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A constructive approach to minimal free resolutions of path ideals of trees

Research Article

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Abstract: For a rooted tree Γ, we consider path ideals of Γ, which are ideals that are generated by all directed paths of a fixed length in Γ. In this paper, we provide a combinatorial description of the minimal free resolution of these path ideals. In particular, we provide a class of subforests of Γ that are in one-to-one correspondence with the multi-graded Betti numbers of the path ideal as well as providing a method for determining the projective dimension and the Castelnuovo-Mumford regularity of a given path ideal.

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Keywords: Betti numbers, Path ideals, Rooted trees, Monomial ideals

1. Introduction

Monomial ideals have been studied extensively in the literature and many applications of monomial ideals have been explored. In this paper, we consider a specific class of monomial ideals, known as *path ideals*. Path ideals are a type of squarefree monomial ideals that are generated in a fixed degree. *Edge ideals*, a specific class of path ideals, were first studied by Conca and De Negri in [4]. Given a graph $\Gamma = (V, E)$ having vertex set V and edge set E, we can form the path ideal of length $(t-1)$ associated to Γ by considering the ideal generated by the monomials corresponding to all $(t-1)$ length paths in Γ. If $V = \{x_1, \dots, x_n\}$, this ideal is considered in the polynomial ring $R := k[x_1, \dots, x_n]$, where k is a field.

In [10], Nagel and Reiner showed in Proposition 6.1 that the Betti numbers of edge ideals of arbitrary graphs can be as complicated as desired. In particular, this proposition implies that the Betti numbers of the edge ideal of an arbitrary graph can depend on the choice of the field, k. For the case of path ideals of directed, rooted trees Bouchat, Há, and O'Keefe showed in Theorem 2.7 of [3] that the Betti

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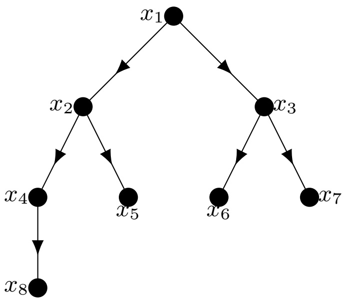


$$\begin{aligned} I_2(\Gamma) &= (x_1x_2, x_2x_4, x_4x_8, x_2x_5, x_1x_3, x_3x_6, x_3x_7) \\ I_3(\Gamma) &= (x_1x_2x_4, x_2x_4x_8, x_1x_2x_5, x_1x_3x_6, x_1x_3x_7) \\ I_4(\Gamma) &= (x_1x_2x_4x_8) \end{aligned}$$

Figure 1. An example of a graph, Γ, and its associated path ideals

numbers of a path ideal associated to a rooted tree are independent of the choice of the field, k. In this paper, we will restrict our focus to the study of path ideals of *rooted trees*. Recall that a *tree* is a simple, connected graph containing no loops or multi-edges. Then a *rooted tree* is a tree together with a fixed vertex, called the *root*. It is natural to consider a rooted tree as a directed graph in which all edges are assigned the direction going away from the root. It should be noted that x_i will denote both the vertex in the graph Γ as well as the monomial in the polynomial ring R.

Definition 1.1. *Given a directed, rooted tree* Γ *and* $t \geq 2$, *the* path ideal *of length* $(t-1)$ *of* Γ *is:*

$$I_t(\Gamma) := (x_{i_1}x_{i_2}\cdots x_{i_t} \mid \{x_{i_1}, x_{i_2}, \ldots, x_{i_t}\} \textit{ forms a directed path in } \Gamma).$$

Thus, for a given directed, rooted tree Γ there can be more than one path ideal associated to Γ as illustrated in Figure 1.

For a given tree Γ, we can successively remove any leaves that occur at level less than $(t-1)$ when considering the path ideal $I_t(\Gamma)$, as these vertices cannot contribute to a minimal generator in $I_t(\Gamma)$. A tree that has had successive removal of all leaves ocurring at level less than $(t-1)$ will be said to be in *clean form*.

The minimal free resolutions of path ideals of trees have been studied by many authors, including Há and Van Tuyl in [7], Katzman in [8], and Kummini in [9]. The basic tools and decompositions that will be used in this paper were introduced by Faridi and Alilooee in [1]. The aim of this paper is to give a constructive description of the multi-graded Betti numbers for path ideals of rooted trees. This constructive description of the Betti numbers corresponding to a path ideal of a rooted tree will also provide a method to compute the projective dimension as well as the Castelnuovo-Mumford regularity.

2. Basic definitions

We begin with the background concepts from commutative algebra.

Let M be a finitely generated graded S-module. Associated to M is a *minimal free resolution*, which is of the form

$$0 \to \bigoplus_{\mathbf{a}} S(-\mathbf{a})^{\beta_{p,\mathbf{a}}(M)} \xrightarrow{\delta_p} \bigoplus_{\mathbf{a}} S(-\mathbf{a})^{\beta_{p-1,\mathbf{a}}(M)} \xrightarrow{\delta_{p-1}} \cdots \xrightarrow{\delta_1} \bigoplus_{\mathbf{a}} S(-\mathbf{a})^{\beta_{0,\mathbf{a}}(M)} \to M \to 0$$

where the maps δ_i are exact and where $S(-\mathbf{a})$ denotes the translation of S obtained by shifting the degree of elements of S by $\mathbf{a} \in (\mathbb{N} \cup \{0\})^n$. The numbers $\beta_{i,\mathbf{a}}(M)$ are called the *multi-graded Betti numbers* (or $\mathbb{N}^n$*-graded Betti numbers*) of M, and they correspond to the number of minimal generators of degree $\mathbf{a}$ occurring in the i^{th}-syzygy module of M. It should be noted that the *graded Betti numbers* (or $\mathbb{N}$*-graded Betti numbers*) of M can be defined as $\beta_{i,j}(M) := \bigoplus_{a_1+\cdots+a_n=j} \beta_{i,\mathbf{a}}(M)$ where $\mathbf{a} = (a_1, \ldots, a_n)$.

There are two invariants corresponding to the minimal free resolution of M which measure the "size" of the resolution.

Definition 2.1. *Let M be a finitely generated graded S-module.*

1. *The* projective dimension *of M, denoted* $\mathrm{pd}(M)$, *is the length of the minimal free resolution associated to M.*
2. *The* Castelnuovo-Mumford regularity *(or* regularity*), denoted* $\mathrm{reg}(M)$, *is*

$$\mathrm{reg}(M) := \max\{j - i \mid \beta_{i,j}(M) \neq 0\}.$$

We also need the following definitions from simplicial topology.

Definition 2.2.

1. *An abstract* simplicial complex, *Δ, on a vertex set $\mathcal{X} = \{x_1, \dots, x_n\}$ is a collection of subsets of $\mathcal{X}$ satisfying:*

   (a) $\{x_i\} \in \Delta$ for all i, and
   (b) $F \in \Delta$, $G \subset F \Longrightarrow G \in \Delta$.

   The elements of Δ are called faces *of Δ, and the maximal faces (under inclusion) are called* facets *of Δ. The simplicial complex Δ with faces $F_1, \dots, F_s$ will be denoted by $\langle F_1, \dots, F_s \rangle$.*
2. *If Δ is an abstract simplicial complex, then a face $\tau \in \Delta$ is called a* free face *if it is contained in a unique facet of Δ.*
3. *For any $\mathcal{Y} \subseteq \mathcal{X}$, an* induced subcollection *of Δ on $\mathcal{Y}$, denoted by $\Delta_{\mathcal{Y}}$, is the simplicial complex whose vertex set is a subset of $\mathcal{Y}$ and whose facet set is given by*

$$\{F \mid F \subseteq \mathcal{Y} \text{ and } F \text{ is a facet of } \Delta\}.$$

4. *If F is a face of $\Delta = \langle F_1, \dots, F_s \rangle$, the* complement *of F in Δ is given by*

$$F^c_{\mathcal{X}} = \mathcal{X} \setminus F$$

   and

$$\Delta^c_{\mathcal{X}} = \langle (F_1)^c_{\mathcal{X}}, \dots, (F_s)^c_{\mathcal{X}} \rangle.$$

It should be noted that if $F_1, \dots, F_s$ are facets in Definition 2.2(1), then $\langle F_1, \dots F_s \rangle$ is a minimal representation of Δ. In particular, the *complementary complex* $\Delta^c_{\mathcal{X}}$, described in the fourth part of Definition 2.2, is heavily utilized within this paper.

Definition 2.3.

1. *Let Δ be a simplicial complex with vertex set $x_1, \dots, x_n$. Then the* facet ideal *of Δ is defined as*

$$I(\Delta) := \left(\prod_{x \in F} x \mid F \text{ is a facet of } \Delta \right).$$

   Moreover, I is an ideal in the polynomial ring $R := k[x_1, \dots, x_n]$.
2. *Let I be an ideal in $R := k[x_1, \dots, x_n]$ minimally generated by square-free monomials $m_1, \dots, m_s$. The* facet complex *$\Delta(I)$ associated to I has vertex set $\{x_1, \dots, x_n\}$ and is defined by*

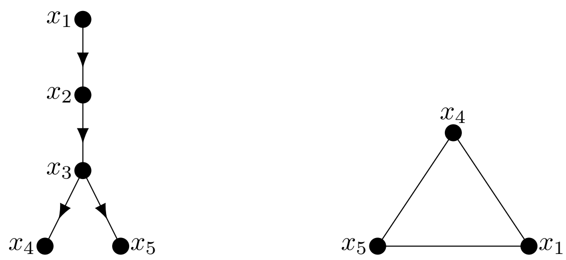
$$\Delta(I) := \langle F_1, \dots, F_s \rangle$$

   where $F_i = \{x_j \mid x_j | m_i \ , 1 \leq j \leq n\}$ for $1 \leq i \leq s$.

The above constructions provide a one-to-one correspondence between simplicial complexes on the vertex set $\{x_1, \ldots, x_n\}$ and the square-free monomial ideals in $R := k[x_1, \ldots, x_n]$. We illustrate this construction with the following example.

Example 2.4. *Consider the path ideal $I_3(\Gamma) = (x_1x_2x_3, x_2x_3x_4, x_2x_3x_5)$ associated to the tree, Γ, pictured below. Then the facet complex associated to Γ with respect to $t = 3$ is*

$$\Gamma^c_{\mathrm{Vert}(\Gamma)} = \langle (x_1x_2x_3)^c_{\mathrm{Vert}(\Gamma)}, (x_2x_3x_4)^c_{\mathrm{Vert}(\Gamma)}, (x_2x_3x_5)^c_{\mathrm{Vert}(\Gamma)} \rangle = \langle x_4x_5, x_1x_5, x_1x_4 \rangle$$



Γ $\qquad\qquad$ $\Gamma^c_{\mathrm{Vert}(\Gamma)}$

The dimension of the reduced homology group of the complementary complex of a path ideal is instrumental in determining the Betti numbers of the corresponding minimal free resolution. We will use a corollary of Theorem 2.8 of Alilooee and Faridi in [1] to help determine the multi-graded Betti numbers for path ideals of rooted trees. This corollary, found below, appears in [2] as Corollary 2.11.

Corollary 2.5. *Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k, and let I be a pure, squarefree monomial ideal in S. Then the multi-graded Betti numbers of I are given by*

$$\beta_{i,\boldsymbol{a}}(I) = \dim_k \widetilde{H}_{i-1}\left(\Gamma^c_{\mathrm{Vert}(\Gamma)}\right)$$

for $i \geq 1$ where Γ is an induced subcollection of $\Delta(I)$ with $\mathrm{Vert}(\Gamma) = \{x_i \mid a_i = 1\}$ where $\boldsymbol{a} = (a_1, \ldots, a_n)$.

It should be noted that the Betti numbers $\beta_{0,\mathbf{a}}(I)$ correspond to the minimal generating set of I. Finally, we will rely upon Lemma 4.1 in [1] of Alilooee and Faridi, which is a tool that allows us to determine the i^{th} reduced homology group of the complementary complex using the $(i-1)^{\text{st}}$ reduced homology group of a smaller complex, which we call the *deleted complementary complex*, that is,

Lemma 2.6 ([Lemma 4.1, Alilooe and Faridi])**.** *Suppose Γ is a tree generated by the paths $P_1, P_2, \ldots, P_k$ and suppose $P_1 \cap (P_2 \cup P_3 \cup \cdots \cup P_k) \neq \emptyset$. Then*

$$\dim_k \widetilde{H}_i\left(\langle P_1^c, P_2^c, \ldots, P_k^c \rangle\right) = \dim_k \widetilde{H}_{i-1}\left(\langle (P_2)^c_{\mathrm{Vert}(\Gamma)\setminus P_1}, (P_3)^c_{\mathrm{Vert}(\Gamma)\setminus P_1}, \ldots, (P_k)^c_{\mathrm{Vert}(\Gamma)\setminus P_1} \rangle\right).$$

It should further be noted that in Corollary 6.1 of [5], it is shown that the independence complex of graph forests are simple-homotopy equivalent to a vertex or to a sphere. Thus by Corollary 2.5, the multi-graded Betti numbers of path ideals of directed, rooted trees correspond via simple-homotopy equivalence to a vertex or to a sphere. We conclude this section with some graph theoretic definitions before stating our three line graph constructions on directed, rooted trees.

Definition 2.7. *Let Γ be a rooted tree with root x and vertex set $\mathrm{Vert}(\Gamma)$, and let $y \in \mathrm{Vert}(\Gamma)$.*

1. *The* level *of y, denoted* $\mathrm{level}(y)$*, is the length of the unique path in Γ from x to y. The* height *of Γ is the maximal level over all vertices in Γ.*

2. *The* parent *vertex of the non-root vertex y is the unique vertex z such that yz forms an edge in Γ and* $\mathrm{level}(z) = \mathrm{level}(y) - 1$.

3. *A subgraph H of Γ is called an* induced subgraph *if for every pair of vertices $x, y \in H$ the following condition holds: if $\{x, y\}$ is an edge of Γ, then it is also an edge of H.*

Now we can define the essential constructions for this paper. Starting with a tree Γ which is composed of generating paths of length $(t-1)$ in the set $\{P_1, P_2, \dots, P_k\}$, we assume $\Gamma^c_{\mathrm{Vert}(\Gamma)}$ is not contractible, that is, there exists an $i \geq 0$ for which $\dim_k \widetilde{H}_i\left(\Gamma^c_{\mathrm{Vert}(\Gamma)}\right) = 1$. We create a new tree Γ' such that all edges and vertices of Γ are also contained in Γ' as follows.

Definition 2.8. *Let Γ be a tree with vertex set* $\mathrm{Vert}(\Gamma)$*and edge set $E(\Gamma)$ and assume the vertex set $\{y_1, y_2, \dots, y_{t+1}\} \cap \mathrm{Vert}(\Gamma) = \emptyset$.*

1. *Let Γ' be the tree whose vertex set is $\mathrm{Vert}(\Gamma') = \mathrm{Vert}(\Gamma) \cup \{y_1, y_2, \dots, y_{t+1}\}$ and whose edge set is $E(\Gamma') = E(\Gamma) \cup \{(x_0, y_{t+1}), (y_{t+1}, y_t), (y_t, y_{t-1}), \dots, (y_2, y_1)\}$ for some vertex $x_0 \in \mathrm{Vert}(\Gamma)$. We say Γ' is a **glue** of Γ.*
2. *Let Γ' be the tree whose vertex set is $\mathrm{Vert}(\Gamma') = \mathrm{Vert}(\Gamma) \cup \{y_1, y_2, \dots, y_n\}$ for some $1 \leq n \leq t$ and whose edge set is $E(\Gamma') = E(\Gamma) \cup \{(x_0, y_n), (y_n, y_{n-1}), (y_{n-1}, y_{n-2}), \dots, (y_2, y_1)\}$ for some vertex $x_0 \in \mathrm{Vert}(\Gamma)$. If $\mathrm{level}(y_1) = \mathrm{level}(x_\ell)$ for at least one leaf vertex $x_\ell \in \Gamma$ which lies on a directed path from x_0, we say Γ' is a **split** of Γ.*
3. *Let Γ' be the tree whose vertex set is $\mathrm{Vert}(\Gamma') = \mathrm{Vert}(\Gamma) \cup \{y_1, y_2, \dots, y_n\}$ for some $1 \leq n \leq t$ and whose edge set is $E(\Gamma') = E(\Gamma) \cup \{(x_0, y_n), (y_n, y_{n-1}), (y_{n-1}, y_{n-2}), \dots, (y_2, y_1)\}$ for some vertex $x_0 \in \mathrm{Vert}(\Gamma)$. If $\mathrm{level}(y_1) \neq \mathrm{level}(x_\ell)$ for at least one leaf vertex $x_\ell \in \Gamma$ which lies on a directed path from x_0, we say Γ' is a **contraction** of Γ.*

Remark 2.9. *We will also use the terms glue and split as constructions on the empty graph. Consider $t \geq 2$. Starting with the empty graph, a split will result in the line graph having exactly t vertices, and a glue from the empty graph will result in the line graph having exactly $t+1$ vertices. It should be noted that adding less than t new vertices to the empty graph will result in a graph having no minimal generators for the ideal $I_t(\Gamma)$.*

Definition 2.8 along with Lemma 2.6 will allow us to begin with a tree whose complementary complex is homotopic to a sphere in some dimension, adjoin a line graph, and consequently determine the homotopy type of the resulting complementary simplicial complex of the modified tree. In particular, we will show that these three cases, glue, split, and contraction, are the only needed constructions and consequently the new simplicial complex will be homotopic to a sphere in either one or two dimensions higher, or it will be contractible.

3. Constructions and Betti numbers

In this section, we will explore how a glue, split, and contraction of a directed graph Γ will determine the non-zero Betti number of top dimension in the corresponding minimal free resolution of the path ideal of the newly constructed graph, Γ' and consequently will determine the projective dimension.

When we create a new tree, Γ', from Γ by adding n new vertices $\{y_1, y_2, \dots, y_n\}$ through a glue, split, or contraction, at most n new generating paths of length $(t-1)$ will be added to the path ideal of $I_t(\Gamma)$; depending on the level of the vertex y_1. Let E_i be the unique directed path of length $(t-1)$ which terminates in vertex y_i for $1 \leq i \leq j$ where $j = \min\{\mathrm{level}(y_1) - t + 2, n\}$. The number j of new generating paths depends on the level of x_0 in the tree Γ and the number of vertices added, as there may not exist a path of length $(t-1)$ which terminates in vertex y_i. Thus Γ' is generated by the paths $\{P_1, \dots, P_k, E_1, \dots, E_j\}$.

In each of the following cases we consider the deleted complementary complex that was described in Section 2. Since E_1 always contains a leaf of Γ', the conditions of Lemma 2.6 are satisfied and

$$\Gamma'^c_{\mathrm{Vert}(\Gamma')\setminus E_1} = \langle (P_1)^c_{\mathrm{Vert}(\Gamma')\setminus E_1}, \dots, (P_k)^c_{\mathrm{Vert}(\Gamma')\setminus E_1}, (E_2)^c_{\mathrm{Vert}(\Gamma')\setminus E_1}, \dots, (E_j)^c_{\mathrm{Vert}(\Gamma')\setminus E_1} \rangle.$$

We begin with the case of a glue.

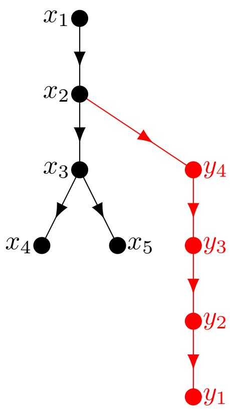
Proposition 3.1. *Let Γ be a tree in clean form with respect to $t \geq 2$ such that there exists $i \geq 0$ for which $\beta_{i,\boldsymbol{a}}(I_t(\Gamma)) = 1$, where $\boldsymbol{a} = (a_1, \dots, a_m)$ and $a_j = 1$ if and only if $x_j \in \Gamma$. Let Γ' be a glue of Γ, then*

$$\beta_{i+2,a'}(I_t(\Gamma')) = 1$$

where $\boldsymbol{a}' = (a_1, \dots, a_{m+t+1})$ and $a_j = 1$ if and only if $x_j \in \Gamma'$.

Before the proof, we illustrate the deletion methods of Lemma 2.6 in the case of a glue with the following example.

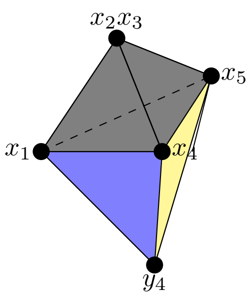
Example 3.2. *Consider the tree Γ_1, occurring as a glue of Γ:*



Then using the deletion method of Lemma 2.6, we set $E_1 = \{y_1, y_2, y_3\}$, $E_2 = \{y_2, y_3, y_4\}$, $E_3 = \{x_2, y_3, y_4\}$, $E_4 = \{x_1, x_2, y_4\}$, $E_5 = \{x_1, x_2, x_3\}$, $E_6 = \{x_2, x_3, x_4\}$, and $E_7 = \{x_2, x_3, x_5\}$. It follows that:

$$\begin{aligned} \dim_k \widetilde{H}_i\left((\Gamma_1)^c_{\mathrm{Vert}(\Gamma_1)}\right) &= \dim_k \widetilde{H}_{i-1}\langle (E_2)^c_{\mathrm{Vert}(\Gamma_1)\setminus E_1}, \dots, (E_7)^c_{\mathrm{Vert}(\Gamma_1)\setminus E_1}\rangle \\ &= \dim_k \widetilde{H}_{i-1}\langle x_1x_2x_3x_4x_5, x_1x_3x_4x_5, x_3x_4x_5, x_4x_5y_4, x_1x_5y_4, x_1x_4y_4\rangle \\ &= \dim_k \widetilde{H}_{i-1}\langle x_1x_2x_3x_4x_5, x_4x_5y_4, x_1x_5y_4, x_1x_4y_4\rangle \end{aligned}$$

After identifying the vertices x_2 and x_3, the simplicial complex can be visualized as,



where the tetrahedron labeled by $x_1x_2x_3x_4x_5$ is solid, but the tetrahedron labeled by $x_1x_4x_5y_4$ is hollow. Thus $\dim_k \widetilde{H_2}(\langle x_1x_2x_3x_4x_5, x_4x_5y_4, x_1x_5y_4, x_1x_4y_4\rangle) = 1$, and thus, by Lemma 2.6, we have $\dim_k \widetilde{H_3}\left((\Gamma_1)^c_{\mathrm{Vert}(\Gamma_1)}\right) = 1$.

Proof of Proposition 3.1. Let Γ' be a glue of the tree Γ in clean form with respect to $t \geq 2$ such that there exists $i \geq 0$ for which $\beta_{i,\mathbf{a}}(I_t(\Gamma)) = 1$, where $\mathbf{a} = (a_1, \ldots, a_m)$ and $a_i = 1$ if and only if $x_i \in \Gamma$. Further, let $E_1, E_2, \ldots, E_j$ be the new generating paths of length $t-1$ that are in Γ' but not in Γ, where E_1 is the directed path $E_1 = y_t \ldots y_2 y_1$. To proceed, we will delete the path E_1 and apply Lemma 2.6.

First, the sole vertex in the set $\mathrm{Vert}(E_2) \setminus \mathrm{Vert}(E_1)$ is y_n, so $\langle (E_2)^c_{\mathrm{Vert}(\Gamma')\setminus E_1} \rangle = \langle (y_n)^c_{\mathrm{Vert}(\Gamma'\setminus E_1)} \rangle$. Further y_n is a vertex contained in every generating path $E_3, E_4, \ldots E_j$, thus $\langle (y_n)^c_{\mathrm{Vert}(\Gamma'\setminus E_1)} \rangle$ contains $\langle (E_i)^c_{\mathrm{Vert}(\Gamma')\setminus E_1} \rangle$ for all $3 \leq i \leq j$. Therefore,

$$\Gamma'^c_{\mathrm{Vert}(\Gamma')\setminus E_1} = \langle (P_1)^c_{\mathrm{Vert}(\Gamma')\setminus E_1}, \ldots, (P_k)^c_{\mathrm{Vert}(\Gamma')\setminus E_1}, (y_n)^c_{\mathrm{Vert}(\Gamma'\setminus E_1)} \rangle.$$

We observe, for all $1 \leq i \leq k$, the complex $\langle (P_i)^c_{\mathrm{Vert}(\Gamma'\setminus E_1)} \rangle$ is equal to the simplicial join $\langle (P_i)^c_{\mathrm{Vert}(\Gamma)} * y_n \rangle$, and thus y_n must be a vertex of the simplex $\langle (P_i)^c_{\mathrm{Vert}(\Gamma')\setminus E_1} \rangle$. Therefore the complex $\langle (P_1)^c_{\mathrm{Vert}(\Gamma')\setminus E_1}, \ldots, (P_k)^c_{\mathrm{Vert}(\Gamma')\setminus E_1} \rangle$ is exactly the cone of y_n over $\Gamma^c_{\mathrm{Vert}(\Gamma)}$. In particular, because of our assumption of a non-zero Betti number in dimension $i \geq 0$, there exists a union of faces of $\Gamma^c_{\mathrm{Vert}(\Gamma)}$ that is homotopic to an i-dimensional sphere, and thus the simplicial complex $\langle (P_1)^c_{\mathrm{Vert}(\Gamma')\setminus E_1}, \ldots, (P_k)^c_{\mathrm{Vert}(\Gamma')\setminus E_1} \rangle$ is homotopic to a cone over an i-dimensional sphere. The complex $\langle (y_n)^c_{\mathrm{Vert}(\Gamma'\setminus E_1)} \rangle$ is the simplex whose vertices are all the vertices in $\mathrm{Vert}(\Gamma)$. Any vertex in $\mathrm{Vert}(\Gamma)$ that in not contained in the resulting i-dimensional sphere is a free face of the complex $\Gamma^c_{\mathrm{Vert}(\Gamma)}$ as well as of the complex $\Gamma'^c_{\mathrm{Vert}(\Gamma')\setminus E_1}$, and hence can be collapsed into the face not containing the vertex y_n. In particular it can be retracted to the face whose boundary is contained in the union of faces of $\Gamma^c_{\mathrm{Vert}(\Gamma)}$ that is homotopic to an i-dimensional sphere. Therefore $\Gamma'^c_{\mathrm{Vert}(\Gamma')\setminus E_1}$ is homotopic to the cone over an open i-dimensional sphere which has been filled with the simplex $\langle (y_n)^c_{\mathrm{Vert}(\Gamma'\setminus E_1)} \rangle$, thus creating a sphere of dimension i.

We now have

$$\dim_k \widetilde{H}_{i+1}((\Gamma')^c_{\mathrm{Vert}(\Gamma')}) = \dim_k \widetilde{H}_i((\Gamma')^c_{\mathrm{Vert}(\Gamma')\setminus E_1}) = 1.$$

Then the complex $(\Gamma')^c_{\mathrm{Vert}(\Gamma')}$ is homotopic to a sphere of dimension $i+1$; that is, of dimension two more than the sphere describing the complementary complex of Γ and further

$$\beta_{i+2,a'}(I_t(\Gamma')) = 1$$

where $\mathbf{a}' = (a_1, \ldots, a_{m'})$ and $a_i = 1$ if and only if $x_i \in \Gamma'$. □

Next, we consider the split construction from a given tree.

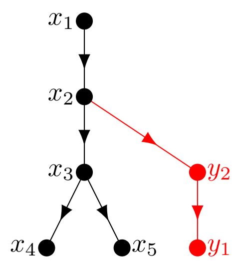
Proposition 3.3. *Let Γ be a tree in clean form with respect to $t \geq 2$ such that there exists $i \geq 0$ for which $\beta_{i,\boldsymbol{a}}(I_t(\Gamma)) = 1$, where $\boldsymbol{a} = (a_1, \ldots, a_m)$ and $a_j = 1$ if and only if $x_j \in \Gamma$. Let Γ' be a split of Γ, then*

$$\beta_{i+1,a'}(I_t(\Gamma')) = 1$$

where $\boldsymbol{a}' = (a_1, \ldots, a_{m'})$ and $a_j = 1$ if and only if $x_j \in \Gamma'$.

As before, we illustrate the deletion methods of Lemma 2.6 in the case of a split with the following example before providing the proof.

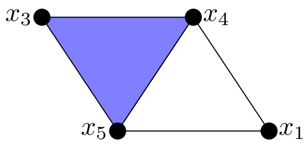
Example 3.4. *Consider the tree Γ_2, occuring as a split of Γ:*



Using the deletion method, we set $E_1 = \{x_2, y_1, y_2\}$, $E_2 = \{x_1, x_2, y_2\}$, $E_3 = \{x_1, x_2, x_3\}$, $E_4 = \{x_2, x_3, x_4\}$, *and* $E_5 = \{x_2, x_3, x_5\}$. *It follows that*

$$\begin{aligned} \dim_k \widetilde{H}_i \left((\Gamma_2)^c_{\mathrm{Vert}(\Gamma_2)} \right) &= \dim_k \widetilde{H}_{i-1} \langle (E_2)^c_{\mathrm{Vert}(\Gamma_2)\setminus E_1}, \ldots, (E_5)^c_{\mathrm{Vert}(\Gamma_2)\setminus E_1} \rangle \\ &= \dim_k \widetilde{H}_{i-1} \langle x_3x_4x_5, x_4x_5, x_1x_5, x_1x_4 \rangle \\ &= \dim_k \widetilde{H}_{i-1} \langle x_3x_4x_5, x_1x_5, x_1x_4 \rangle. \end{aligned}$$

The complex can be visualized as:



Hence $\dim_k \widetilde{H_1}(\langle x_3x_4x_5, x_1x_5, x_1x_4 \rangle) = 1$, *and by Lemma 2.6* $\dim_k \widetilde{H_2} \left((\Gamma_2)^c_{\mathrm{Vert}(\Gamma_2)} \right) = 1$.

Proof of Proposition 3.3. Let Γ be a tree in clean form such that $\beta_{i,\mathbf{a}}(I_t(\Gamma)) = 1$ for some $i \geq 0$ where $\mathbf{a} = (a_1, \ldots, a_m)$ and $a_i = 1$ if and only if $x \in \mathrm{Vert}(\Gamma)$, and let Γ' be a split of Γ. Then Γ' is formed from Γ by attaching a line graph with vertices $y_n, \ldots, y_1$ where $n \leq t$ via the edge $\{x_0, y_n\}$ for some $x_0 \in \mathrm{Vert}(\Gamma)$. Further there exists a leaf vertex x_ℓ of Γ that is on a directed path from x_0 such that $\mathrm{level}(x_\ell) = \mathrm{level}(y_1)$. Let E_i be the length $t-1$ path in Γ' terminating at vertex y_i while $P_1, \ldots, P_k$ will denote the paths of length $t-1$ in Γ.

If $E_1 \cap \mathrm{Vert}\,\Gamma = \emptyset$, the result follows directly from Lemma 2.6. Suppose to the contrary, and let x be a vertex in the intersection of E_1 and $\mathrm{Vert}(\Gamma)$. We note for any $1 \leq i \leq k$, if $x \in (P_i)^c_{\mathrm{Vert}(\Gamma)}$, then $x_\ell \in (P_i)^c_{\mathrm{Vert}(\Gamma)}$. This is because $\mathrm{level}(x_\ell) - \mathrm{level}(x) \leq t$, so all paths containing x_ℓ must also contain x. Thus every face in $\Gamma^c_{\mathrm{Vert}(\Gamma)}$ containing x must also contain x_ℓ. Hence, vertex x is a free vertex in the complex $\Gamma^c_{\mathrm{Vert}(\Gamma)}$ which may be retracted onto x_ℓ, and the complex $\Gamma^c_{\mathrm{Vert}(\Gamma)}$ is collapsed onto the complex $\langle (P_1)^c_{\mathrm{Vert}\,\Gamma'\setminus E_1}, \ldots, (P_k)^c_{\mathrm{Vert}\,\Gamma'\setminus E_1} \rangle \subseteq (\Gamma')^c_{\mathrm{Vert}(\Gamma'\setminus E_1)}$. As retractions preserve homotopy, we have

$$\Gamma^c_{\mathrm{Vert}(\Gamma)} \sim \langle (P_1)^c_{\mathrm{Vert}\,\Gamma'\setminus E_1}, \ldots, (P_k)^c_{\mathrm{Vert}\,\Gamma'\setminus E_1} \rangle.$$

If Γ' only contains one additional path when compared to Γ, then the paths making up Γ' are $E_1, P_1, \ldots, P_k$. It follows that

$$\beta_{i+1,\mathbf{a}'}\left(I_t(\Gamma')\right) = \dim_k \widetilde{H}_i \left((\Gamma')^c_{\mathrm{Vert}(\Gamma')} \right) = \dim_k \widetilde{H}_{i-1} \left(\langle (P_1)^c_{\mathrm{Vert}\,\Gamma'\setminus E_1}, \ldots, (P_k)^c_{\mathrm{Vert}\,\Gamma'\setminus E_1} \rangle \right)$$

where $\mathbf{a}' = (a_1, \ldots, a_{m'})$ such that $a_i = 1$ if and only if $x_i \in \Gamma'$. It follows that

$$\beta_{i,\mathbf{a}}\left(I_t(\Gamma)\right) = \dim_k \widetilde{H}_{i-1} \left((\Gamma)^c_{\mathrm{Vert}(\Gamma)} \right) = 1$$

and hence the claim is proven.

Now assume that Γ' contains two or more new paths when compared to Γ. It follows that

$$\begin{aligned} \beta_{i+1,\mathbf{a}'}\left(I_t(\Gamma')\right) &= \dim_k \widetilde{H}_i \left((\Gamma')^c_{\mathrm{Vert}(\Gamma')} \right) \\ &= \dim_k \widetilde{H}_{i-1} \left(\langle (E_2)^c_{\mathrm{Vert}\,\Gamma'\setminus E_1}, \ldots, (E_j)^c_{\mathrm{Vert}\,\Gamma'\setminus E_1}, (P_1)^c_{\mathrm{Vert}\,\Gamma'\setminus E_1}, \ldots, (P_k)^c_{\mathrm{Vert}\,\Gamma'\setminus E_1} \rangle \right). \end{aligned}$$

Let x' denote the sole vertex in the set $\mathrm{Vert}(E_2) - \mathrm{Vert}(E_1)$. Then $(E_2)^c_{\mathrm{Vert}(\Gamma')\setminus E_1} = (x')^c_{\mathrm{Vert}(\Gamma')\setminus E_1}$. Furthermore, notice that $(E_i)^c_{\mathrm{Vert}(\Gamma')\setminus E_1} \subseteq (E_2)^c_{\mathrm{Vert}(\Gamma')\setminus E_1}$ for all $3 \leq i \leq j$. Hence,

$$\beta_{i+1,\mathbf{a}'}\left(I_t(\Gamma')\right) = \dim_k \widetilde{H}_{i-1} \left(\langle (x')^c_{\mathrm{Vert}\,\Gamma'\setminus E_1}, (P_1)^c_{\mathrm{Vert}\,\Gamma'\setminus E_1}, \ldots, (P_k)^c_{\mathrm{Vert}\,\Gamma'\setminus E_1} \rangle \right).$$

Furthermore, because the complex $\langle (P_1)^c_{\mathrm{Vert}\,\Gamma'\setminus E_1}, \ldots, (P_k)^c_{\mathrm{Vert}\,\Gamma'\setminus E_1}\rangle$ is just the complex formed by retracting the vertices in $\mathrm{Vert}(E_1) \cap \mathrm{Vert}(\Gamma)$ onto the leaf vertex x_ℓ in the complex $\Gamma^c_{\mathrm{Vert}(\Gamma)}$, it is left to check that the face $(x')^c_{\mathrm{Vert}(\Gamma')\setminus E_1}$ of $(\Gamma')^c_{\mathrm{Vert}(\Gamma'\setminus E_1)}$ does not close the i-dimensional hole that we know by assumption is in $\Gamma^c_{\mathrm{Vert}(\Gamma)}$. Suppose to the contrary, when the simplex $(x')^c_{\mathrm{Vert}(\Gamma')\setminus E_1}$ is inserted in the complex $\langle (P_1)^c_{\mathrm{Vert}\,\Gamma'\setminus E_1}, \ldots, (P_k)^c_{\mathrm{Vert}\,\Gamma'\setminus E_1}\rangle$, the newly formed complex $(\Gamma')^c_{\mathrm{Vert}(\Gamma'\setminus E_1)}$ is contractible. First observe that the height of the subtree rooted at x' is t. We first assume that Γ' has exactly two leaves, x_ℓ and y_1 on directed paths from x'. Therefore, without loss of generality, we assume the directed path from x' to x_ℓ was glued onto some subtree of Γ whose leaves are all at a level less than $\mathrm{level}(x_\ell)$. We know from the proof of Proposition 3.1 that in this case, the vertex x' is on the boundary of the complex homotopic to the i-dimensional sphere formed by the glue. Hence, x' is not a free vertex and cannot be retracted onto another vertex in the complex. Therefore, the simplex $(x')^c_{\mathrm{Vert}(\Gamma')\setminus E_1}$ cannot fill this hole and the thus the complex $(\Gamma')^c_{\mathrm{Vert}(\Gamma'\setminus E_1)}$ is not contractible. By extension, if Γ' has more than two leaves on directed paths from x', each additional split still has a face determined by the complement of x', and hence the complex cannot be contractible. $\square$

In the final proposition of this section, we consider a contraction of a given tree.

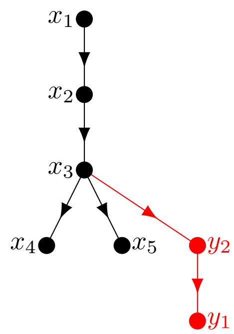
Proposition 3.5. *Let Γ be a tree in clean form with respect to $t \geq 2$ such that there exists $i \geq 0$ for which $\beta_{i,\boldsymbol{a}}(I_t(\Gamma)) = 1$, where $\boldsymbol{a} = (a_1, \ldots, a_m)$ and $a_j = 1$ if and only if $x_j \in \Gamma$. Let Γ' be a contraction of Γ, then*

$$\beta_{i,a'}(I_t(\Gamma')) = 0$$

for all i where $\boldsymbol{a'} = (a_1, \ldots, a_{m'})$ and $a_j = 1$ if and only if $x_j \in \Gamma'$.

We precede the proof with an example illustrating the deletion methods of Lemma 2.6 in the case of a contraction.

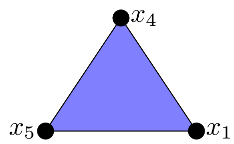
Example 3.6. *Consider the tree Γ_3, occurring as a contraction of Γ:*



Then using the deletion method, we set $E_1 = \{x_3, y_1, y_2\}$, $E_2 = \{x_2, x_3, y_2\}$, $E_3 = \{x_1, x_2, x_3\}$, $E_4 = \{x_2, x_3, x_4\}$, and $E_5 = \{x_2, x_3, x_5\}$. It follows that:

$$\begin{aligned}
\dim_k \widetilde{H}_i\left((\Gamma_3)^c_{\mathrm{Vert}(\Gamma_3)}\right) &= \dim_k \widetilde{H}_{i-1}\langle (E_2)^c_{\mathrm{Vert}(\Gamma_3)\setminus E_1}, \ldots, (E_5)^c_{\mathrm{Vert}(\Gamma_3)\setminus E_1}\rangle \\
&= \dim_k \widetilde{H}_{i-1}\langle x_1x_4x_5, x_4x_5, x_1x_5, x_1x_4\rangle \\
&= \dim_k \widetilde{H}_{i-1}\langle x_1x_4x_5\rangle
\end{aligned}$$

The complex can be visualized as:



Notice that $\dim_k \widetilde{H}_i(\langle x_1x_4x_5\rangle) = 0$ *for all* i*. Hence, by Lemma 2.6* $\beta_{i,\mathbf{a}}(I_t(\Gamma)) = 0$ *for all* i*.*

Proof of Proposition 3.5. Let Γ be a tree in clean form with respect to $t \geq 2$ such that there exists $i \geq 0$ for which $\beta_{i,\mathbf{a}}(I_t(\Gamma)) = 1$, where $\mathbf{a} = (a_1, \dots, a_m)$ and $a_i = 1$ if and only if $x_i \in \Gamma$ and let Γ' be a contraction of Γ, also in clean form. (If Γ' is not in clean form, the vertices lying on the paths of length less than $t-1$ do not contribute to the minimal free resolution and hence trivially $\beta_{i,\mathbf{a'}}(I_t(\Gamma')) = 0$ for all i where $\mathbf{a}' = (a_1, \dots, a_{m'})$ and $a_i = 1$ if and only if $x_i \in \Gamma'$.) Assume $E_1, E_2, \dots, E_j$ are the new paths of length $(t-1)$ found in Γ' but not in Γ. In this final case, we still assume $1 \leq n \leq t$, but now let $\text{level}(y_1) \neq \text{level}(x)$ for all leaf vertices x of Γ which lie on a directed path from x_0. First we look at the situation where $\text{level}(y_1) < \text{level}(x_\ell)$ for some leaf vertex x_ℓ. As before we want to apply Lemma 2.6, but this time we delete the path $P = x_t x_{t-1} \cdots x_1$ which terminates at the leaf $x_\ell \in \Gamma$ which is on a directed path from x_0. Let x_{t+1} be the unique parent vertex of the initial vertex x_t of the path P, which must exist because Γ' is in clean form. Then $x_{t+1}x_t \cdots x_2 = P_m$ was a generating path of Γ for some m, and so $\langle (P_m)^c_{\text{Vert}(\Gamma')\setminus P}\rangle = \langle (x_{t+1})^c_{\text{Vert}(\Gamma')\setminus P}\rangle$ is a part of the deleted complementary complex

$$\Gamma'^c_{\text{Vert}(\Gamma')\setminus P} = \langle (P_1)^c_{\text{Vert}(\Gamma')\setminus P}, \dots, (P_k)^c_{\text{Vert}(\Gamma')\setminus P}, (E_1)^c_{\text{Vert}(\Gamma')\setminus P}, \dots, (E_j)^c_{\text{Vert}(\Gamma')\setminus P}\rangle.$$

Any other generating paths that contain x_{t+1}, will have complementary complexes contained in the simplex $\langle (x_{t+1})^c_{\text{Vert}(\Gamma')\setminus P}\rangle$, and because $\text{level}(y_1) < \text{level}(x_1)$, this includes all of the new generators $E_1, E_2, \dots, E_j$, so

$$\Gamma'^c_{\text{Vert}(\Gamma')\setminus P} = \langle (P_1)^c_{\text{Vert}(\Gamma')\setminus P}, \dots, (P_k)^c_{\text{Vert}(\Gamma')\setminus P}\rangle.$$

Now, the vertex y_1 is a vertex of the simplex $\langle x^c_{t+1}\rangle$, but y_1 is also a vertex of every simplex generated from the original paths $P_1^c, P_2^c, \dots, P_k^c$ in Γ. Thus $(\Gamma')^c_{\text{Vert}(\Gamma')\setminus P}$ is homotopic to a cone of the vertex y_1 over some simplicial complex and hence is contractible.

On the other hand, if the level of y_1 is greater than the level of some leaf vertex x_ℓ, apply Lemma 2.6 and delete the unique path E_1 of length $t-1$ terminating at y_1. In this final situation, we must be careful of the case where x_0 is a leaf. If x_0 is not a leaf, as above, let x_{t+1} be the parent of the initial vertex of the path E_1. Proceeding as before we see

$$\Gamma'^c_{\text{Vert}(\Gamma')\setminus E_1} = \langle (P_1)^c_{\text{Vert}(\Gamma')\setminus E_1}, \dots, (P_k)^c_{\text{Vert}(\Gamma')\setminus E_1}\rangle$$

and that $\langle (P_m)^c_{\text{Vert}(\Gamma')\setminus E_1}\rangle = \langle (x_{t+1})^c_{\text{Vert}(\Gamma')\setminus P}\rangle$ for some m. This time x_ℓ is a vertex of the simplex $\langle (x_{t+1})^c_{\text{Vert}(\Gamma')\setminus P}\rangle$ as well as of every other simplex $\langle (P_j)^c_{\text{Vert}(\Gamma')\setminus E_1}\rangle$ not contained in $\langle (x_{t+1})^c_{\text{Vert}(\Gamma')\setminus P}\rangle$. Therefore $(\Gamma')^c_{\text{Vert}(\Gamma')\setminus E_1}$ is a cone over the vertex x_ℓ, and thus is contractible.

If x_0 is a leaf, then $y_1y_2 \cdots y_n$ is a part of a longer line graph containing x_0. In this case the attachment of this longer line graph should be considered as whole: as either a glue or a contraction with a different attaching vertex x_0 which is not a leaf. □

In the final section, we state our main result and apply the constructions above to an example, completely determining the Betti numbers of the minimal free resolution of its path ideal.

4. Minimal free resolutions

We observe, any tree Γ' can be constructed from one of its subtrees, Γ, through a sequence of line graph attachments. Therefore, we only need to consider the situation where Γ' differs from Γ by exactly one line graph. If the line graph consists of an attachment of more than $t+1$ vertices, we further decompose the attachment into a sequence of some number of glues as well as possibly either a split or a contraction. We will also impose order on the creation of Γ' such that when a new line graph is adjoined, the level of x_0 (i.e. the level of the glue, split, or contraction) is greater than $\text{ht}(\Gamma) - t$. Thus, under this model, any directed graph Γ' containing the directed graph Γ as a subgraph, could have been created from Γ through a sequence of glues, splits, and contractions.

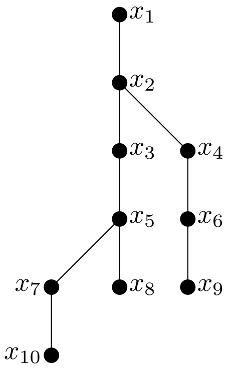
By extension and given the conventions of a glue and split to an empty graph, any clean graph can be created from a sequence of glues, splits, and contractions starting with an empty graph for some $t \geq 2$. As the empty graph trivially satisfies the conditions on Propositions 3.1, 3.3, and 3.5, we have that all clean, directed graphs can be constructed from a sequence of glues, splits, and contractions. Given a subforest Λ of Γ, let $g(\Lambda)$ and $s(\Lambda)$ denote the number of glues and splits, respectively, required to construct Λ from an empty graph. Then we have the following theorem.

Theorem 4.1. *Let Γ be a directed, rooted tree, and let $t \geq 2$. Then $\beta_{i,\mathbf{a}}(I_t(\Gamma)) = 1$ precisely when $\{x_j \mid a_j \neq 0\}$ corresponds to the vertex set of an induced subforest, Λ, of Γ constructed from the empty graph by a sequence of only glues and splits. Moreover, in this case $i = 2g(\Lambda) + s(\Lambda) - 1$.*

Proof. Since multigraded Betti numbers correspond to induced subforests of Γ, the result follows immediately from Propositions 3.1, 3.3, and 3.5. □

We illustrate Theorem 4.1 with an example.

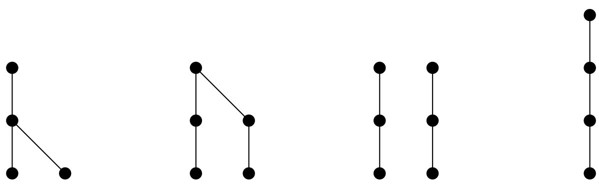
Example 4.2. *Consider the tree Γ depicted below. The graded minimal free resolution of $I_3(\Gamma)$ is also provided below and was obtained using Macaulay 2 (see [6]).*



$$I_3(\Gamma) = (x_1x_2x_3, x_2x_3x_5, x_3x_5x_7, x_3x_5x_8, x_5x_7x_{10}, x_1x_2x_4, x_2x_4x_6, x_4x_6x_9)$$

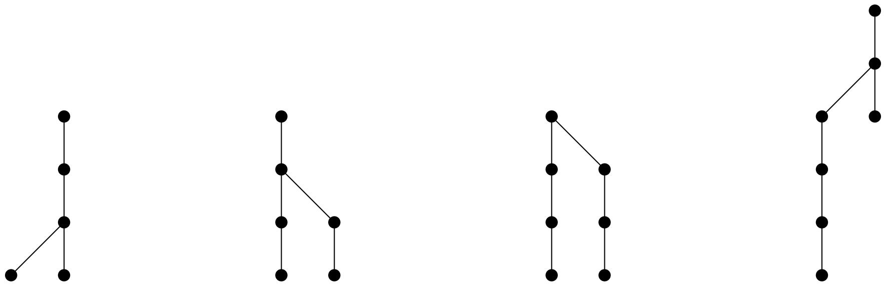
$$0 \to R^3(-8) \to \begin{matrix} R(-5) \\ \oplus \\ R(-6) \\ \oplus \\ R^8(-7) \end{matrix} \to \begin{matrix} R^8(-4) \\ \oplus \\ R(-5) \\ \oplus \\ R^5(-6) \end{matrix} \to R^8(-3) \to I_3(\Gamma) \to 0$$

The Betti numbers, $\beta_{0,3}(I_3(\Gamma))$ correspond to all paths in Γ of length 2, which are all induced subforests of Γ formed as a split from the empty graph. The Betti numbers, $\beta_{1,j}(I_3(\Gamma))$, correspond to all induced subtrees of Γ formed either by two splits from the empty graph or by one glue from the empty graph. Hence, these Betti numbers correspond to induced subforests of Γ of the following form:

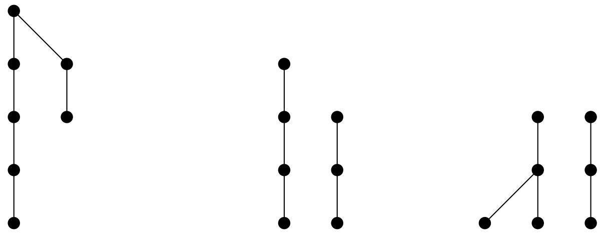


Two Splits (2 of this type) *Two Splits (1 of this type)* *Two Splits (5 of this type)* *One Glue (6 of this type)*

The Betti numbers, $\beta_{2,j}(I_3(\Gamma))$, correspond to all induced subforests of Γ formed either by three splits from the empty graph or by one glue and one split from the empty graph.

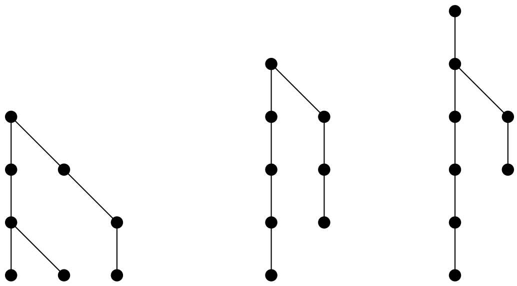


One Glue & One Split (1 of this type) *One Glue & One Split (1 of this type)* *One Glue & One Split (2 of this type)* *One Glue & One Split (1 of this type)*



One Glue & One Split (1 of this type) *One Glue & One Split (3 of this type)* *Three Splits (1 of this type)*

Lastly, the Betti numbers, $\beta_{3,j}(I_3(\Gamma))$, correspond to all induced subforests of Γ formed from either four splits, from one glue and two splits, or from two glues. The only possible induced subforests of these types in Γ are depicted below:



One Glue & Two Splits (1 of this type) *Two Glues (1 of this type)* *Two Glues (1 of this type)*

It should be noted that the vertex sets of the induced subforests correspond to the multi-graded Betti numbers in the minimal free resolution of $I_3(\Gamma)$.

Theorem 4.1 also implicitly describes the projective dimension and regularity of the $I_t(\Gamma)$. We have the following corollaries.

Corollary 4.3. *Let Γ be a directed, rooted tree and let $t \geq 2$, and let*

$$\begin{aligned}\mathcal{C} \ := \ \{\Lambda \ \mid \ \Lambda \textit{ is an induced subforest of } \Gamma \textit{ constructible from} \\ \textit{the empty graph using only splits and glues}\}.\end{aligned}$$

Then the projective dimension of $I_t(\Gamma)$ is given by

$$\mathrm{pd}(I_t(\Gamma))) = \max\{2g(\Lambda) + s(\Lambda) \ \mid \ \Lambda \in \mathcal{C}\} - 1.$$

and the Castelnuovo-Mumford regularity of $I_t(\Gamma)$ is given by

$$\mathrm{reg}(I_t(\Gamma)) = \max\{|\operatorname{Vert}(\Lambda)| - (2g(\Lambda) + s(\Lambda)) \ \mid \ \Lambda \in \mathcal{C}\} + 1.$$

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