J. Algebra Comb. Discrete Appl.
4(2) • 155–164

Received: 12 June 2015
Accepted: 20 February 2016

Journal of Algebra Combinatorics Discrete Structures and Applications

Strongly nil $*$-clean rings

Research Article

Abdullah Harmancı, Huanyin Chen, A. Çiğdem Özcan

Abstract: A $*$-ring R is called *strongly nil $*$-clean* if every element of R is the sum of a projection and a nilpotent element that commute with each other. In this paper we investigate some properties of strongly nil $*$-rings and prove that R is a strongly nil $*$-clean ring if and only if every idempotent in R is a projection, R is periodic, and $R/J(R)$ is Boolean. We also prove that a $*$-ring R is commutative, strongly nil $*$-clean and every primary ideal is maximal if and only if every element of R is a projection.

2010 MSC: 16W10, 16U99

Keywords: Rings with involution, Strongly nil $*$-clean ring, $*$-Boolean ring, Boolean ring

1. Introduction

Let R be an associative ring with unity. A ring R is called *strongly nil clean* if every element of R is the sum of an idempotent and a nilpotent that commute. These rings were first considered by Hirano-Tominaga-Yakub [9] and refered to as [E-N]-representable rings. In [7], Diesl introduces this class and studies their properties. The class of strongly nil clean rings lies between the class of Boolean rings and strongly π-regular rings (i.e. for every $a \in R$, $a^n \in Ra^{n+1} \cap a^{n+1}R$ for some positive integer n) [7, Corollary 3.7].

An *involution* of a ring R is an operation $* : R \to R$ such that $(x + y)^* = x^* + y^*$, $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in R$. A ring R with an involution $*$ is called a *$*$-ring*. An element p in a $*$-ring R is called a *projection* if $p^2 = p = p^*$ (see [2]). Recently the concept of strongly clean rings were considered for any $*$-ring. Vaš [12] calls a $*$-ring R *strongly $*$-clean* if each of its elements is the sum of a projection and a unit that commute with each other (see also [10]).

In this paper, we adapt strongly nil cleanness to $*$-rings. We call a $*$-ring R *strongly nil $*$-clean* if every element of R is the sum of a projection and a nilpotent element that commute. The paper consists

Abdullah Harmancı (Corresponding Author), A. Çiğdem Özcan; Hacettepe University, Department of Mathematics, 06800 Beytepe, Ankara, Turkey (email: harmanci@hacettepe.edu.tr, ozcan@hacettepe.edu.tr).
Huanyin Chen; Hangzhou Normal University, Department of Mathematics, 310036, Hangzhou, China (email: huanyinchen@aliyun.com).

of three parts. In Section 2, we characterize the class of strongly nil $*$-clean rings in several different ways. For example, we show that a ring R is a strongly nil $*$-clean ring if and only if every idempotent in R is a projection, R is periodic, and $R/J(R)$ is Boolean. Also, if R is a commutative $*$-ring and $R[i] = \{a + bi \mid a, b \in R,\ i^2 = -1\ \}$, then with the involution $*$ defined by $(a + bi)^* = a^* + b^*i$, the ring $R[i]$ is strongly nil $*$-clean if and only if R is strongly nil $*$-clean. Foster [8] introduced the concept of Boolean-like rings as a generalization of Boolean rings. In Section 3, we adapt the concept of Boolean-like rings to rings with involution and prove that a $*$-ring R is $*$-Boolean-like if and only if R is strongly nil $*$-clean and $\alpha\beta = 0$ for all nilpotent elements α, β in R. In the last section, we investigate submaximal ideals (see [11]) of strongly nil $*$-clean rings. We also define $*$-Boolean rings as $*$-rings over which every element is a projection and characterize them in terms of strongly nil $*$-cleanness. As a corollary, we get that R is a Boolean ring if and only if R is commutative, strongly nil clean and every primary ideal of R is maximal. Other characterizations of Boolean rings by means of (strongly) nil clean rings can be found in [7].

Throughout this paper all rings are associative with unity (unless otherwise noted). We write $J(R)$, $N(R)$ and $U(R)$ for the Jacobson radical of a ring R, the set of all nilpotent elements in R and the set of all units in R, respectively. The ring of all polynomials in one variable over R is denoted by $R[x]$.

2. Characterization theorems

The main purpose of this section is to provide several characterizations of strongly nil $*$-clean rings.

First recall some definitions. A ring R is called *uniquely nil clean* if, for any $x \in R$, there exists a unique idempotent $e \in R$ such that $x - e \in N(R)$ [7]. If, in addition, x and e commute, R is called *uniquely strongly nil clean* [9]. Strongly nil cleanness and uniquely strongly nil cleanness are equivalent by [9, Theorem 3].

Analogously, for a $*$-ring, we define *uniquely strongly nil $*$-clean rings* by replacing "idempotent" with "projection" in the definition of uniquely strongly nil clean rings.

We will use the following lemma frequently.

Lemma 2.1. [10, Lemma 2.1] *Let R be a $*$-ring. If every idempotent in R is a projection, then R is abelian, i.e. every idempotent in R is central.*

Proposition 2.2. *Let R be a $*$-ring. Then the following are equivalent.*

(i) *R is strongly nil $*$-clean;*

(ii) *R is strongly nil clean and every idempotent in R is a projection;*

(iii) *R is uniquely strongly nil $*$-clean.*

Proof. (i) ⇒ (ii) Assume that R is strongly nil $*$-clean. Then R is strongly $*$-clean as can be seen in the proof of [7, Proposition 3.4], i.e. if $x \in R$, there exist a projection e and a nilpotent w in R such that $x - 1 = e + w$ and $ew = we$. This gives that $x = e + (1 + w)$ where e is a projection, $1 + w$ is invertible and $e(1 + w) = (1 + w)e$. Now, by [10, Theorem 2.2], every idempotent in R is a projection and central. Hence R is uniquely nil clean by [9, Theorem 3].

(ii) ⇒ (iii) If R is uniquely nil clean, then R is uniquely strongly nil clean by Lemma 2.1. Hence R is uniquely strongly nil $*$-clean.

(iii) ⇒ (i) Clear. □

We note that the condition "every idempotent in R is a projection" in Proposition 2.2 is necessary as the following example shows.

Example 2.3. Let $R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}$ where $0, 1 \in \mathbb{Z}_2$. Define $* : R \to R$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a+b & b \\ a+b+c+d & b+d \end{pmatrix}$. Then R is a commutative $*$-ring with the usual matrix addition and multiplication. In fact, R is Boolean. Thus, for any $x \in R$, there exists a unique idempotent $e \in R$ such that $x - e \in R$ is nilpotent. But it is not strongly nil $*$-clean because the only projections are the trivial projections and there does not exist a projection e in R such that $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} - e$ is nilpotent.

In [9, Theorem 3], it is proved that R is strongly nil clean if and only if $N(R)$ is an ideal and $R/N(R)$ is Boolean. Also, R is uniquely nil clean if and only if R is abelian, $N(R)$ is an ideal and $R/N(R)$ is Boolean [9, Theorem 4]. So if we adapt these results to rings with involution, immediately we have the following proposition by using Proposition 2.2.

Proposition 2.4. *Let R be a $*$-ring. Then R is strongly nil $*$-clean if and only if*

(1) *Every idempotent in R is a projection;*

(2) *$N(R)$ forms an ideal;*

(3) *$R/N(R)$ is Boolean.*

A ring R is called *strongly J-$*$-clean* if for any $x \in R$ there exists a projection $e \in R$ such that $x - e \in J(R)$ and $ex = xe$ [6], equivalently, for any $x \in R$ there exists a unique projection $e \in R$ such that $x - e \in J(R)$ [6, Theorem 3.2]. We call R *uniquely nil $*$-clean ring* if for any $a \in R$, there exists a unique projection $e \in R$ such that $a - e \in N(R)$.

Proposition 2.5. *Let R be a $*$-ring. Then the following are equivalent.*

(i) *R is strongly nil $*$-clean;*

(ii) *R is strongly J-$*$-clean and $J(R)$ is nil;*

(iii) *R is uniquely nil $*$-clean and $J(R)$ is nil.*

Proof. (i) $\Rightarrow$ (ii) Suppose that R is strongly nil $*$-clean. In view of Proposition 2.4, $N(R)$ forms an ideal of R, and this gives that $N(R) \subseteq J(R)$ (see also [7, Proposition 3.18]). By [7, Proposition 3.16], $J(R)$ is nil, and so $N(R) = J(R)$. Hence R is strongly J-$*$-clean.

(ii) $\Rightarrow$ (i) is obvious.

(i) and (ii) $\Rightarrow$ (iii) Since R is strongly J-$*$-clean, there exists a unique projection $e \in R$ such that $x - e \in J(R)$ by [6, Theorem 3.2]. Since $J(R) = N(R)$, R is uniquely nil $*$-clean.

(iii) $\Rightarrow$ (ii) Since $J(R) \subseteq N(R)$, R is strongly J-$*$-clean rings by [6, Theorem 3.2]. □

From Proposition 2.5 and [6, Proposition 2.1], it follows that

$$\{\text{strongly nil } *\text{-clean}\} \subset \{\text{strongly } J\text{-}*\text{-clean}\} \subset \{\text{strongly } *\text{-clean}\}.$$

The first inclusion is strict because, for example, the power series ring $\mathbb{Z}_2[[x]]$ with the identity involution is strongly J-$*$-clean but not strongly nil $*$-clean by [4, Example 2.5(5)]. The second inclusion is also strict by [6, Example 2.2(2)].

We should note that a strongly nil clean ring may not be strongly J-clean (see [4, Example on p. 3799]). Hence strongly nil clean and strongly nil $*$-clean classes have different behavior when compared to classes of strongly J-clean and strongly J-$*$-clean classes respectively.

Lemma 2.6. *Let R be a $*$-ring. Then R is strongly nil $*$-clean if and only if*

(1) *Every idempotent in R is a projection;*

(2) *$J(R)$ is nil;*

(3) *$R/J(R)$ is Boolean.*

Proof. Assume that (1), (2) and (3) hold. For any $x \in R$, $x + J(R) = x^2 + J(R)$. As $J(R)$ is nil, every idempotent in R lifts modulo $J(R)$. Thus, we can find an idempotent $e \in R$ such that $x - e \in J(R) \subseteq N(R)$. By Lemma 2.1, $xe = ex$, and so the result follows. The converse is by Propositions 2.4 and 2.5. □

Recall that a ring R is *periodic* if for any $x \in R$, there exist distinct $m, n \in \mathbb{N}$ such that $x^m = x^n$. With this information we can now prove the following.

Theorem 2.7. *Let R be a $*$-ring. Then R is strongly nil $*$-clean if and only if*

(1) *Every idempotent in R is a projection;*

(2) *R is periodic;*

(3) *$R/J(R)$ is Boolean.*

Proof. Suppose that R is strongly nil $*$-clean. By virtue of Lemma 2.6, every idempotent in R is a projection and $R/J(R)$ is Boolean. For any $x \in R$, $x - x^2 \in N(R)$. Write $(x - x^2)^m = 0$, and so $x^m = x^{m+1}f(x)$, where $f(x) \in \mathbb{Z}[x]$. According to Herstein's Theorem (cf. [3, Proposition 2]), R is periodic. Conversely, $J(R)$ is nil as R is periodic. Therefore the proof is completed by Lemma 2.6. □

Proposition 2.8. *A $*$-ring R is strongly nil $*$-clean if and only if*

(1) *R is strongly $*$-clean;*

(2) *$N(R) = \{x \in R \mid 1 - x \in U(R)\}$.*

Proof. Suppose that R is strongly nil $*$-clean. By the proof of Proposition 2.5, $N(R) = J(R)$. Since R is strongly J-$*$-clean, $N(R) = \{x \in R \mid 1 - x \in U(R)\}$ by [6, Theorem 3.4].

Conversely, assume that (1) and (2) hold. Let $a \in R$. Then we can find a projection $e \in R$ such that $(a - 1) - e \in U(R)$ and $e(a - 1) = (a - 1)e$. That is, $(1 - a) + e \in U(R)$. As $1 - (a - e) \in U(R)$, by hypothesis, $a - e \in N(R)$. In addition, $ea = ae$. Accordingly, R is strongly nil $*$-clean. □

Let R be a $*$-ring. Define $* : R[x]/(x^n) \to R[x]/(x^n)$ by $a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + (x^n) \mapsto a_0^* + a_1^*x + \cdots + a_{n-1}^*x^{n-1} + (x^n)$. Then $R[x]/(x^n)$ is a $*$-ring (cf. [10]).

Corollary 2.9. *Let R be a $*$-ring. Then R is strongly nil $*$-clean if and only if so is $R[x]/(x^n)$ for every $n \geq 1$.*

Proof. One direction is obvious. Conversely, assume that R is strongly nil $*$-clean. Clearly, $N\big(R[x]/(x^n)\big) = \{a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + (x^n) \mid a_0 \in N(R), a_1, \cdots, a_{n-1} \in R\}$. In view of Proposition 2.8, $N\big(R[x]/(x^n)\big) = \{a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + (x^n) \mid 1 - a_0 \in U(R), a_1, \cdots, a_{n-1} \in R\}$. Also note that R is abelian. Thus, it can be easily seen that every element in $R[x]/(x^n)$ can be written as the sum of a projection and a nilpotent element that commute. □

Let R be a commutative $*$-ring and consider the ring $R[i] = \{a + bi \mid a, b \in R,\ i^2 = -1\ \}$ and i commutes with elements of R. Then $R[i]$ is a $*$-ring, where the involution is $* : R[i] \to R[i]$, $a + bi \mapsto a^* + b^*i$.

Note that if x and y are idempotent elements that commute, then $(x-y)^3 = x-3xy+3xy-y = x-y$. This argument will also be used in Lemma 4.6.

Proposition 2.10. *Let R be a commutative $*$-ring. Then with the involution $(a + bi)^* = a^* + b^*i$, $R[i]$ is strongly nil $*$-clean if and only if R is strongly nil $*$-clean.*

Proof. Suppose that $R[i]$ is strongly nil $*$-clean. Then every idempotent in R is a projection. Since R is commutative, $N(R)$ forms an ideal. For any $a \in R$, we see that $a-a^2 \in N\big(R[i]\big)$, and so $a-a^2 \in N(R)$. Thus, $R/N(R)$ is Boolean. Therefore R is strongly nil $*$-clean by Proposition 2.4.

Conversely, assume that R is strongly nil $*$-clean. As R is commutative, $N\big(R[i]\big)$ forms an ideal of $R[i]$. Let $a + bi \in R[i]$ be an idempotent. Then we can find projections $e, f \in R$ and nilpotent elements $u, v \in R$ such that $a = e + u$, $b = f + w$. Then $a - a^*, b - b^* \in N(R)$. This shows that $(a + bi) - (a + bi)^* = (a - a^*) + (b - b^*)i \in N\big(R[i]\big)$. As $a + bi, (a + bi)^* \in R[i]$ are idempotents, we see that $\big((a+bi) - (a+bi)^*\big)^3 = (a+bi) - (a+bi)^*$ by the above argument. Hence, $\big((a+bi) - (a+bi)^*\big)\big(1 - ((a + bi) - (a + bi)^*)^2\big) = 0$, therefore $(a + bi) - (a + bi)^* = 0$. That is, $a + bi \in R[i]$ is a projection.

Since R is strongly nil $*$-clean, it follows from Proposition 2.4 that $2 - 2^2 \in N(R)$, and so $2 \in N(R)$. For any $a + bi \in R[i]$, it is easy to verify that

$$\begin{aligned}(a + bi) - (a + bi)^2 &= (a - a^2) - 2abi + bi - b^2i^2 \\ &\equiv b^2 + bi \\ &\equiv b + bi \ \big(\mathrm{mod}\ N\big(R[i]\big)\big).\end{aligned}$$

This shows that $\big((a+bi)-(a+bi)^2\big)^2 \equiv 2b^2i \equiv 2b \equiv 0 \ \big(\mathrm{mod}\ N\big(R[i]\big)\big)$. Hence, $(a+bi)-(a+bi)^2 \in N(R[i])$. That is, $R[i]/N\big(R[i]\big)$ is Boolean. According to Proposition 2.4, we complete the proof. □

3. $*$-Boolean like rings

In this section, we consider a subclass of strongly nil $*$-clean rings consisting of rings which we call $*$-Boolean-like. First recall that a ring R is called *Boolean-like* if it is commutative with unit and is of characteristic 2 with $ab(1+a)(1+b) = 0$ for every $a, b \in R$ [8]. Any Boolean ring is clearly a Boolean-like ring but not conversely (see [8]). Any Boolean-like ring is uniquely nil clean by [8, Theorem 17]. Also, R is Boolean-like if and only if (1) R is a commutative ring with unit; (2) It is of characteristic 2; (3) It is nil clean; (4) $ab = 0$ for every nilpotent element a, b in R [8, Theorem 19].

Definition 3.1. A $*$-ring R is said to be *$*$-Boolean-like* provided that every idempotent in R is a projection and $(a - a^2)(b - b^2) = 0$ for all $a, b \in R$.

The following is an example of a $*$-Boolean-like ring.

Example 3.2. Let $R = \{\begin{pmatrix} a & b \\ c & a \end{pmatrix} \mid a, b, c \in \mathbb{Z}_2\}$. Define $\begin{pmatrix} a & b \\ c & a \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & a' \end{pmatrix} = \begin{pmatrix} a+a' & b+b' \\ c+c' & a+a' \end{pmatrix}$, $\begin{pmatrix} a & b \\ c & a \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & a' \end{pmatrix} = \begin{pmatrix} aa' & ab'+ba' \\ ca'+ac' & aa' \end{pmatrix}$ and $* : R \to R$, $\begin{pmatrix} a & b \\ c & a \end{pmatrix} \mapsto \begin{pmatrix} a & c \\ b & a \end{pmatrix}$. Then R is a $*$-ring. Let $\begin{pmatrix} a & b \\ c & a \end{pmatrix} \in R$ be an idempotent. Then $a = a^2$ and $(2a-1)b = (2a-1)c = 0$. As $(2a-1)^2 = 1$, we see that $b = c = 0$, and so the set of all idempotents in R is $\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}$. Thus, every idempotent in R is a projection. For any $A, B \in R$, we see that $(A - A^2)(B - B^2) = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} = 0$. Therefore R is $*$-Boolean-like.

Theorem 3.3. *Let R be a $*$-ring. Then R is $*$-Boolean-like if and only if*

(1) *R is strongly nil $*$-clean;*

(2) *$\alpha\beta = 0$ for all nilpotent elements $\alpha, \beta \in R$.*

Proof. Suppose that R is $*$-Boolean-like. Then every idempotent in R is a projection; hence, R is abelian. For any $a \in R$, $(a - a^2)^2 = 0$, and so $a^2 = a^3 f(a)$ for some $f(t) \in \mathbb{Z}[t]$. This implies that R is strongly π-regular, and so it is π-regular. It follows from [1, Theorem 3] that $N(R)$ forms an ideal. Further, $a - a^2 \in N(R)$. Therefore $R/N(R)$ is Boolean. According to Proposition 2.4, R is strongly nil $*$-clean. For any nilpotent elements $\alpha, \beta \in R$, we can find some $m, n \in \mathbb{N}$ such that $\alpha^m = \beta^n = 0$. Since $\alpha^2 = \alpha^3 g(\alpha)$ for some $g(t) \in \mathbb{Z}[t]$, $\alpha^2 = 0$. Likewise, $\beta^2 = 0$. This shows that $\alpha\beta = (\alpha - \alpha^2)(\beta - \beta^2) = 0$.

Conversely, assume that (1) and (2) hold. By Proposition 2.4, every idempotent is a projection, and for any $a \in R$, $a - a^2$ is nilpotent. Hence for any $a, b \in R$, $(a - a^2)(b - b^2) = 0$. Therefore R is $*$-Boolean-like. □

Corollary 3.4. *$*$-Boolean-like rings are commutative rings.*

Proof. Let $x, y \in R$. In view of Theorem 3.3, $x - e$ and $y - f$ are nilpotent for some projections $e, f \in R$. Again by Theorem 3.3, $(x - e)(y - f) = 0 = (y - f)(x - e)$. Since R is abelian, it follows that $xy = yx$. Hence R is commutative. □

Example 3.5. Let R be the ring

$$\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\},$$

where $0, 1 \in \mathbb{Z}_2$. Define $* : R \to R, A \mapsto A^T$, the transpose of A. Then R is a $*$-ring in which $(a - a^2)(b - b^2) = 0$ for all $a, b \in R$. Further, $\alpha\beta = 0$ for all nilpotent elements $\alpha, \beta \in R$. But R is not $*$-Boolean-like.

We end this section with an example showing that strongly nil clean rings need not be strongly nil $*$-clean.

Example 3.6. Consider the ring

$$R = \{ \begin{pmatrix} a & 2b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z}_4 \}.$$

Then for any $x, y \in R$, $(x - x^2)(y - y^2) = 0$. Obviously, R is not commutative. This implies that R is not a $*$-Boolean-like ring for any involution $*$. Accordingly, R is not strongly nil $*$-clean for any involution $*$; otherwise, every idempotent in R is a projection, a contradiction (see Lemma 2.1). We can also consider the involution $* : R \to R$, $\begin{pmatrix} a & 2b \\ 0 & c \end{pmatrix} \mapsto \begin{pmatrix} c & -2b \\ 0 & a \end{pmatrix}$ and the idempotent $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ which is not a projection. On the other hand, since $(x - x^2)^2 = 0$ and so $x - x^2 \in N(R)$ for all $x \in R$, we get that R is strongly nil clean by [9, Theorem 3].

4. Submaximal ideals and $*$-Boolean rings

An ideal I of a ring R is called a *submaximal ideal* if I is covered by a maximal ideal of R. That is, there exists a maximal ideal I_1 of R such that $I \subsetneqq I_1 \subsetneqq R$ and for any ideal K of R such that $I \subseteq K \subseteq I_1$, we have $I = K$ or $K = I_1$. This concept was initially introduced to study Boolean-like rings (cf. [11]).

A $*$-ring R is called a *$*$-Boolean ring* if every element of R is a projection.

The purpose of this section is to characterize submaximal ideals of strongly nil $*$-clean rings, and $*$-Boolean rings by means of strongly nil $*$-cleanness. We begin with the following lemma.

Lemma 4.1. *Let R be strongly nil $*$-clean. Then an ideal M of R is maximal if and only if*

(1) *M is prime;*

(2) *For any $a \in R, n \geq 1$, $a^n \in M$ implies that $a \in M$.*

Proof. Suppose that M is maximal. Obviously, M is prime. Let $a \in R$ and $a^n \in M$. If $a \notin M$, $RaR + M = R$. Thus, $\overline{R}\overline{a}\overline{R} = \overline{R}$ where $\overline{R} = R/M$ and $\overline{a} = a + M$. Clearly, R is an abelian clean ring, and so it is an exchange ring by [5, Theorem 17.2.2]. This implies that R/M is an abelian exchange ring. As in the proof of [5, Proposition 17.1.9], there exists a nonzero idempotent $\overline{e} \in \overline{R}$ such that $\overline{e} \in \overline{a}\overline{R}$ and $\overline{1} - \overline{e} \in (\overline{1} - \overline{a})\overline{R}$. Since $\overline{R}\overline{e}\overline{R}$ is a nonzero ideal of simple ring $\overline{R}$, $\overline{R}\overline{e}\overline{R} = \overline{R}$. Thus $1 - e \in M$. Hence, $1 - ar \in M$ for some $r \in R$. This implies that $a^{n-1} - a^n r \in M$, and so $a^{n-1} \in M$. By iteration of this process, we see that $a \in M$, as required.

Conversely, assume that (1) and (2) hold. Assume that M is not maximal. Then we can find a maximal ideal I of R such that $M \subsetneqq I \subsetneqq R$. Choose $a \in I$ while $a \notin M$. By hypothesis, there exists a projection $e \in R$ and a nilpotent $u \in R$ such that $a = e + u$. Write $u^m = 0$. Then $u^m \in M$. By hypothesis, $u \in M$. This shows that $e \notin M$. Clearly, R is abelian. Thus $eR(1-e) \subseteq M$. As M is prime, we deduce that $1 - e \in M$. As a result, $1 - a = (1-e) - u \in M$, and so $1 = (1-a) + a \in I$. This gives a contradiction. Therefore M is maximal. □

Let R be a strongly nil $*$-clean ring, and let $x \in R$. Then there exists a unique projection $e \in R$ such that $x - e \in N(R)$. We denote e by x_P and $x - e$ by x_N.

Lemma 4.2. *Let I be an ideal of a strongly nil $*$-clean ring R, and let $x \in R$ be such that $x \notin I$. If $x_P \notin I$, then there exists a maximal ideal J of R such that $I \subseteq J$ and $x \notin J$.*

Proof. Let $\Omega = \{K \mid K \text{ is an ideal in } R, I \subseteq K, x_P \notin K\}$. Then $\Omega \neq \emptyset$. Given $K_1 \subseteq K_2 \subseteq \cdots$ in Ω, we set $Q = \bigcup_{i=1}^{\infty} K_i$. Then Q is an ideal of R. If $Q \notin \Omega$, then $x_P \in Q$, and so $x_P \in K_i$ for some i. This gives a contradiction. Thus, Ω is inductive. By using Zorn's Lemma, there exists an ideal L of R which is maximal in Ω. Let $a, b \in R$ such that $a, b \notin L$. By the maximality of L, we see that $RaR + L, RbR + L \notin \Omega$. This shows that $x_P \in (RaR + L) \cap (RbR + L)$. Hence, $x_P = x_P^2 \in RaRbR + L$. This yields that $aRb \nsubseteq L$; otherwise, $x_P \in L$, a contradiction. Hence, L is prime. Assume that L is not maximal. Then we can find a maximal ideal M of R such that $L \subsetneqq M \subsetneqq R$. Clearly, R is abelian. By the maximality, we see that $x_P \in M$, and so $1 - x_P \notin M$. This implies that $1 - x_P \notin L$. As $x_P R(1 - x_P) = 0 \subseteq L$, we have that $x_P \in L$, a contradiction. Therefore L is a maximal ideal, as asserted. □

Proposition 4.3. *Let R be strongly nil $*$-clean. Then the intersection of two maximal ideals is submaximal and it is covered by each of these two maximal ideals. Further, there is no other maximal ideals containing it.*

Proof. Let I_1 and I_2 be two distinct maximal ideals of R. Then $I_1 \cap I_2 \subsetneqq I_1$. Suppose $I_1 \cap I_2 \subseteq L \subsetneqq I_1$. Then we can find some $x \in I_1$ while $x \notin L$. Write $x_N^n = 0$. Then $x_N^n \in I_1$. In light of Lemma 4.1, $x_N \in I_1$. Likewise, $x_N \in I_2$. Thus, $x_N \in I_1 \cap I_2 \subseteq L$. This shows that $x_P \notin L$. By virtue of Lemma 4.2, there exists a maximal ideal M of R such that $L \subseteq M$ and $x \notin M$. Hence, $I_1 \cap I_2 \subseteq M$ and $I_1 \neq M$. If $I_2 \neq M$, then $I_2 + M = R$. Write $t + y = 1$ with $t \in I_2, y \in M$. Then for any $z \in I_1$, $z = zt + zy \in I_1 \cap I_2 + M = M$, and so $I_1 = M$. This gives a contradiction. Thus $I_2 = M$, and then $L \subseteq M \subseteq I_2$. As a result, $L \subseteq I_1 \cap I_2$, and so $I_1 \cap I_2 = L$. Therefore $I_1 \cap I_2$ is a submaximal ideal of R. We claim that $I_1 \cap I_2$ is semiprime. If $K^2 \subseteq I_1 \cap I_2$, then for any $a \in K$, we see that $a^2 \in I_1 \cap I_2$. In view of Lemma 4.1, $a \in I_1 \cap I_2$. This implies that $K \subseteq I_1 \cap I_2$. Hence, $I_1 \cap I_2$ is semiprime. Therefore $I_1 \cap I_2$ is the intersection of maximal ideals containing $I_1 \cap I_2$.

Assume that K is a maximal ideal of R such that $I_1 \cap I_2 \subseteq K$. If $K \neq I_1, I_2$, then $I_1 + K = I_2 + K = R$. This implies that $I_1 \cap I_2 + K = R$, and so $K = R$, a contradiction. Thus, $K = I_1$ or $K = I_2$, and so the proof is completed. □

We call a local ring R *absolutely local* provided that for any $0 \neq x \in J(R)$, $J(R) = RxR$.

Corollary 4.4. *Let R be strongly nil $*$-clean, and let I be an ideal of R. Then I is a submaximal ideal if and only if R/I is Boolean with four elements or R/I is absolutely local.*

Proof. Let I be a submaximal ideal of R.

Case I. I is contained in more than one maximal ideal. Then I is contained in two distinct maximal ideals of R. Since I is submaximal, there exists a maximal ideal L of R such that I is covered by L. Thus, we have a maximal ideal L' such that $L' \neq L$ and $I \subsetneqq L'$. Hence, $I \subseteq L \cap L' \subseteq L$. Clearly, $L \cap L' \neq L$ as $L + L' = R$, and so $I = L \cap L'$. In view of Proposition 4.3, there is no maximal ideal containing I except for L and L'. This shows that R/I has only two maximal ideals covering $\{0 + I\}$. For any $a \in R$, it follows from Proposition 2.4 that $a - a^2 \in R$ is nilpotent. Write $(a - a^2)^n = 0$. Then $(a - a^2)^n \in L$. According to Lemma 4.1, $a - a^2 \in L$. Likewise, $a - a^2 \in L'$. Thus, $a - a^2 \in L \cap L'$, and so $a - a^2 \in I$. This shows that R/I is Boolean. Therefore R/I is Boolean with four elements.

Case II. Suppose that I is contained in only one maximal ideal L of R. Then R/I has only one maximal ideal L/I. Clearly, R is an abelian exchange ring, and then so is R/I. Let $\overline{e} \in R/I$ be a nontrivial idempotent. Then $I \subseteq I + ReR \subseteq L$ or $I + ReR = R$. Likewise, $I \subseteq I + R(1-e)R \subseteq L$ or $I + R(1-e)R = R$. This shows that $I + ReR = R$ or $I + R(1-e)R = R$ Thus, $(R/I)(e+I)(R/I) = R/I$ or $(R/I)(1-e+I)(R/I) = R/I$, a contradiction. Therefore all idempotents in R/I are trivial. It follows from [5, Lemma 17.2.1] that R/I is local. For any $\overline{0} \neq \overline{x} \in L/I$, we see that $0 \neq I \subseteq RxR \subseteq L$. As I is submaximal, we deduce that $L = RxR$. Therefore R is absolutely local.

Conversely, assume that R/I is Boolean with four elements. Then R/I has precisely two maximal ideals covering $\{0 + I\}$, and so R has precisely two maximal ideals covering I. Thus, we have a maximal ideal L such that $I \subsetneqq L$. If $I \subseteq K \subseteq L$. Then $K = I$ or K is maximal, and so $K = L$. Consequently, I is submaximal. Assume that R/I is absolutely local. Then R/I has a unique maximal ideal L/I. Hence, L is a maximal ideal of R such that $I \subsetneqq L$. Assume that $I \subsetneqq K \subseteq L$. Choose $a \in K$ while $a \notin I$. Then $L = RaR \subseteq K$, and so $K = L$. Therefore I is submaximal, as required. □

Corollary 4.5. *Let R be strongly nil $*$-clean. If I_1 and I_2 are distinct maximal ideals of R, then $R/(I_1 \cap I_2)$ is Boolean.*

Proof. Since $I_1/(I_1 \cap I_2)$ and $I_2/(I_1 \cap I_2)$ are distinct maximal ideals, $R/(I_1 \cap I_2)$ is not local. In view of Proposition 4.3, $I_1 \cap I_2$ is a submaximal ideal of R. Therefore Corollary 4.4 yields the proof. □

Recall that an ideal I of a commutative ring R is *primary* provided that for any $x, y \in R$, $xy \in I$ implies that $x \in I$ or $y^n \in I$ for some $n \in \mathbb{N}$. Clearly, every maximal ideal of a commutative ring is primary. We end this article by giving the relation between strongly nil $*$-clean rings and $*$-Boolean rings.

Lemma 4.6. *Let R be a commutative strongly nil $*$-clean ring. Then the intersection of all primary ideals of R is zero.*

Proof. Let a be in the intersection of all primary ideal of R. Assume that $a \neq 0$. Let $\Omega = \{I \mid I \text{ is an ideal of } R \text{ such that } a \notin I\}$. Then $\Omega \neq \emptyset$ as $0 \in \Omega$. Given any ideals $I_1 \subseteq I_2 \subseteq \cdots$ in Ω, we set $M = \bigcup\limits_{i=1}^{\infty} I_i$. Then $M \in \Omega$. Thus, Ω is inductive. By using Zorn's Lemma, we can find an ideal Q which is maximal in Ω. It will suffice to show that Q is primary. If not, we can find some $x, y \in R$ such that $xy \in Q$, but $x \notin Q$ and $y^n \notin Q$ for any $n \in \mathbb{N}$. This shows that $a \in Q + (x)$, and so $a = b + cx$ for some $b \in Q, c \in R$. Since R is strongly nil $*$-clean, it follows from Theorem 2.7 that there are some distinct $k, l \in \mathbb{N}$ such that $y^k = y^l$. Say $k > l$. Then $y^l = y^k = y^{l+1}y^{k-l-1} = y^l y y^{k-l-1} = y^{l+2}y^{2(k-l-1)} = \cdots = y^{2l}y^{l(k-l-1)}$. Hence, $y^{l(k-l)} = y^l(y^{l(k-l-1)}) = y^{2l}y^{2l(k-l-1)} = \left(y^{l(k-l)}\right)^2$. Choose $s = l(k-l)$. Then y^s is an idempotent. Write $y = y_P + y_N$. Then $y^s - y_P = (y_P + y_N)^s - y_P = y_N\left(sy_P + \cdots + y_N^{s-1}\right) \in N(R)$. As R is a commutative ring, we see that $(y^s - y_P)^3 = y^s - y_P$. This implies that $y^s = y_P$. Since $xy \in Q$, we have that $xy^s \in Q$, and so $xy_P \in Q$. It follows from $a = b + cx$ that $ay_P = by_P + cxy_P \in Q$. Clearly, $y^s \notin Q$, and so $a \in Q + (y_P)$. Write $a = d + ry_P$ for some $d \in Q, r \in R$. We see that $ay_P = dy_P + ry_P$, and so $ry_P \in Q$. This implies that $a \in Q$, a contradiction. Therefore Q is primary, a contradiction. Consequently, the intersection of all primary ideals of R is zero. □

Theorem 4.7. *Let R be a $*$-ring. Then R is a $*$-Boolean ring if and only if*

(1) *R is commutative;*

(2) *Every primary ideal of R is maximal;*

(3) *R is strongly nil $*$-clean.*

Proof. Suppose that R is a $*$-Boolean ring. Clearly, R is a commutative strongly nil $*$-clean ring. Let I be a primary ideal of R. If I is not maximal, then there exists a maximal ideal M such that $I \subsetneqq M \subsetneqq R$. Choose $x \in M$ while $x \notin I$. As x is an idempotent, we see that $xR(1-x) \subseteq I$, and so $(1-x)^m \in I \subset M$ for some $m \in \mathbb{N}$. Thus, $1-x \in M$. This implies that $1 = x + (1-x) \in M$, a contradiction. Therefore I is maximal, as required.

Conversely, assume that $(1), (2)$ and (3) hold. Clearly, every maximal ideal of R is primary, and so $J(R) = \bigcap\{P \mid P \text{ is primary}\}$. In view of Lemma 4.6, $J(R) = 0$. Hence every element is a projection i.e. R is $*$-Boolean. □

Corollary 4.8. *A ring R is a Boolean ring if and only if*

(1) *R is commutative;*

(2) *Every primary ideal of R is maximal;*

(3) *R is strongly nil clean.*

Proof. Choose the involution as the identity. Then the result follows from Theorem 4.7. □

Acknowledgment: The authors are grateful to the referee for his/her careful reading and valuable comments on this manuscript. This research was supported by the Scientific and Technological Research Council of Turkey (TUBITAK-2221 Visiting Scientists Fellowship Programme). Huanyin Chen is thankful for the support by the Natural Science Foundation of Zhejiang Province (No. LY17A010018).

References

[1] A. Badawi, On abelian π–regular rings, Comm. Algebra 25(4) (1997) 1009–1021.
[2] S. K. Berberian, Baer $*$–Rings, Springer-Verlag, Heidelberg, London, New York, 2011.
[3] M. Chacron, On a theorem of Herstein, Canad. J. Math. 21 (1969) 1348–1353.
[4] H. Chen, On strongly J–clean rings, Comm. Algebra 38(10) (2010) 3790–3804.
[5] H. Chen, Rings Related Stable Range Conditions, Series in Algebra 11, World Scientific, Hackensack, NJ, 2011.
[6] H. Chen, A. Harmancı A. Ç. Özcan, Strongly J–clean rings with involutions, Ring theory and its applications, Contemp. Math. 609 (2014) 33–44.
[7] A. J. Diesl, Nil clean rings, J. Algebra 383 (2013) 197–211.
[8] A. L. Foster, The theory of Boolean–like rings, Trans. Amer. Math. Soc. 59 (1946) 166–187.
[9] Y. Hirano, H. Tominaga, A. Yaqub, On rings in which every element is uniquely expressable as a sum of a nilpotent element and a certain potent element, Math. J. Okayama Univ. 30 (1988) 33–40.
[10] C. Li, Y. Zhou, On strongly $*$–clean rings, J. Algebra Appl. 10(6) (2011) 1363–1370.

[11] V. Swaminathan, Submaximal ideals in a Boolean–like rings, Math. Sem. Notes Kobe Univ. 10(2) (1982) 529–542.
[12] L. Vaš, ∗–Clean rings; some clean and almost clean Baer ∗–rings and von Neumann algebras, J. Algebra 324(12) (2010) 3388–3400.