - 5q^2 - sq$ with $0 \leq s \leq q-4$, $q \geq 7$ [21]. The existence of a $[g_q(4,d)+1,4,d]_q$ code for $d = 2q^3 - 3q^2 - 2q$ and for $d = q^3 - q^2 - rq$ with $1 \leq r \leq q-1$ follows from the recent result from [10]. It is also known that $n_q(4,d) = g_q(4,d)$ for $d \geq 2q^3 - 3q^2 + 1$ for all q and that $n_q(4,d) = g_q(4,d) + 1$ for $2q^3 - 3q^2 - 2q + 1 \leq d \leq 2q^3 - 3q^2$ for $q \geq 5$ [12]. Since the existence of an $[n,k,d]_q$ code implies the existence of an $[n-1,k,d-1]_q$ code by puncturing, we get the following results from Theorems 1.2-1.5.

Corollary 1.6. *$n_q(4,d) = g_q(4,d) + 1$ for*

(1) $2q^3 - 3q^2 - 3q + 1 \leq d \leq 2q^3 - 3q^2$ *for* $q \geq 9$,

(2) $2q^3 - 5q^2 - 2q + 1 \leq d \leq 2q^3 - 5q^2$ *for* $q \geq 9$,

(3) $q^3 - q^2 - rq - 2 \leq d \leq q^3 - q^2 - rq$ *with* $4 \leq r \leq 6$ *for* $q \geq 9$.

Corollary 1.7. *$n_q(4,d) \geq g_q(4,d) + 1$ for*

(1) $2q^3 - rq^2 - 2q + 1 \leq d \leq 2q^3 - rq^2 - q$ *for* $4 \leq r \leq (q+1)/2$, $q \geq 7$,

(2) $2q^3 - 4q^2 - 3q + 1 \leq d \leq 2q^3 - 4q^2 - 2q$ *for* $q \geq 9$.

The remainder of the paper is organized as follows. In Section 2, we give the geometric preliminaries and some results on linear codes of dimension 3. We prove Theorems 1.2, 1.3 and 1.5 in Sections 3, 4 and 5, respectively. The proof of Theorem 1.4 is similar to that of Theorem 1.3 and therefore skipped. We give some remarks in Section 6 as Conclusion.

2. Preliminaries

In this section, we give the geometric method and preliminary results to prove the non–existence of some Griesmer codes. We denote by $\mathrm{PG}(r,q)$ the projective geometry of dimension r over $\mathbb{F}_q$. The 0-flats, 1-flats, 2-flats, $(r-2)$-flats and $(r-1)$-flats in $\mathrm{PG}(r,q)$ are called *points, lines, planes, secundums* and *hyperplanes*, respectively.

Let $\mathcal{C}$ be an $[n,k,d]_q$ code having no coordinate which is identically zero. The columns of a generator matrix of $\mathcal{C}$ can be considered as a multiset of n points in $\Sigma = \mathrm{PG}(k-1,q)$, denoted by $\mathcal{M}_{\mathcal{C}}$. An *$i$-point* is a point of Σ which has multiplicity i in $\mathcal{M}_{\mathcal{C}}$. Denote by γ_0 the maximum multiplicity of a point from Σ in $\mathcal{M}_{\mathcal{C}}$ and let C_i be the set of i-points in Σ, $0 \le i \le \gamma_0$. For any subset S of Σ, *the multiplicity of S with respect to $\mathcal{M}_{\mathcal{C}}$*, denoted by $m_{\mathcal{C}}(S)$, is defined as $m_{\mathcal{C}}(S) = \sum_{i=1}^{\gamma_0} i \cdot |S \cap C_i|$, where $|T|$ denotes the number of elements in a set T. A line l with $t = m_{\mathcal{C}}(l)$ is called a *t-line*. A *t-plane* and so on are defined similarly. Then we obtain the partition $\Sigma = \bigcup_{i=0}^{\gamma_0} C_i$ such that $n = m_{\mathcal{C}}(\Sigma)$ and

$$n - d = \max\{m_{\mathcal{C}}(\pi) \mid \pi \in \mathcal{F}_{k-2}\},$$

where $\mathcal{F}_j$ denotes the set of j-flats of Σ. Conversely, such a partition $\Sigma = \bigcup_{i=0}^{\gamma_0} C_i$ as above gives an $[n,k,d]_q$ code in the natural manner. For an m-flat Π in Σ, we define

$$\gamma_j(\Pi) = \max\{m_{\mathcal{C}}(\Delta) \mid \Delta \subset \Pi,\ \Delta \in \mathcal{F}_j\} \quad \text{for} \quad 0 \le j \le k-2.$$

We denote simply by γ_j instead of $\gamma_j(\Sigma)$. Then $\gamma_{k-2} = n-d$, $\gamma_{k-1} = n$. For a Griesmer $[n,k,d]_q$ code, it is known (see [19]) that

$$\gamma_j = \sum_{u=0}^{j} \left\lceil \frac{d}{q^{k-1-u}} \right\rceil \quad \text{for} \quad 0 \le j \le k-1. \tag{1}$$

So, every Griesmer $[n,k,d]_q$ code is projective if $d \le q^{k-1}$. We denote by λ_s the number of s-points in Σ. Note that we have

$$\lambda_2 = \lambda_0 + n - \theta_{k-1} \tag{2}$$

when $\gamma_0 = 2$. Denote by a_i the number of i-hyperplanes in Σ. The list of a_i's is called the *spectrum* of $\mathcal{C}$. We usually use τ_j's for the spectrum of a hyperplane of Σ to distinguish from the spectrum of $\mathcal{C}$. Let θ_j be the number of points in a j-flat, i.e., $\theta_j = (q^{j+1}-1)/(q-1)$. Simple counting arguments yield the following.

Lemma 2.1 ([15]). (1) $\displaystyle\sum_{i=0}^{n-d} a_i = \theta_{k-1}$. (2) $\displaystyle\sum_{i=1}^{n-d} i a_i = n\theta_{k-2}$.

(3) $\displaystyle\sum_{i=2}^{n-d} i(i-1)a_i = n(n-1)\theta_{k-3} + q^{k-2}\sum_{s=2}^{\gamma_0} s(s-1)\lambda_s$.

When $\gamma_0 \le 2$, the above three equalities yield the following:

$$\sum_{i=0}^{n-d-2} \binom{n-d-i}{2} a_i = \binom{n-d}{2}\theta_{k-1} - n(n-d-1)\theta_{k-2} + \binom{n}{2}\theta_{k-3} + q^{k-2}\lambda_2. \tag{3}$$

If $a_i = 0$ for all $i < n-d$, then every point in Σ is an s-point for some integer s. This fact is known as follows.

Lemma 2.2 ([2]). *Any linear code over a finite field with constant Hamming weight is a replication of simplex (i.e., dual Hamming) codes.*

Lemma 2.3 ([27])**.** *Let Π be an w-hyperplane through a t-secundum δ. Then*

(1) $t \le \gamma_{k-2} - (n-w)/q = (w + q\gamma_{k-2} - n)/q$.

(2) $a_w = 0$ *if a* $[w, k-1, d_0]_q$ *code with* $d_0 \ge w - \left\lfloor \frac{w+q\gamma_{k-2}-n}{q} \right\rfloor$ *does not exist, where* $\lfloor x \rfloor$ *denotes the largest integer less than or equal to* x.

(3) $\gamma_{k-3}(\Pi) = \left\lfloor \frac{w+q\gamma_{k-2}-n}{q} \right\rfloor$ *if a* $[w, k-1, d_1]_q$ *code with* $d_1 \ge w - \left\lfloor \frac{w+q\gamma_{k-2}-n}{q} \right\rfloor + 1$ *does not exist.*

(4) *Let* c_j *be the number of* j*-hyperplanes through* δ *other than* Π*. Then* $\sum_j c_j = q$ *and*

$$\sum_j (\gamma_{k-2} - j)c_j = w + q\gamma_{k-2} - n - qt. \tag{4}$$

(5) *For a* γ_{k-2}*-hyperplane* Π_0 *with spectrum* $(\tau_0, \cdots, \tau_{\gamma_{k-3}})$*,* $\tau_t > 0$ *holds if* $w + q\gamma_{k-2} - n - qt < q$.

The next two lemmas are needed to prove Theorems 1.3 and 1.4.

Lemma 2.4 ([11])**.** *The spectrum of a* $[2q^2 - 2q - 4, 3, 2q^2 - 4q - 2]_q$ *code with* $q \ge 8$ *is one of the followings:*

(a) $(a_{q-4}, a_{q-2}, a_{2q-3}, a_{2q-2}) = (1, 3, 2q, q^2 - q - 3)$,

(b) $(a_{q-3}, a_{q-2}, a_{2q-4}, a_{2q-3}, a_{2q-2}) = (2, 2, 1, 2q - 2, q^2 - q - 2)$,

(c) $(a_{q-3}, a_{q-2}, a_{2q-4}, a_{2q-3}, a_{2q-2}) = (1, 3, 1, 2q - 1, q^2 - q - 3)$,

(d) $(a_{q-2}, a_{2q-4}, a_{2q-3}, a_{2q-2}) = (4, 1, 2q, q^2 - q - 4)$ *or*

(e) $(a_{q-2}, a_{2q-4}, a_{2q-2}) = (4, q + 1, q^2 - 4)$.

Lemma 2.5 ([11])**.** *The spectrum of a* $[2q^2 - q - 3, 3, 2q^2 - 3q - 2]_q$ *code with* $q \ge 7$ *is one of the followings:*

(a) $(a_{q-3}, a_{q-1}, a_{2q-2}, a_{2q-1}) = (1, 2, 2q, q^2 - q - 2)$,

(b) $(a_{q-2}, a_{q-1}, a_{2q-3}, a_{2q-2}, a_{2q-1}) = (2, 1, 1, 2q - 2, q^2 - q - 1)$,

(c) $(a_{q-2}, a_{q-1}, a_{2q-3}, a_{2q-2}, a_{2q-1}) = (1, 2, 1, 2q - 1, q^2 - q - 2)$,

(d) $(a_{q-1}, a_{2q-3}, a_{2q-2}, a_{2q-1}) = (3, 1, 2q, q^2 - q - 3)$ *or*

(e) $(a_{q-1}, a_{2q-3}, a_{2q-1}) = (3, q + 1, q^2 - 3)$.

An n-set K in $\mathrm{PG}(2, q)$ is an (n, r)*-arc* if every line meets K in at most r points and if some line meets K in exactly r points. Let $m_r(2, q)$ denote the largest value of n for which an (n, r)-arc exists in $\mathrm{PG}(2, q)$. See Table 1 for the known values and bounds on $m_r(2, q)$ for $3 \le q \le 13$ [1]. An $(n, 2)$-arc is simply called an n-arc in $\mathrm{PG}(2, q)$, see [8]. A set $\mathcal{L}$ of s lines in $\Sigma = \mathrm{PG}(2, q)$ is called an *s-arc of lines* in Σ if $\mathcal{L}$ forms an s-arc in the dual space Σ^* of Σ, that is, no three lines of $\mathcal{L}$ are concurrent.

Lemma 2.6 ([9])**.** (1) $m_r(2, q) \le (r - 1)q + r$.

(2) $m_r(2, q) \le (r - 1)q + r - 3$ *for* $4 \le r < q$ *with* $r \not| q$.

(3) $m_r(2, q) \le (r - 1)q + r - 4$ *for* $9 \le r < q$ *with* $r \not| q$.

(4) $m_{q-2}(2, q) = q^2 - 2q - 3\sqrt{q} - 2$ *for odd square* $q > 11^2$.

(5) $m_{q-2}(2, q) \le q^2 - 2q - p^e\lceil \frac{p^{e+1}+1}{p^e+1} \rceil - 2$ *for* $q = p^{2e+1} > 17$.

Table 1. The known values and bounds on $m_r(2,q)$.

q / r	3	4	5	7	8	9	11	13
2	4	6	6	8	10	10	12	14
3		9	11	15	15	17	21	23
4			16	22	28	28	32	38–40
5				29	33	37	43–45	49–53
6				36	42	48	56	64–66
7					49	55	67	79
8						65	78	92
9							89–90	105
10							100–102	118–119
11								132–133
12								145–147

(6) $m_{q-2}(2,q) \le q^2-2q-2\sqrt{q}-2$ *for* $q=2^{2e}>4$ *or* $q \in \{5^2,7^2,9^2,11^2\}$.

(7) $m_{q-1}(2,q) = q^2-q-2\sqrt{q}-1$ *for square* $q>4$.

(8) $m_{q-1}(2,q) \le q^2-q-p^e\lceil \frac{p^{e+1}+1}{p^e+1} \rceil -1$ *for* $q=p^{2e+1}>19$.

Lemma 2.7 ([12]). *Let $\mathcal{C}$ be a $[g_q(3,d),3,d]_q$ code for $d=2q^2-rq$, $3 \le r \le q-q/p$, $q=p^h$ with p prime. Then,*

(1) *the multiset $\mathcal{M}_\mathcal{C}$ consists of two copies of the plane with an r-arc of lines deleted,*

(2) *$\mathcal{C}$ has spectrum $(a_{q-r+2}, a_{2q-r+2}) = (r, \theta_2 - r)$.*

Lemma 2.8 ([25]). *Let $\mathcal{C}$ be an $[n,k,d]_q$ code and let $\cup_{i=0}^{\gamma_0} C_i$ be the partition of $\Sigma = \mathrm{PG}(k-1,q)$ obtained from $\mathcal{C}$. If $\cup_{i\ge 1} C_i$ contains a t-flat Δ and if $d > q^t$, then there exists an $[n-\theta_t, k, d']_q$ code $\mathcal{C}'$ with $d' \ge d - q^t$.*

The punctured code $\mathcal{C}'$ in Lemma 2.8 can be constructed from $\mathcal{C}$ by removing the t-flat Δ from the multiset $\mathcal{M}_\mathcal{C}$. The method to construct new codes from a given $[n,k,d]_q$ code by deleting the coordinates corresponding to some geometric object in $\mathrm{PG}(k-1,q)$ is called *geometric puncturing*, see [21].

An $[n,k,d]_q$ code with generator matrix G is called *extendable* if there exists a vector $h \in \mathbb{F}_q^k$ such that the extended matrix $[G, h^{\mathrm{T}}]$ generates an $[n+1,k,d+1]_q$ code. The following theorems will be applied to prove that a $[g_q(3,d),3,d=2q^2-rq-1]_q$ code is extendable in Lemma 2.12.

Theorem 2.9 ([6, 7]). *Let $\mathcal{C}$ be an $[n,k,d]_q$ code with $d \equiv -1 \pmod q$, $k \ge 3$. Then $\mathcal{C}$ is extendable if $A_i = 0$ for all $i \not\equiv 0, -1 \pmod q$.*

Theorem 2.10 ([20, 28]). *Let $\mathcal{C}$ be an $[n,k,d]_q$ code with $d \equiv -2 \pmod q$, $k \ge 3$, $q \ge 5$. Then $\mathcal{C}$ is extendable if $A_i = 0$ for all $i \not\equiv 0,-1,-2 \pmod q$.*

Theorem 2.11 ([26]). *Let $\mathcal{C}$ be an $[n,k,d]_q$ code with $\gcd(d,q) = 1$. Then $\mathcal{C}$ is extendable if $\sum_{i \not\equiv n, n-d \pmod q} a_i < q^{k-2}$.*

Lemma 2.12. *The spectrum of a $[2q^2-(r-2)q-(r-1), 3, 2q^2-rq-1]_q$ code for $3 \le r \le \frac{q+1}{2}$, $q=p^h$ with p prime is $(a_{q-r+1}, a_{q-r+2}, a_{2q-r+1}, a_{2q-r+2}) = (1, r-1, q, q^2-r+1)$ or $(a_{q-r+2}, a_{2q-r+1}, a_{2q-r+2}) = (r, q+1, q^2-r)$.*

Proof. Let $\mathcal{C}$ be an $[n = 2q^2 - (r-2)q - (r-1), 3, d = 2q^2 - rq - 1]_q$ code for $3 \le r \le \frac{q+1}{2}$, $q = p^h$ with p prime. Note that $\mathcal{C}$ is extended to the code in Lemma 2.7 if $\mathcal{C}$ is extendable. From (1), we have $\gamma_0 = 2$ and $\gamma_1 = 2q - (r-2)$. Since $(\gamma_1 - \gamma_0)\theta_1 + \gamma_0 - 1 = n$, the lines through a fixed 2-point is one $(\gamma_1 - 1)$-line and q γ_1-lines. Hence $a_i = 0$ for $\theta_1 + 1 \le i \le \gamma_1 - 2$. Let l be a t-line containing a 1-point P. Considering the lines through P, we get $n = 2q^2 - (r-2)q - (r-1) \le (\gamma_1 - 1)q + t$, giving $q - (r-1) \le t$. So, $a_i = 0$ for $1 \le i \le q - r$.

Suppose $a_{\theta_1} > 0$. Then, $\mathcal{C}$ is not extendable by Lemma 2.7. Let l be a θ_1-line. Since $n = (\gamma_1 - 1)q + \theta_1 - r$, the lines ($\ne l$) through a fixed 1-point on l are r $(\gamma_1 - 1)$-lines and $(q-r)$ γ_1-lines if $q \ge 2r$. Then, $\mathcal{C}$ is extendable from Theorem 2.11, a contradiction. When $q = 2r - 1$, the lines ($\ne l$) through a fixed 1-point on l are either "one θ_1-line and $(q-1)$ γ_1-lines" or "r $(\gamma_1 - 1)$-lines and $(q-r)$ γ_1-lines". If a 0-point exists, we have $n \ge (\gamma_1 - 1)\theta_1 = n + q$, a contradiction. Hence $[q^2 - (r-1)q - r, 3, q^2 - rq - 1]_q$ code exists by Lemma 2.8. However, there exists no $(q^2 - (r-1)q - r, q - (r-1))$-arc from Lemma 2.6 (2) when $q = 2r - 1 \ge 7$ and from Table 1 when $(q, r) = (5, 3)$, a contradiction. Thus $a_{\theta_1} = 0$. Next, suppose $a_0 > 0$. Then, $\mathcal{C}$ is not extendable by Lemma 2.7. Let l be a 0-line. Since $n = \gamma_1 q + 0 - (r-1)$ and $\gamma_1 - (r-1) > \theta_1$, the lines ($\ne l$) through a fixed 0-point on l are $(\gamma_1 - 1)$-lines or γ_1-lines. Hence $a_j > 0$ implies $j \in \{0, \gamma_1 - 1, \gamma_1\}$ and $a_0 = 1$. Then, $\mathcal{C}$ is extendable by Theorem 2.11, a contradiction. Hence $a_0 = 0$. Finally, suppose $a_i > 0$ for some $q - r + 3 \le i \le q$. Then, $\mathcal{C}$ is not extendable by Lemma 2.7. Let l be a $(q-e)$-line with $0 \le e \le r - 3$ and let Q be a 0-point on l. If four of the lines through Q have multiplicities at most q, then we have $n \le 4q + (q-3)\gamma_1 = n - 2q + 4(r-2) < n$, a contradiction. So, at most two of the lines ($\ne l$) through Q have no 2-point and

$$\sum_{i \not\equiv n, n-d \pmod q} a_i \le 2(e+1) + 1 \le 2r - 3 < 2r - 1 \le q.$$

Then, applying Theorem 2.11, $\mathcal{C}$ is extendable, a contradiction. Hence $a_i = 0$ for all $i \notin \{q - r + 1, q - r + 2, 2q - r + 1, 2q - r + 2\}$. Applying Theorem 2.9, $\mathcal{C}$ is extendable. Hence $\mathcal{C}$ can be obtained from a $[2q^2 - (r-2)q - (r-2), 3, 2q^2 - rq]_q$ code $\mathcal{C}'$ by removing one coordinate. Let R be the point corresponding to the coordinate. There are two possible spectra $(a_{q-r+1}, a_{q-r+2}, a_{2q-r+1}, a_{2q-r+2}) = (1, r-1, q, q^2 - r + 1)$ or $(a_{q-r+2}, a_{2q-r+1}, a_{2q-r+2}) = (r, q+1, q^2 - r)$, according to the cases R is a 1-point or a 2-point, respectively. $\square$

Lemma 2.13. *The spectrum of a $[q^2 - r, 3, q^2 - q - r]_q$ code with $1 \le r \le q - 2$ satisfies $a_i = 0$ for $1 \le i \le q - r - 1$.*

Proof. Let $\mathcal{C}$ be a $[q^2 - r, 3, q^2 - q - r]_q$ code, which is Griesmer. From (1), we have $\gamma_0 = 1$ and $\gamma_1 = q$. Let l be an i-line with $i > 0$ containing a 1-point P. Counting the 1-points on the lines through P, we get $n = q^2 - r \le (q-1)q + i$, whence $q - r \le i$. $\square$

3. Proof of Theorem 1.2

We assume $q \ge 7$ since the theorem is already known for $(r, q) = (3, 5)$ [14]. We first prove the non–existence of $[g_q(4, d), 4, d]_q$ code for $d = 2q^3 - rq^2 - 2q + 2$.

Lemma 3.1. *There exists no $[n = 2\theta_3 - r\theta_2 - 2\theta_1 + 3, 4, d = 2q^3 - rq^2 - 2q + 2]_q$ code for $3 \le r \le \frac{q+1}{2}$, $q = p^h \ge 7$ with p prime.*

Proof. Let $\mathcal{C}$ be a putative $[n = 2q^3 - (r-2)q^2 - rq - (r-3), 4, d = 2q^3 - rq^2 - 2q + 2]_q$ code with $3 \le r \le (q+1)/2$, $q \ge 5$. Note that $n = g_q(4, d)$ and hence $\gamma_0 = 2$, $\gamma_1 = 2\theta_1 - r$, $\gamma_2 = n - d = 2\theta_2 - r\theta_1 - 1$ from (1). Let Δ be a γ_2-plane. Since $\gamma_2 = (\gamma_1 - 2)(q+1) + 2 - 1$ and $n = (\gamma_1 - 2)\theta_2 + 2 - (2q - 1)$, every line on Δ through a 2-point is a γ_1-line or a $(\gamma_1 - 1)$-line, and any i-plane through a 2-point satisfies $(\gamma_1 - 2)(q+1) + 2 - (2q-1) = \gamma_2 - 2(q-1) \le i \le \gamma_2$. By Lemma 2.3 (1), any t-line in an i-plane satisfies

$$t \le \frac{i + r - 3}{q} + 1. \tag{5}$$

The spectrum of Δ is either (A) $(\tau_{q-r+1}, \tau_{q-r+2}, \tau_{2q-r+1}, \tau_{2q-r+2}) = (1, r-1, q, q^2-r+1)$ or (B) $(\tau_{q-r+2}, \tau_{2q-r+1}, \tau_{2q-r+2}) = (r, q+1, q^2-r)$ by Lemma 2.12.

Let δ be an i-plane. It follows from (5) and Δ's possible spectra that $q-r+1 \le \frac{i+q+r-3}{q}$, i.e., $q^2-rq-(r-3) \le i$. Assume $i \le \theta_2$. Since δ has no 2-point, $\delta \cap \Delta$ is a $(q-r+1)$-line or a $(q-r+2)$-line. So, $i \le \theta_2 - r + 1$. Now, let $i = q^2 - uq - (r-3) + s$ with $0 \le u \le r-2$, $0 \le s \le q-1$. From (5), we have $t \le q-u+1$. If $t = q-u+1$, then $i+q+r-3-qt = s \le q-1$, and the γ_2-plane Δ contains a t-line by Lemma 2.3 (5), a contradiction. Hence $t \le q-u$. Considering the lines in δ through a fixed 1-point of $\delta \cap \Delta$, $i \le (q-u-1)q + (q-r+2) = q^2 - uq - (r-2) < i$, a contradiction. Thus, $a_i = 0$ for $q^2-(r-2)q-(r-3) \le i \le \theta_2$, and $a_i > 0$ implies

$$q^2-rq-(r-3) \le i \le q^2-(r-2)q-(r-2) \ \text{ or } \ \gamma_2 - 2q + 2 \le i \le \gamma_2.$$

From (3), we get

$$\sum_i \binom{\gamma_2 - i}{2} a_i = q^2\lambda_2 - q^5 + \frac{3r-2}{2}q^4 - \frac{r^2-3r-4}{2}q^3 - \frac{r^2+6}{2}q^2 - 2q + 3. \tag{6}$$

For any w-plane through a t-line, (4) gives $\sum_j c_j = q$ and

$$\sum_j (2q^2 - (r-2)q - (r-1) - j)c_j = w + q + r - 3 - qt. \tag{7}$$

Suppose $a_i > 0$ for $i = q^2 - rq - (r-3) + e$ with $0 \le e \le q-1$. Since $\delta \cap \Delta$ is a $(q-r+1)$-line by (5), Δ has spectrum (A). If $a_i > 0$, the RHS of (7) is $q^2-(r-1)q+e-qt \le q^2-(r-2)q-1$. Since the coefficient of $c_{q^2-(r-2)q-(r-2)}$ in (7) is $q^2-1 > q^2-(r-2)q-1$, we get $a_i = 1$ and $a_j = 0$ for $q^2-rq-(r-3) \le j \le q^2-(r-2)q-(r-2)$ with $j \ne i$. Setting $w = n-d$, the maximum possible contributions of c_j's in (7) to the LHS of (6) on Δ are $(c_{q^2-rq-(r-3)+e}, c_{n-d-e}, c_{n-d}) = (1, 1, q-2)$ for $t = q-r+1$; $(c_{\gamma_2-2(q-1)}, c_{\gamma_2-(q-1)}, c_{n-d}) = (\frac{q+1}{2}, 1, \frac{q-3}{2})$ if q is odd and $(c_{\gamma_2-2(q-1)}, c_{n-d}) = (\frac{q}{2}+1, \frac{q}{2}-1)$ if q is even for $t = q-r+2$; $(c_{\gamma_2-2(q-1)}, c_{n-d}) = (1, q-1)$ for $t = 2q-r+1$; $(c_{\gamma_2-(q-2)}, c_{n-d}) = (1, q-1)$ for $t = 2q-r+2$. Hence, when q is odd, we get

$$\begin{aligned}
(\text{LHS of (6)}) \le & \left(\binom{q^2+2q-2-e}{2} + \binom{e}{2}\right)\tau_{q-r+1} + \left(\frac{q+1}{2}\binom{2(q-1)}{2} + \binom{q-1}{2}\right)\tau_{q-r+2} + \binom{2(q-1)}{2}\tau_{2q-r+1} + \binom{q-2}{2}\tau_{2q-r+2} \\
\le & \binom{q^2+2q-2}{2}\tau_{q-r+1} + \left(\frac{q+1}{2}\binom{2(q-1)}{2} + \binom{q-1}{2}\right)\tau_{q-r+2} \\
& + \binom{2(q-1)}{2}\tau_{2q-r+1} + \binom{q-2}{2}\tau_{2q-r+2},
\end{aligned}$$

giving

$$\lambda_2 < q^3 - \frac{3r-4}{2}q^2 + \frac{r^2-r-3}{2}q + \frac{r^2-3r+4}{2}.$$

When q is even, we can similarly obtain

$$\lambda_2 < q^3 - \frac{3r-4}{2}q^2 + \frac{r^2-r-3}{2}q + \frac{r^2-2r+3}{2}.$$

On the other hand, since $\lambda_0 \ge |\delta \cap C_0|$, we have

$$\lambda_2 = n - \theta_3 + \lambda_0 \ge n - \theta_3 + \theta_2 - (q^2-rq-(r-3)+e) \ge q^3 - (r-1)q^2 - q + 1$$

giving a contradiction for $q \geq 2r-1$ with $q \geq 7$ and $r \geq 3$. Thus, $a_i = 0$ for $q^2 - rq - (r-3) \leq i \leq q^2 - (r-1)q - (r-2)$.

By a similar argument using Lemma 2.3, (6) and (7), we can get $a_i = 0$ for all $q^2 - (r-1)q - (r-3) \leq i \leq q^2 - (r-2)q - (r-2)$. Hence, $a_i > 0$ implies $\gamma_2 - 2q + 2 \leq i \leq \gamma_2$.

Finally, we investigate (6) and (7) with $i = n - d$ again. We only give the proof when Δ has spectrum (A) since one can prove similarly for spectrum (B). Assume q is odd. The maximum possible contributions of c_j's in (7) to the LHS of (6) on Δ are $(c_{\gamma_2-2(q-1)}, c_{n-d-1}, c_{n-d}) = (\frac{q+3}{2}, 1, \frac{q-5}{2})$ for $t = q - r + 1$; $(c_{\gamma_2-2(q-1)}, c_{\gamma_2-(q-1)}, c_{n-d}) = (\frac{q+1}{2}, 1, \frac{q-3}{2})$ for $t = q - r + 2$; $(c_{\gamma_2-2(q-1)}, c_{n-d}) = (1, q-1)$ for $t = 2q - r + 1$; $(c_{\gamma_2-(q-2)}, c_{n-d}) = (1, q-1)$ for $t = 2q - r + 2$. Hence we get

$$\begin{aligned}(\text{LHS of (6)}) \leq \frac{q+3}{2}\binom{2(q-1)}{2}\tau_{q-r+1} + (\frac{q+1}{2}\binom{2(q-1)}{2} + \binom{q-1}{2})\tau_{q-r+2} \\ + \binom{2(q-1)}{2}\tau_{2q-r+1} + \binom{q-2}{2}\tau_{2q-r+2},\end{aligned}$$

giving

$$\lambda_2 < q^3 - \frac{3r-3}{2}q^2 + \frac{r^2-r-5}{2}q + \frac{r^2-3r+6}{2}.$$

On the other hand, we have

$$\lambda_2 = n - \theta_3 + \lambda_0 \geq (2\theta_3 - r\theta_2 - 2\theta_1 + 3) - \theta_3 = q^3 - (r-1)q^2 - (r+1)q - (r-2),$$

giving a contradiction for $q \geq 2r-1$. One can get a contradiction similarly when q is even. This completes the proof. □

In the above proof, we often obtain a contradiction to rule out the existence of some i-plane by eliminating the value of λ_2 using (4), (3) and the possible spectra for a fixed w-plane. We refer to this proof technique as "(λ_2, w)-**ruling out method** ((λ_2, w)-**ROM**)" in what follows.

Proof of Theorem 1.2. Let $\mathcal{C}$ be a putative $[n = 2q^3 - (r-2)q^2 - rq - (r-2), 4, d = 2q^3 - rq^2 - 2q + 1]_q$ code with $3 \leq r \leq (q+1)/2$, $q \geq 5$. By Lemma 1, $\gamma_0 = 2$, $\gamma_1 = 2q - (r-2)$, $\gamma_2 = n - d = 2\theta_2 - r\theta_1 - 1$. By Lemma 2.12, the spectrum of a γ_2-plane Δ is (A) $(\tau_{q-r+1}, \tau_{q-r+2}, \tau_{2q-r+1}, \tau_{2q-r+2}) = (1, r-1, q, q^2-r+1)$ or (B) $(\tau_{q-r+2}, \tau_{2q-r+1}, \tau_{2q-r+2}) = (r, q+1, q^2-r)$. So a j-line on Δ satisfies

$$j \in \{q-r+1, q-r+2, 2q-r+1, 2q-r+2\}. \tag{8}$$

By Lemma 2.3, an i-plane satisfies $i \geq (q-r+1)q - (q+r-2) = q^2 - rq - (r-2)$. Hence $a_i = 0$ for any $i < q^2 - rq - (r-2)$. Assume that an i-plane contains a 2-point. Since $(\gamma_1 - 2)\theta_2 + 2 = n + 2q$, we have

$$i \geq (\gamma_1 - 2)\theta_1 + 2 - 2q = (2q - r)\theta_1 + 2 - 2q = 2q^2 - rq - (r-2) > \theta_2$$

for $q \geq 2r-1$. Hence an i-plane with $i \leq \theta_2 = q^2 + q + 1$ has no 2-point. Thus $a_i = 0$ if $i < q^2 - rq - (r-2)$ or $\theta_2 < i < 2q^2 - rq - (r-2)$.

Let δ be a i-plane, $s = \gamma_1(\delta)$. Then, $\delta \cap \mathcal{C}$ is an (i, s)-arc, corresponding to an $[i, 3, i-s]_q$ code. By Lemma 2.3(1),

$$s \leq \frac{i+r-2}{q} + 1 \tag{9}$$

By Lemma 2.3 (5), δ contains a t-line if

$$i + q + (r-2) - qt < q. \tag{10}$$

(Case 1) Assume $q^2 - rq - (r-2) \leq i < q^2 - (r-1)q - (r-2)$.
We have $s \leq q - (r-1)$ by (9). Since $\delta \cap \Delta$ is a j-plane satisfying (8), we get $s = q - (r-1)$. By Lemma

2.6 (2), $i \le (q-r)q + (q-r+1) - 3 = q^2 - (r-1)q - (r+2)$.

(Case 2) Assume $q^2 - (r-1)q - (r-2) \le i < q^2 - (r-2)q - (r-2)$.
By (9), $s \le q-(r-2)$. It follows from (Case 1) that $s = q-(r-2)$. By Lemma 2.6 (2), we get $i \le q^2 - (r-2)q - (r+1)$.

(Case 3) Assume $q^2 - (r-2)q - (r-2) \le i < q^2 - (r-3)q - (r-2)$.
By (9), $s \le q-(r-3)$. It follows from (Case 2) that $s = q-(r-3)$. Then, by Lemma 2.3 (5), Δ has a $(q-(r-3))$-line, a contradiction. Hence $a_i = 0$.

(Case 4) Assume $q^2 - uq - (r-2) \le i < q^2 - (u-1)q - (r-2)$, $0 \le u \le r-3$.
By (9), $s \le q-u+1$. If $s = q-u+1$, then Δ contains a $(q-u+1)$-line by Lemma 2.3 (5), a contradiction. Hence $s \le q-u$, and $\delta \cap \Delta$ is a $(q-r+1)$-line or a $(q-r+2)$-line. Considering the lines in δ through a fixed 1-point on $\delta \cap \Delta$, we have $i \le (q-u-1)q + q - r + 2 = q^2 - uq - (r-2)$. Hence $i = q^2 - uq - (r-2)$, and $\delta \cap \Delta$ is a $(q-r+2)$-line. Let P be any 1-point in δ. Then, there exists a γ_2-plane through P meeting δ in a $(q-r+2)$-line. Otherwise, one can get an $[n+1, 4, d+1]_q$ code by adding P to the multiset for $\mathcal{C}$, which contradicts Theorem 3.1. Thus, the lines through P in δ are one $(q-r+2)$-line and q $(q-u)$-lines, and other possible lines in δ are 0-lines. Let $\mathcal{C}_i$ be the code corresponding to δ. Then $\mathcal{C}_i$ is an $[i, 3, i-(q-u) = q^2 - (u+1)q - (r-2-u)]_q$ code with spectrum

$$(\mu_0, \mu_{q-r+2}, \mu_{q-u}) = \left(q\left(\frac{r-2}{q-u} - \frac{r-u-3}{q-r+2}\right), \frac{i}{q-r+2}, \frac{iq}{q-u} \right), \tag{11}$$

where μ_j is the number of j-lines in δ. Since $(q-u)(q-1) < i$ from the assumption $q \ge 2r-1$, we get $\mu_0 = 0$ or 1. Take a 0-point Q not on a 0-line in δ. It follows from $(q-u)q + q - r + 2 = i + q$ that $r-2-u$ divides q. So,

$$r - 2 - u = p^m \tag{12}$$

for some integer $m \ge 0$. If $m = 0$, then $u = r-3$ and $i = q^2 - (r-3)q - (r-2)$. Since $gcd(q-r+2, q-r+3) = 1$, $(q-r+2)|i$ implies $(q-r+3)|q$. From (11), $\mu_0 = \dfrac{q}{q-r+3}(r-2) \ne 0$. Hence $\mu_0 = 1$, $r = 3$, $u = 0$, and $i = q^2 - 1$. Assume $m > 0$. Then $h \ge 2$ and $1 \le m \le h-1$, for $r - 2 \le (q-3)/2$.
Suppose $\mu_0 = 0$. From (11) and (12), we have

$$(u+1)q = p^{2m} + u(r-1). \tag{13}$$

If $h \le 2m$, then, from (12) and (13), q divides either u or $r-1$, a contradiction. Hence $2m \le h-1$. From (13), we get

$$q = \frac{p^{2m}}{u+1} + \frac{u(r-1)}{u+1} < p^{h-1} + r - 1 \le \frac{q}{2} + \frac{q-1}{2} < q,$$

a contradiction. Hence $\mu_0 = 1$. Since $(q-u)(q-1) + q - r + 2 = i + u$, the number of $(q-r+2)$-lines through a fixed 0-point on the 0-line in δ is $1 + u/(r-2-u)$. So, p^m divides u and $r-2$ also from (12). From $\mu_0 = 1$ and (11), we have

$$\frac{(q-u)(q-r+2)}{q} = q(u+1) - u(r-1) - p^{2m}. \tag{14}$$

Suppose $h \le 2m$. Then, from (14), we obtain

$$(r-2)((1-u)q - u) \equiv 0 \pmod{q^2}. \tag{15}$$

Since q divides $u(r-2)$, (15) yields $(r-2)(q-u) \equiv 0 \pmod q$, a contradiction. Hence $2m \le h-1$. If $u = 0$, then (14) gives $r - 2 = p^{2m}$, which contradicts (12). Thus, $u > 0$. Then, from (14), we have $q(u+1) - u(r-1) - p^{2m} < q - r + 2$, giving $qu < u(r-1) - (r-1) + 1 + p^{2m}$, i.e.,

$$q \le \frac{(u-1)(r-1)}{u} + \frac{p^{2m}}{u} < p^m + r - 1 \le \sqrt{\frac{q}{p}} + \frac{q-1}{2} < q,$$

a contradiction. Hence $a_i = 0$ except for the case $(r, u) = (3, 0)$.

(Case 5) Assume $q^2 + q - (r-2) \le i \le \theta_2$.
By (9), $s \le q+2$. If $s = q+2$, then Δ contains a $(q+2)$-line by Lemma 2.3 (5), a contradiction. Hence $s \le q+1$. So, $\delta \cap \Delta$ is a $(q-r+1)$-line or a $(q-r+2)$-line. Considering the lines through a fixed 1-point on $\delta \cap \Delta$, we get $i \le q \cdot q + (q-r+2) = q^2 + q - (r-2)$. Hence $i = q^2 + q - (r-2)$. Since $\theta_2 - (q^2 + q - (r-2)) = r-1$ and $\theta_1 - (r-1) = q - r + 2$, a t-line on δ satisfies $\theta_1 \ge t \ge q - r + 2$. So, $\delta \cap \Delta$ is a $(q-r+2)$-line.
Hence, the spectrum of δ is $(\tau_{q-r+2}, \tau_q, \tau_{q+1}) = (1, (r-1)q, (q-r+2)q)$. Then any point of $\delta \backslash \Delta$ is not contained in a γ_2-plane, and $\mathcal{C}$ is extendable, which contradicts Lemma 2.11. Hence $a_i = 0$.

From the above (Case 1) - (Case 5), $a_i > 0$ implies

$$\begin{aligned} i \in \{&q^2 - rq - (r-2), \cdots, q^2 - (r-1)q - (r+2), q^2 - (r-1)q - (r-2), \cdots, \\ &q^2 - (r-2)q - (r+1), 2q^2 - rq - (r-2), \cdots, 2q^2 - (r-2)q - (r-1)\}, \end{aligned}$$

or $i = q^2 - 1$ when $r = 3$. By (3), we get

$$\sum_j \binom{\gamma_2 - j}{2} = q^2\lambda_2 - q^5 + \frac{3r-2}{2}q^4 - \frac{r^2 - 3r - 4}{2}q^3 - \frac{r^2+2}{2}q^2 - 2q + 1. \tag{16}$$

Note that the LHS of (16) contains the term $\binom{q^2-q-1}{2}a_{q^2-1}$ only for $r = 3$. For any w-plane through a t-line, (4) gives $\sum_j c_j = q$ and

$$\sum_j (2q^2 - (r-2)q - (r-1) - j)c_j = w + q + (r-2) - qt. \tag{17}$$

Now, we rule out the possible i-planes for $q^2 - rq - (r-2) \le i \le q^2 - (r-1)q - r - 2$ by (λ_2, γ_2)-ROM. Suppose $a_i > 0$ for $i = q^2 - rq - (r-2) + e$ with $0 \le e \le q - 4$ and let δ be an i-plane. We may assume that Δ has spectrum (A) since $\delta \cap \Delta$ is a $(q-r+1)$-line. It follows from (4) that $a_i = 1$ and that $a_j = 0$ for $q^2 - rq - (r-2) \le j \le q^2 + q - (r-2)$ with $j \ne i$. Assume q is odd. Setting $w = n - d$, the maximum possible contributions of c_j's in (17) to the LHS of (16) are $(c_{q^2-rq-(r-2)+e}, c_{n-d-e}, c_{n-d}) = (1, 1, q-2)$ for $t = q - r + 1$; $(c_{2q^2-rq-(r-2)}, c_{2q^2-(r-\frac{3}{2})q-(r-\frac{3}{2})}, c_{n-d}) = (\frac{q+1}{2}, 1, \frac{q-3}{2})$ for $t = q - r + 2$; $(c_{2q^2-rq-(r-2)}, c_{n-d}) = (1, q-1)$ for $t = 2q - r + 1$; $(c_{2q^2-(r-1)q-(r-2)}, c_{n-d}) = (1, q-1)$ for $t = 2q - r + 2$. Hence we get

$$\begin{aligned} (\text{LHS of } (16)) \le &\left(\binom{q^2+2q-1-e}{2} + \binom{e}{2}\right)\tau_{q-r+1} + \left(\frac{q+1}{2}\binom{2q-1}{2} + \binom{\frac{q-1}{2}}{2}\right)\tau_{q-r+2} \\ &+ \binom{2q-1}{2}\tau_{2q-r+1} + \binom{q-1}{2}\tau_{2q-r+2} \\ \le &\binom{q^2+2q-1}{2}\tau_{q-r+1} + \left(\frac{q+1}{2}\binom{2q-1}{2} + \binom{\frac{q-1}{2}}{2}\right)\tau_{q-r+2} \\ &+ \binom{2q-1}{2}\tau_{2q-r+1} + \binom{q-1}{2}\tau_{2q-r+2}, \end{aligned}$$

giving

$$\lambda_2 < q^3 + \frac{4-3r}{2}q^2 + \frac{r^2-r-1}{2}q + \frac{4r^2-7r+3}{8}.$$

On the other hand, since $\lambda_0 \ge |\delta \cap C_0| = \theta_2 - i$, we have

$$\lambda_2 = n - \theta_3 + \lambda_0 \ge n - \theta_3 + (\theta_2 - (q^2 - (r-1)q - (r+2))) = q^3 + (1-r)q^2 - q + 4,$$

giving a contradiction. One can get a contradiction similarly when q is even. Hence $a_i = 0$.

One can also rule out possible i-planes for $i = q^2 - (r-1)q - (r-2) + e$ with $0 \le e \le q-3$ by (λ_2, γ_2)-ROM.

Next, we rule out the possible (q^2-1)-plane by (λ_2, q^2-1)-ROM. Suppose $a_{q^2-1} > 0$ for $r = 3$. The spectrum of a (q^2-1)-plane is $(\tau_0, \tau_{q-1}, \tau_q) = (1, q+1, q^2-1)$ since it corresponds to a $[q^2-1, 3, q^2-q-1]_q$ code. From (17) we have $a_{q^2-1} = 1$ and $a_j = 0$ for $q^2-2q-1 \le j \le q^2-q-5$. Then, the maximum possible contributions of c_j's in (17) with $w = q^2-1$ to the LHS of (16) are $(c_i, c_{2q^2-2q-5}, c_{n-d-1}) = (1, 1, q-2)$ for $t = 0$; $(c_{2q^2-3q-1}, c_{n-d-1}, c_{n-d}) = (1, 1, q-2)$ for $t = q-1$; $c_{2q^2-q-3} = q$ for $t = q$. Hence we get

$$(\text{LHS of } (16)) \le \binom{q^2-q-1}{2} + (\binom{q^2-q-1}{2} + \binom{q+3}{2})\tau_0 + \binom{2q-1}{2}\tau_{q-1} + 0 \cdot \tau_q$$

giving $\lambda_2 < q^3 - 5q^2/2 - 2q + 4$. On the other hand, since $\lambda_0 \ge \theta_2 - i$, we have

$$\lambda_2 = n - \theta_3 + \lambda_0 \ge 2q^3 - q^2 - 3q - 1 - \theta_3 + \theta_2 - (q^2-1) = q^3 - 2q^2 - 3q,$$

giving a contradiction. Hence $a_{q^2-1} = 0$.

Finally, we apply (λ_2, γ_2)-ROM for $i = \gamma_2$ to get a contradiction. We only give the proof when Δ has spectrum (A) since one can prove similarly for spectrum (B). Assume q is odd. The maximum possible contributions of c_j's in (17) to the LHS of (16) on Δ are $(c_{2q^2-rq-(r-2)}, c_{2q^2-(r-\frac{1}{2})q-(r-\frac{3}{2})}, c_{n-d}) = (\frac{q+1}{2}, 1, \frac{q-3}{2})$ for $t = q-r+1$; $(c_{2q^2-rq-(r-2)}, c_{2q^2-(r-\frac{3}{2})q-(r-\frac{3}{2})}, c_{n-d}) = (\frac{q+1}{2}, 1, \frac{q-3}{2})$ for $t = q-r+2$; $(c_{2q^2-rq-(r-2)}, c_{n-d}) = (1, q-1)$ for $t = 2q-r+1$; $(c_{2q^2-(r-1)q-(r-2)}, c_{n-d}) = (1, q-1)$ for $t = 2q-r+2$. Hence

$$\begin{aligned}(\text{LHS of } (16)) \le {} & (\frac{q+1}{2}\binom{2q-1}{2} + \binom{\frac{3q-1}{2}}{2})\tau_{q-r+1} + (\frac{q+1}{2}\binom{2q-1}{2} + \binom{\frac{q-1}{2}}{2})\tau_{q-r+2} \\ & + \binom{2q-1}{2}\tau_{2q-r+1} + \binom{q-1}{2}\tau_{2q-r+2},\end{aligned}$$

giving

$$\lambda_2 < q^3 + \frac{3-3r}{2}q^2 + \frac{r^2-r-3}{2}q + \frac{4r^2-7r+4}{4}.$$

On the other hand, it follows from $\lambda_0 \ge \theta_2 - i$ that

$$\lambda_2 = n - \theta_3 + \lambda_0 \ge n - \theta_3 = q^3 - (r-1)q^2 - (r+1)q - (r-1),$$

giving a contradiction. One can get a contradiction similarly when q is even. This completes the proof. □

4. Proof of Theorem 1.3

To prove Theorems 1.3 and 1.4, the possible spectra of some 3-dimensional codes in Table 2 are needed. We omit the proof of Theorem 1.4 as noted in Section 1.

See [13] for the proof of Theorem 1.3 for $q = 9$. Let $\mathcal{C}$ be a putative $[n = 2q^3-q^2-4q-2, 4, d = 2q^3-3q^2-3q+1]_q$ code for $q \ge 11$. It follows from (1) that $\gamma_0 = 2$, $\gamma_1 = 2q-1$, $\gamma_2 = 2q^2-q-3$. The spectrum of a γ_2-plane Δ is one of the followings by Lemma 2.5: (A) $(\tau_{q-3}, \tau_{q-1}, \tau_{2q-2}, \tau_{2q-1}) = (1, 2, 2q, q^2-q-2)$, (B) $(\tau_{q-2}, \tau_{q-1}, \tau_{2q-3}, \tau_{2q-2}, \tau_{2q-1}) = (2, 1, 1, 2q-2, q^2-q-1)$, (C) $(\tau_{q-2}, \tau_{q-1}, \tau_{2q-3}, \tau_{2q-2}, \tau_{2q-1}) = (1, 2, 1, 2q-1, q^2-q-2)$, (D) $(\tau_{q-1}, \tau_{2q-3}, \tau_{2q-2}, \tau_{2q-1}) = (3, 1, 2q, q^2-q-3)$, or (E) $(\tau_{q-1}, \tau_{2q-3}, \tau_{2q-1}) = (3, q+1, q^2-3)$. Hence, a j-line on Δ satisfies

$$j \in \{q-3, q-2, q-1, 2q-3, 2q-2, 2q-1\}. \tag{18}$$

Table 2. The spectra of some $[n, 3, d]_q$ codes for $q \geq 9$ ([5, 8]).

parameters	possible spectra
$[q^2-3, 3, q^2-q-3]_q$, $q \geq 11$	$(a_0, a_{q-3}, a_{q-1}, a_q) = (1, 1, 3q, q^2-2q-1)$
	$(a_0, a_{q-2}, a_{q-1}, a_q) = (1, 3, 3q-3, q^2-2q)$
$[q^2-3, 3, q^2-q-3]_q$, $q = 9$	$(a_0, a_6, a_8, a_9) = (1, 1, 27, 62)$
	$(a_0, a_7, a_8, a_9) = (1, 3, 24, 63)$
	$(a_6, a_9) = (13, 78)$
$[q^2-2, 3, q^2-q-2]_q$	$(a_0, a_{q-2}, a_{q-1}, a_q) = (1, 1, 2q, q^2-q-1)$
$[q^2-1, 3, q^2-q-1]_q$	$(a_0, a_{q-1}, a_q) = (1, q+1, q^2-1)$
$[q^2, 3, q^2-q]_q$	$(a_0, a_q) = (1, q^2+q)$
$[q^2+q-3, 3, q^2-4]_q$	$(a_{q-3}, a_q, a_{q+1}) = (1, 4q, q^2-3q)$
	$(a_{q-2}, a_{q-1}, a_q, a_{q+1}) = (1, 3, 4q-5, q^2-3q+2)$
	$(a_{q-1}, a_q, a_{q+1}) = (6, 4q-8, q^2-3q+3)$
$[q^2+q-2, 3, q^2-3]_q$	$(a_{q-2}, a_q, a_{q+1}) = (1, 3q, q^2-2q)$
	$(a_{q-1}, a_q, a_{q+1}) = (3, 3q-3, q^2-2q+1)$
$[q^2+q-1, 3, q^2-2]_q$	$(a_{q-1}, a_q, a_{q+1}) = (1, 2q, q^2-q)$
$[q^2+q, 3, q^2-1]_q$	$(a_q, a_{q+1}) = (q+1, q^2)$
$[q^2+q+1, 3, q^2]_q$	$a_{q+1} = q^2+q+1$

From Lemma 2.1 (3), we have $\lambda_0(\Delta) = 5, 5, 4, 3, 4$ for the cases A,B,C,D,E, respectively. By Lemma 2.3, an i-plane satisfies $i \geq q(q-3)-(q+2) = q^2-4q-2$. Hence $a_i = 0$ for any $i < q^2-4q-2$. Assume that an i-plane contains a 2-point. Since $(\gamma_1-2)\theta_2+2 = n+3q+1$, we have $i \geq (\gamma_1-2)\theta_1+2-(3q+1) = 2q^2-4q-2$. Let δ be an i-plane, $r = \gamma_1(\delta)$. Then, $\delta \cap \mathcal{C}$ is an (i, r)-arc, corresponding to an $[i, 3, i-r]_q$ code. Lemma 2.3 (1) gives

$$r \leq \frac{i+2}{q} + 1. \tag{19}$$

For any w-plane through a t-line, (4) gives

$$\sum_j (\gamma_2 - j)c_j = w + q + 2 - qt \tag{20}$$

with $\sum_j c_j = q$. The equality (2) yields:

$$\lambda_2 = q^3 - 2q^2 - 5q - 3 + \lambda_0. \tag{21}$$

Assume $q^2-4q-2 \leq i < q^2-3q-2$. From (19), and (18) we have $r = q-3$. Then, $i \leq (q-4)q+(q-3)-4 = q^2-3q-7$ for $q \geq 13$ by Lemma 2.6 (3) and $i \leq 78$ for $q = 11$ by Table 1. We also have that $q^2-3q-2 \leq i < q^2-2q-2$ implies $r = q-2$ and $i \leq (q-3)q+(q-2)-4 = q^2-2q-6$ and that $q^2-2q-2 \leq i < q^2-q-2$ implies $r = q-1$ and $i \leq (q-2)q+(q-1)-4 = q^2-q-5$. Hence, $i > q^2-q-5$ implies $r \geq q$. Assume $q^2-q-2 \leq i < q^2-2$. By (19), $r = q$. (20) with $(w, t) = (i, q)$ yields that $c_{\gamma_2} > 0$, which contradicts that a γ_2-plane has no q-line. Hence $a_i = 0$. Similarly, $q^2-2 \leq i < q^2+q-2$ implies $r = q$ and $i \leq q^2$. The spectrum of a q^2-plane is $(\tau_0, \tau_q) = (1, q^2+q)$ from Table 2, which contradicts (18). Hence $q_{q^2} = 0$. We have $a_{q^2+q} = a_{\theta_2} = 0$ similarly. Thus, we have $a_i = 0$ for all

$$i \notin \{q^2-4q-2, \ldots, q^2-3q-7, q^2-3q-2, \ldots, q^2-2q-6, q^2-2q-2, \ldots,$$

$$q^2-q-5, q^2-2, q^2-1, q^2+q-2, q^2+q-1, 2q^2-4q-2, \ldots, 2q^2-q-3\}.$$

Note that $a_{79} = a_{80} = a_{81} = 0$ for $q = 11$. From (3), we get

$$\sum_{i=0}^{\gamma_2-2} \binom{\gamma_2-i}{2} a_i = q^2\lambda_2 - (q^5 - \frac{7}{2}q^4 - \frac{7}{2}q^3 + \frac{13}{2}q^2 + \frac{7}{2}q - 1). \tag{22}$$

We first rule out possible (q^2+q-2)-planes by (λ_2, q^2+q-2)-ROM. Suppose $a_{q^2+q-2} > 0$. The spectrum of a $[q^2+q-2, 3, q^2-3]_q$ code is (X) $(\tau_{q-2}, \tau_q, \tau_{q+1}) = (1, 3q, q^2-2q)$ or (Y) $(\tau_{q-1}, \tau_q, \tau_{q+1}) = (3, 3q-3, q^2-2q+1)$ from Table 2. Setting $w = q^2+q-2$ in (20), the maximum possible contributions of c_j's to the LHS of (22) are $(c_{2q^2-4q-2}, c_{2q^2-2q-4}, c_{n-d}) = (1, 1, q-2)$ for $t = q-2$; $(c_{2q^2-4q-2}, c_{n-d-1}, c_{n-d}) = (1, 1, q-2)$ for $t = q-1$; $(c_{2q^2-2q-4}, c_{n-d-1}) = (1, q-1)$ for $t = q$; $c_{n-d-1} = q$ for $t = q+1$. Estimating the LHS of (22) for spectrum (X), we get

$$(\text{LHS of (22)}) \le \binom{q^2-2q-1}{2} + (\binom{3q-1}{2} + \binom{q+1}{2})\tau_{q-2} + \binom{3q-1}{2}\tau_{q-1} + \binom{q+1}{2}\tau_q,$$

giving $\lambda_2 \le (2q^3 - 6q^2 - 8q + 27)/2$. On the other hand, (21) gives $\lambda_2 \ge q^3 - 2q^2 - 5q$, a contradiction. We also get a contradiction similarly for spectrum (Y). Hence $a_{q^2+q-2} = 0$. One can prove $a_{q^2+q-1} = a_{q^2-2} = a_{q^2-1} = 0$ for $q \ge 11$ using the spectra in Table 2, similarly.

Next, we rule out the possible i-planes for $q^2-4q-2 \le i \le q^2-3q-7$ for $q \ge 13$ and for $75 \le i \le 78$ for $q \ge 11$ by (λ_2, γ_2)-ROM. Suppose $a_i > 0$ for $i = q^2-4q-2+e$ with $0 \le e \le q-5$, $q \ge 13$. Then, we may assume that Δ has spectrum (A). Note that RHS of (20) is at most $q^2-3q+e-qt \le q^2-4q-5$. Since Δ has no 0-line and the coefficient of c_{q^2-q-5} in (20) is q^2+2, we get $a_i = 1$ and $a_j = 0$ for $j \le q^2-q-5$ with $j \ne i$. Setting $w = n-d$ in (20), the maximum possible contributions of c_j's to the LHS of (22) are $(c_i, c_{n-d-e}, c_{n-d}) = (1, 1, q-2)$ for $t = q-3$; $(c_{2q^2-4q-2}, c_{n-d-y}, c_{n-d}) = (x, 1, q-x-1)$ for $t = q-1$; $(c_{2q^2-3q-2}, c_{n-d}) = (1, q-1)$ for $t = 2q-2$; $(c_{2q^2-2q-2}, c_{n-d}) = (1, q-1)$ for $t = 2q-1$, where $(x, y) = (q/3, 4q/3-1)$, $(x, y) = ((q-1)/3, (7q-4)/3)$, $(x, y) = ((q+1)/3, (q-2)/3)$ if $q \equiv 0, 1, 2 \pmod 3$, respectively. Estimating the LHS of (22), we get

$$\begin{aligned}(\text{LHS of (22)}) &\le (\binom{q^2+3q-1-e}{2} + \binom{e}{2})\tau_{q-3} + (\binom{3q-1}{2}x + \binom{y}{2})\tau_{q-1} \\ &\quad + \binom{2q-1}{2}\tau_{2q-2} + \binom{q-1}{2}\tau_{2q-1} \\ &\le \binom{q^2+3q-1}{2}\tau_{q-3} + (\binom{3q-1}{2}\frac{q+1}{3} + \binom{\frac{q-2}{3}}{2})\tau_{q-1} + \binom{2q-1}{2}\tau_{2q-2} + \binom{q-1}{2}\tau_{2q-1},\end{aligned}$$

giving $\lambda_2 \le (18q^3 - 45q^2 + 81q + 92)/18$. On the other hand, since Δ has five 0-points and one $(q-3)$-line, say l, $\Delta \backslash l$ has one 0-point. Since $c_{n-d} \ge q-e-1 \ge 4$ for $t = q-3$, there are at least four γ_2-planes with spectrum (A) through l and (21) yields

$$\lambda_2 \ge q^3 - 2q^2 - 5q - 3 + (\theta_2 - (q^2 - 3q - 7)) + 4 = q^3 - 2q^2 - q + 5,$$

giving a contradiction for $q \ge 13$. For $q = 11$, we consider a putative i-plane with $i = q^2-4q-2+e$ with $0 \le e \le 3$ in the same way. Since $c_{n-d} \ge q-e-1 \ge 7$ for $t = q-3$, we can get a contradiction as above. Hence $a_i = 0$ for $q^2-4q-2 \le i \le q^2-3q-7$. One can similarly prove that $a_i = 0$ for $q^2-3q-2 \le i \le q^2-2q-6$ and for $q^2-2q-2 \le i \le q^2-q-5$ by (λ_2, γ_2)-ROM.

Thus, we have proved that $a_i = 0$ for all $i < 2q^2-4q-2$. Finally, applying (λ_2, γ_2)-ROM for $i = \lambda_2$, we get a contradiction as follows. Setting $w = n-d$, the maximum possible contributions of c_j's in (20) to the LHS of (22) are $(c_{2q^2-4q-2}, c_{n-d-w}, c_{n-d}) = (z, 1, q-z-1)$ for $t = q-3$; $(c_{2q^2-4q-2}, c_{n-d-b}, c_{n-d}) = (a, 1, q-a-1)$ for $t = q-2$; $(c_{2q^2-4q-2}, c_{n-d-y}, c_{n-d}) = (x, 1, q-x-1)$ for $t = q-1$; $(c_{2q^2-4q-2}, c_{n-d}) = (1, q-1)$ for $t = 2q-3$; $(c_{2q^2-3q-2}, c_{n-d}) = (1, q-1)$ for $t = 2q-2$; $(c_{2q^2-2q-2}, c_{n-d}) = (1, q-1)$ for $t = 2q-1$, where $(a, b, x, y, z, w) = (q/3, 7q/3-1, q/3, 4q/3-1, q/3+1, q/3)$, $((q+2)/3, (q-1)/3, (q-1)/3, (7q-4)/3, (q+2)/3, (4q-1)/3)$, $((q+1)/3, (4q-2)/3, (q+1)/3, (q-2)/3, (q+1)/3, (7q-2)/3)$ if $q \equiv 0, 1, 2 \pmod 3$, respectively. Estimating the LHS of (22), we get

$$\begin{aligned}(\text{LHS of (22)}) &\le (\binom{3q-1}{2}z + \binom{w}{2})\tau_{q-3} + (\binom{3q-1}{2}a + \binom{b}{2})\tau_{q-2} \\ &\quad + (\binom{3q-1}{2}x + \binom{y}{2})\tau_{q-1} + \binom{3q-1}{2}\tau_{2q-3} + \binom{2q-1}{2}\tau_{2q-2} + \binom{q-1}{2}\tau_{2q-1}\end{aligned}$$

giving $\lambda_2 \le (6q^3 - 18q^2 + 24q + 37)/6$ if Δ has spectrum (D) and if $q \equiv 2 \bmod 3$. On the other hand, (21) yields $\lambda_2 \ge q^3 - 2q^2 - 5q - 3$, giving a contradiction for $q \ge 11$. One can get a contradiction similarly for the other cases. This completes the proof. □

5. Proof of Theorem 1.5

Lemma 5.1. *There exists no $[g_q(4,d), 4, d]_q$ code for $d = q^3 - q^2 - 4q - 2$ for $q \ge 9$.*

Proof. Let $\mathcal{C}$ be a putative $[n = q^3 - 4q - 6, 4, d = q^3 - q^2 - 4q - 2]_q$ code. Note that $n = g_q(4,d)$ and hence $\gamma_0 = 1$, $\gamma_1 = q$, $\gamma_2 = n - d = q^2 - 4$ from (1). Let Δ be a γ_2-plane and let δ be an i-plane. By Lemma 2.13, the spectrum of Δ satisfies $\tau_j = 0$ for $1 \le j \le q-5$. Since a t-line in δ satisfies $t \le (i+6)/q$, we have $a_i = 0$ for $1 \le i \le q - 7$. Assume $i = sq - 6 + e$ with $0 \le e \le q - 1$. For $2 \le s \le q - 5$, we have $\gamma_1(\delta) \le s - 1$ by Lemma 2.3 (5). Then, it follows from Lemma 2.6 (1) that $i \le (s-2)q + s - 1 < sq - 6 + e$, a contradiction. For $s = q - 4$, from Lemma 2.6 (2), we have $i \le (q-5)q + q - 4 - 3 < i$, a contradiction again. Similarly, using Lemma 2.6 and Table 1, we can deduce that $a_i = 0$ for all $i \notin \{0, q^2 - 6, q^2 - 5, q^2 - 4\}$ for $q \ge 11$ and that $a_i = 0$ for all $i \notin \{0, 48, 75, 76, 77\}$ for $q = 9$. For $q = 9$, a 48-plane has a 0-line [24], but the equation (4) with $(i,t) = (48, 0)$ has no solution. Hence $a_{48} = 0$. From (4), we have $a_0 = 0$ or 1. The equality (3) gives

$$a_{q^2-6} = (q^4 + 4q^3 - 9q^2 + 14q + 2)/2 - \binom{q^2-4}{2} a_0 \ge 2q^3 + 7q - 9 > \theta_3,$$

a contradiction. □

Lemma 5.2. *There exists no $[g_q(4,d), 4, d]_q$ code for $d = q^3 - q^2 - 6q$ for $q \ge 9$.*

Proof. Let $\mathcal{C}$ be a putative $[n = q^3 - 6q - 6, 4, d = q^3 - q^2 - 6q]_q$ code. Then, $n = g_q(4,d)$ and $\gamma_0 = 1$, $\gamma_1 = q$, $\gamma_2 = n - d = q^2 - 6$ from (1). Let Δ be a γ_2-plane and let δ be an i-plane. By Lemma 2.13, the spectrum of Δ satisfies $\tau_j = 0$ for $1 \le j \le q - 7$. Since a t-line in δ satisfies $t \le (i+6)/q$, we have $a_i = 0$ for $1 \le i \le q - 7$. Using Lemmas 2.3, 2.6 and Table 1 similarly to the proof of Lemma 5.1, we can deduce that $a_i = 0$ for all $i \notin \{0, q^2 - 6\}$ for $q \ge 11$ and that $a_i = 0$ for all $i \notin \{0, 48, 75\}$ for $q = 9$. Since the equation (4) with $(i,t) = (0,0)$ has no solution for $q \ge 9$, we obtain $a_0 = 0$. Then, the three equations in Lemma 2.1 have no solution for $q = 9$, a contradiction. We also get a contradiction for $q \ge 11$ from Lemma 2.2. □

Lemma 5.3. *There exists no $[g_q(4,d), 4, d]_q$ code for $d = q^3 - q^2 - 6q - 1$ for $q \ge 9$.*

Proof. Let $\mathcal{C}$ be a putative $[n = q^3 - 6q - 7, 4, d = q^3 - q^2 - 6q - 1]_q$ code. Then, $n = g_q(4,d)$, $\gamma_0 = 1$, $\gamma_1 = q$, $\gamma_2 = n - d = q^2 - 6$ from (1). Let Δ be a γ_2-plane and let δ be an i-plane. By Lemma 2.13, the spectrum of Δ satisfies $\tau_j = 0$ for $1 \le j \le q - 7$. Since a t-line in δ satisfies $t \le (i+7)/q$, we have $a_i = 0$ for $1 \le i \le q - 8$. Using Lemmas 2.3, 2.6 and Table 1, it can be shown that $a_i = 0$ for all $i \notin \{0, q^2 - 3q - 7, q^2 - 7, q^2 - 6\}$ for $q \ge 11$ and that $a_i = 0$ for all $i \notin \{0, 47, 48, 65, 74, 75\}$ for $q = 9$.

Suppose $a_0 > 0$. It follows from (4) that $a_0 = 1$ and that $a_j = 0$ for $0 < j < q^2 - 7$. Then, $\mathcal{C}$ is extendable by Theorem 2.11, contradicting Lemma 5.2. Hence $a_0 = 0$. Then, $\mathcal{C}$ is extendable by Theorem 2.9, a contradiction again. □

Lemma 5.4. *There exists no $[g_q(4,d), 4, d]_q$ code for $d = q^3 - q^2 - 6q - 2$ for $q \ge 9$.*

Proof. Let $\mathcal{C}$ be a putative $[n = q^3 - 6q - 8, 4, d = q^3 - q^2 - 6q - 2]_q$ code. Then, $n = g_q(4,d)$, $\gamma_0 = 1$, $\gamma_1 = q$, $\gamma_2 = n - d = q^2 - 6$ from (1). Let Δ be a γ_2-plane and let δ be an i-plane. By Lemma 2.13, the spectrum of Δ satisfies $\tau_j = 0$ for $1 \le j \le q - 7$. Since a t-line in δ satisfies $t \le (i+8)/q$, we have $a_i = 0$ for $1 \le i \le q - 9$ for $q \ge 11$. Using Lemmas 2.3, 2.6 and Table 1, it can be shown that

$a_i = 0$ for all $i \notin \{0, q^2-4q-8, q^2-3q-8, q^2-3q-7, q^2-8, q^2-7, q^2-6\}$ for $q \geq 13$,

$a_i = 0$ for all $i \notin \{0, 102, 113, 114, 115\}$ for $q = 11$,

$a_i = 0$ for all $i \notin \{0, 28, 37, 46, 47, 48, 55, 64, 65, 73, 74, 75\}$ for $q = 9$.

Suppose $a_0 > 0$. It follows from (4) that $a_0 = 1$ and that $a_j = 0$ for $0 < j < q^2-8$ for $q \geq 9$. Then, the equality (3) gives $a_{q^2-8} = 3q^3 + 10q - 20 > \theta_3$, a contradiction. Hence $a_0 = 0$. Then, $\mathcal{C}$ is extendable by Theorem 2.10, a contradiction again. □

The following three lemmas can be proved similarly to Lemmas 5.2, 5.3, 5.4, respectively.

Lemma 5.5. *There exists no $[g_q(4,d), 4, d]_q$ code for $d = q^3 - q^2 - 5q$ for $q \geq 9$.*

Lemma 5.6. *There exists no $[g_q(4,d), 4, d]_q$ code for $d = q^3 - q^2 - 5q - 1$ for $q \geq 9$.*

Lemma 5.7. *There exists no $[g_q(4,d), 4, d]_q$ code for $d = q^3 - q^2 - 5q - 2$ for $q \geq 9$.*

Now, Theorem 1.5 follows from Lemmas 5.1, 5.4, 5.7. This completes the proof. □

6. Conclusion

To solve the problem finding the exact values of $n_q(k,d)$ for all d for fixed q and k, it is sufficient to determine $n_q(k,d)$ for finite values of d since $n_q(k,d) = g_q(k,d)$ for all $d \geq (k-2)q^{k-1} - (k-1)q^{k-2} + 1$, $k \geq 3$ for all q [17]. For $k = 4$, it is known that $n_q(4,d) = g_q(4,d)$ for $q^3 - q^2 - q + 1 \leq d \leq q^3 + q^2 + q$, $d \geq 2q^3 - 3q^2 + 1$ for all q and for $2q^3 - 5q^2 + 1 \leq d \leq 2q^3 - 5q^2 + 3q$ for $q \geq 7$ ([18, 21]). The key contribution here is showing the non–existence of $[g_q(4,d), 4, d]_q$ codes for many values of d close to these "Griesmer area", and it seems reasonable to seek a generalization for larger k. To this direction, see [3] and [4].

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